Lecture 6: More on holonomic $\mathcal{D}$-modules

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<u>Recall</u>: for $M \in \mathrm{Mod}_c(\mathcal{D}_X)$, we have $\mathrm{char}(\mathcal{M}) \subset T^*X$, and M is hol. $\Leftrightarrow$ $\mathrm{char}(\mathcal{M})$ is Lagrangian subvariety.

Let $\mathrm{Mod}_h(\mathcal{D}_X)$ full subcat. of $\mathrm{Mod}_c(\mathcal{D}_X)$ of hol. $\mathcal{D}_X$-mod.

easy: $\mathrm{Mod}_h(\mathcal{D}_X)$ is abelian (since for $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{L} \to 0$, we have $\mathrm{char}(\mathcal{N}) = \mathrm{char}(\mathcal{M}) \cup \mathrm{char}\,\mathcal{L}$).

<u>Lemma</u>: $\mathcal{M} \in \mathrm{Mod}_h(\mathcal{D}_X) \Rightarrow \exists\, \mathcal{U} \subset X$ open s.t. $\mathcal{M}_{|\mathcal{U}}$ is integrable connection (possibly of $rk = 0$).

<u>Proof</u>: Put $S := \mathrm{char}(\mathcal{M}) \setminus T^*_X X$. If $S = \emptyset \Rightarrow \mathrm{char}(\mathcal{M}) = T^*_X X$
$\Rightarrow \mathcal{M}$ is $\mathcal{O}_X$-coherent, i.e. $\mathcal{M}$ is int. connection.

Let $S \neq \emptyset$, consider projection $\pi_{|S}: S \to \pi(S)$.

$\mathrm{char}(\mathcal{M})$ conical in fibres of π + $S \cap T^*_X X = \emptyset \Rightarrow$
$\dim$ fibres $\pi_{|S} > 0$, hence $\dim \pi(S) < \dim S \leq \dim \mathrm{char}(\mathcal{M}) = \dim X \Rightarrow \exists\, \mathcal{U} \overset{\neq 0}{\subset} X$ open s.t. $\mathcal{U} \subset X \setminus \pi(S)$

Then $\mathrm{char}(\mathcal{M}_{|\mathcal{U}}) \subset T^*_\mathcal{U}\mathcal{U} \Rightarrow \mathcal{M}_{|\mathcal{U}}$ is $\mathcal{O}_\mathcal{U}$-coherent. $\square$

<u>Def.</u>: $D^b_h(\mathcal{D}_X)$ is full subcat. of $D^b_c(\mathcal{D}_X)$ of complexes of $\mathcal{D}_X$-mod. with holonomic cohomology (is itself triangulated).

Next we want to prove:

<u>The Main Theorem</u>: Let $f: X \to Y$ be a morphism

Then i) $f_+ : D^b_h(\mathcal{D}_X) \longrightarrow D^b_h(\mathcal{D}_Y)$

ii) $f^+[\dim X - \dim Y] : D^b_h(\mathcal{D}_Y) \to D^b_h(\mathcal{D}_X)$

<u>Rk</u>: a) wrong for analytic $\mathcal{D}$-modules

b) $f_+ : D^b_c(\mathcal{D}_X) \to D^b_c(\mathcal{D}_Y)$ only if f is proper (!)

<u>Proof requires many steps:</u>

1.) case i: $X \hookrightarrow Y$ closed embedding. Then $\forall$

$M \in D^b_c(\mathcal{D}_X)$: $\quad M \in D^b_h(\mathcal{D}_X) \Longleftrightarrow i_+ M \in D^b_h(\mathcal{D}_Y)$

<u>Pf</u>: i_+ exact in this case: can assume $M \in \mathrm{Mod}_c(\mathcal{D}_X)$

Consider cotangent map

$$T^*Y \overset{\omega}{\hookleftarrow} X\times_Y T^*Y \overset{T^*i}{\twoheadrightarrow} T^*X$$

then we have: $\mathrm{char}(i_+M) = \omega\left((T^*i)^{-1}(\mathrm{char}(M))\right)$

<u>Proof (sketch)</u>: work locally on X, reduce by induction on $\mathrm{codim}_Y(X)$ to the case where $X=\{x_1=0\}\subset Y$ is hyperplane

notice: $T^*i:(\xi_1,\dots,\xi_n,x_2,\dots,x_n)\mapsto(\xi_2,\dots,\xi_n,x_2,\dots,x_n)$, $\omega:(\xi_1\dots\xi_n,x_2,\dots,x_n)\mapsto(\xi_1\dots\xi_n,0,x_2,\dots,x_n)$

Put $N=i_+M$, then $N=\mathbb{C}[\partial_{x_1}]\otimes_{\mathbb{C}} i_*M$. Choose good filtration $G_\cdot M$ s.t. $G_{-1}M=0$, define $F_jN := \sum_{l=0}^{j}\sum_{k\le l}\mathbb{C}\partial_{x_1}^k\otimes i_*G_{j-l}M$.

<u>Check</u>: - $F_\cdot N$ is good

- $gr_j^F N = \bigoplus_{l=0}^{j}\mathbb{C}\xi_1^l\otimes_{\mathbb{C}} i_*\,gr^G_{j-l}M$; $\xi_1=\sigma(\partial_{x_1})$

hence $gr^F_\cdot N\simeq\mathbb{C}[\xi_1]\otimes_{\mathbb{C}} i_*\,gr^G_\cdot M\simeq\mathbb{C}[x_1,\xi_1]/(x_1)\otimes i_*\,gr^G_\cdot M$

we obtain: $\mathrm{char}(N)=\mathrm{supp}\,gr^F_\cdot N=\omega\left((T^*i)^{-1}(\mathrm{supp}\,gr^G_\cdot M)\right)=\omega\left((T^*i)^{-1}\mathrm{char}M\right)$

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now notice that ω is closed embedding and that $\dim(\text{fibres of } T^*i)=\mathrm{codim}_Y X$, hence $M\in\mathrm{Mod}_h(\mathcal{D}_X)\iff$

$\dim(\mathrm{char}(M))=\dim X\iff\dim\omega\left((T^*i)^{-1}(\mathrm{char}M)\right)=\dim X+\mathrm{codim}_Y X=\dim Y$

$\iff\dim\mathrm{char}(N)=\dim Y\iff N\in\mathrm{Mod}_h(\mathcal{D}_Y)$ $\square$

2.) let $f: X = Z \times Y \to Y$ be the projection. Now the problem is local on Y, so let Y be affine. By some Čech-type argument, we can also assume that X is affine $\left(\begin{array}{l}\text{replace } Z \text{ by } \bigcup Z_\alpha \text{ and } M \text{ by Čech complex} \\ \text{w.r.t. covering } X = \bigcup (Z_\alpha \times Y)\end{array}\right)$

Embed $X \overset{\alpha}{\hookrightarrow} \mathbb{A}^n$ and $Y \overset{\beta}{\hookrightarrow} \mathbb{A}^m$, consider

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook, "i_f"'] & Y \arrow[dd, hook, "\beta"] \\
X \times Y \arrow[d, hook, "\alpha\times\beta"'] & \\
\mathbb{A}^{n+m} \arrow[r, "p"] & \mathbb{A}^m
\end{tikzcd}$$

Then: $f_+ M \in D^b_h(Y) \overset{\text{part 1.)}}{\iff} \beta_+ f_+ M \in D^b_h(\mathcal{D}_{\mathbb{A}^m})$

moreover: $(\alpha\times\beta)\circ i_f$ is closed embedding,

hence: $\big((\alpha\times\beta)\circ i_f\big)_+ M \in D^b_h(\mathcal{D}_{\mathbb{A}^{n+m}})$
part 1.)

<u>hence:</u> it suffices to treat case $X = \mathbb{A}^{n+m} \xrightarrow{f=p} \mathbb{A}^m = Y$

and by induction it suffice to do case $m = n-1$

3.) assume this is done and hence stability $f_+ : D^b_h(\mathcal{D}_X) \to D^b_h(\mathcal{D}_Y)$ is proved for all f.

Then we deduce stability by $f^+[\underbrace{\mathrm{codim}_Y X}_{=:\, c}]$ as follows:

decompose again f into proj. / closed embedding.

If $f: X = Z \times Y \to Y$ is projection, then $f^+[c] = f^*[c]$ $= (\mathcal{O}_Z \boxtimes -)[c]$ is exact (mod shift) and we have

$$\mathrm{char}(\mathcal{O}_Z \boxtimes M) = \mathrm{char}\, \mathcal{O}_Z \times \mathrm{char}\, M = T^*_Z Z \times \mathrm{char}\, M$$

$\Rightarrow f^+ M[c] \in D^b_{h}(\mathcal{D}_X)$.

If $f = i: X \hookrightarrow Y$ is closed embedding, then

<u>Theorem (without proof)</u>: Let $j: U := Y \setminus X \hookrightarrow X$, then $\exists$ distinguished triangle in $D^b_{q.c.}(\mathcal{D}_Y)$:

adjunction triangle

$$i_+ i^+[c] M \to M \to j_+ j^+ M \xrightarrow{+1}$$

$\forall M \in D^b_{q.c.}(\mathcal{D}_Y)$

(idea: derive this from $R\Gamma_X(M) \to M \to Rj_* j^* M \xrightarrow{+1}$ in $D^b(\mathcal{O}_Y)$)

Now if $M \in \mathrm{Mod}_h(\mathcal{D}_Y) \Rightarrow j^+ M = j^* M = M_{|U} \in \mathrm{Mod}_h(\mathcal{D}_U)$

$\xRightarrow[j_+ : D^b_h(\mathcal{D}_U) \to D^b_h(\mathcal{D}_X)]{\text{assuming}}$ $j_+ j^+ M \in D^b_h(\mathcal{D}_Y)$ $\xRightarrow[\text{above triangle}]{\text{by}}$ $i_+ i^+[c] M \in D^b_h(\mathcal{D}_Y)$

$\xLeftrightarrow[\text{part 1.)}]{}$ $i^+[c] M \in D^b_h(\mathcal{D}_X)$

It remains to prove: $p : \mathbb{A}^n \longrightarrow \mathbb{A}^{n-1}$, $(x_1,\dots,x_n) \mapsto (x_2,\dots,x_n)$

then $\forall\, M \in \mathrm{Mod}_h(D_{\mathbb{A}^n}) : p_+ M \in D^b_{h}(D_{\mathbb{A}^{n-1}})$.

<u>Lemma:</u> Consider Fourier transform $\hat{\ }: \mathrm{Mod}(D_{\mathbb{A}^n}) \to \mathrm{Mod}(D_{(\mathbb{A}^n)^\vee})$ (and similarly $\mathrm{Mod}(D_{\mathbb{A}^{n-1}}) \to \mathrm{Mod}(D_{(\mathbb{A}^{n-1})^\vee})$). Let $i : (\mathbb{A}^{n-1})^\vee \hookrightarrow (\mathbb{A}^n)^\vee$, $(y_2,\dots,y_n) \mapsto (0, y_2,\dots,y_n)$.

Then: $\widehat{H^k p_+ M} \simeq H^k i^+ \hat{M} \qquad \forall\, k \in \mathbb{Z}$.

<u>Proof</u> (extended exercise, 2 ways to prove it using the two definitions of the Fourier transform)

<u>Theorem</u>: Let $M \in \mathrm{Mod}_c(D_{\mathbb{A}^n})$. Then M is holonomic iff $\hat{M}$ is so.

<u>Proof:</u> Consider Bernstein filtration on $D_{\mathbb{C}^n} =: D_n$

$$B_k D_n := \sum_{|\alpha|+|\beta| \le k} \mathbb{C}\, \underline{x}^\alpha \underline{\partial}_x^\beta \qquad \left(\text{recall: } F_k D_n := \sum_{|\alpha| \le k} \mathbb{C}[\underline{x}]\, \underline{\partial}_x^\alpha\right)$$

<u>ex.:</u> $\mathrm{gr}^B_\cdot D_n \simeq \mathbb{C}[\xi_1,\dots,\xi_n, x_1,\dots,x_n] \simeq \mathrm{gr}^F_\cdot D_n$

similar to $F_\cdot D_n$: good B-filtrations on $M \in \mathrm{Mod}(D_n)$

notice: if $B_\cdot M$ is good B-filt. $\Rightarrow \dim_{\mathbb{C}} B_k M < \infty \;\forall k$

7

Lemma: Let $B_\bullet M$ good B-filtration on $M \in \mathrm{Mod}_{f.g.}(D_n)$.

Then: 1.) $\exists\, h_{M,B}(t) \in \mathbb{Q}[t]$: $h_{M,B}(k) = \dim_{\mathbb{C}} B_k M$ $\forall k \gg 0$

2.) write $h_{M,B}(t) = \frac{m}{d!} \cdot t^d + a_{d-1} \cdot t^{d-1} + \ldots + a_0$

Then $\overset{\text{dimension}}{d =: d(M)}$ and $\overset{\text{multiplicity}}{m =: m(M)}$ do <u>not</u> depend on B.

3.) Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be exact sequence in $\mathrm{Mod}_{f.g.}(D_n)$. Then $d(M_2) = \max(d(M_1), d(M_3))$

$$m(M_2) = \begin{cases} m(M_1) + m(M_3) & d(M_1) = d(M_3) \\ m(M_1) & d(M_1) > d(M_3) \\ m(M_3) & d(M_3) > d(M_1) \end{cases}$$

Pf.: clear for graded modules, use $\dim B_k M = \sum_{\ell \le k} \dim \mathrm{gr}^B_\ell M$ and by using $0 \to \mathrm{gr}^B_\bullet M_1 \to \mathrm{gr}^B_\bullet M_2 \to \mathrm{gr}^B_\bullet M_3 \to 0$ if $B_\bullet M_1$ resp. $B_\bullet M_3$ are the filtrations induced from $B_\bullet M_2$

Prop: Let $0 \neq M \in \mathrm{Mod}_{f.g.}(D_n)$, then $\dim \mathrm{char}(M) = d(M)$

Proof: „known": $\overbrace{\dim \mathrm{supp}\, \mathrm{gr}^B_\bullet M}$ $= d(M)$.

general fact: $G.D_n$ any filtration s.t. $\mathrm{gr}^G_\bullet D$ is a regular commutative ring of dimension m.

$G.M$ G-filtration on $M \in \mathrm{Mod}_{f.g}(D_n)$, then

$$\dim \mathrm{supp}\, \mathrm{gr}^G_\bullet M = 2n - \min \{ i \mid \mathrm{Ext}^i_{D_n}(M, D_n) \neq 0 \}$$

(follows from corresponding fact in comm. algebra).

hence $\dim \mathrm{char}(M) = \dim \overline{\mathrm{supp}\, \mathrm{gr}^B_\bullet M}$. $\square$

end of proof of Prop.

<u>Back to the proof of the theorem:</u> by definition of Bernstein filtration $B.D_n$, we have $d(M) = d(\hat{M})$. Hence $\dim \mathrm{char}(M) = \dim \mathrm{char}(\hat{M})$ by Prop. $\square$

We are now reduced to prove: $M \in \mathrm{Mod}_{h}(D_{\mathbb{A}^n})$, $i : \mathbb{A}^{n-1} \hookrightarrow \mathbb{A}^n$, then $H^k(i^+[1] M) \in \mathrm{Mod}_{h}(D_{\mathbb{A}^{n-1}})$ $\forall k$

Consider again adjunction triangle:

$$i_+ i^+[1] M \to M \to j_+ j^+ M \xrightarrow{+1}, \text{ where } j : \mathbb{G}_m \times \mathbb{A}^{n-1} \hookrightarrow \mathbb{A}^n$$

$$\Rightarrow 0 \to H^0 i_+ i^+[+1] M \to M \to H^0 j_+ j^+ M \to H^1 i_+ i^+[1] M \to 0$$

Lemma: In this situation, $H^0 j_+ j^+ M \in \mathrm{Mod}_h(D_{/\mathbb{A}^n})$
(and all $H^k j_+ j^+ M = 0$ $\forall k \neq 0$ since j is affine)

Assuming Lemma, we get $M, j_+ j^+ M \in D^b_h(D_{/\mathbb{A}^n})$,
hence $i_+ i^+[1] M \in D^b_h(/\mathbb{A}^n)$, but i_+ is exact:

$$H^k i_+ i^+[1] M = i_+ (H^k i^+[1] M) \in \mathrm{Mod}_h(D_{/\mathbb{A}^n})$$

$$\overset{\Longleftrightarrow}{\text{part 1.)}} H^k i_+[1] M \in \mathrm{Mod}_h(D_{/\mathbb{A}^{n-1}}) \quad \begin{pmatrix}\text{non-zero only} \\ \text{for } k \neq 0,1\end{pmatrix}$$

Proof of the lemma: first show: $j_+ j^+ M = \mathbb{C}[x_1^{\pm}] \underset{\mathbb{C}[x]}{\otimes} M =: M[x_1^{-1}]$
(exercise). Let $B_\bullet M$ be a Bernstein filtration on M.
Put $\tilde{B}_k M[x_1^{-1}] := \{ x_1^{-k} \cdot m \mid m \in B_{2k} M \}$, check that
this is B-filtration. We have

we have $B_{2k} M \twoheadrightarrow \tilde{B}_k M[x_1^{-1}]$

$$\dim_{\mathbb{C}} \tilde{B}_k M[x^{-1}] \leq \dim_{\mathbb{C}} B_{2k} M = \frac{m(M)}{n!} (2k)^n + \text{l.o.t}$$

$$= \frac{m(M) \cdot 2^n}{n!} \cdot k^n + \text{l.o.t.}$$

$\overset{\text{to check}}{\Longrightarrow} M[x_1^{-1}] \in \mathrm{Mod}_{f.g}(D_{/\mathbb{A}^n})$ & $d(M[x_1^{-1}]) = n$ $\square$

The proof of the Main Theorem: $f_+ : D^b_q(\mathcal{D}_X) \to D^b_q(\mathcal{D}_Y)$; $f^+[c] : D^b_q(\mathcal{D}_Y) \to D^b_q(\mathcal{D}_X)$ is finished

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<u>Corollary:</u> $\mathcal{M}^\cdot \in D^b_q(\mathcal{D}_X) \Rightarrow \forall k$:

$$\dim_{\mathbb{C}}\left(\mathbb{H}^k\, DR^\cdot_X(\mathcal{M}^\cdot)\right) < \infty$$

<u>Proof:</u> Let $a : X \to \{pt\}$, then we have

$$\begin{aligned} Ra_* DR^{\cdot+n}_X(\mathcal{M}^\cdot) &\simeq a_+ \mathcal{M}^\cdot \in D^b(\mathbb{C}_X) \\ \| \\ R\Gamma(DR^{\cdot+n}_X(\mathcal{M}^\cdot)) \end{aligned}$$

Now by the Main theorem, $a_+\mathcal{M} \in D^b_q(\{pt\})$ $= D^b_c(\{pt\}) =$ finite-dim Vect$/\mathbb{C}$

and $\mathbb{H}^k DR^\cdot_X(\mathcal{M}^\cdot) = H^k R\Gamma(DR^\cdot_X(\mathcal{M}^\cdot))$

$\square$