Singularity theory - Exercise class 5

(1) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $\quad f(x,y) = x^2$.

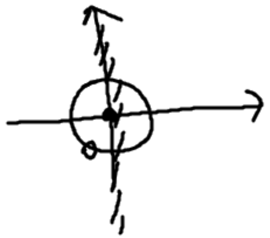
(a) $\frac{\partial f}{\partial x} = 2x$, $\quad \frac{\partial f}{\partial y} = 0$ $\quad \rightsquigarrow Df = (2x, 0)$

$\rightsquigarrow Df(0) = (0,0) \Rightarrow 0$ is a singularity for f.

Here: $Df = (2x, 0) = (0,0) \iff x = 0$

$\rightsquigarrow$ every point on the y-axis $\{x = 0\}$ is a singular pt

$\Rightarrow 0$ is a non-isolated singularity.



(b) $f \in R = \begin{cases} \mathcal{E}_n \\ \mathcal{O}_n \end{cases}$ is k-determined iff $\forall g \in R$, $T^k_f = T^k_g \Rightarrow f \sim g$.

they are equal after a change of coordinates

Show: f of point (a) is not k-determined, $\forall k \geq 1$.

$f(x,y) = x^2$

* What is T^k_f ?

$$T^k_f = \sum_{\substack{\nu \in \mathbb{N}^2 \\ |\nu| \leq k}} \frac{1}{\nu!} D^\nu_0 f \, \underline{x}^\nu \quad , \quad \nu = (\nu_1, \nu_2), \quad D^\nu_0 f = \frac{\partial^{\nu_1+\nu_2} f}{\partial x_1^{\nu_1} \partial x_2^{\nu_2}}(0)$$

$\underline{x} = (x,y)$

$\frac{\partial f}{\partial x}(0) = 0$, $\frac{\partial f}{\partial y}(0) = 0$, $\frac{\partial^2 f}{\partial^2 x}(0) = 2$, all the other derivatives, in 0, are 0.

$$T^0_f = f(0) = 0 \quad , \quad T^1_f = 0 \quad , \quad T^2_f = \frac{1}{2!} \frac{\partial^2 f}{\partial^2 x}(0) x^2 = x^2 = T^k_f \quad \forall k \geq 2.$$

$T_f^0 = T_f^1 = 0, \quad T_f^k = x^2 \qquad \forall k \geq 2.$

* $\forall \ell \in \mathbb{N}, \quad g_\ell : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad g_\ell(x,y) = x^2 - y^{2\ell}$

Fix k, take $\ell > \frac{k}{2} \rightsquigarrow k < 2\ell$

then: $\begin{cases} T^k_{g_\ell} = 0 & \text{if } k < 2 \\ T^k_{g_\ell} = x^2 & \text{if } k \geq 2 \end{cases} \Rightarrow T^k_{g_\ell} = T^k_f$

<u>Conclusion</u>: $\forall k \geq 0$ we can find ℓ s.th. $T^k_f = T^k_{g_\ell}$ (enough to take $\ell > \frac{k}{2}$)

<u>Then</u>: f k-determined $\Rightarrow f \sim g_\ell$ for that choice of ℓ. Is this true?

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be a local diffeo around 0, $\varphi = (\varphi_1, \varphi_2)$, $\varphi_i : \mathbb{R}^2 \to \mathbb{R}$.
$(\varphi(0) = 0)$

Then: $f(\varphi(x,y)) = f(\varphi_1(x,y), \varphi_2(x,y)) = \varphi_1(x,y)^2 \stackrel{?}{=} g_e(x,y)$

take $x = 0$ $\rightsquigarrow$ $f(\varphi(0,y)) = \varphi_1(0,y)^2$

$\neq$ for $y \neq 0$

but: $g_e(0,y) = -y^{2e}$

$(g_e(x,y) = x^2 - y^{2e})$

$\Rightarrow$ we do not find a change of coordinates s.th. $f \circ \varphi = g_e$.

2) $\mathcal{G} = \{$ germs around 0 of local diffeo/biholo. at 0, leaving the origin invariant $\}$

a change of coordinates

$\mathbb{R}^n / \mathbb{C}^n$

(a) • $\mathcal{G}$ is a group wrt composition:

the composition is well defined; the neutral element is the germ of the identity.

• $$\begin{aligned} R \times \mathcal{G} &\longrightarrow R \\ (f, \varphi) &\longmapsto f \circ \varphi \end{aligned}$$

this composition makes sense because $\varphi(0) = 0$

is a well-def right group action. Write: $f * \varphi = f \circ \varphi$

we need to check:

$$(f * \varphi_1) * \varphi_2 \overset{?}{=} f * (\varphi_1 \circ \varphi_2)$$

$$(f \circ \varphi_1) * \varphi_2 = f \circ \varphi_1 \circ \varphi_2 \overset{\checkmark}{=} f \circ \varphi_1 \circ \varphi_2$$

(b) $f \sim g \iff f$ and g are in the same orbit under the action of $\mathcal{G}$.
$(f, g \in R)$

$\Rightarrow f \sim g$ is an equivalence relation, called "right-equivalence".

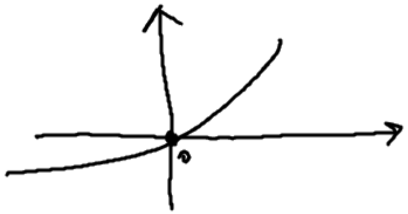
$\underline{\underline{R = \mathcal{E}_n}}$: relation between right-equivalence and k-equivalence ?

$$f \sim_k g \iff T_f^k = T_g^k .$$

(i) Ex 1 : $f \not\sim g_\ell \quad \forall \ell \in \mathbb{N}$ but $f \sim_k g_\ell$ when $\ell > \frac{k}{2}$.

$\rightsquigarrow$ k-equivalence $\not\Rightarrow$ righ-equiv.

(ii) $f(x) = 2(e^x - 1)$
$f : \mathbb{R} \to \mathbb{R}$



f is invertible around 0 !
the inverse is a local diffeo φ, $\varphi(0) = 0$

$\Rightarrow f \circ \varphi = id$

$\Rightarrow f \sim id$.

But: $T_{id} = 0 + 1\cdot x = x \Rightarrow T^k_{id} = x \quad \forall k \geq 1$

$\left(\begin{array}{l} id : \mathbb{R} \to \mathbb{R} \\ \quad x \mapsto x \end{array}\right)$

$T_f = 0 + 2\cdot x + \frac{2}{2!}x^2 + \dots \Rightarrow \forall k \geq 1 \quad T^k_f \neq T^k_{id}$

$(f(x) = 2(e^x - 1))$

$\Rightarrow f \not\sim_k id \quad \forall k \geq 1 \quad \rightsquigarrow$ right-equiv $\not\Rightarrow$ k-equiv, $\forall k \geq 1$.

(c) $m \subseteq R$ max ideal ; $\quad m^\infty * \mathcal{G} \subseteq m^\infty$

$m = \{ f \in R \mid f(0) = 0 \}$, $\quad m^\infty = \bigcap_{k \geq 1} m^k$

* $\mathcal{G}$ preserves m : $\quad f \in m, \ \varphi \in \mathcal{G} \quad f \circ \varphi(0) = f(\overset{0}{\overset{\shortparallel}{\varphi(0)}}) = f(0) = 0$ ✓
* $\forall \varphi \in \mathcal{G}, \quad \begin{array}{c} R \xrightarrow{\varphi^*} R \\ f \longmapsto f \circ \varphi \end{array}$ is an algebra homo (ex 4.3.(a)) $\quad [\Rightarrow \mathcal{G}$ preserves $m^k \ \forall k$ $\Rightarrow$ preserves $m^\infty]$

$$\varphi^*: R \longrightarrow R \qquad f \longmapsto f\circ\varphi$$

is a algebra homo :

- $\varphi^*(f+g) = (f+g)\circ\varphi = f\circ\varphi + g\circ\varphi = \varphi^*(f) + \varphi^*(g)$
- $\varphi^*(f\cdot g) = (f\cdot g)\circ\varphi = (f\circ\varphi)\cdot(g\circ\varphi) = \varphi^*(f)\cdot\varphi^*(g)$
- $\varphi^*(\alpha f) = \alpha f\circ\varphi = \alpha\,\varphi^*(f)$, $\alpha \in \mathbb{R}/\mathbb{C}$

$\underline{\underline{m}}^\infty$ is preserved by the action of $\mathcal{G}$ $\Longrightarrow$ $\mathcal{G}$ acts on the quotient $R/_{m^\infty}$.

(d) $R = \mathcal{E}_1$; what are the orbits of the action of $\mathcal{G}$ on $\mathcal{E}/_{m^\infty_\varepsilon}$?

Lemma 1.6: $0 \in U \subseteq \mathbb{R}$, $f: U \to \mathbb{R}$ smooth s.th. $f^{(k)}(0) \neq 0$ for some $k \geq 0$. If k is the smallest k s.th. $f^{(k)}(0) \neq 0$, $\exists\, \varphi$ local diffeo at 0, $\varphi(0) = 0$, s.th $f\circ\varphi = \pm x^k$.

To show: $\pm x^k$ are representatives of the orbits.

* $[\alpha] \in \mathcal{E}/\mathfrak{m}^\infty$, $[\alpha] \neq [0]$ $\rightsquigarrow$ $\alpha \notin \mathfrak{m}^\infty$, then $\exists k$ s.th. $\alpha^{(k)}(0) \neq 0$
$\Rightarrow$ we can apply the Lemma, to map $[\alpha]$ to $[\pm x^k]$

* $[x^k] \neq [x^\ell]$ $\quad k \neq \ell$
$[x^k] = [x^\ell]$ $\Leftrightarrow$ they have the same derivatives in 0 $\Leftrightarrow$ $k = \ell$.

3) $m_\varepsilon^\infty \subseteq \mathcal{E}$ is not finitely generated.

Nakayama's Lemma: (R, m) local ring, $M =$ f.g. R-mod s.th. $M = mM$ $(M \subset mM)$.

Then: $M = \{0\}$.

We want to use it in the following way:

* $M = m_\varepsilon^\infty$ is a module: m_ε is an ideal $\Rightarrow m_\varepsilon^k$ is an ideal $\Rightarrow m^\infty = \bigcap m_\varepsilon^k$ is an ideal ($\Rightarrow$ it is a $\mathcal{E}$-mod).
* $m_\varepsilon^\infty = m_\varepsilon \cdot m_\varepsilon^\infty$ $(M \subset mM)$
* we know $m_\varepsilon^\infty \neq \{0\}$

$\Rightarrow m_\varepsilon^\infty$ is not f.g.

Let's check it: $m_\varepsilon^\infty \subset m_\varepsilon \cdot m_\varepsilon^\infty$

f

Lemma 1.4: $\exists\, f(x) = \sum_{i=1}^{n} g_i(x)\, x_i$, where $g_i(x) = \int_0^1 \frac{\partial}{\partial x_i} f(tx)\,dt$

$\mathbb{R}^n$ $\quad m_\varepsilon$ $\quad$ Taylor formula

we only need to check: $g_i(x) \in m_\varepsilon^\infty$.

We want to check that $g_i \in m_\varepsilon^\infty = \bigcap m_\varepsilon^k \iff D^\nu g_i(0) = 0 \quad \forall \nu \in \mathbb{N}^n$

$\left[\; g_i(x) = \int_0^1 \frac{\partial}{\partial x_i} f(tx)\, dt \; \right]$, $f \in m_\varepsilon^\infty$

$g_i(0) = 0$ because $\frac{\partial}{\partial x_i} f(0) = 0$ $(\Leftarrow f \in m_\varepsilon^\infty)$

and all the derivatives of g_i are 0 in 0 (def g_i + $f \in m_\varepsilon^\infty$)

$\Rightarrow \quad g_i \in m_\varepsilon^\infty$.