Theorietag 2026
TU Ilmenau
April 1, 2026

Minimizing a word by permuting its coordinates

Dominik Scheder, TU Chemnitz

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Based on joint work with Johannes Tantow, also TU Chemnitz

A Concrete Example

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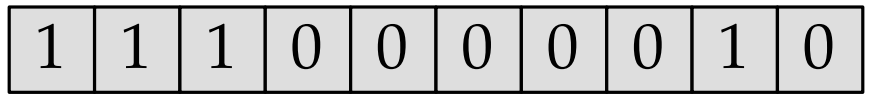

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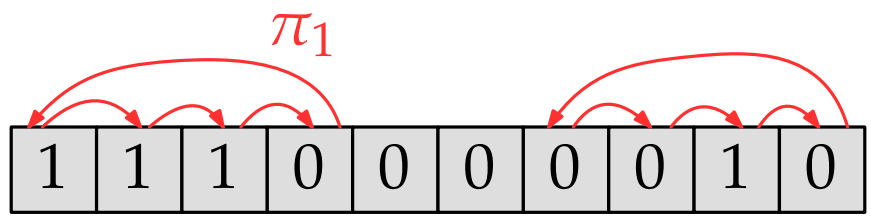

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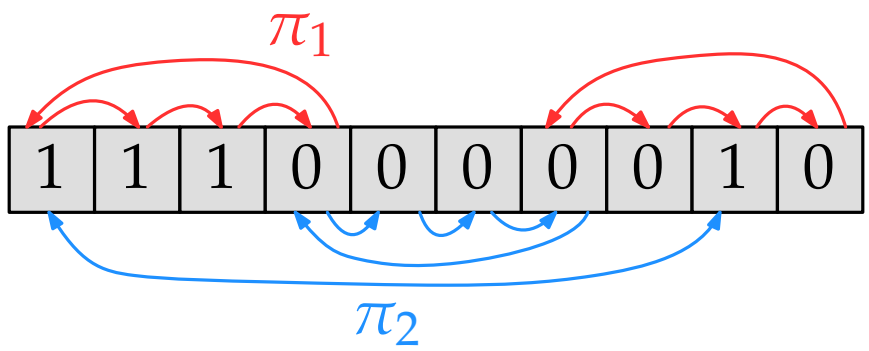

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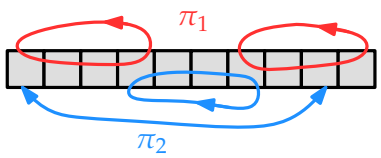

1	1	1	0	0	0	0	0	1	0

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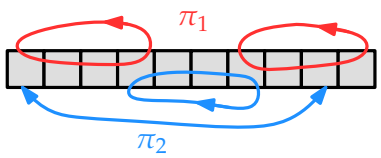

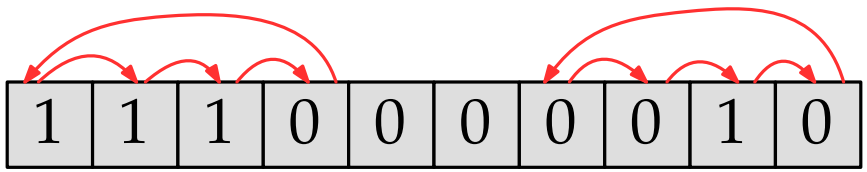

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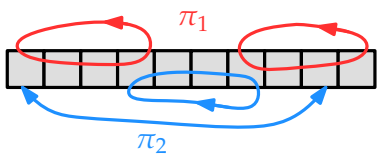
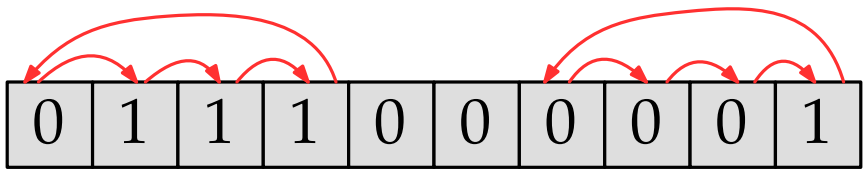


applying π_1 makes the word smaller



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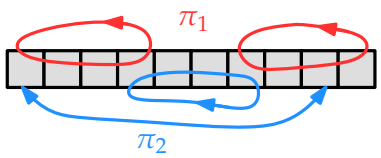
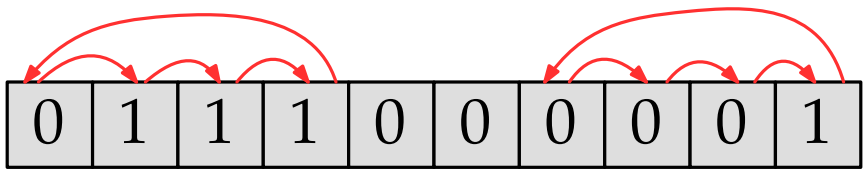


applying π_1 makes the word smaller

applying π_1 again would make word larger



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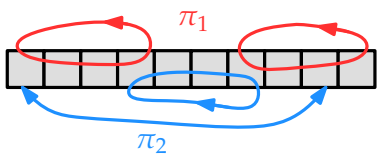

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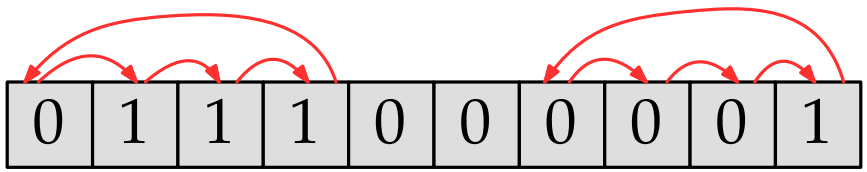
applying π_1 again would make word larger

cannot apply π_1 again



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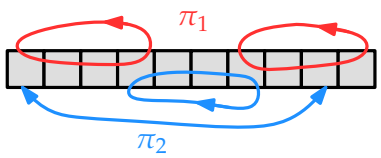
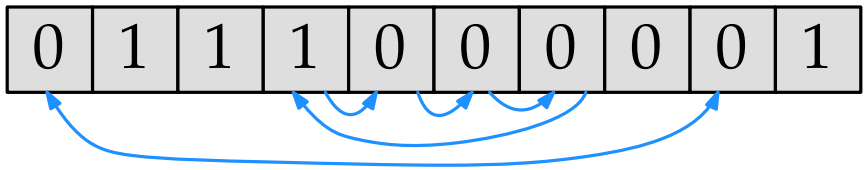
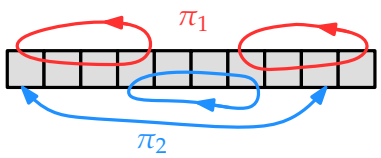
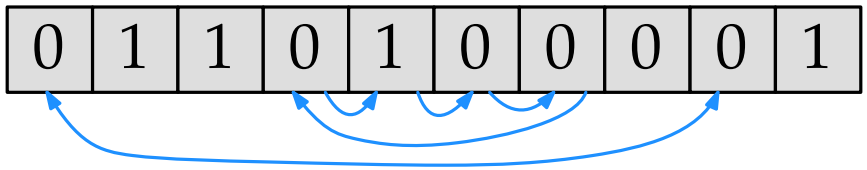




but can apply π_2

A Concrete Example





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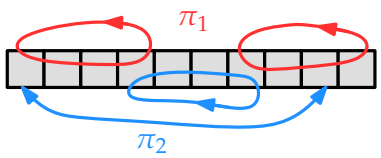

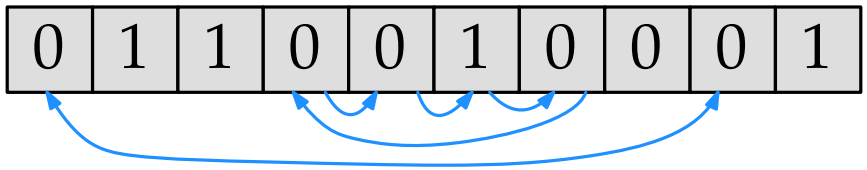

but can apply π_2
and again

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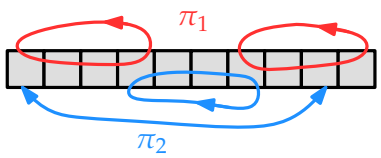

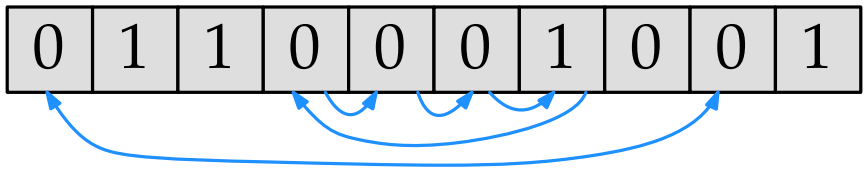

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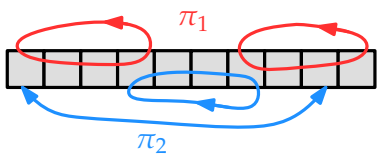

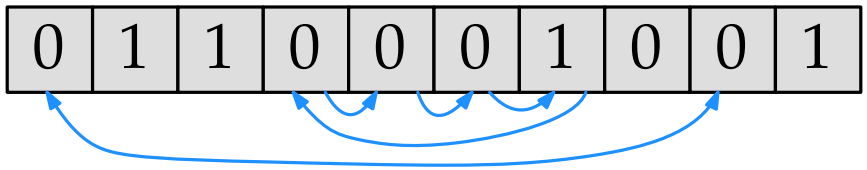

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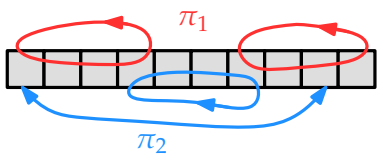

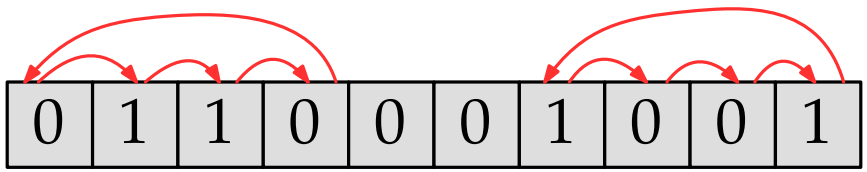

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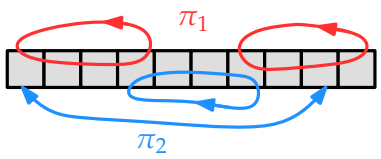

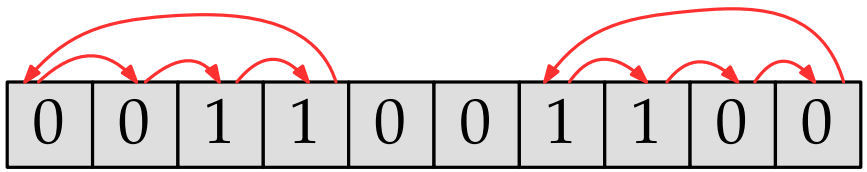

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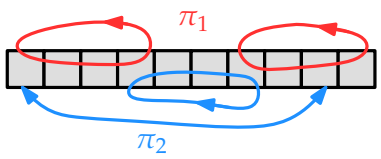

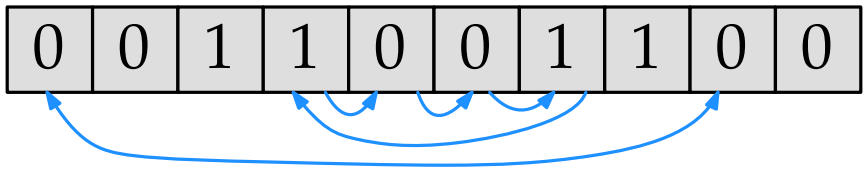

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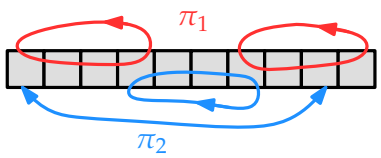

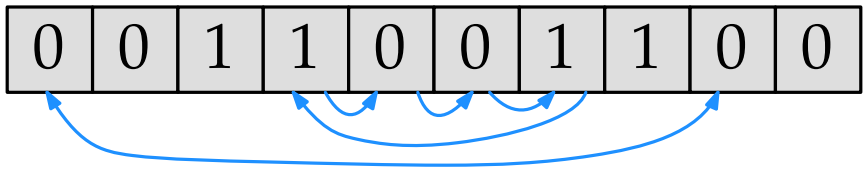

Applying π_2 would make the word larger

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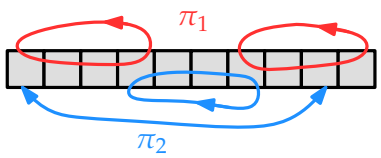

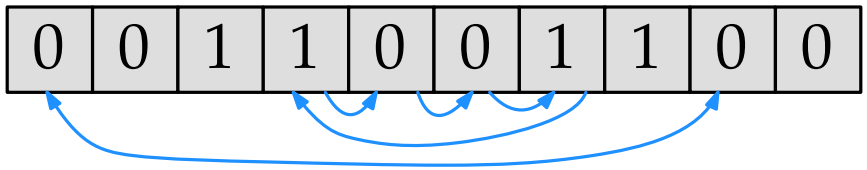

Applying π_2 would make the word larger

This is a local minimum.

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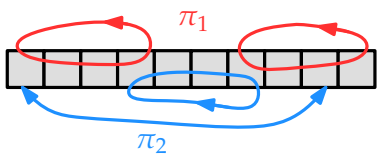

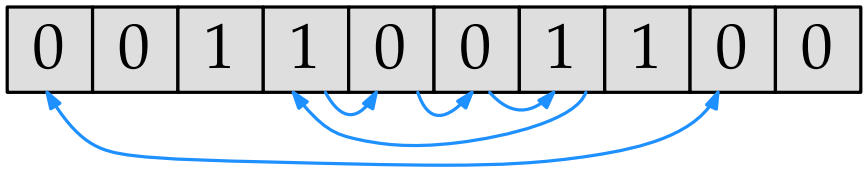

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But not a global minimum.

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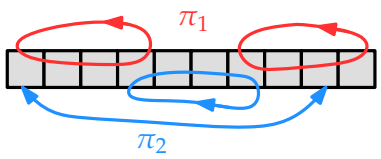

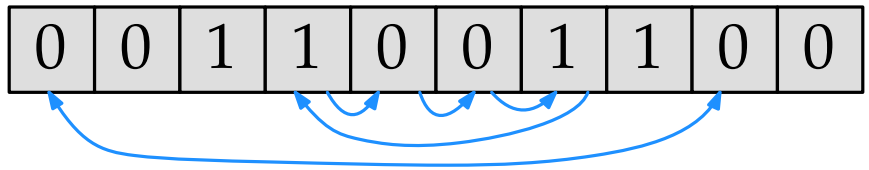

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This "greedy" procedure always finds a local minimum.

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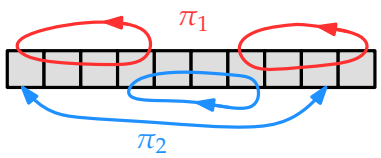

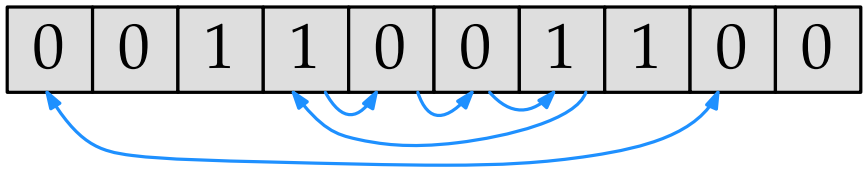

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How long can it take?

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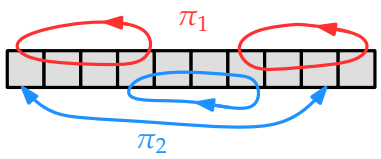

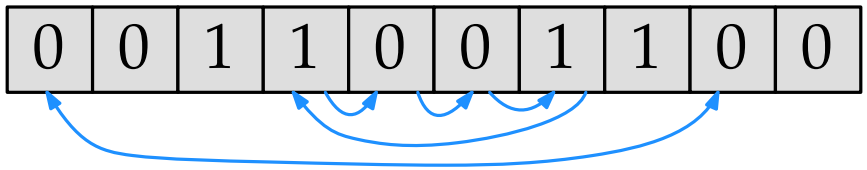

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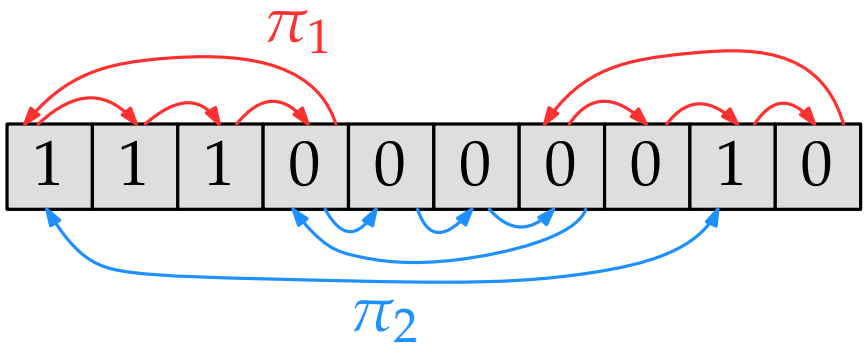
How long can it take?

Can we compute a local minimum directly?

The Formal Setup



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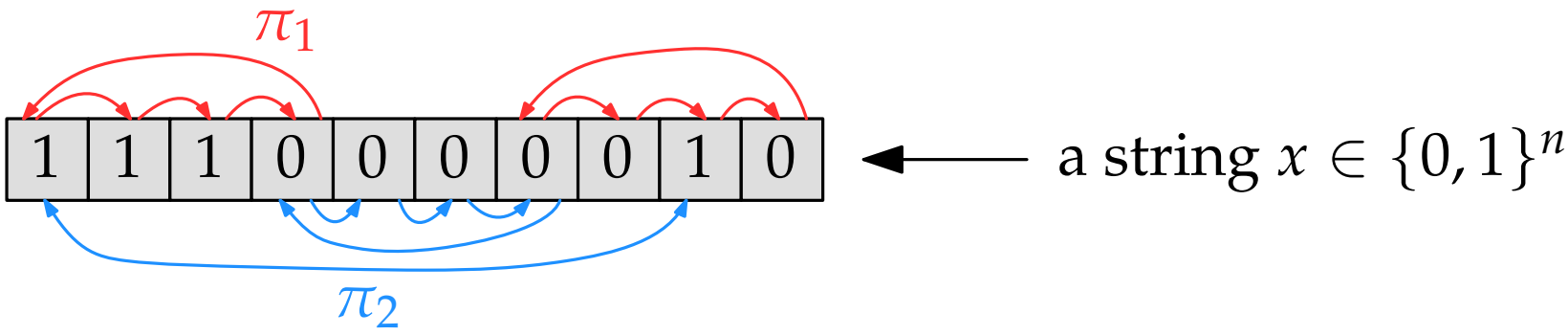

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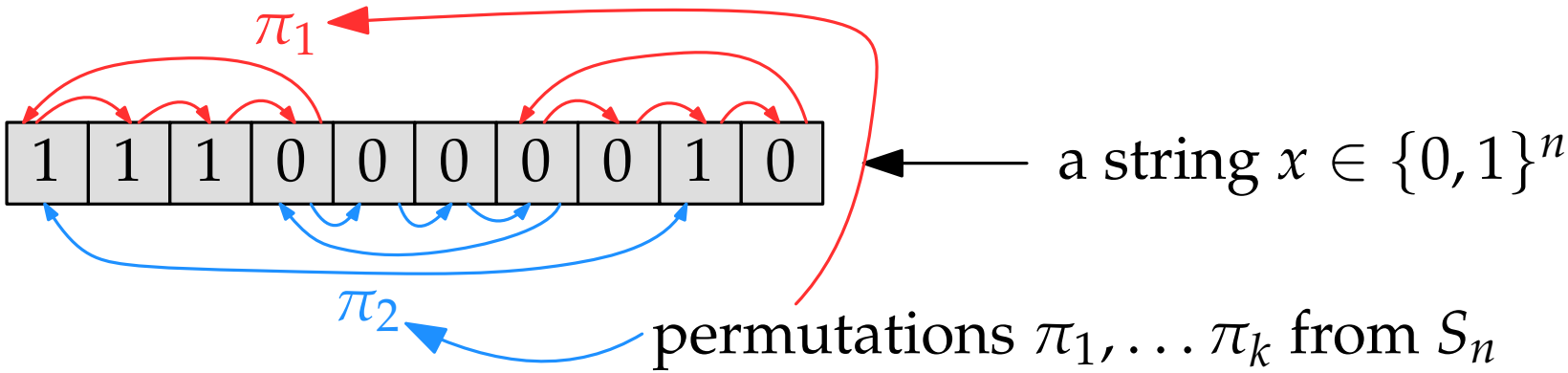

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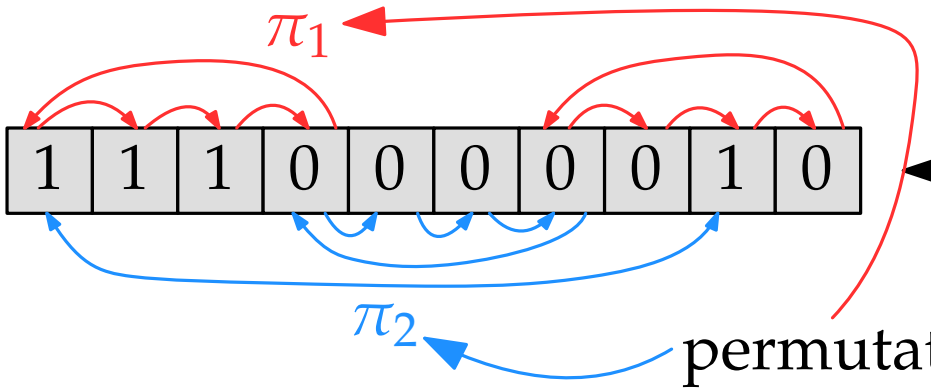

a string $x \in \{0,1\}^n$

permutations $\pi_1, \ldots \pi_k$ from S_n

they generate a subgroup $G = \langle \pi_1, \ldots, \pi_k \rangle$

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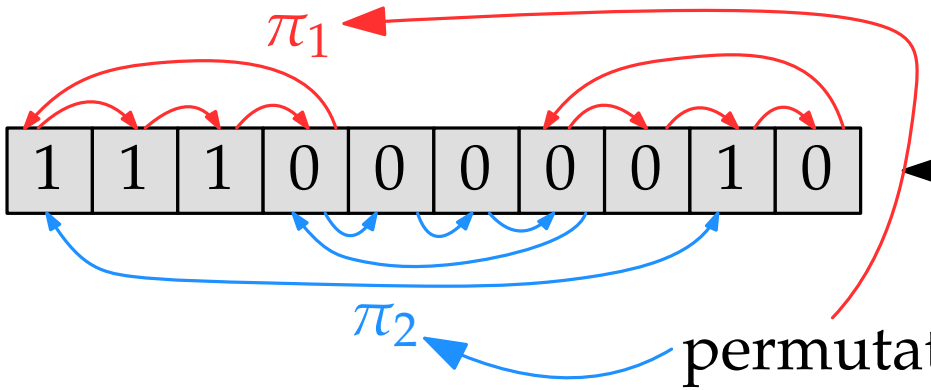

a string $x \in \{0,1\}^n$

permutations $\pi_1, \ldots \pi_k$ from S_n

they generate a subgroup $G = \langle \pi_1, \ldots, \pi_k \rangle$

they send x into the orbit $Gx = \{x \circ \pi \mid \pi \in G\}$

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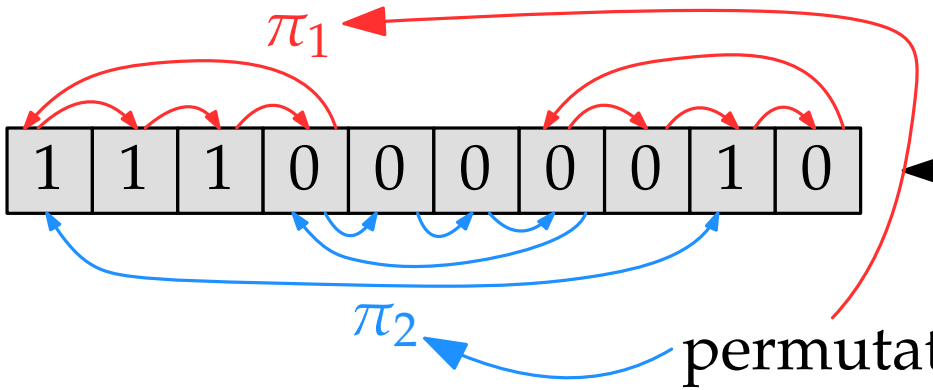

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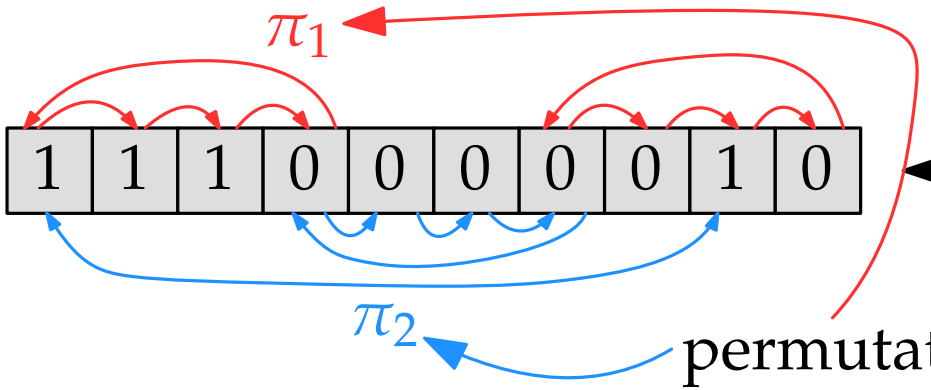
that's the correct order of composition by the way, since $\pi : [n] \to [n]$ and $x : [n] \to \{0,1\}$, so $x(\pi(i))$ makes sense syntactically

The Formal Setup



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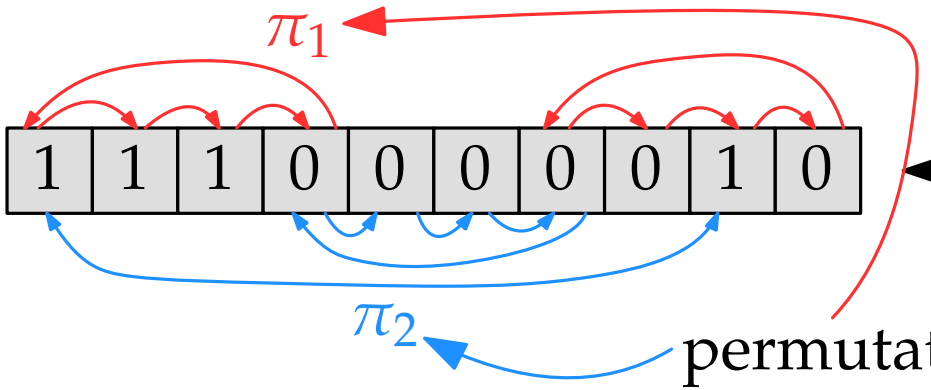

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global optimum: smallest element $y \in Gx$

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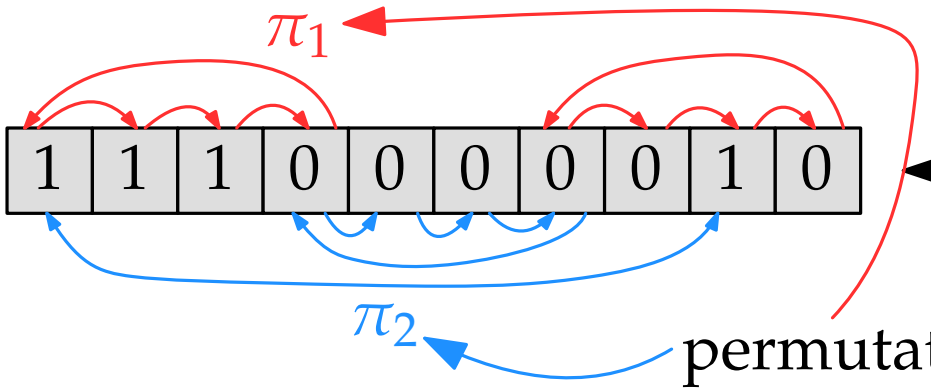

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global optimum: smallest element $y \in Gx$

in the lexicographic order on $\{0,1\}^n$

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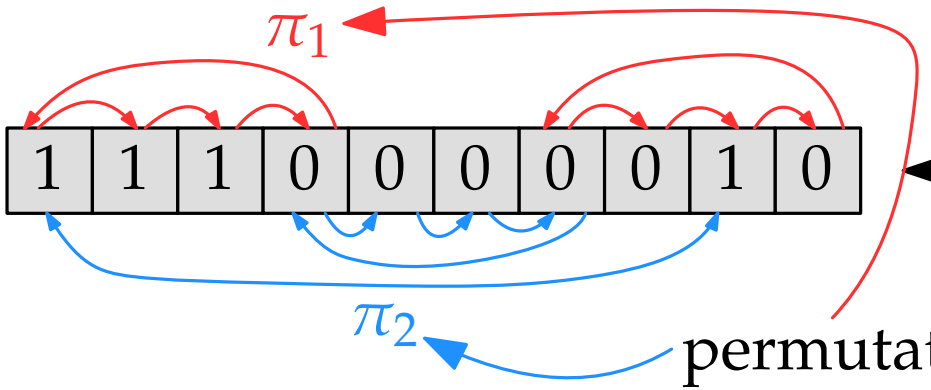

a string $x \in \{0,1\}^n$

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global optimum: smallest element $y \in Gx$

local optimum: $y \in Gx$ such that $y \circ \pi_i \succeq y$ for all $1 \leq i \leq k$

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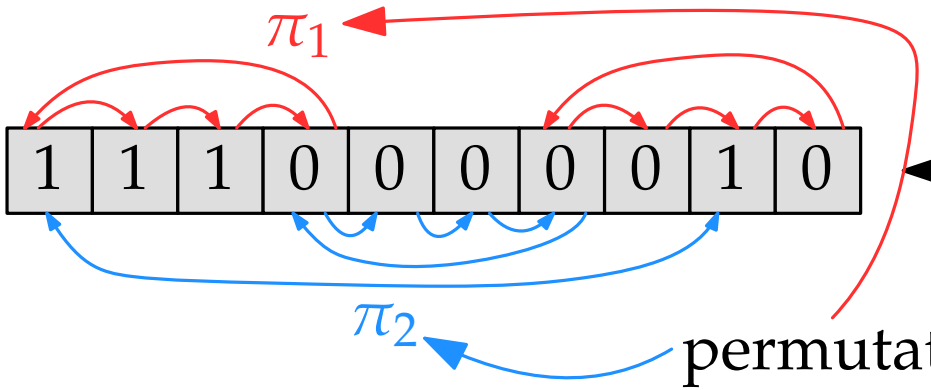

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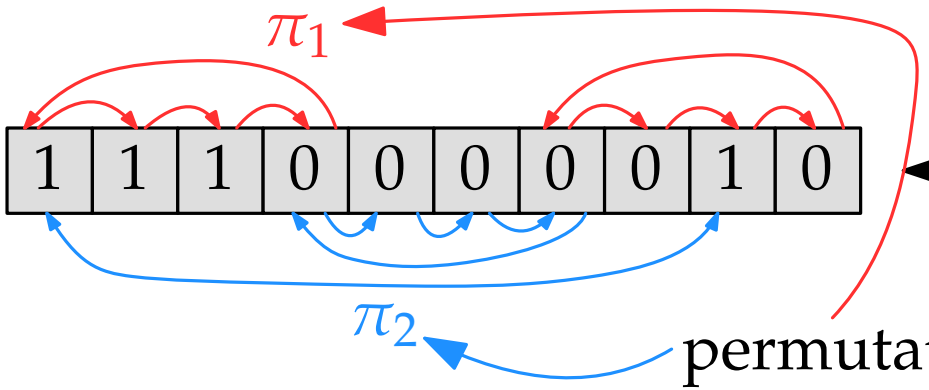
Motivation. This question was posed by Kołodziejczyk and Thapen in the context of simulating symmetry-breaking rules in certain proof systems.

About the Global Optimum



a string $x \in \{0,1\}^n$

permutations $\pi_1, \ldots \pi_k$ from S_n

global optimum: smallest element of orbit $\{x \circ \pi \mid \pi \in \langle \pi_1, \ldots, \pi_k \rangle\}$

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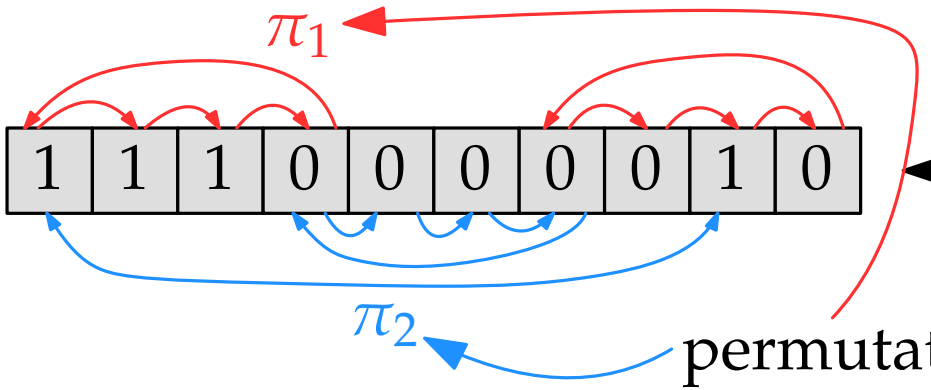

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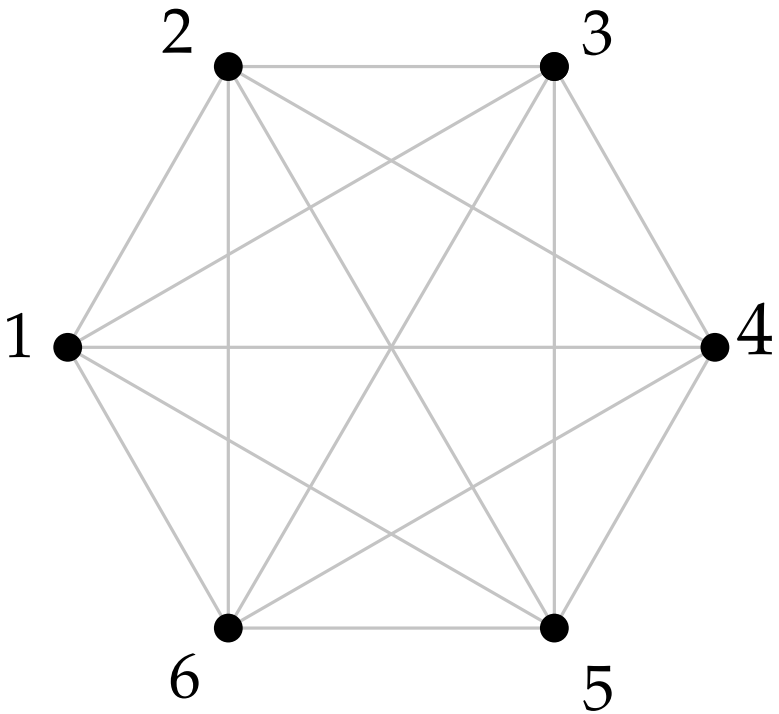

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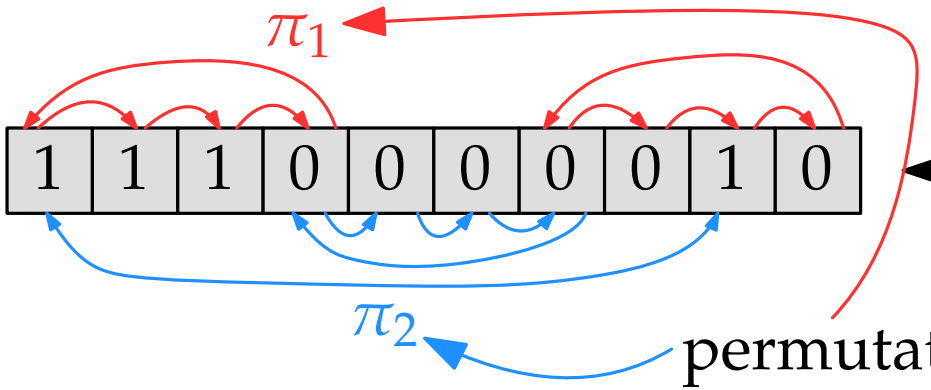

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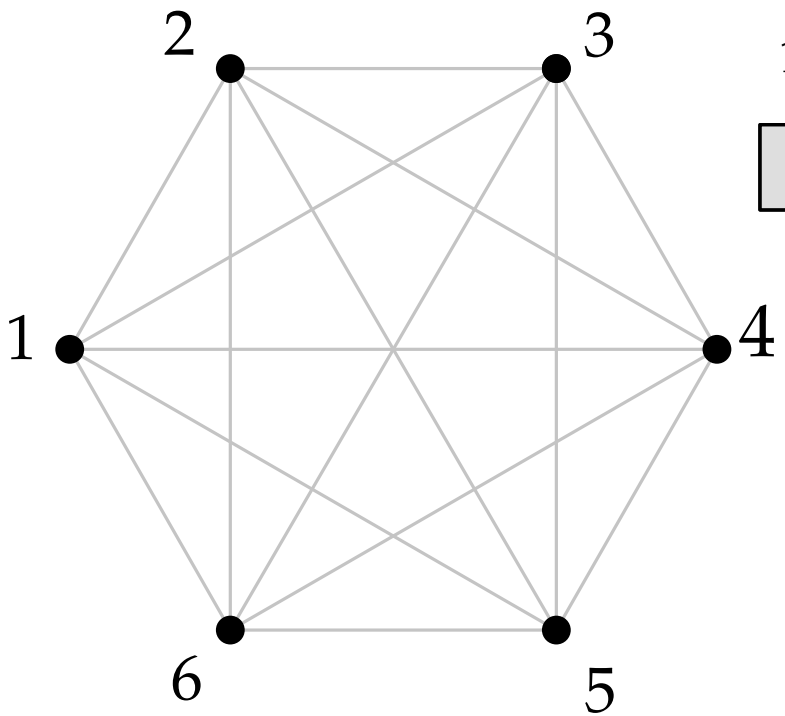

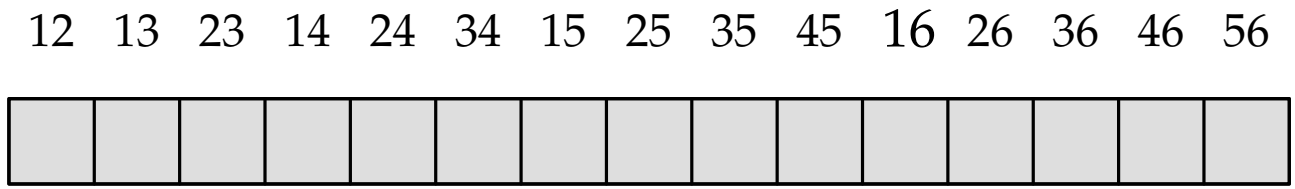

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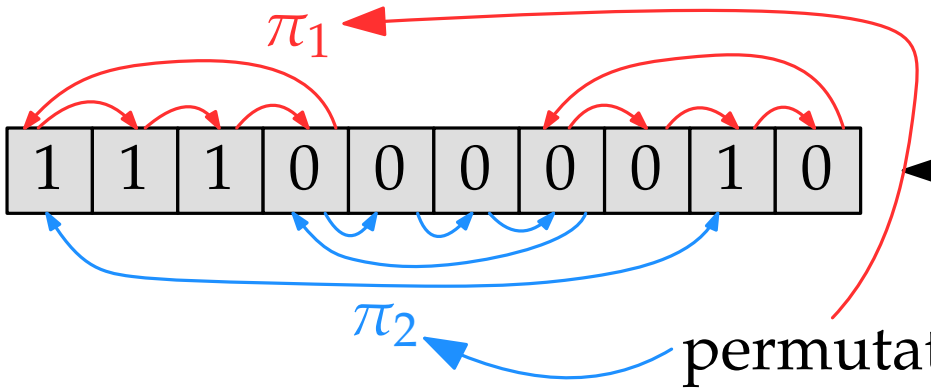

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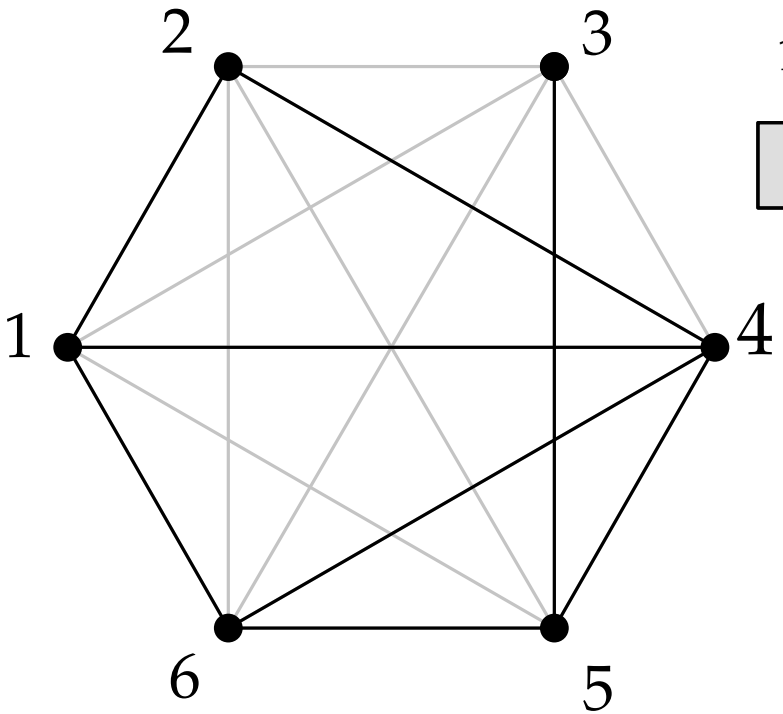

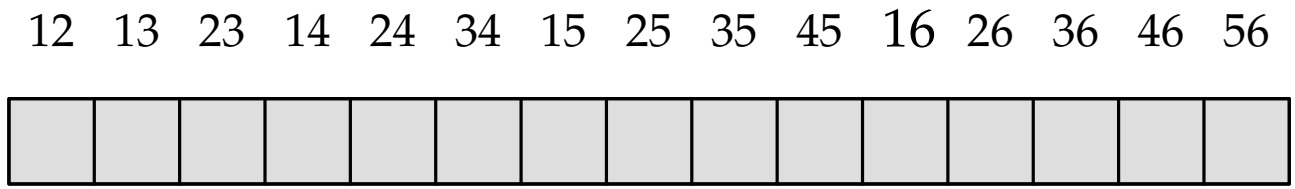

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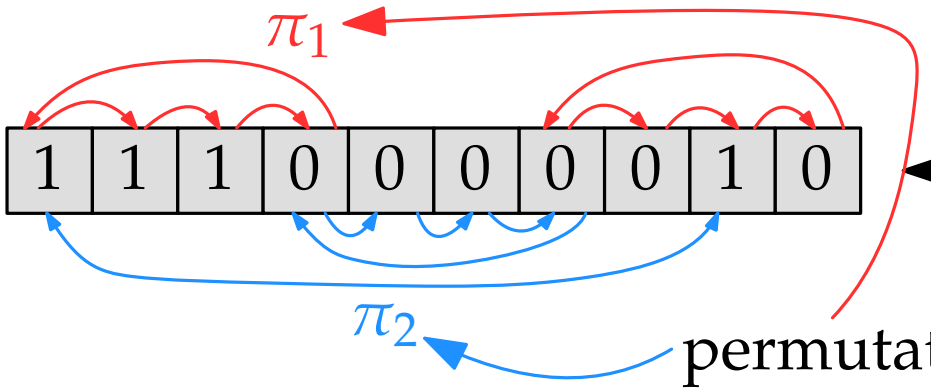

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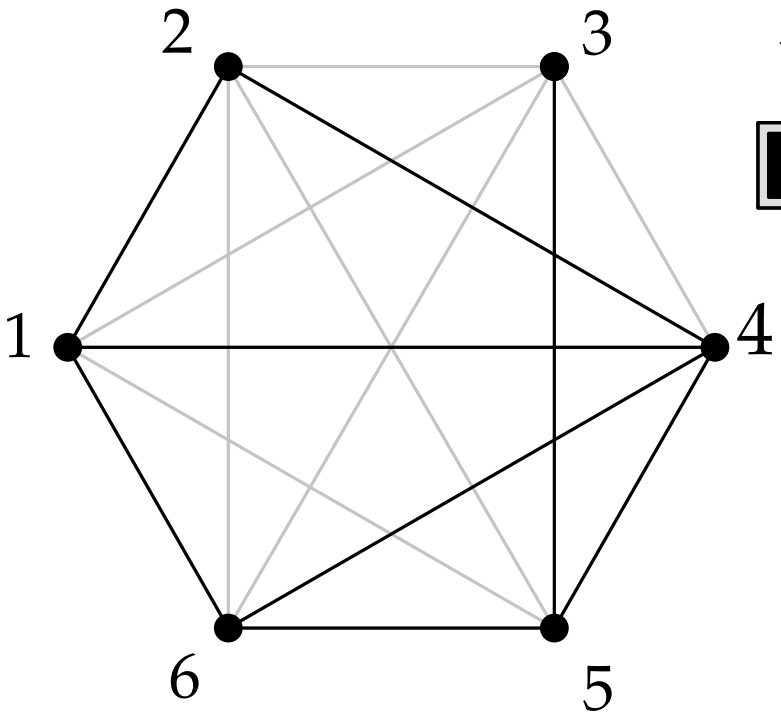

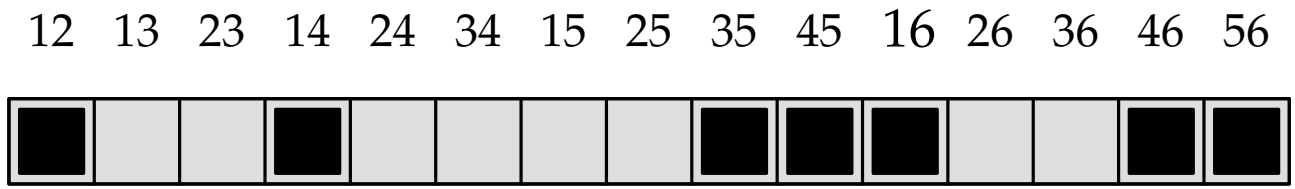

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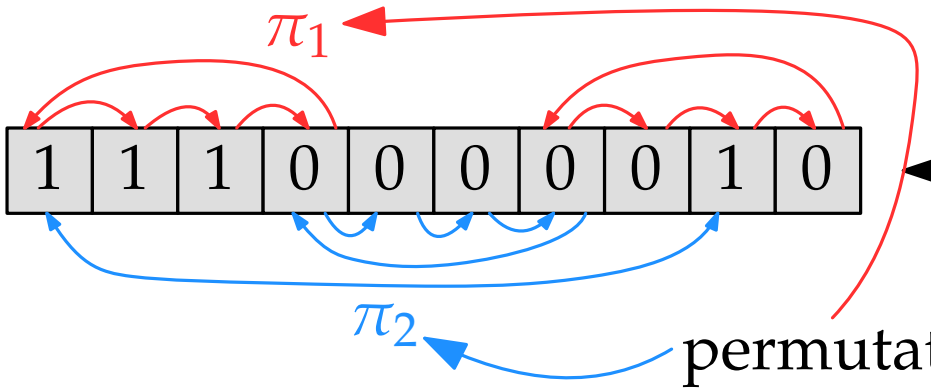

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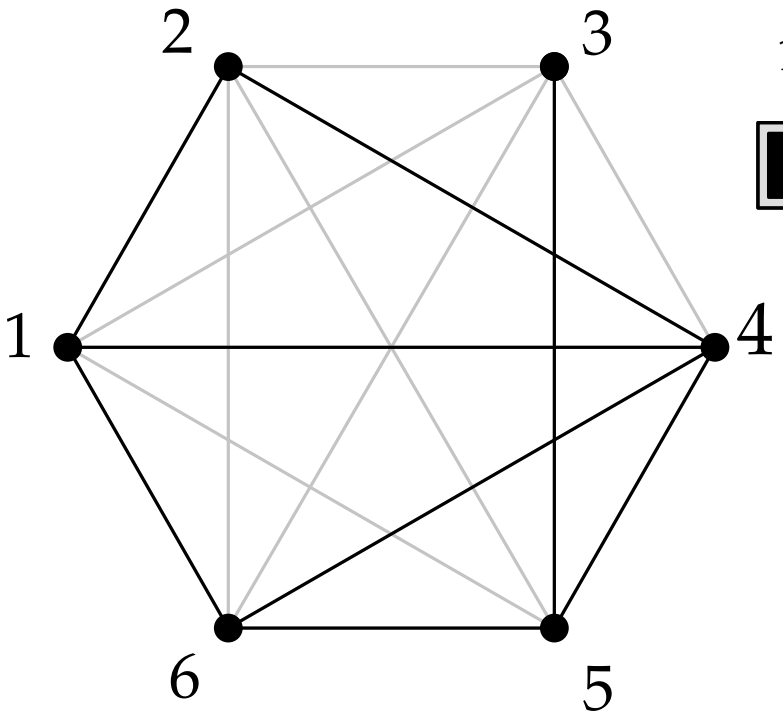
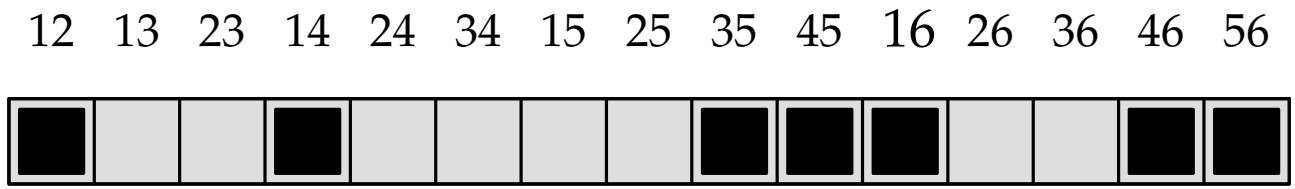
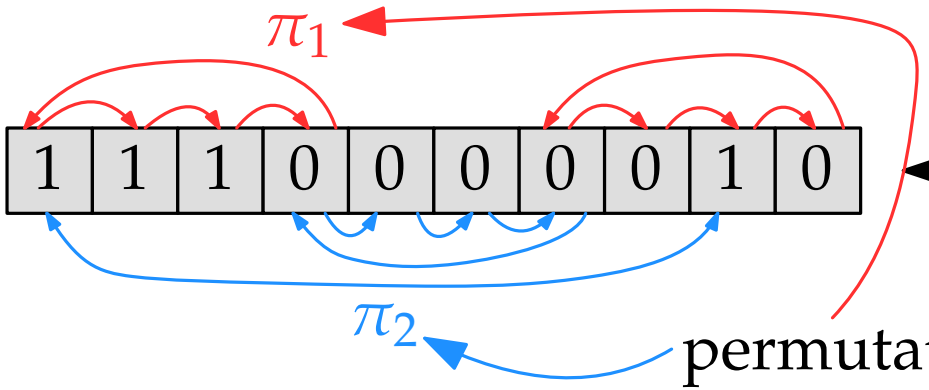




π_1 := switch vertex 1 and 2

About the Global Optimum



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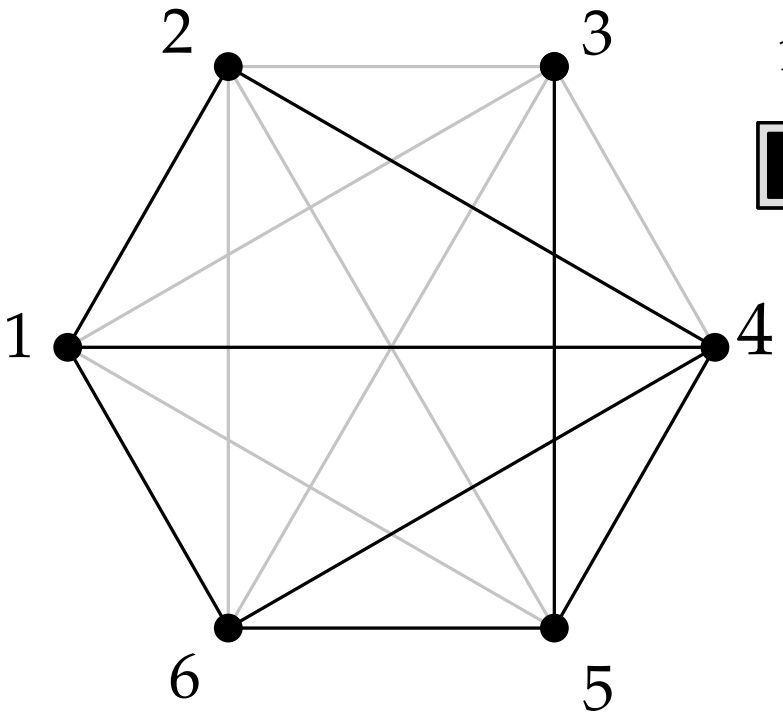
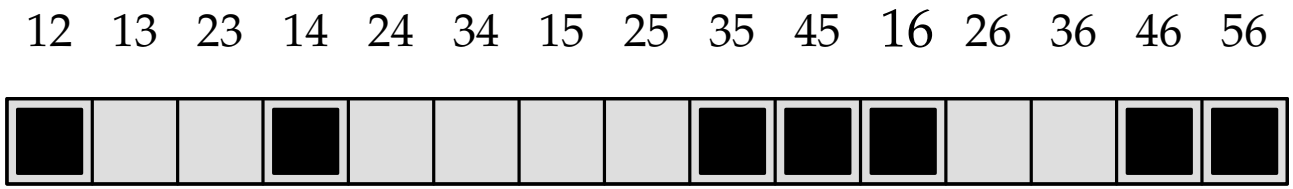
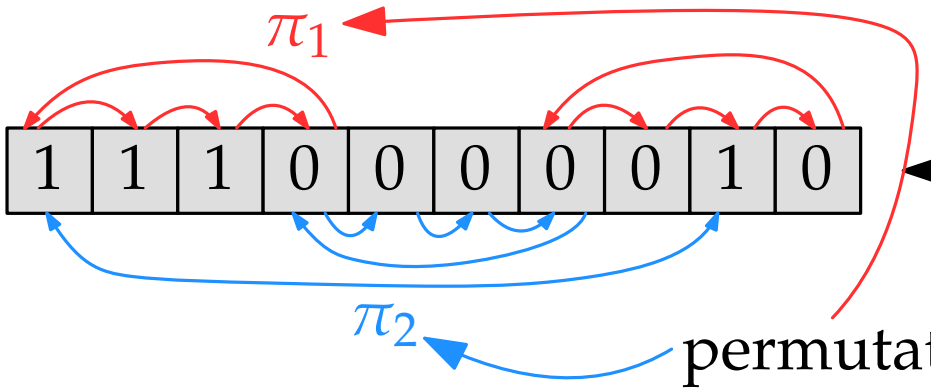




π_1 := switch vertex 1 and 2

π_2 := rotate vertices clockwise

About the Global Optimum



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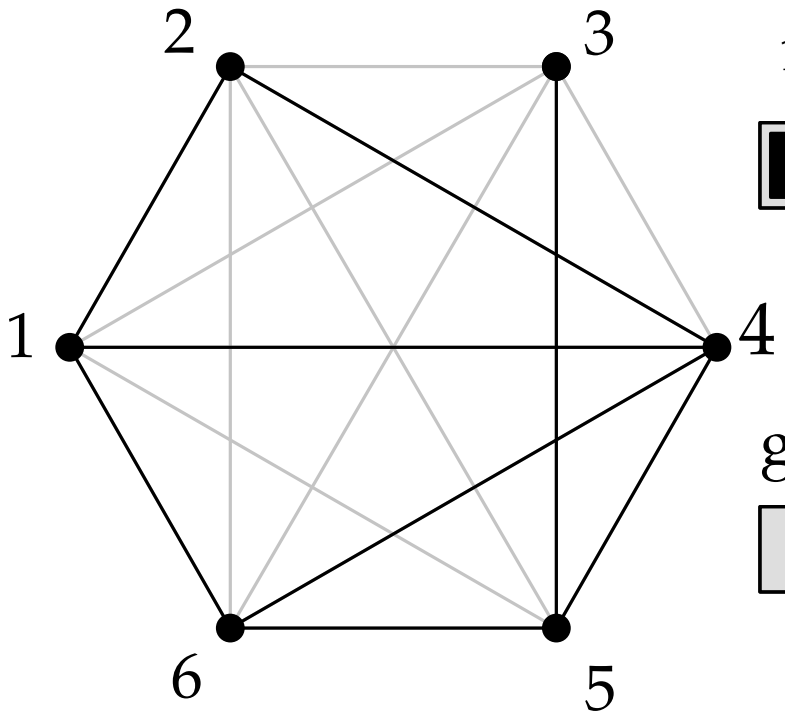

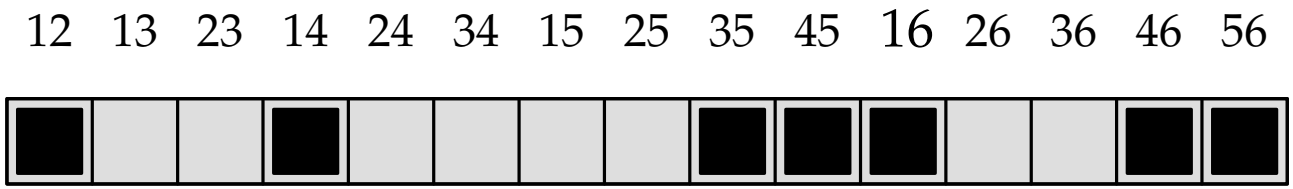

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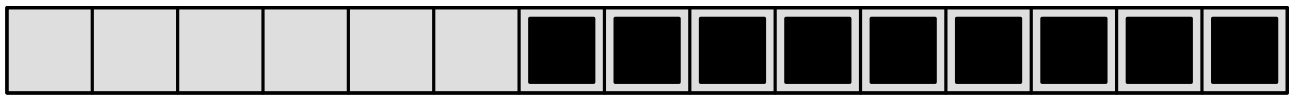

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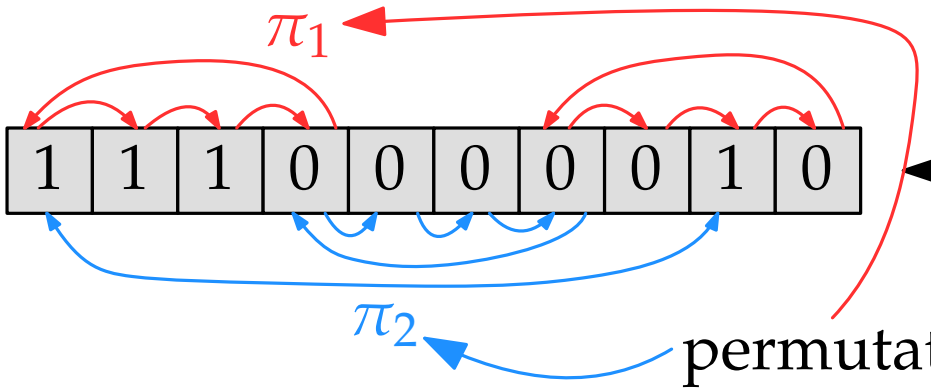

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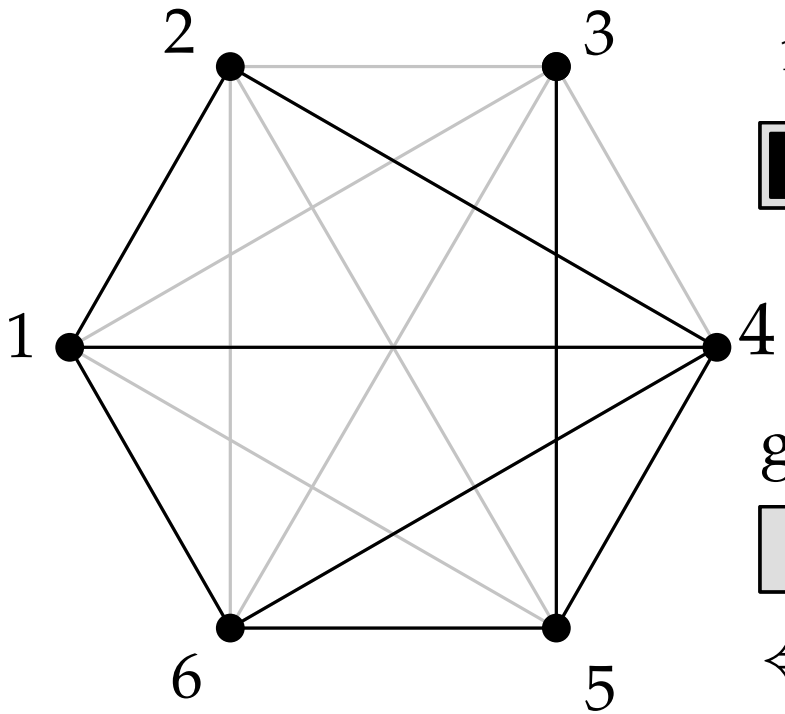

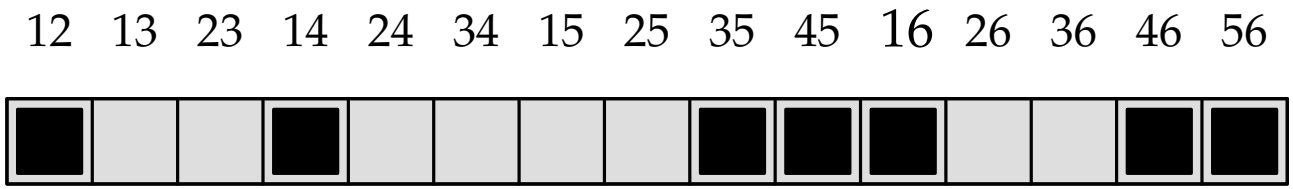

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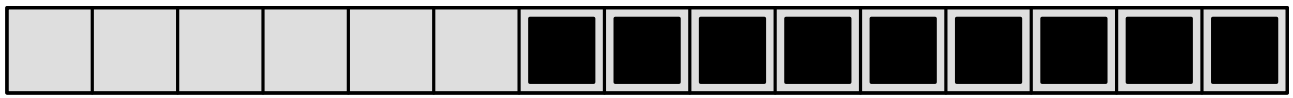
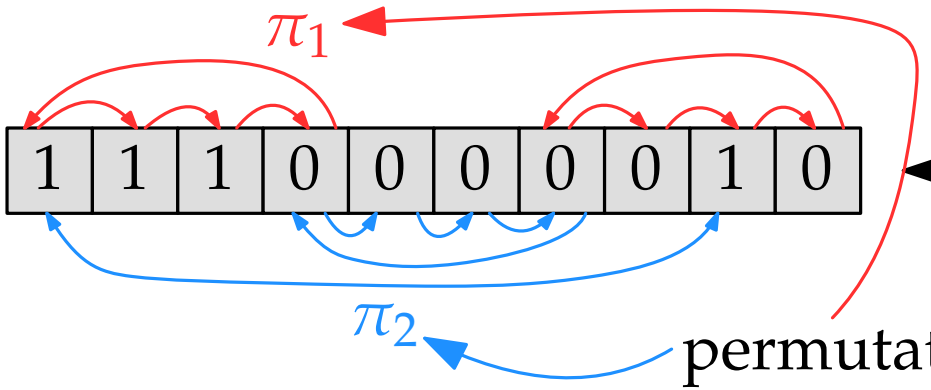


$\Longleftrightarrow$ G has independent set of size k

About the Global Optimum



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Theorem (Babai and Luks, 1983). This is NP-hard, even for $k = 2$.

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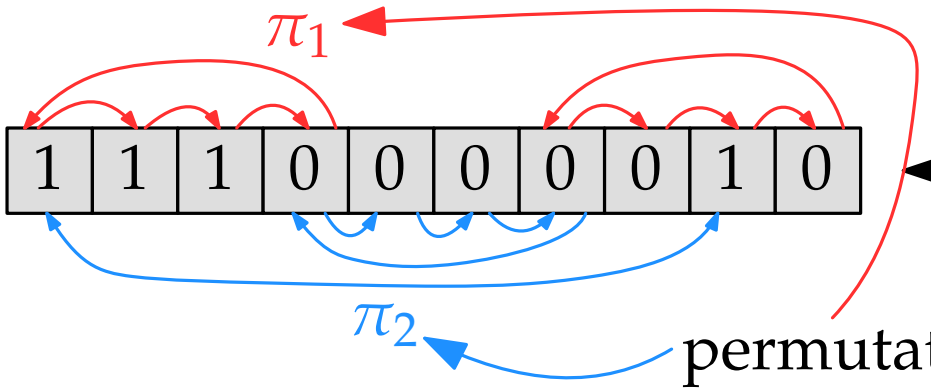

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Theorem (S. and Tantow, 2025). This is NP-hard, even for $k = 1$.

Finding a Local Optimum

a string $x \in \{0,1\}^n$

permutations $\pi_1, \ldots \pi_k$ from S_n

Finding a Local Optimum

a string $x \in \{0,1\}^n$

permutations $\pi_1, \ldots \pi_k$ from S_n

x

Finding a Local Optimum

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permutations $\pi_1, \ldots \pi_k$ from S_n

$x \prec x \circ \pi_{i_1}$

Finding a Local Optimum

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Finding a Local Optimum

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We call this an *improving path*.

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then all is well

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shortcut to finding a local minimum?

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shortcut to finding a local minimum? ↓ clever algorithms

↓ only polynomially many steps then all is well

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clever algorithms

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then all is well

clever algorithms

(few and far between)

hardness results

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We call this an *improving path*.

An improving path will arrive at a local optimum.

How long can this take?

- only polynomially many steps then all is well
- exponentially many steps shortcut to finding a local minimum?
  - clever algorithms (few and far between)
  - hardness results (S. and Tantow, 2025)

Finding a Local Optimum for $k = 1$

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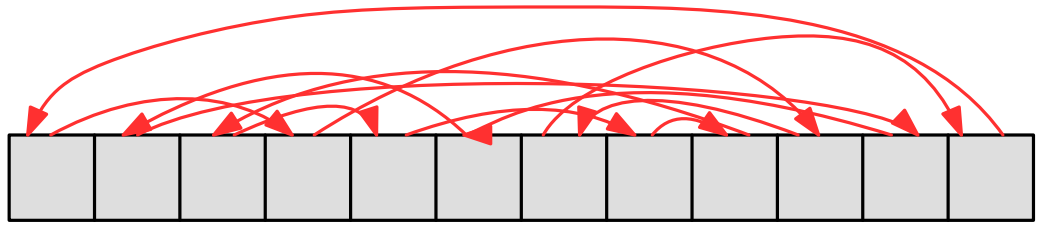

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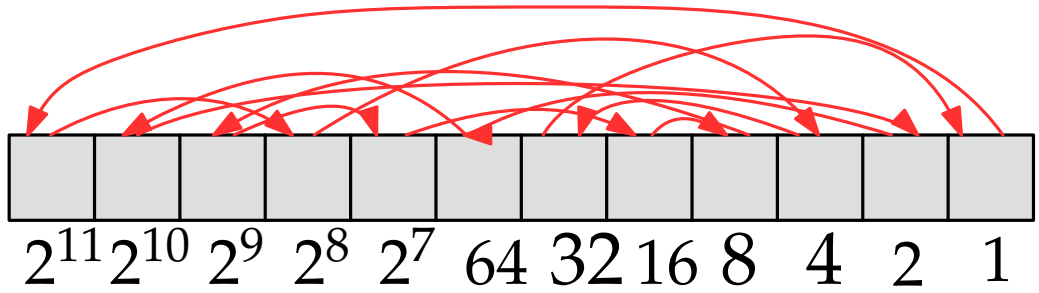

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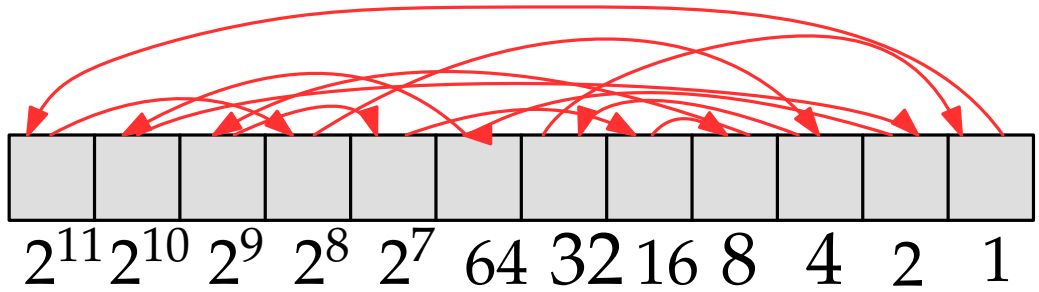
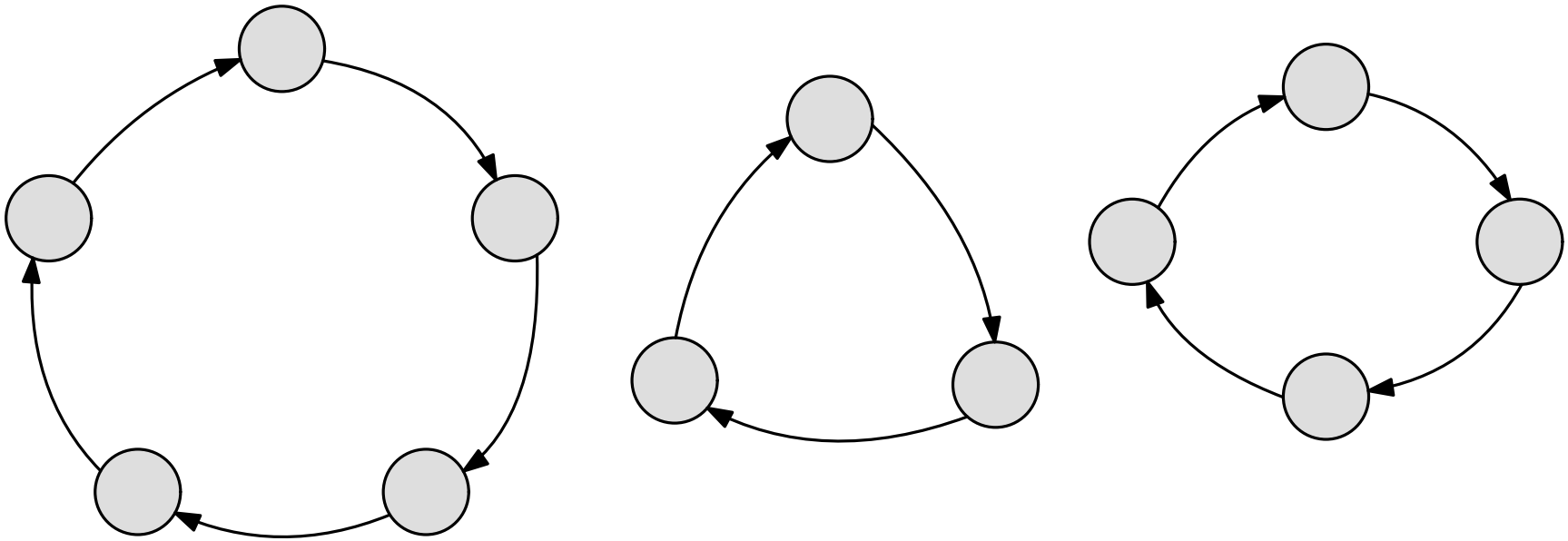
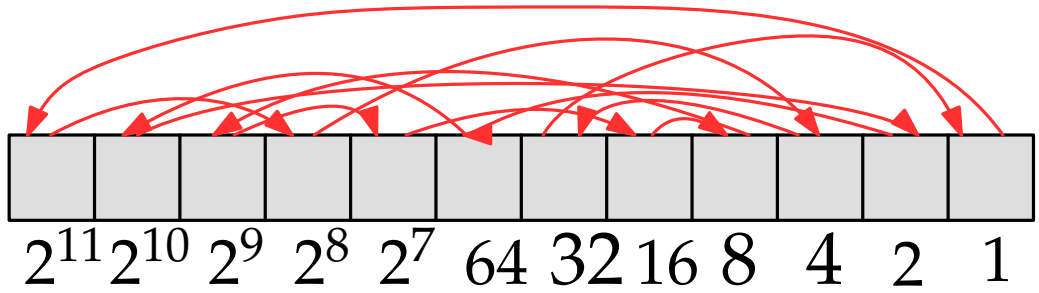
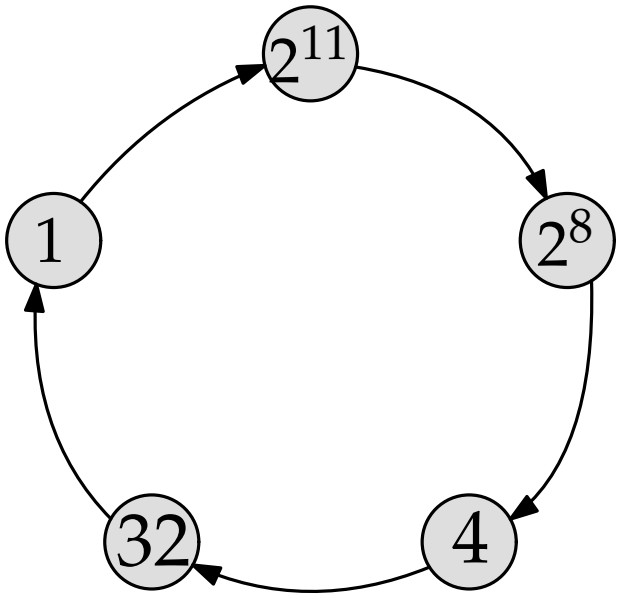
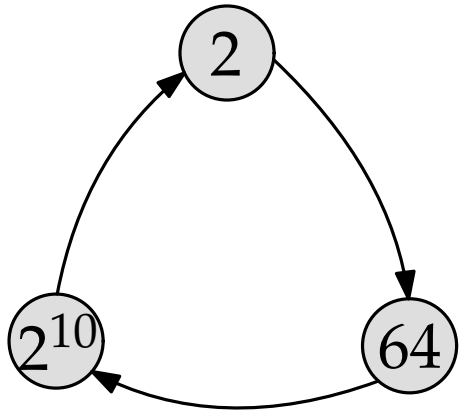


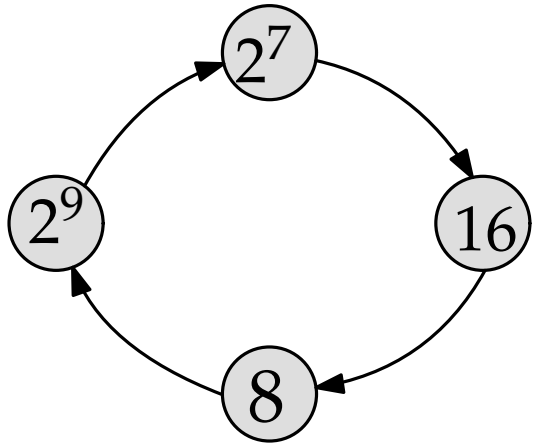


Finding a Local Optimum for $k = 1$

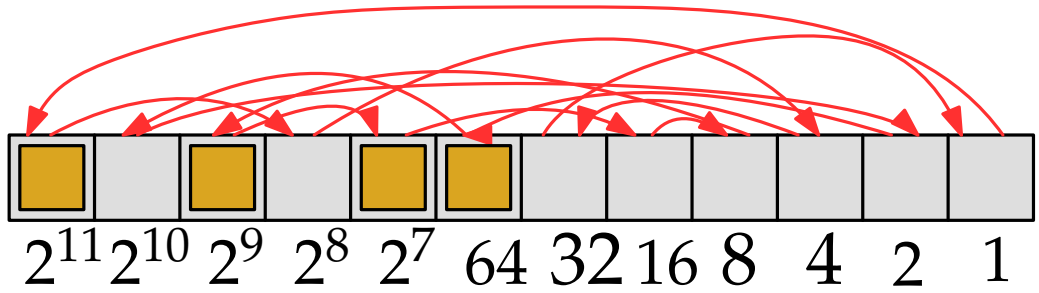


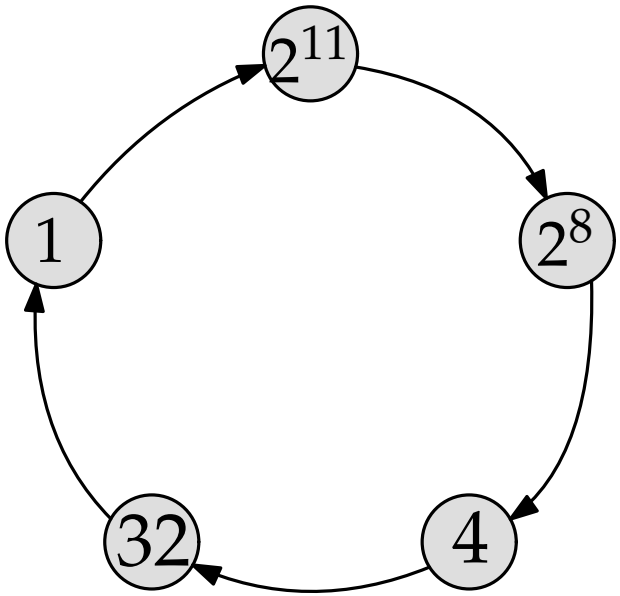


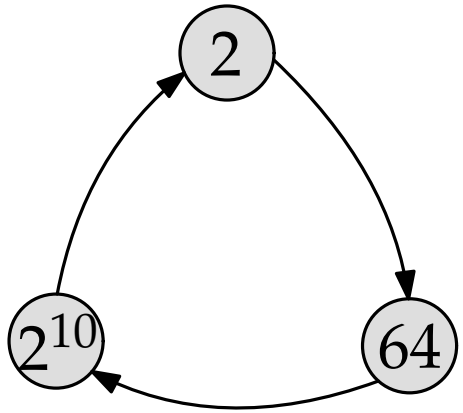


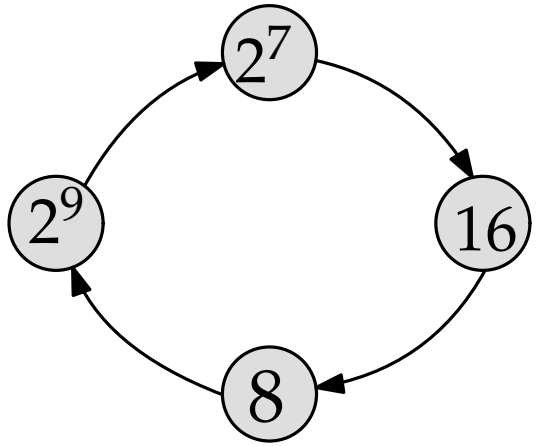


Finding a Local Optimum for $k = 1$









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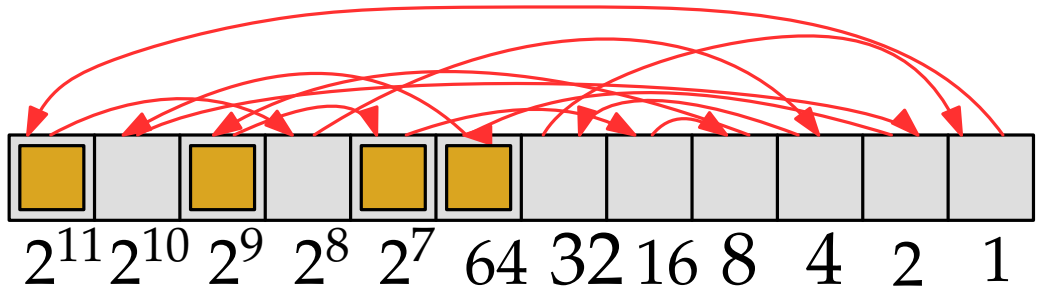

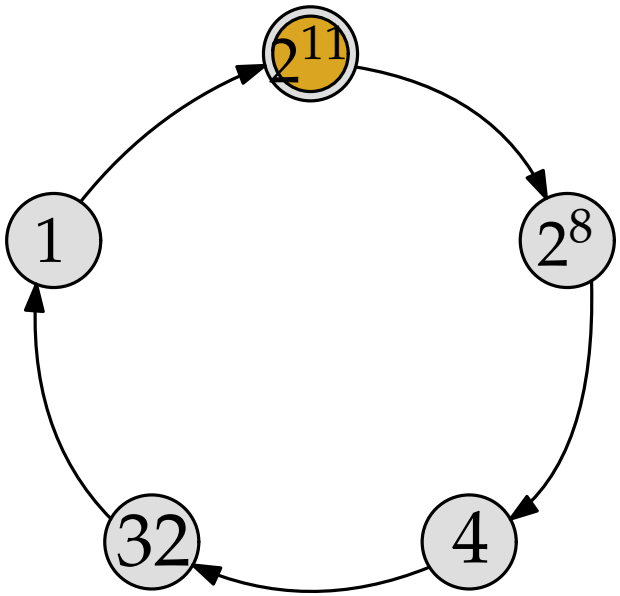

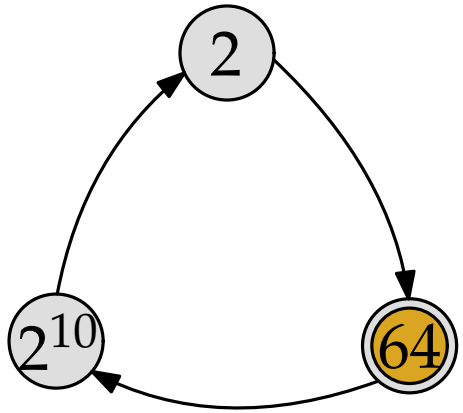

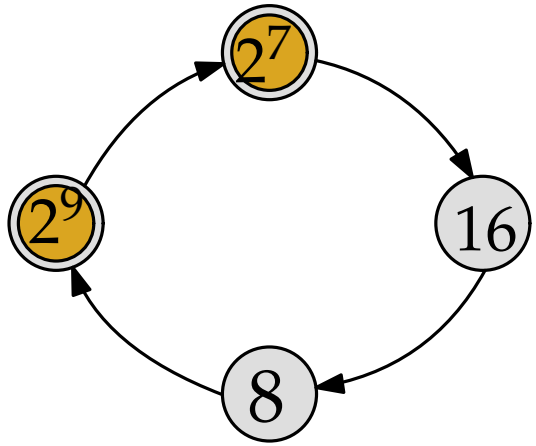

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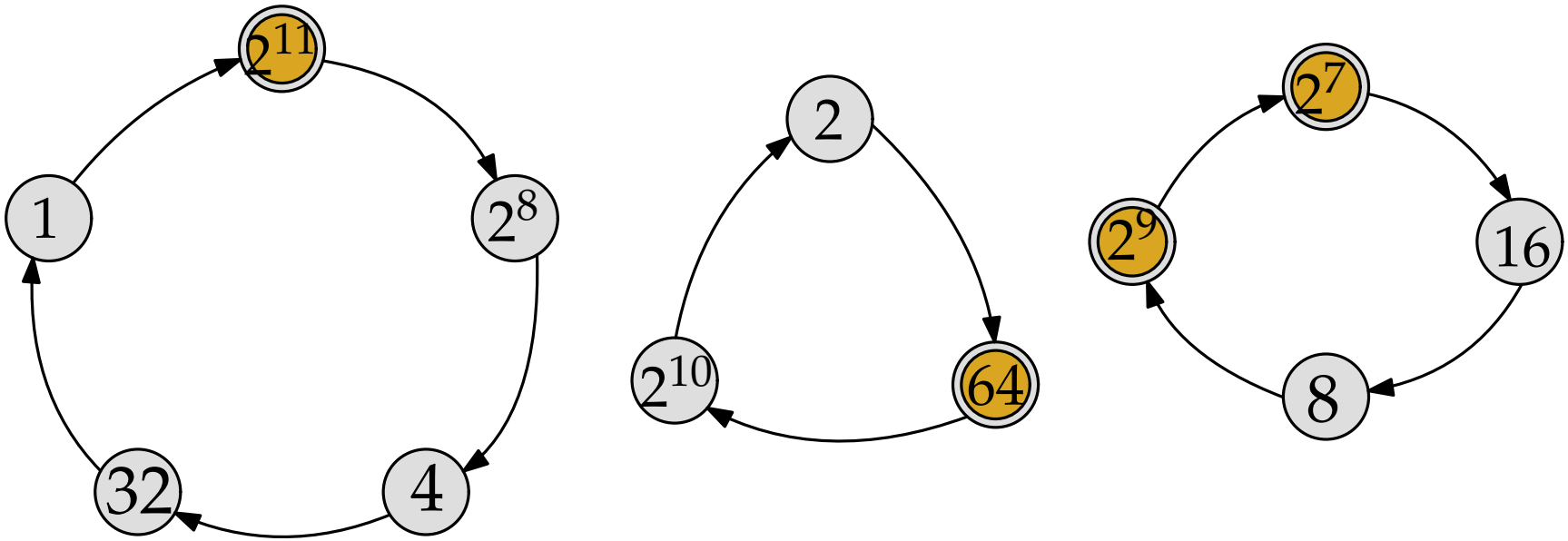

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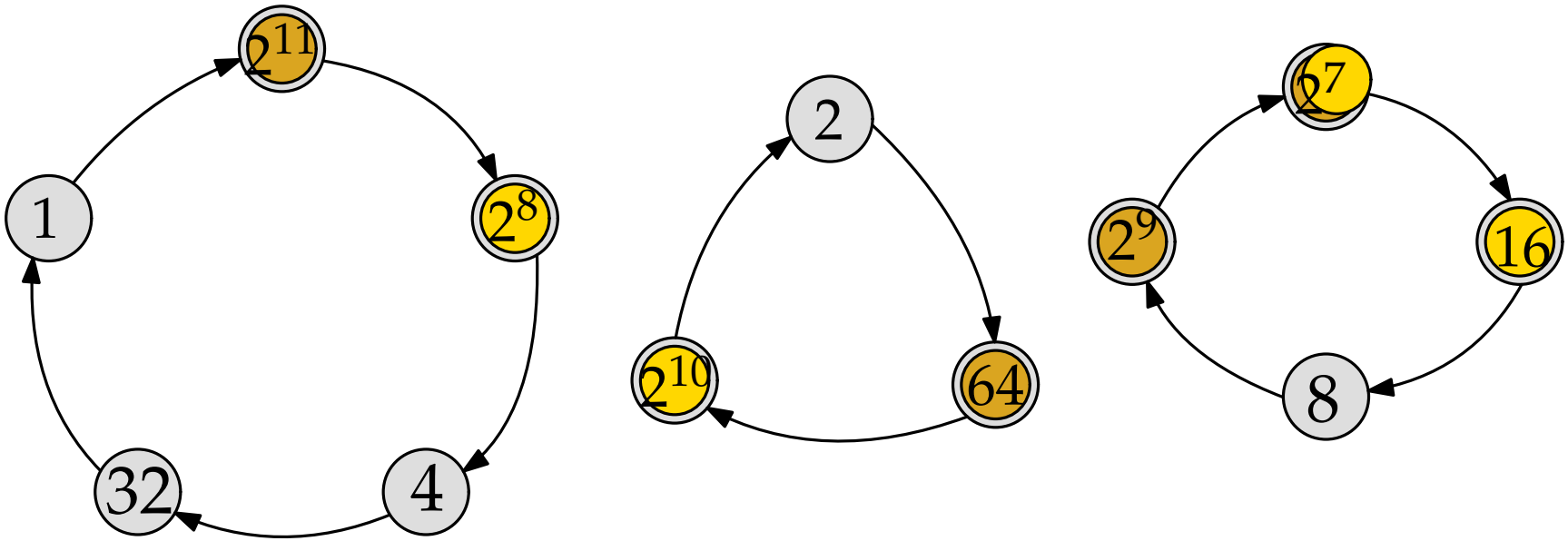

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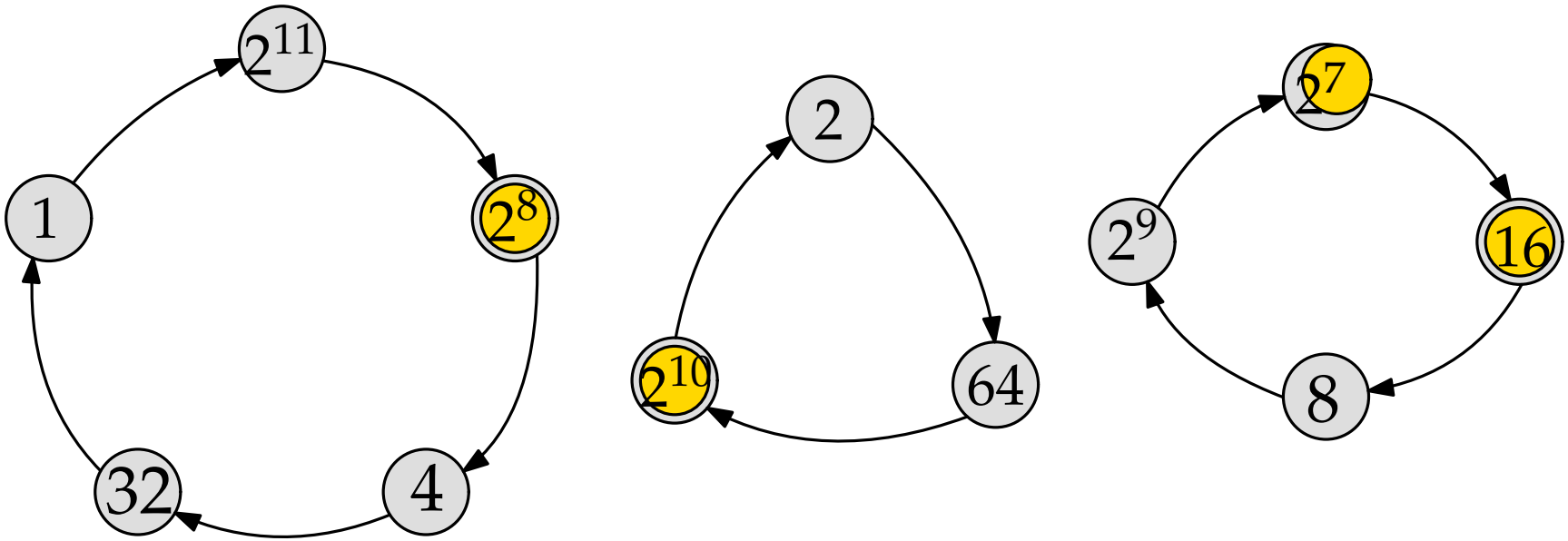

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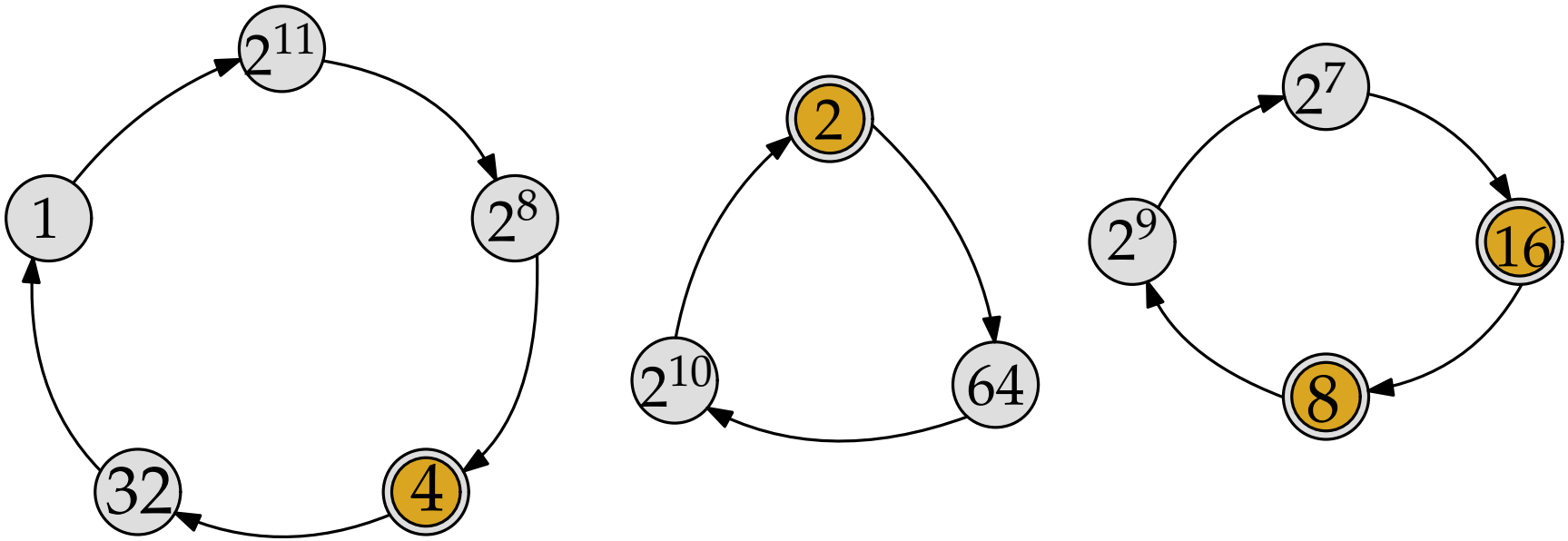

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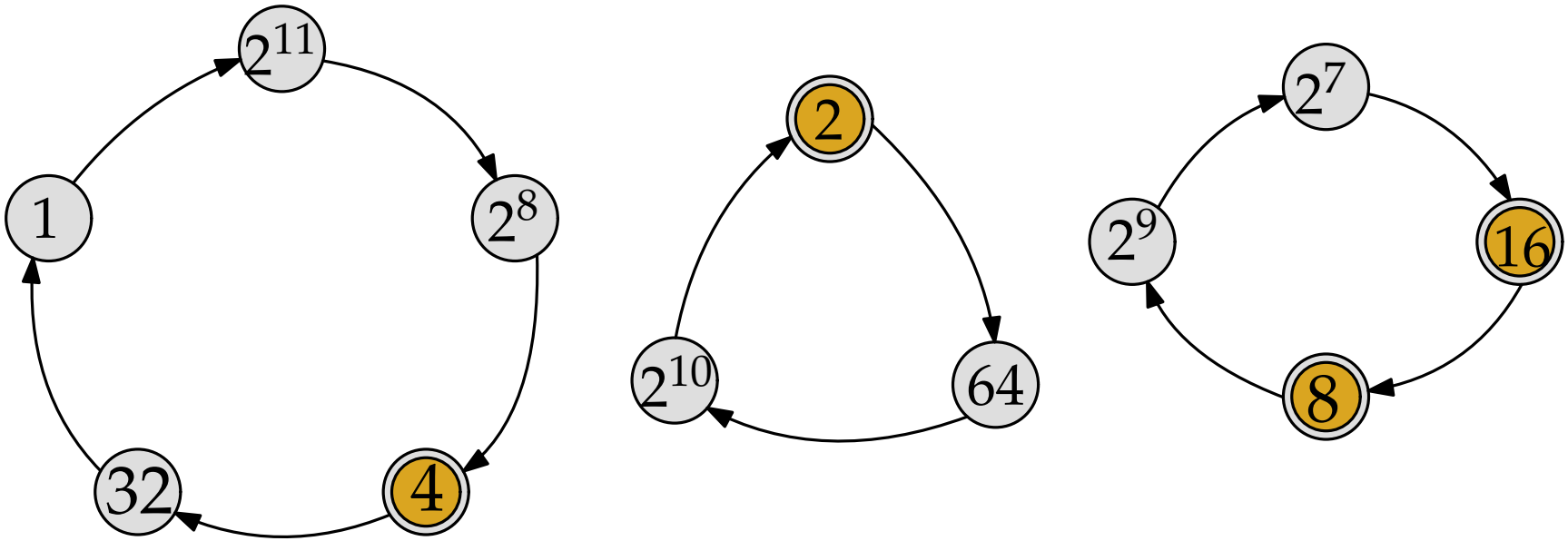
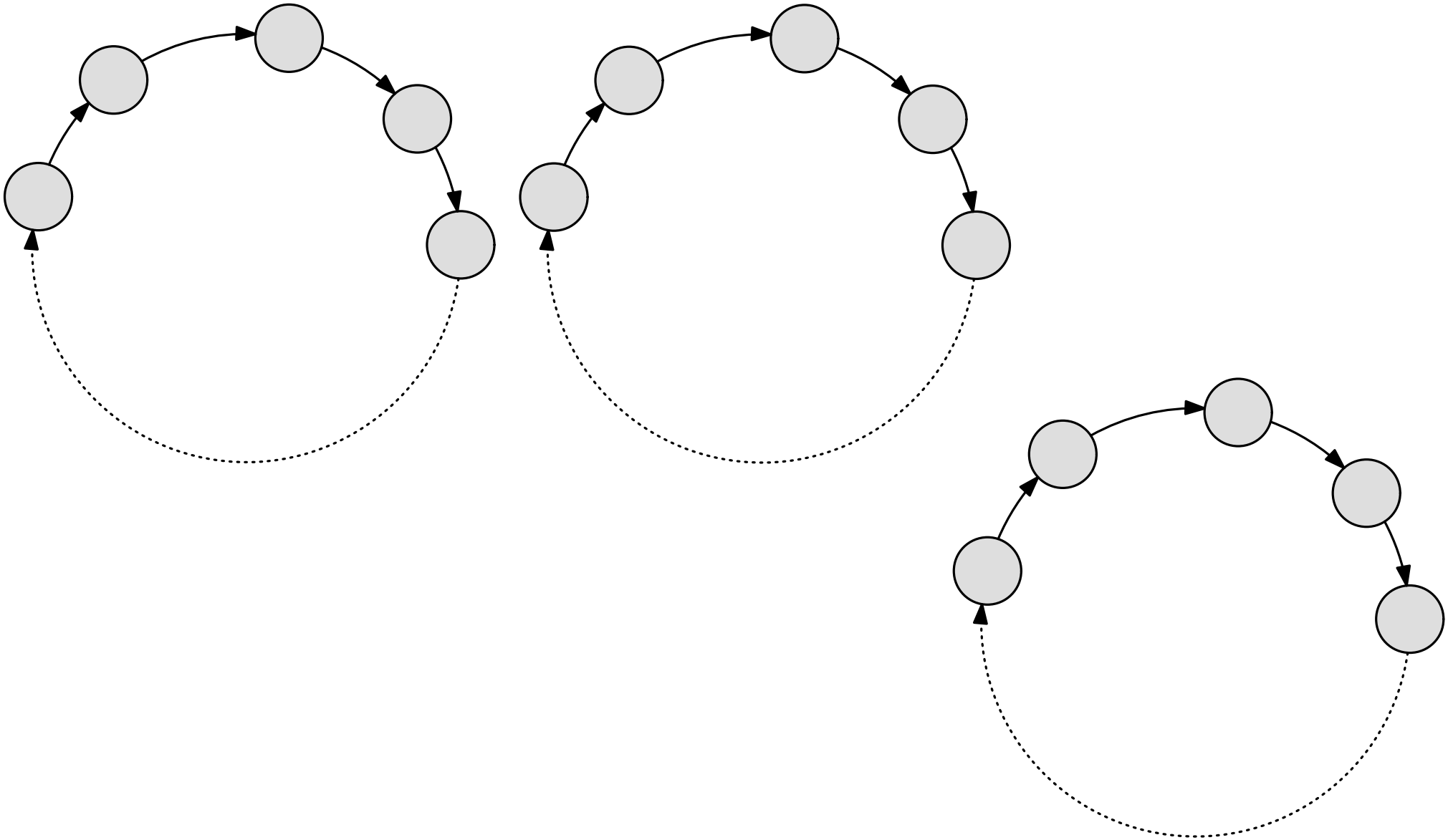


After two steps, the improving path has reached a local minimum

Finding a Local Optimum for $k = 1$



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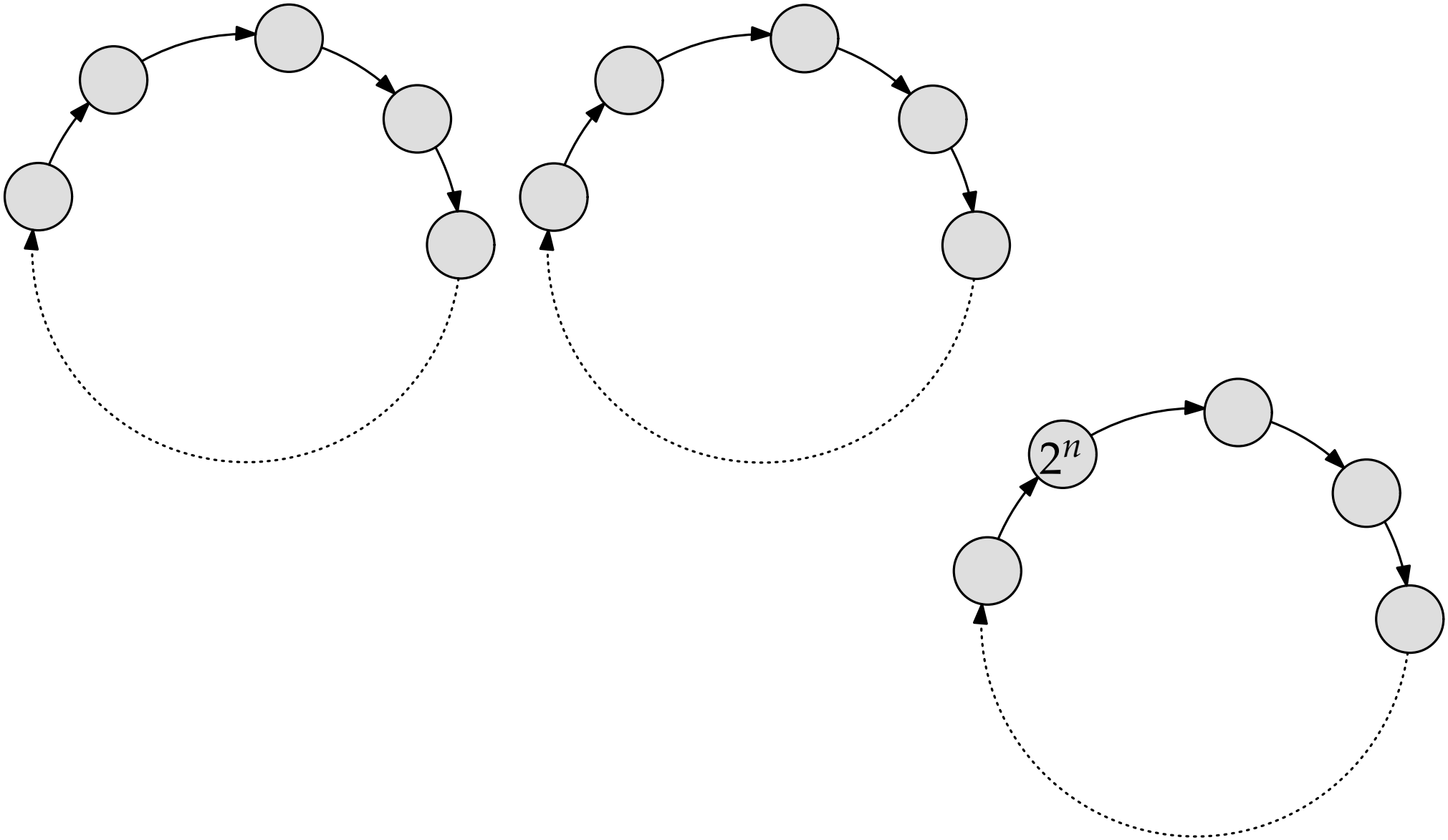

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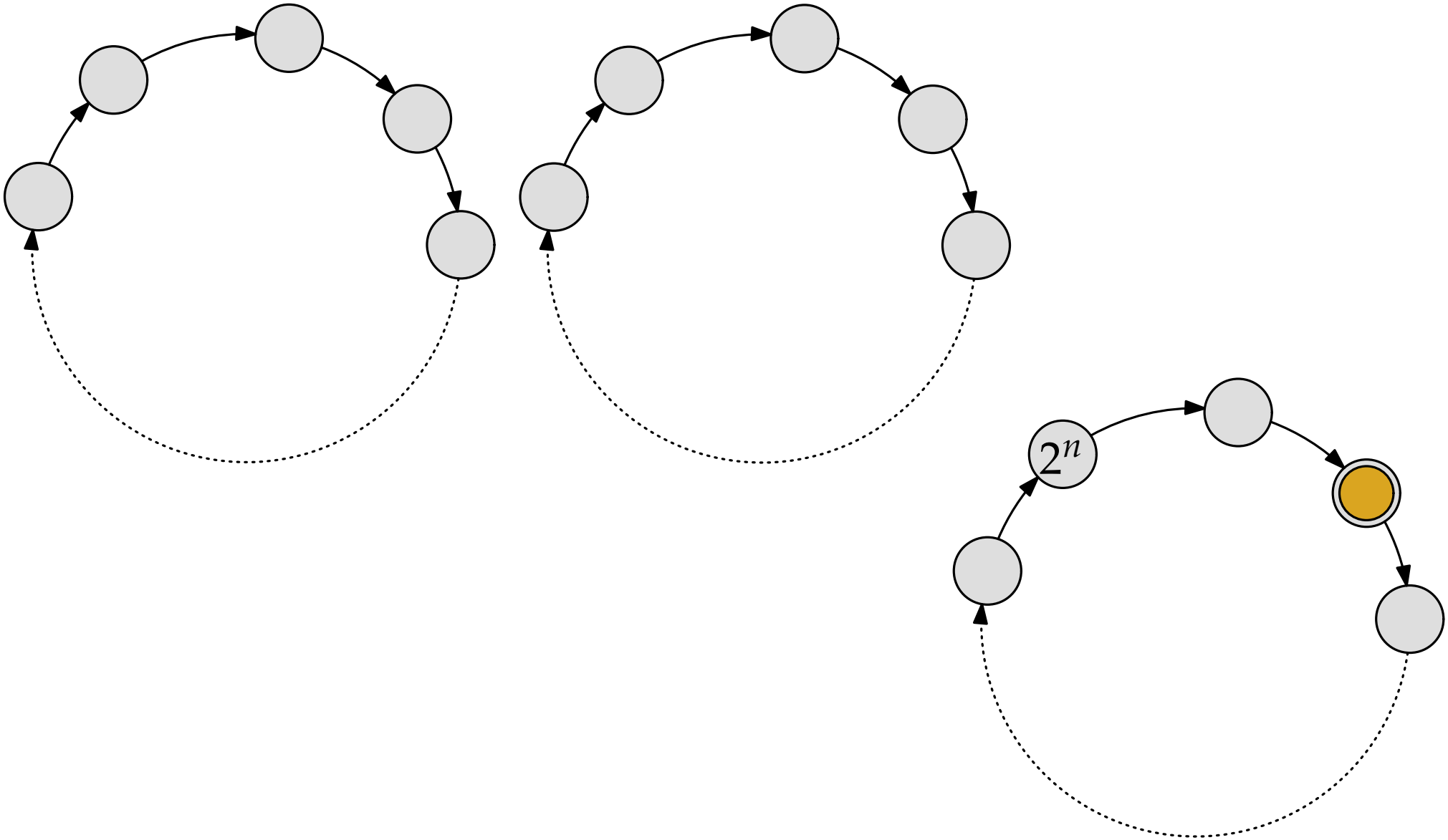
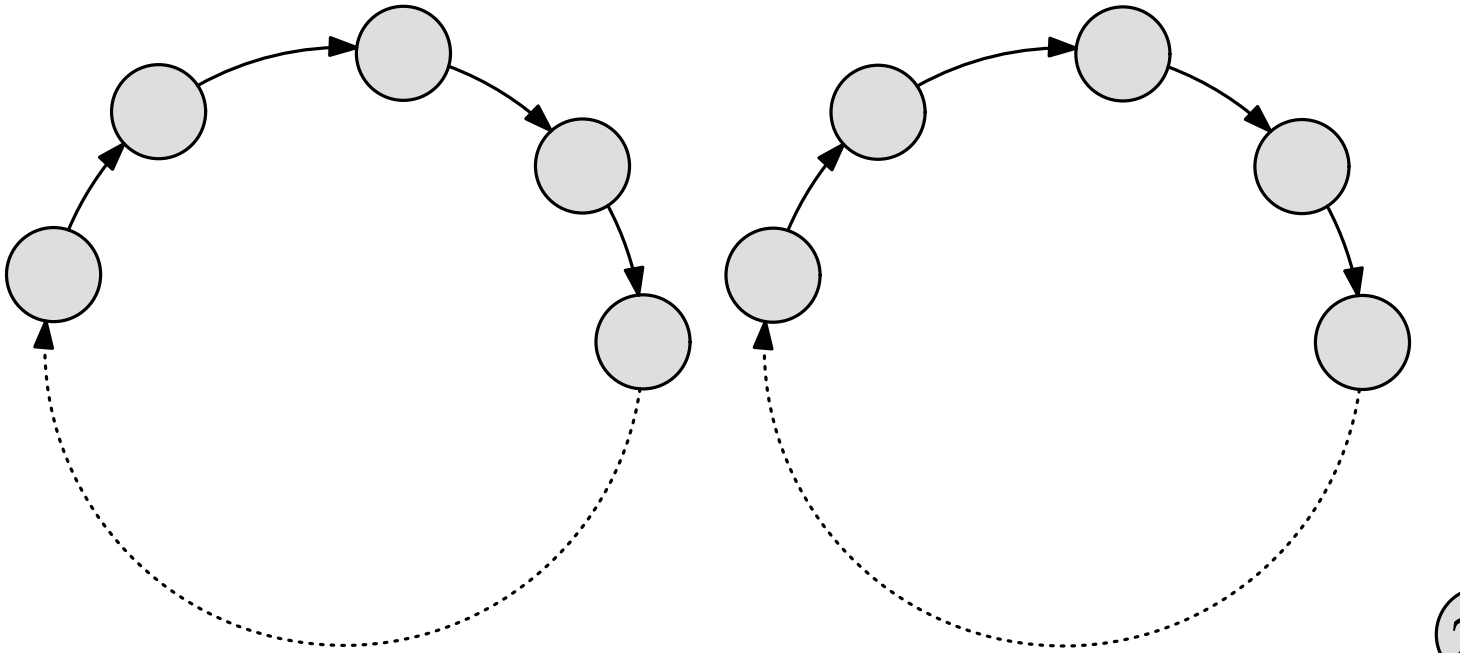
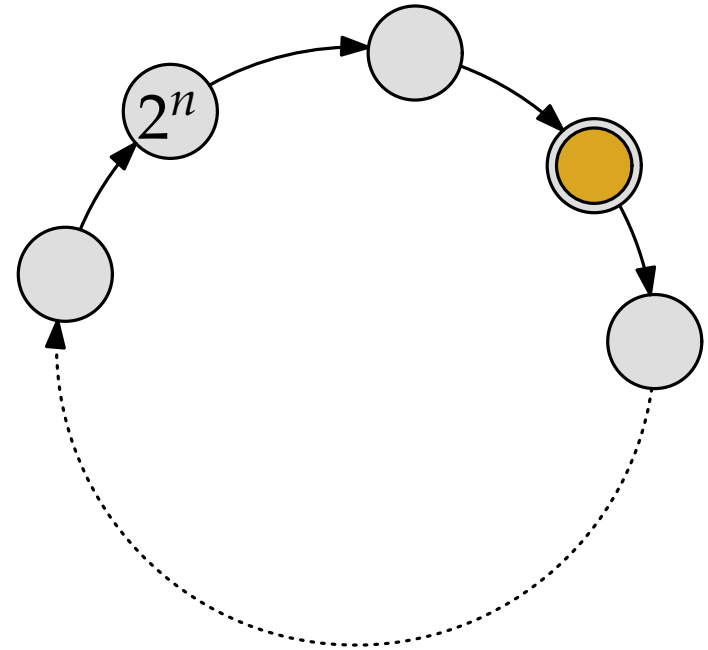


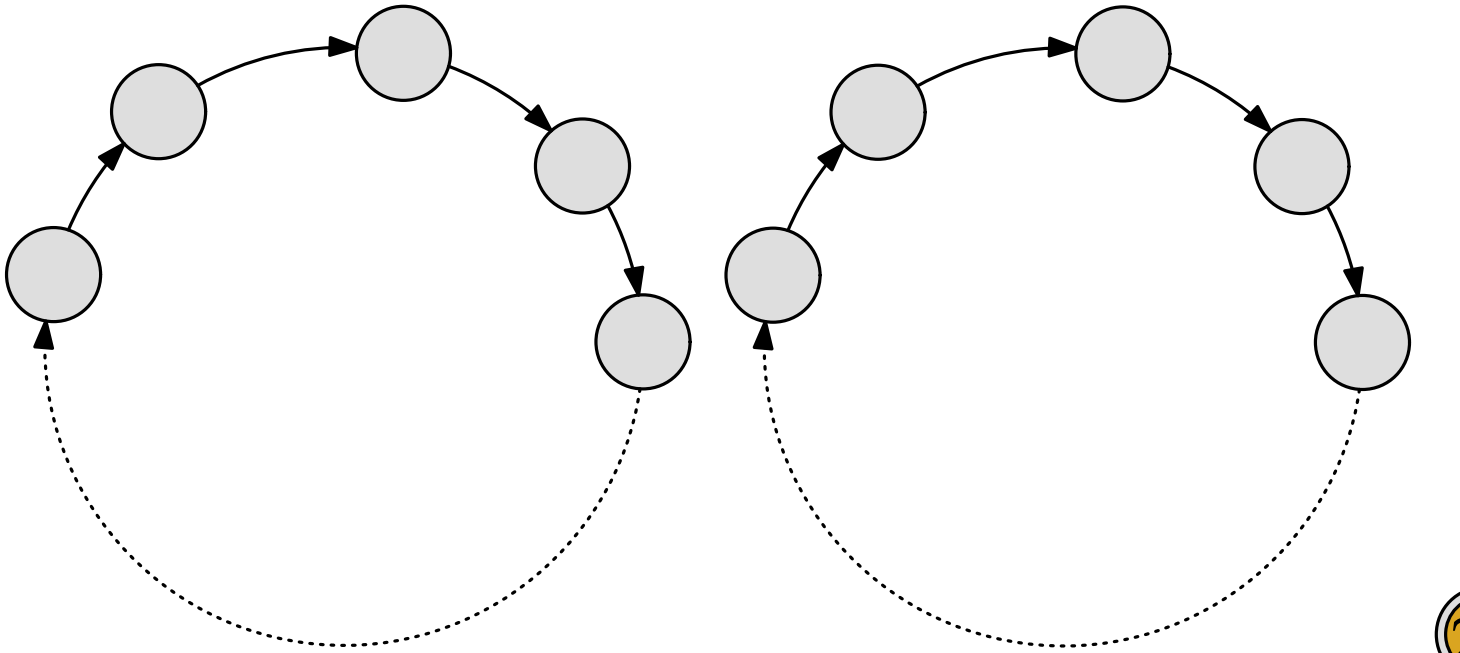
Finding a Local Optimum for $k = 1$

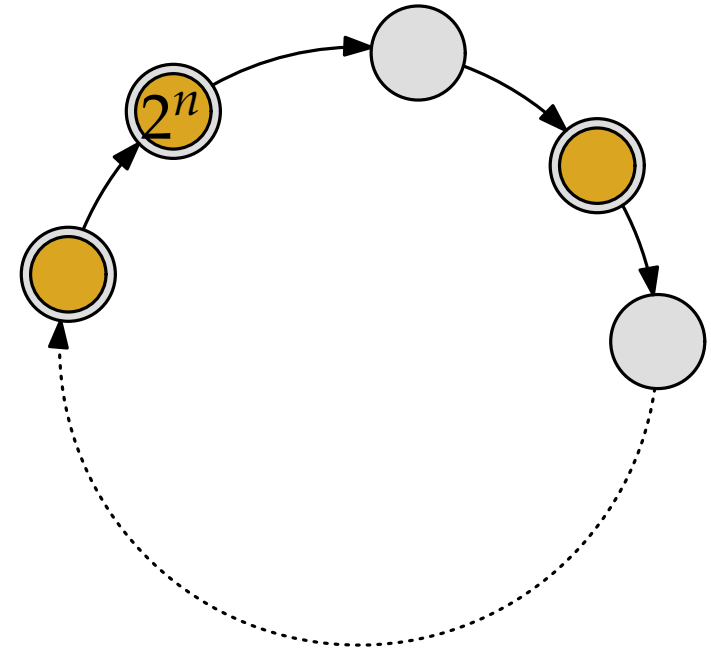




Within at most $n - 1$ steps, the 2^n-position will change from no-coin to yes-coin and the weight of the string $x \circ \pi$ will increase!

Finding a Local Optimum for $k = 1$





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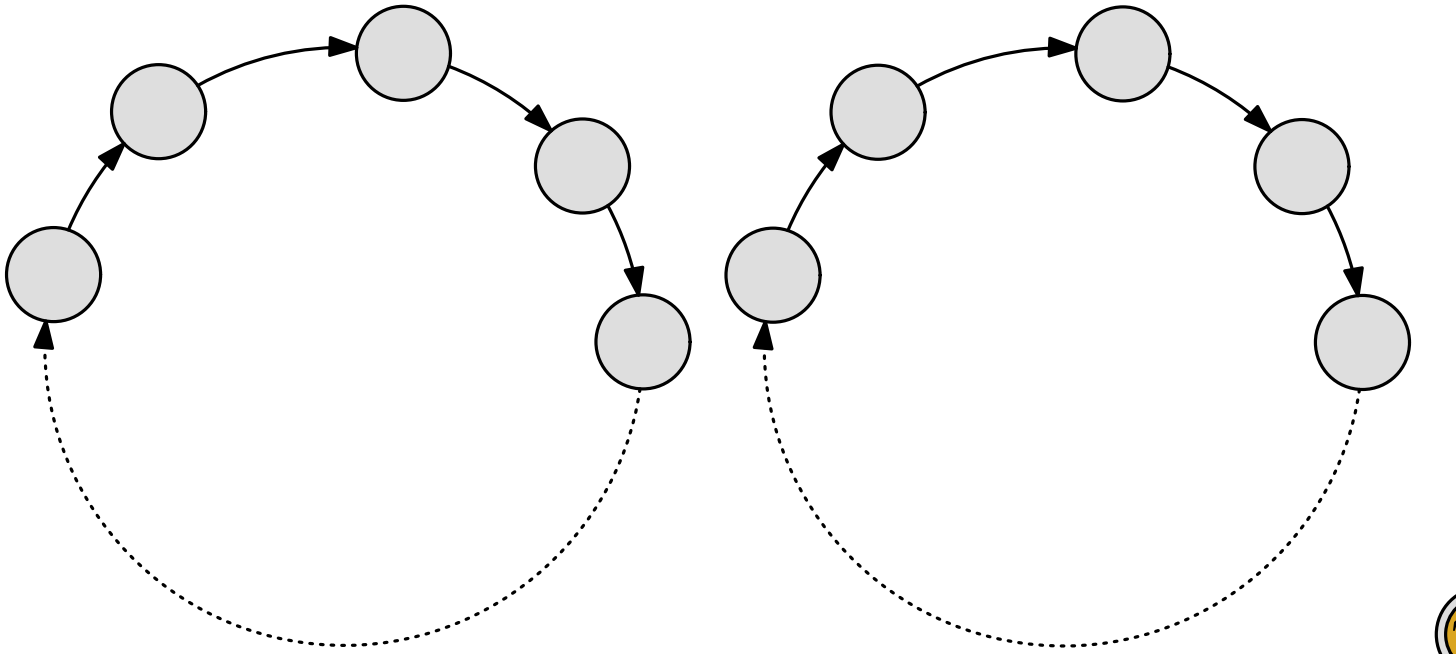
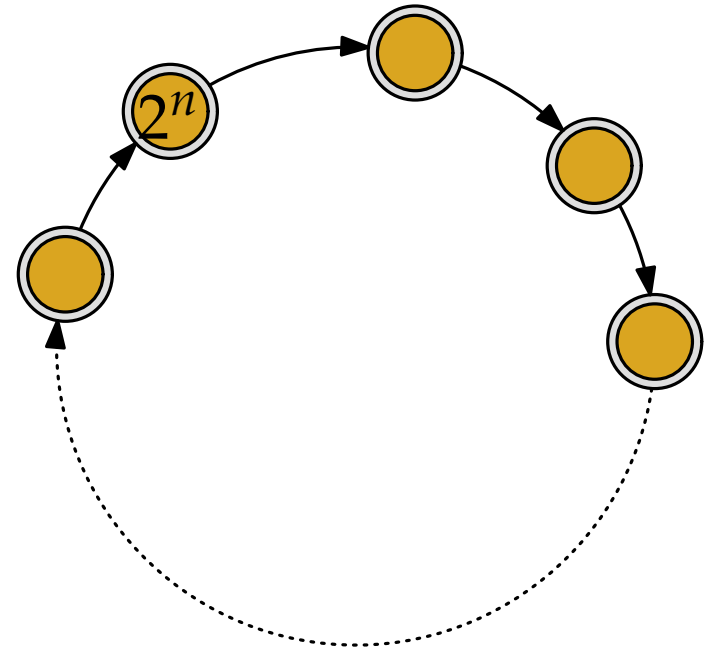
Finding a Local Optimum for $k = 1$

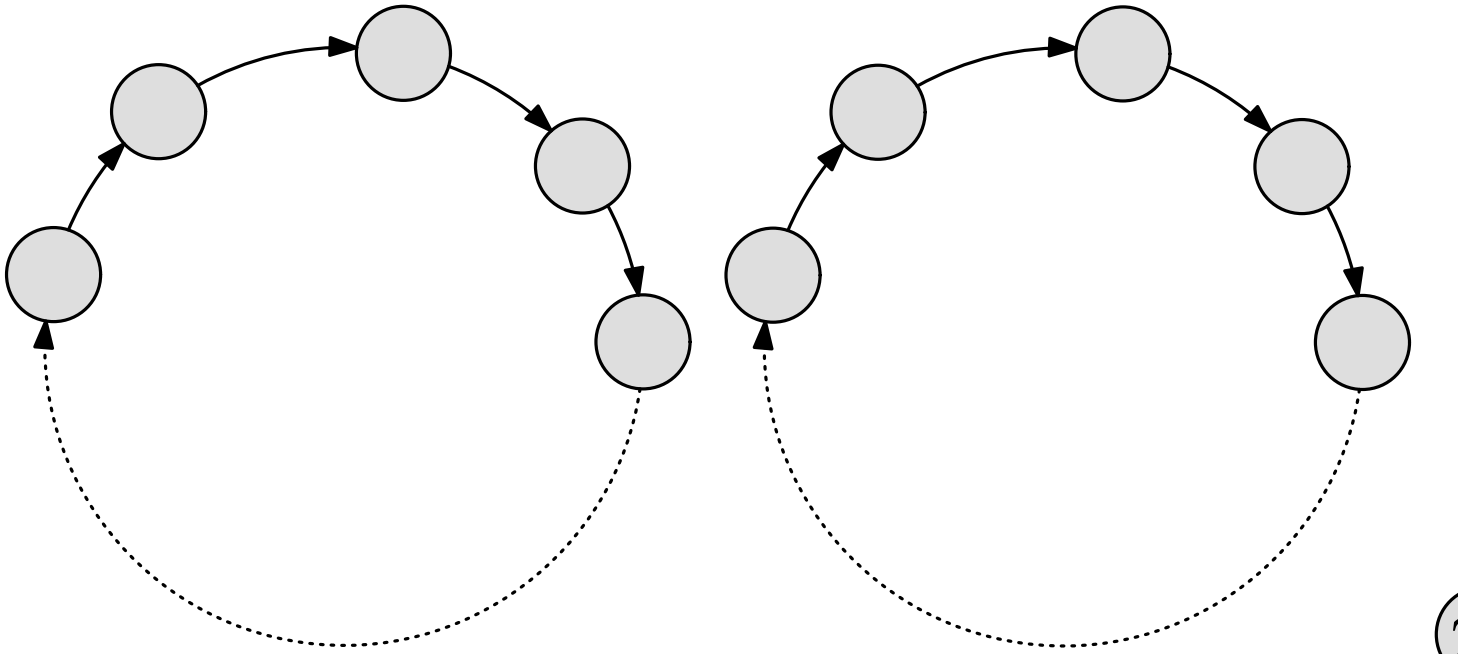


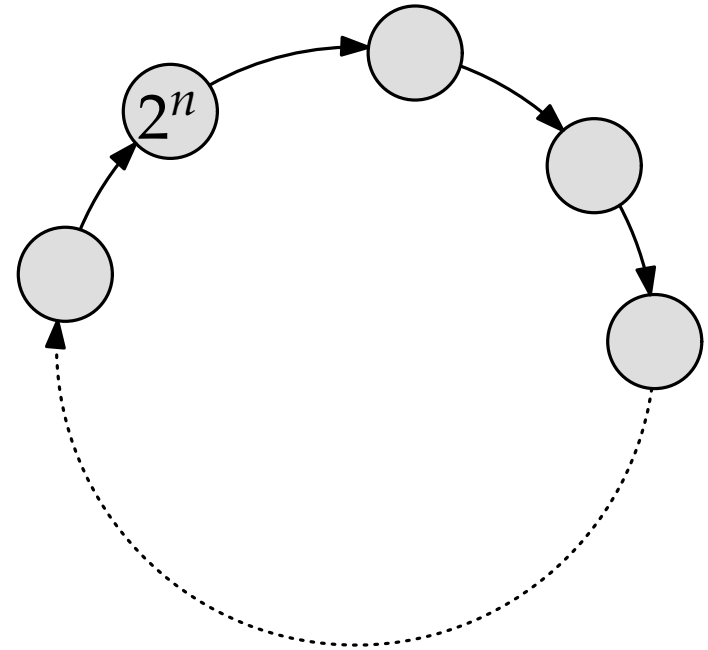


Within at most $n - 1$ steps, the 2^n-position will change from no-coin to yes-coin and the weight of the string $x \circ \pi$ will increase!

Except if cycles is all-coins

Finding a Local Optimum for $k = 1$



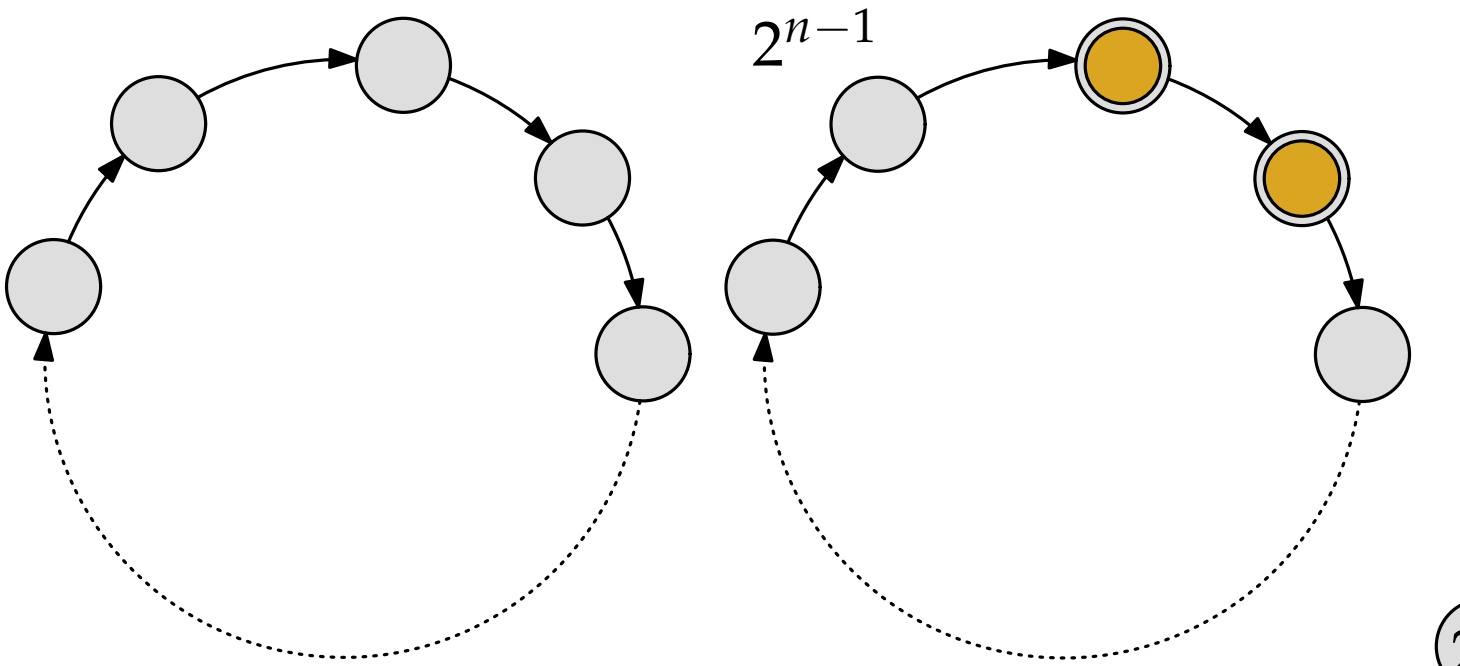


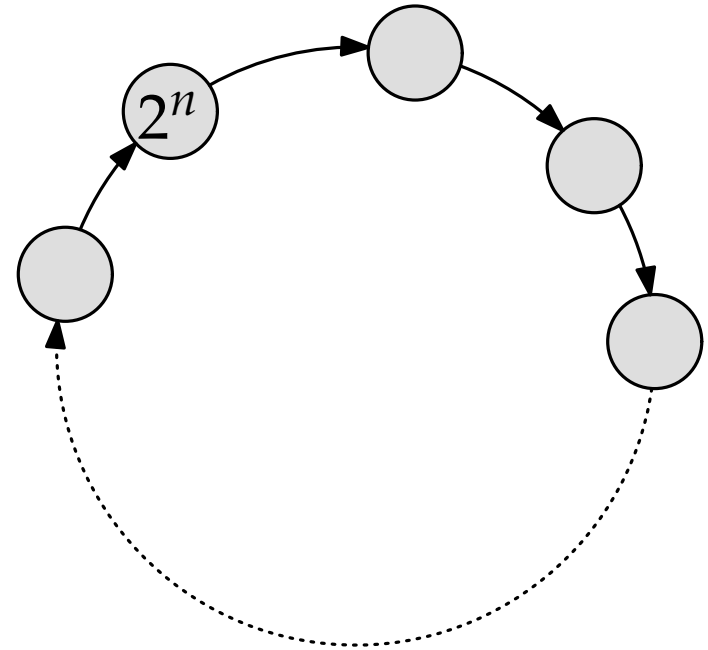
Within at most $n - 1$ steps, the 2^n-position will change from no-coin to yes-coin and the weight of the string $x \circ \pi$ will increase!

Except if cycles is all-coins

Or no-coins

Finding a Local Optimum for $k = 1$





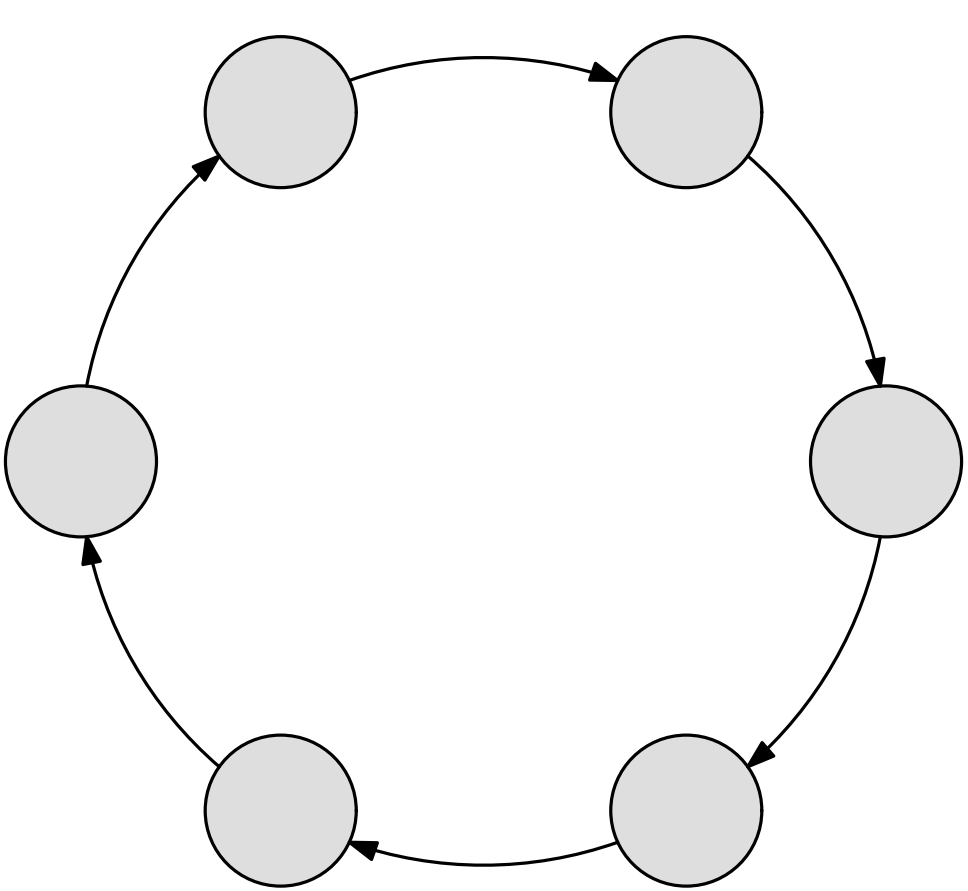
Within at most $n - 1$ steps, the 2^n-position will change from no-coin to yes-coin and the weight of the string $x \circ \pi$ will increase!

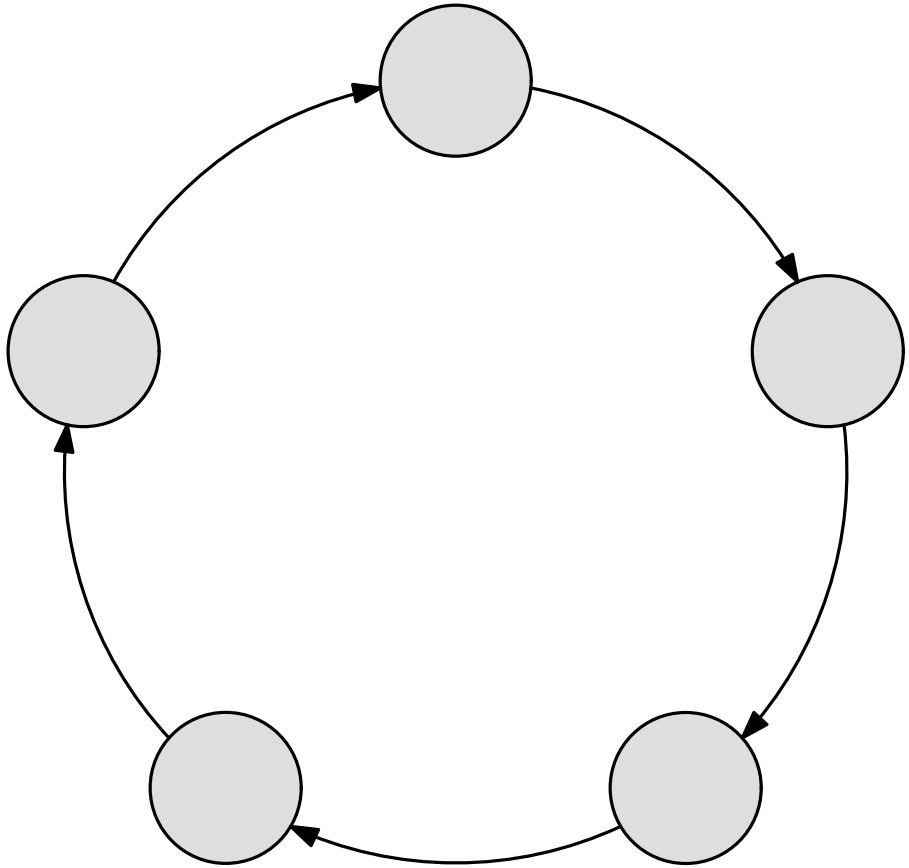
Except if cycles is all-coins

Or no-coins

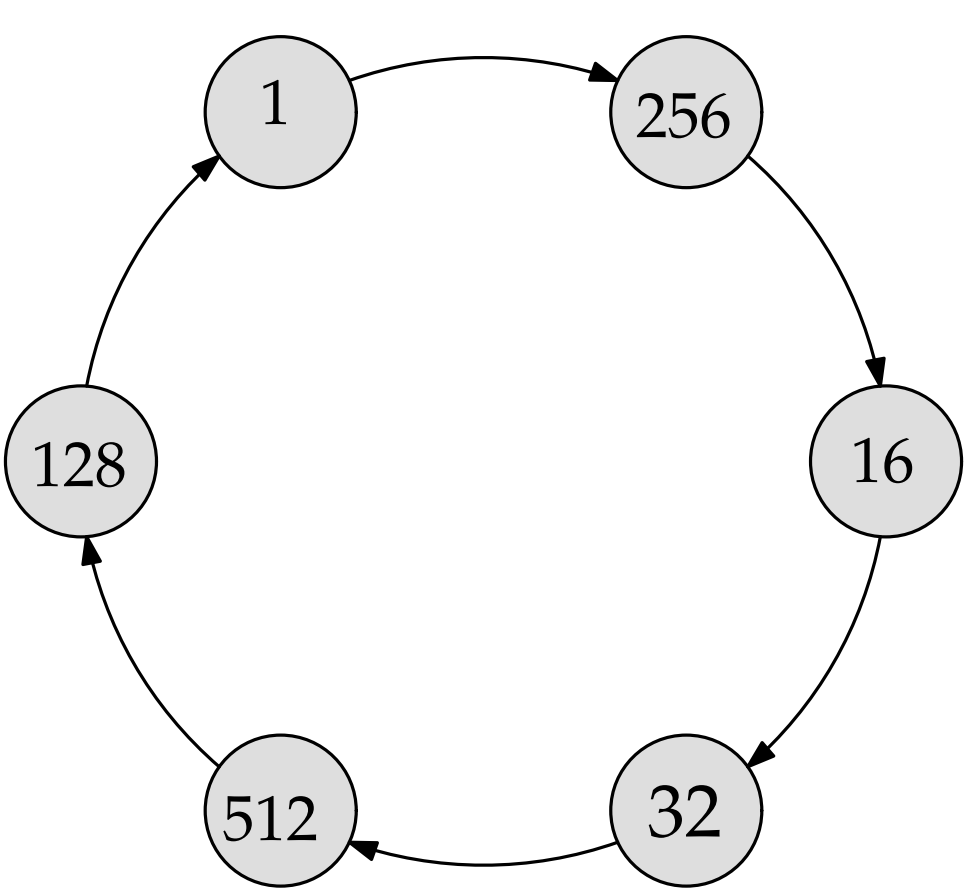
Then just look at second most important cycle.

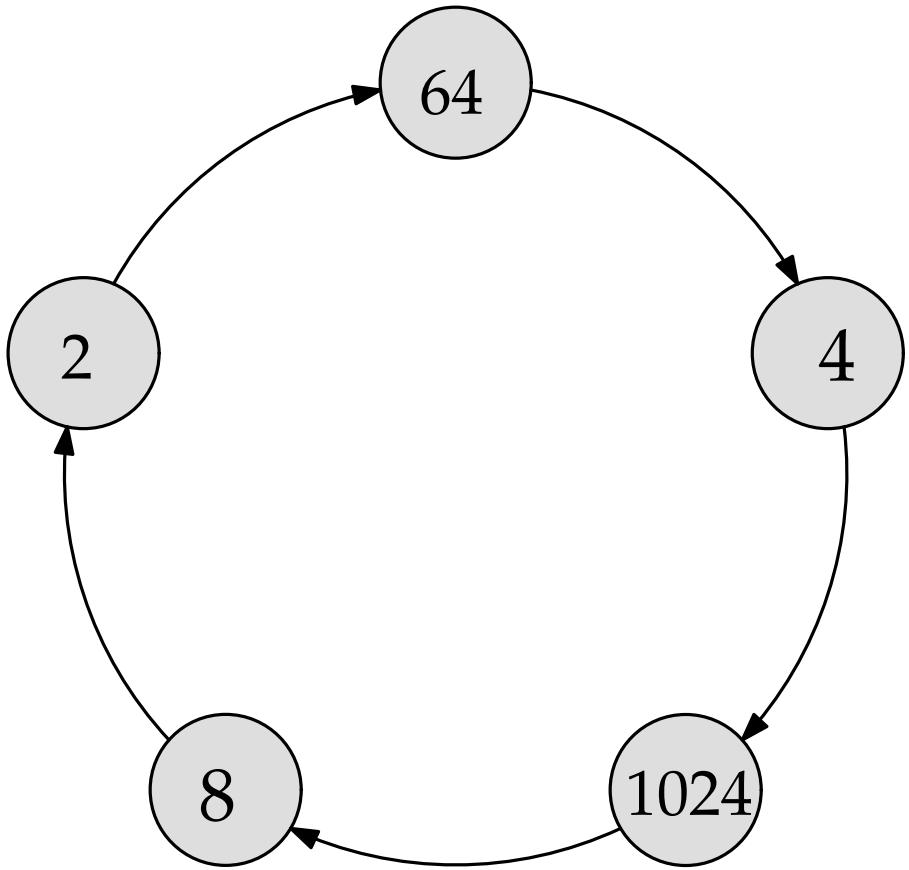
Finding a Local Optimum for $k = 1$



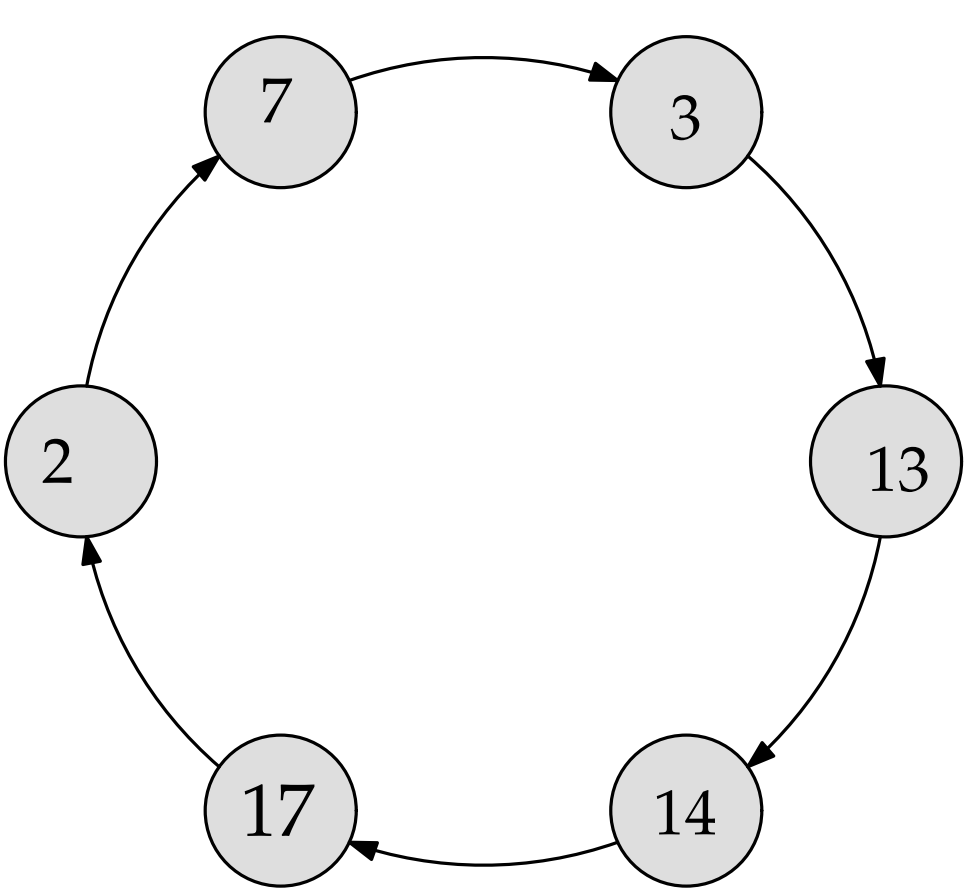


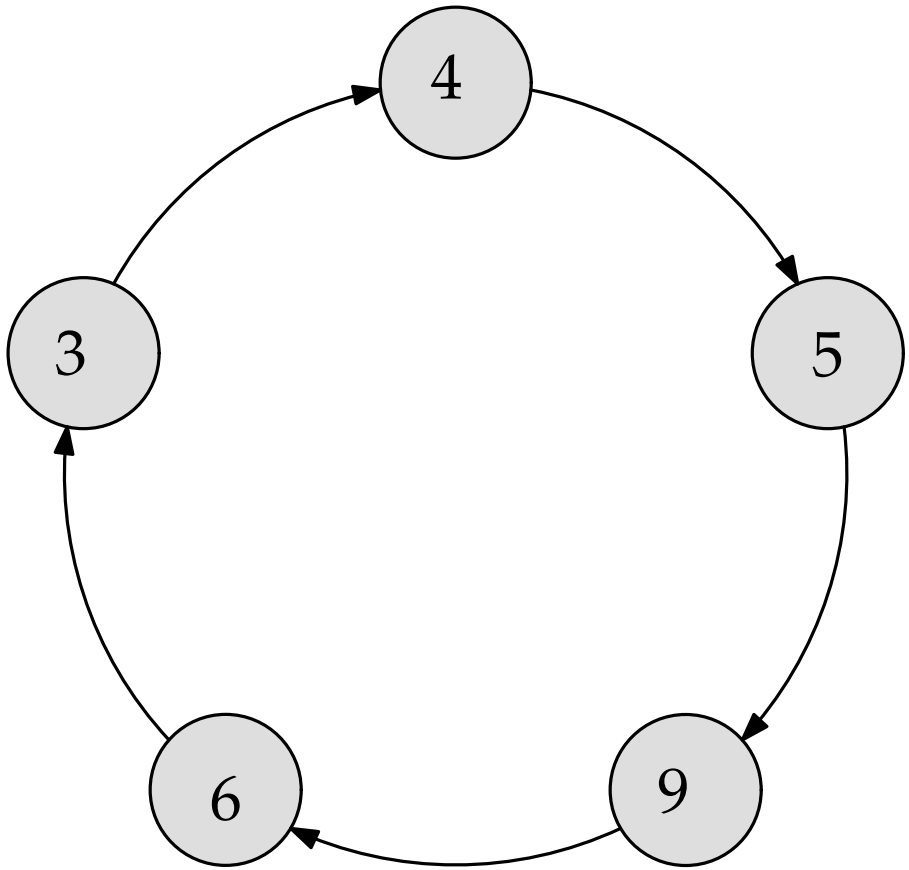
Finding a Local Optimum for $k = 1$





Finding a Local Optimum for $k = 1$





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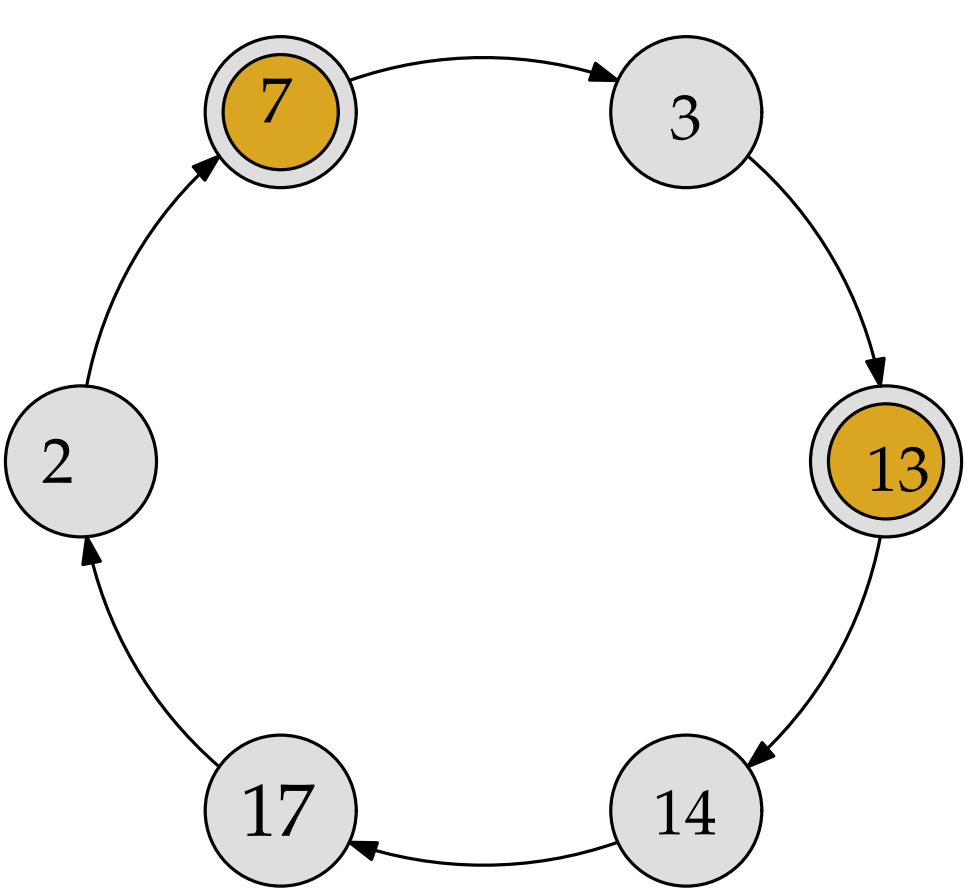

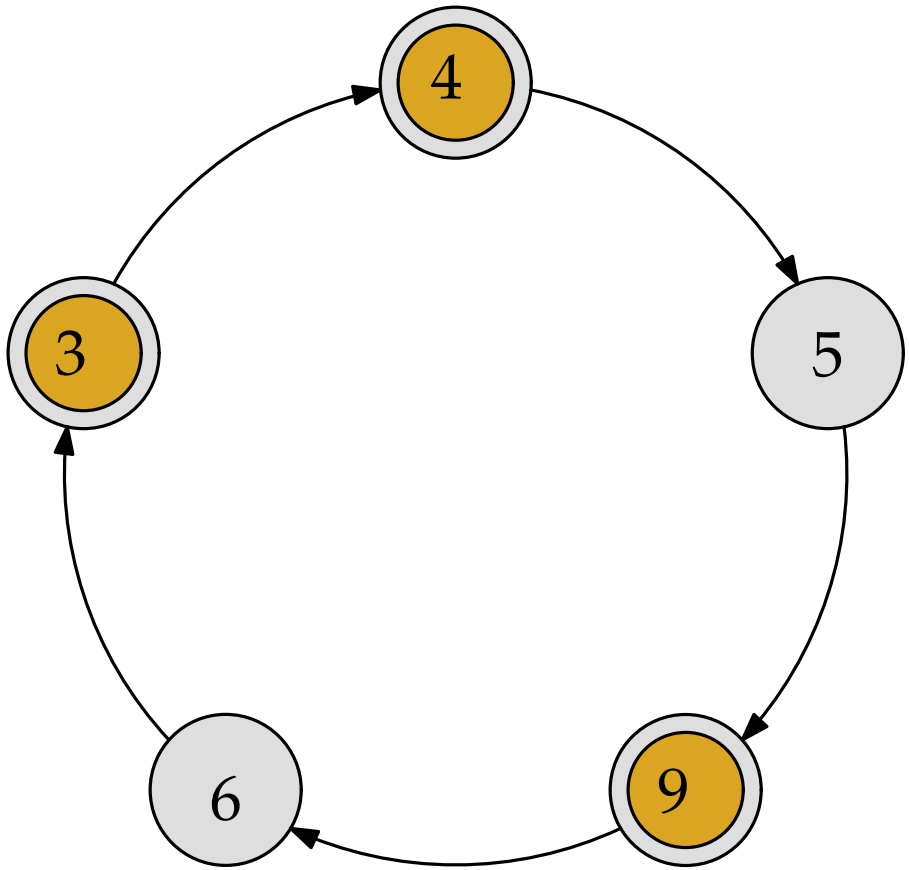

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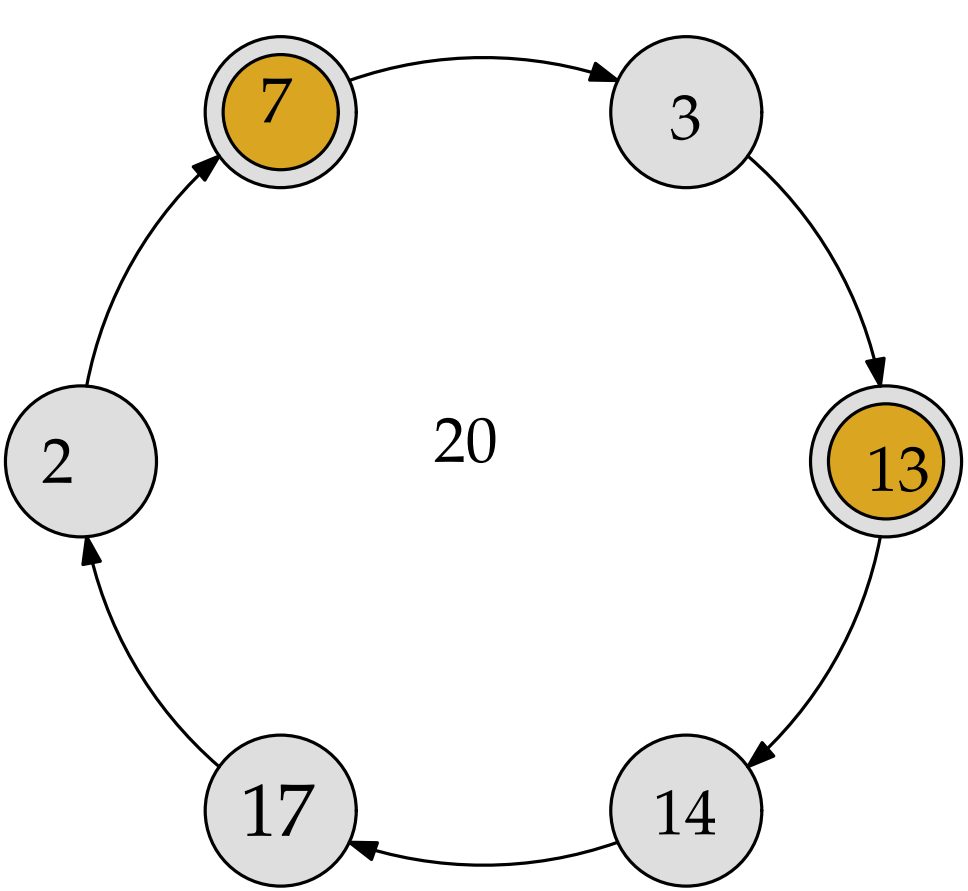

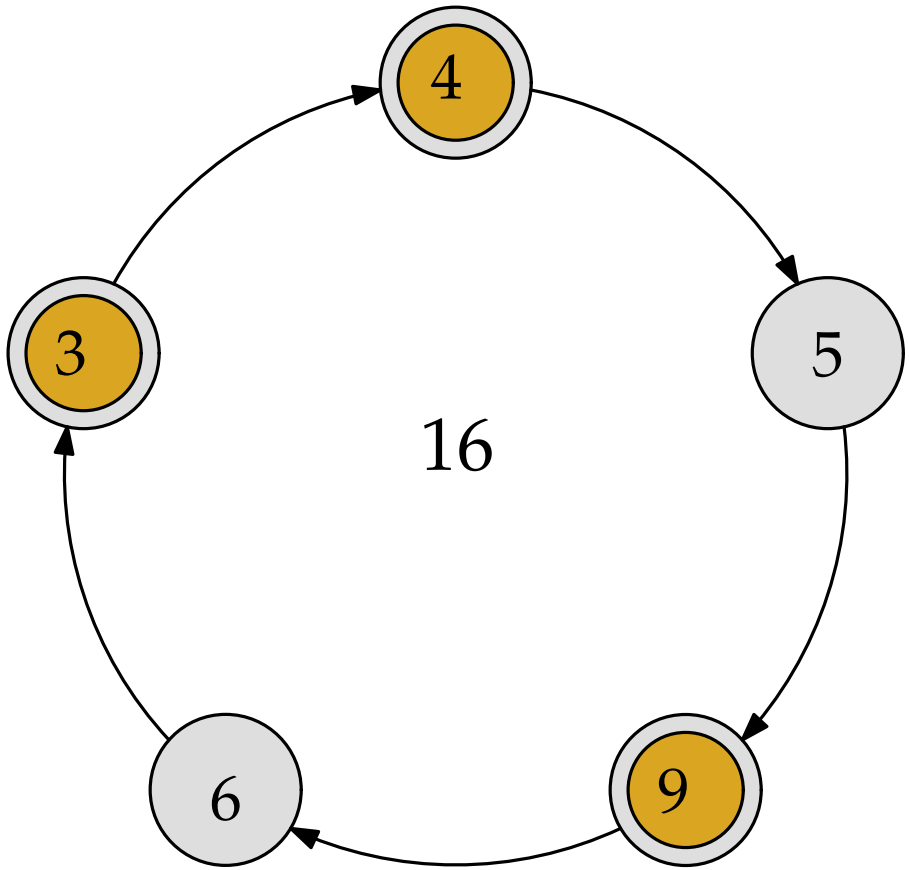

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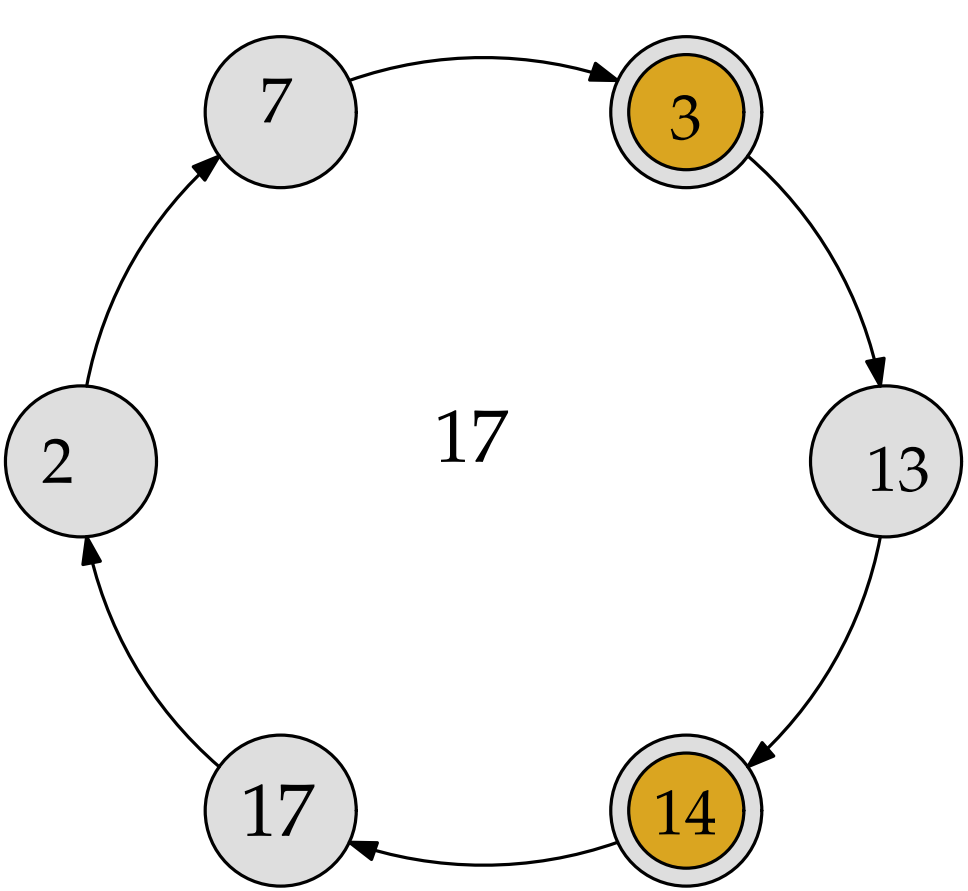

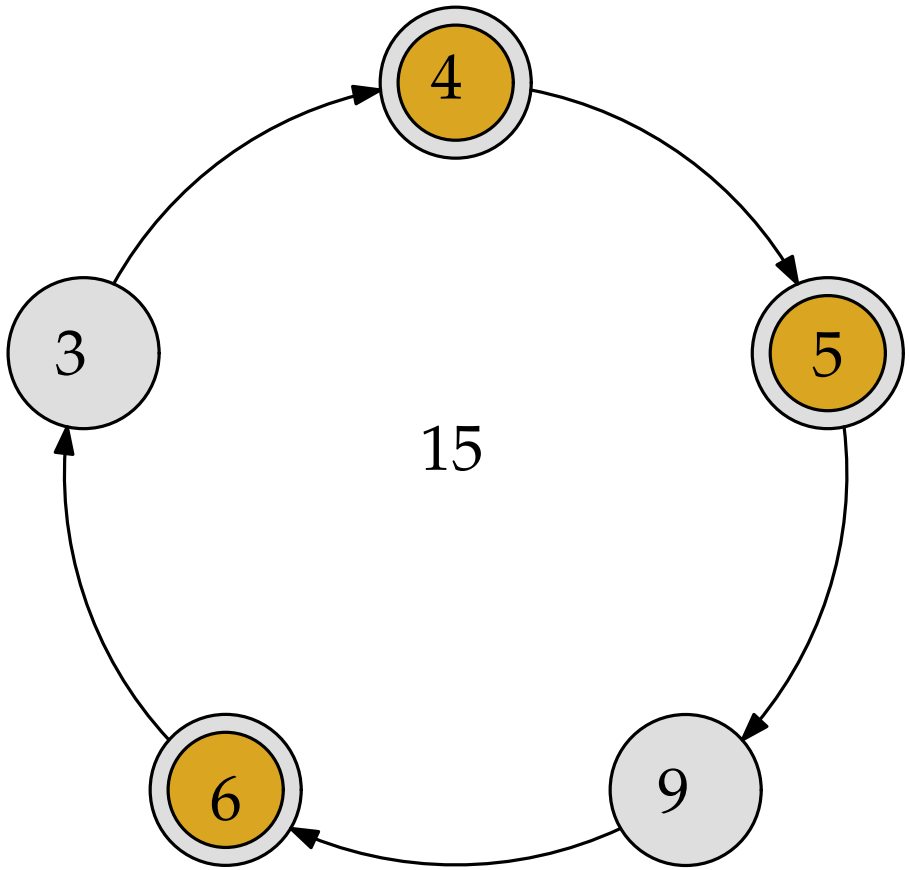

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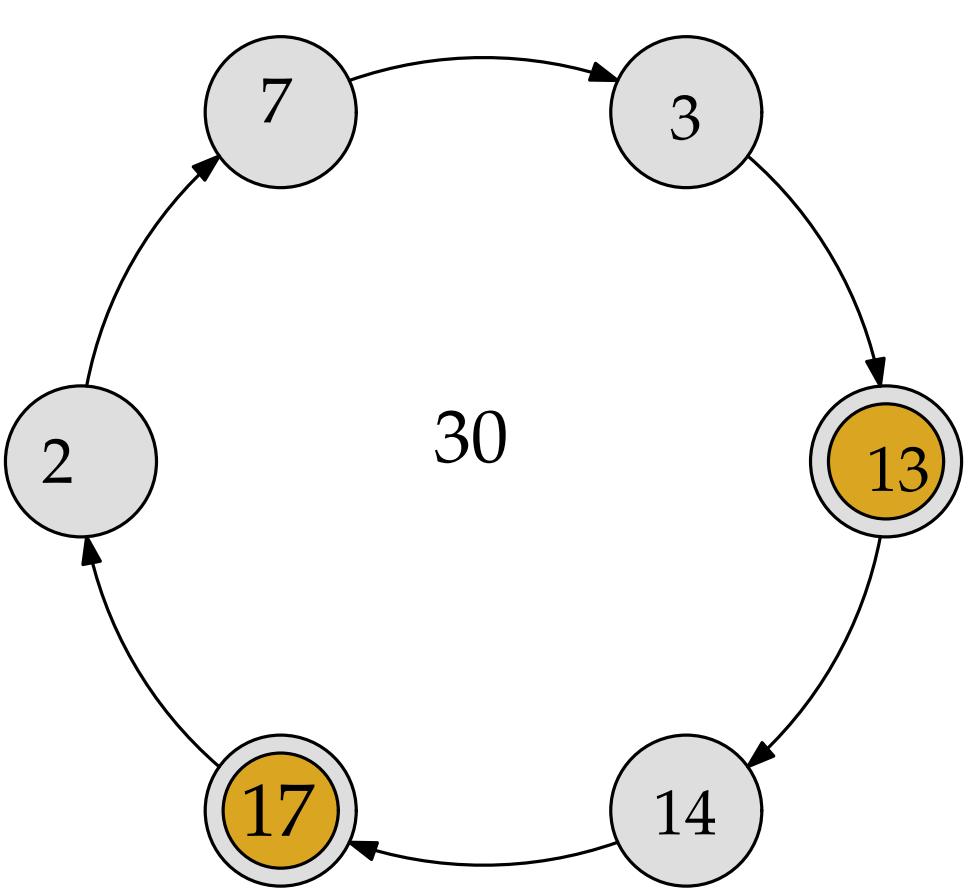

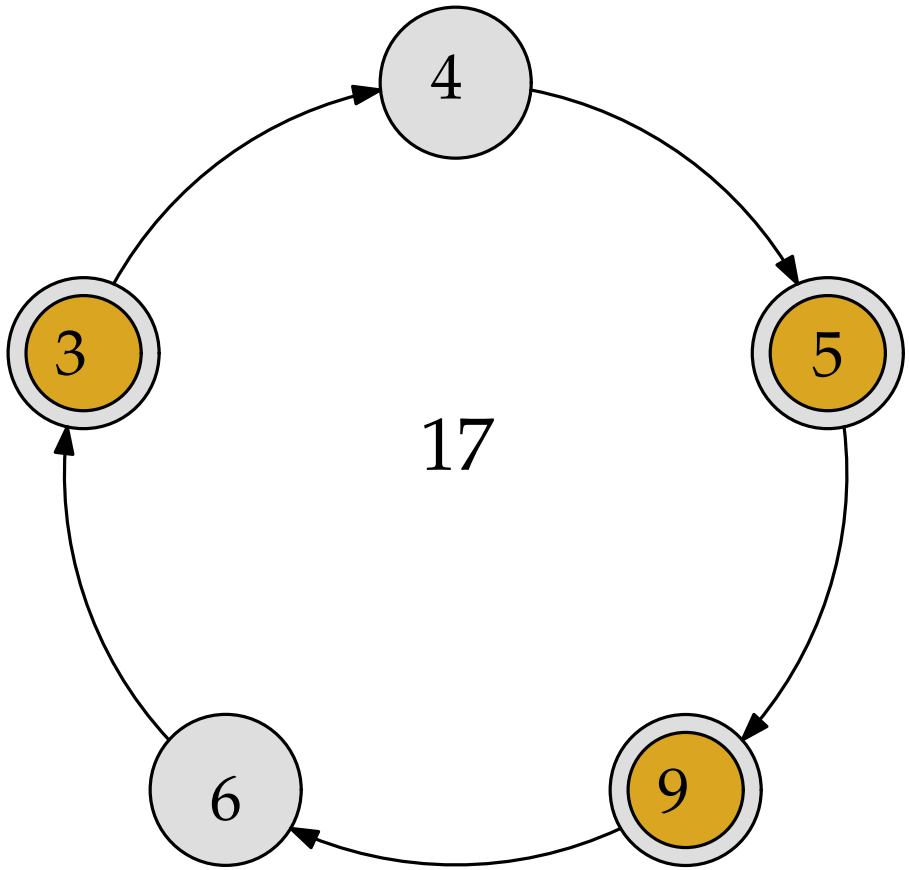

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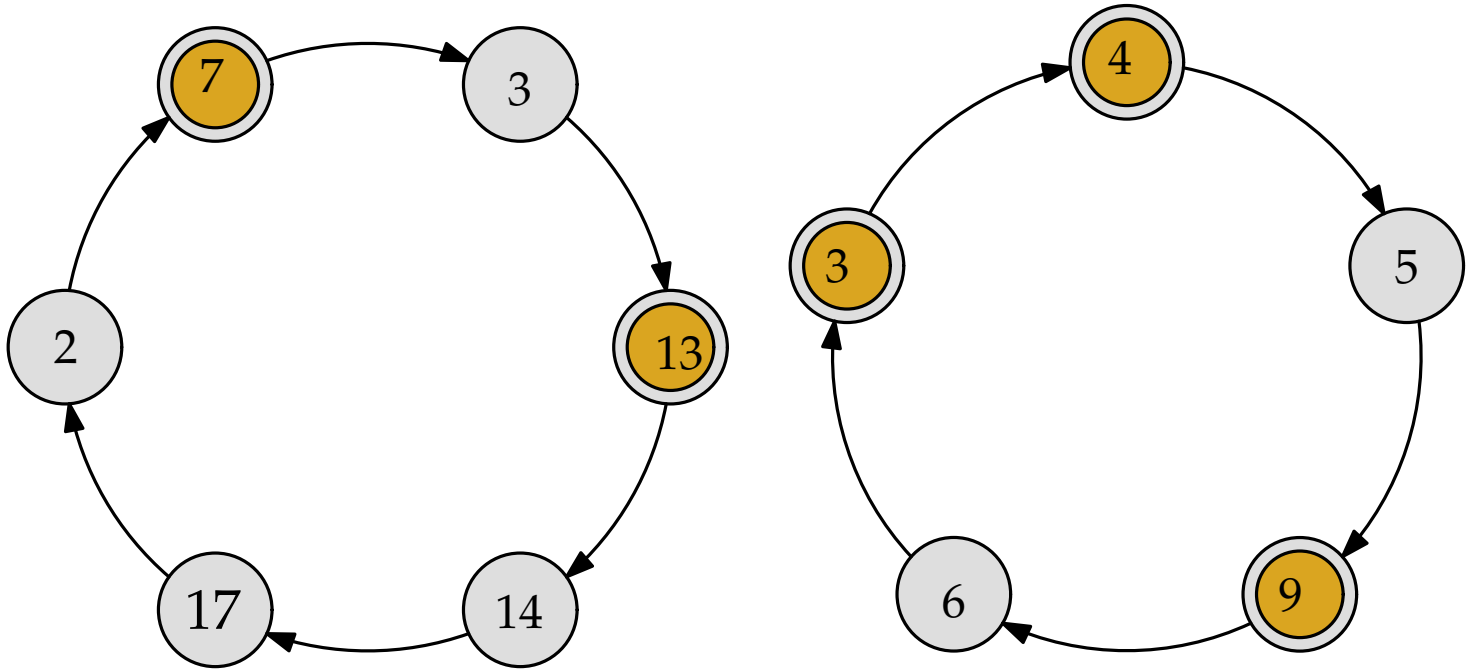

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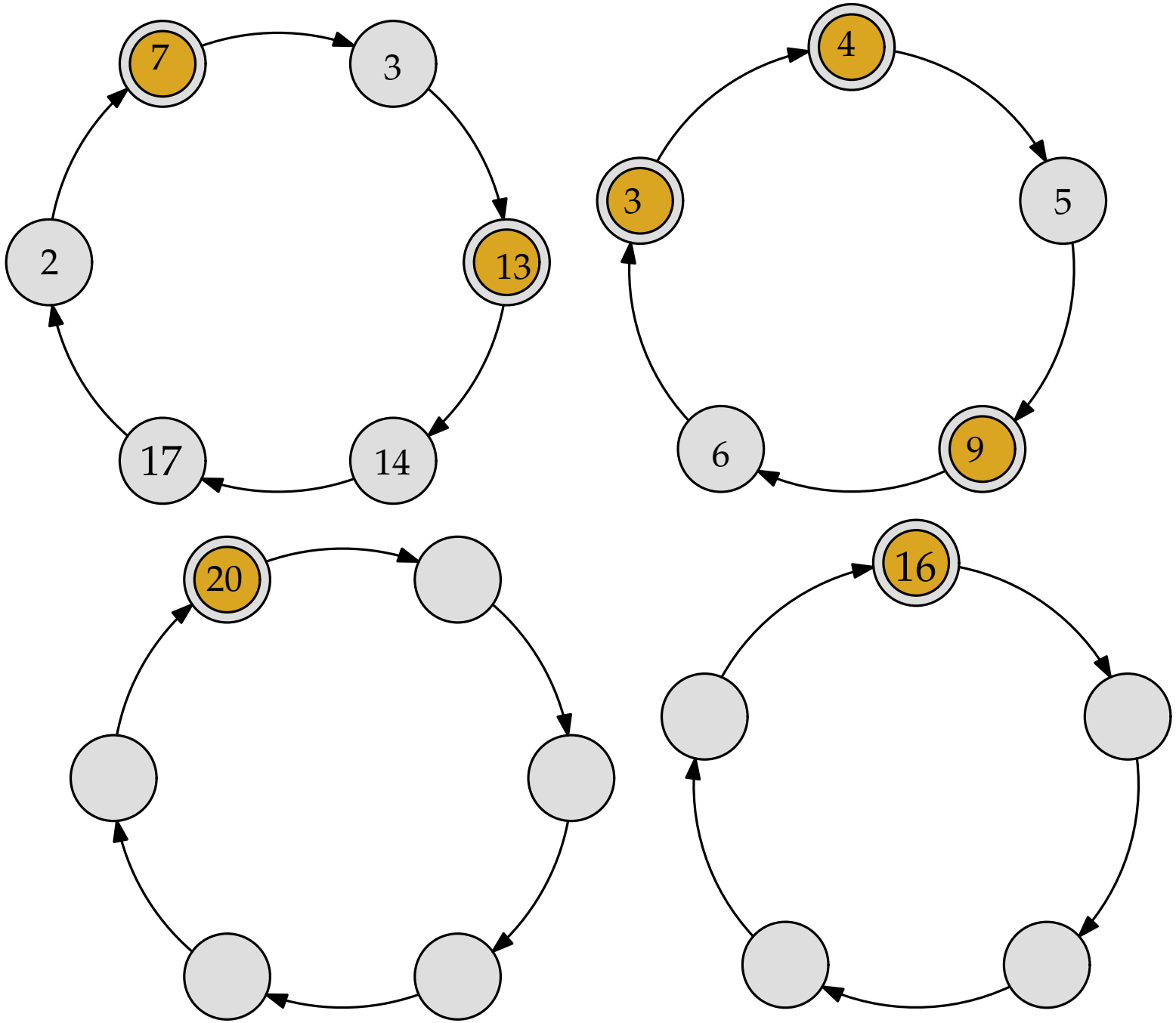

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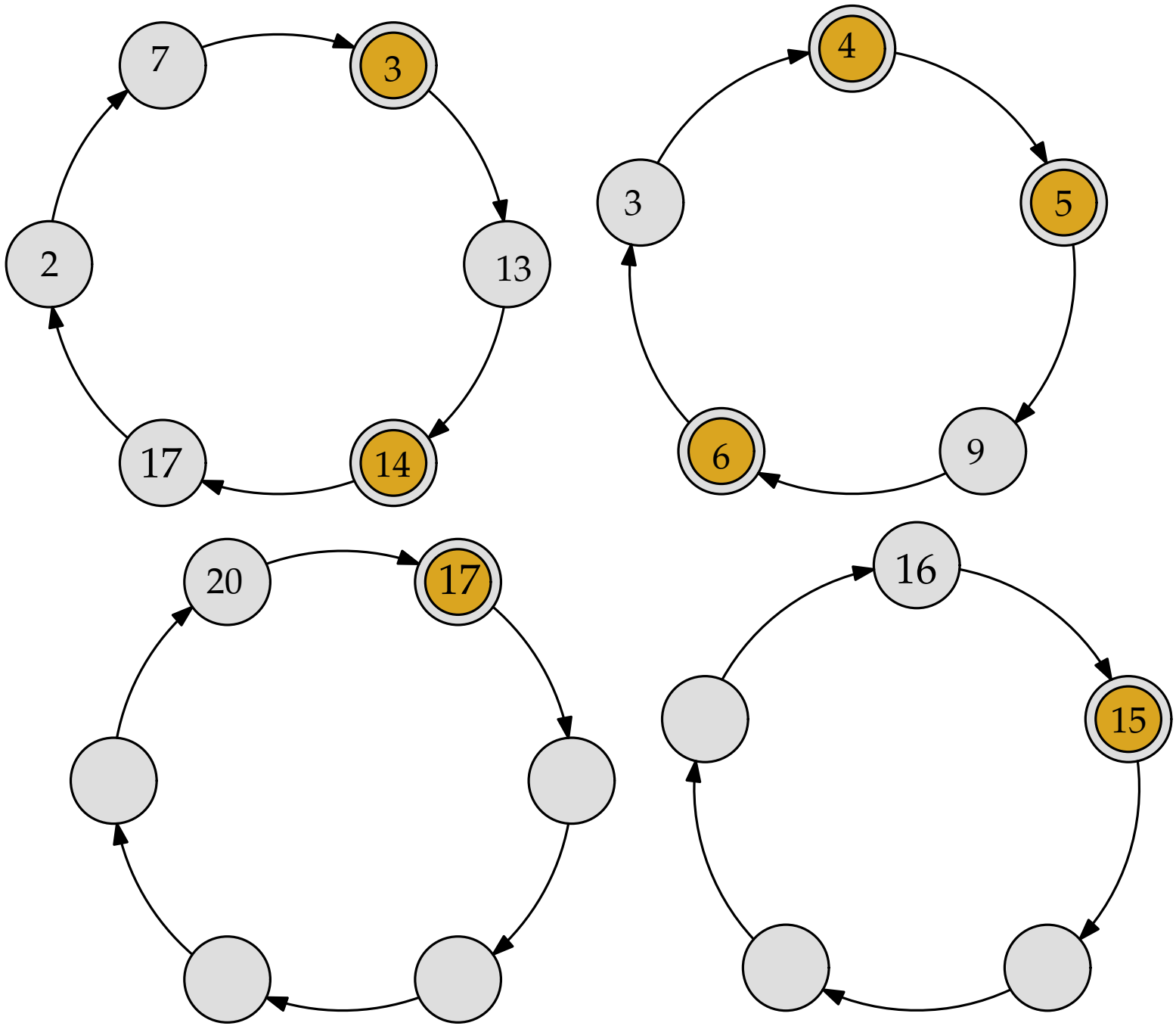

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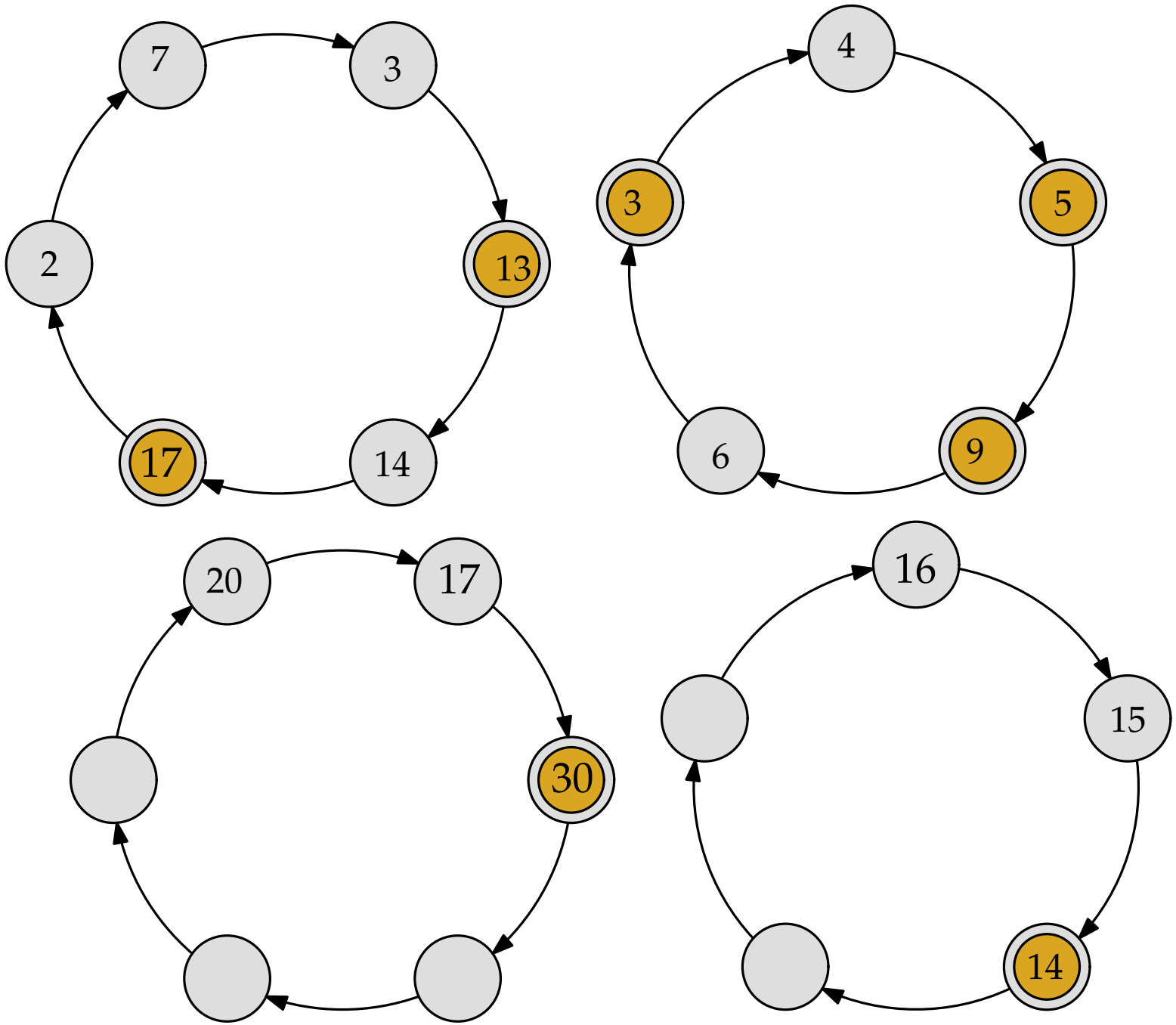

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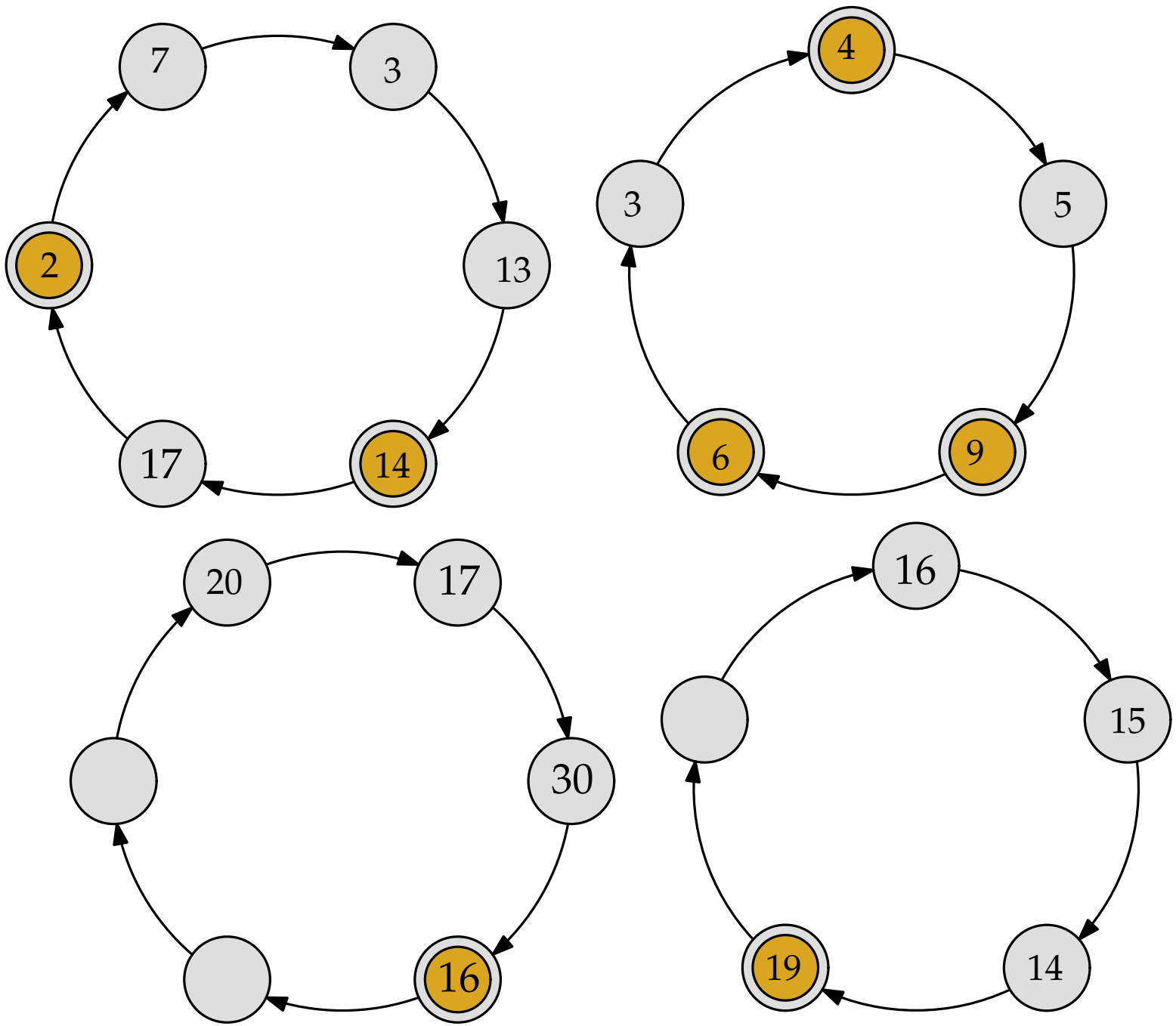

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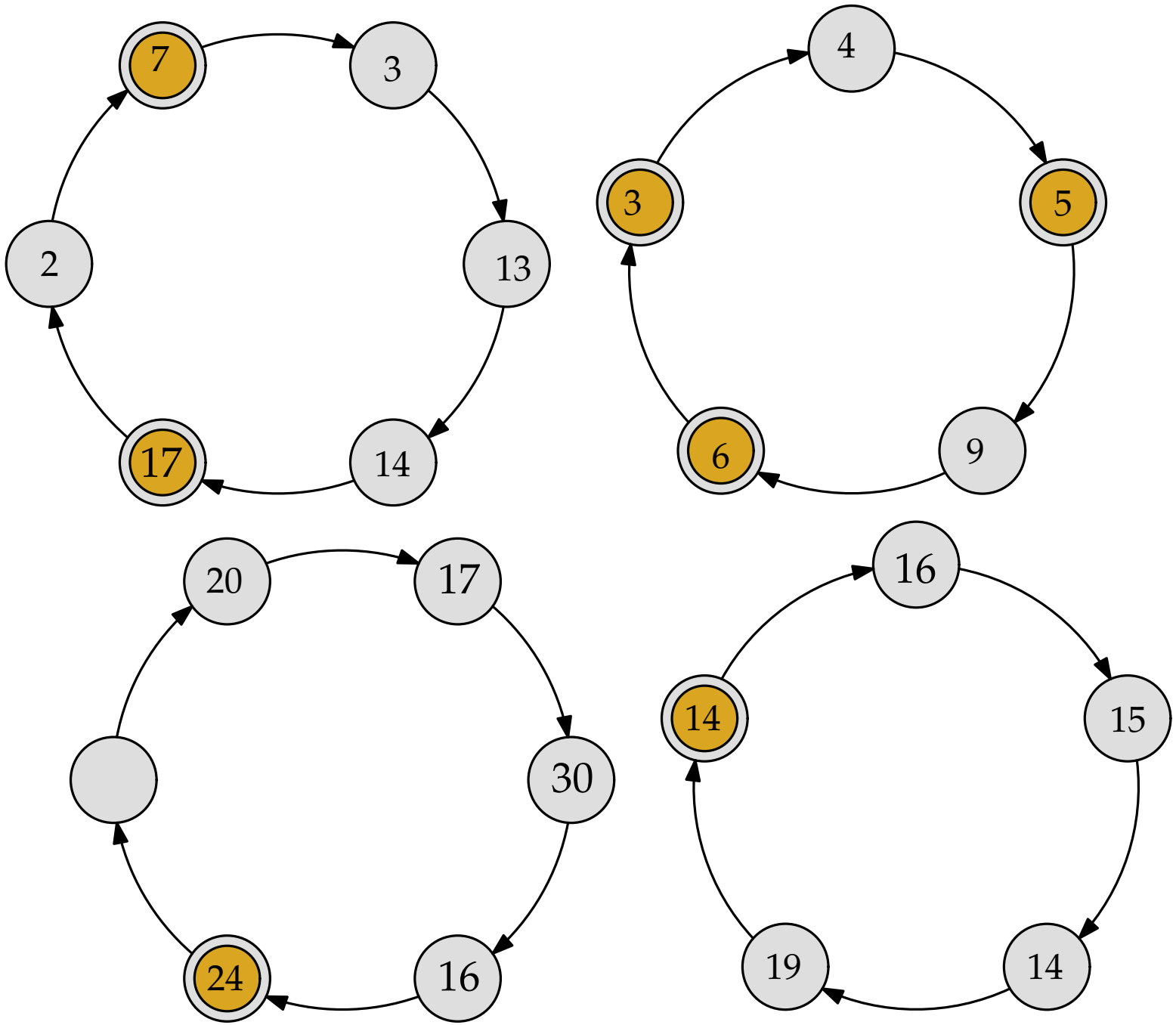

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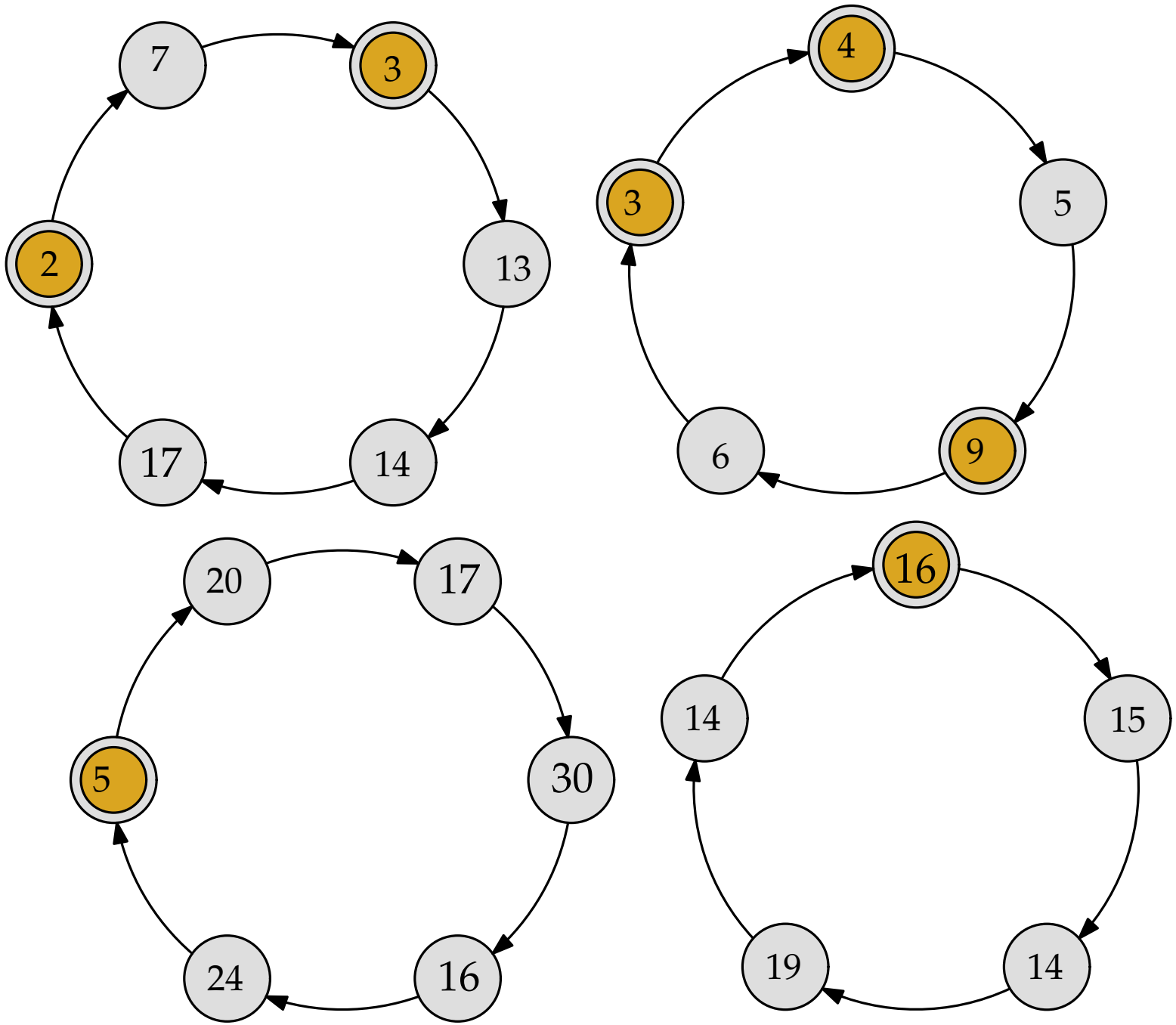

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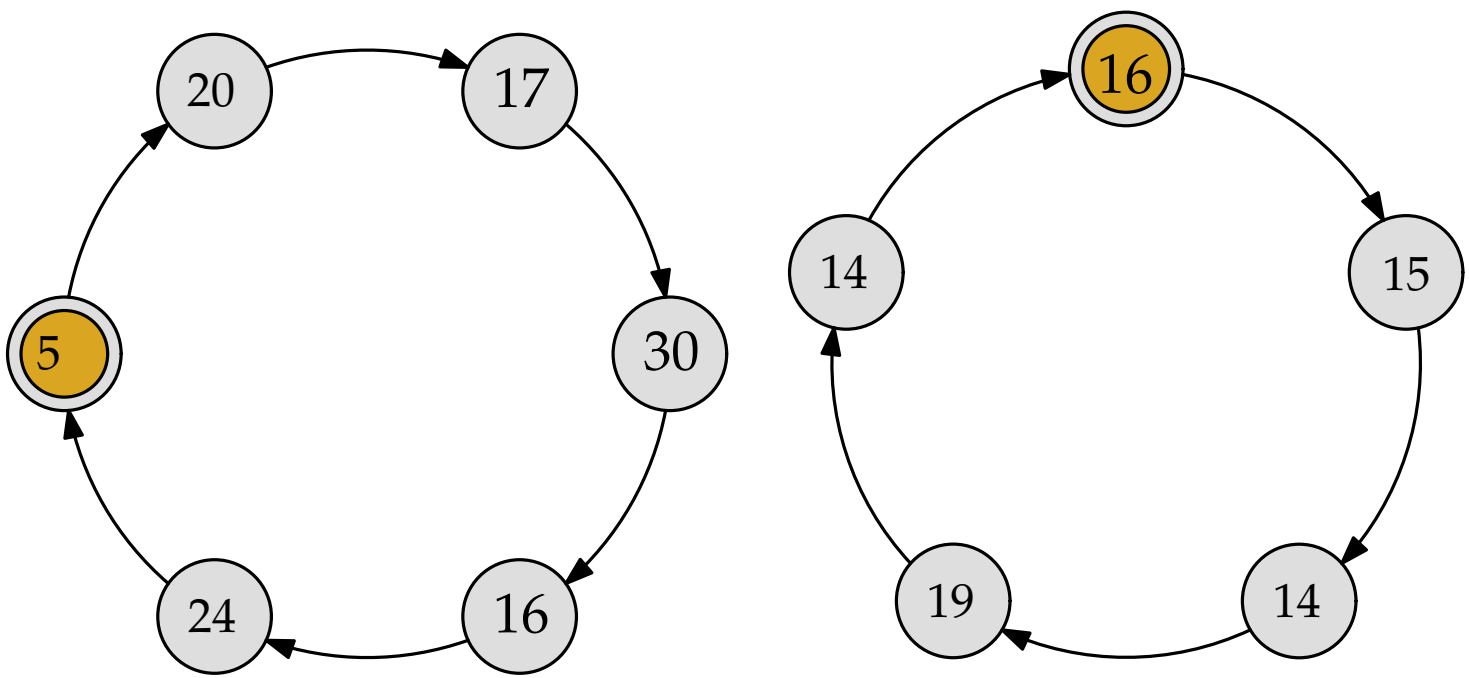
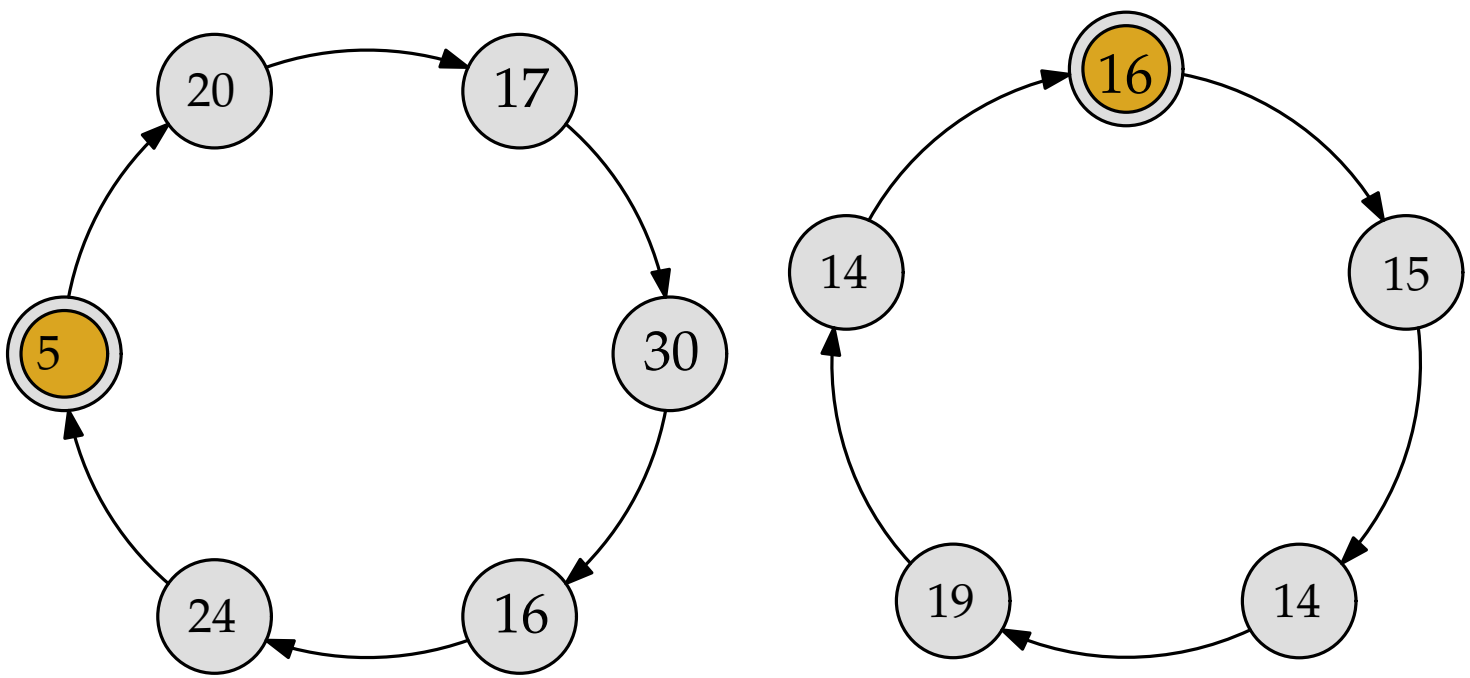


Finding a Local Optimum for $k = 1$

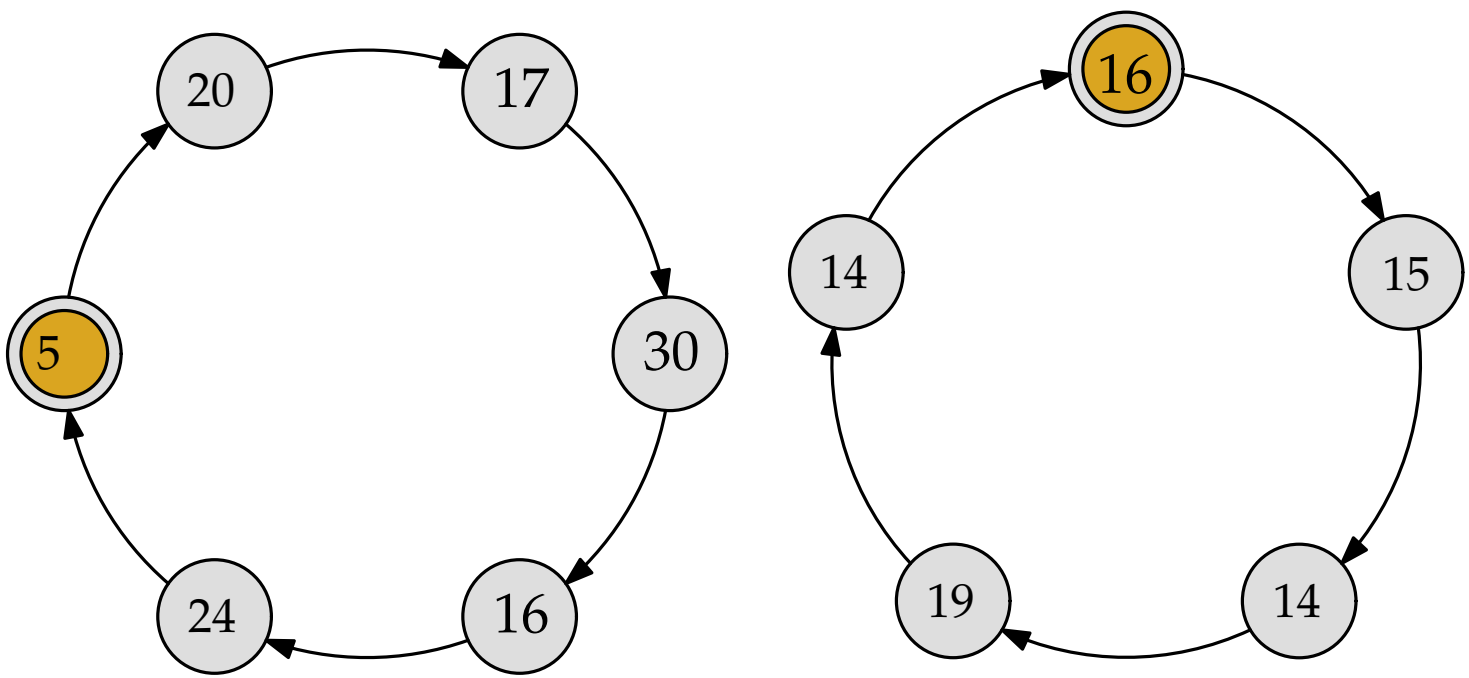
Without loss of generality only one "coin" per cycle.



Finding a Local Optimum for $k = 1$

Without loss of generality only one "coin" per cycle.

Functions $f_1, f_2, \ldots, f_l : \mathbb{Z} \to \mathbb{Z}$



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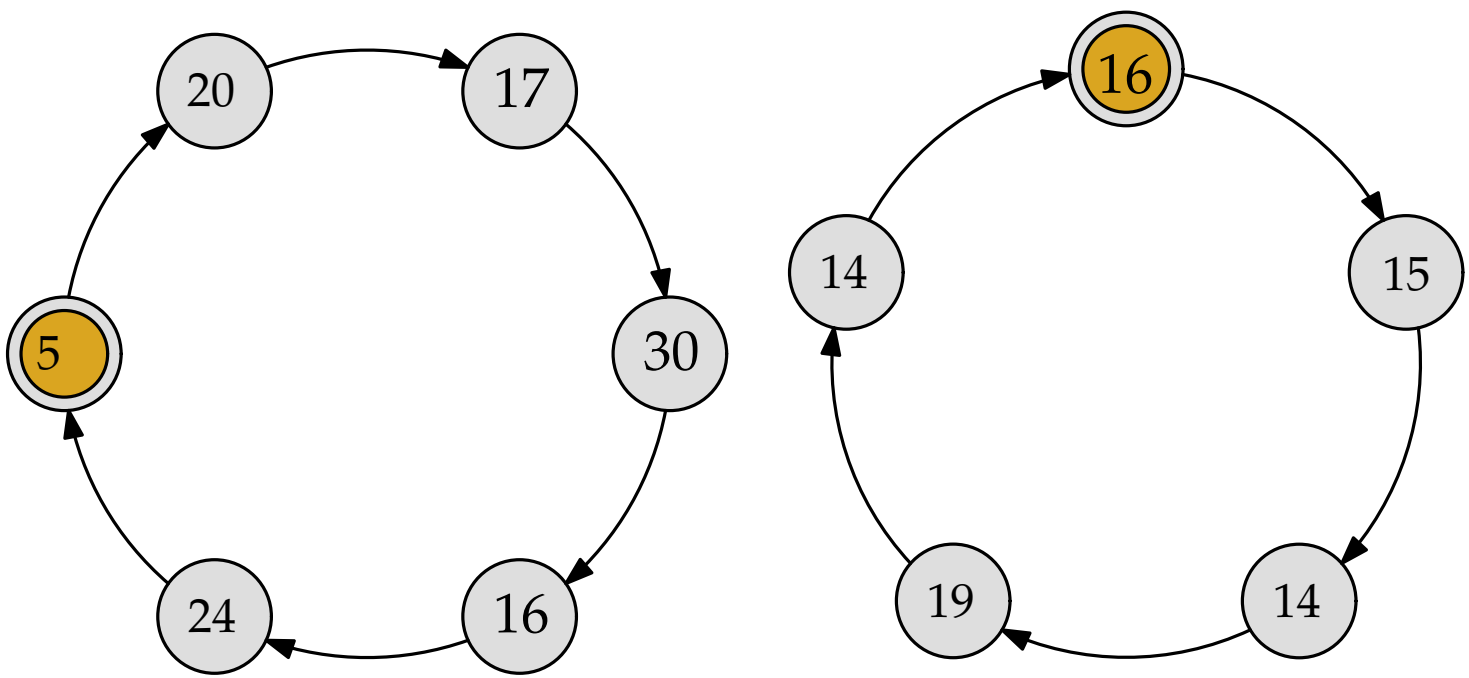
with f_i being periodic with period p_i



Finding a Local Optimum for $k = 1$

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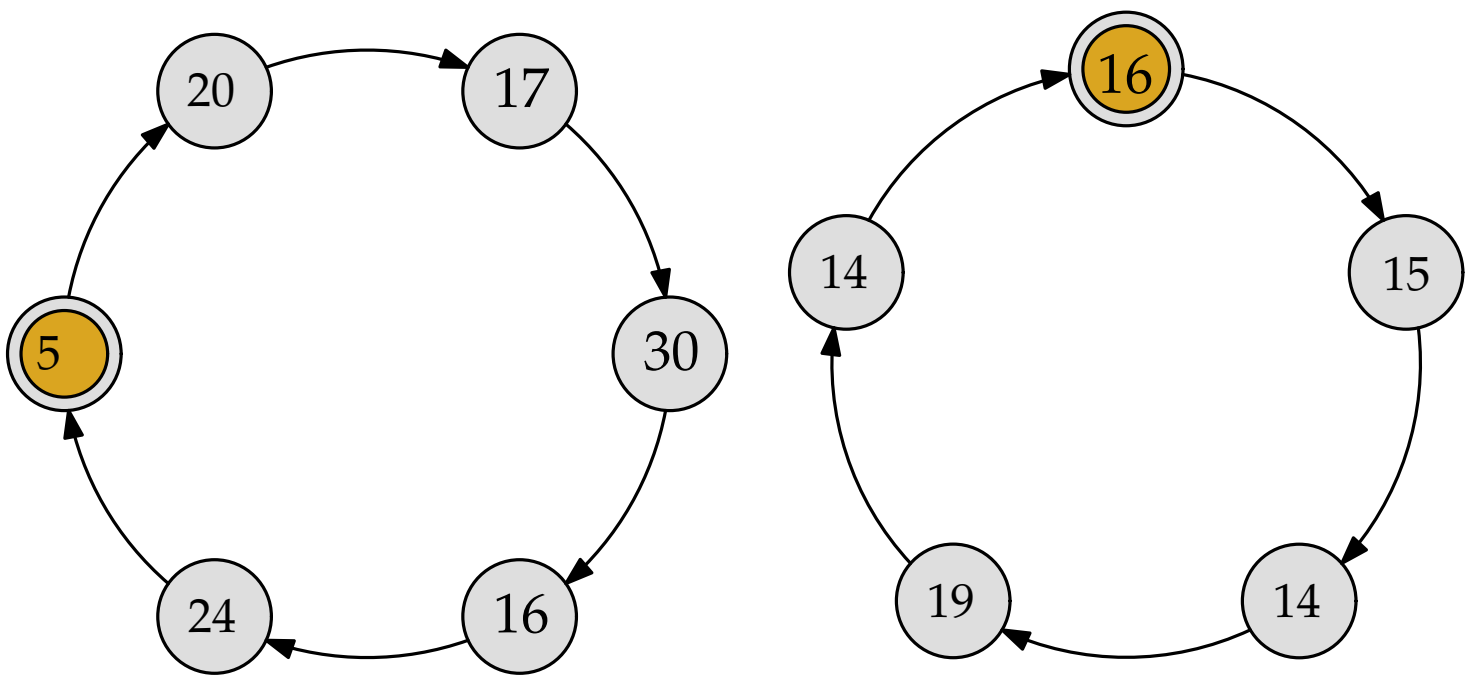
with f_i being periodic with period p_i and $p_1 + \cdots + p_l = n$



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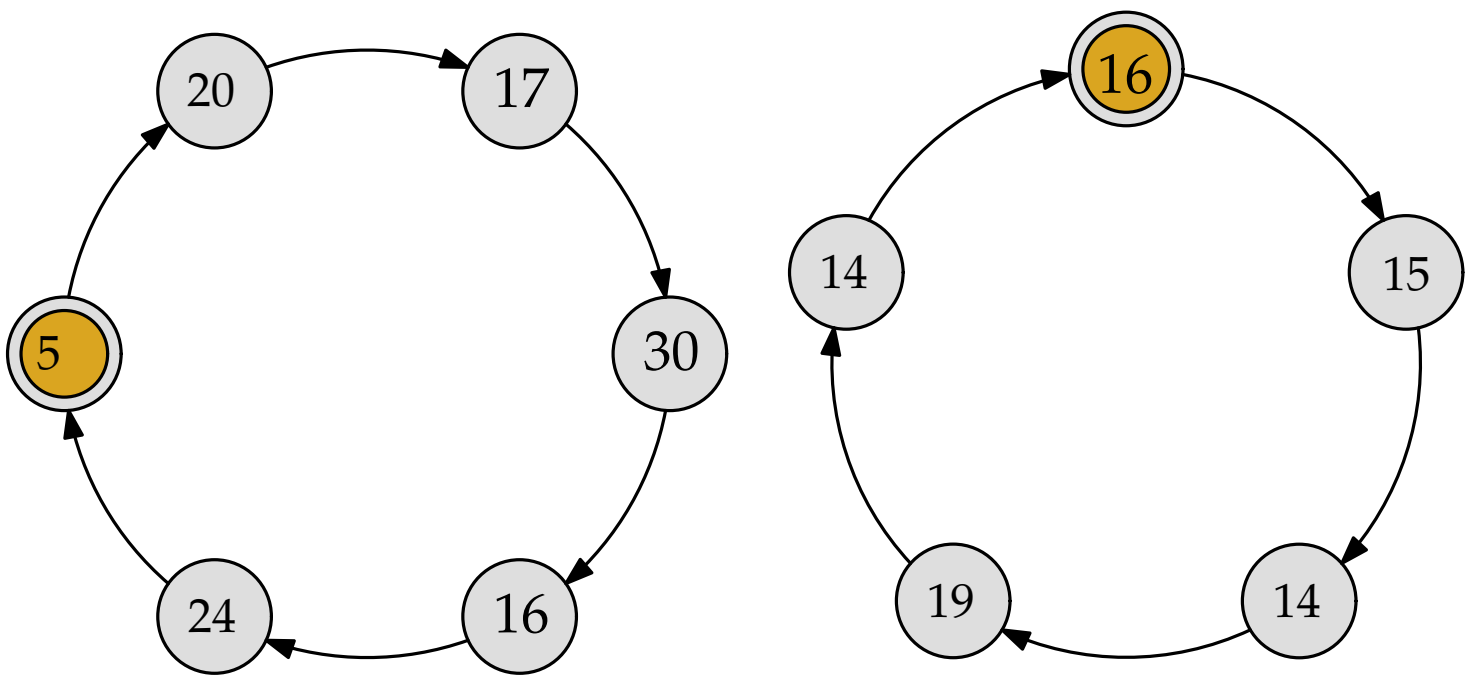

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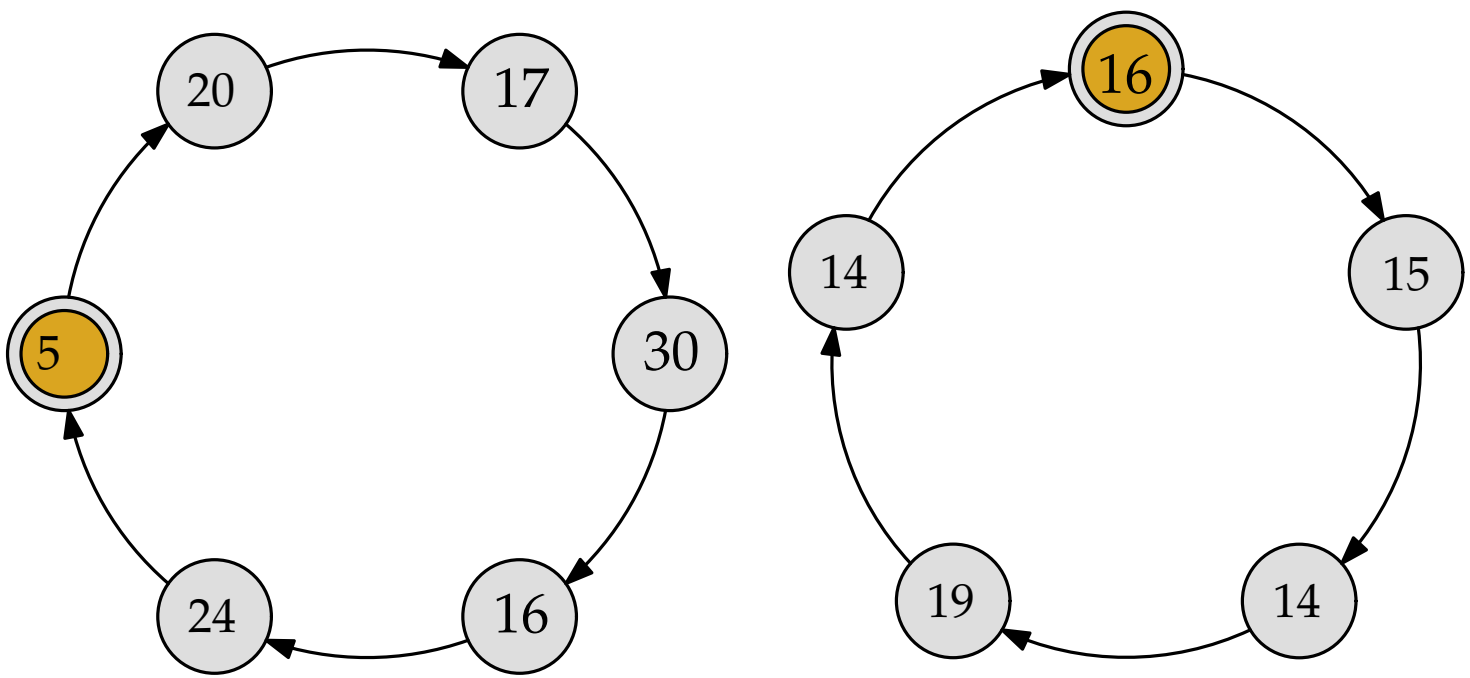

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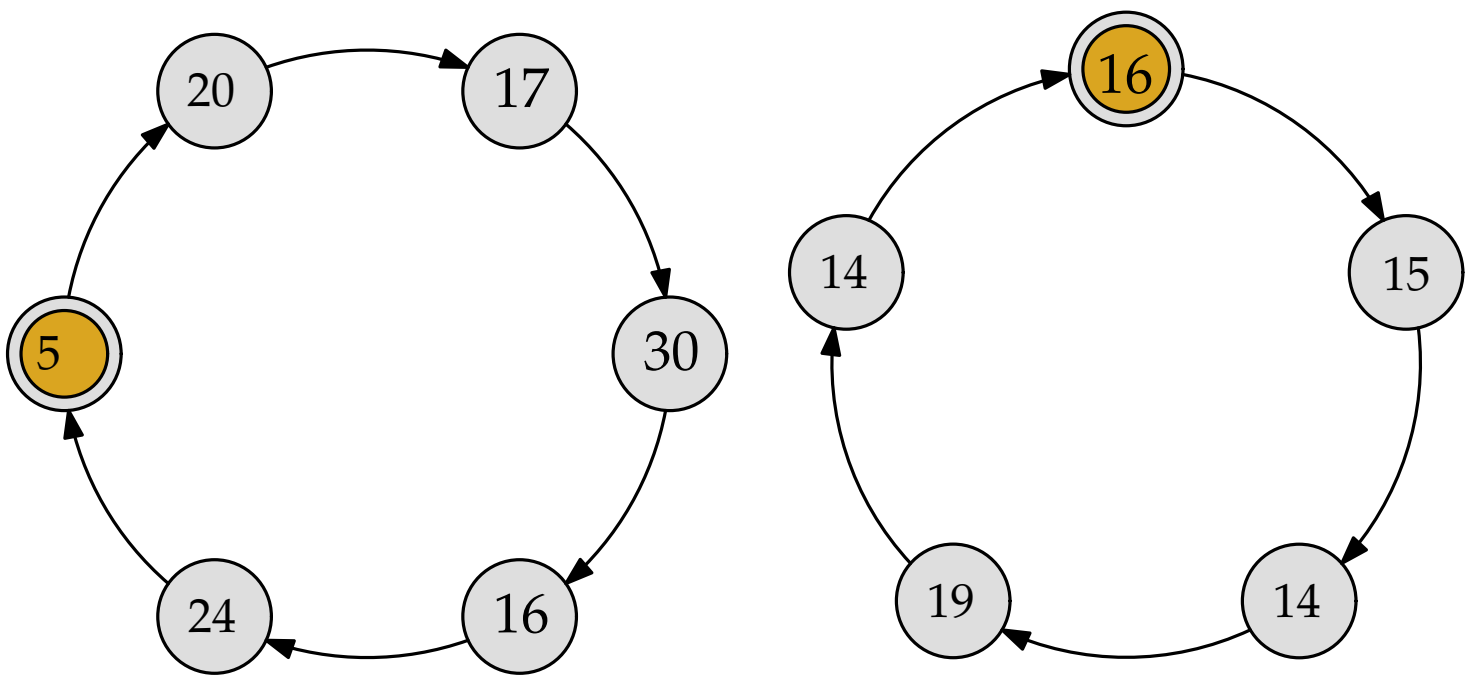
On how large an interval $[a, b]$ can f be strictly decreasing?



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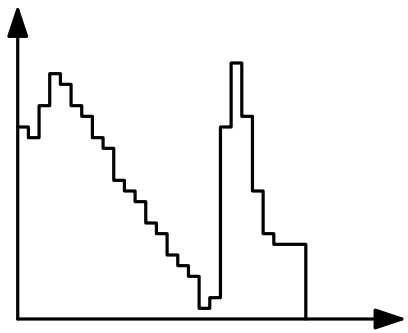

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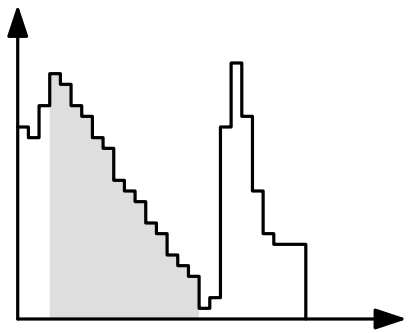

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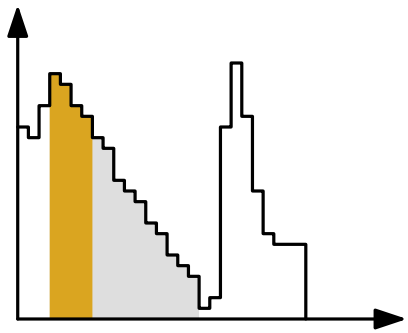

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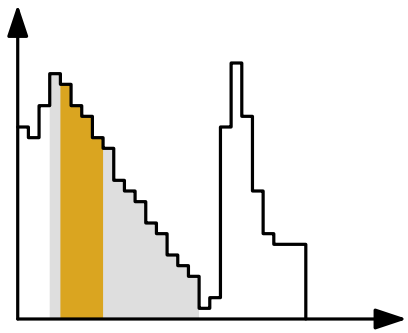

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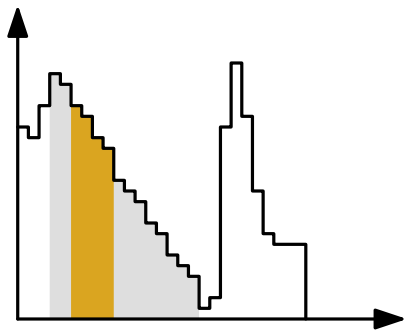

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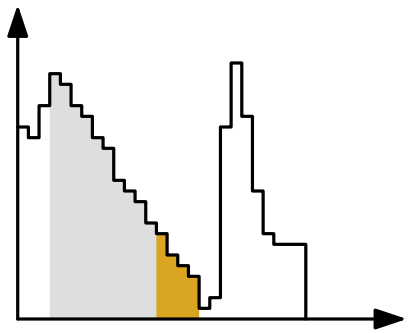

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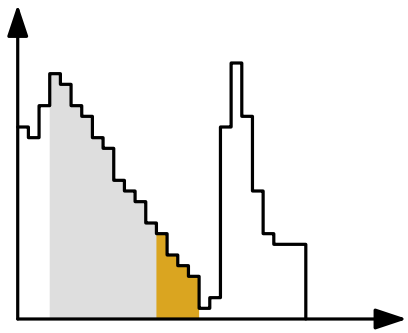

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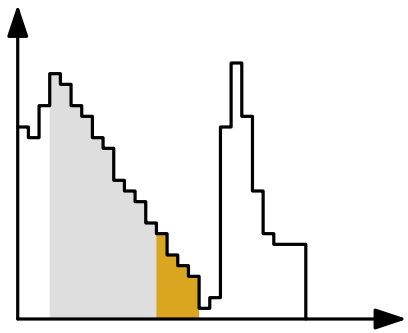

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Now use induction on $F_2, \ldots, F_l$.

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Corollary. For $k = 1$, the improving path for $x \in \{0,1\}^n$ under the action of π_1 contains at most n strings.
This is quite obvious for the lexicographic weight function and follows from the lemma for general weight functions.

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This talk: not complexity theory but how long can improving path get?

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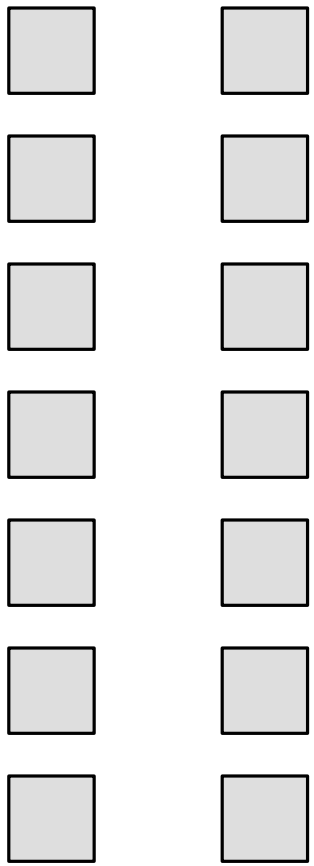

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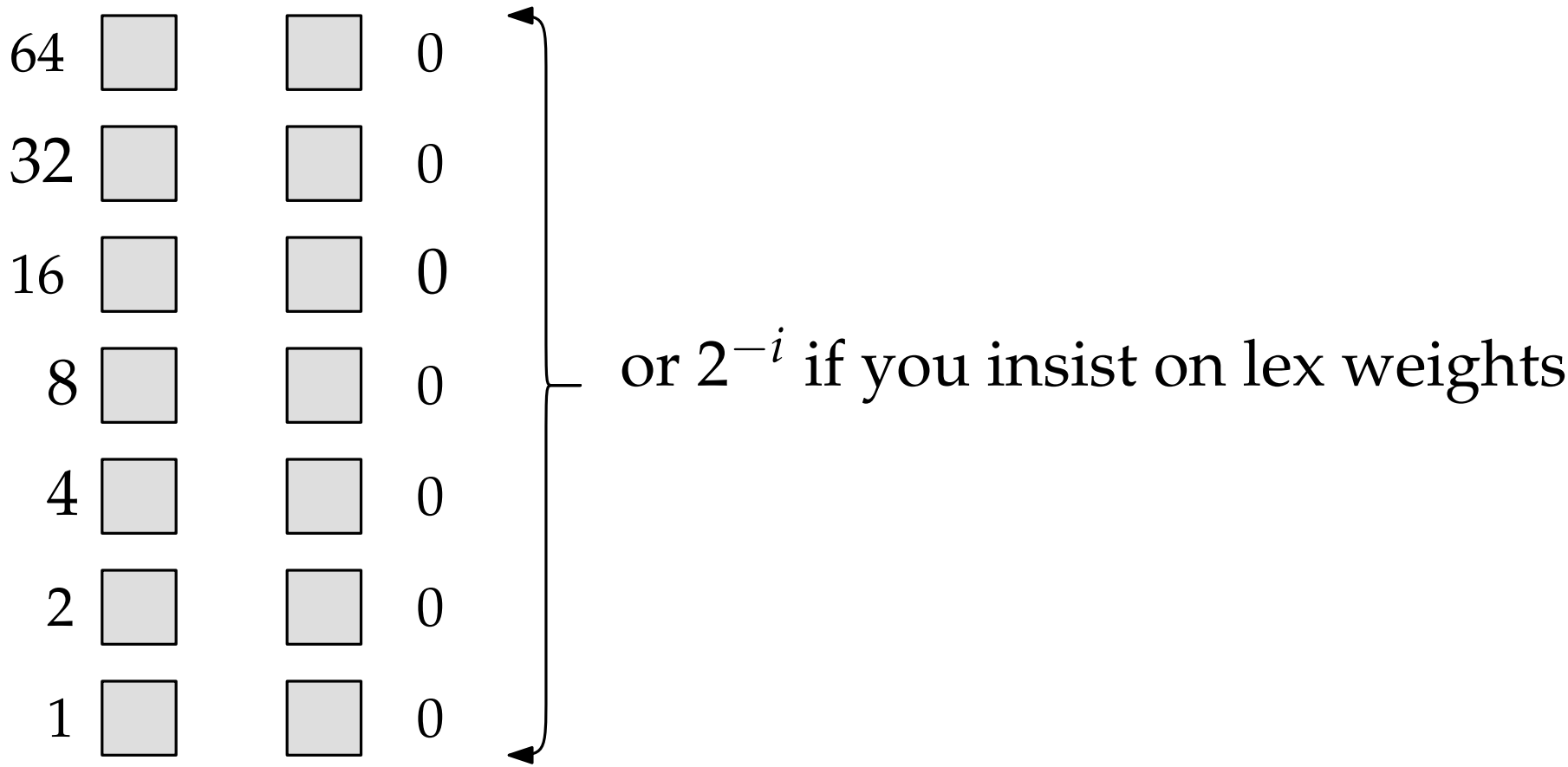

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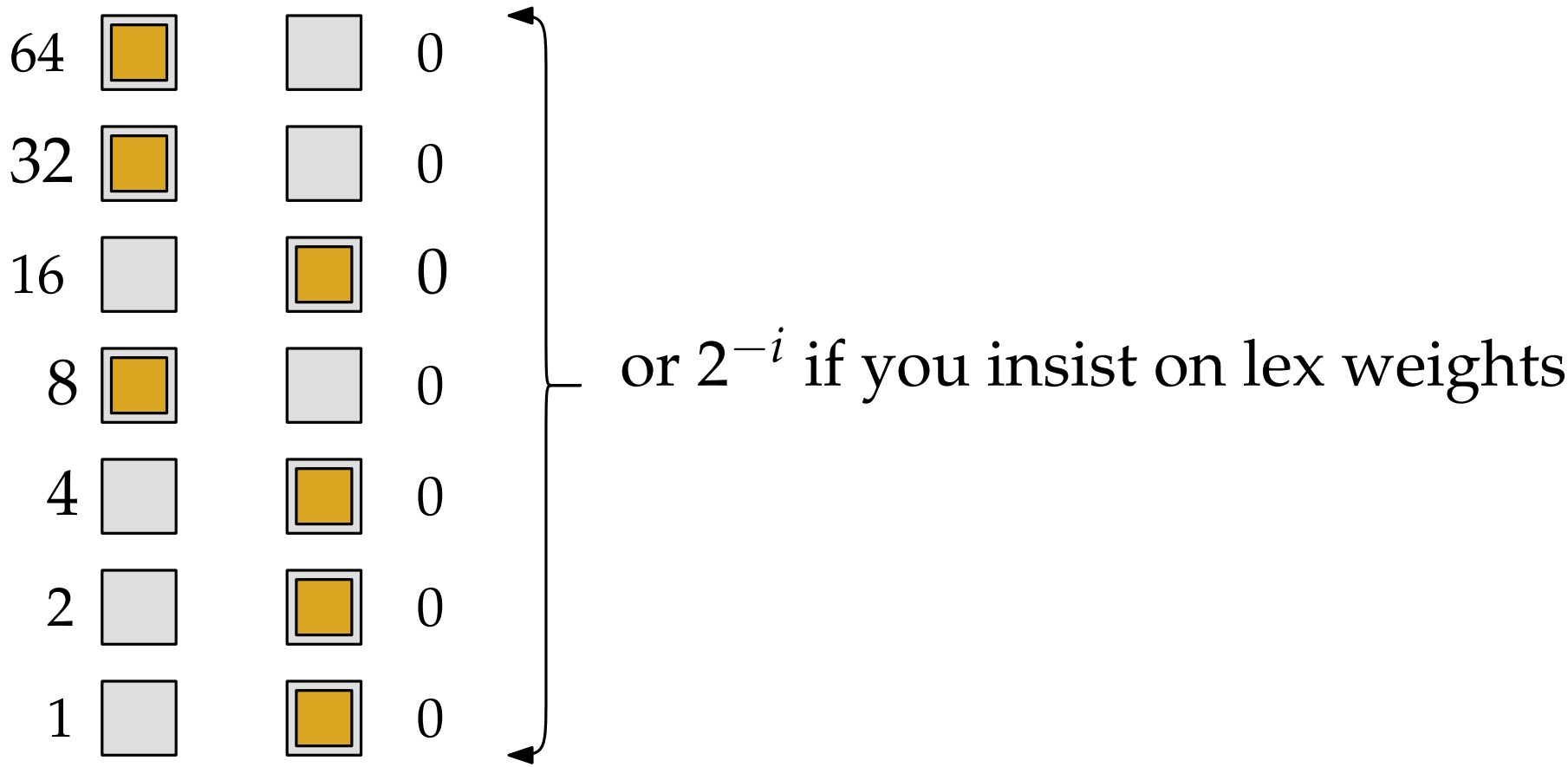

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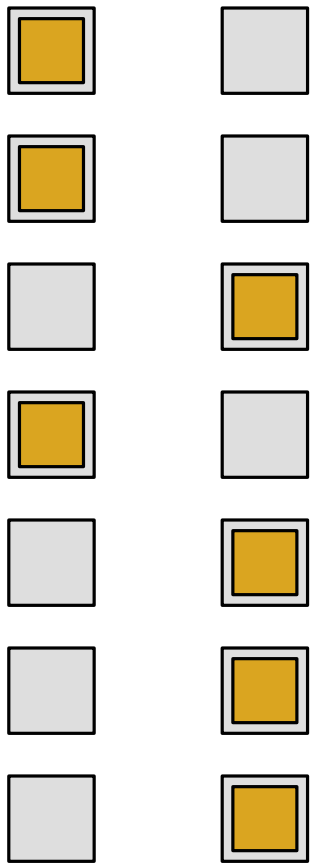

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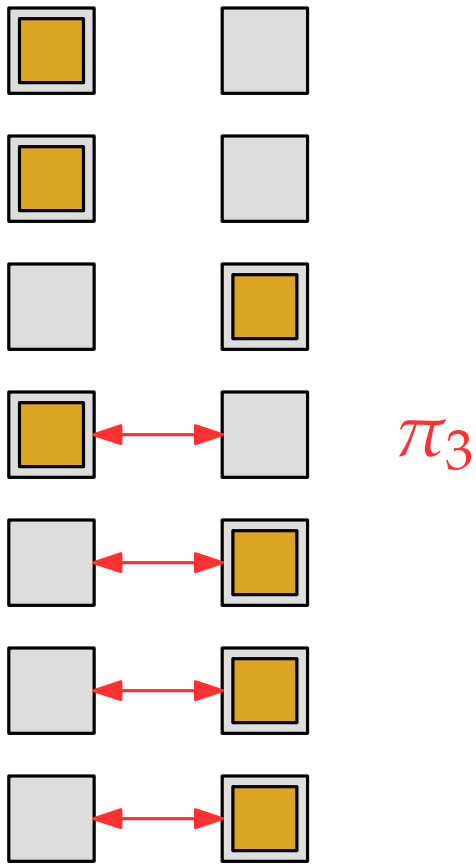

π_3

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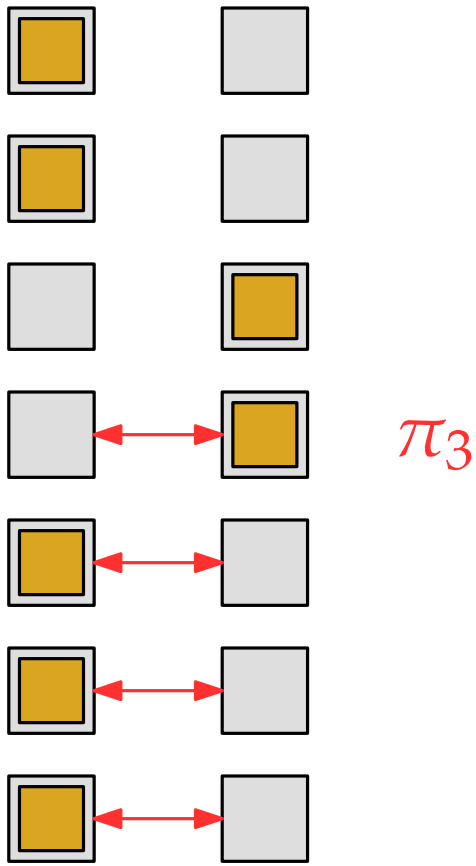

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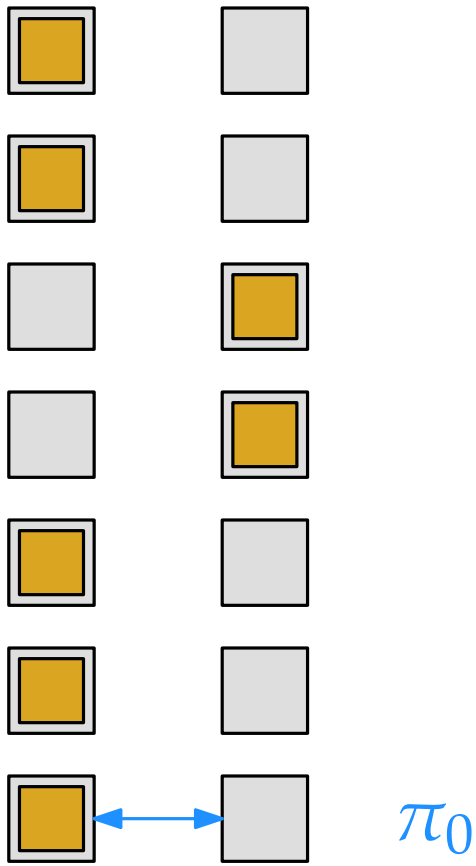

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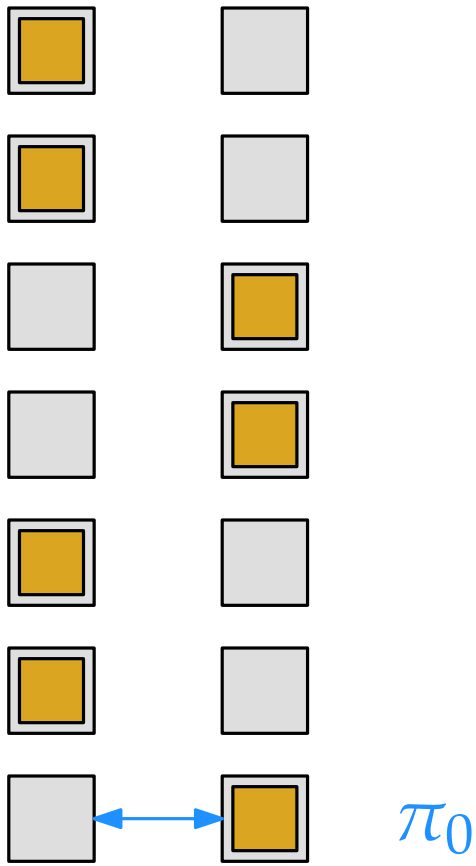

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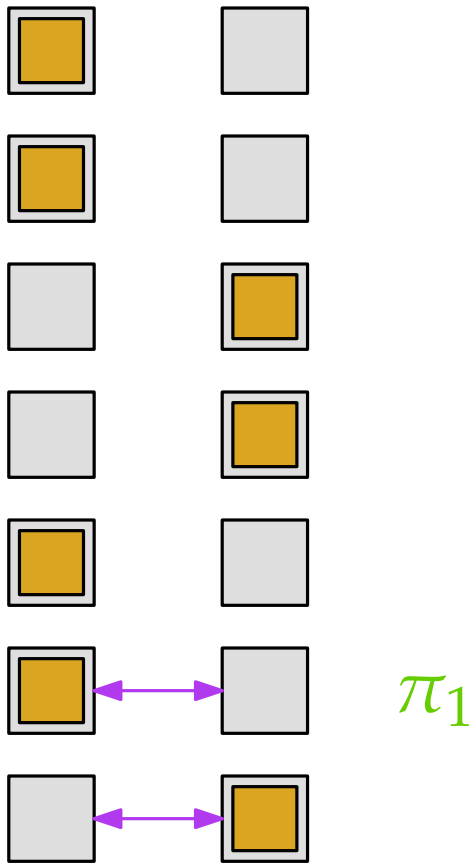

π_1

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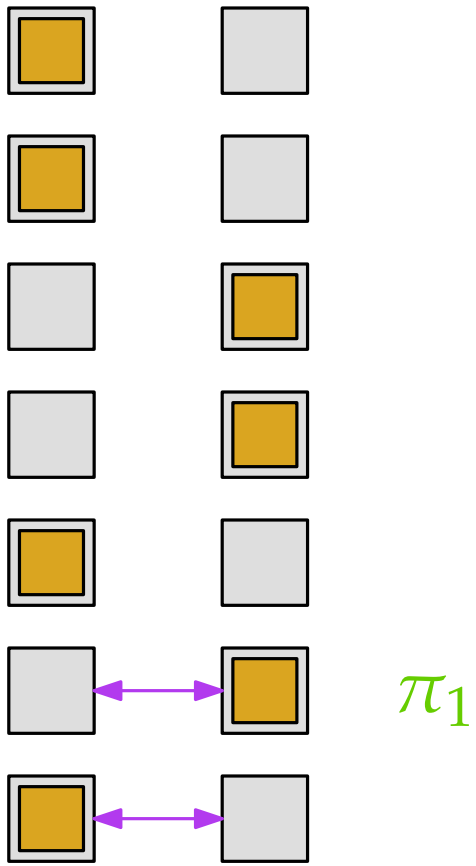

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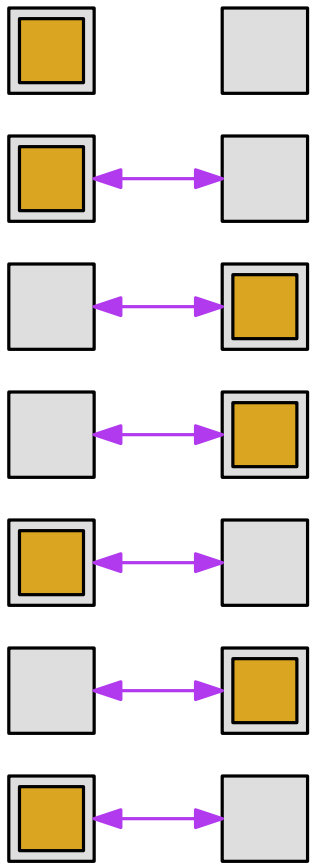

π_5

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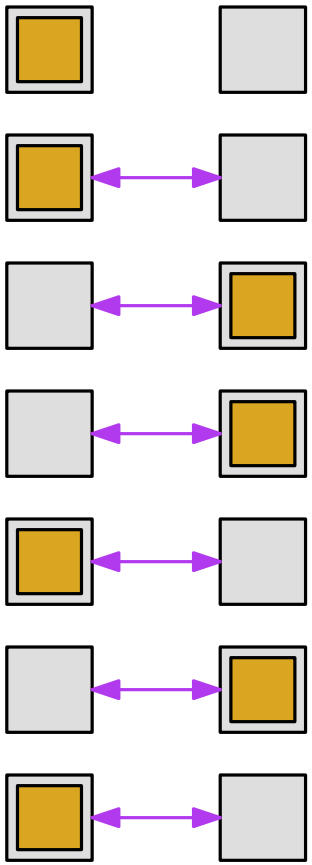

π_5

Wait! We have to prevent the "player" to apply π_5 prematurely.

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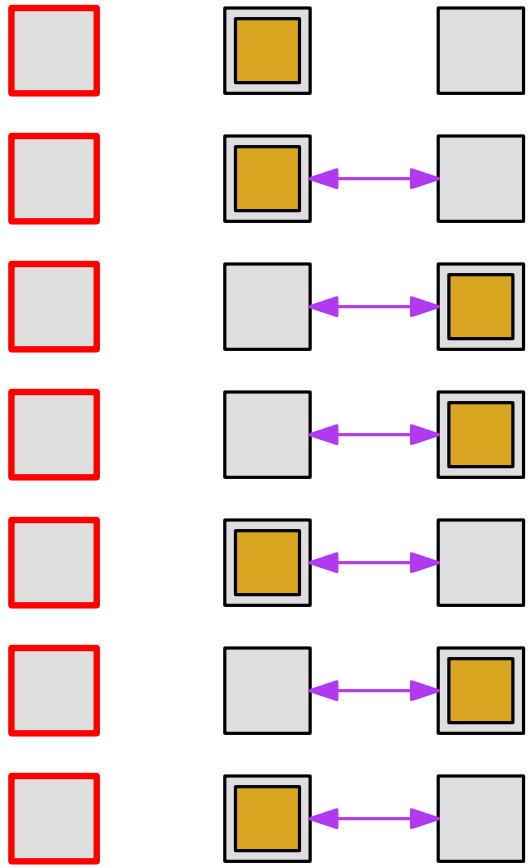

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Death positions of infinite weight (or 2^i for $i \geq n$ if you insist)

They will never carry a coin during an improving path.

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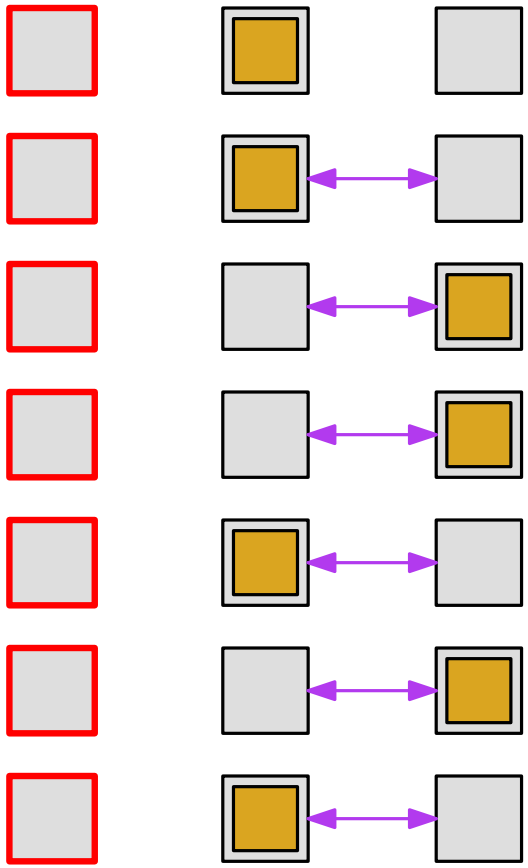

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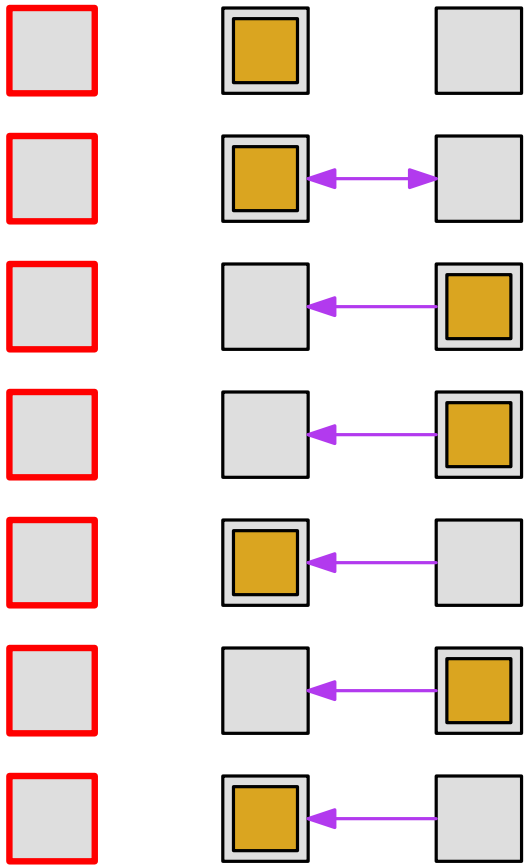

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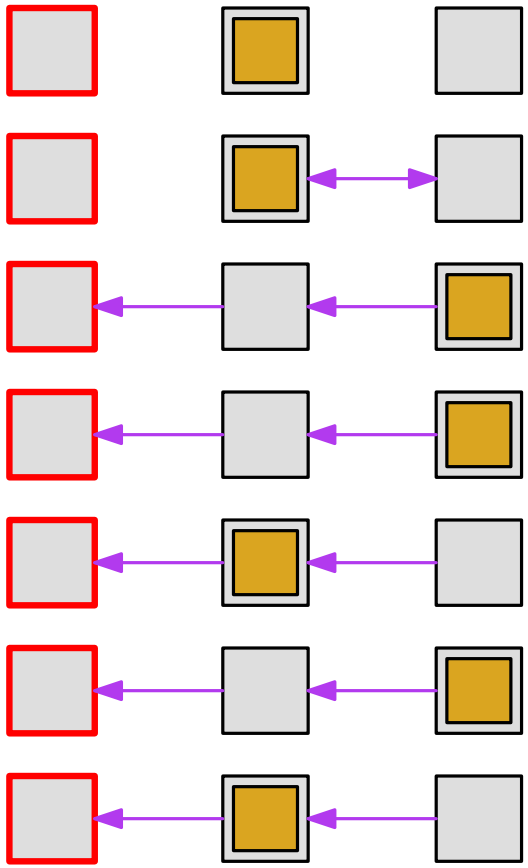

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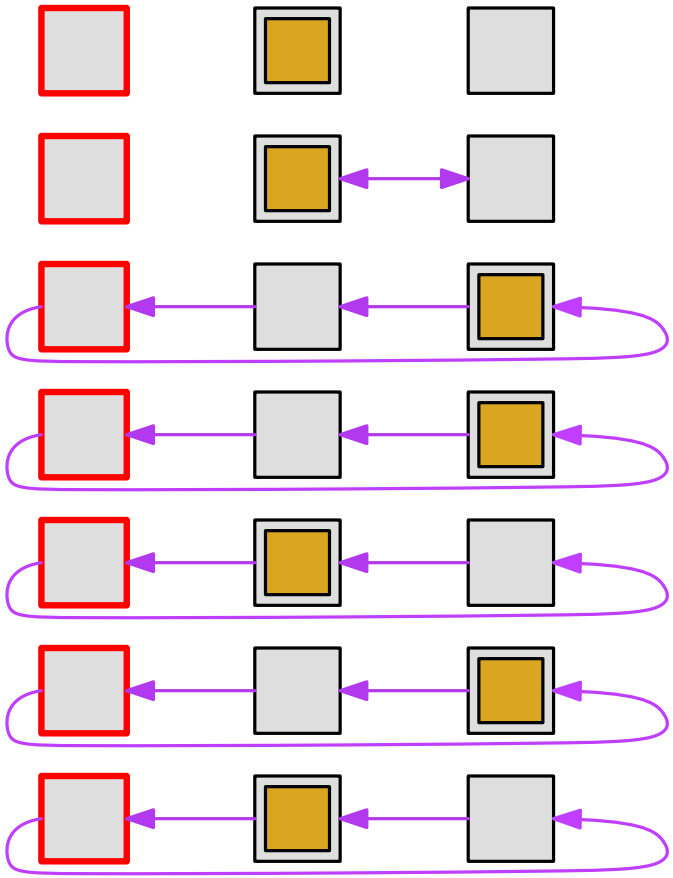

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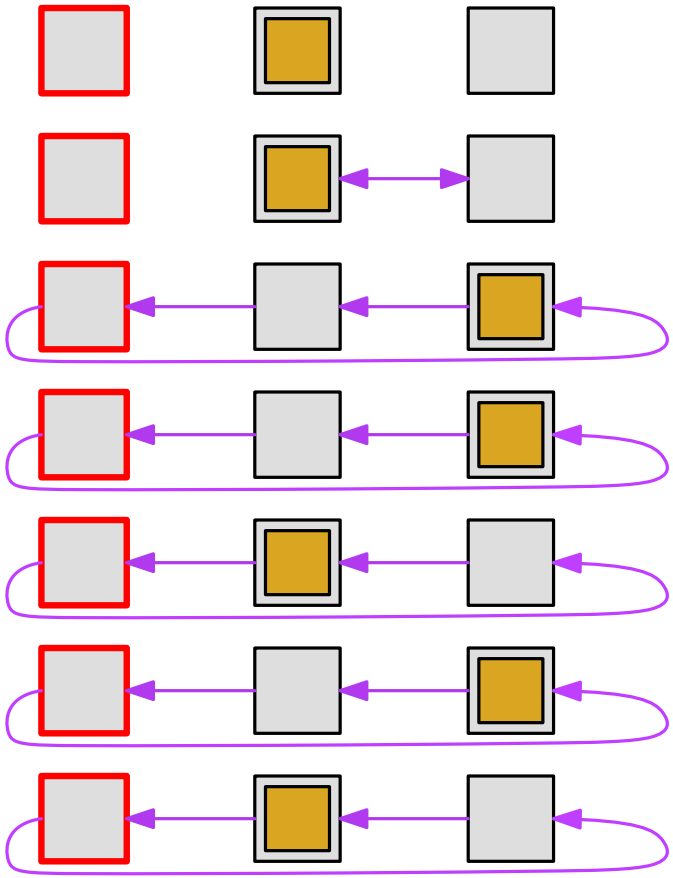

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Only once $b_0 = b_1 = \cdots = b_4 = 0$ can we apply π_5!

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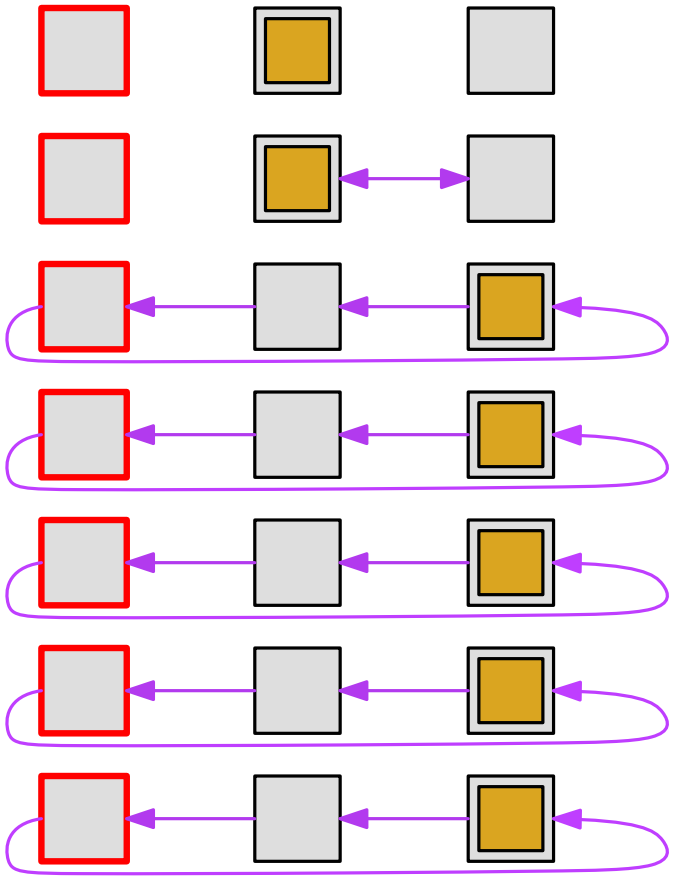

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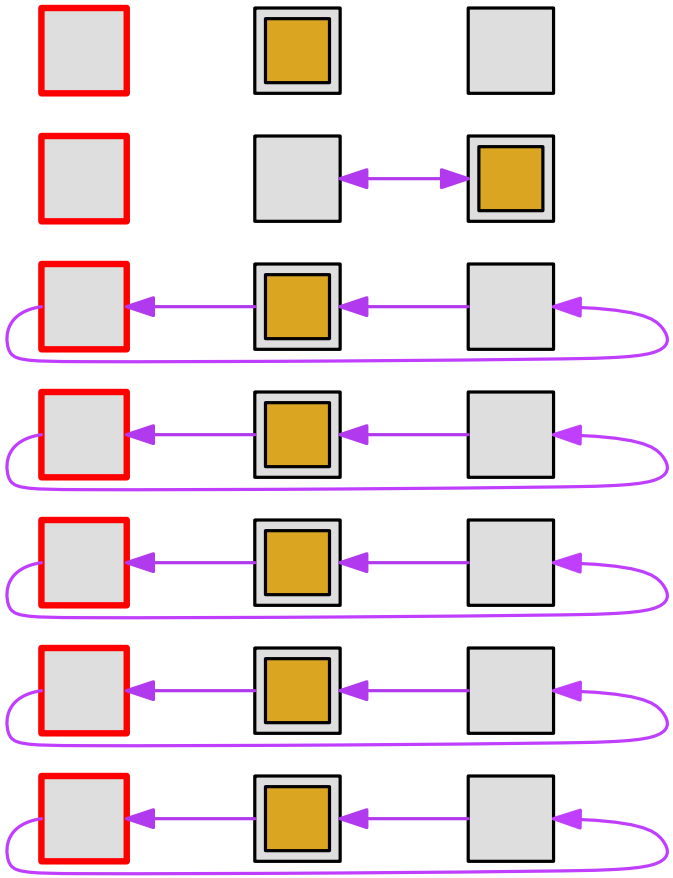

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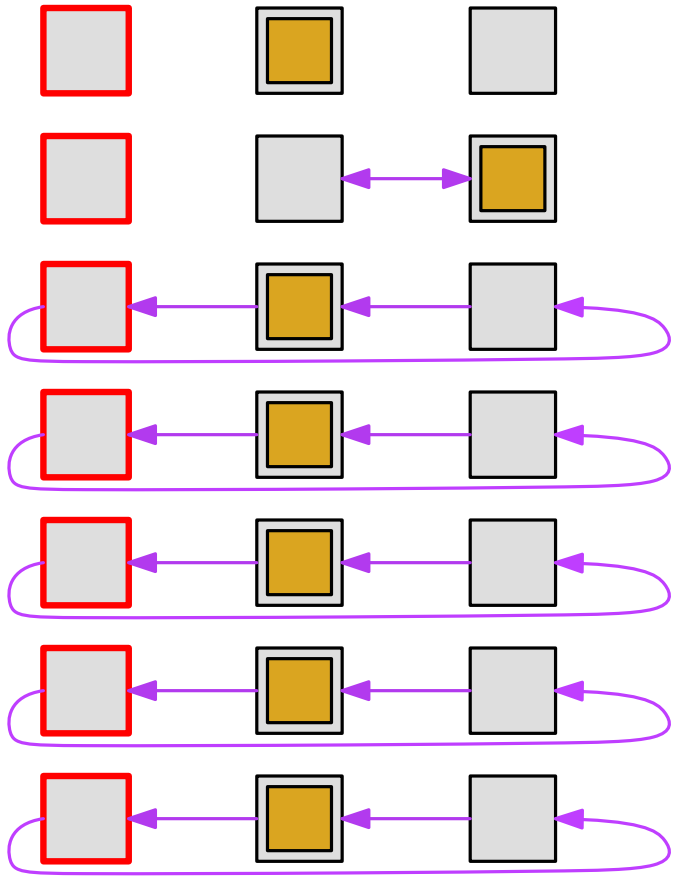


Wait! We have to prevent the "player" to apply π_5 prematurely.

Only once $b_0 = b_1 = \cdots = b_4 = 0$ can we apply π_5!

With n permutations and $3n$ positions we can build a counter that counts from $2^n - 1$ to 0.

Constant k: Improving Path can be exponential?

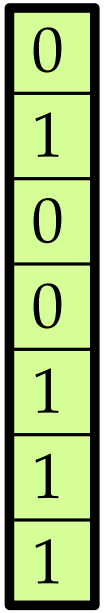
Surely for *constant* k the length of an improving path must be polynomial?

Constant k: Improving Path can be exponential?

Surely for *constant* k the length of an improving path must be polynomial?

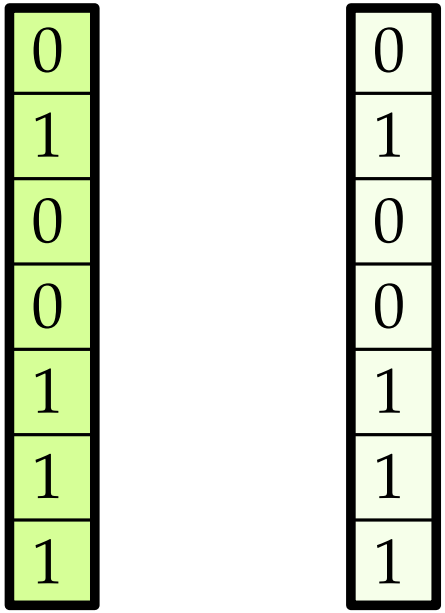
Construction. A machine with a constant number of permutations implementing a binary counter.

A counter with constantly many permutations



A register containing the current number in binary. These positions are pretty valuable.

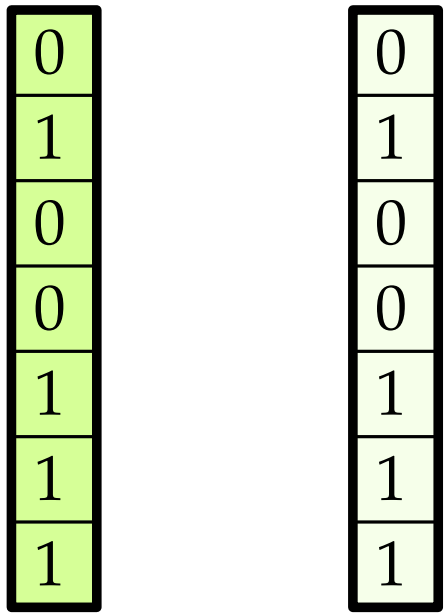
A counter with constantly many permutations



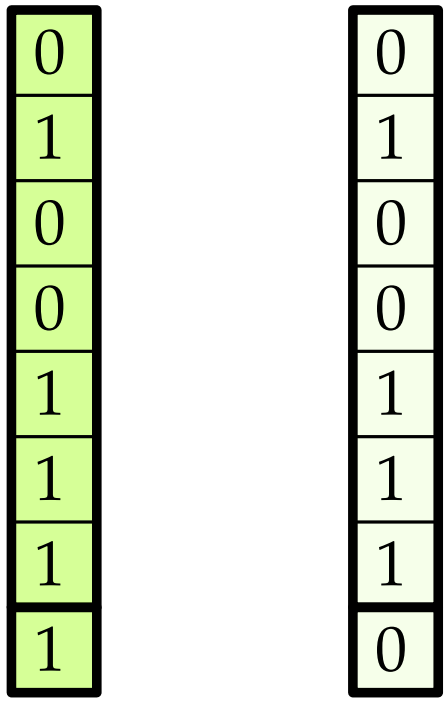
A copy of it, but the positions here are basically worthless.

A register containing the current number in binary. These positions are pretty valuable.

A counter with constantly many permutations

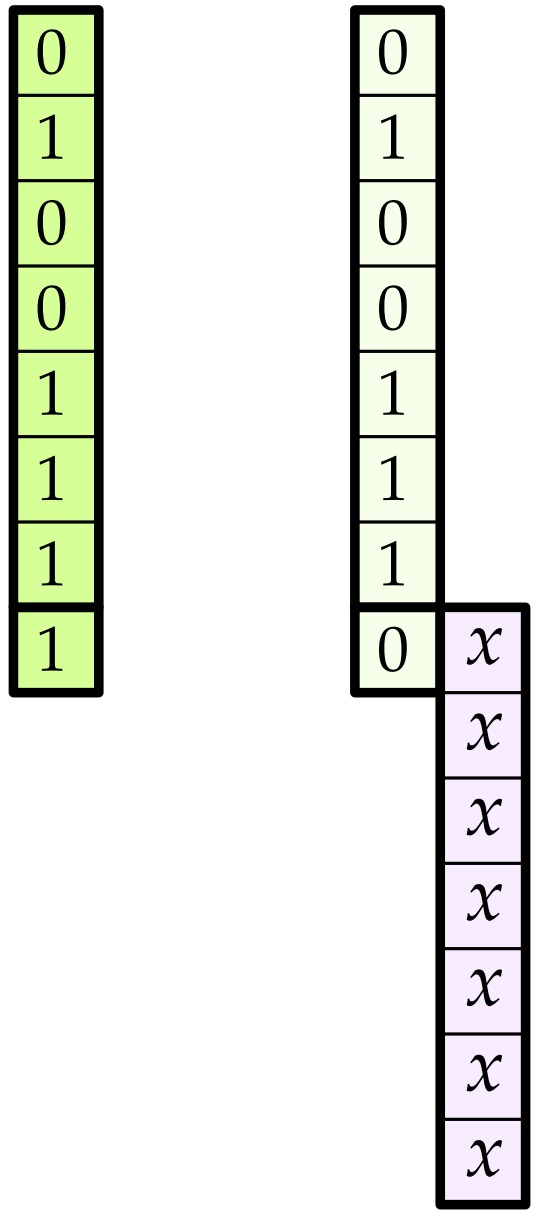


A counter with constantly many permutations



The two numbers are not equal: they differ by $\frac{1}{2}$ because there is one more position "after the dot".

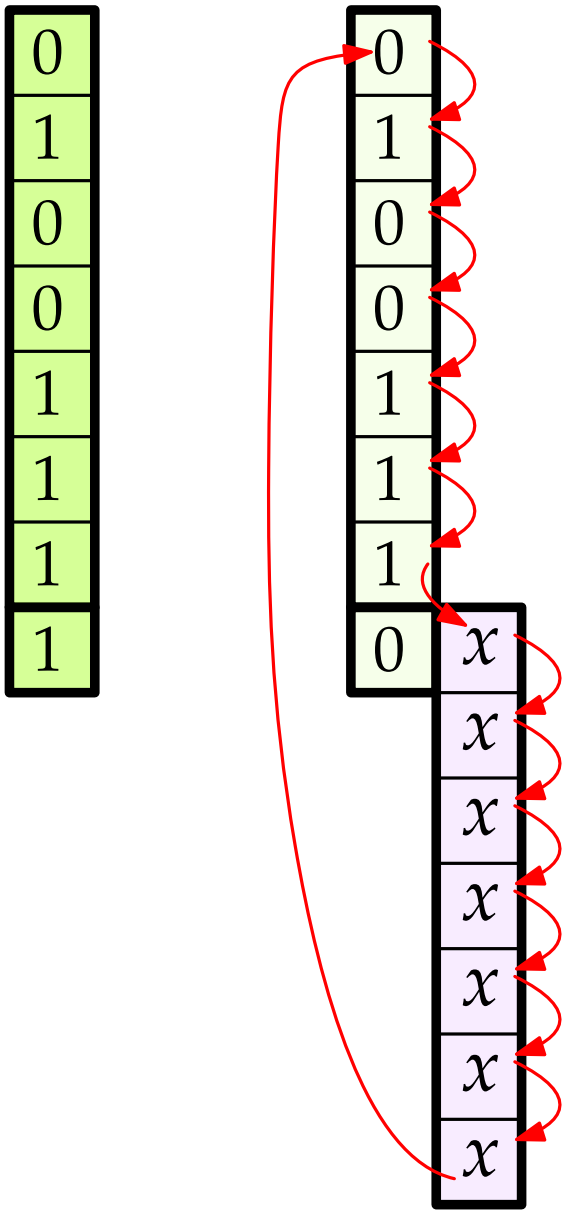
A counter with constantly many permutations



Some manoeuvering space filled with a new symbol, x. For example, we can encode $0, 1, x$ using three positions per cell.

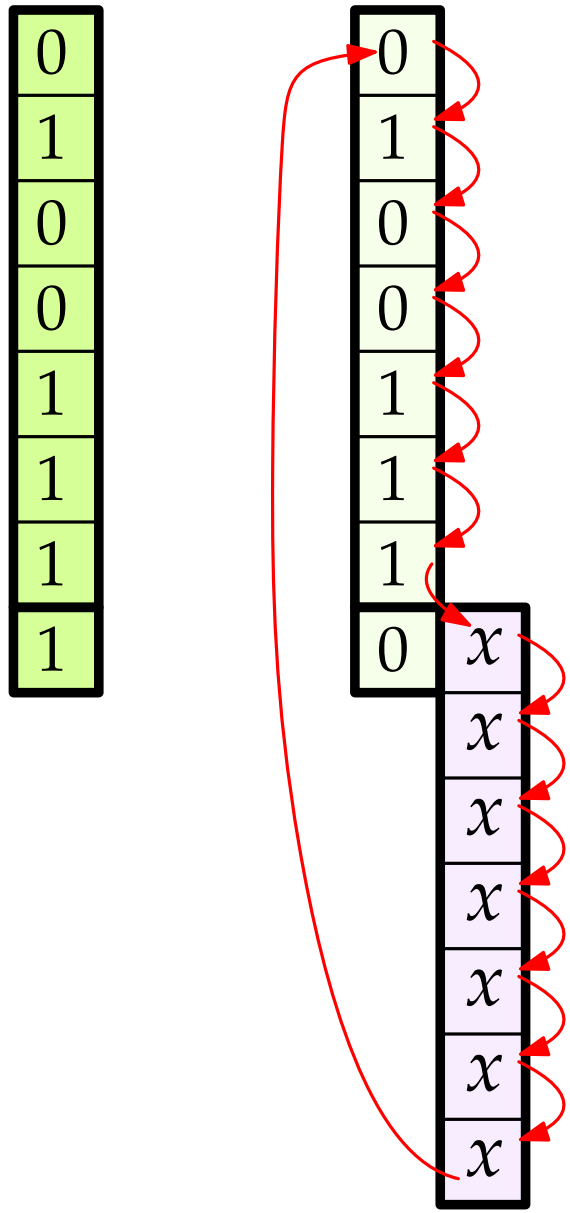
This space "likes" to have 1's but does not tolerate 0's. A 0 here would give payoff $-\infty$.

A counter with constantly many permutations



The permutation π

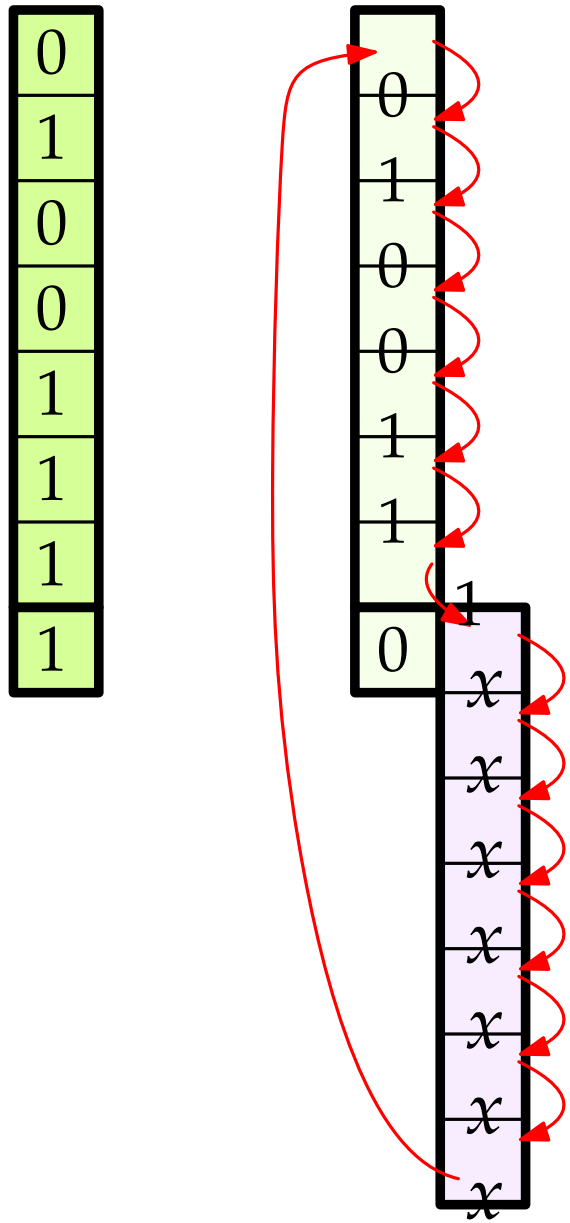
A counter with constantly many permutations



The permutation π

Applying π increases the payoff because the manoeuvering buffer gives us a payoff for each 1.

A counter with constantly many permutations



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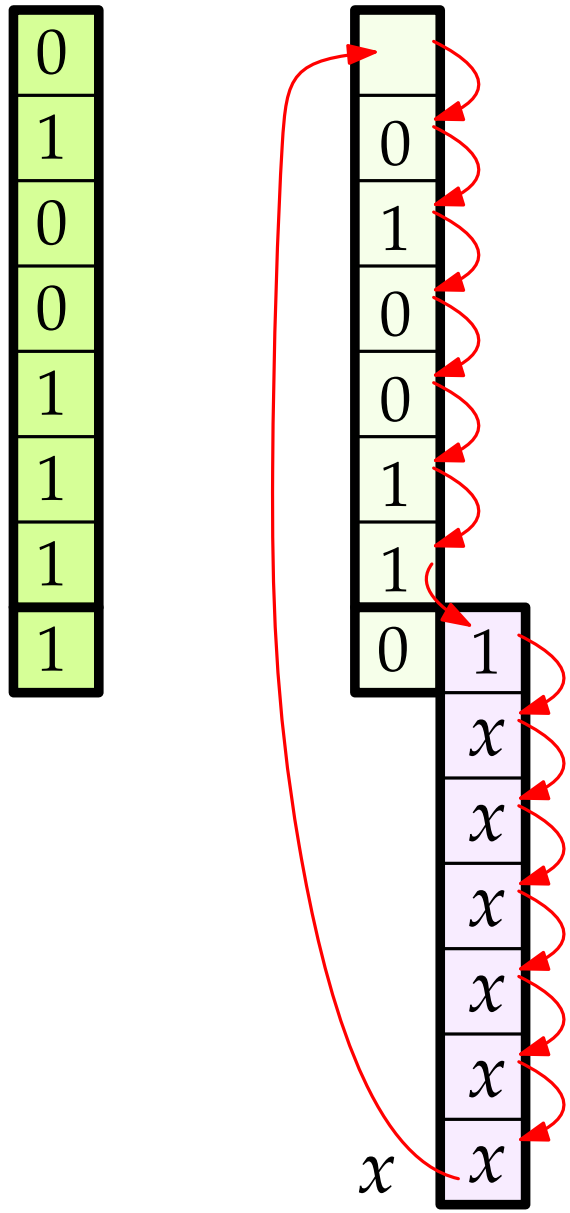

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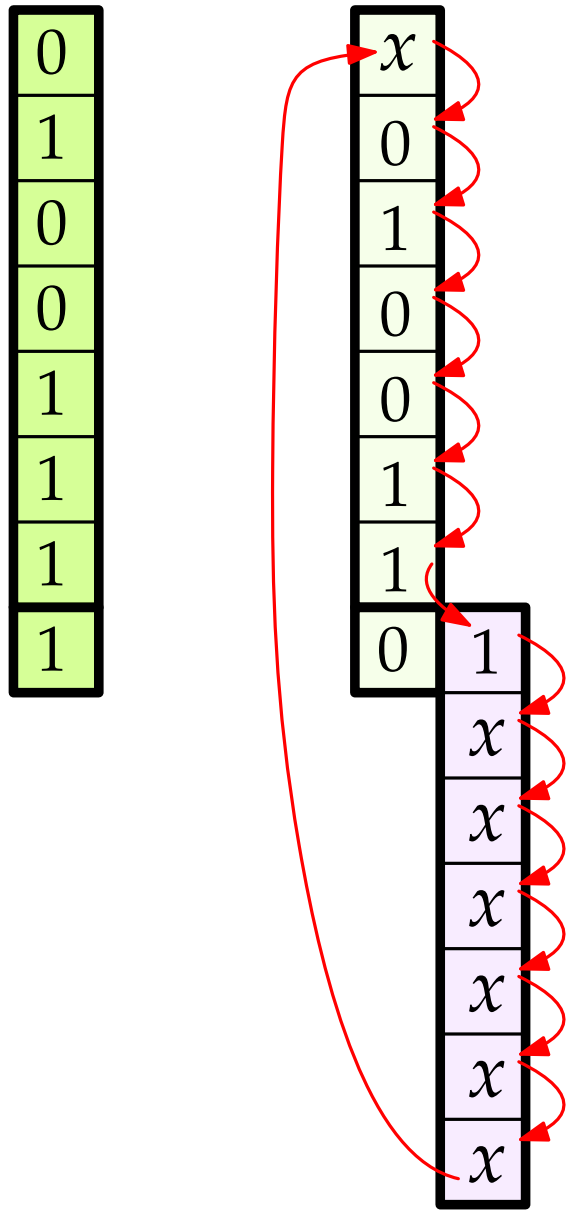

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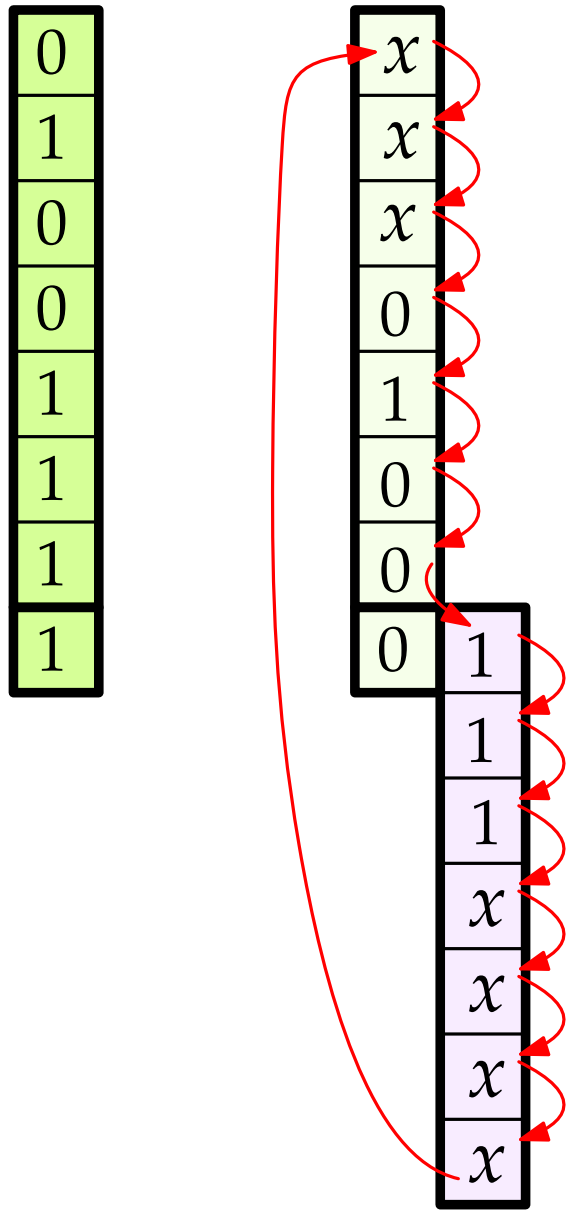

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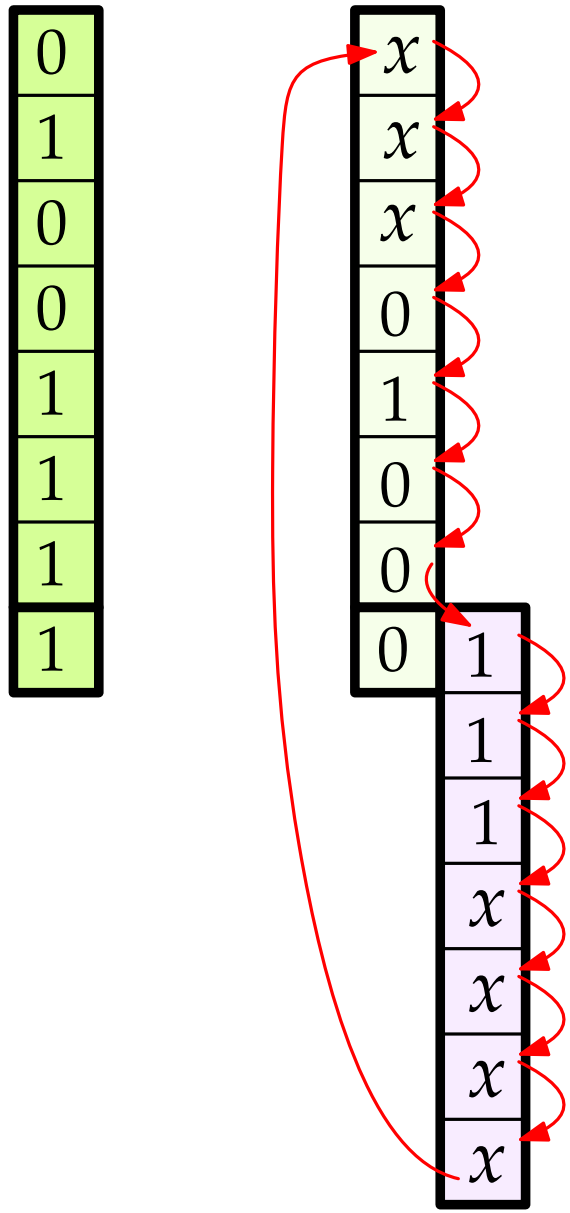
Applying π increases the payoff because the manoeuvering buffer gives us a payoff for each 1.

...and twice more

A counter with constantly many permutations



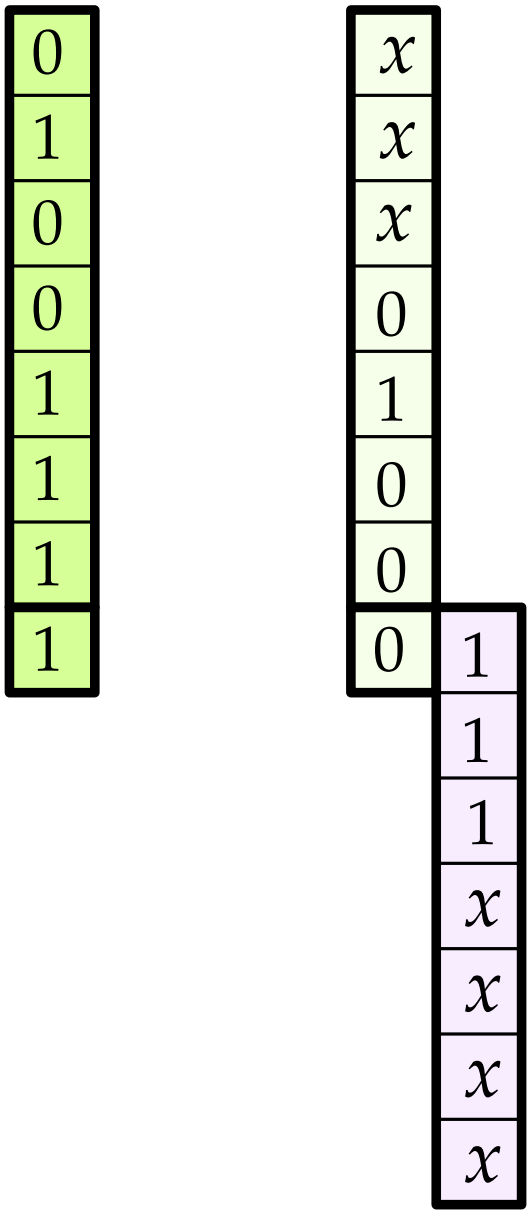
The permutation π

Applying π increases the payoff because the manoeuvering buffer gives us a payoff for each 1.

...and twice more

Cannot do it again because a 0 on violet carries an extreme penalty.

A counter with constantly many permutations



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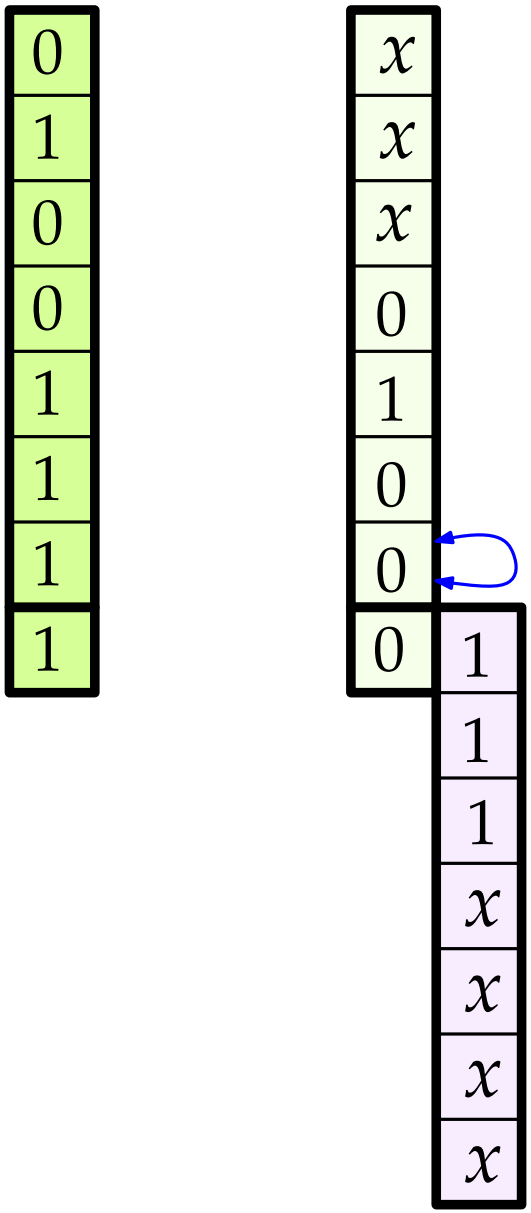
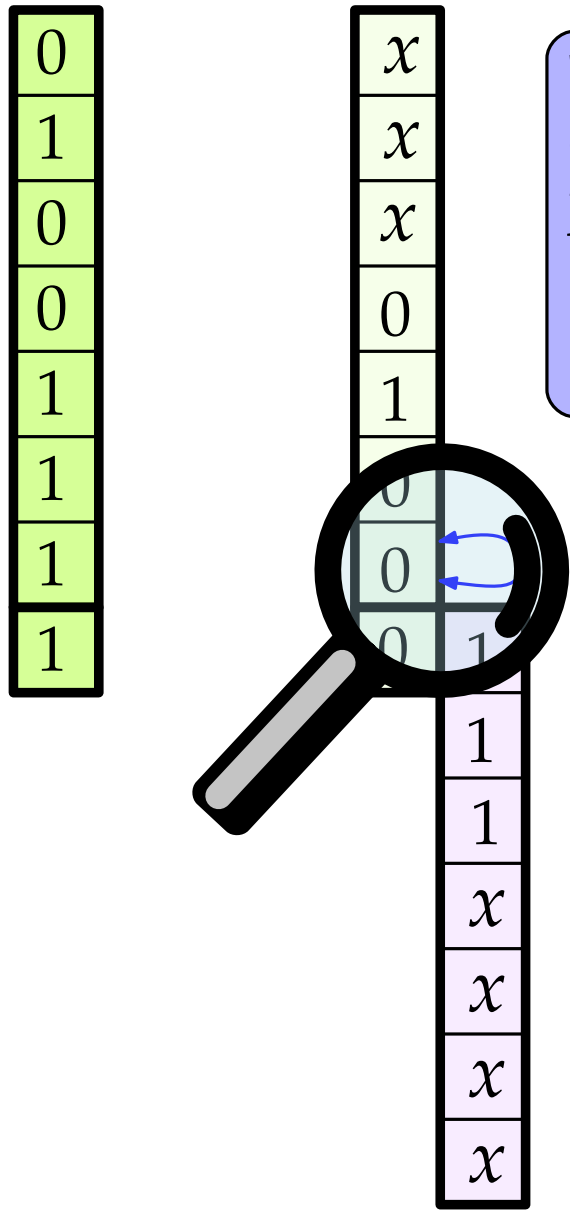


The permutation σ inverts the lowest cell in the copy register, but can only be applied if it has a 0 there.

A counter with constantly many permutations



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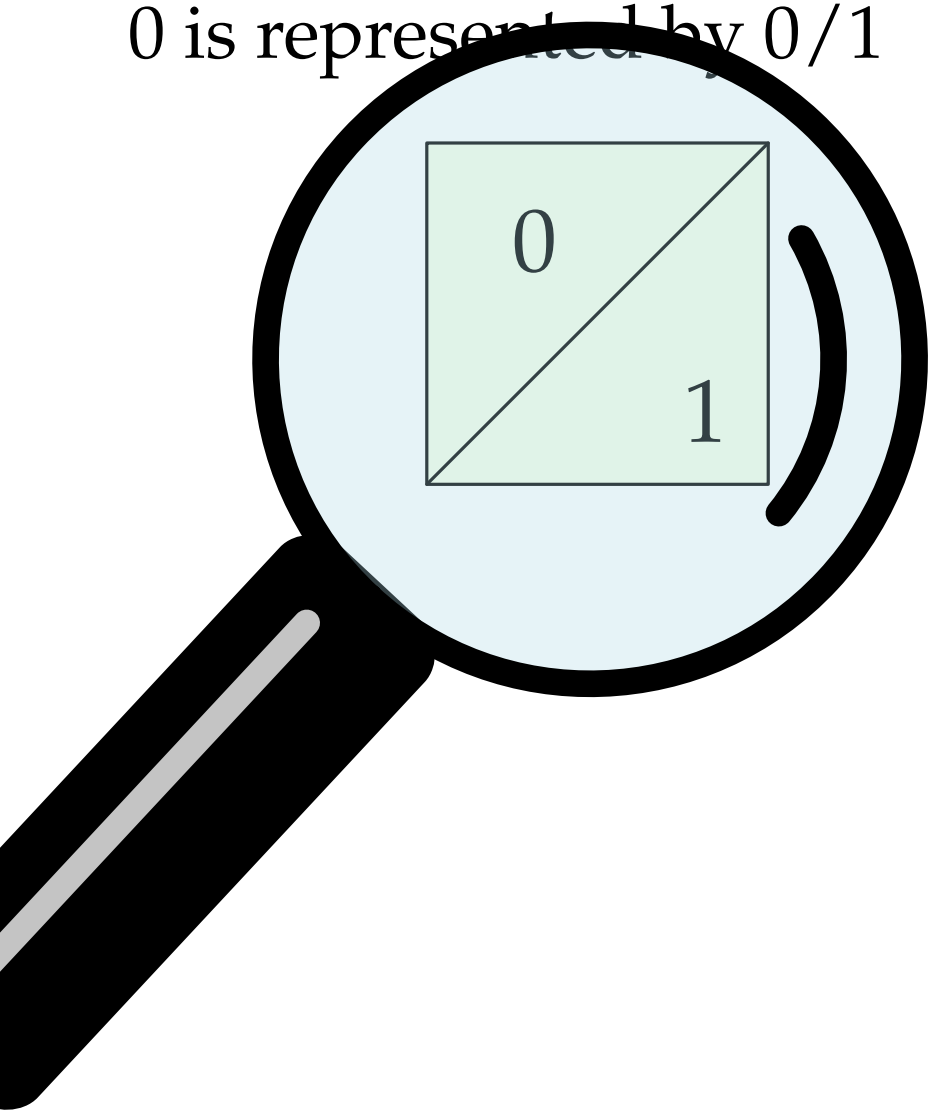
A counter with constantly many permutations

0 is represented by 0/1



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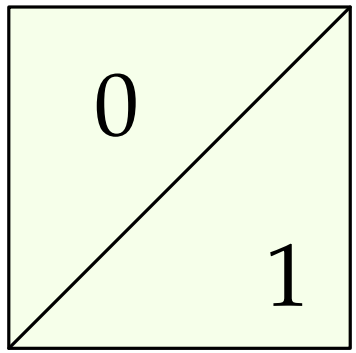

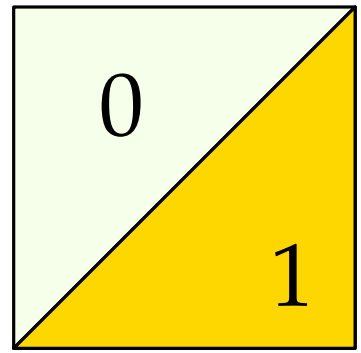

The golden 1 is extremely valuable

A counter with constantly many permutations

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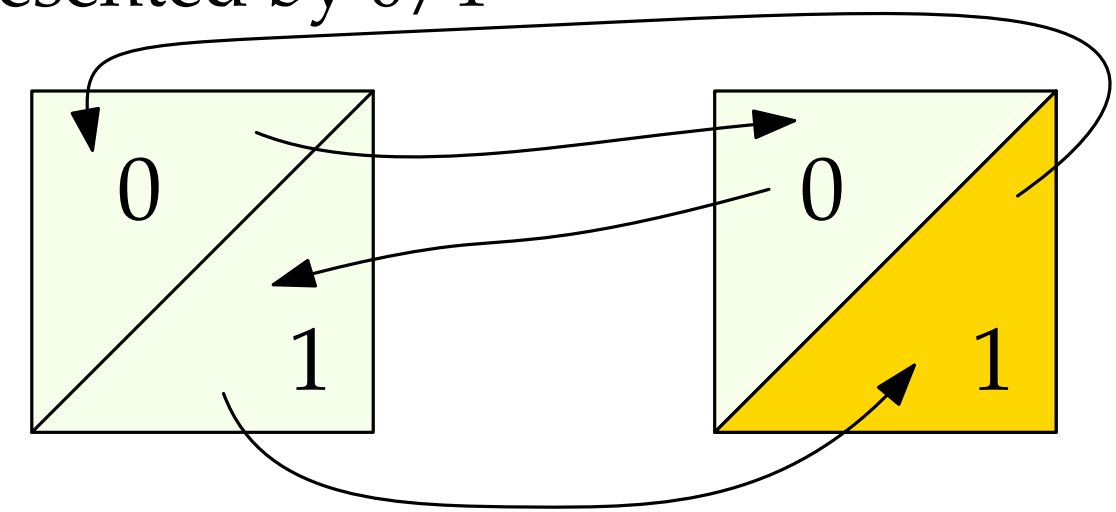
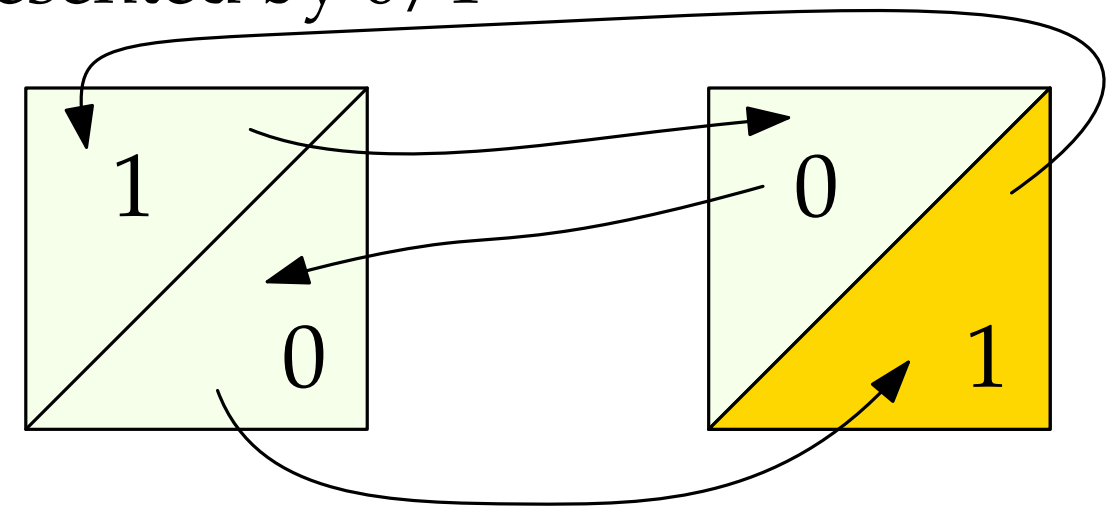


The golden 1 is extremely valuable

If the left square is in state 0 (represented by 0/1) then applying σ inverts the left cell and leaves the right cell unchanged.

A counter with constantly many permutations

0 is represented by 0/1

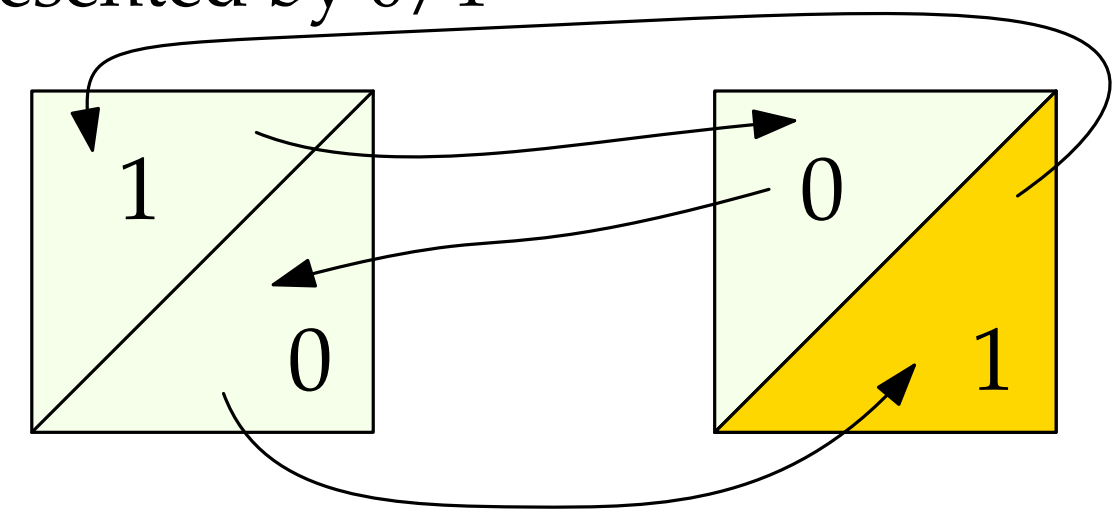


The golden 1 is extremely valuable

If the left square is in state 1 (represented by 1/0) then applying σ would put a 0 into the golden cell.

A counter with constantly many permutations

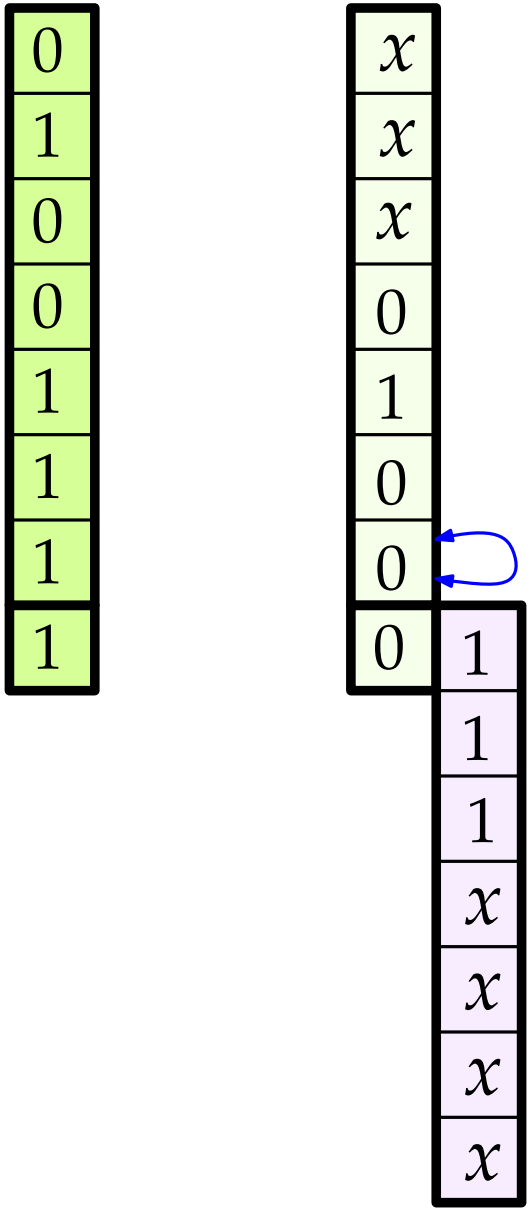
0 is represented by 0/1



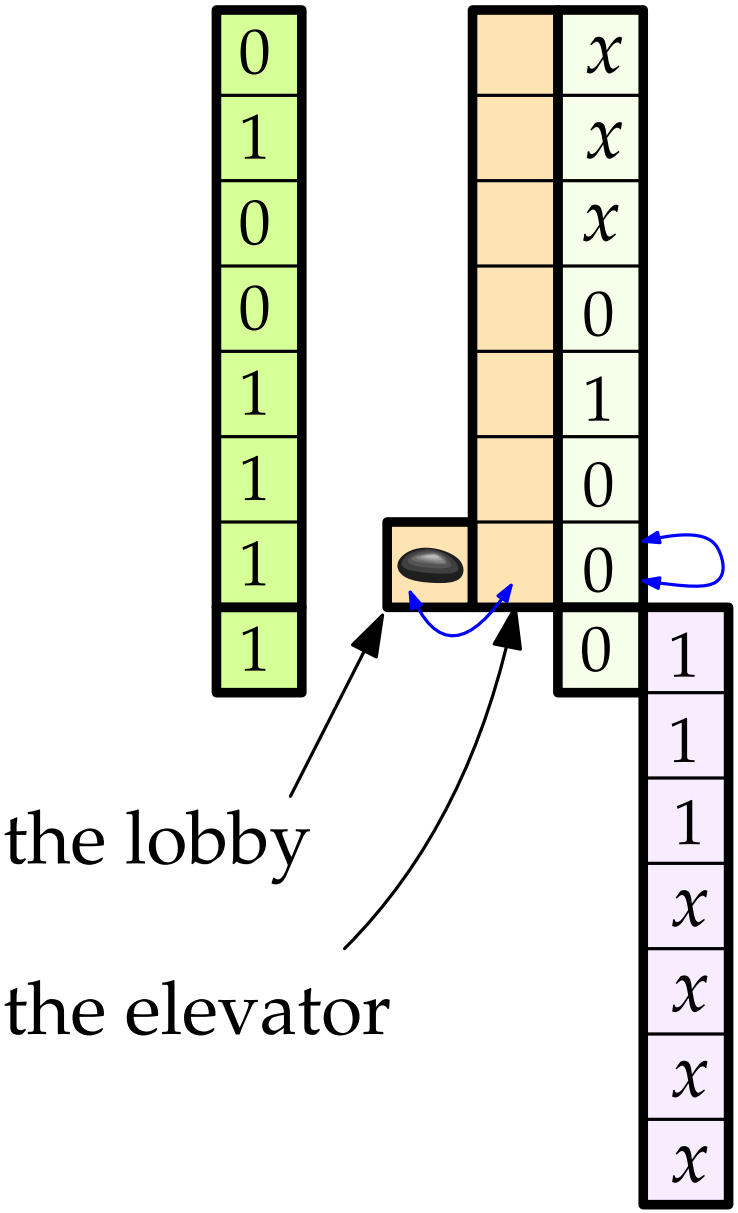
The golden 1 is extremely valuable

Actually we will use more than two positions per cell, to represent the value x. Maybe 3 positions are enough. A similar construction will work in this case.

A counter with constantly many permutations



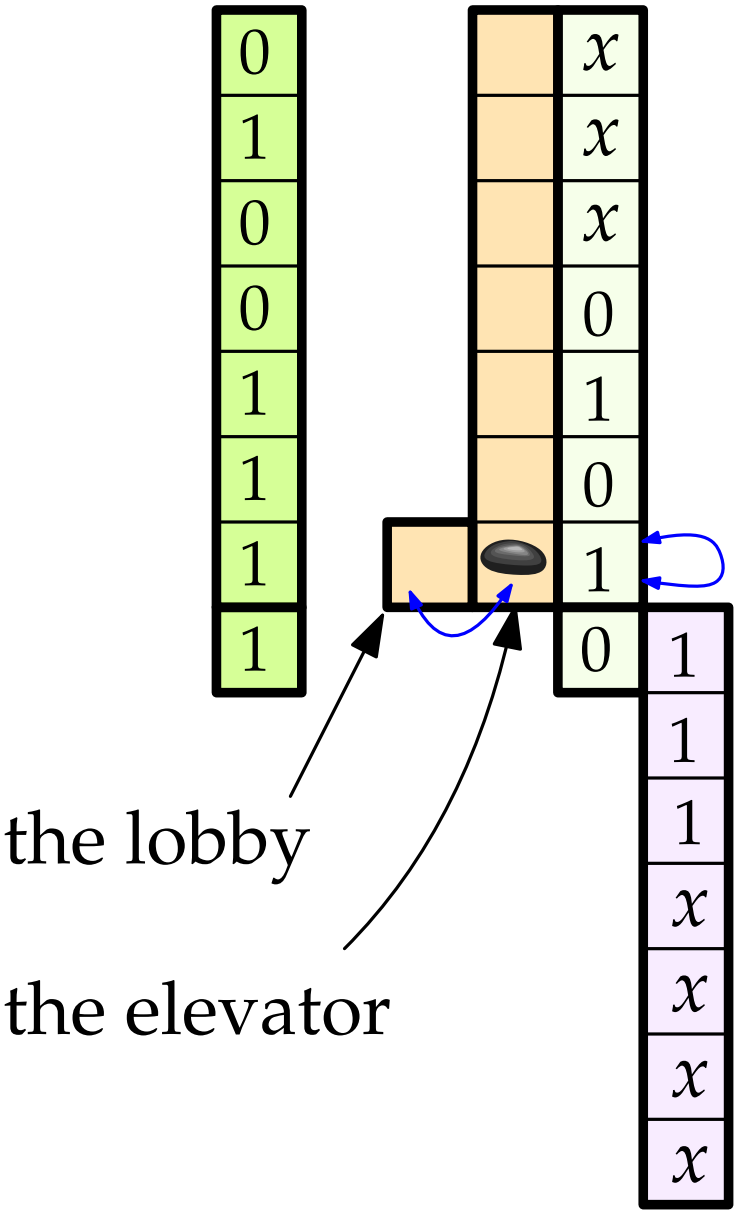
A counter with constantly many permutations



Furthermore, σ moves the pebble from the "lobby" into the "elevator".

This is only possible if pebble really is in the lobby. Otherwise it would give a payoff of $-\infty$.

A counter with constantly many permutations



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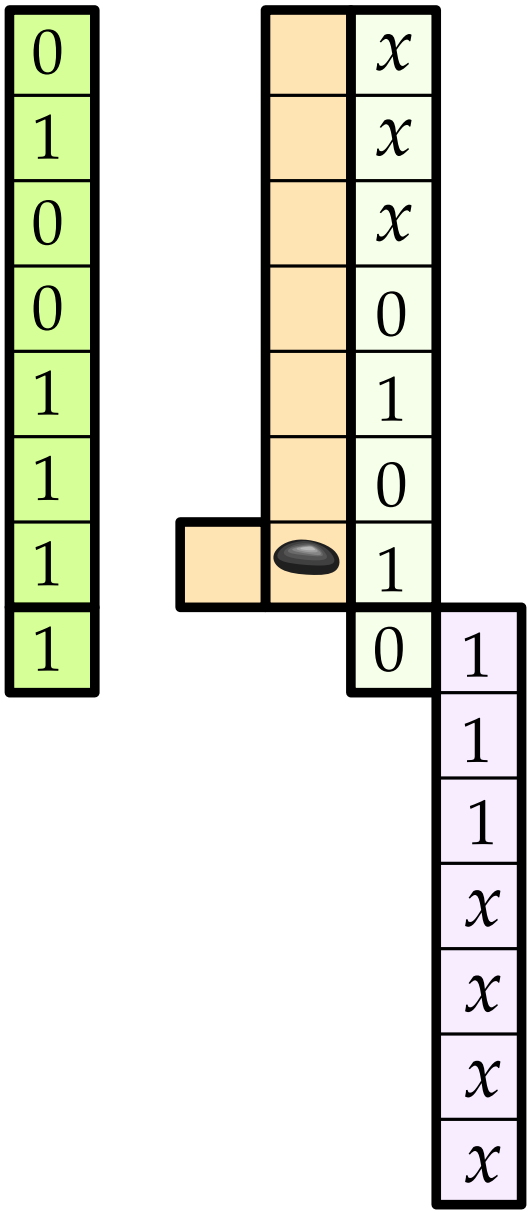
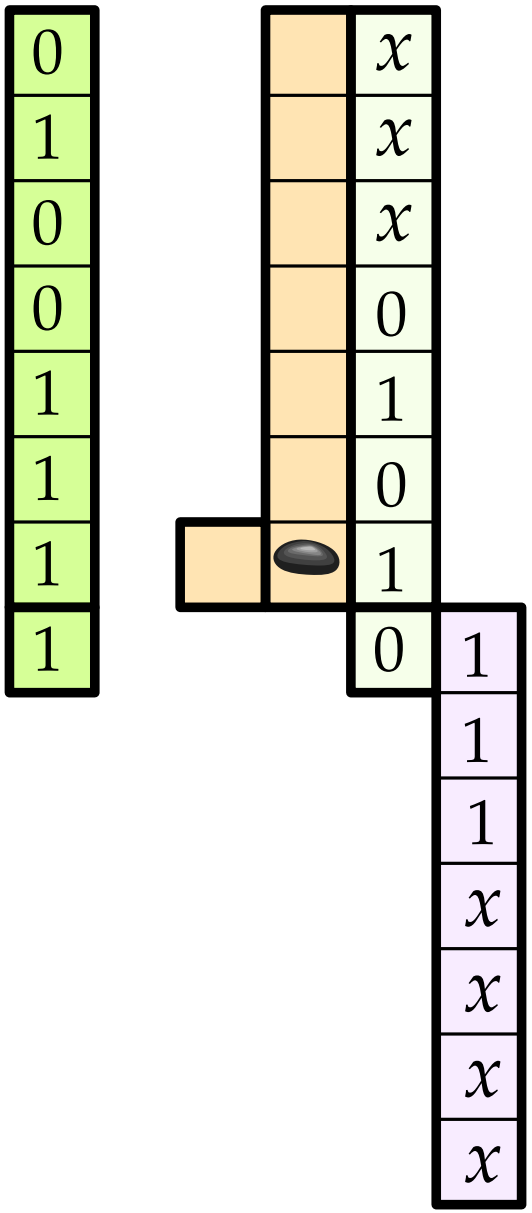
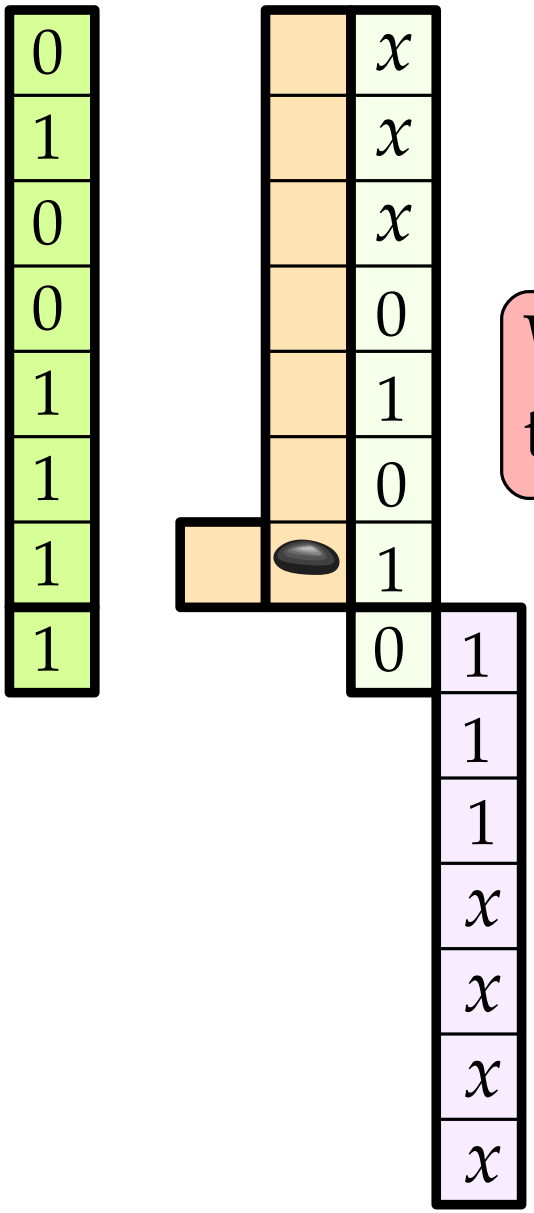


A counter with constantly many permutations



Wait, can't we apply π again und push the new 1 down?

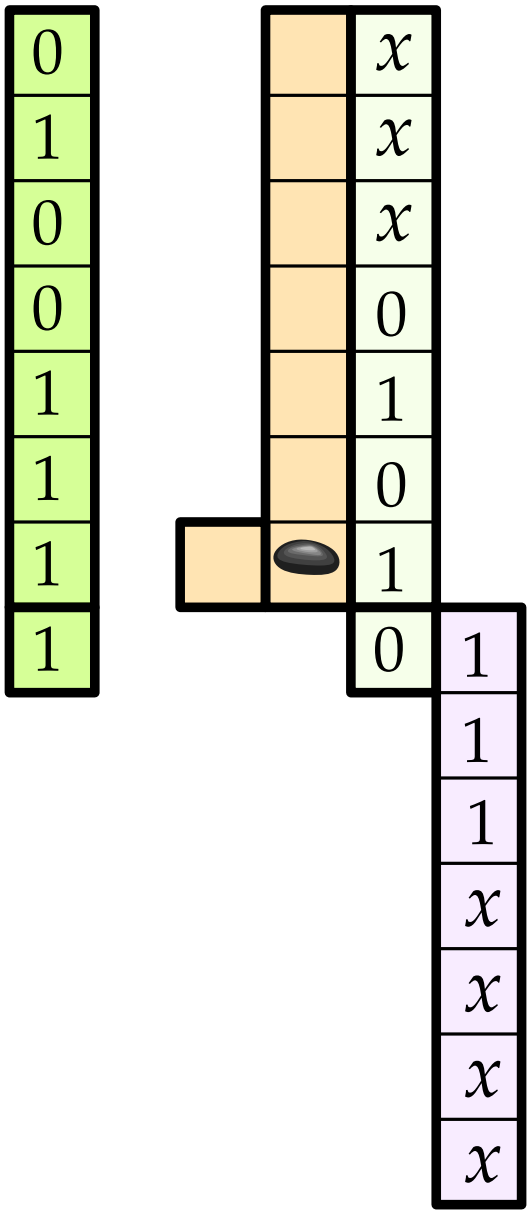
A counter with constantly many permutations



Wait, can't we apply π again und push the new 1 down?

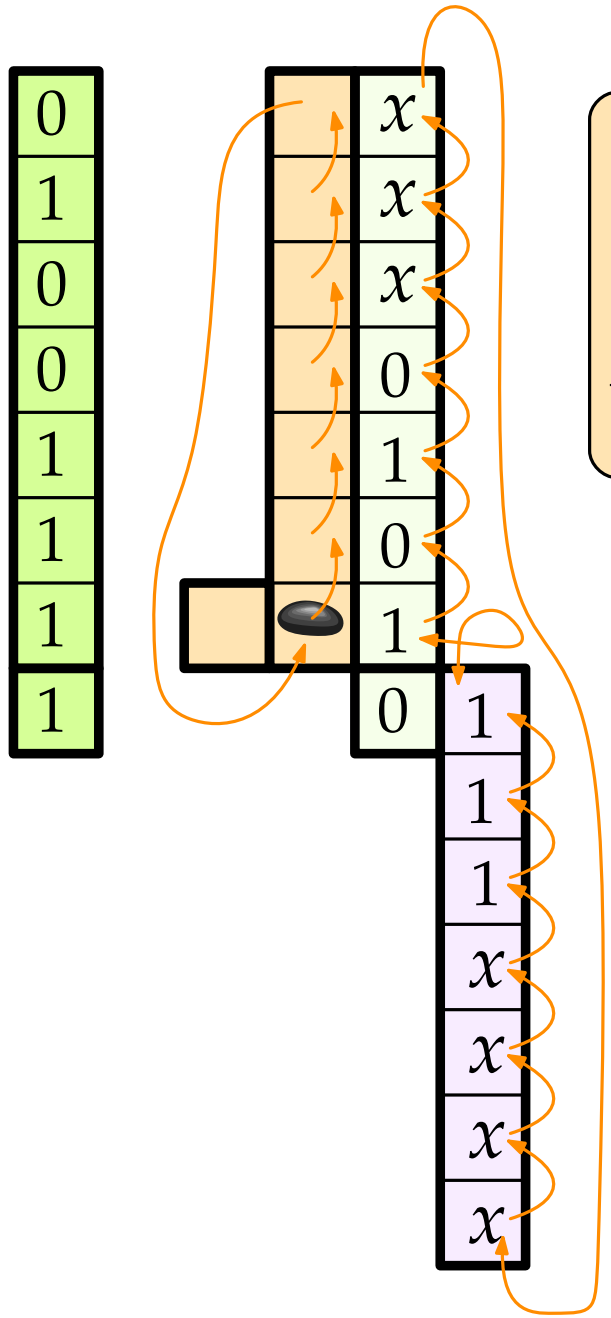
We need to add to π a check that the elevator is completely empty!

A counter with constantly many permutations



The higher the stone, the higher the payoff. This is stronger than the violet buffer wanting 1's.

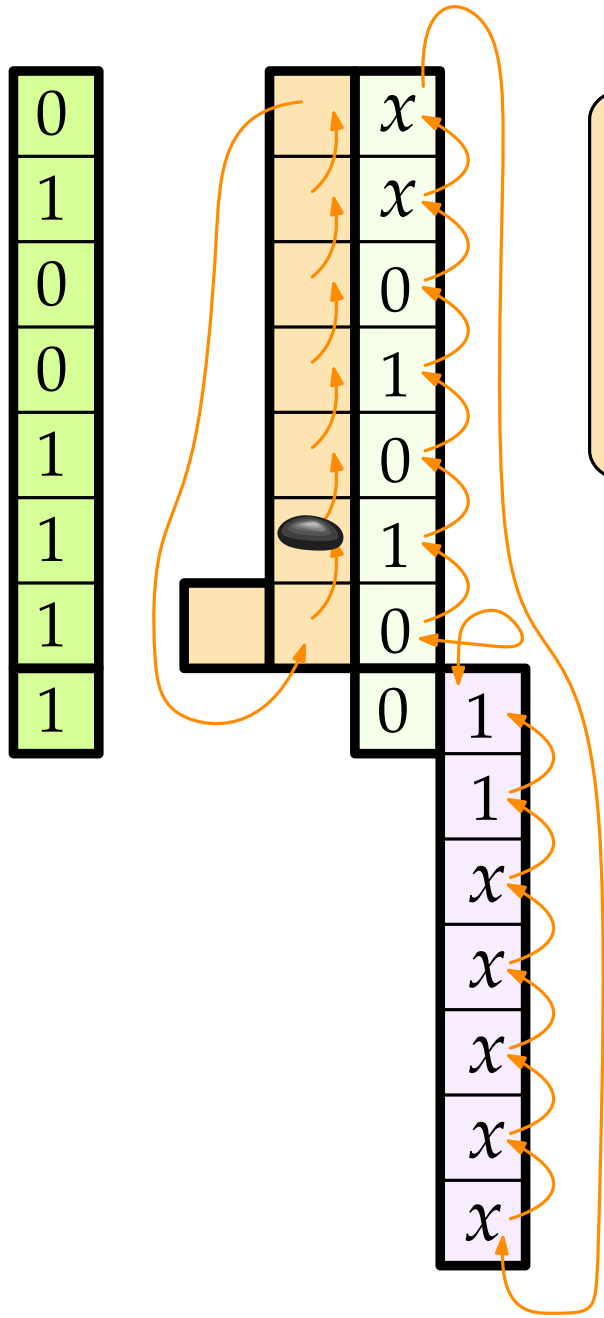
A counter with constantly many permutations



The higher the stone, the higher the payoff. This is stronger than the violet buffer wanting 1's.

Permutation τ pushes up the pebble, the buffer, and the right register. It can work against the buffer's love for 1's because lifting up the pebble creates a higher payoff. Also, it inverts the 1 that goes from buffer to register.

A counter with constantly many permutations



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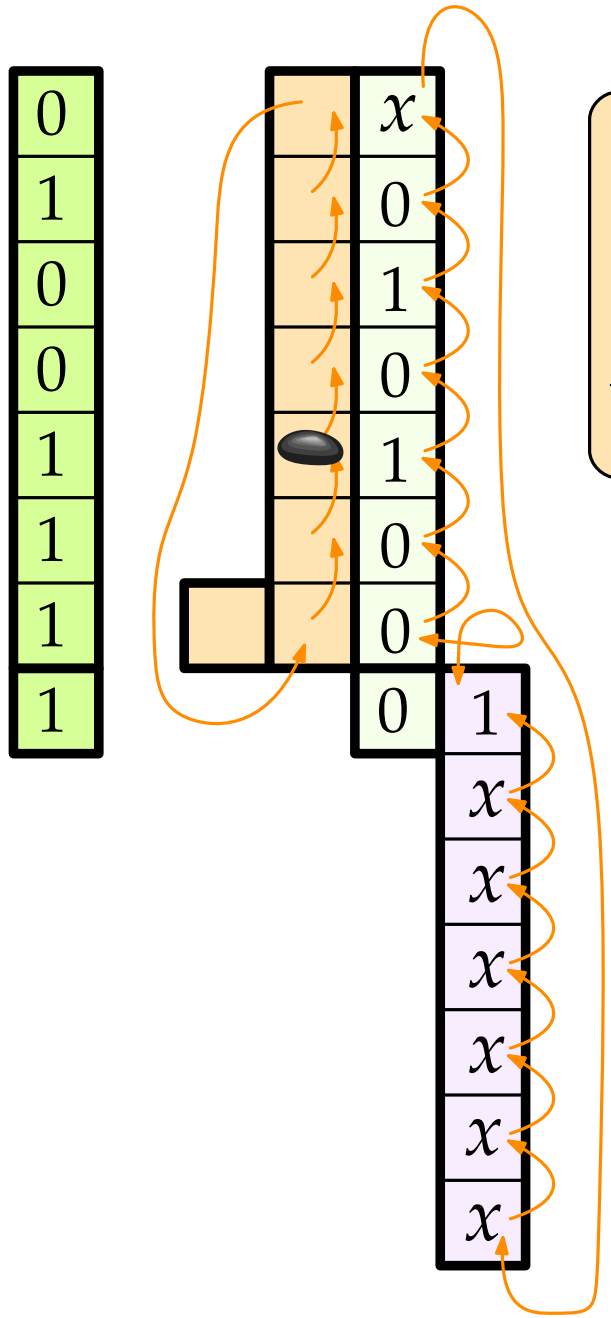

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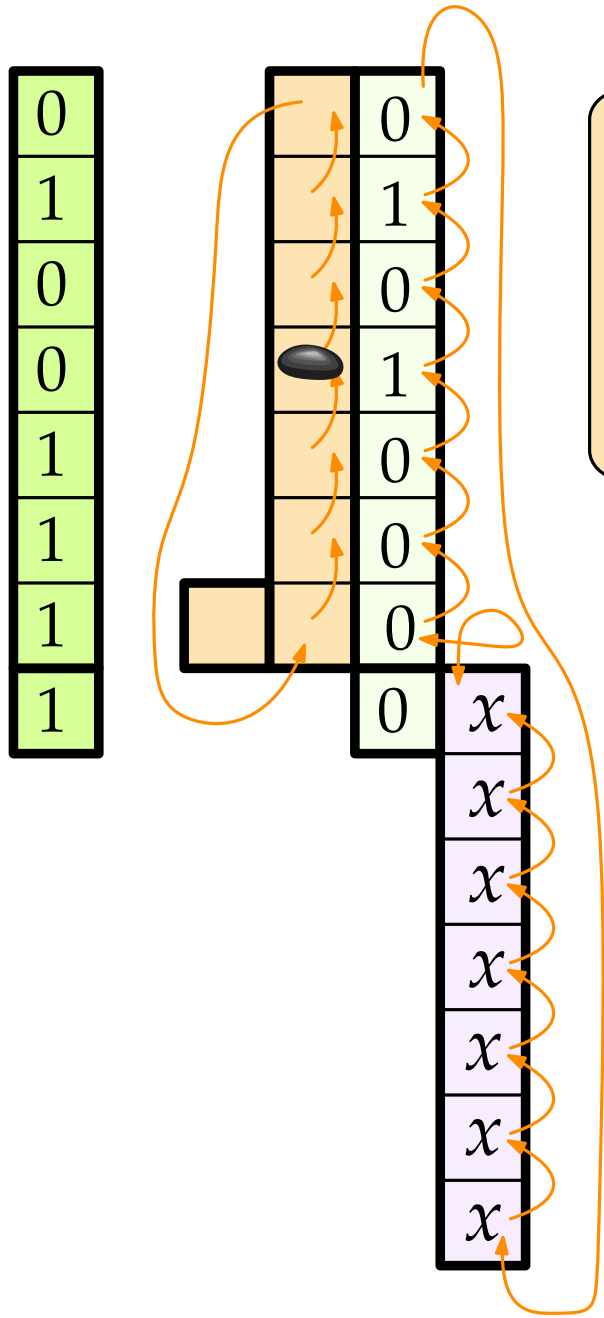

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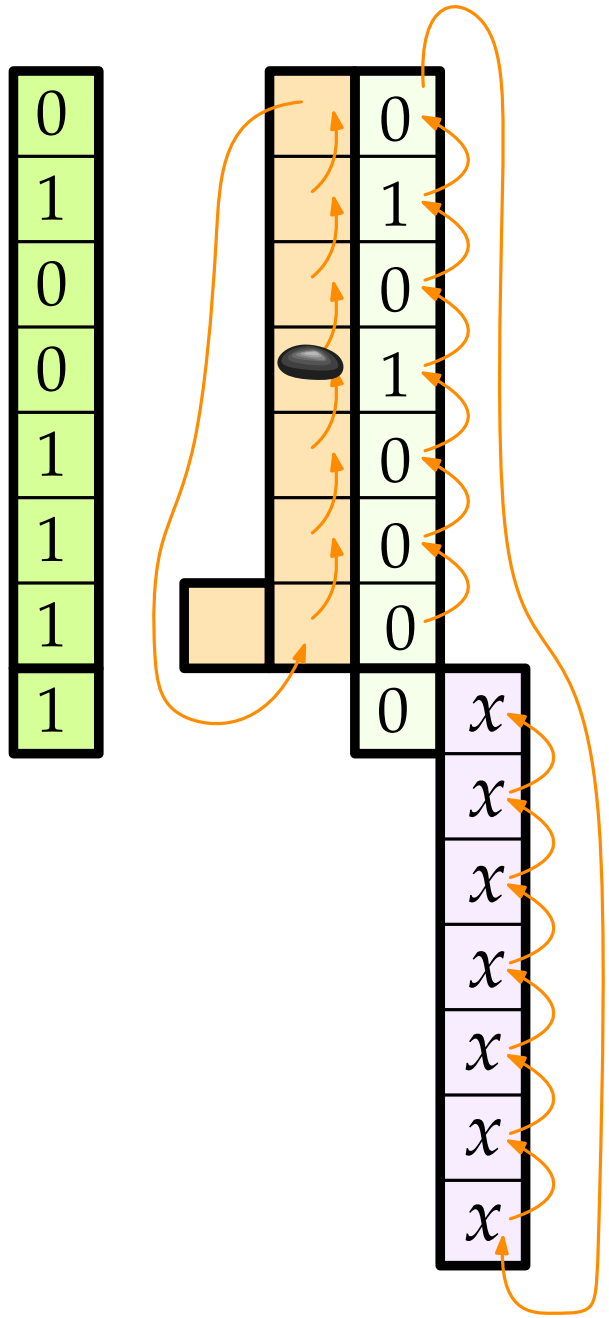

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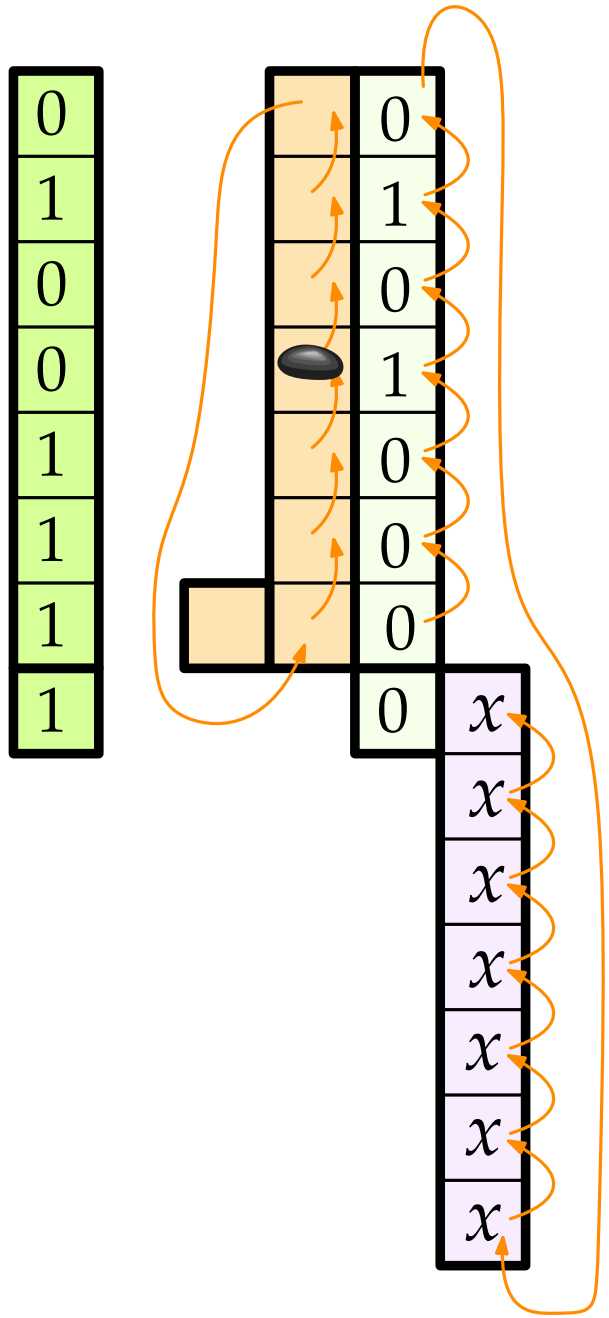
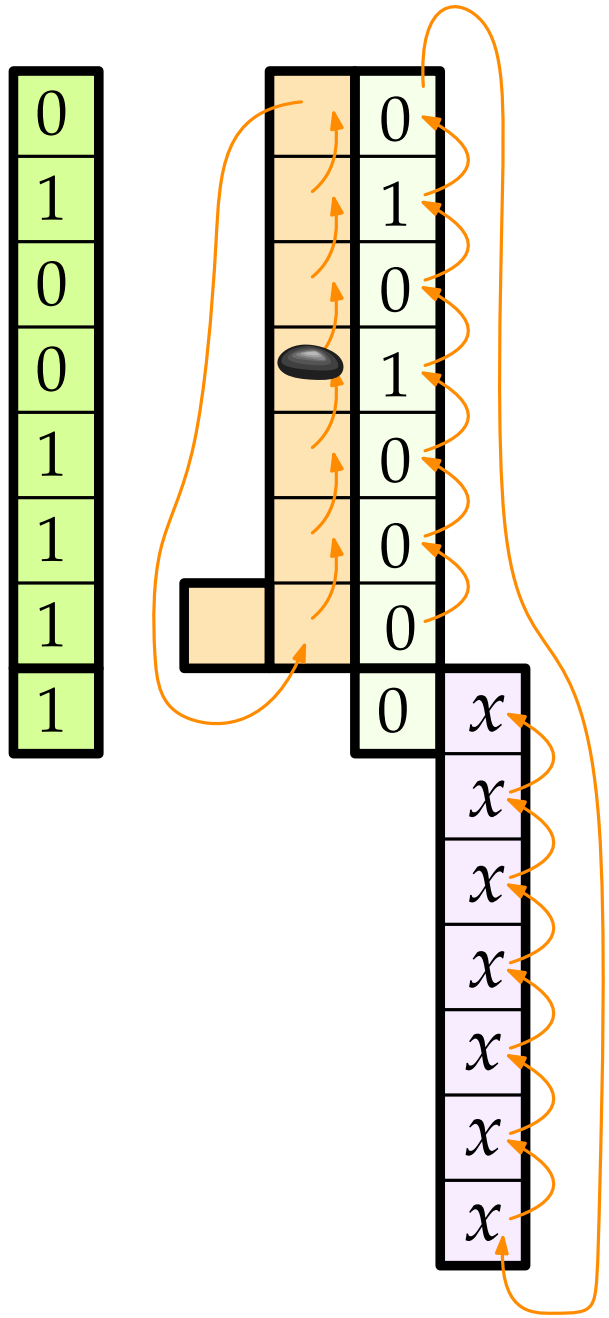


Cannot apply τ again because x in the right register causes payoff $-\infty$.

A counter with constantly many permutations



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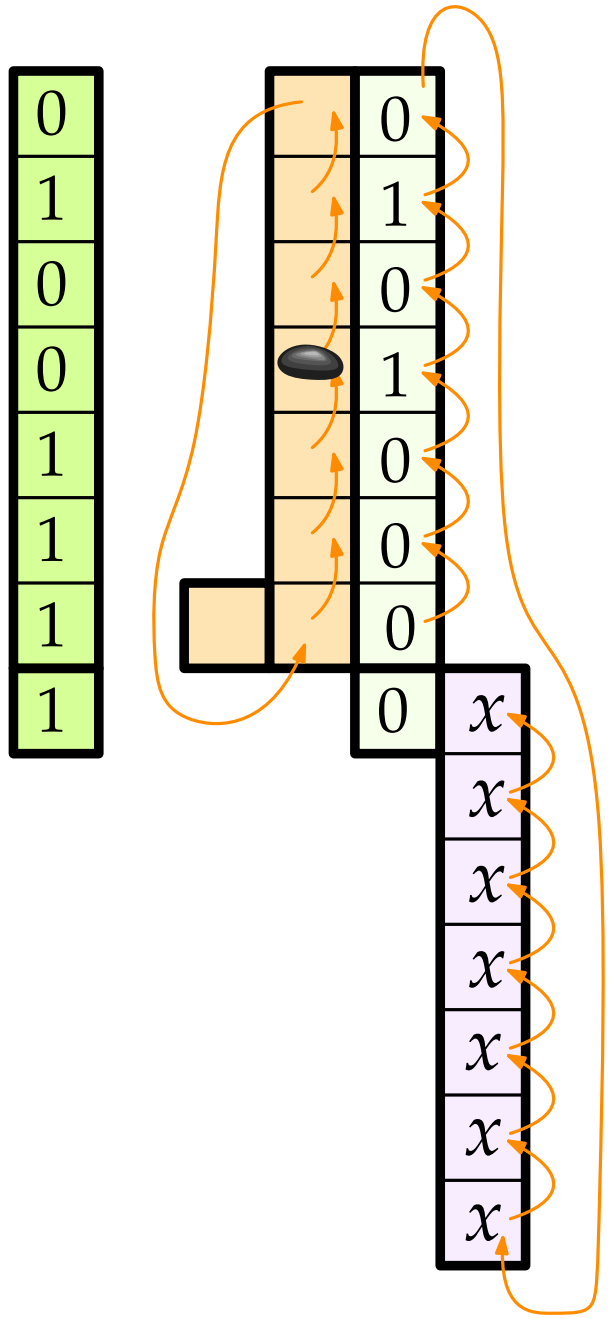

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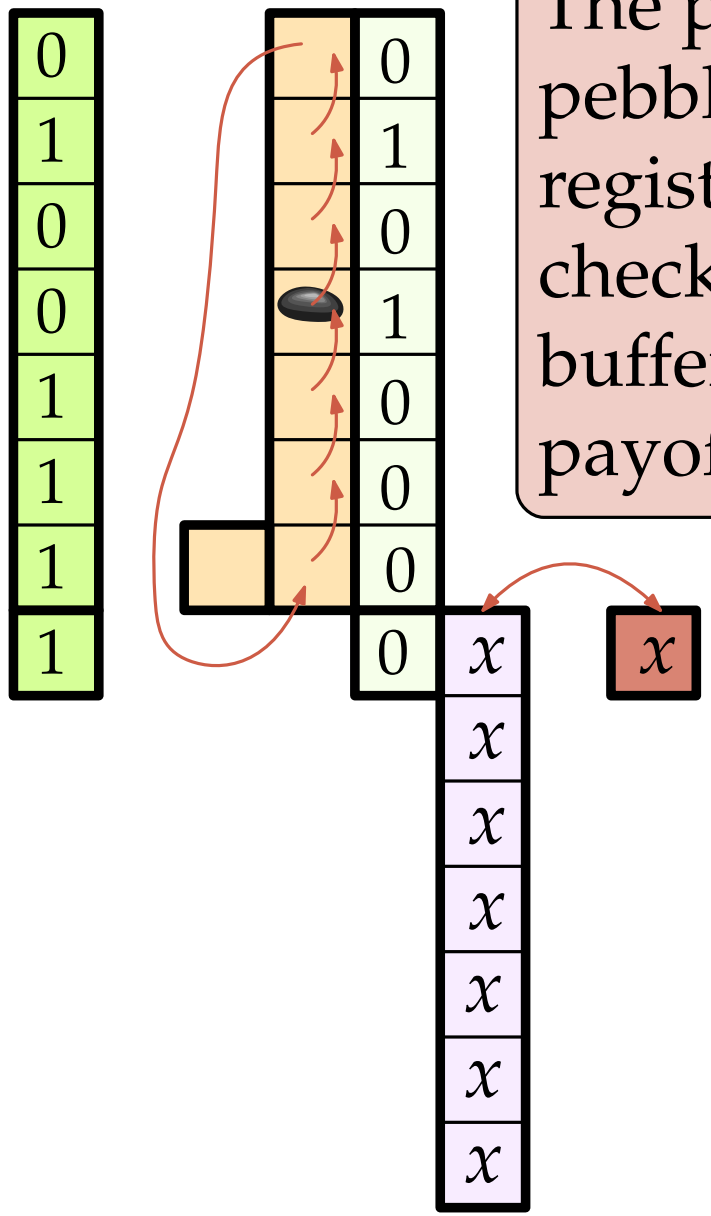
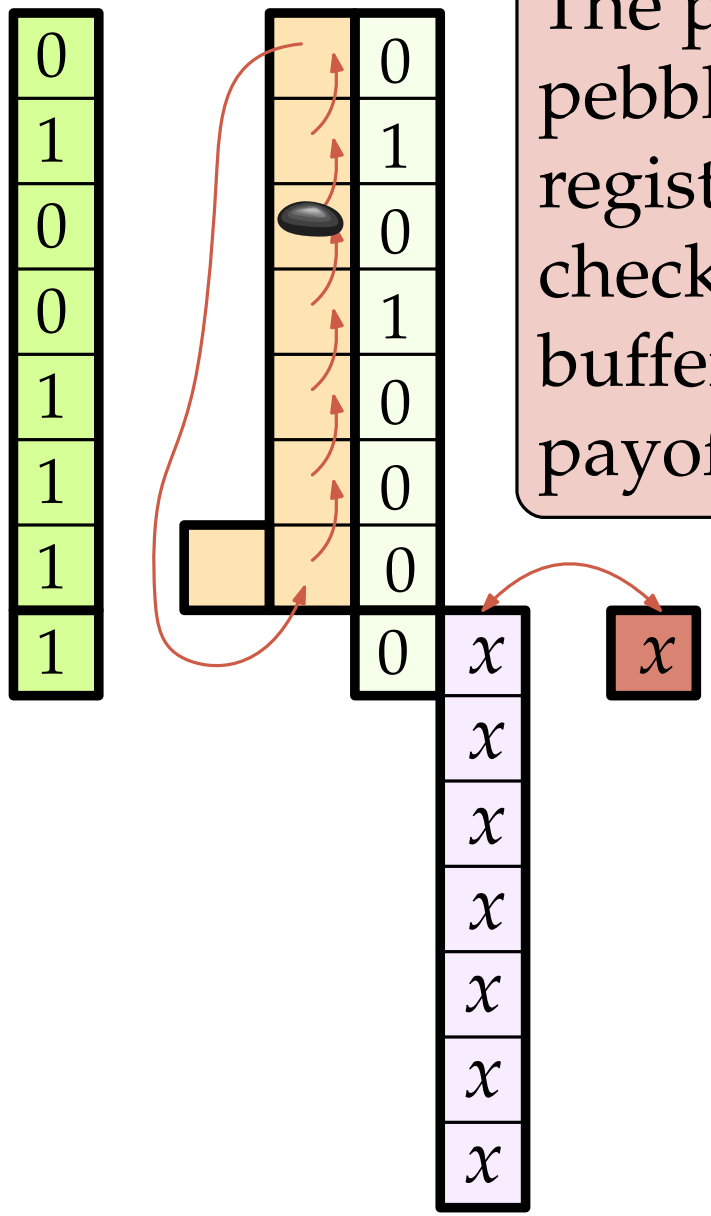
Cannot apply π because π would push the pebble down the elevator (we didn't show the elevator when introducing π).

A counter with constantly many permutations



The permutation π' pushes the pebble up the elevator but leaves register and buffer unchanged. It checks whether the top cell of the buffer really is an x (if not then payoff $-\infty$).

A counter with constantly many permutations



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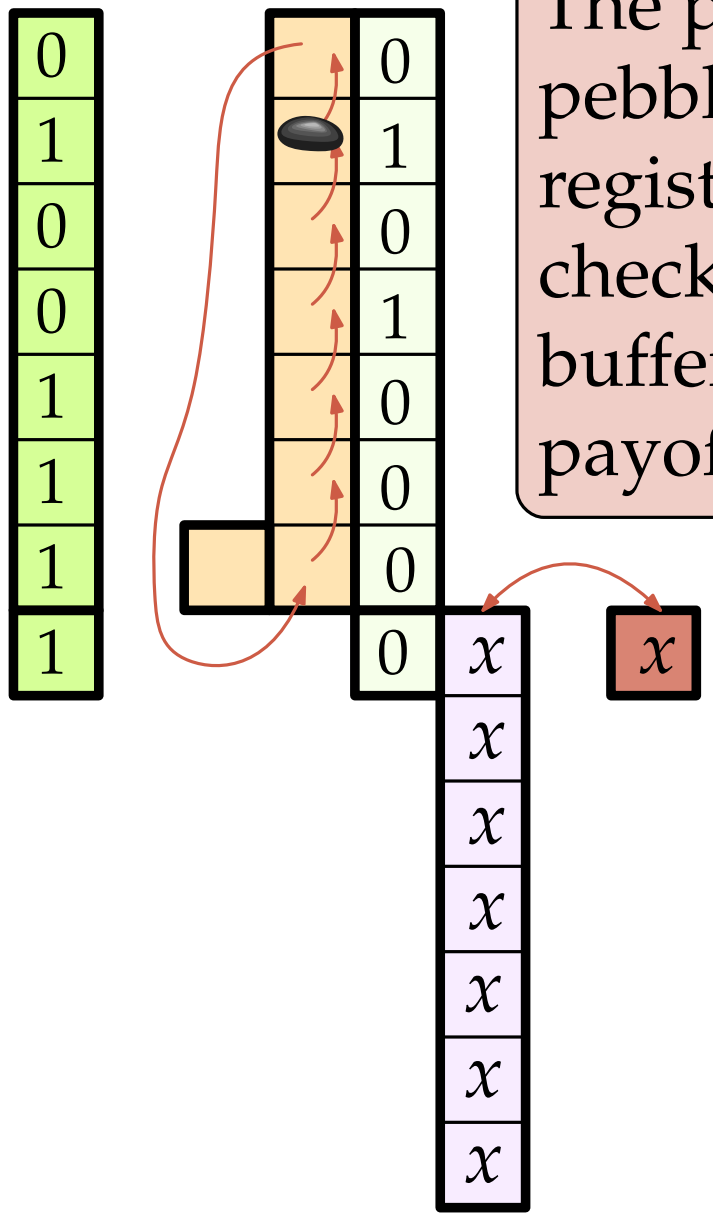

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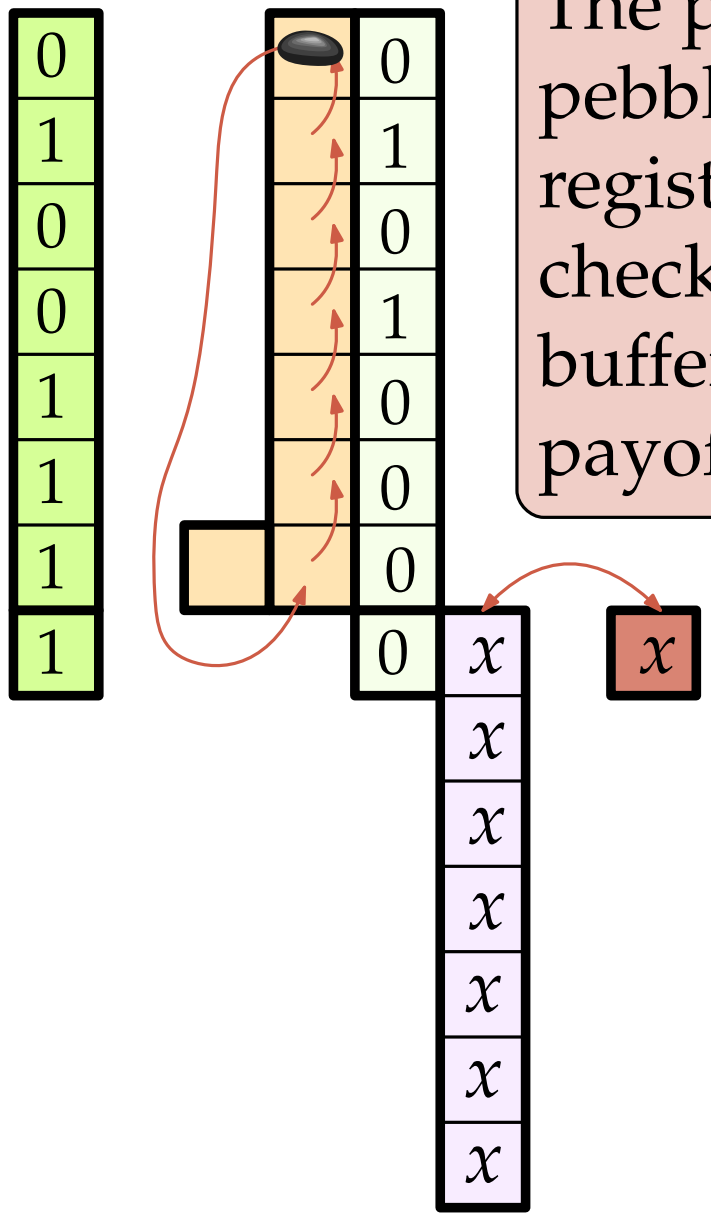

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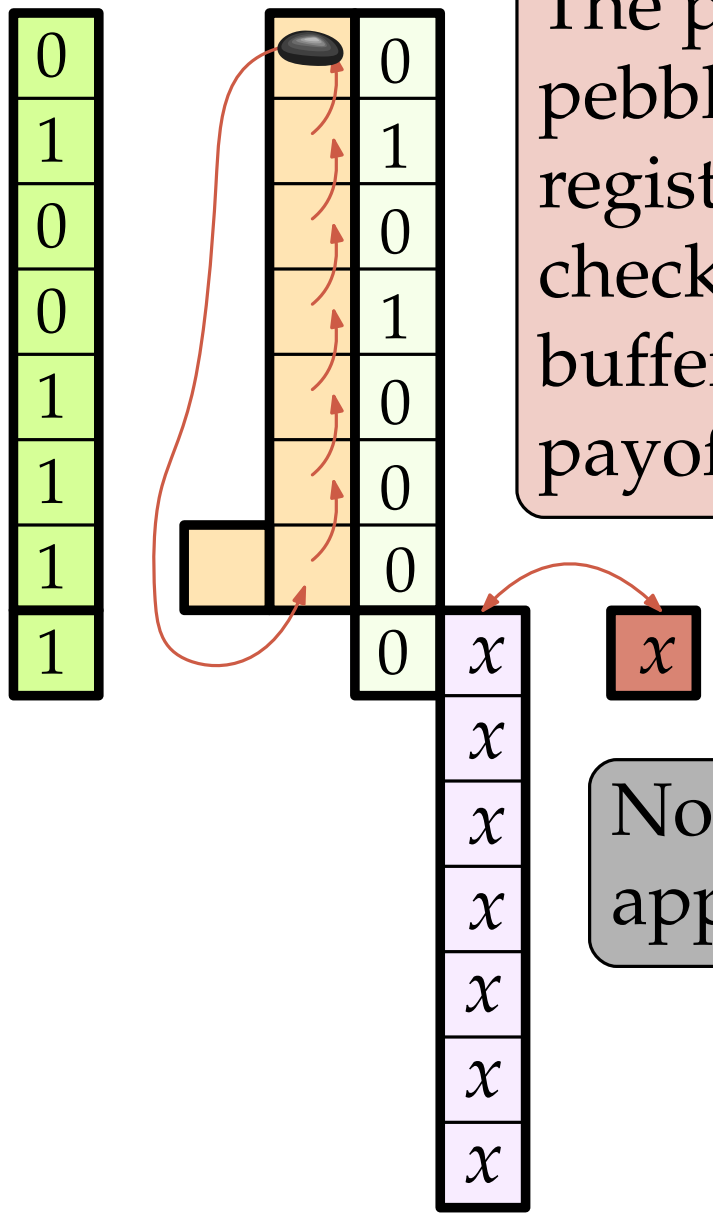
A counter with constantly many permutations



The permutation π' pushes the pebble up the elevator but leaves register and buffer unchanged. It checks whether the top cell of the buffer really is an x (if not then payoff $-\infty$).

No permutation can be applied anymore.

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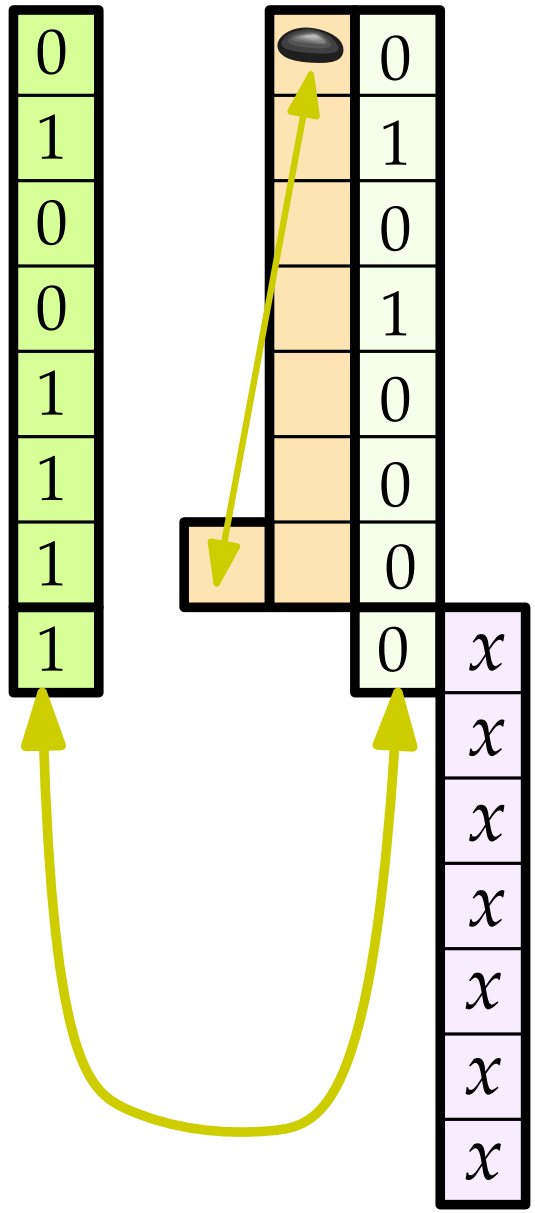

Now comes the final permutation: α. It exchanges left and right register and places the pebble back into the lobby. This increases the value if the right register encodes a higher value than the left register; note that the elevator is worth less than the left register.

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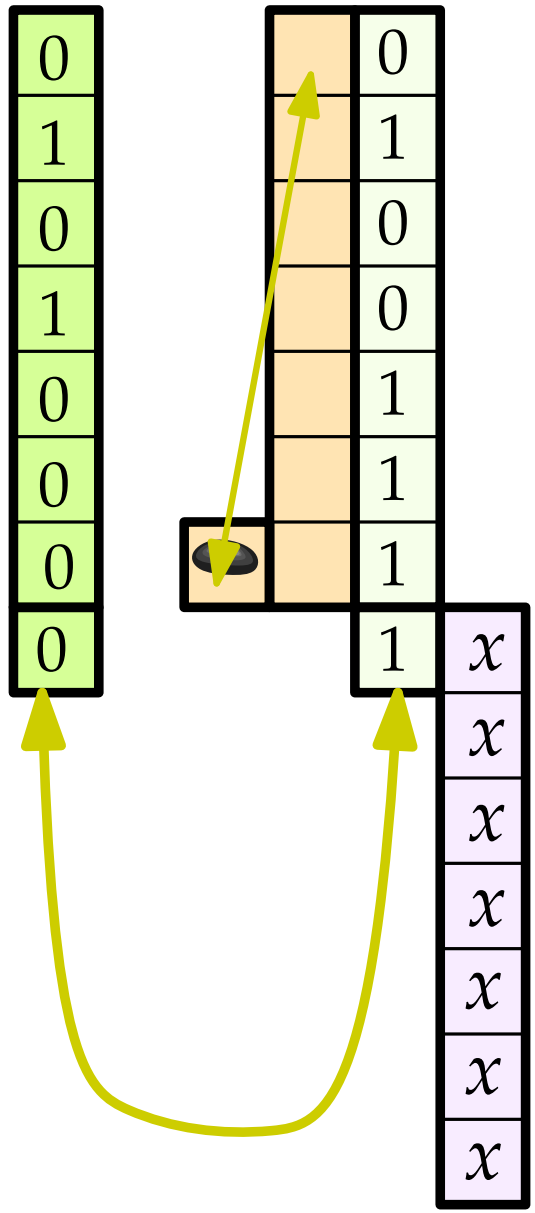

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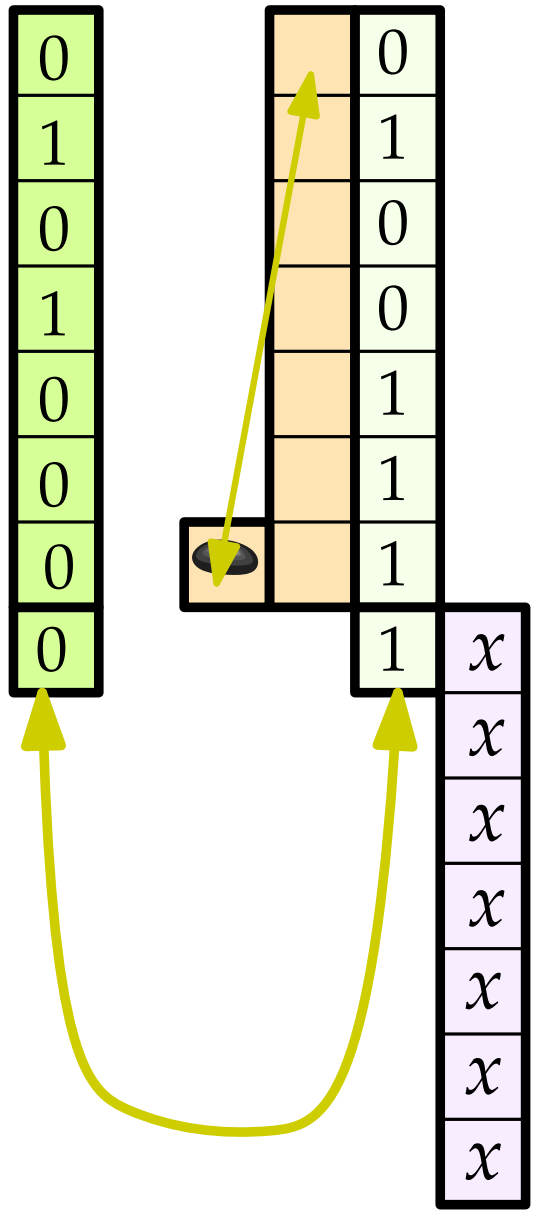

Now repeat

1. π: pushing down
2. σ: inverting the buffer and lowest register cell
3. τ, τ': pushing up

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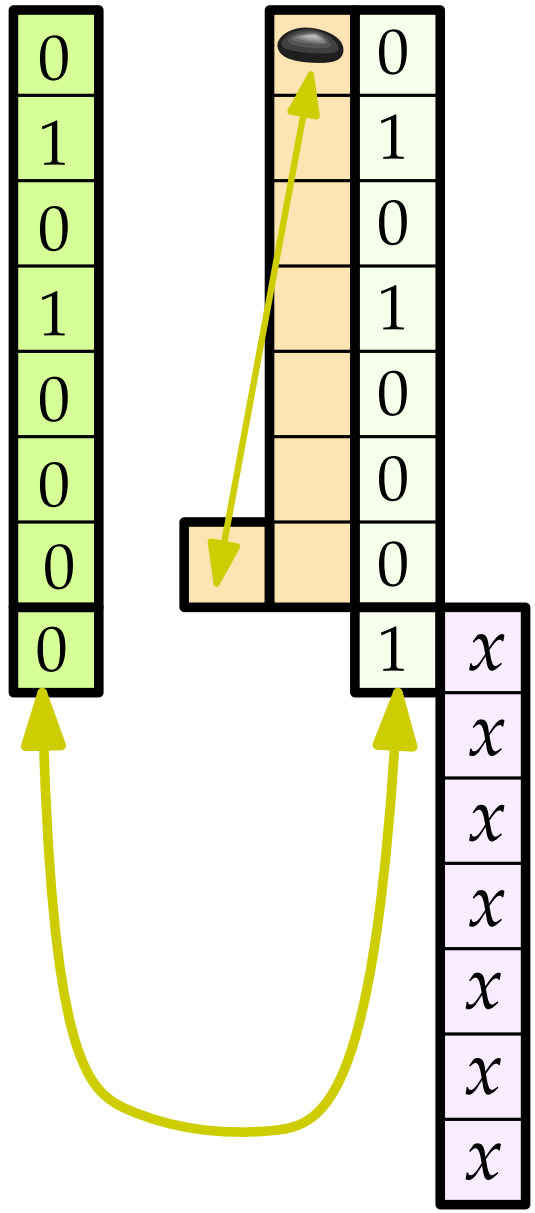

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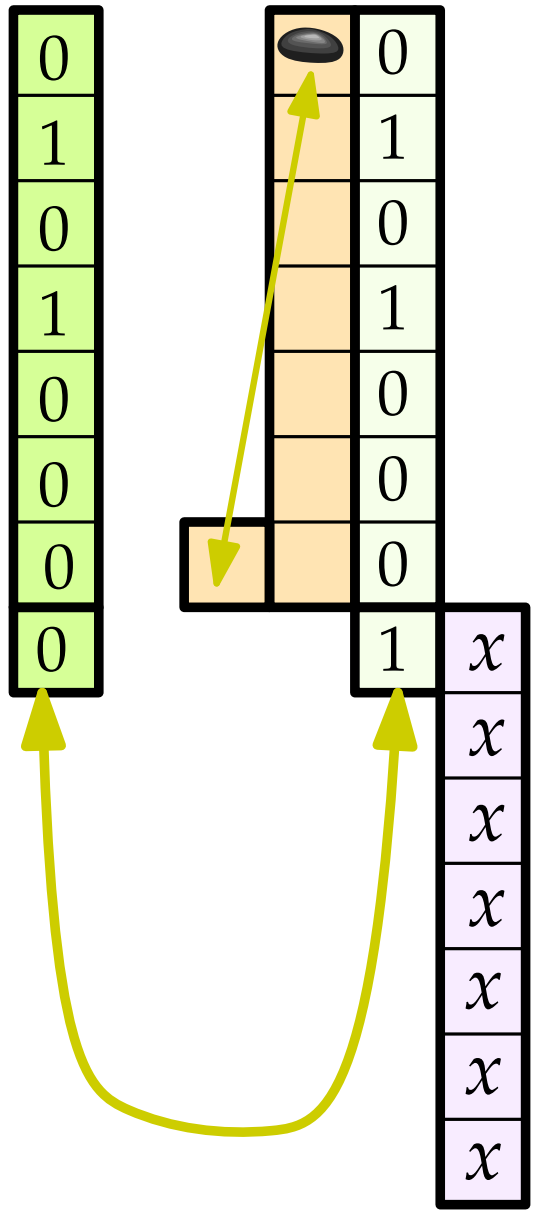

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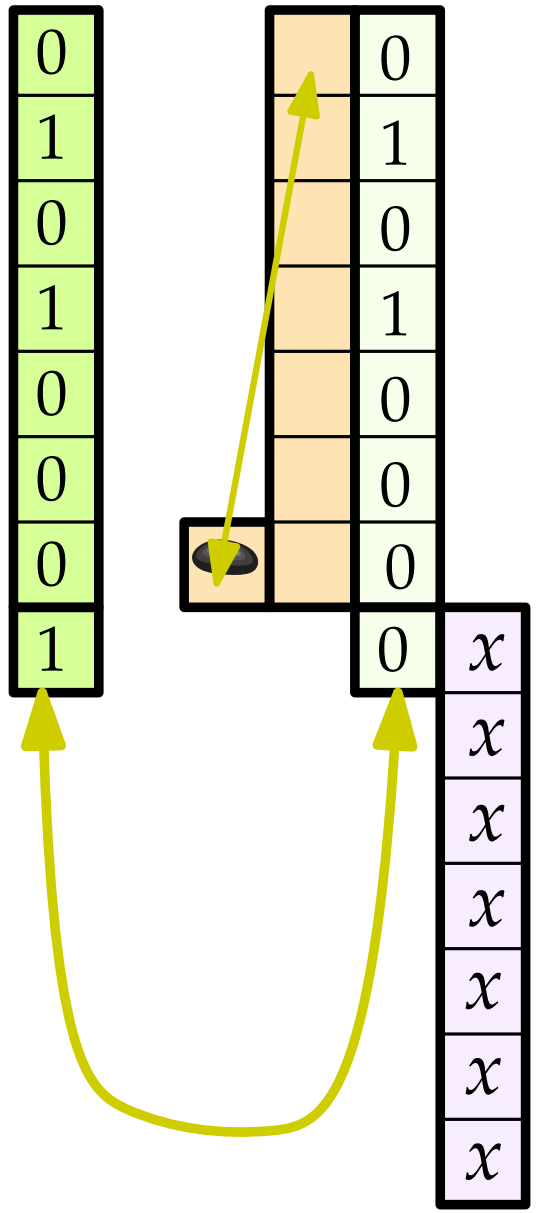

Now repeat

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This is still an improving switch, but only because of the bit after the dot

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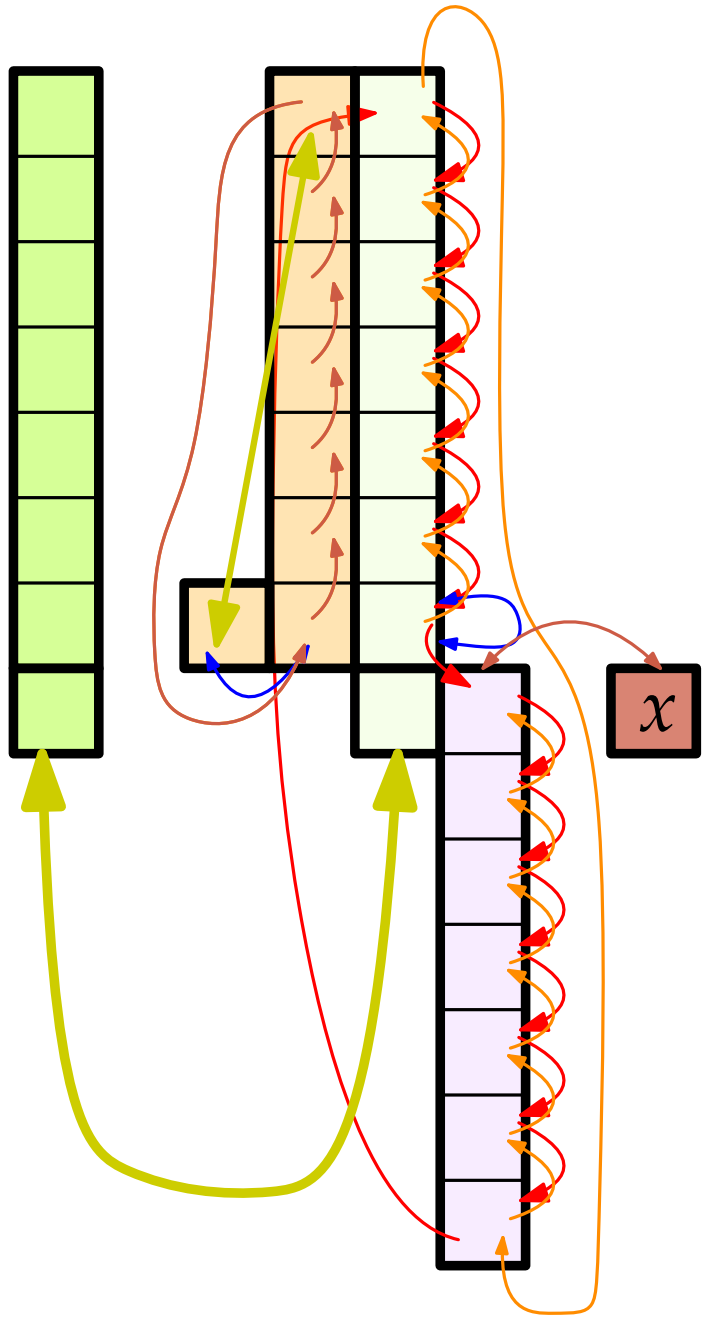

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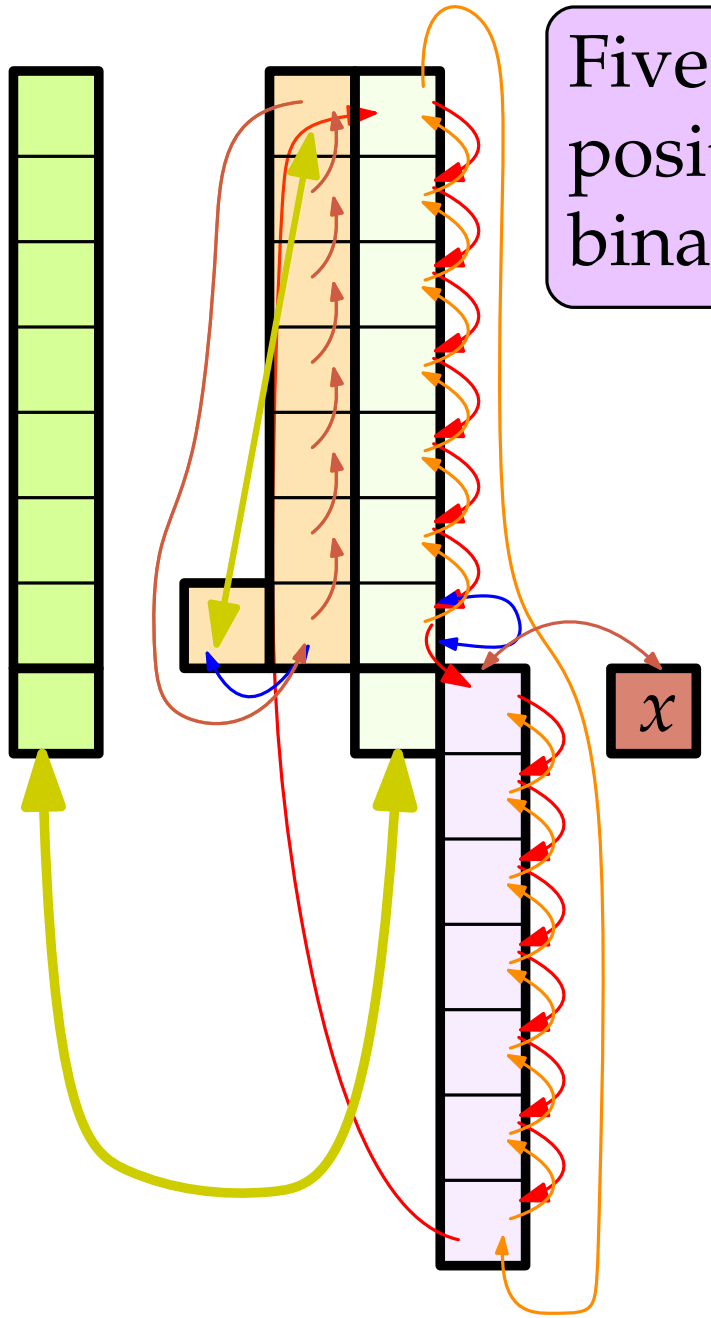

Five permutations and $O(n)$ positions suffice to build an n-bit binary counter.

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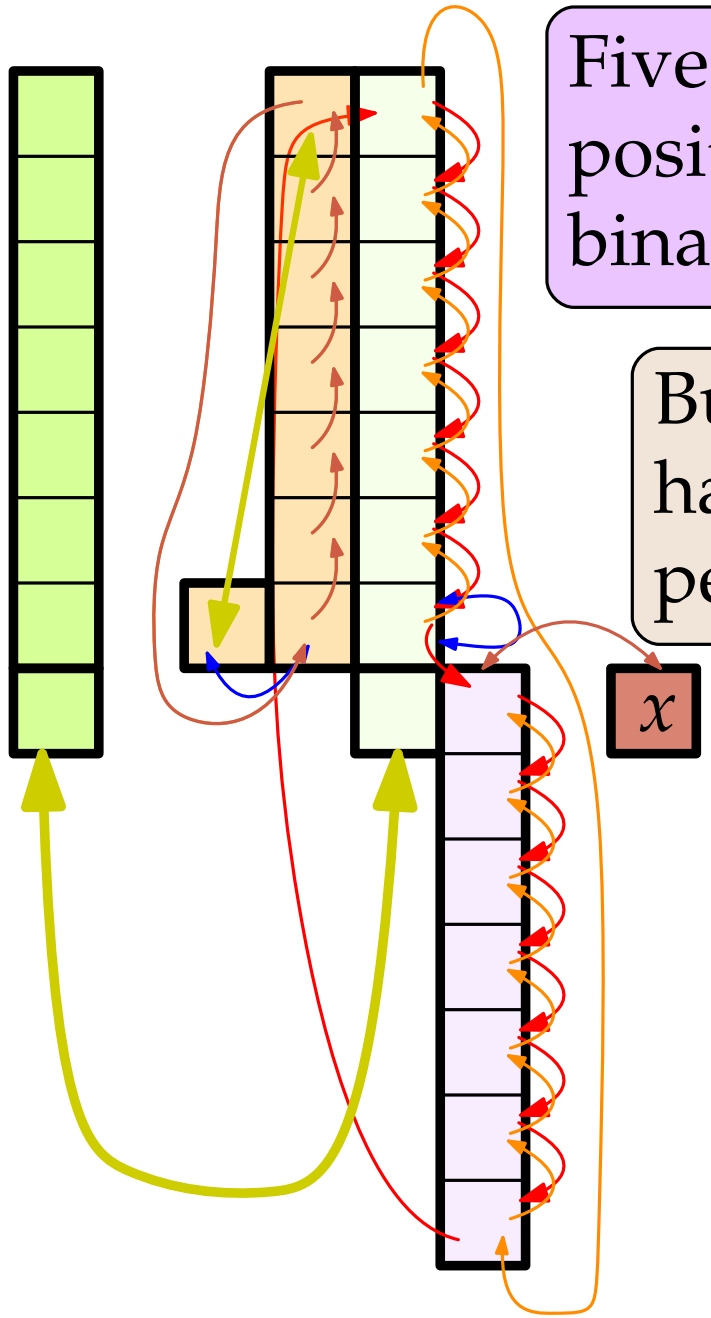

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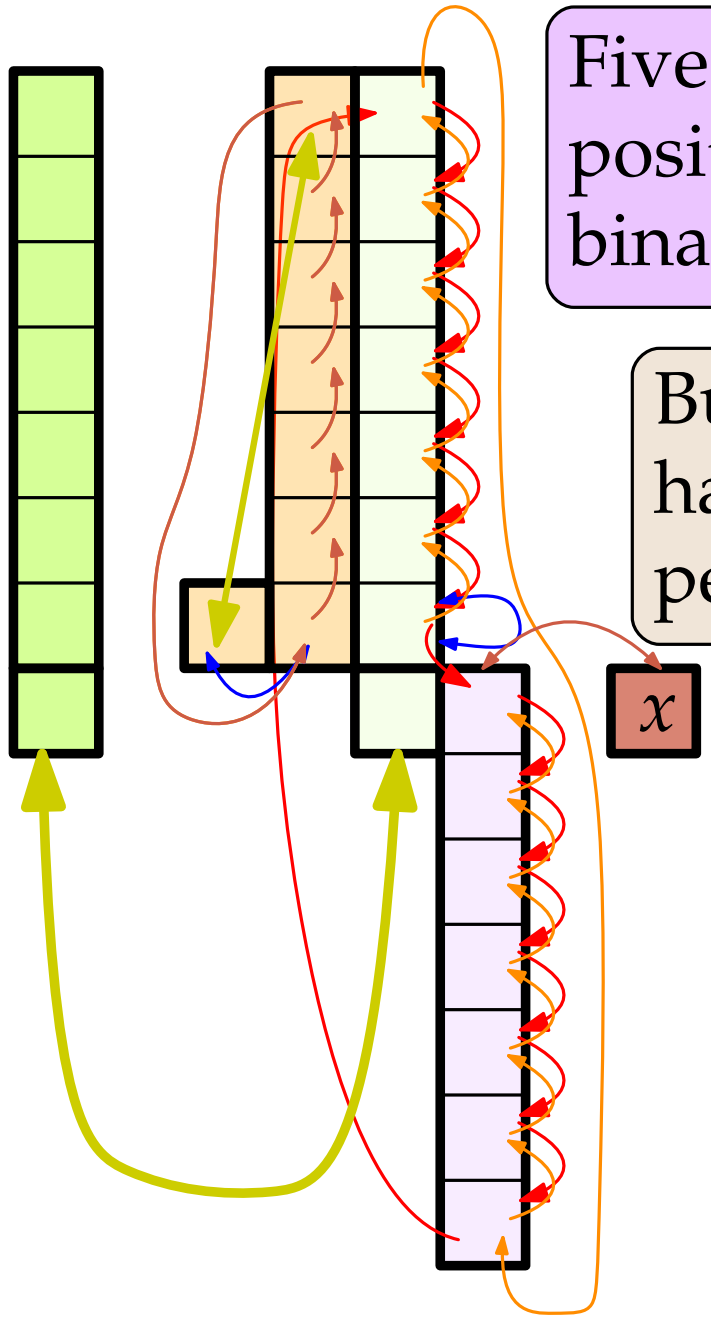

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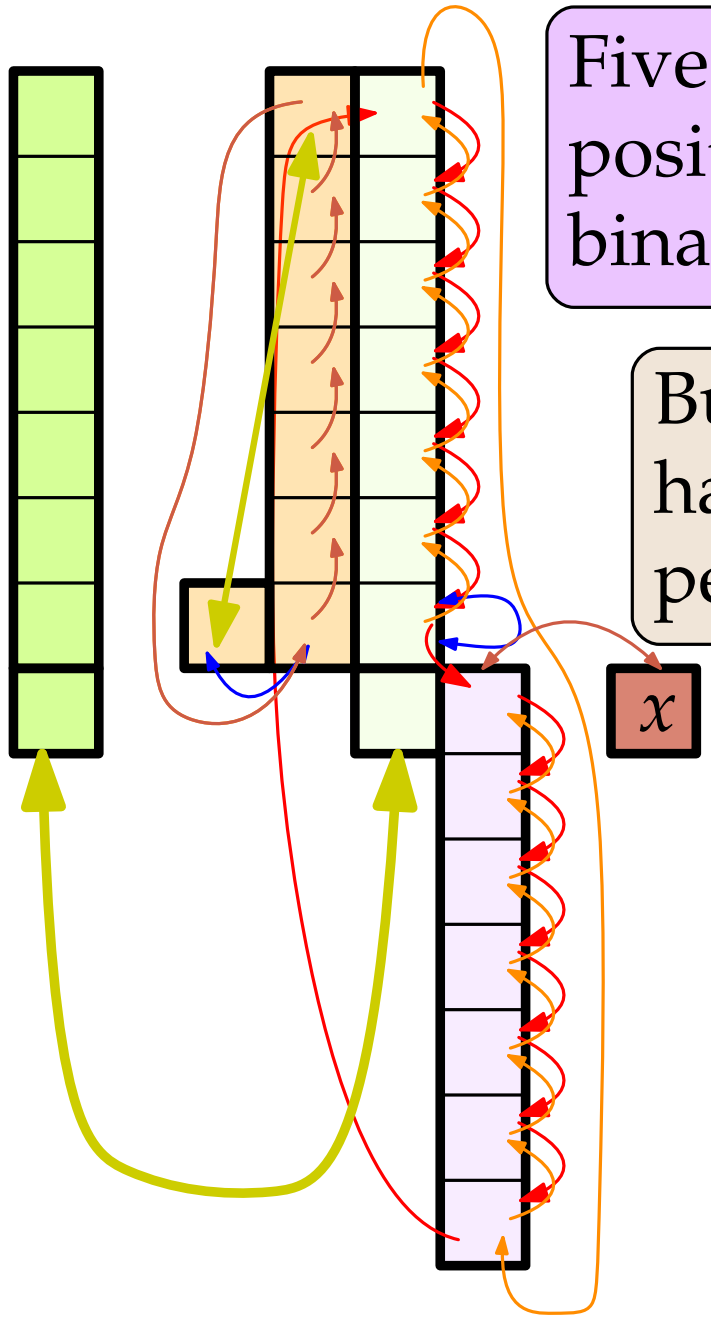

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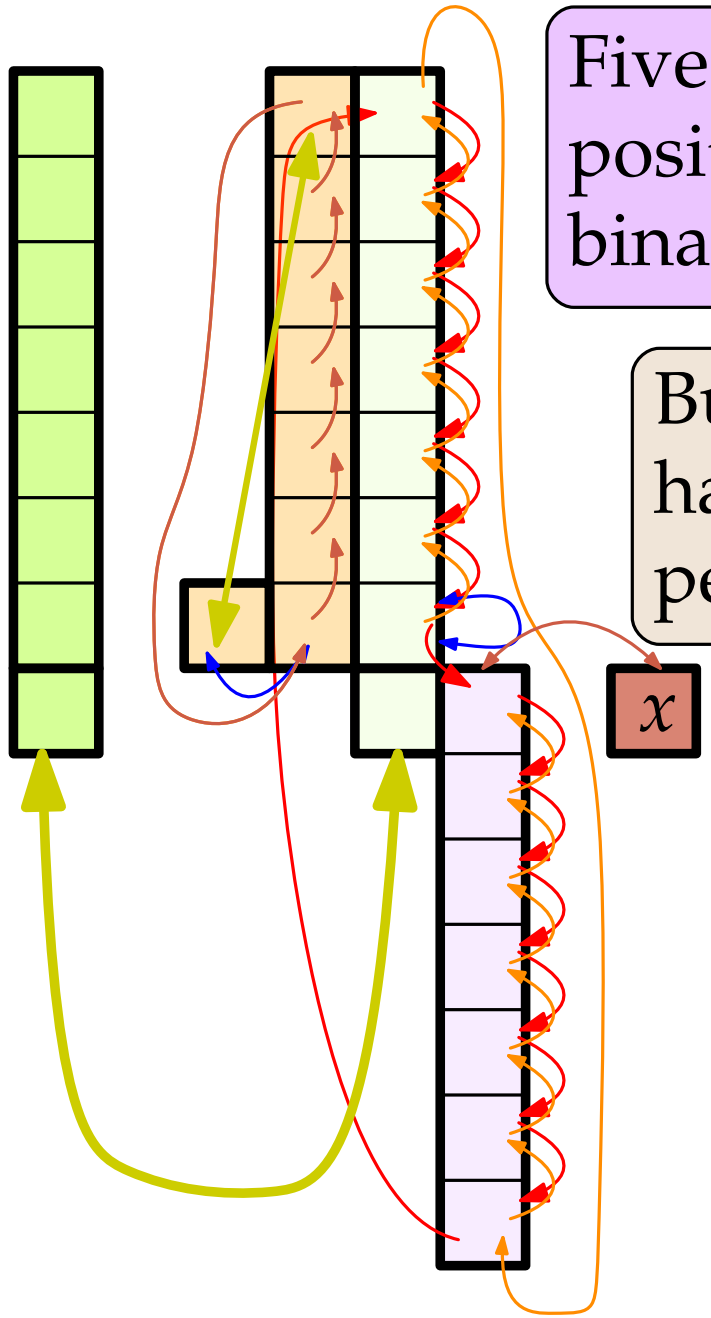

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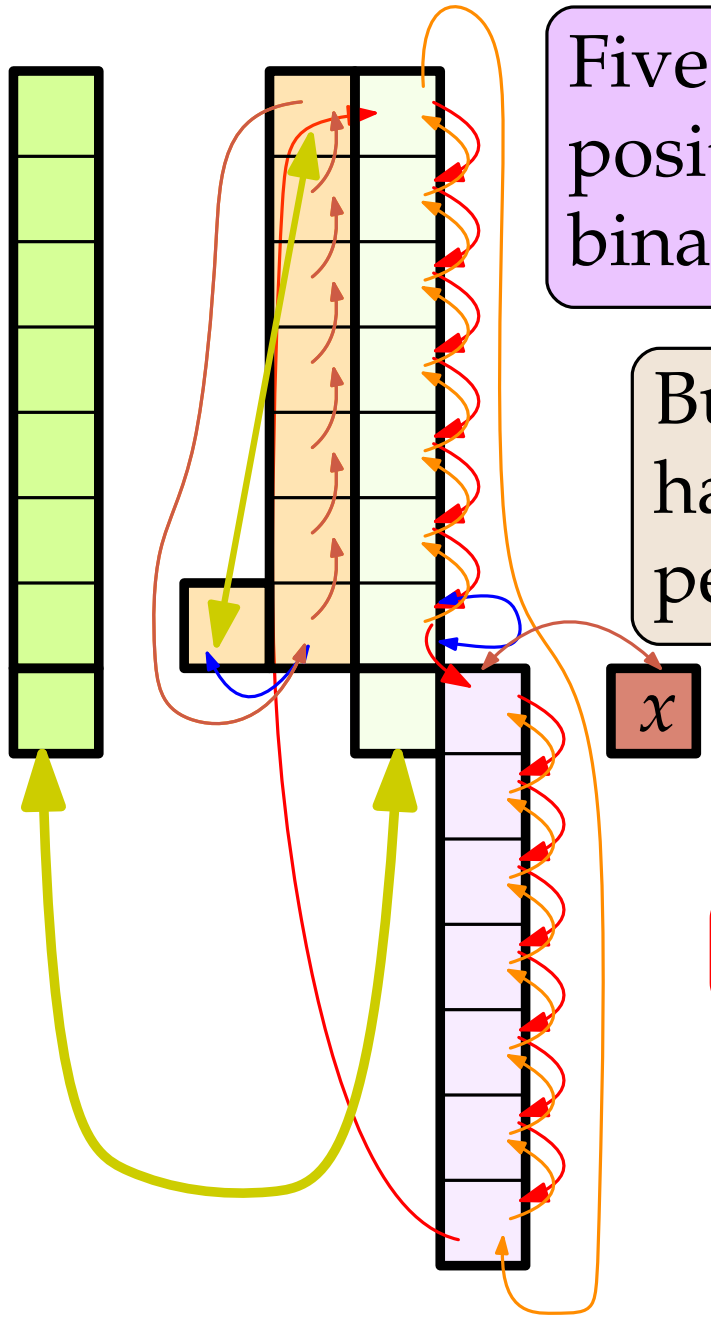

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Quadratic Path for 2 Commuting Permutations

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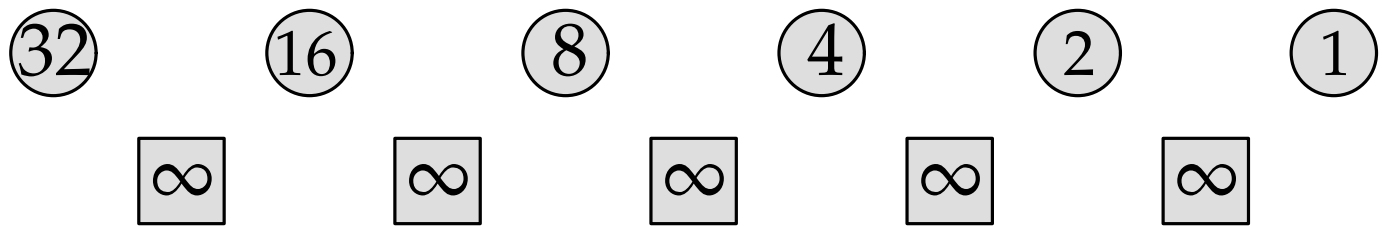

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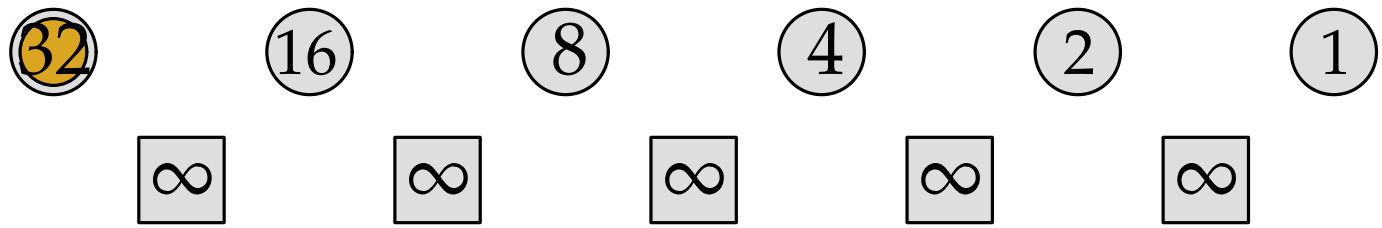

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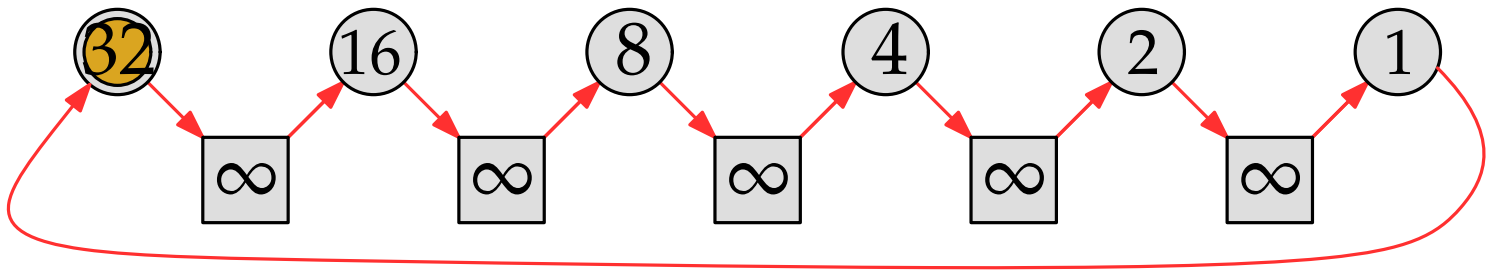

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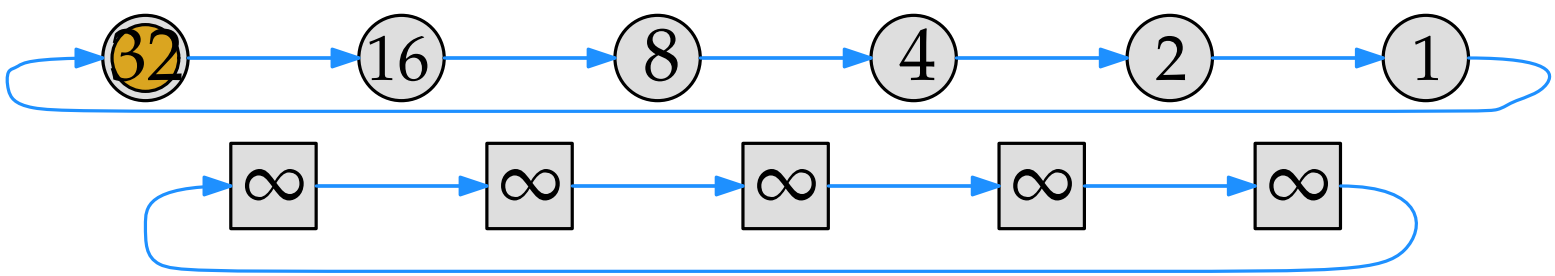

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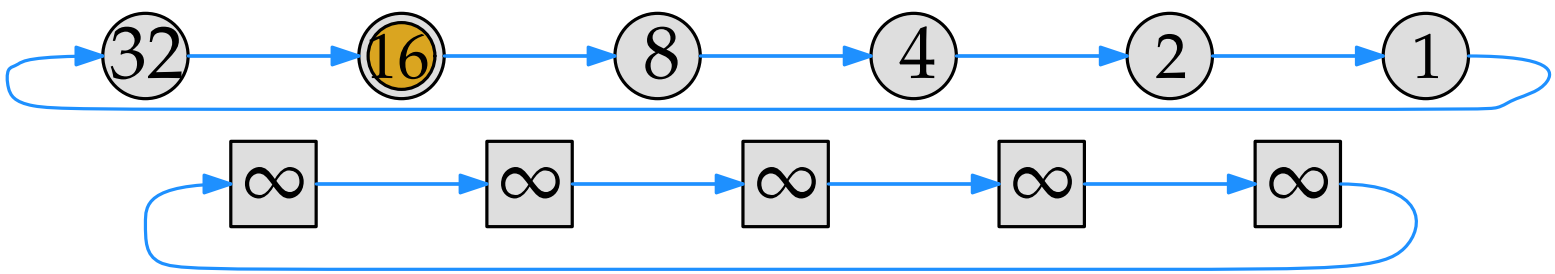

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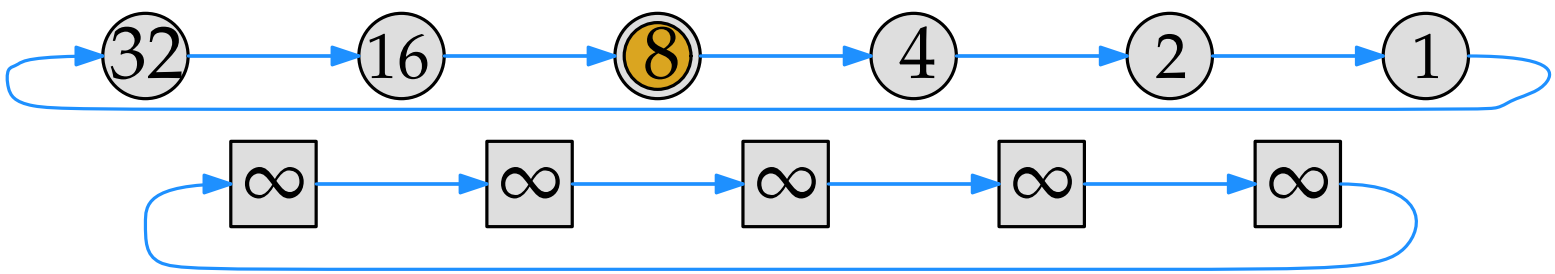

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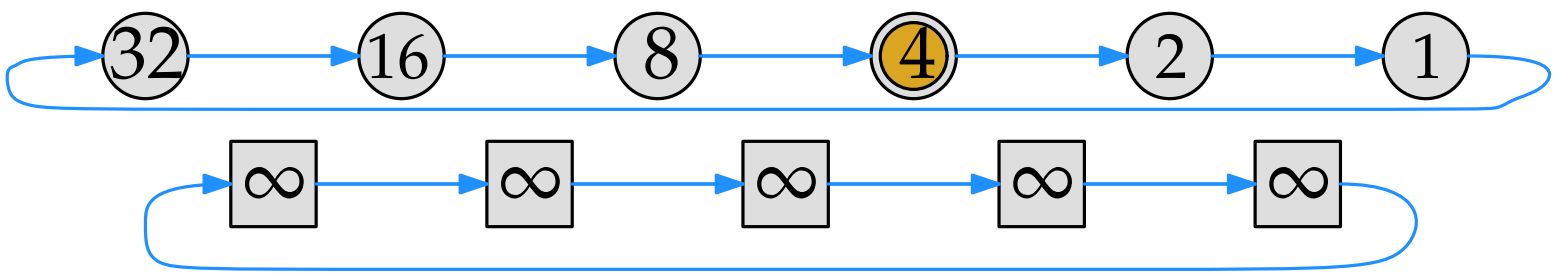

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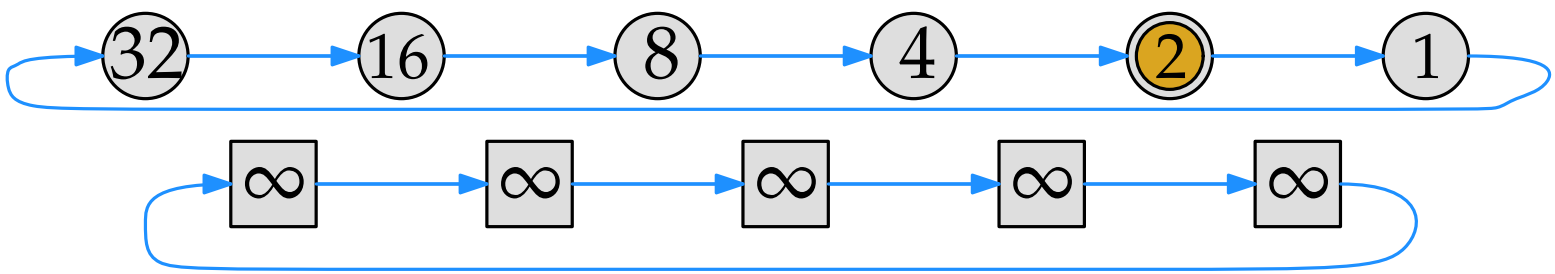

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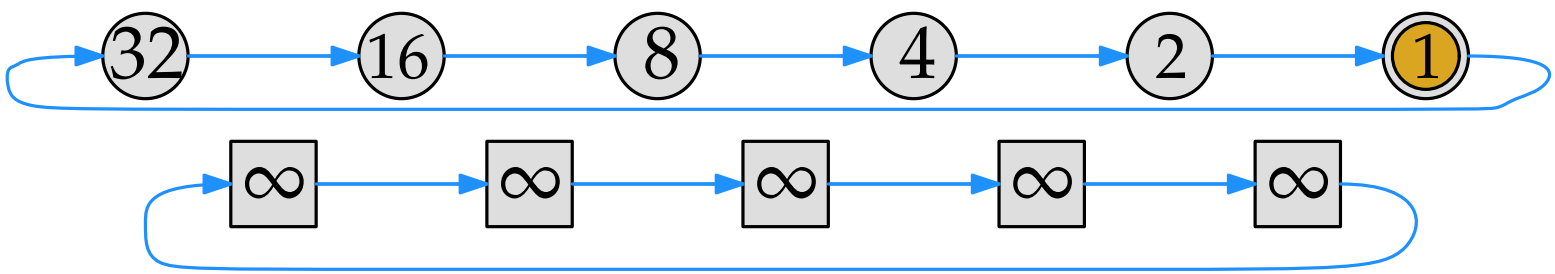

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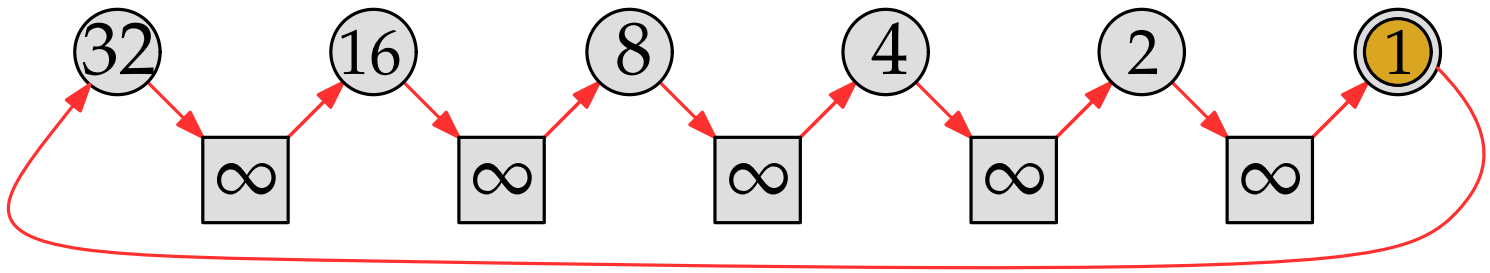

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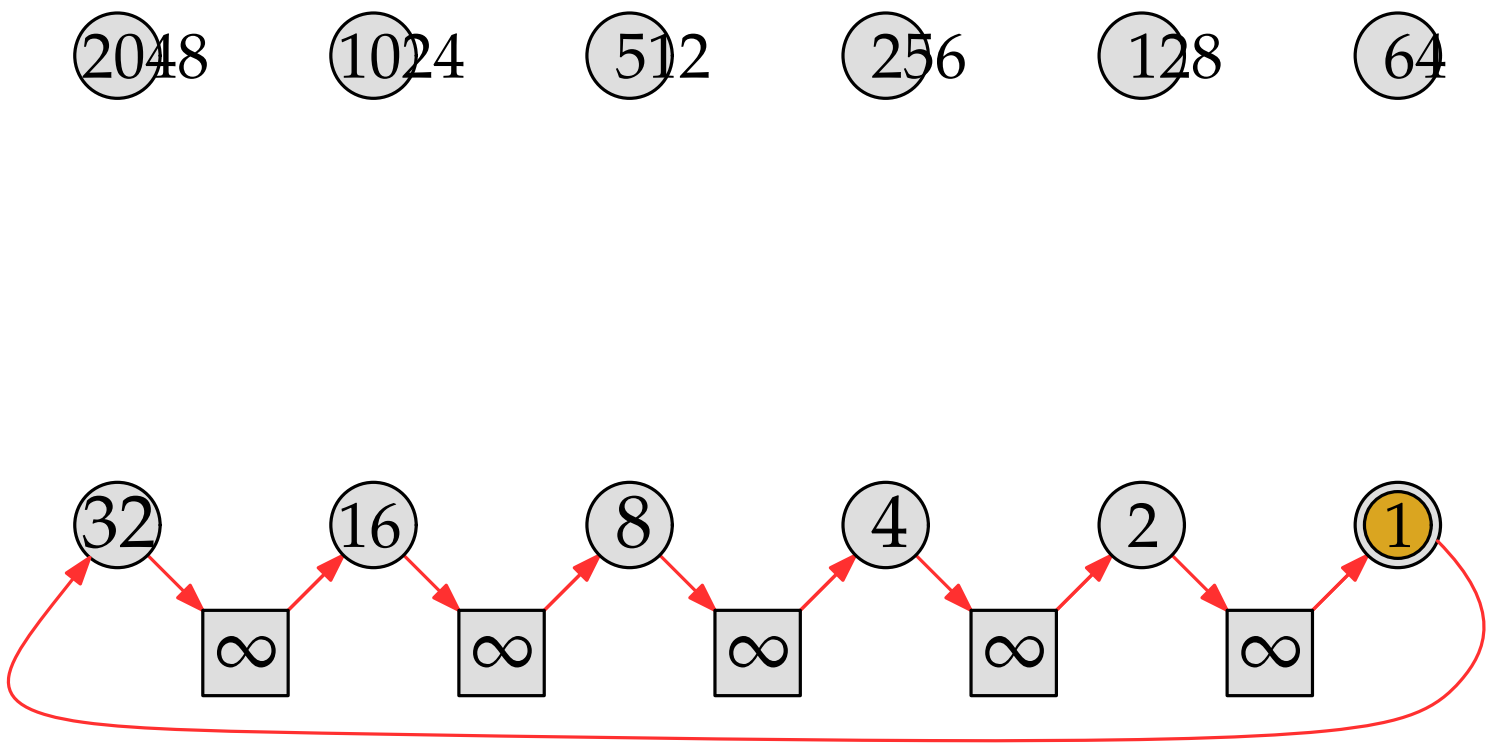

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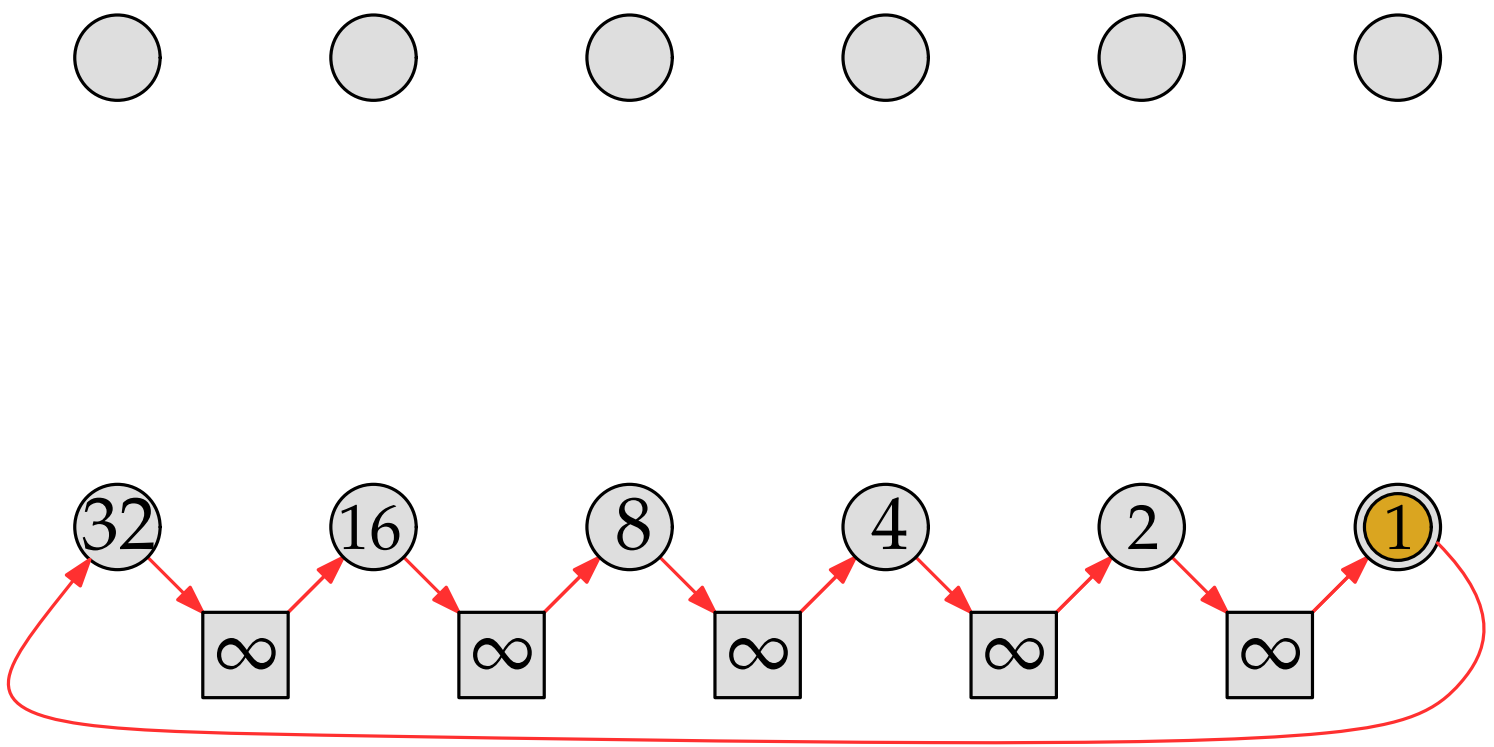

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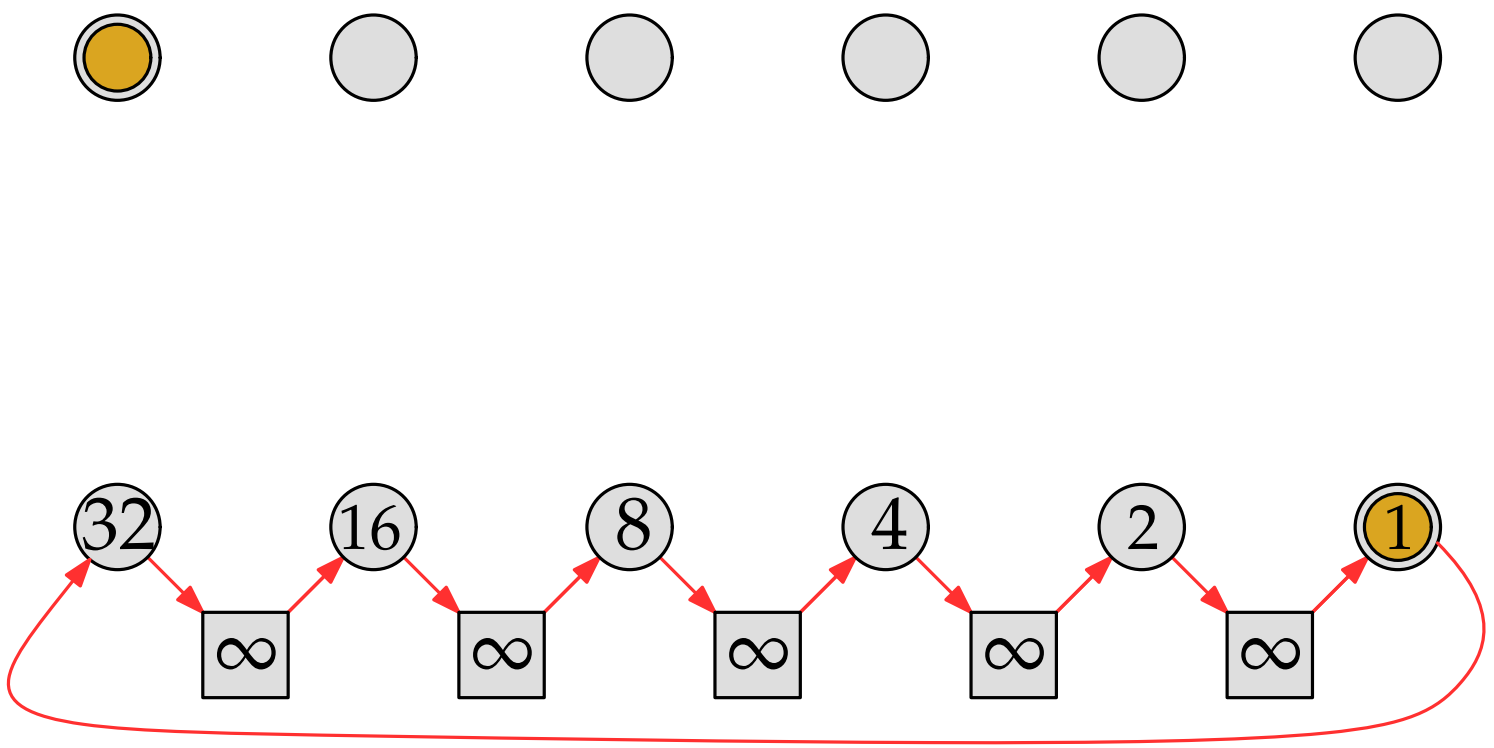

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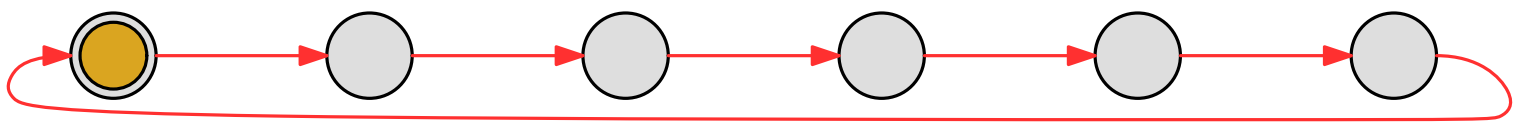

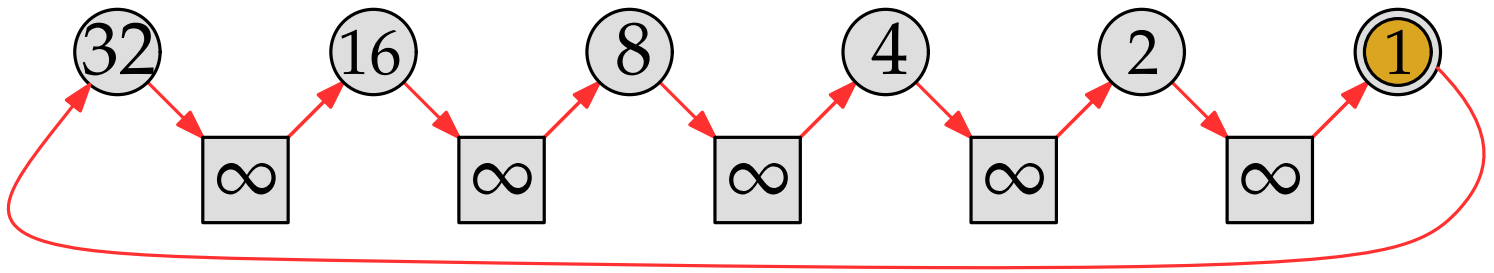

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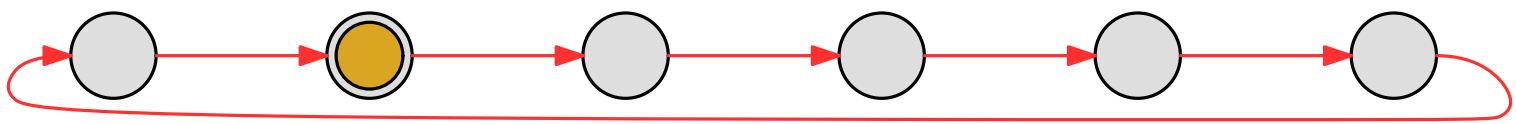

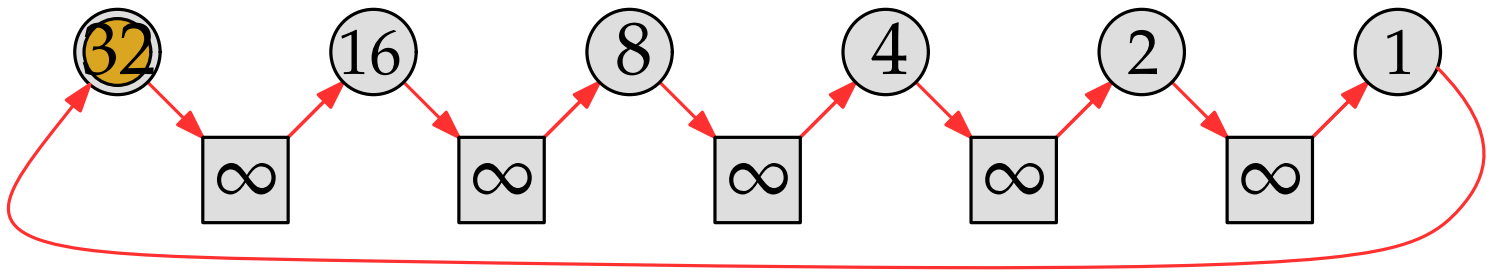

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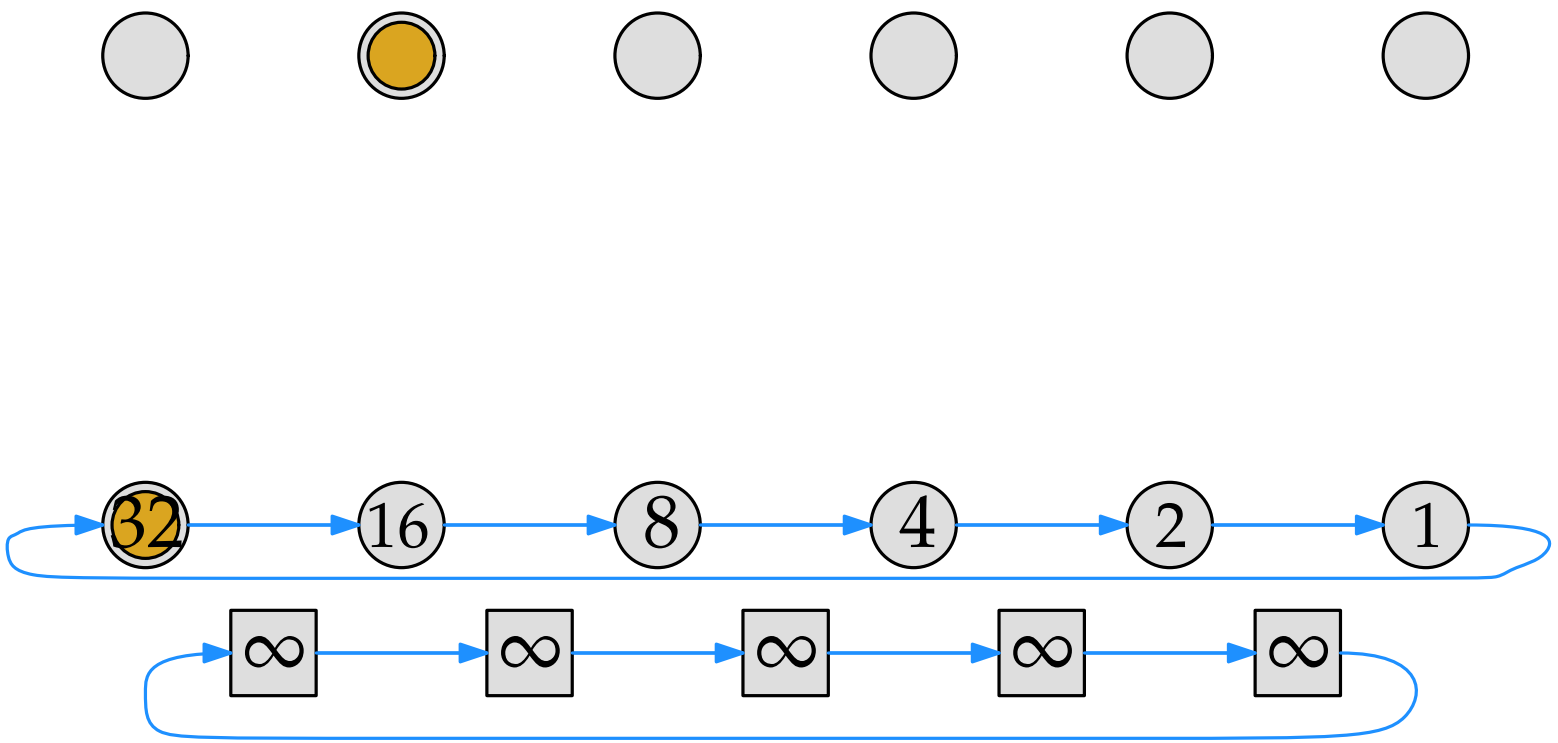

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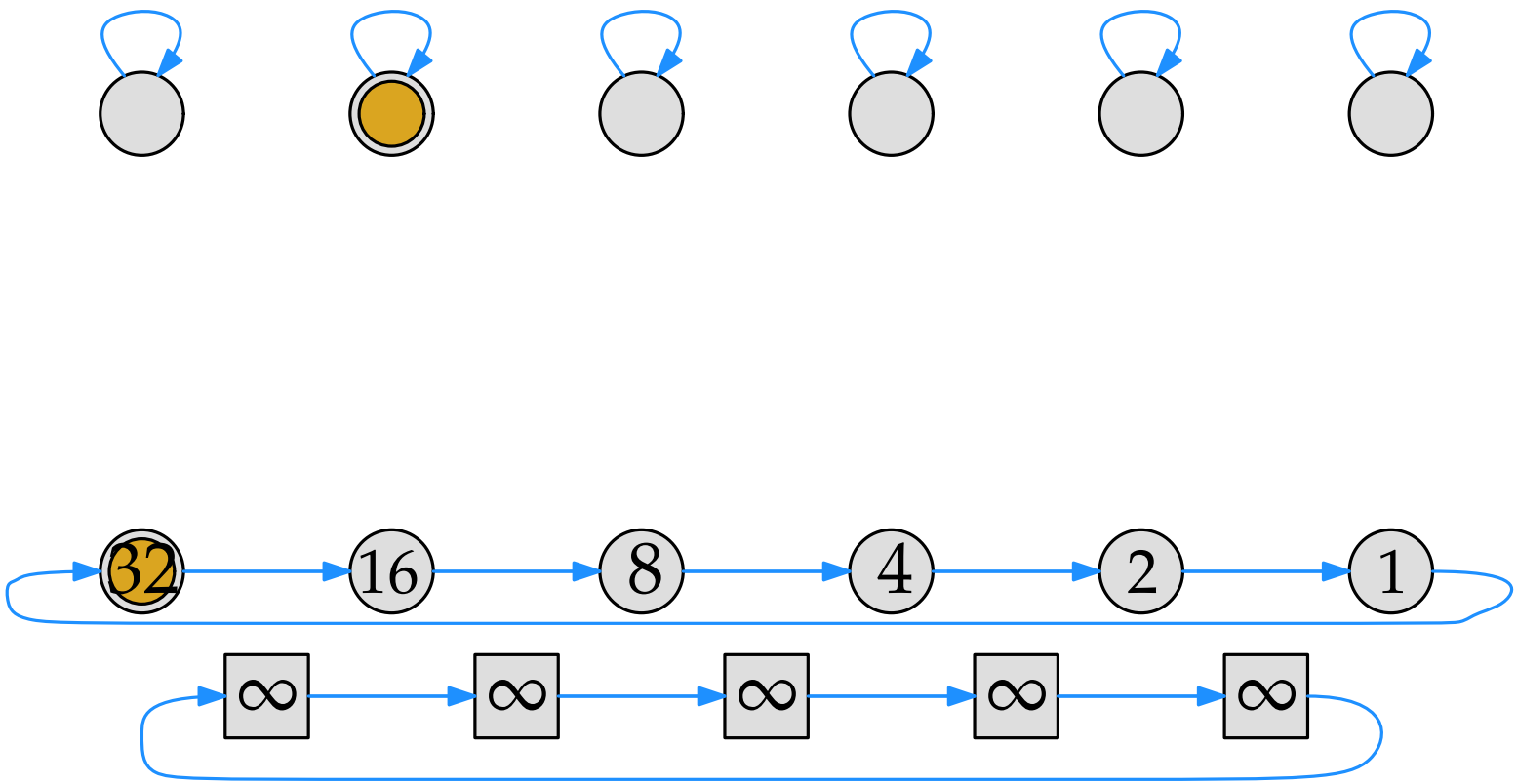
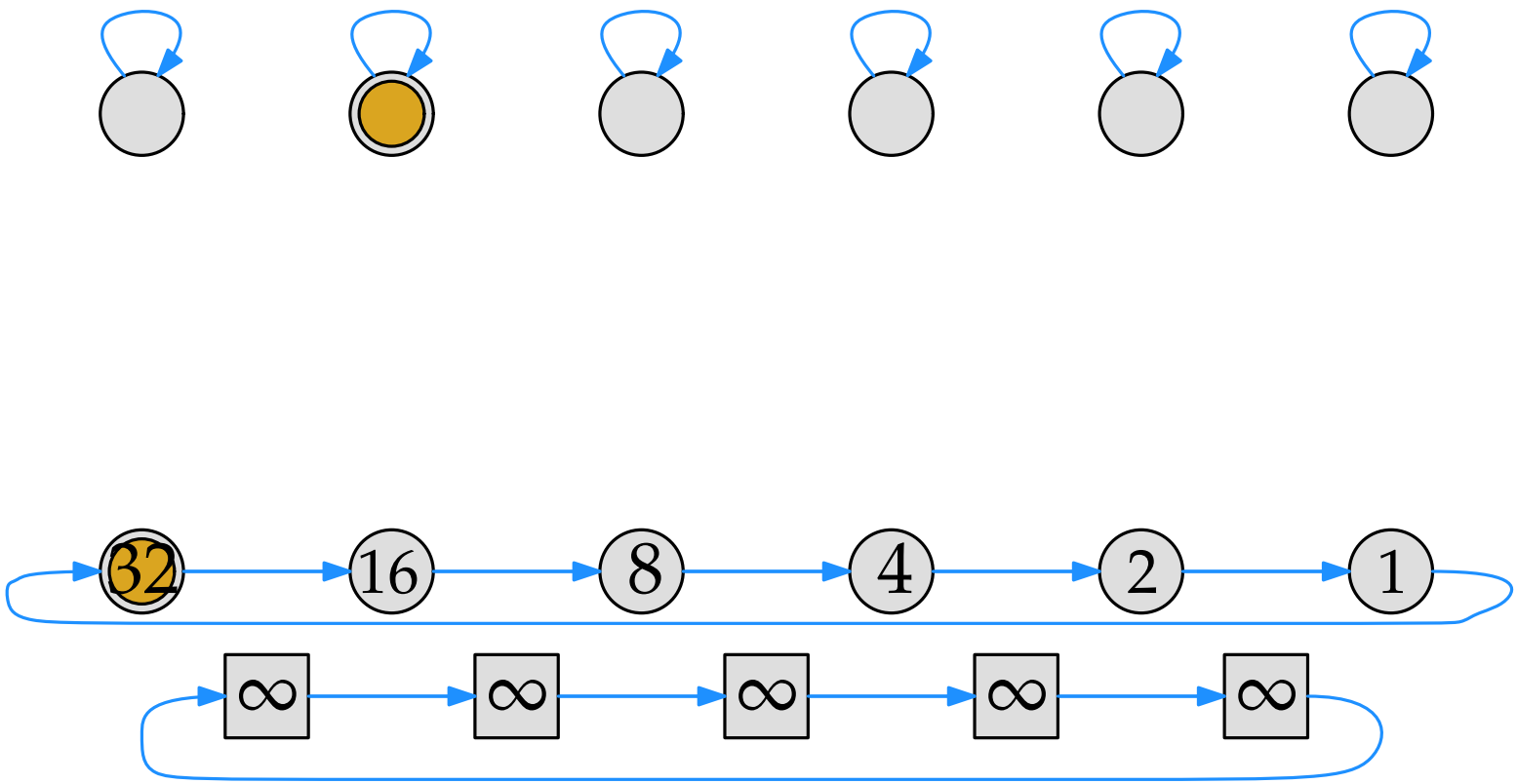


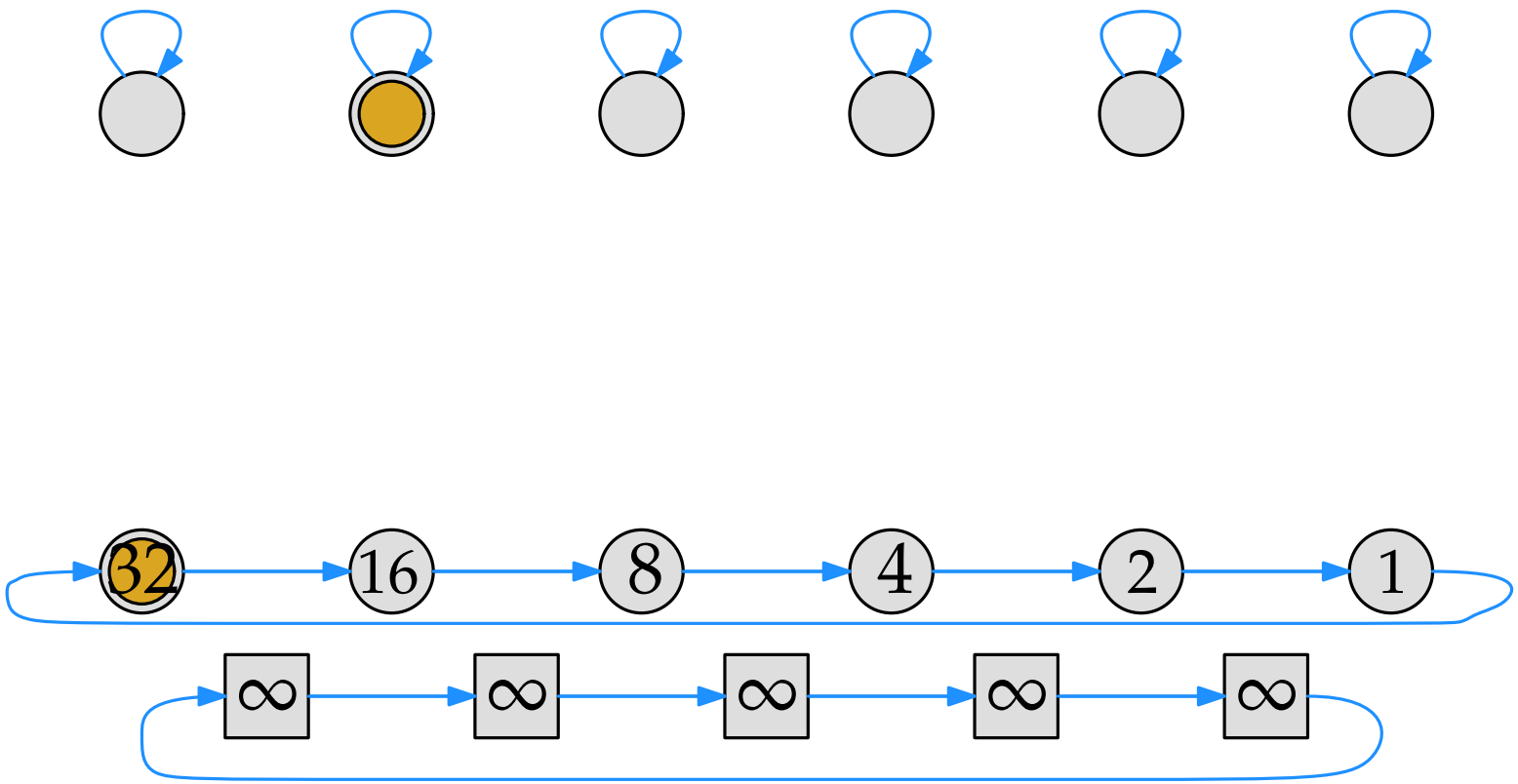
Quadratic Path for 2 Commuting Permutations

Lemma. The improving path under 2 commuting permutations on n positions can have length $\Omega(n^2)$.



Quadratic Path for 2 Commuting Permutations

Lemma. The improving path under 2 commuting permutations on n positions can have length $\Omega(n^2)$. In general: k commuting permutations can give an improving path of length $\Omega(n^k)$.



Two permutations π and π^2

Simplest Open Problem. What about having just two permutations, π and π^2?

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Proof. We leave it as an exercise.

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But can we actually show that the path has at most polynomial length?

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Conjecture. Suppose $f_1, \ldots, f_l : \mathbb{Z} \to \mathbb{Z}$ are periodic functions with f_i having period p_i. Set $f := f_1 + \cdots f_l$.
If $f(1), f(2), f(3), \ldots, f(b)$ is 2-jump decreasing
then b is polynomial in n.