Technische Universität Chemnitz, Fakultät für Mathematik
Christian Sevenheck

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Exercises to "Introduction to $\mathcal{D}$-modules"

1. Let $M \in \mathrm{Mod}(\mathbb{A}^n)$ and consider the projection $p : \mathbb{A}^n \twoheadrightarrow \mathbb{A}^{n-1}$ given by $(x_1, \ldots, x_n) \mapsto (x_2, \ldots, x_n)$. Let $(\mathbb{A}^n)^\vee$ be the dual space and let $i : (\mathbb{A}^{n-1})^\vee \hookrightarrow (\mathbb{A}^n)^\vee$ be the closed embedding given by $(y_2, \ldots, y_n) \mapsto (0, y_2, \ldots, y_n)$. Recall that $\widehat{\ } : \mathrm{Mod}(\mathcal{D}_{\mathbb{A}^r}) \to \mathrm{Mod}(\mathcal{D}_{(\mathbb{A}^r)^\vee})$ denotes the algebraic Fourier transformation. Show that
$$\widehat{\mathcal{H}^k p_+ M} \cong \mathcal{H}^k i^+ \widehat{M}$$
holds for all $k \in \mathbb{Z}$. There are two (slightly) different proofs of this fact, one using the direct definition of the Fourier transformation (i.e. where for $N \in \mathrm{Mod}(\mathcal{D}_{\mathbb{A}^n})$ we let $\widehat{N} = N$ as $\mathbb{C}$-vector spaces with actions $y_i \cdot := -\partial_{x_i}$ and $\partial_{y_i} := x_i$, if $(y_1, \ldots, y_n)$ are coordinates on $(\mathbb{A}^n)^\vee$ dual to coordinates $(x_1, \ldots, x_n)$ of $\mathbb{A}^n$), the other one using the defintion $\widehat{M} := \pi_{2,+}\left(\pi_1^+ M \otimes \mathcal{E}^{can}\right)$, where
$$\begin{array}{ccc} & \mathbb{A}^n \times (\mathbb{A}^n)^\vee & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ \mathbb{A}^n & & (\mathbb{A}^n)^\vee \end{array}$$
and where $can : \mathbb{A}^n \times (\mathbb{A}^n)^\vee$ is the canonical pairing (given by $can = \sum_{i=1}^n x_i y_i$)). For this second proof, use the base change formula (see Hotta, Theorem 1.7.3).

2. Let $(\mathcal{M}, \nabla)$ be an integrable connection on an algebraic manifold X. Consider $\mathcal{M}$ as a holonomic left $\mathcal{D}_X$-module. Show that its holonomic dual (denoted by $\mathbb{D}\mathcal{M}$ in the lecture) is the dual connection, i.e. its underlying $\mathcal{O}_X$-module is $\mathcal{M}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X)$ (this was already shown in the lecture) together with its dual connection $-\nabla^{tr}$, which is the natural connection on the Hom-sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X)$ (or otherwise said which is given in a local basis $f_1, \ldots, f_n$ of $\mathcal{M}^\vee$ dual to a basis $e_1, \ldots, e_n$ of $\mathcal{M}$ by the matrix $-A^{tr}$ if $A \in \mathrm{Mat}(d \times d, \mathcal{O}_X)$ is the matrix of the connection ∇, that is, if we have $\nabla \underline{e} = \underline{e} \cdot A$). Hint: go through the proof in the lecture and follow carefully the left- resp. right-module structures.

3. Consider the algebraic manifold $X = (\mathbb{C}^*)^2$ with coordinates (z, t) and write $\mathcal{D} := \mathbb{C}[t^\pm, z^\pm]\langle \partial_t, \partial_z \rangle = \Gamma(X, \mathcal{D}_X)$. Since X is affine, we work with $\mathcal{D}$-modules instead of sheaves of $\mathcal{D}_X$-modules. Put
$$P_1 := z^n \prod_{i=1}^n (t\partial_t - \alpha_i) - t \qquad\qquad P_2 := z\partial_z + nt\partial_t$$
and consider the left $\mathcal{D}$-module $\mathcal{M} := \mathcal{D}/\mathcal{D}(P_1, P_2)$.

   (a) Show that $\mathcal{M}$ is holonomic.

   (b) Calculate $\mathbb{D}\mathcal{M}$ by exhibiting a cyclic presentation for it, i.e. give an isomorphism $\mathbb{D}\mathcal{M} = \mathcal{D}/J$ for some left ideal $\mathcal{J} \subset \mathcal{D}$ (hint: calculate explicitly a resolution of $\mathcal{M}$ by free left $\mathcal{D}$-modules and apply $Hom_{\mathcal{D}}(-, \mathcal{D})$ to it).

Lecture notes, exercise etc. to be found at

`https://www.tu-chemnitz.de/mathematik/algebra/DModules-SS19/DModules.php`