Singularity theory - Exercise class 6

1) R = comm. k-alg, $R = I_0 \supset I_1 \supset I_2 \supset \dots = I_\bullet$ ← ideals

$J \subseteq R$ ideal s.th R/J = finite dim k-v.sp.

Then: $\dim_k(R/J) = \sum_{i \geq 0} \dim_k \left(\frac{I_i + J}{I_{i+1} + J} \right)$.

(a) $I, J \subseteq R$ ideals $\Rightarrow$ $\frac{I}{I \cap J} \cong \frac{I+J}{J}$ (2° thm of iso) :

$I + J = \{ i + j \mid i \in I, j \in J \} \supseteq I, J, I \cap J$

$I + J \subseteq R$ ideal

Consider the quotient morphism $q : R \twoheadrightarrow R/J$

$q' : I \to R/J$ ($I \subseteq R$) restriction of q to I

* $\ker(q') = I \cap J$
* $\mathrm{Im}(q') = \{\text{cosets } i + J, i \in I\} = \frac{I+J}{J}$ $\Rightarrow$ $I/_{I \cap J} \cong \frac{I+J}{J}$.

(b) $R \overset{q}{\twoheadrightarrow} R/J \leftarrow$ f.d. k-v.sp.

$\cup\!|$

$I_\bullet$

<u>To show</u> : $I_\bullet$ induces a filtration of k-v.sp. on R/J.

$$\begin{array}{ccc} R & \xrightarrow{q} & R/J \\ \cup\!| & & \\ I_i & \longrightarrow & q(I_i) \\ \cup\!| & & \cup\!| \\ I_{i+1} & \longrightarrow & q(I_{i+1}) \end{array}$$

$\rightsquigarrow$ $q(I_\bullet)$ is a filtration on R/J of R-mod

$\Rightarrow$ it is a filtration of k-v.sp., of <u>finite dim</u> ($\subseteq R/J$)

$$q(I_\bullet) = I_\bullet / I_\bullet \cap J \overset{(a)}{=} \frac{I_\bullet + J}{J}.$$

(c) $V = F_0 V \supsetneq F_1 V \supsetneq F_2 V \supsetneq \dots$ is finite for dimensional reasons: $V = F_0 V \supset \dots \supset F_k V \supset 0 = F_{k+1} V$

$gr_i^F V := F_i V / F_{i+1} V$. To show: $\dim_k(V) \overset{(*)}{=} \sum_{i \geq 0} \dim_k gr_i^F(V)$.

Then: $R/J \supset \frac{I_\bullet + J}{J}$ filtration of k-v.sp

(if $(*)$ holds true)

$\Rightarrow \dim_k(R/J) = \sum_i \dim_k gr_i\left(\frac{I_\bullet + J}{J}\right) = \sum_i \dim_k \underbrace{\left(\frac{I_i + J}{J} \Big/ \frac{I_{i+1} + J}{J}\right)}_{\text{3° thm of iso } \cong \frac{I_i + J}{I_{i+1} + J}}$.

$0 \to F_k V \to F_{k-1} V \to \underbrace{F_{k-1}V / F_k V}_{gr_{k-1}} \to 0$

$F_k V = F_k V / F_{k+1} V = gr_k$

$\Rightarrow \dim_k F_{k-1} = \dim_k gr_k + \dim_k gr_{k-1}$.

$0 \to F_{k-1} V \to F_{k-2} V \to gr_{k-2} \to 0 \Rightarrow \dim_k F_{k-2} V = \dim_k gr_{k-2} + \dim_k gr_{k-1} + \dim_k gr_k$

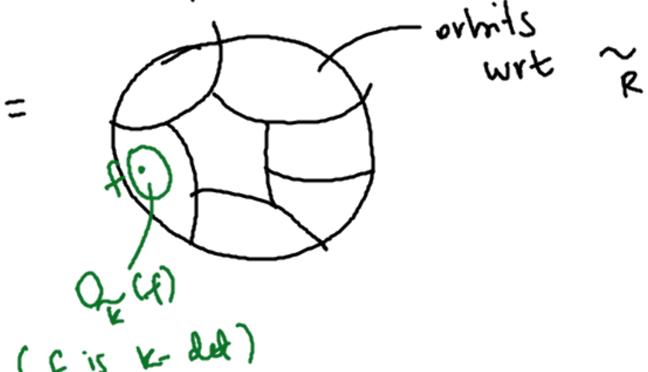
Go on ... last step: $0 \to F_1 V \to V \to gr_0 \to 0 \rightsquigarrow (*)$.

2) R_n the usual local algebra

We have two equiv. relations: • $\underset{R}{\sim}$ = right equiv.

• $\underset{k}{\sim}$ = k-equiv: $g \sim_k f$ iff $T^k_g = T^k_f$.

$f \in R$, $O_{\underset{R}{\sim}}(f)$ = orbit of f wrt. $\underset{R}{\sim}$, $O_{\underset{k}{\sim}}(f)$ = orbit of f wrt $\underset{k}{\sim}$

$R =$ 

f is k-determined $\iff O_{\underset{k}{\sim}}(f) \subset O_{\underset{R}{\sim}}(f)$

$$O_{\underset{k}{\sim}}(f) \subset O_{\underset{R}{\sim}}(f)$$
$$\parallel$$
$$O_{\underset{k}{\sim}}(T^k_f) \implies O_{\underset{k}{\sim}}(T^k_f) \subset O_{\underset{R}{\sim}}(T^k_f)$$

Conclusion: f is k-det $\iff T^k_f$ is k-determined. But we know that some polyn. are k-determined:

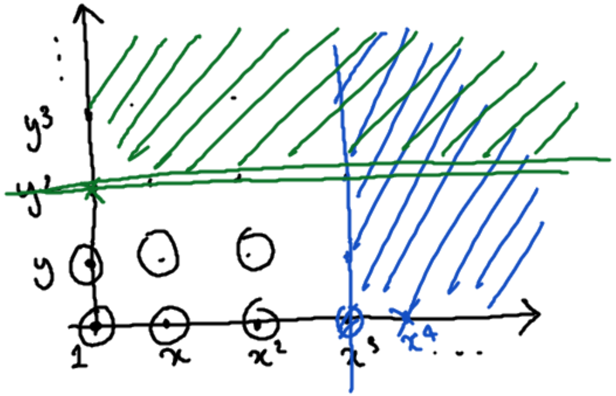
* x is 1-determ in R_n
* $\pm x_1^2 \pm x_2^2 \pm \dots \pm x_n^2$ is 2-det in R_n

(a) $f = x^4 + y^3 \in R_2$

$Df = (4x^3, 3y^2) \overset{!}{=} (0,0) \iff (x,y) = (0,0)$ isolated sing.

$J_f = (x^3, y^2)$ Jacobi ideal. $\mu(f) := \dim_k \left(R_2 / J_f \right)$. Milnor number

$$R_2/J_f \cong k[x,y]/(x^3, y^2)$$



$\Rightarrow \mu(f) = 6.$

in $k[x,y]$: $1, x, y, xy, x^2, \dots$ linearly indep over k

in $k[x,y]/(x^3, y^2)$: $[1], [x], [y], [xy], \dots$ are generators over k, but maybe not linearly independent.

Then: Corollary 3.7 $\Rightarrow$ f is $(\underbrace{\mu(f)+1}_{7})$ - determined - Is f k-det. for $k<7$?

* Look at T^k_f: $T^1_f = T^2_f = 0$ $\Rightarrow$ f is not 1/2-determ.

$f = x^4 + y^3$ not k-det

* Also: assume f is 3-det; $f \sim_3 T^3_f \Rightarrow f \underset{R}{\sim} T^3_f \overset{\text{Lemma 3.8}}{\Longrightarrow} \mu(f) = \mu(T^3_f)$.

$\mu(T^3_f) = ?$ $T^3_f = y^3$, $DT^3_f = (0, 3y^2) \rightsquigarrow J_{T^3_f} = (y^2)$

$\dim_k R_2/(y^2) = \infty \Rightarrow \mu(T^3_f) = \infty \neq \mu(f) = 6$ ↯.

* Is f 4-det? Thm 3.6: $m^k \subset m^2 J_f \Rightarrow f$ is $(k-1)$-det.

$m^2 \cdot J_f = (x^2, xy, y^2)\cdot(x^3, y^2) = (x^5, x^2y^2, x^4y, xy^3, y^2x^3, y^4) \supseteq m^5 = (x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5)$

$\Rightarrow$ f is 4-determ. $\Longrightarrow$ 4 = determinacy of f.

(b) $f = x^3 + y^5 + x \in R_2$

$Df = (3x^2+1, 5y^4) = (0,0)$ isolated sings. (0 is not a sing)

$J_f = (3x^2+1, y^4) = R_2 \Rightarrow \mu(f) = \dim_k \left(R_2/J_f \right)^{0} = 0$

invertible in R_2

$\Rightarrow f$ is 1-determined $\Rightarrow$ 1 = determinacy of f.

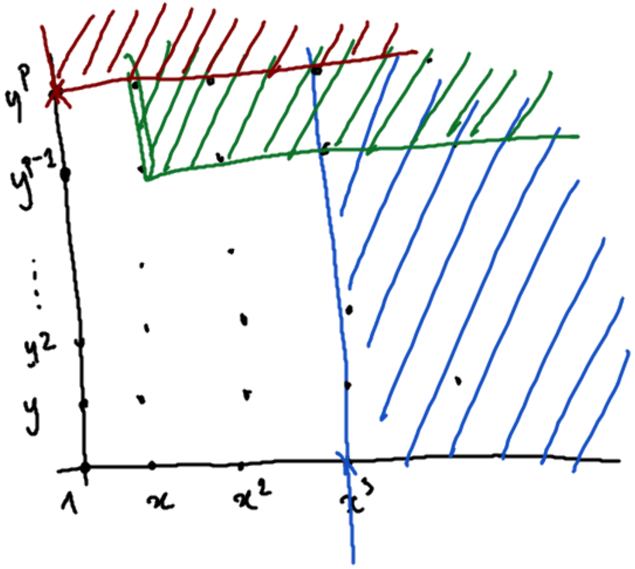
Also: $T^1_f = x$ is 1-determined $\Leftrightarrow$ f is 1-determined.

(c) $f = x^2 - y^2 \in R_3$

$Df = (2x, -2y, 0)$ non isolated sing! The sing's are $\{(0,0,z)\}$ = z-axis.

3) (a) $f = x^3 + xy^p \in R_2$, $p > 2$ $(p \geq 2)$

$Df = (3x^2 + y^p, \; pxy^{p-1}) \rightsquigarrow J_f = (3x^2 + y^p, \; xy^{p-1})$



$\cancel{x^3}$: $\begin{cases} x(3x^2 + y^p) = 3x^3 + xy^p \in J_f \\ y(xy^{p-1}) = xy^p \in J_f \end{cases}$

$3x^2 = -y^p$

no relations left

$\rightsquigarrow R_2/J_f$ has as k-basis

$$\begin{array}{llll} 1 & y & y^2 \ldots & y^{p-1} \\ x & xy & \ldots\ldots & xy^{p-2} \\ x^2 & \ldots\ldots & & x^2y^{p-2} \end{array}$$

$\Rightarrow \dim_k R_2/J_f = p + 2(p-1) = 3p-2.$
$= \mu(f).$

(c) $f = x^2 + y^2 + 2xy \in R_2$

$Df = (2x+2y,\ 2y+2x) \rightsquigarrow J_f = (x+y)$. $\quad R_2/(x+y) = ?$

Consider: $R_2 \xrightarrow{\alpha} R_1$ morph. of k-algebras

$f(x,y) \longmapsto f(x,-x)$

* $\alpha(x+y) = 0 \Rightarrow (x+y) \subseteq \ker\alpha$
* α is surj: $g \in R_1 \rightsquigarrow g(x) \in R_2,\ \alpha(g(x)) = g(x)$
  $\quad$ "$g(x)$

$\Rightarrow R_2/(x+y) \twoheadrightarrow R_1$ morph. of k-alg $\Rightarrow$ k-v.sp.

$\uparrow$ because $(x+y) \subseteq \ker\alpha$

$\Rightarrow \dim_k(R_2/(x+y)) \geq \dim_k(R_1) = \infty$

$\Rightarrow \mu(f) = \infty$.