Exercises to "Introduction to $\mathcal{D}$-modules"

1. Let $f : \mathbb{A}^n \to \mathbb{A}^1_t$ be a polynomial and denote by $i_f : \mathbb{A}^n \hookrightarrow \mathbb{A}^1_t \times \mathbb{A}^n$ its graph embedding.
   (a) Verify (using the definition of the direct image discussed before) the two expressions given in the lecture for the graph embedding module $i_{f,+}\mathcal{O}_{\mathbb{A}^n}$.
   (b) Check (along the lines given in the lecture, i.e, by decomposing $f = p \circ i_f$) that the direct image complex $f_+\mathcal{O}_{\mathbb{A}^n} \in D^b(\mathcal{D}_{\mathbb{A}^1_t})$ is represented by
   $$\left(f_*\Omega^{\bullet+n}_{\mathbb{A}^n}[\partial_t], d - (df \wedge - \otimes \partial_t)\right).$$

2. Let $M \in \mathrm{Mod}(\mathcal{D}_{\mathbb{A}^1_t})$. Check that the Fourier transform $\widehat{M}$ of M (i.e. the $\mathbb{C}$-vector space M together with the action of $\tau\cdot := -\partial_t\cdot$ and $\partial_\tau\cdot := t\cdot$) can be defined by
   $$\widehat{M} := H^0 q_+ \left(p^+ M \otimes_{\mathcal{O}_{\mathbb{A}^1_t \times \mathbb{A}^1_\tau}} \mathcal{E}^{t\cdot\tau}\right),$$
   where $p : \mathbb{A}^1_\tau \times \mathbb{A}^1_t \twoheadrightarrow \mathbb{A}^1_t$, where $q : \mathbb{A}^1_\tau \times \mathbb{A}^1_t \twoheadrightarrow \mathbb{A}^1_\tau$ and where $\mathcal{E}^{t\cdot\tau}$ is the free rank 1 $\mathcal{O}_{\mathbb{A}^1_t \times \mathbb{A}^1_\tau}$-module with connection $\nabla := d + d(t\cdot\tau)$. Check also that $H^i q_+ \left(p^+ M \otimes_{\mathcal{O}_{\mathbb{A}^1_t \times \mathbb{A}^1_\tau}} \mathcal{E}^{t\cdot\tau}\right) = 0$ for $i \neq 0$.

3. Show that for $\mathcal{M} \in \mathrm{Mod}_c(\mathcal{D}_X)$, the characteristic variety $\mathrm{char}(\mathcal{M})$ does not depend on the choice of a good filtration $F_\bullet\mathcal{M}$. Hints:
   (a) First show that if we have two filtration $F_\bullet\mathcal{M}, G_\bullet\mathcal{M}$ such that $F_\bullet$ is good, then there is some $a \in \mathbb{N}$ such that $F_i\mathcal{M} \subset G_{i+a}\mathcal{M}$ for all $i \in \mathbb{Z}$.
   (b) Conclude that any two good filtrations $F_\bullet\mathcal{M}, G_\bullet\mathcal{M}$ are contained in each other in the sense that there is some $i_0 \in \mathbb{N}$ such that $G_{i-i_0}\mathcal{M} \subset F_i\mathcal{M} \subset G_{i+i_0}\mathcal{M}$ for all $i \in \mathbb{Z}$.
   (c) Now deduce the independence of the characteristic variety from the choice of a good filtration.

4. (a) Let $i : X \hookrightarrow Y$ be a smooth algebraic subvariety (of a smooth algebraic variety Y). Show that we have $\mathrm{char}(i_+\mathcal{O}_X) = T^*_X Y$, where
   $$T^*_X Y := \{(\xi, x) \in T^*Y \,|\, \xi(v) = 0 \;\; \forall v \in T_x X\}$$
   is the *conormal bundle* of Y in X.
   (b) Show that $T^*_X Y \subset T^*X$ is a lagrangian submanifold, thus proving that $i_+\mathcal{O}_X$ is holonomic.

Lecture notes, exercise etc. to be found at

`https://www.tu-chemnitz.de/mathematik/algebra/DModules-SS19/DModules.php`