Lecture 4: Derived categories of $\mathcal{D}$-modules, inverse/direct images II

Leipzig MPI, 24.5.19

<u>recall:</u> X, Y smooth alg. varieties, $f: X \to Y \rightsquigarrow$

$\mathcal{D}_{X\to Y}$, $\mathcal{D}_{Y\leftarrow X}$ and functors $f^+ : Mod(\mathcal{D}_Y) \to Mod(\mathcal{D}_X)$,

$f_+ : Mod(\mathcal{D}_X) \to Mod(\mathcal{D}_Y)$. Pb: f_+ neither left nor right exact $\rightsquigarrow$ go to derived categories.

Some facts without proof:

<u>Lemma:</u> - $A = \begin{cases} \mathcal{D}_{X,x} \\ \mathcal{D}_X(U) \end{cases} \Rightarrow \mathrm{gldim}\, A \leq 2n$

- $M \in Mod_{q.c.}(\mathcal{D}_X) \Rightarrow \exists$ resolution $\ldots \to P_1 \to P_0 \to M \to 0$

P_i $\mathcal{D}_X$-locally free AND $\exists$ finite resolution,

$0 \to P_m \to \ldots \to P_1 \to P_0 \to M \to 0$, P_i locally projective

if $M \in Mod_c(\mathcal{D}_X) \Rightarrow$ all P_i's finite rk

Def.: Let $D^b(\mathcal{D}_X)$ resp. $D^b_{qc}(\mathcal{D}_X)$ resp. $D^b_c(\mathcal{D}_X)$ be the derived cat. of bounded complexes of left $\mathcal{D}_X$-mod. resp. of $\mathcal{D}_X$-mod with q.c. resp. coh. cohomology.

(note: (Bernstein) this is equ. to derived cat. of q.c. / coh. $\mathcal{D}_X$-mod.)

Def.: (inverse image): $f: X \to Y$, then

$$f^+: D^b(\mathcal{D}_Y) \longrightarrow D^b(\mathcal{D}_X)$$

$$M^\cdot \longmapsto \mathcal{D}_{X\to Y} \overset{L}{\underset{f^{-1}\mathcal{D}_Y}{\otimes}} f^{-1}M^\cdot$$

(can be computed by flat resolution of M resp. of $\mathcal{D}_{X\to Y}$)

notice: f^+ sends $D^b_{q.c.}(\mathcal{D}_Y)$ to $D^b_{q.c.}(\mathcal{D}_X)$

(since $\mathrm{Mod}_{q.c}(\mathcal{D}_X)$ means q.c. as $\mathcal{O}_X$-mod, use

$\mathcal{D}_{X\to Y} \overset{L}{\otimes}_{f^{-1}\mathcal{D}_Y} f^{-1}M \simeq (\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y) \overset{L}{\otimes}_{f^{-1}\mathcal{D}_Y} f^{-1}M \simeq \mathcal{O}_X \overset{L}{\otimes}_{f^{-1}\mathcal{O}_Y} f^{-1}M$ as $\mathcal{O}_X$-mod)

<u>However</u>: for $M = \mathcal{D}_Y \Rightarrow f^+ M = \mathcal{D}_{X\to Y} \overset{L}{\underset{f^{-1}\mathcal{D}_Y}{\otimes}} f^{-1}\mathcal{D}_Y = \mathcal{D}_{X\to Y}$

if f is closed embedding of coord. subspace $\{y_{r+1} = \ldots = y_n = 0\} \Rightarrow \mathcal{D}_{X\to Y} = \mathcal{D}_X[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \notin \mathrm{Mod}_c(\mathcal{D}_X)$.

<u>Lemma</u>: $f: X \to Y$, $g: Y \to Z \Rightarrow (g\circ f)^+ = f^+ \circ g^+$

(use $\mathcal{D}_{X\to Y} \overset{L}{\underset{f^{-1}\mathcal{D}_Y}{\otimes}} f^{-1}\mathcal{D}_{Y\to Z} = \mathcal{D}_{X\to Z}$) (see Hotta 1.5.11.)

<u>exercise</u>: $j: U \hookrightarrow X$ open embedding, then $j^+ M = j^{-1} M$.

(in particular, $H^i j^+ M = 0 \ \forall i \neq 0$)

<u>Prop.</u>: $f: X \to Y$ smooth (e.g. $X = Z \times Y \xrightarrow{f = p_2} Y$),

then: 1.) $H^i f^+ M = 0 \quad \forall i \neq 0$

2.) $M \in \mathrm{Mod}_c(\mathcal{D}_Y) \Rightarrow f^* M := H^0 f^+ M \in \mathrm{Mod}_c(\mathcal{D}_X)$

<u>Proof</u>: 1.) $\mathcal{O}_X$ is $f^{-1}\mathcal{O}_Y$-flat and $f^+ M \underset{\text{as } \mathcal{O}_X\text{-mod}}{=} \mathcal{O}_X \overset{L}{\otimes}_{f^{-1}\mathcal{O}_Y} \mathcal{D}$

2.) locally we have $f(x_1 \ldots x_n) = (x_1 \ldots x_m) =: (y_1 \ldots y_m)$

and $\mathcal{D}_{X\to Y} = \bigoplus \mathcal{O}_X \partial_{y_1}^{r_1} \ldots \partial_{y_m}^{r_m} \Rightarrow \mathcal{D}_X \to \mathcal{D}_{X\to Y}$, $P \mapsto P(\underset{\mathcal{O}_X}{1} \otimes \underset{f^{-1}\mathcal{D}_Y}{1})$

is given by $\partial_{x_1}^{r_1} \dots \partial_{x_n}^{r_n} \mapsto \begin{cases} \partial_{y_1}^{r_1} \dots \partial_{y_m}^{r_m} & \text{if } r_i = 0 \ \forall i > m \\ 0 & \text{else} \end{cases}$

$\Rightarrow \mathcal{D}_X \twoheadrightarrow \mathcal{D}_{X\to Y}$ surjective.

Take free $\mathcal{D}_Y$-resolution of $M \rightsquigarrow f^*M = \operatorname{coker}(\mathcal{D}_{X\to Y}^k \to \mathcal{D}_{X\to Y}^{\ell})$

$\cong \operatorname{coker}(\mathcal{D}_X^{k'} \to \mathcal{D}_X^{\ell'}) \Rightarrow f^*M \in \operatorname{Mod}_c(\mathcal{D}_X)$.

Prop.: $i: X \hookrightarrow Y$ closed embedding, $d = \operatorname{codim}_Y(X)$ then for $M \in \operatorname{Mod}_{q.c.}(\mathcal{D}_Y)$, we have: $H^k(i^+M) = 0 \ \forall k \notin [-d, 0]$

Pf: $\mathcal{O}_X$ has resolution as $i^{-1}\mathcal{O}_Y$-module by Koszul-complex: locally $i(x_1 \dots x_r) = (x_1, \dots, x_r, 0 \dots 0) =: (y_1 \dots y_n)$:

$K_j := \Lambda^j \bigoplus_{k=r+1}^{n} i^{-1}\mathcal{O}_Y \, dy_k$, $K_j \to K_{j-1}$, $f dy_{i_1} \wedge \dots \wedge dy_{i_j} \longmapsto$

$\sum_{p=1}^{j} (-1)^{p+1} f \, dy_{i_1} \wedge \dots \widehat{dy_{i_p}} \wedge \dots dy_{i_j}$ (this is coordinate ind.)

(clear: length is d). Then $\mathcal{D}_{X\to Y} \simeq [\dots \to K_{n-j} \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}\mathcal{D}_Y \to \dots]$

and $i^+M = [\dots \to K_{n-j} \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}M \to \dots]$

<u>exercise</u>: make left $\mathcal{D}_X$-structure on $H^i(i^+M)$ explicit

direct image: $f : X \to Y$, define

$$D^b(\mathcal{D}_X) \longrightarrow D^b(f^{-1}\mathcal{D}_Y)$$

$$M^\cdot \longmapsto \mathcal{D}_{Y\leftarrow X} \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} M^\cdot$$

using flat resolution of $M^\cdot$

and $$D^b(f^{-1}\mathcal{D}_Y) \longrightarrow D^b(\mathcal{D}_Y)$$

$$N^\cdot \longmapsto Rf_*(N^\cdot)$$

using injective resolution of $N^\cdot$ (works for any sheaf of rings)

and: $$f_+ : D^b(\mathcal{D}_X) \longrightarrow D^b(\mathcal{D}_Y)$$

$$M^\cdot \longmapsto Rf_*\left(\mathcal{D}_{Y\leftarrow X} \overset{\mathbb{L}}{\otimes} M^\cdot\right)$$

Theorem: - f_+ sends $D^b_{q.c}(\mathcal{D}_X)$ into $D^b_{q.c}(\mathcal{D}_Y)$

- f proj. $D^b_c(\mathcal{D}_X)$ — " — $D^b_c(\mathcal{D}_Y)$

Fact: This holds for the functor Rf_*

(Hartshorne, ...)

exercise: $f: X \to Y$, $g: Y \to Z \Rightarrow (g \circ f)_+ = g_+ \circ f_+$

(Hotta, 1.5.21.)

exercise: $j: U \hookrightarrow X$ open embedding, then

$$j_+ M = Rj_* M \quad (\text{use } \mathcal{D}_{X \leftarrow U} = j^{-1}\mathcal{D}_X = \mathcal{D}_U)$$

(however, j_+ is not exact in general in this case)

strategy to proof Theorem above: decompose into open embedding and projection:

Let $i: X \hookrightarrow Y$, $(x_1, \dots, x_r) \mapsto (x_1 \dots x_r, 0 \dots 0) = (y_1 \dots y_n)$

Recall that $\mathcal{D}_{Y \leftarrow X} = \mathbb{C}[\partial_{y_{r+1}}, \dots \partial_{y_n}] \otimes_{\mathbb{C}} \mathcal{D}_X$ (flat over $\mathcal{D}_X$)

and $Ri_* = i_*$ (i is affine), hence: $H^k(i_+ M) = 0 \ \forall i \neq 0$

$$H^0(i_+ M) = i_*\left(\mathcal{D}_{Y \leftarrow X} \overset{\otimes}{\mathcal{D}_X} M\right) = \mathbb{C}\left[\partial_{y_{r+1}}, \dots, \partial_{y_n}\right] \otimes_{\mathbb{C}} i_* M.$$

left action of $\mathcal{D}_Y$: $\partial_{y_k}(1 \otimes m) = \partial_{y_k} \otimes m \quad \forall k \in \{r+1, \dots, n\}$

$\partial_{y_k}(1 \otimes m) = 1 \otimes \partial_{y_k} m, \ \forall k \in \{1, \dots, r\}$, $\varphi(1 \otimes m) = 1 \otimes \varphi_{|X} \cdot m \quad \forall \varphi \in \mathcal{O}_Y$

(exercise!)

obvious: $\mathrm{Mod}_{q.c}(\mathcal{D}_X) \xrightarrow{i_+} \mathrm{Mod}_{q.c.}(\mathcal{D}_Y)$

<u>Case of projection</u>: Let $p : X = Y \times Z \to Y$

Preliminary construction: Spencer & deRham complexes:

<u>Lemma</u>: $\exists$ resolution of $\mathcal{O}_X$ resp. of ω_X by locally free left resp. right $\mathcal{D}_X$-modules:

$$\overbrace{0 \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^n \Theta_X \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^{n-1}\Theta_X \to \cdots\cdots \to \mathcal{D}_X \otimes \Theta_X \to \mathcal{D}_X}^{\text{Spencer complex (of } \mathcal{D}_X\text{)} : Sp(\mathcal{D}_X)} \xrightarrow{\alpha} \mathcal{O}_X \to 0$$

$$\underbrace{0 \to \Omega^0_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \cdots\cdots \to \Omega^n_X \otimes_{\mathcal{O}_X} \mathcal{D}_X}_{\text{de Rham complex (of } \mathcal{D}_X\text{)} \; DR(\mathcal{D}_X)} \xrightarrow{\beta} \omega_X \to 0$$

where $\alpha(P) := P(1)$, $\beta(\omega \otimes P) := \omega . P$, and

$$\mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^k \Theta_X \longrightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^{k-1} \Theta_X$$

$$P \otimes v_1 \wedge \ldots \wedge v_k \mapsto \sum_{i=1}^{k} (-1)^{i-1} P v_i \otimes v_1 \wedge \ldots \wedge \hat{v}_i \wedge \ldots \wedge v_k$$

$$+ \sum_{i<j} (-1)^{i+j} P \otimes [v_i, v_j] \otimes v_1 \wedge \ldots \wedge \hat{v}_i \wedge \ldots \wedge \hat{v}_j \wedge \ldots \wedge v_k$$

and: $\Omega^k_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \longrightarrow \Omega^{k+1}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$

$$\omega \otimes P \longmapsto d\omega \otimes P + \sum_{i=1}^n dx_i \wedge \omega \otimes \partial_{x_i} P$$

(where $(x_i)_{i=1..n}$ are local coordinates on X)

Pf: recall that $\mathrm{Mod}(\mathcal{D}_X) \longrightarrow \mathrm{Mod}(\mathcal{D}_X^{opp})$

$$M \longmapsto \omega_X \otimes_{\mathcal{O}_X} M$$

moreover we have contraction (iso of right $\mathcal{D}_X$-mod)

$$\omega_X \otimes_{\mathcal{O}_X} (\mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^k \Theta_X) \longrightarrow \Omega^{n-k}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$$

$$\omega \otimes 1 \otimes v \longmapsto (-1)^{(n-k)\frac{n-k-1}{2}} \cdot \omega(v \wedge -) \otimes 1$$

inducing $[\mathrm{Sp}^\cdot(\mathcal{D}_X)]^r \xrightarrow{\sim} \mathrm{DR}^\cdot(\mathcal{D}_X)$. Hence show Lemma only for $\mathrm{Sp}^\cdot(\mathcal{D}_X)$: Define $F_p \mathrm{Sp}^\cdot(\mathcal{D}_X)$:

$$F_{p-n}\mathcal{D}_X \otimes \Lambda^n \Theta_X \to F_{p-n+1}\mathcal{D}_X \otimes \Lambda^{n-1}\Theta_X \to \dots \to F_p \mathcal{D}_X \to F_p \mathcal{O}_X$$

where $F_p \mathcal{O}_X = \begin{cases} 0 & \forall p < 0 \\ \mathcal{O}_X & \forall p \geq 0 \end{cases}$. Notice that this is well defined. Sufficient to show: $\mathrm{gr}^F_\cdot \mathrm{Sp}^\cdot(\mathcal{D}_X)$ is acyclic

$$gr_\bullet^F(Sp(\mathcal{D}_X)) : \pi_* \mathcal{O}_{T^*X} \otimes_{\mathcal{O}_X} \Lambda^n \Theta_X \to \dots \to \pi_* \mathcal{O}_{T^*X} \otimes \Theta_X \to \pi_* \mathcal{O}_{T^*X} \to \mathcal{O}_X$$

with $\pi_* \mathcal{O}_{T^*X} \otimes \Lambda^k \Theta_X \to \pi_* \mathcal{O}_{T^*X} \otimes \Lambda^{k-1} \Theta_X$

$$\varphi \otimes v_1 \wedge \dots \wedge v_k \mapsto \sum_{i=1}^k (-1)^{k+1} \varphi \underbrace{\sigma(v_i)}_{\text{Im in } F_1\mathcal{D}_X / F_0\mathcal{D}_X} \otimes v_1 \wedge \dots \hat{v}_i \wedge \dots v_k$$

This is Koszul complex of $X \subset T^*X$ (compl. int.) hence acyclic. $\square$

Back to $p : X = Y \times Z \to Y$. Let $M \in Mod(\mathcal{D}_X)$.

want to compute $Rp_*(\mathcal{D}_{Y \leftarrow X} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} M)$.

<u>Def.</u>: Let $N \in Mod(\mathcal{D}_X)$ then put

$$DR^k_{X/Y}(N) := \Omega^{n+k}_{X/Y} \otimes_{\mathcal{O}_X} N \quad \forall k \in [-r, 0], \; r = \dim Z$$

with differential: $d(\omega \otimes s) = d\omega \otimes s + \sum_{i=1}^r (dz_i \wedge \omega) \otimes \partial_{z_i} s$

(relativ de Rham complex) $\quad$ (z_i) loc. coord. on Z

Claim: If $p: X = Y \times Z \to Y$ is proj.,

then $DR_{X/Y}(\mathcal{D}_X) \xrightarrow{\sim} \mathcal{D}_{Y \leftarrow X}$ as right $\mathcal{D}_X$-mod.

Proof: similar to absolute case (exercise)

Hence: $\mathcal{D}_{Y \leftarrow X} \overset{L}{\otimes}_{\mathcal{D}_X} M \simeq DR_{X/Y}(\mathcal{D}_X) \otimes_{\mathcal{D}_X} M$

$\overset{(*)}{\simeq} DR_{X/Y}(M)$, but
$\uparrow$
$\mathcal{O}_X$-mod

$\Omega^{n+k}_{X/Y} \otimes M$ has left $f^{-1}\mathcal{D}_Y$-structure by

$P(\omega \otimes m) := \omega \otimes \bar{P} m$ where $\bar{P}$ is image

of $P \in f^{-1}\mathcal{D}_Y$ in $\mathcal{D}_X$. This is left $f^{-1}\mathcal{D}_Y$-str.

on $\mathcal{D}_{Y \leftarrow X} \overset{L}{\otimes}_{\mathcal{D}_X} M$. Hence $(*)$ holds in $D^b(f^{-1}\mathcal{D}_Y)$

so: $f_+ M = Rf_* DR_{X/Y}(M) \in D^b_{q.c.}(\mathcal{D}_Y)$

(since $\Omega^{n+k}_{X/Y}$ is $\mathcal{O}_Y$-quasi-coherent).