Lecture 7: Duality and solution complexes

Let $M \in \mathrm{Mod}(\mathcal{D}_X)$ be a left $\mathcal{D}_X$-module.

From analogy with $\mathcal{O}_X$-module theory, one would like to consider $\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{D}_X)$ as its dual.

<u>However</u>: 1.) This is only a right $\mathcal{D}_X$-module (using right structure on $\mathcal{D}_X$) $\rightsquigarrow$ one should consider $\omega_X^\vee \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{D}_X)$

2.) <u>example</u>: let $X = \mathbb{A}^1$, $M = D_X/(P)$, $P \overset{\neq 0}{\in} D_X$

we have exact sequence of left D_X-modules

$$0 \to D_X \xrightarrow{\cdot P} D_X \to M \to 0$$

and by applying $\mathcal{H}om_{D_X}(-, D_X)$ we obtain exact sequence of right D_X-modules:

$$0 \to \mathcal{H}om_{D_X}(M, D_X) \to D_X \xrightarrow{P\cdot} D_X \to \mathcal{E}xt^1_{\mathcal{D}_X}(M, \mathcal{D}_X) \to 0$$

(since $\mathcal{E}xt^i_{D_X}(D_X, D_X) = \begin{cases} D_X & i=0 \\ 0 & \text{else} \end{cases}$)

BUT: $\ker(P\cdot : D_X \to D_X) = 0$!

Hence we have: $0 \to D_X \xrightarrow{P\cdot} D_X \to \mathcal{E}xt^1(\mathcal{M}, D_X) \to 0$

moreover, we have $\omega_X^\vee \otimes_{\mathcal{O}_X} \mathcal{E}xt_{D_X}(\mathcal{M}, D_X) = D_X / D_X \cdot {}^tP$,
where ${}^tP$ is the adjoint / transpose of P (see lecture 2)

So it makes sense to call the left D_X-module

$\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}xt^1_{D_X}(M, D_X)$ the dual of M.

Def.: Let $M \in D^b(\mathcal{D}_X)$, then define

$$\mathbb{D}M := R\mathcal{H}om_{\mathcal{D}_X}(M^\cdot, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \omega_X^\vee [\dim X]$$

then $\mathbb{D}M \in D^b(\mathcal{D}_X)$

ex.: $M = \mathcal{D}_X \Rightarrow \mathcal{H}^k \mathbb{D}M = \begin{cases} \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^\vee & k = -\dim X \\ 0 & \text{else} \end{cases}$

Fact: $\mathbb{D} : D^b_c(\mathcal{D}_X) \circlearrowleft$: use resolution by $<\infty$ rk modules

Lemma: $\mathbb{D}^2 \simeq \mathrm{Id}$ on $D^b_c(\mathcal{D}_X)$

<u>Pf</u>.: check that $\mathbb{D}^2\mathcal{M} \simeq R\mathcal{H}om_{\mathcal{D}_X^{opp}}(\underbrace{R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X)}_{\text{right mod}},\mathcal{D}_X)$

left-mod

put $\mathcal{N} := R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X)$, then we have can. morphism $\mathcal{M}\otimes_{\mathbb{C}}\mathcal{N} \to \mathcal{D}_X$, this is morphism in $D^b(\mathcal{D}_X\otimes_{\mathbb{C}}\mathcal{D}_X^{opp})$

we have: $R\mathcal{H}om_{\mathcal{D}_X\otimes_{\mathbb{C}}\mathcal{D}_X^{opp}}(\mathcal{M}\otimes_{\mathbb{C}}\mathcal{N},\mathcal{D}_X) \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\mathcal{H}om_{\mathcal{D}_X^{op}}(\mathcal{N},\mathcal{D}_X))$

hence (apply $H^0(-)$): $\mathrm{Hom}_{\mathcal{D}_X\otimes\mathcal{D}_X^{opp}}(\mathcal{M}\otimes_{\mathbb{C}}\mathcal{N},\mathcal{D}_X) = \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, R\mathcal{H}om_{\mathcal{D}_X^{op}}(\mathcal{N},\mathcal{D}_X))$

hence can. morphism gives $\mathcal{M}\to\mathbb{D}^2\mathcal{M}$.

To show: this is isomorphism: do it locally and replace $\mathcal{M}$ by (direct summand of) $\mathcal{D}_X^k$, i.e. by $\mathcal{D}_X$. Then statement is obvious. $\mathcal{M}\in D^b_c(\mathcal{D}_X)$

Relation: duality $\leftrightarrow$ characteristic variety

<u>Theorem</u>: $\mathcal{M}\in \mathrm{Mod}_c(\mathcal{D}_X)$, $\dim X = n$,

i) $\dim \mathrm{char}(\omega_X^\vee\otimes_{\mathcal{D}_X}\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X)) \le 2n-i$

ii) $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X) = 0 \quad \forall i < 2n-\dim\mathrm{char}(\mathcal{M})$

<u>Sketch of proof:</u> recall that for A reg. comm.

and $S \in \mathrm{Mod}_{f.g}(A)$, we have:

$$\underbrace{\dim S}_{\dim(\mathrm{ann}_A S)} + \overbrace{\min\{ i \mid \mathrm{Ext}^i_A(S, A) \neq 0 \}}^{= \operatorname{depth}(\mathrm{ann}_A S)} = \dim A \quad (*)$$

also: $\dim \mathrm{Ext}^i_A(S,A) \leq \dim A - i \quad (**)$

work locally and take good filtration $F_\cdot M$

Then one can show that $\mathrm{char}(\omega_X^\vee \otimes \mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)) \subset$ $\mathrm{supp}(\mathcal{E}xt^i_{gr^F_\cdot \mathcal{D}_X}(gr^F_\cdot \mathcal{M}, gr^F_\cdot \mathcal{D}_X))$. Hence

$$\dim \mathrm{char}(\omega_X^\vee \otimes \mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)) \leq \dim \mathrm{supp}(\mathcal{E}xt^i_{gr\mathcal{D}_X}(gr\mathcal{M}, gr\mathcal{D}_X)) \overset{(**)}{\leq} 2n - i$$

and if $i < 2n - \dim \mathrm{char}(\mathcal{M}) \overset{(*)}{\Longrightarrow} \mathcal{E}xt^i_{gr^F\mathcal{D}_X}(gr^F\mathcal{M}, gr^F\mathcal{D}_X) = 0$

$$\mathrm{char}(\omega_X^\vee \otimes \mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)) = \emptyset \Rightarrow \mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0$$

$\square$

Corollary: $\mathcal{M} \in \mathrm{Mod}_c(\mathcal{D}_X)$ holonomic $\iff \mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0 \ \forall i \neq n$

$$\left(\iff H^i(\mathbb{D}\mathcal{M}) = 0 \quad \forall i \neq 0 \right)$$

$$\iff \mathbb{D}\mathcal{M} = H^0(\mathbb{D}\mathcal{M}) \text{ is holonomic}$$

Proof: $\mathcal{M}$ holonomic $\iff \dim \mathrm{char}(\mathcal{M}) = n$

$$\overset{\text{Th.}}{\underset{ii)}{\Longrightarrow}} \mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0 \quad \forall i < n$$

on the other hand (Th. i)) $\dim \mathrm{char}\left(\omega_X \otimes \mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)\right) = n$

hence $\mathcal{M}$ hol. $\Rightarrow \mathbb{D}\mathcal{M}$ hol.

suppose: $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0 \ \forall i < n \implies \mathbb{D}\mathcal{M} = H^0(\mathbb{D}\mathcal{M}) =: \mathcal{M}^*$

by Th., 1): $\dim \mathrm{char}\left(\omega_X^\vee \otimes \mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}^*, \mathcal{D}_X)\right) = n$

Since $\mathcal{M} = \mathbb{D}^2\mathcal{M} = \mathbb{D}(\mathbb{D}\mathcal{M}) = \mathbb{D}(\mathcal{M}^*)$, we know that $\mathbb{D}\mathcal{M}^* = H^0\,\mathbb{D}\mathcal{M}^* = \omega_X^\vee \otimes \mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}^*, \mathcal{D}_X) \Rightarrow \mathcal{M}$ is hol.

$\square$

Example: let $(\mathcal{M},\nabla)$ be an integrable connection.

Then (exercise): $\mathcal{M}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{O}_X)$ is int. conn.

with operator given in local basis $f_1,\dots,f_n$ dual

to basis $e_1,\dots e_n$ of $\mathcal{M}$ by $-A^t$, if $\nabla \underline{e} = \underline{e}\cdot A$.

by definition, $(\mathcal{M},\nabla)$ is also left $\mathcal{D}_X$-module.

$\underline{Q}$: $\mathbb{D}(\mathcal{M},\nabla) = ?$

Recall Spencer complex $Sp^\bullet(\mathcal{D}_X) \xrightarrow{\simeq} \mathcal{O}_X$

$$\dots \to \mathcal{D}_X \otimes \Lambda^p \Theta_X \longrightarrow \mathcal{D}_X \otimes \Lambda^{p-1}\Theta_X \to \dots$$

$\mathcal{M}$ is $\mathcal{O}_X$-loc. free $\Rightarrow Sp^\bullet(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{\simeq} \mathcal{M}$

resolution of $\mathcal{M}$ by locally free left $\mathcal{D}_X$-mod.

$$\Rightarrow \mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X) = \mathcal{H}^n\left(\mathcal{H}om_{\mathcal{D}_X}(Sp^\bullet(\mathcal{D}_X)\otimes\mathcal{M},\mathcal{D}_X)\right)$$

__Now__: $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^p\Theta_X \otimes \mathcal{M}, \mathcal{D}_X) \simeq$

$\mathcal{H}om_{\mathcal{O}_X}(\Lambda^p\Theta_X \otimes \mathcal{M}, \mathcal{D}_X) \simeq$

$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \Omega^p_X \otimes \mathcal{D}_X)$ (iso of right $\mathcal{D}_X$-mod.)

Hence: $\mathbb{D}M = H^0 \mathbb{D}M \simeq \omega_X^\vee \otimes \mathcal{E}xt^n_{\mathcal{D}_X}(M, \mathcal{D}_X)$

$$= \omega_X^\vee \otimes \operatorname{coker}\left(\mathcal{H}om_{\mathcal{O}_X}(M, \Omega^{n-1} \otimes \mathcal{D}_X) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(M, \underset{\omega_X}{\overset{\shortparallel}{\Omega^n_X}} \otimes \mathcal{D}_X) \right)$$

recall that $\Omega^\cdot_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \xrightarrow{\simeq} \omega_X$ $\quad | \mathcal{H}om_{\mathcal{O}_X}(M, -)$ exact

$\Rightarrow \operatorname{coker}(-\shortparallel-) \simeq \mathcal{H}om_{\mathcal{O}_X}(M, \omega_X)$ right $\mathcal{D}_X$-mod

and $\mathbb{D}M \simeq \omega_X^\vee \otimes \mathcal{H}om_{\mathcal{O}_X}(M, \omega_X) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}(M, \mathcal{O}_X)$

$$(\varphi, \phi) \longmapsto (m \mapsto \varphi(\phi(m)))$$

<u>exercise</u>: check that $\mathcal{D}_X$-mod structure on $M^\vee$ is dual connection $-\nabla^\epsilon$.

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some facts without proof:

1.) $\forall M, N \in D^b_c(\mathcal{D}_X)$, we have

$$R\mathcal{H}om_{\mathcal{D}_X}(M, N) \simeq \omega_X \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} (\mathbb{D}M \overset{\mathbb{L}}{\underset{\mathcal{O}_X}{\otimes}} N)[-n]$$

$$\simeq DR^\cdot(\mathbb{D}M \overset{\mathbb{L}}{\underset{\mathcal{O}_X}{\otimes}} N)[-n]$$

$$\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathbb{D}M \overset{\mathbb{L}}{\underset{\mathcal{O}_X}{\otimes}} N)$$

2.) (apply $R\Gamma(-)$ to 1.)):

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{N}) \cong a_+ (\mathbb{D}\mathcal{M} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{N})$$
$$\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathbb{D}\mathcal{M} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{N})$$

where $a : X \to \{pt\}$

in part.: $(\mathcal{N} = \mathcal{O}_X)$: $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$
$= R\Gamma(DR^\cdot \mathbb{D}\mathcal{M})[-n] = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathbb{D}\mathcal{M})$

in part.: $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \cong \mathbb{H}^0(DR^\cdot(\mathbb{D}\mathcal{M}))$

since $\mathbb{D}\mathcal{M}$ is hol. iff $\mathcal{M}$ is so, we get

<u>Corollary:</u> $\mathcal{M}$ hol., put $Sol^\cdot(\mathcal{M}) := DR(\mathbb{D}\mathcal{M})$
$= R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$, then $\dim_{\mathbb{C}} \mathbb{H}^k Sol^\cdot(\mathcal{M}) < \infty$

in part. $\dim_{\mathbb{C}} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) < \infty$

<u>Pf.:</u> have seen last time: $a_+ = R\Gamma(DR^\cdot(-)) : D^b_h(\mathcal{D}_X) \to$
$D^b_h(pt) = D^b_c(pt) = D^b_{f.d}(\mathbb{C}) = \{\text{complexes of } \mathbb{C}\text{-v.sp.}, \dim H^i < \infty\}$