Introduction to algebraic D-modules

today: informal overview

1.) Differential operators: let $n \in \mathbb{N}$ and consider

$A_n := \mathbb{C}[x_1, \dots, x_n]\langle \partial_{x_1}, \dots, \partial_{x_n} \rangle$: non-commutative $\mathbb{C}$-algebra generated by $x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n}$ subject to relations: $[x_i, x_j] = [\partial_{x_i}, \partial_{x_j}] = 0$ and $[\partial_{x_i}, x_j] = \delta_{ij}$ $\forall i, j$. Call it Weyl algebra, write $A_n := \mathbb{C}[x_1, \dots x_n]\langle \partial_{x_1}, \dots, \partial_{x_n} \rangle$. Let F be any space of functions on which x_i, ∂_{x_j} act: e.g.

$F = \mathbb{C}[x], \mathbb{C}\{x\}, \mathbb{C}\{x^{1/p}\} := \mathbb{C}\{t\}$ with $t^p = x$ ......

let $f \in F$, then $\partial_{x_i}(x_i f) = \partial_{x_i}(x_i) \cdot f + x_i \cdot \partial_{x_i}(f)$

$= 1 \cdot f + x_i \cdot \partial_{x_i}(f) \Rightarrow (\partial_{x_i} \cdot x_i)(f) - (x_i \cdot \partial_{x_i})(f) = f$

in A_n

$\Rightarrow F$ is left A_n-module.

other method to get A_n-modules: Consider elements $P_1, \dots, P_k \in A_n$. Remark: Any $Q \in A_n$ has expression $\sum_{(I,J)\in E} c_{IJ} x^I \partial_x^J$ where $E \subset \mathbb{N}^2$ finite.

Consider the <u>left</u> ideal: $I := A_n(P_1, \dots, P_k) := \sum_{i=1}^{k} A_n \cdot P_i \subset A_n$. Can take quotient

$M := A_n / I$, this is a left A_n-module

(but not a ring unless I is two-sided).
which never happens

Consider differential system: $P_1 u = \dots = P_k u = 0$ for $u \in F$ (some function space).

<u>Description of solutions</u>: Claim: $\exists$ $\mathbb{C}$-lin. isomorphism: $\{u \in F \mid P_1(u) = \dots = P_k(u) = 0\} \simeq$

$\mathrm{Hom}_{A_n}(M, F)$

<u>Pf</u>: $\mathrm{Hom}_{A_n}(M,F) = \mathrm{Hom}_{A_n}(A_n/I, F)$

$$= \{\phi \in \mathrm{Hom}_{A_n}(A_n, F) \mid \phi(I) = 0\} \overset{\substack{\phi \text{ left} \\ A_n\text{-lin.}}}{=}$$

$$\{\phi(1) =: u \in F \mid \phi(P_1) = \dots = \phi(P_k) = 0\}$$

but since $\phi(P_i) = \phi(\underset{\text{in } A_n}{P_i \cdot 1}) = \underset{\text{action on } F}{P_i}(\phi(1))$

$$= P_i u$$

we get $\mathrm{Hom}_{A_n}(M,F) \simeq \{u \in F \mid P_1 u = \dots = P_k u = 0\}$

in particular, solutions of $\{P_i u = 0\}$ depend only on I, not on choice of generators $P_1, \dots, P_k$

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more generally: consider exact sequence of left A_n-mod.

$A_n^k \longrightarrow A_n^\ell \longrightarrow M \longrightarrow 0$ (generalizing $A_n^k \to A_n \to M = A_n/I \to 0$)

The map $A_n^k \to A_n^\ell$ is given by right multiplication by a matrix $\underline{P} \in M(k \times \ell, A_n)$

$\underline{P} = (P_{ij})_{\substack{i=1,\dots k \\ j=1,\dots,\ell}}$. This corresponds to the

vector-valued system of PDE's: $\left(\sum_{j=1}^{\ell} P_{ij}(u_j) = 0\right)_{i=1,\dots k}$

$(u_1, \dots, u_\ell) \in F^\ell$. Similarly, we have isomorphism

$$\phi \in \operatorname{Hom}_{A_n}\left(A_n^\ell / \operatorname{Im}(\cdot \underline{P})\right) \simeq \left\{ (u = \phi(e_j))_{j=1,\dots,\ell} \in F^\ell \;\middle|\; \sum_{j=1}^{\ell} P_{ij}(u_j) = 0 \;\; \forall\, i = 1..k \right\}$$

<u>Philosophy:</u> Homogeneous systems of linear PDE's with polynomial coefficients $\Longleftrightarrow$ finitely generated modules over A_n

2.) Coherent and holonomic $\mathcal{D}_X$-modules

X smooth alg. variety $/\mathbb{C}$, $\mathcal{D}_X \subset \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$ subsheaf generated by $\mathcal{O}_X$ & $\mathrm{Der}(\mathcal{O}_X, \mathcal{O}_X)$.

locally on $U \subset X$ with coordinates $x_1 \dots x_n$: $\Gamma(U, \mathcal{D}_X) = \mathcal{O}_U\langle \partial_{x_1}, \dots, \partial_{x_n} \rangle$

if $U = \mathbb{A}^n_{\mathbb{C}}$: $\Gamma(U, \mathcal{D}_X) = A_n$. Differential system $\Leftrightarrow$ coherent sheaf of $\mathcal{D}_X$-modules. Solutions?

<u>ex.:</u> $X = \mathbb{A}^2$, $\Gamma(X, \mathcal{D}_X) = A^2$, $\mathcal{M} = \mathcal{D}_{\mathbb{A}^2} / \mathcal{D}_{\mathbb{A}^2}(\partial_x)$

$\Rightarrow f(y) \in \mathbb{C}[y] \subset \mathbb{C}[x,y]$ is solution

$\rightsquigarrow \dim_{\mathbb{C}} \mathrm{Sol}(\mathcal{M}) = \infty$ 

<u>Holonomic modules:</u> Put $F_k \mathcal{D}_X := \{ P \in \mathcal{D}_X \mid \deg(P) \leq k \}$

<u>Def.:</u> Let $\mathcal{M} \in \mathrm{Coh}(\mathcal{D}_X)$. A filtration $F_\bullet \mathcal{M} := (F_i \mathcal{M})$ is called good iff: 1.) $F_i \mathcal{M} \subset F_{i+1} \mathcal{M}$, 2.) $F_i \mathcal{M}$ is $\mathcal{O}_X$-coherent, 3.) $F_i \mathcal{M} = 0 \ \forall i \ll 0$, 4.) $\mathcal{M} = \bigcup_{i \in \mathbb{Z}} F_i \mathcal{M}$,

5.) $F_i \mathcal{D}_X \cdot F_k \mathcal{M} \subseteq F_{k+i} \mathcal{M} \quad \forall i \in \mathbb{N}, k \in \mathbb{Z}$

6.) $\exists N : \forall k \geq N : F_i \mathcal{D}_X \cdot F_k \mathcal{M} = F_{k+i} \mathcal{M}$

Consider graded sheaf:

$$gr^F_\bullet \mathcal{D}_X = \bigoplus_{k\in\mathbb{N}_0} gr^F_k \mathcal{D}_X = \bigoplus_{k\in\mathbb{N}_0} \left(F_k\mathcal{D}_X / F_{k-1}\mathcal{D}_X\right) \quad F_{-1}\mathcal{D}_X := \{0\}$$

locally on $U\subset X$ with coordinates $x_1 \dots x_n$, for

$$P = \sum_{\substack{(I,J)\in\mathbb{N}^2 \\ \text{finite}}} c_{IJ}\, x^I \partial_x^J, \text{ put } \sigma(P) = \sum_{|J|\,\max} c_{IJ}\, x^I \xi^J \in \mathcal{O}_X(U)[\xi_1 \dots \xi_n]$$

then if $\deg(P) = k$ (i.e. $P\in F_k\mathcal{D}_X \setminus F_{k-1}\mathcal{D}_X$), then

$\sigma(P) \in gr^F_k \mathcal{D}_X$.

<u>Fundamental fact:</u> $\deg(P) = k$, $\deg(Q) = \ell$, then

$\deg \underbrace{[P,Q]}_{P\cdot Q - Q\cdot P} \le k+\ell-1$. Hence by putting

$\sigma(P)\cdot\sigma(Q) := \sigma(P\cdot Q) \rightsquigarrow gr^F_k \mathcal{D}_X$ is commutative

<u>algebra</u>. Actually: $gr^F_k \mathcal{D}_X = \pi_* \mathcal{O}_{T^*X}$, $\pi: T^*X \to X$

Now let $\mathcal{M} \in \mathrm{Coh}(\mathcal{D}_X)$, $F_\bullet\mathcal{M}$ a <u>good</u>

filtration, then: $gr^F_\bullet \mathcal{M} = \bigoplus_{k\in\mathbb{Z}} gr^F_k \mathcal{M}$ is $gr^F_\bullet \mathcal{D}_X$ <u>coherent</u>

Def.: $\mathrm{Char}(\mathcal{M}) := \mathrm{supp}(\mathrm{gr}^F_\bullet \mathcal{M}) \subset T^*X$

characteristic variety

Theorem: 1.) $\mathrm{char}(\mathcal{M})$ does not depend on $F_\bullet\mathcal{M}$

2.) $\dim \mathrm{char}(\mathcal{M}) \geq n$

Def.: $\mathcal{M} \in \mathrm{Coh}(\mathcal{D}_X)$ is called holonomic $\iff$

$\dim \mathrm{char}(\mathcal{M}) = n$

ex.: $n=1$, $\mathcal{M} = \mathcal{D}_{\mathbb{A}^1} / \mathcal{D}_{\mathbb{A}^1} \cdot P$, $P = \sum_{i=0}^{n} a_i(x) \cdot \partial_x^i$ $a_i \in \mathbb{C}[x]$

$\Rightarrow \Gamma(\mathbb{A}^1, \mathrm{gr}^F_\bullet \mathcal{M}) = \mathbb{C}[x,\xi] / \sigma(P) = \mathbb{C}[x,\xi] / a_n(x) \cdot \xi^n$

if $P \not\equiv 0 \Rightarrow$ this is 1-dimensional $\Rightarrow \mathcal{M}$ holonomic

Moreover: $\{x \in \mathbb{C} : a_n(x) = 0\}$ are singular pts of $\mathcal{M}$

in the sense that $\mathcal{M}_{|\mathbb{A}^1 \setminus \{x \mid a_n(x)=0\}}$ is $\mathcal{O}$-coherent

and actually $\mathcal{O}$-free, i.e. a vector bundle.

Thm (Kashiwara): $\mathcal{M} \in \mathrm{Coh}(\mathcal{D}_X)$ holonomic, then
$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) < \infty$.

3.) Vector bundles with connection

Let $E \to X$ alg. VB, i.e. the sheaf of sections $\mathcal{E}: U \mapsto \Gamma(U, E)$ is $\mathcal{O}_X$-locally free. A connection on E is a $\mathbb{C}$-linear sheaf homomorphism $\nabla: \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X$ s.t. $\nabla(f \cdot e) = f \cdot \nabla e + e \otimes df$, $\forall$ local sections e of $\mathcal{E}$ and $f \in \mathcal{O}_X$. $\exists$ extension $\nabla^{(2)}: \mathcal{E} \otimes \Omega^1_X \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^2_X$. ∇ is called flat $\Leftrightarrow \nabla^{(2)} \circ \nabla = 0$

<u>Lemma:</u> A coherent $\mathcal{D}_X$-module $\mathcal{M}$ is a VB with connection iff it is also coherent over $\mathcal{O}_X$ (then it is automatically $\mathcal{O}_X$-locally free).

<u>Philosophy:</u> holonomic $\mathcal{D}_X$-modules are „vector bundles with connection aquiring singularities"