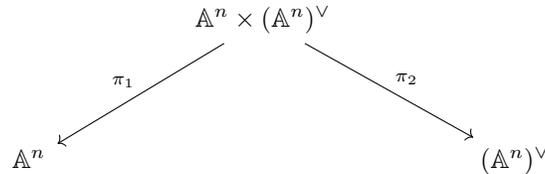


Exercises to “Introduction to \mathcal{D} -modules”

- Let $M \in \text{Mod}(\mathbb{A}^n)$ and consider the projection $p : \mathbb{A}^n \rightarrow \mathbb{A}^{n-1}$ given by $(x_1, \dots, x_n) \mapsto (x_2, \dots, x_n)$. Let $(\mathbb{A}^n)^\vee$ be the dual space and let $i : (\mathbb{A}^{n-1})^\vee \hookrightarrow (\mathbb{A}^n)^\vee$ be the closed embedding given by $(y_2, \dots, y_n) \mapsto (0, y_2, \dots, y_n)$. Recall that $\widehat{} : \text{Mod}(\mathcal{D}_{\mathbb{A}^r}) \rightarrow \text{Mod}(\mathcal{D}_{(\mathbb{A}^r)^\vee})$ denotes the algebraic Fourier transformation. Show that

$$\widehat{\mathcal{H}^k p_+ M} \cong \mathcal{H}^k i^+ \widehat{M}$$

holds for all $k \in \mathbb{Z}$. There are two (slightly) different proofs of this fact, one using the direct definition of the Fourier transformation (i.e. where for $N \in \text{Mod}(\mathcal{D}_{\mathbb{A}^n})$ we let $\widehat{N} = N$ as \mathbb{C} -vector spaces with actions $y_i \cdot := -\partial_{x_i}$ and $\partial_{y_i} := x_i$, if (y_1, \dots, y_n) are coordinates on $(\mathbb{A}^n)^\vee$ dual to coordinates (x_1, \dots, x_n) of \mathbb{A}^n), the other one using the definition $\widehat{M} := \pi_{2,+}(\pi_1^+ M \otimes \mathcal{E}^{can})$, where



and where $can : \mathbb{A}^n \times (\mathbb{A}^n)^\vee$ is the canonical pairing (given by $can = \sum_{i=1}^n x_i y_i$). For this second proof, use the base change formula (see Hotta, Theorem 1.7.3).

- Let (\mathcal{M}, ∇) be an integrable connection on an algebraic manifold X . Consider \mathcal{M} as a holonomic left \mathcal{D}_X -module. Show that its holonomic dual (denoted by $\mathbb{D}\mathcal{M}$ in the lecture) is the dual connection, i.e. its underlying \mathcal{O}_X -module is $\mathcal{M}^\vee := \text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X)$ (this was already shown in the lecture) together with its dual connection $-\nabla^{tr}$, which is the natural connection on the Hom-sheaf $\text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X)$ (or otherwise said which is given in a local basis f_1, \dots, f_n of \mathcal{M}^\vee dual to a basis e_1, \dots, e_n of \mathcal{M} by the matrix $-A^{tr}$ if $A \in \text{Mat}(d \times d, \mathcal{O}_X)$ is the matrix of the connection ∇ , that is, if we have $\nabla \underline{e} = \underline{e} \cdot A$). Hint: go through the proof in the lecture and follow carefully the left- resp. right-module structures.
- Consider the algebraic manifold $X = (\mathbb{C}^*)^2$ with coordinates (z, t) and write $\mathcal{D} := \mathbb{C}[t^\pm, z^\pm]\langle \partial_t, \partial_z \rangle = \Gamma(X, \mathcal{D}_X)$. Since X is affine, we work with \mathcal{D} -modules instead of sheaves of \mathcal{D}_X -modules. Put

$$P_1 := z^n \prod_{i=1}^n (t \partial_t - \alpha_i) - t \qquad P_2 := z \partial_z + nt \partial_t$$

and consider the left \mathcal{D} -module $\mathcal{M} := \mathcal{D}/\mathcal{D}(P_1, P_2)$.

- Show that \mathcal{M} is holonomic.
- Calculate $\mathbb{D}\mathcal{M}$ by exhibiting a cyclic presentation for it, i.e. give an isomorphism $\mathbb{D}\mathcal{M} = \mathcal{D}/J$ for some left ideal $J \subset \mathcal{D}$ (hint: calculate explicitly a resolution of \mathcal{M} by free left \mathcal{D} -modules and apply $\text{Hom}_{\mathcal{D}}(-, \mathcal{D})$ to it).