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# Exercises to "Introduction to $\mathcal{D}$-modules"

1. Show that if $j : U \hookrightarrow X$ is an open embedding of smooth algebraic varieties then:

   (a) For any $M \in \mathrm{Mod}(\mathcal{D}_X)$, we have
   $$j^+M = j^*M = j^{-1}M \in \mathrm{Mod}(\mathcal{D}_U)$$

   (b) For any $N \in \mathrm{Mod}(\mathcal{D}_U)$, we have
   $$j_+N = Rj_*N \in D^b(\mathcal{D}_X).$$

2. Let $i : X \hookrightarrow Y$ be a closed embedding, given in local coordinates by $(x_1, \ldots, x_r) \hookrightarrow (x_1, \ldots, x_r, 0, \ldots, 0) =: (y_1, \ldots, y_n)$. Use the computation of the transfer module $\mathcal{D}_{Y \leftarrow X}$ from the last sheet to check that for $M \in \mathrm{Mod}(\mathcal{D}_X)$, we have $i_+M \cong (i_*M)[\partial_{y_{r+1}}, \ldots, \partial_{y_n}]$ as $\mathbb{C}$-vector spaces. Make the left action of $\mathcal{D}_Y$ on $i_+M$ explicit.

3. Consider the Spencer complex
   $$\mathrm{Sp}^\bullet(\mathcal{D}_X) = (\ldots \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^k \Theta_X \to \ldots)$$
   resp. the de Rham complex
   $$\mathrm{DR}^\bullet(\mathcal{D}_X) = (\ldots \to \Omega^k_X \otimes_{\mathcal{O}_X} \mathcal{D}_X X \to \ldots)$$
   of $\mathcal{D}_X$. Show that the contraction
   $$\begin{array}{rcl} \omega_X \otimes_{\mathcal{O}_X} \left(\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^k \Theta_X\right) & \longrightarrow & \Omega^{n-k}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \\ \omega \otimes 1 \otimes \theta & \longmapsto & (-1)^{(n-k)\frac{n-k-1}{2}} \omega(\theta \wedge -) \otimes 1 \end{array}$$
   is an isomorphism of right $\mathcal{D}_X$-modules (make the right structure on both sides explicit) and induces an isomorphism of complexes $[\mathrm{Sp}^\bullet(\mathcal{D}_X)]^r \cong \mathrm{DR}^\bullet(\mathcal{D}_X)$, where $[\mathrm{Sp}^\bullet(\mathcal{D}_X)]^r$ denotes the complex of right $\mathcal{D}_X$-modules associated to the Spencer complex of $\mathcal{D}_X$ (which is a complex of left $\mathcal{D}_X$-modules).

4. Let $X = Y \times Z \twoheadrightarrow Y$ be a projection. Show that the relative de Rham complex $\mathrm{DR}^\bullet_{X/Y}(\mathcal{D}_X)$ of $\mathcal{D}_X$ is a resolution by free right $\mathcal{D}$-modules of the transfer module $\mathcal{D}_{Y \leftarrow X}$. Deduce that for any left $\mathcal{D}_X$-module $M$, we have
   $$\mathcal{D}_{Y \leftarrow X} \otimes^{\mathbb{L}}_{\mathcal{D}_X} M \cong \mathrm{DR}^\bullet_{X/Y}(M)$$
   as left $f^{-1}\mathcal{D}_Y$-modules.

Lecture notes, exercise etc. to be found at

`https://www.tu-chemnitz.de/mathematik/algebra/DModules-SS19/DModules.php`