Exercises to "Introduction to $\mathcal{D}$-modules"

1. Consider the filtration $F_k\mathcal{D}_X$ defined locally by

$$F_k\mathcal{D}_X(U) = \sum_{|\underline{\alpha}|\leq k} \mathcal{O}_X(U)\partial_{\underline{x}}^{\underline{\alpha}}$$

for an affine open set $U \subset X$ with local coordinate system $x_1, \ldots, x_n$. Show

   (a) $F_k\mathcal{D}_X \subset F_{k+1}\mathcal{D}_X$,
   (b) $\mathcal{D}_X = \bigcup_{k\in\mathbb{N}} F_k\mathcal{D}_X$,
   (c) each $F_k\mathcal{D}_X$ is $\mathcal{O}_X$-locally free,
   (d) $F_0\mathcal{D}_X = \mathcal{O}_X$,
   (e) $F_k\mathcal{D}_X \cdot F_l\mathcal{D}_X = F_{k+l}\mathcal{D}_X$,
   (f) for all local sections $P \in F_k\mathcal{D}_X$, $Q \in F_l\mathcal{D}_X$ we have $[P, Q] \in F_{k+l-1}\mathcal{D}_X$,
   (g) we have the charactarization

$$F_k\mathcal{D}_X = \{P \in \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X) \,|\, \forall f \in \mathcal{O}_X : [P, f] \in F_{k-1}\mathcal{D}_X\}$$

Deduce that (b)+(g) gives an alternative definition of $\mathcal{D}_X$ which makes sense for singular varieties X (Grothendieck's definition).

2. Show that $\mathrm{Gr}^F_\bullet \mathcal{D}_X$ is a commutative sheaf of rings (graded by degree of symbols of operators). Show that it is a (sheaf of) $\mathcal{O}_X$-algebra(s) that can be identified with $\pi_*\mathcal{O}_{T^*X}$, where $\pi : T^*X \twoheadrightarrow X$ is the cotangent bundle of X.

3. Let $\mathcal{M}$ be an $\mathcal{O}_X$-module. Show that a left $\mathcal{D}_X$-module structure on $\mathcal{M}$ is uniquely determined by a morphism $\nabla : \Theta_X \to \mathcal{E}nd_{\mathbb{C}}(\mathcal{M})$ satisfying

   (a) $\nabla_{f\vartheta}(s) = f\nabla_\vartheta(s)$,
   (b) $\nabla_\vartheta(fs) = f\nabla_\vartheta(s) + \vartheta(f)\nabla_\vartheta(s)$,
   (c) $\nabla_{[\vartheta,\rho]}(s) = [\nabla_\vartheta, \nabla_\rho](s)$

for all local sections $f \in \mathcal{O}_X, \vartheta, \rho \in \Theta_X, s \in \mathcal{M}$. Show also that this is equivalent to having a morphism

$$\nabla : \mathcal{M} \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

which is $\mathbb{C}$-linear, such that $\nabla(fs) = f\nabla s + df \otimes s$ and such that $\nabla^{(2)} \circ \nabla = 0$, where $\nabla^{(2)} : \Omega^1_X \otimes \mathcal{M} \to \Omega^2_X \otimes \mathcal{M}$ denotes the extension sending $\alpha \otimes s$ to $d\alpha \otimes s - \alpha \wedge \nabla s$.

Similarly, show that a right $\mathcal{D}_X$-module structure on $\mathcal{M}$ is uniquely determined by a morphism $\nabla' : \Theta_X \to \mathcal{E}nd_{\mathbb{C}}(\mathcal{M})$ satisfying

   (a) $\nabla'_{f\vartheta}(s) = \nabla'_\vartheta(fs)$,
   (b) $\nabla'_\vartheta(fs) = f\nabla'_\vartheta(s) + \vartheta(f)\nabla'_\vartheta(s)$,
   (c) $\nabla'_{[\vartheta,\rho]}(s) = [\nabla'_\vartheta, \nabla'_\rho](s)$.

4. Show that the map

$$\Theta_X \times \Omega^n_X \longrightarrow \Omega_X$$

sending (ϑ, ω) to $-\mathrm{Lie}_\vartheta(\omega)$ puts a uniquely determined right $\mathcal{D}_X$-module structure on the canonical sheaf $\omega_X := \Omega^n_X$.

5. Let $\mathcal{M}, \mathcal{N}$ be left $\mathcal{D}_X$-modules and let $\mathcal{M}'$, $\mathcal{N}'$ be right $\mathcal{D}_X$-modules. Verify that

   (a) $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$,
   (b) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$,
   (c) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}', \mathcal{N}')$,

   are left $\mathcal{D}_X$-modules and that

   (a) $\mathcal{M}' \otimes_{\mathcal{O}_X} \mathcal{N}$,
   (b) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}')$,

   are right $\mathcal{D}_X$-modules, where in all cases we endow tensor products resp. homorphism sheaves with the action by Θ_X given in the lecture.

6. Check that putting
$$\mathcal{M}' := \omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$$
   for a left $\mathcal{D}_X$-module $\mathcal{M}$ and putting
$$\mathcal{N} := \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{N}')$$
   for a right $\mathcal{D}_X$-module $\mathcal{N}'$ gives an equivalence between the categories of left and right $\mathcal{D}_X$-modules.

Lecture notes, exercise etc. to be found at

https://www.tu-chemnitz.de/mathematik/algebra/DModules-SS19/DModules.php