Exercise 1:

V = real vector space, $h: V \times V \to \mathbb{R}$ symm. bilinear map. Show that there exists a basis $\{v_1, \dots, v_n\}$ of V s.th.

$$H = \left(h(v_i, v_j)\right)_{\substack{i=1,\dots,n \\ j=1,\dots,n}} = \begin{pmatrix} \boxed{Id} & & 0 \\ & \boxed{-Id} & \\ 0 & & \boxed{0} \end{pmatrix}$$

(a) $Rad(h) := \{ v \in V \mid h(v, w) = 0 \quad \forall w \in V \}$

<u>To show</u>: $V = Rad(h) \oplus U \Rightarrow h|_U$ is non deg.

Assume $\exists u \in U$ s.th. $h(u, u') = 0 \quad \forall u' \in U$.

For any $v \in V$, write $v = v_1 + v_2$, $v_1 \in Rad(h)$, $v_2 \in U$.

$h(u,v) = h(u,v_1) + h(u,v_2) = 0$, and this holds true $\forall v \in V$

$v_1 \in Rad(h) \to$ $h(u,v_1) = 0$; $v_2 \in U \to$ $h(u,v_2) = 0$

$\Rightarrow u \in Rad(h)$

But $u \in U$, and $U \cap Rad(h) = (0) \Rightarrow u = 0$.

(b) h = non-deg., $w \in V$ s.th. $h(w,w) \neq 0$; $W = (\mathbb{R}w)^{\perp}$

Goal: $h|_W$ is non deg.

First step: $V = \mathbb{R}w \oplus W$, i.e.:

orthogonal (obviously, by def)

1. $\langle \mathbb{R}w, W \rangle = V$
2. $\mathbb{R}w \cap W = (0)$.

1. Take $v \in V$. Since $h(w,w) \neq 0$, we can write:

$$v = \underbrace{\frac{h(v,w)}{h(w,w)} w}_{\in \mathbb{R}w} + \underbrace{\left(v - \frac{h(v,w)}{h(w,w)} w\right)}_{\in W:}$$

$$\boxed{h\left(v - \frac{h(v,w)}{h(w,w)} w, w\right) = h(v,w) - \frac{h(v,w)}{\cancel{h(w,w)}} \cancel{h(w,w)} = 0 \quad \checkmark}$$

2. $v \in \mathbb{R}w \cap W \iff \begin{cases} h(v,w) = 0 \\ v = aw \end{cases} \iff a \underbrace{h(w,w)}_{\neq 0} = 0 \Rightarrow a = 0$

$\Rightarrow v = 0$.

$\Rightarrow$ We have the direct sum. Is $h|_W$ non deg?

assume that $h|_W$ is deg $\Rightarrow \exists \underset{\substack{\neq \\ 0}}{u} \in W$ s.th $h(u,u')=0$
$\forall u' \in W$.

Take $v \in V$ (any) $\rightsquigarrow v = aw + \underbrace{u'}_{\in W}$ (because $V = \mathbb{R}w \oplus W$).

$h(v,u) = \underset{\substack{\parallel \\ 0}}{h(aw,u)} + \underset{\substack{\parallel \\ 0}}{h(u,u')} = 0 \Rightarrow u \in Rad(h) = (0)$. ⚡

$u \in W$ $u' \in W$

(u was $\neq 0$ by assumption)

(c) <u>Goal</u>: show that there exists a basis $\{v_1, \dots, v_n\}$ of V s.th. H has the form

matrix representing h in the basis $\{v_1, \dots, v_n\}$

$$\begin{pmatrix} Id_{k\times k} & & 0 \\ & -Id_{l\times l} & \\ 0 & & 0_{r\times r} \end{pmatrix}.$$

Also, show that such k, l, r don't dep on the choice of the basis.

Start writing: $V = \underbrace{\mathrm{Rad}(h)} \oplus U \overset{(a)}{\Longrightarrow} h|_U$ is non deg.

here the matrix of h is the 0-matrix! It corresp. to the last block in the previous page

Then: $\exists\, u \in U$ s.th $h(u,u) \neq 0$.

Why? If $h(u,u) = 0 \ \forall u \in U$, then $h(u,v) = 0 \ \forall v \in U$:

$$h(u,v) = \frac{1}{2}\Big(h(u+v,u+v) - h(u,u) - h(v,v)\Big) \ ↯$$

$h(u,u) \begin{cases} > 0 \\ < 0 \end{cases}$ $\rightsquigarrow$ up to divid. by $\sqrt{|h(u,u)|}$, we get $h(u,u) = \begin{cases} 1 \\ -1 \end{cases}$,

and $U = \mathbb{R}\cdot u \oplus u^{\perp}$.

(b)

$\Rightarrow$ $h|_{u^\perp}$ is non-deg $\Rightarrow$ we can go on by induction,

until $\dim(u^\perp) = 1$ ($\leftarrow$ base of the induction).

In this case, it is enough to do a rescaling.

As conclusion, we have a basis s.th.

$$H = \begin{pmatrix} \boxed{Id_{k\times k}} & & 0 \\ & \boxed{-Id_{\ell\times\ell}} & \\ 0 & & \boxed{0_{r\times r}} \end{pmatrix}.$$

We want to show that (k,ℓ,r) does not depend on the choice of the basis.

1. $r = \dim \mathrm{Rad}(h) \Rightarrow r$ is an invariant.

Why?

For any basis $\{v_1, \ldots, v_n\}$ of V giving "such an H":

$$\mathrm{Rad}(h) = \mathrm{Span}_{\mathbb{R}}\{v_{k+\ell+1}, \ldots, v_n\} =: S$$

<u>Indeed</u>: "$\supseteq$" trivial ; "$\subseteq$" ?

assume $\exists u \in Rad(h)$, $u \notin S \rightsquigarrow u = a_1 v_1 + \ldots + a_n v_n$,

where $\exists a_i \neq 0$, $i \leq k+l$

$\rightsquigarrow h(u, v_i) = \pm a_i \neq 0$ $\lightning$.

$i \leq k+l$

2. What about k and l?

<u>Rmk</u>: it is not true that $k = \dim\{\underbrace{v \in V \text{ s.th. } h(v,v) > 0}_{V^+}\}$

or better: V^+ is not a v.sp.!

<u>Ex</u>: $V \cong \mathbb{R}^2$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ wrt $\{v_1, v_2\}$

then: $v_1 \in V^+$, $\underbrace{2v_1 + v_2}_{v_3} \in V^+$, but $v_3 - 2v_1 = v_2 \notin V^+$!

Then: let's go back to the exercise.

Assume to have:

$\{v_1, \ldots, v_n\}$ s.th. $\qquad \{v'_1, \ldots, v'_n\}$ s.th

$$H = \begin{pmatrix} \boxed{Id_{k\times k}} & & 0 \\ & \boxed{-Id_{\ell\times\ell}} & \\ 0 & & \boxed{0_{r\times r}} \end{pmatrix}, \text{ and } \begin{pmatrix} \boxed{Id_{k'\times k'}} & & 0 \\ & \boxed{-Id_{\ell'\times\ell'}} & \\ 0 & & \boxed{0_{r\times r}} \end{pmatrix} = H$$

Consider: $\mathrm{Span}_{\mathbb{R}}\{v_1, \ldots, v_k\} =: \underset{\substack{\cap \\ V}}{W}$, $\mathrm{Span}_{\mathbb{R}}\{v'_{k+1}, \ldots, v'_n\} =: \underset{\substack{\cap \\ V}}{W'}$

$W \cap W' = (0)$: $\forall \underset{\neq 0}{u} \in W \cap W'$, $\begin{cases} h(u,u) > 0 \leftarrow u \in W \\ h(u,u) \leq 0 \leftarrow u \in W' \end{cases}$ $\lightning$

$\Rightarrow k + \ell' + r \leq n = k + \ell + r \quad \Rightarrow \ell' \leq \ell$

But also $\ell \leq \ell'$ (reversing) $\Rightarrow \ell = \ell' \Rightarrow k = k'$.

Exercise 3:

Take: $g \in \mathbb{R}[x]$, $p \in \mathbb{N}$, $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) := y^p - g(x)$$

(a) Show: $\mathrm{Crit}(f) \cap f^{-1}(0) = \{(a,0) \mid g \text{ has a multiple root at } a\}$

$(p \geq 2)$

$$\mathrm{Crit}(f) := \left\{(a,b) \in \mathbb{R}^2 \;\middle|\; \frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0\right\}.$$

$$(a,b) \in \mathrm{Crit}(f) \cap f^{-1}(0) \iff \begin{cases} f(a,b) = 0 = b^p - g(a) \\ \dfrac{\partial f}{\partial x}(a,b) = -g'(a) = 0 \\ \dfrac{\partial f}{\partial y}(a,b) = p\, b^{p-1} = 0 \quad (p-1 \geq 1) \\ \quad \Leftrightarrow b = 0 \end{cases}$$

then: $\begin{cases} f(a,0) = -g(a) = 0 \\ -g'(a) = 0 \end{cases} \iff \begin{cases} g(a) = 0 \\ g'(a) = 0 \end{cases}$ $(*)$

<u>To show:</u>

$(*) \iff$ a is a multiple root of g:

"$\Leftarrow$": $g(x) = (x-a)^{m} \cdot h(x)$, $m \geq 2$

$\rightsquigarrow g'(x) = m(x-a)^{m-1} h(x) + (x-a)^m h'(x)$, $m-1 \geq 1$

$\Rightarrow$ a is a root of g'

"$\Rightarrow$": assume that a is a root, but not a multiple one

$\rightsquigarrow g(x) = (x-a)h(x)$, where $h(a) \neq 0$

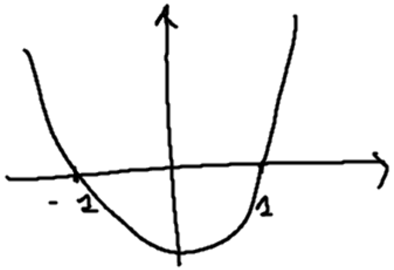
$\rightsquigarrow g'(x) = h(x) + (x-a)h'(x) \rightsquigarrow g'(a) = h(a) \neq 0$ $\lightning$.

(b) $f: \mathbb{R}^2 \to \mathbb{R}$, $\quad f(x,y) = y^p - g(x)$, $\quad p \in \mathbb{N}$, $g \in \mathbb{R}[x]$.

Draw $f^{-1}(0) \subseteq \mathbb{R}^2$:

1) $p = 1$, $\quad g = x^2 - 1 \quad \rightsquigarrow \quad y = x^2 - 1$

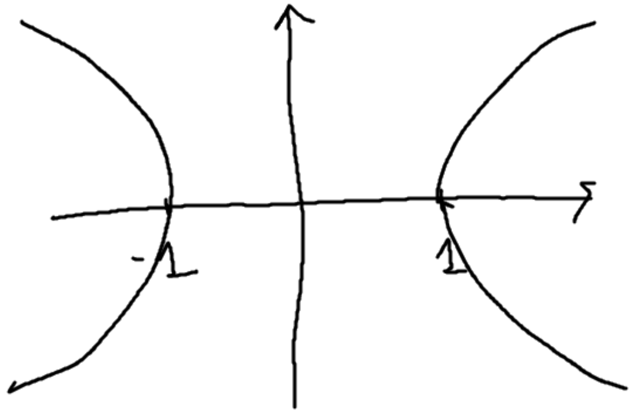
parabola



2) $p = 2$, $\quad g = x^2 - 1$

$\rightsquigarrow \quad y^2 - x^2 + 1 = 0$

hyperbola



3) $p = 2$, $g = x^2(1 - x^2)^3$

$y^2 = x^2(1-x^2)^3$

$f^{-1}(0) \ni (x,y) \Leftrightarrow (x,-y) \in f^{-1}(0)$

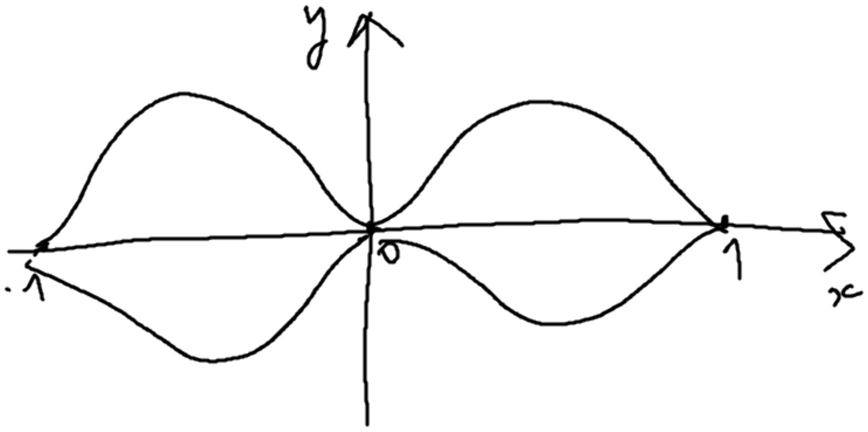
$\Leftrightarrow (-x,y) \in f^{-1}(0)$

$\rightsquigarrow$ assume $x \geq 0$ & $y \geq 0$

$\rightsquigarrow$ we want to understand

$y = x \cdot (1-x^2)^{3/2}$

$y \geq 0$, $x \in [-1,1]$



Exercise 1

- $x^3, \; x^4, \; x^5, \; x^6 \quad = \quad f(x)$

we need to compute the derivative :

$3x^2, \; 4x^3, \; 5x^4, \; 6x^5 \quad = f'(x)$

$f'(x) = 0 \iff x = 0.$

- $f(x,y) = x^3 + y^3$

$\frac{\partial f}{\partial x} = 3x^2 \quad , \quad \frac{\partial f}{\partial y} = 3y^2$

$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (0,0) \iff (x,y) = (0,0)$

- $f(x,y) = x^3 - xy^2$

$\frac{\partial f}{\partial x} = 3x^2 - y^2$, $\frac{\partial f}{\partial y} = -2xy$

$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = 0 \iff \begin{cases} xy = 0 \\ 3x^2 - y^2 = 0 \end{cases} \Rightarrow x \text{ or } y = 0$

$\Downarrow$

both $x = y = 0$.

- $f(x,y) = x^2 y - y^4$

$\frac{\partial f}{\partial x} = \underset{0}{\underset{\shortparallel!}{2xy}}$, $\frac{\partial f}{\partial y} = \underset{0}{\underset{\shortparallel!}{x^2 - 4y^3}}$ $\Rightarrow x = y = 0$.