HW2 - Singularity theory

Ex 1: $U \subseteq \mathbb{R}^n$ open, $0 \in U$, $f: U \to \mathbb{R}$ smooth, $\varphi: U \to U$ local diffeo + $\underline{\underline{\varphi(0) = 0}}$.

0 is a critical point of f, i.e. $Df(0) = 0$.

$g: U \to \mathbb{R}$, $\quad g(y) := f(\varphi(y))$.

Show: $D^2 g(0)$ and $D^2 f(0)$ have the same rank and index.

Goal: $D^2(f \circ \varphi)(0) = \underbrace{(D\varphi)^t(0)}_{\text{inv}} D^2 f(0) \underbrace{D\varphi(0)}_{\text{inv}}$. $\quad (*)$

$$(C^t A C)_{ij} = ?$$

$$(AC)_{ij} = \sum_k a_{ik} \cdot c_{kj} \quad \rightsquigarrow \quad (C^t A C)_{ij} = \sum_\ell \underbrace{c^t_{i\ell}}_{= c_{\ell i}} \cdot \underbrace{\sum_k a_{\ell k} c_{kj}}_{(AC)_{\ell j}} = \sum_{k,\ell} c_{\ell i}\, a_{\ell k}\, c_{kj}.$$

Then, since $(D\varphi)_{i,j} = \dfrac{\partial \varphi_i}{\partial x_j}$, on the right of $\circledast$ we have:

$$\sum_{k,l} \frac{\partial \varphi_l}{\partial x_i}(0)\, \frac{\partial^2 f}{\partial x_l \partial x_k}(0)\, \frac{\partial \varphi_k}{\partial x_j}(0).$$

On the other hand, on the left of $\circledast$ we have:

$$\left(D^2(f\circ\varphi)(0) \right)_{i,j} = \left(\frac{\partial^2 (f\circ\varphi)}{\partial x_i \partial x_j}(0) \right) = \left(\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}(f\circ\varphi) \right)(0) \right) =$$

$$= \left(\frac{\partial}{\partial x_i}\left(\sum_k \frac{\partial f}{\partial x_k}\circ\varphi \cdot \frac{\partial \varphi_k}{\partial x_j} \right)(0) \right) =$$

$$= \left(\sum_k \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_k}\circ\varphi \right)(0) \cdot \frac{\partial \varphi_k}{\partial x_j}(0) + \overbrace{\frac{\partial f}{\partial x_k}\circ\underbrace{\varphi(0)}_{0}\cdot(\dots)}^{0} \right) =$$

$$= \left(\sum_{l,k} \left(\frac{\partial}{\partial x_l}\frac{\partial f}{\partial x_k} \right)\varphi(0) \cdot \frac{\partial \varphi_l}{\partial x_i}(0) \cdot \frac{\partial \varphi_k}{\partial x_j}(0) \right) = \left(\sum_{k,l} \frac{\partial^2 f}{\partial x_l \partial x_k}(0)\, \frac{\partial \varphi_l}{\partial x_i}(0)\, \frac{\partial \varphi_k}{\partial x_j}(0) \right).$$

Exercise 2: $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 - y^2 - xy^2 + x^2 y$.

(a) What are the rank and the index of $D^2 f(0)$?

$$\frac{\partial f}{\partial x} = 2x - y^2 + 2xy \,, \quad \frac{\partial f}{\partial y} = -2y - 2xy + x^2$$

$$\rightsquigarrow D^2 f(x,y) = \begin{pmatrix} 2+2y & -2y+2x \\ -2y+2x & -2-2x \end{pmatrix}$$

$$\Rightarrow D^2 f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \leftarrow \begin{array}{l} \mathrm{rk} = 2 \\ \text{index} = 1. \end{array}$$

(b) Show that $\exists$ local diffeo φ at 0 in $\mathbb{R}^2$ which transforms f into $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x,y) = x^2 - y^2$.

Morse Lemma: $f : U \to \mathbb{R}$ smooth at 0 with $f(0)=0$. Then:
$U \subset \mathbb{R}^n$

0 is a non-deg critical point $\iff \exists\, \varphi$ local diffeo of $\mathbb{R}^n$ around 0, $\varphi(0)=0$, s.th.

$$f(\varphi(x_1, \dots, x_n)) = x_1^2 + \dots + x_k^2 - x_{k+1}^2 - \dots - x_n^2$$

where $n-k$ = index of $(D^2 f)(0)$.

In our case: $(x,y) \rightsquigarrow f(\varphi(x,y)) = x^2 - y^2$.

Are we under the hypotheses of Morse Lemma?

- f smooth in 0 ✓
- $f(0)=0$ ✓
- $Df(0) = (0,0)$, $D^2 f(0)$ has max rk ✓

$\Rightarrow$ we get the statement.

Explicit diffeo?

$f(x,y) = x^2 - y^2 - xy^2 + x^2 y = x^2(1+y) - y^2(1+x)$

$\rightsquigarrow$ define $\varphi: B_\varepsilon(0) \longrightarrow B_\varepsilon(0)$ $\qquad |\varepsilon| < 1$

$$(x,y) \longmapsto \left(x \cdot \sqrt{1+y}\,,\ y\sqrt{1+x}\right)$$

$$g(x,y) = x^2 - y^2, \qquad g: \mathbb{R}^2 \to \mathbb{R}$$

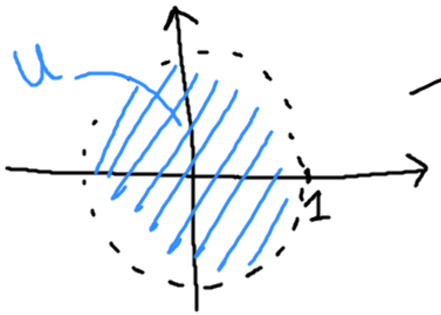
$$g \circ \varphi(x,y) = g\left(x\sqrt{1+y}\,,\ y\sqrt{1+x}\right) = x^2(1+y) - y^2(1+x) = f(x,y).$$

Is φ a local diffeo around 0? Compute $D\varphi$:

$$D\varphi = \begin{pmatrix} (1+y)^{1/2} & \frac{1}{2} x (1+y)^{-1/2} \\ \frac{1}{2} y (1+x)^{-1/2} & (1+x)^{1/2} \end{pmatrix} \rightsquigarrow D\varphi(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \checkmark.$$

$\llcorner$ $\det \neq 0$

Exercise 3 :



$U \xrightarrow{f} \mathbb{R}$

$$f(x,y) = \sqrt{1-x^2-y^2} = (1-x^2-y^2)^{1/2}$$

(a) $$\frac{\partial f}{\partial x} = \frac{1}{\cancel{2}}(1-x^2-y^2)^{-1/2}(-\cancel{2}x) = \frac{-x}{\sqrt{1-x^2-y^2}}$$

$$\frac{\partial f}{\partial y} = -\frac{y}{\sqrt{1-x^2-y^2}}$$

$$(Df)(\bar{x},\bar{y}) = (0,0) \iff \bar{x}=\bar{y}=0 \iff \underset{(0,0)}{\overset{}{0}} \in U \text{ is the only critical point.}$$

(b) $(x_0,y_0) \in U \setminus \mathrm{Crit}(f)$. Construct a local diffeo around (x_0,y_0) which transforms f into a linear function.

$(x_0,y_0) \neq (0,0)$, wlog $y_0 \neq 0$.

We would like to apply:

<u>Lemma 1.7</u> (lecture 3): $U \subseteq \mathbb{R}^n$ star shaped with center $0 \in U$. $f: U \to \mathbb{R}$ smooth, $f(0)=0$. Then:

$$Df(0) \neq 0 \iff \exists \text{ local diffeo } \varphi \text{ at } 0,\ \varphi(0)=0, \text{ s.th. } f(\varphi(\underline{y})) = y_1 \quad \forall\, \underline{y}=(y_1,\dots,y_n) \in \underset{\substack{\downarrow \\ 0}}{V} \subseteq U.$$

<u>Rmk</u>: $U =$  is star shaped wrt any $(x_0,y_0) \in U$ ✓.

<u>Rmk</u>: Our f does not fit the hypotheses of Lemma 1.7 ($Df(0)=0$).

$(x_0,y_0)\in U$ as before $(y_0\neq 0)$.

Consider: $F(x,y) := f(x+x_0, y+y_0) - \overbrace{f(x_0,y_0)}^{c\in\mathbb{R}}$

then: (1) $F(0,0)=0$

$$(2)\quad DF = \begin{pmatrix} \frac{\partial f}{\partial x}(x+x_0, y+y_0) \\ \frac{\partial f}{\partial y}(x+x_0, y+y_0) \end{pmatrix} \Rightarrow DF(0) = Df(x_0,y_0) \neq (0,0).$$

$(x_0,y_0)\notin \mathrm{Crit}(f)$

Lemma 1.7

$\Longrightarrow \exists\, \varphi: V\to V$, $\varphi(0)=0$, $F(\varphi(x,y)) = x \quad \forall (x,y)\in V.$

0

Expl: $\varphi(x,y) = (\underbrace{F(x,y)}_{\varphi_1(x,y)}, \underbrace{x}_{\varphi_2(x,y)})$.

(look inside the proof of Lemma 1.7)

Then: $x = F(\varphi(x,y)) = f(\varphi_1(x,y) + x_0 , \varphi_2(x,y) + y_0) - c$

Call: $\tilde{\varphi}(x,y) = (\varphi_1(x,y) + x_0 , \varphi_2(x,y) + y_0)$ local diffeo

and

$(f \circ \tilde{\varphi})(x,y) = x + c$ linear. ("affine" in the language of linear algebra)

where: $\tilde{\varphi}(x,y) = (\underbrace{f(x+x_0, y+y_0)}_{\sqrt{1-(x+x_0)^2-(y+y_0)^2}} - c + x_0 , x + y_0).$