4th Workshop on
Graphs, Algorithms and Machine Learning
Cala Millor, Mallorca
March 24, 2026

Polynomial Local Search: The complexity of finding a local optimum

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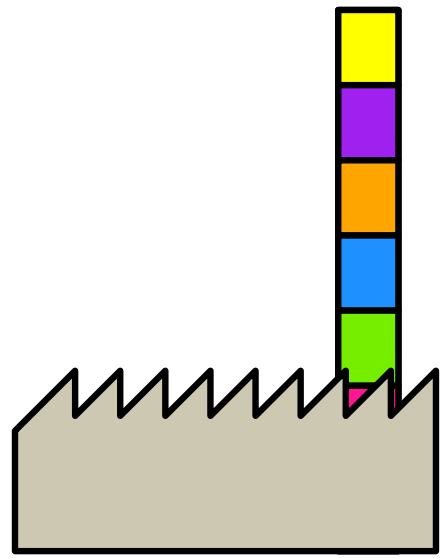
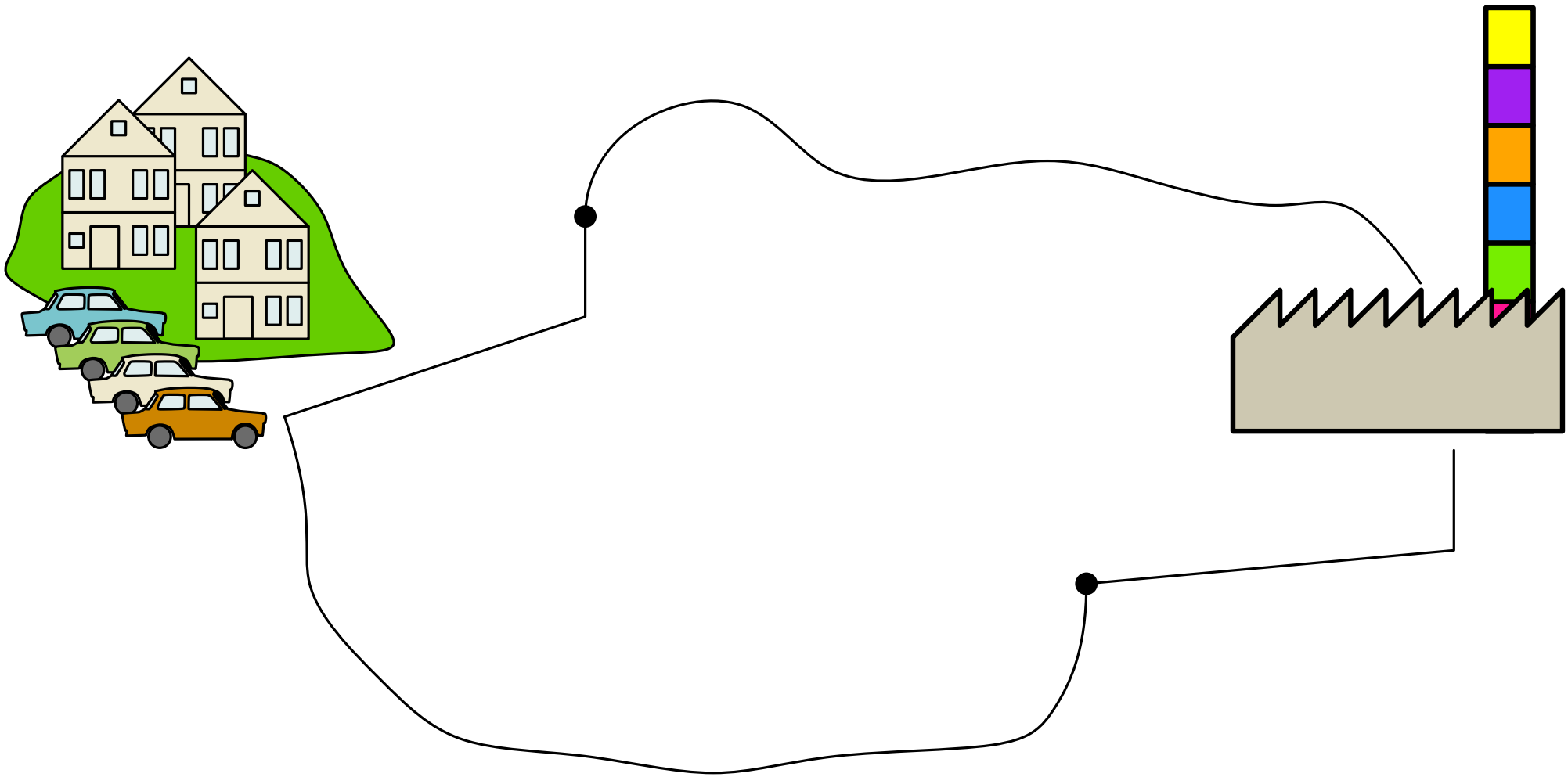
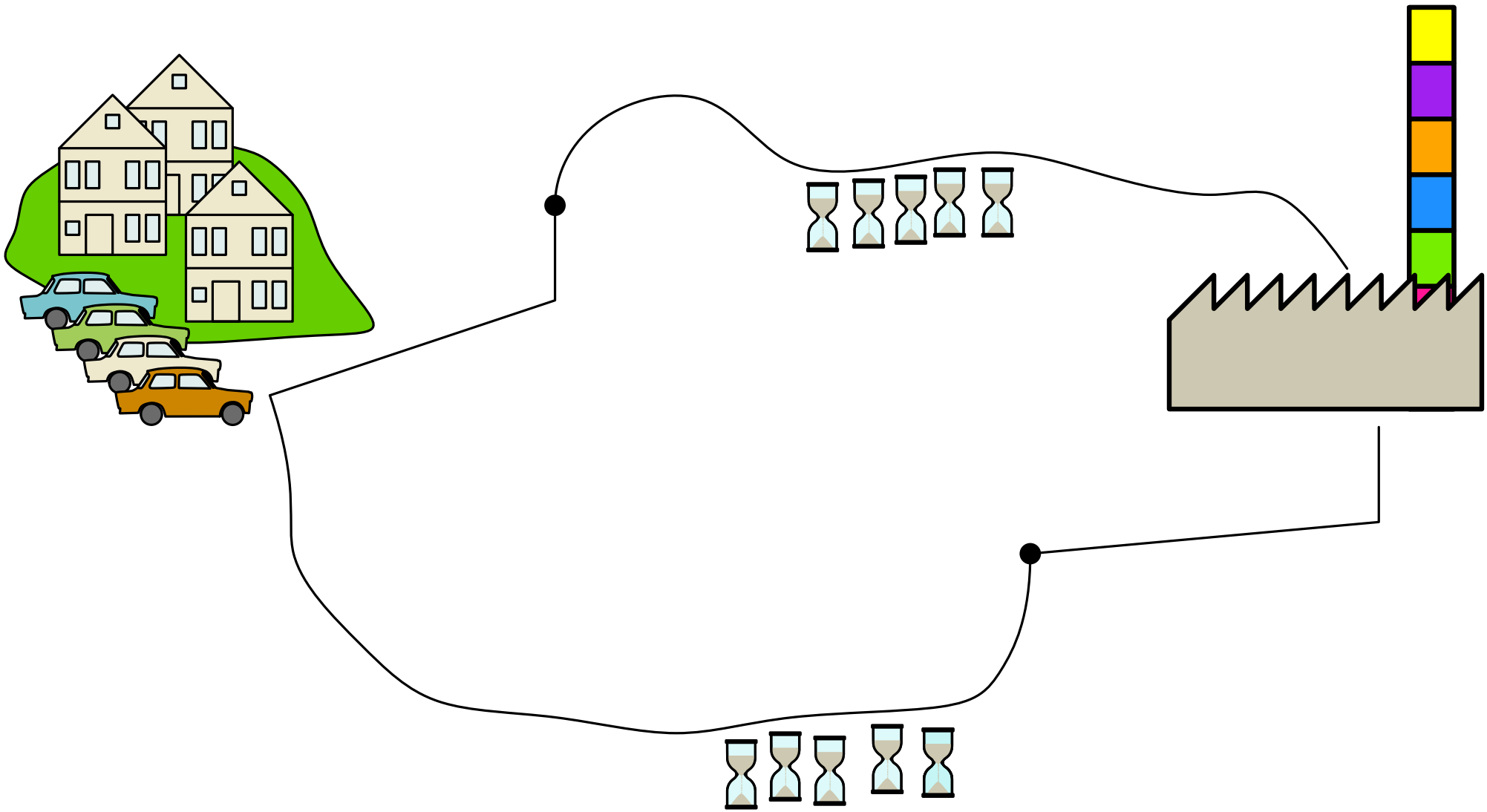
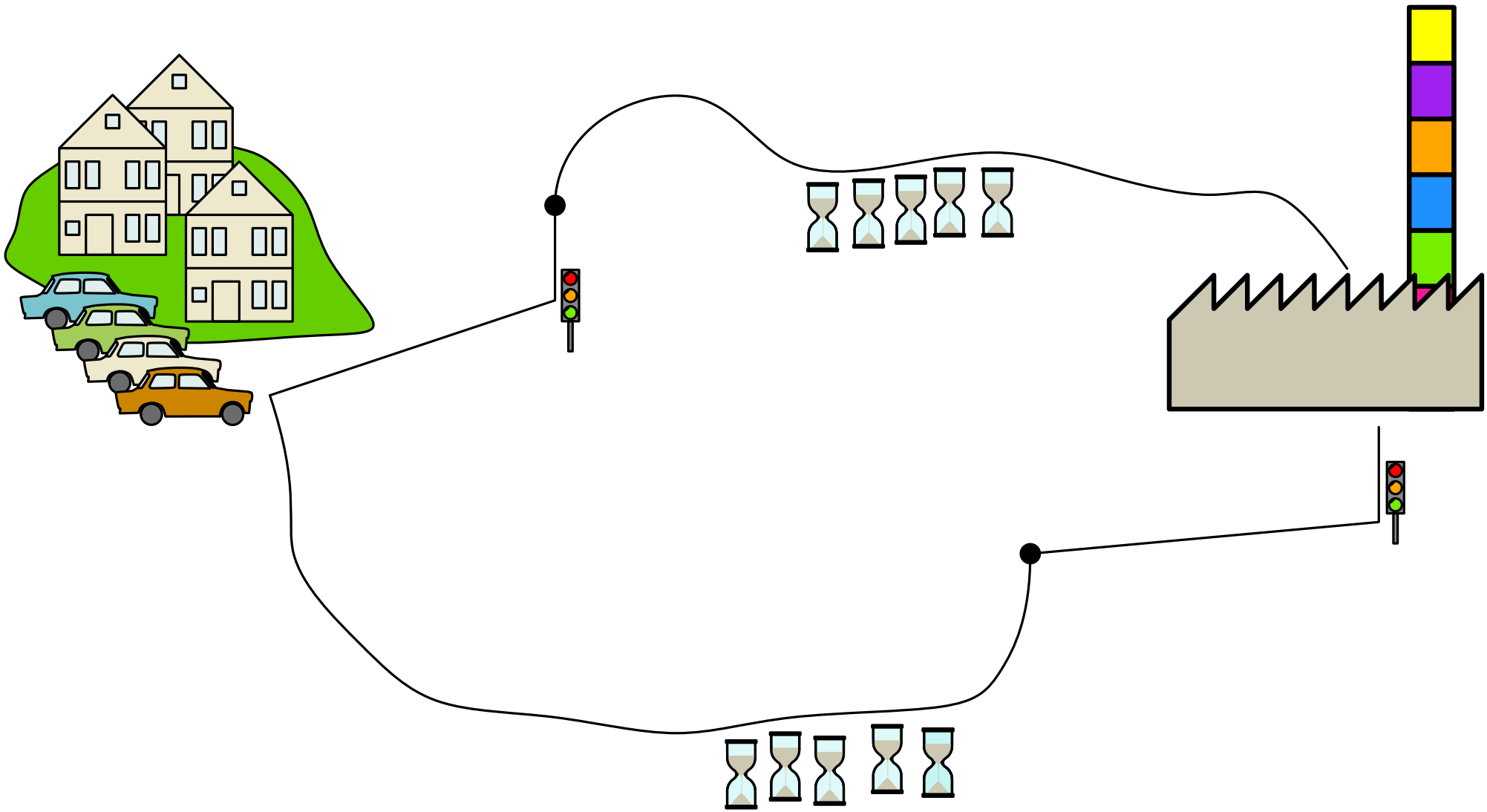
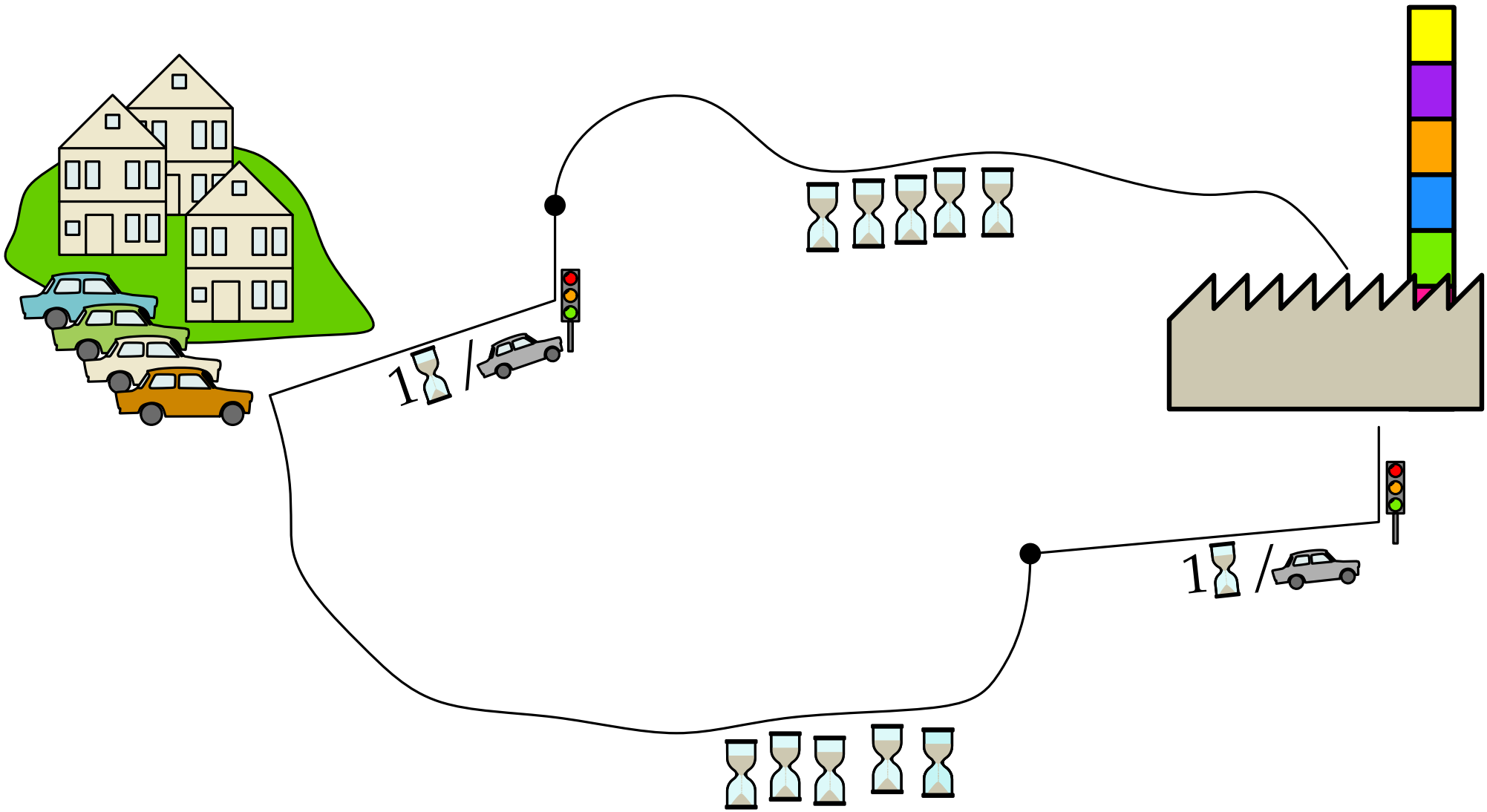
Based on joint work with Johannes Tantow, also TU Chemnitz

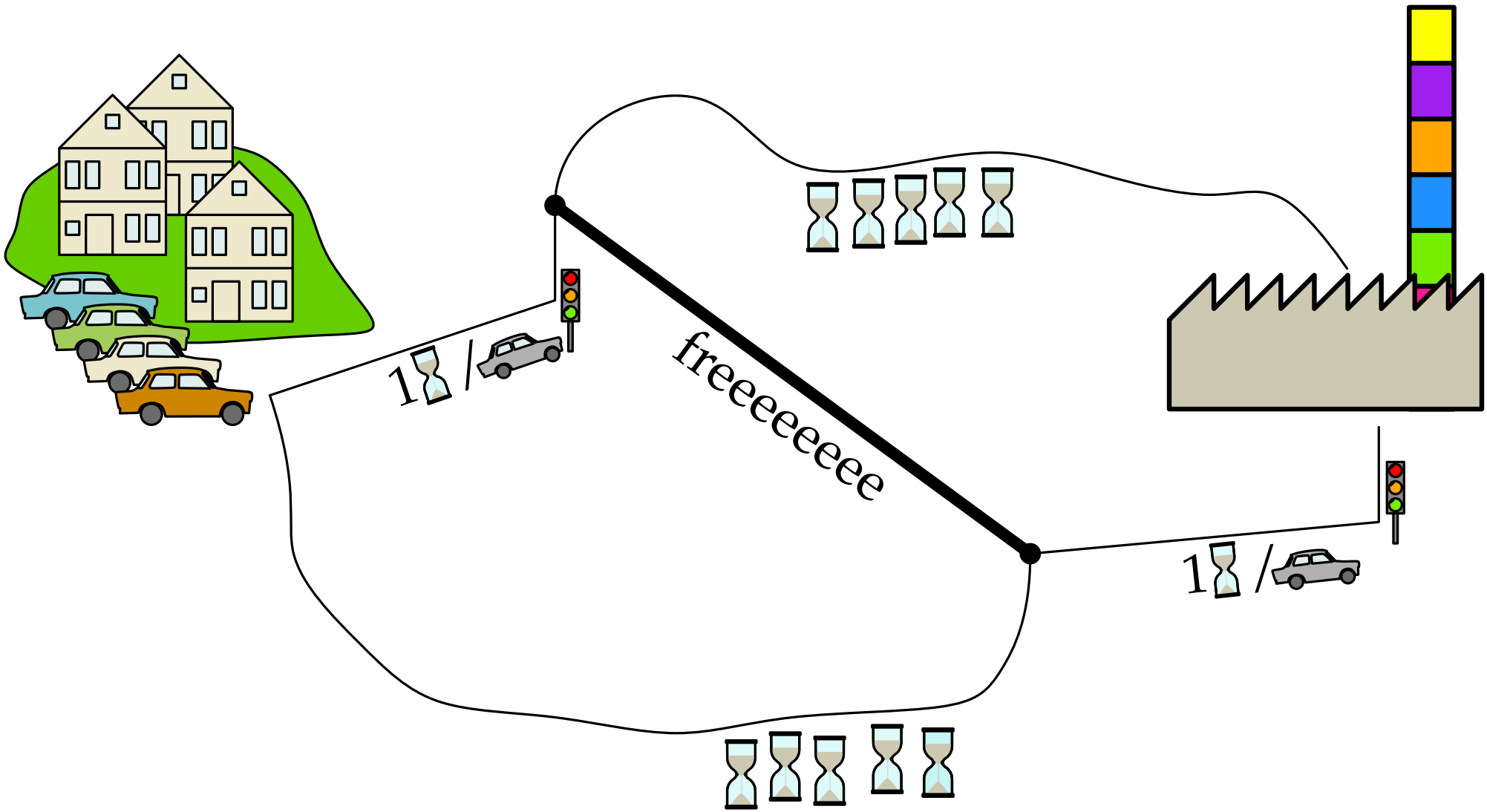








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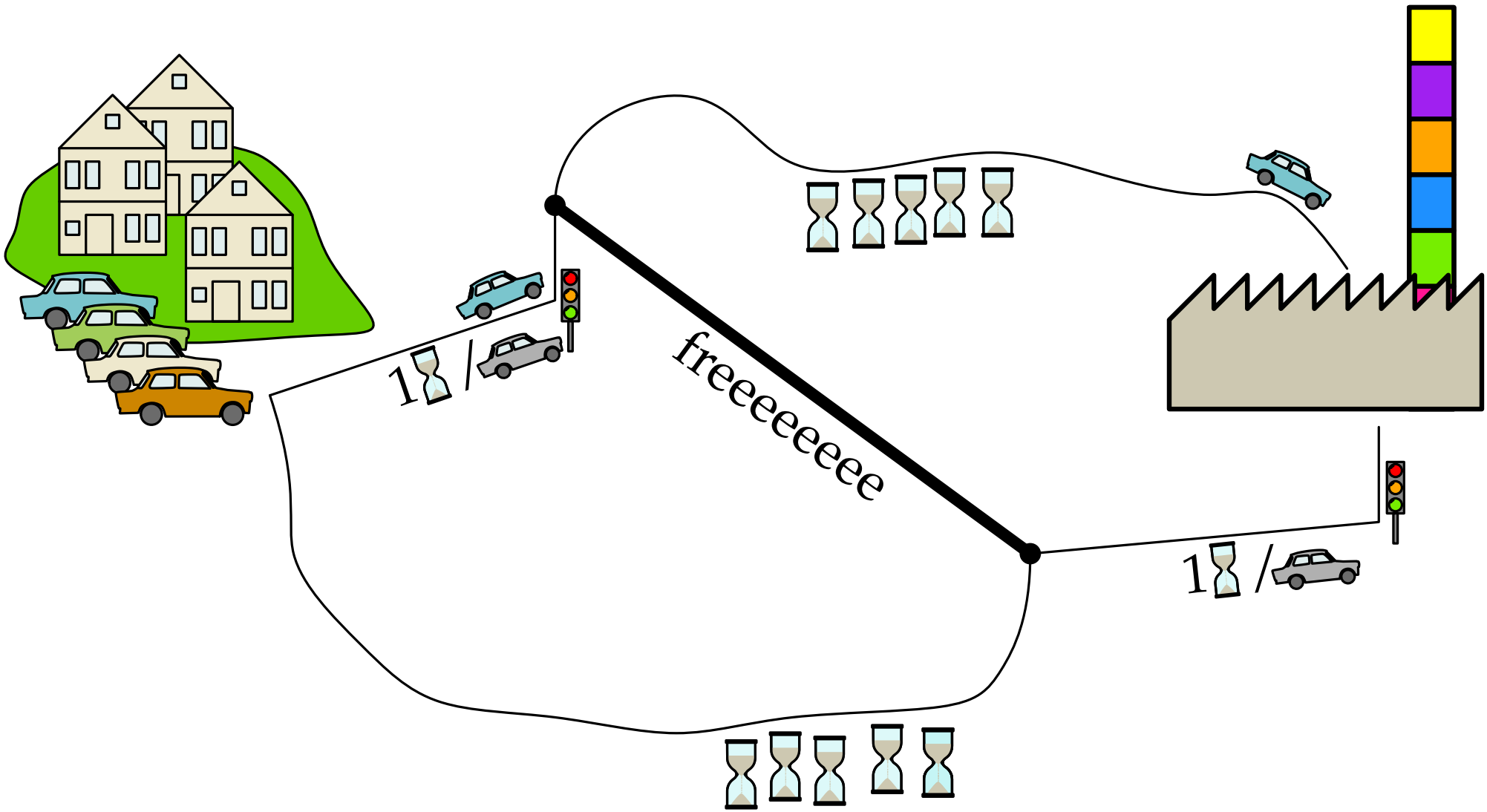
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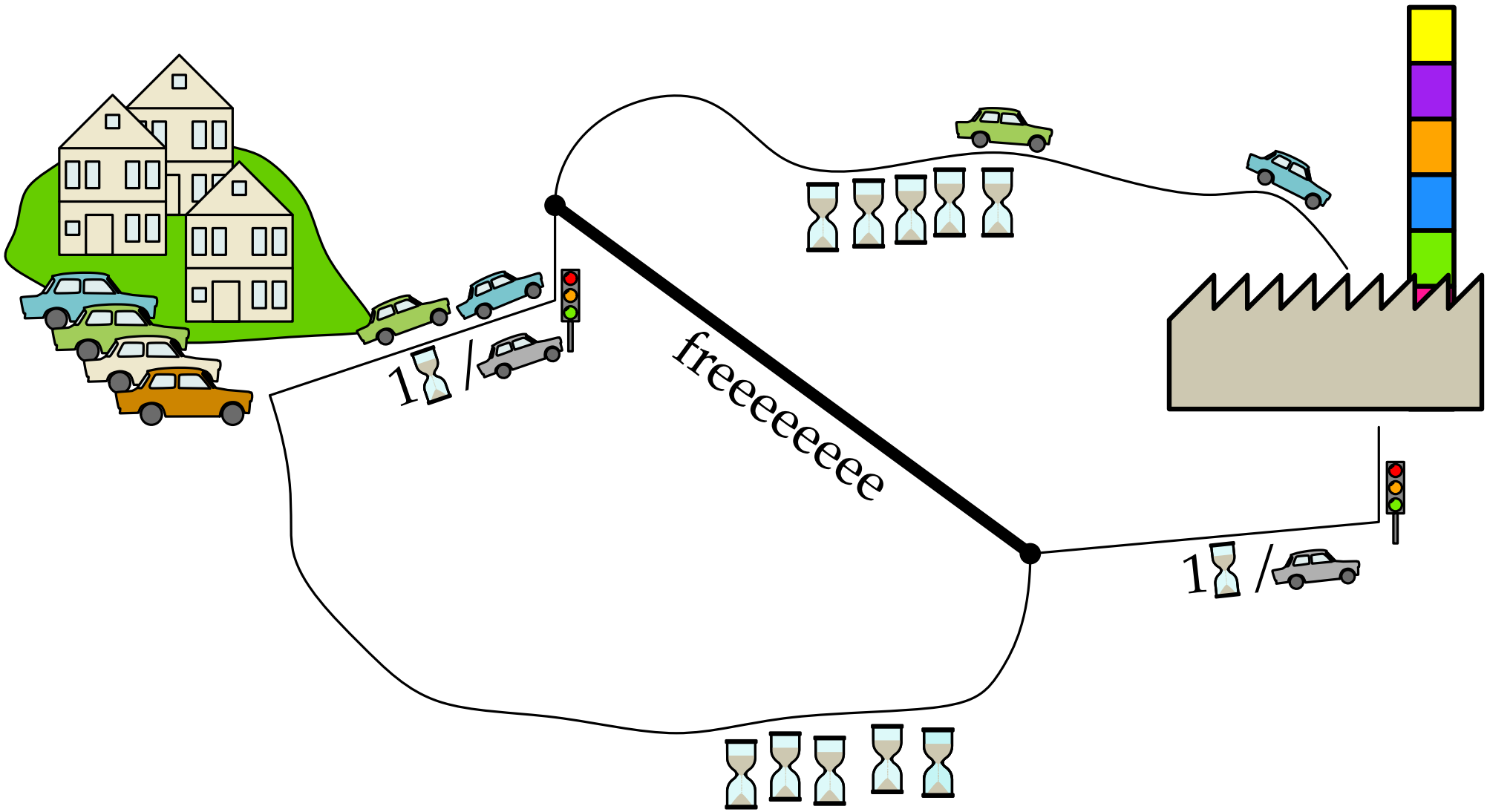
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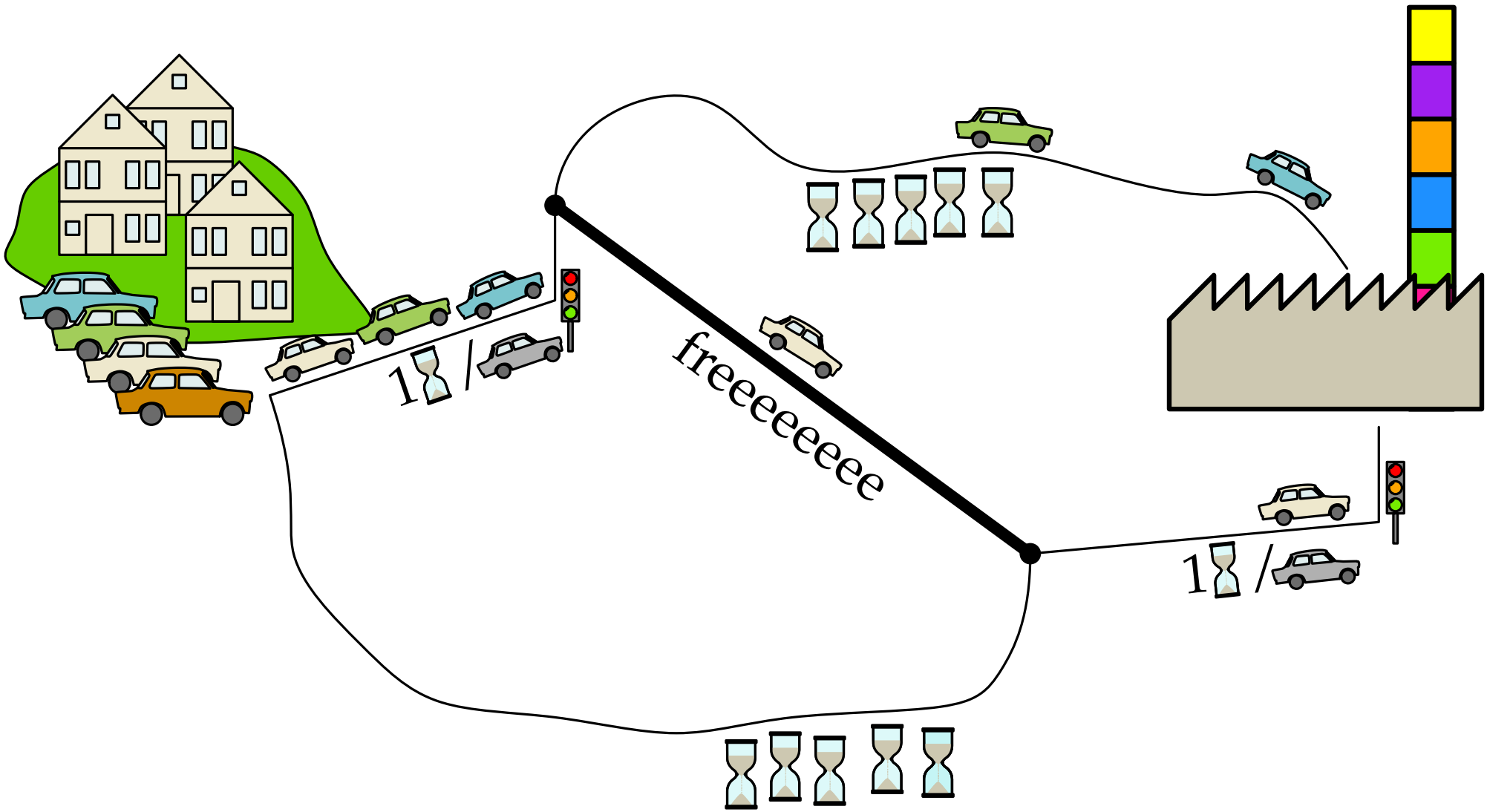
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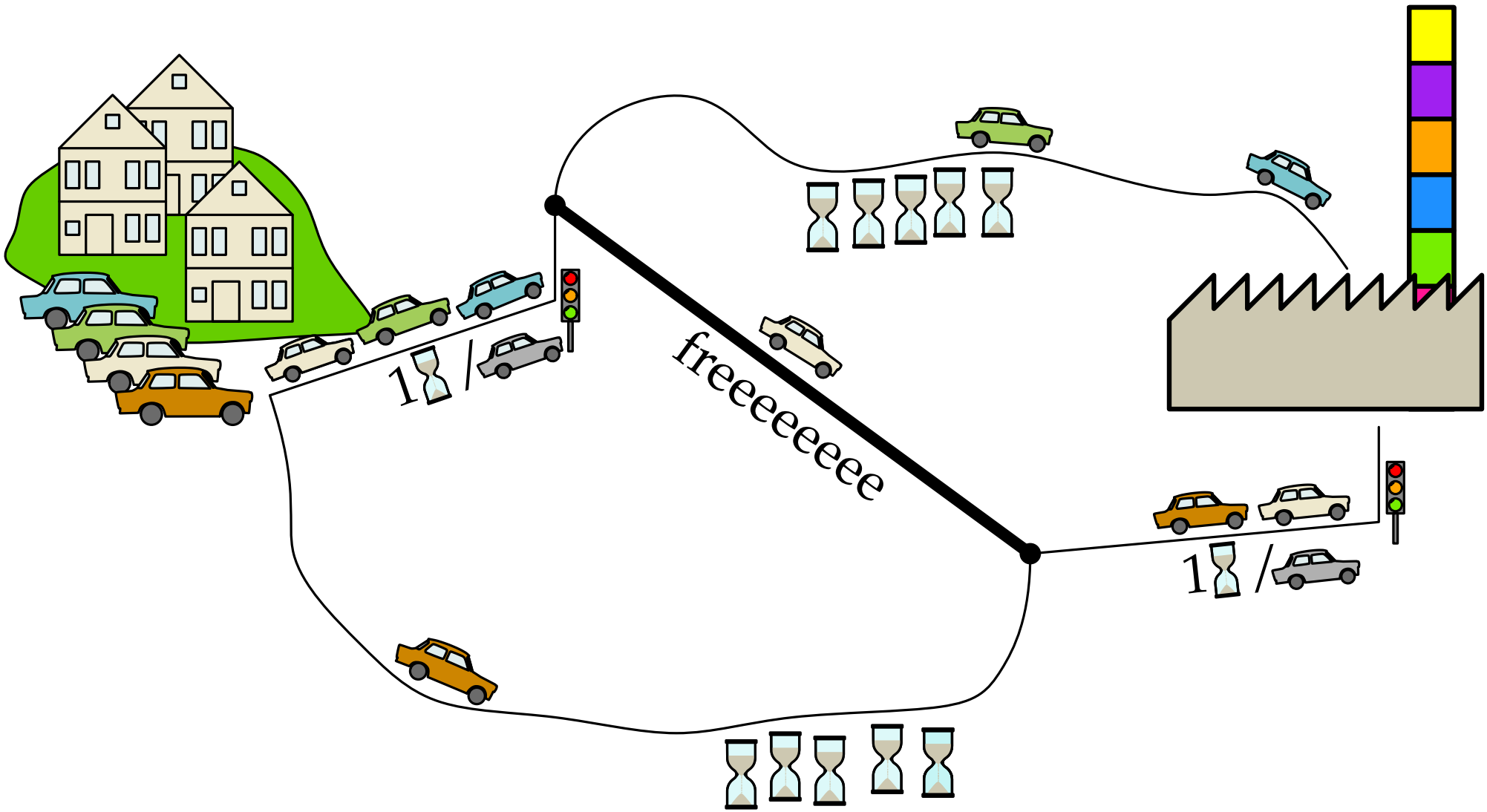
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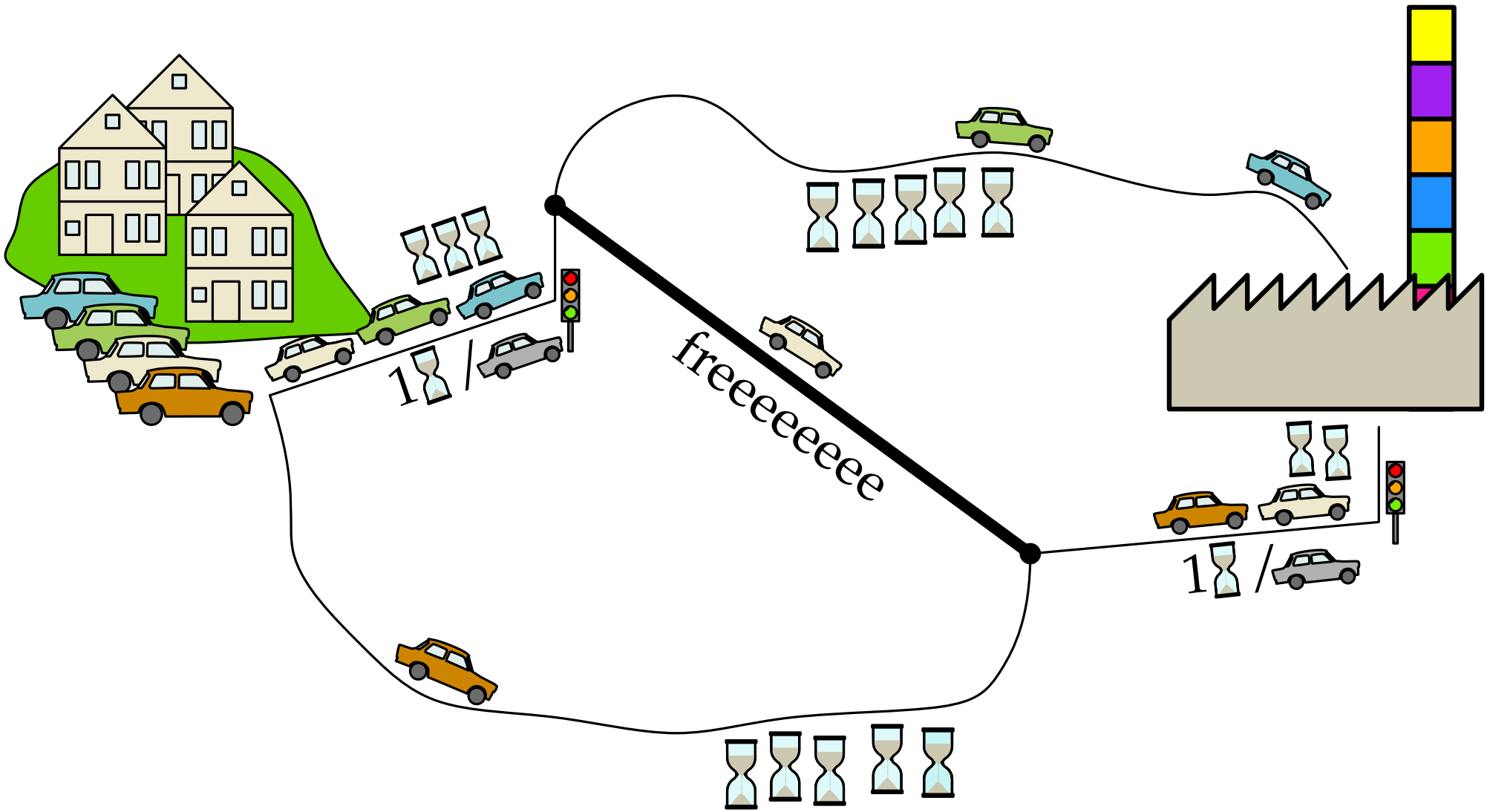
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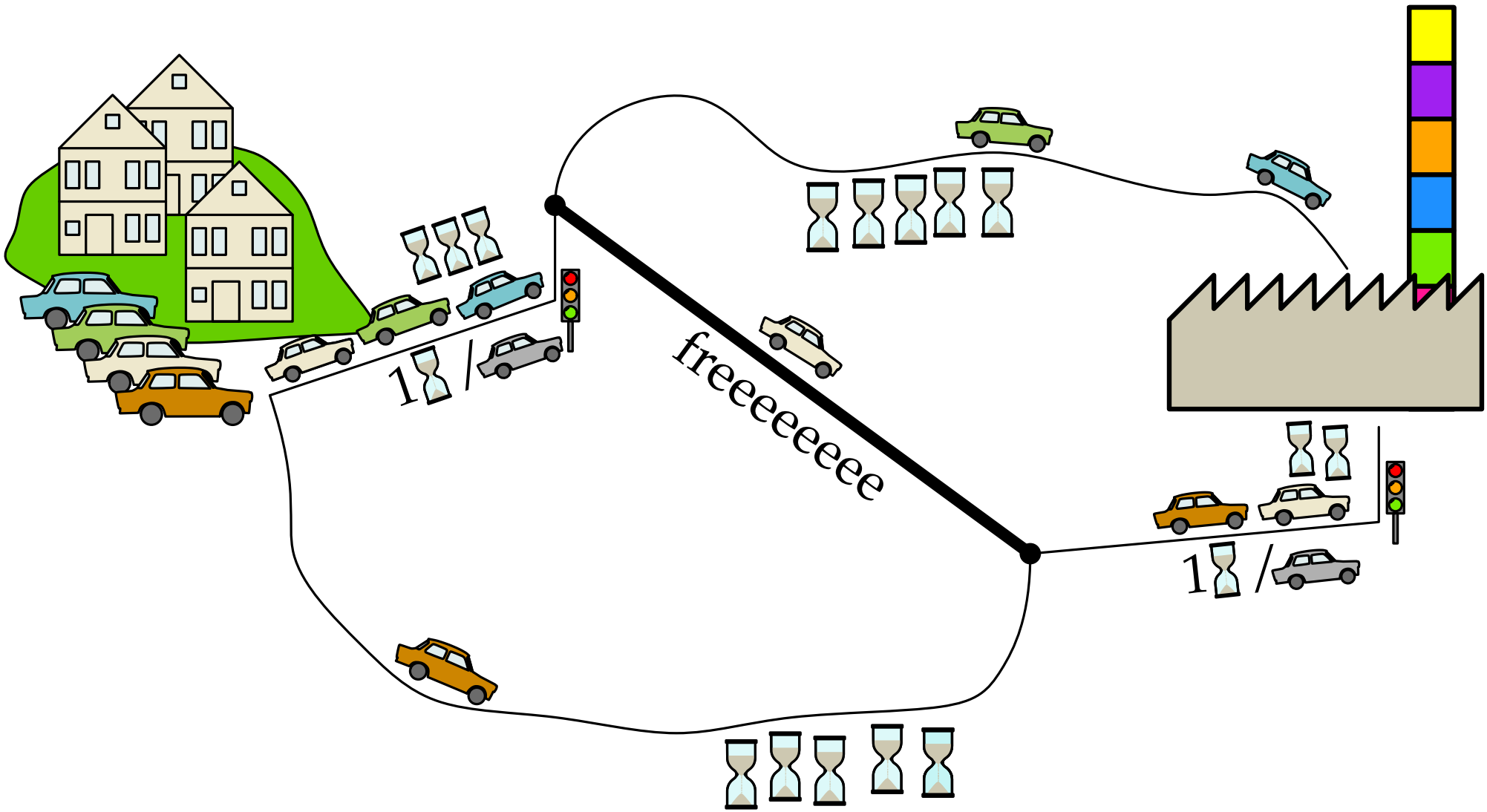
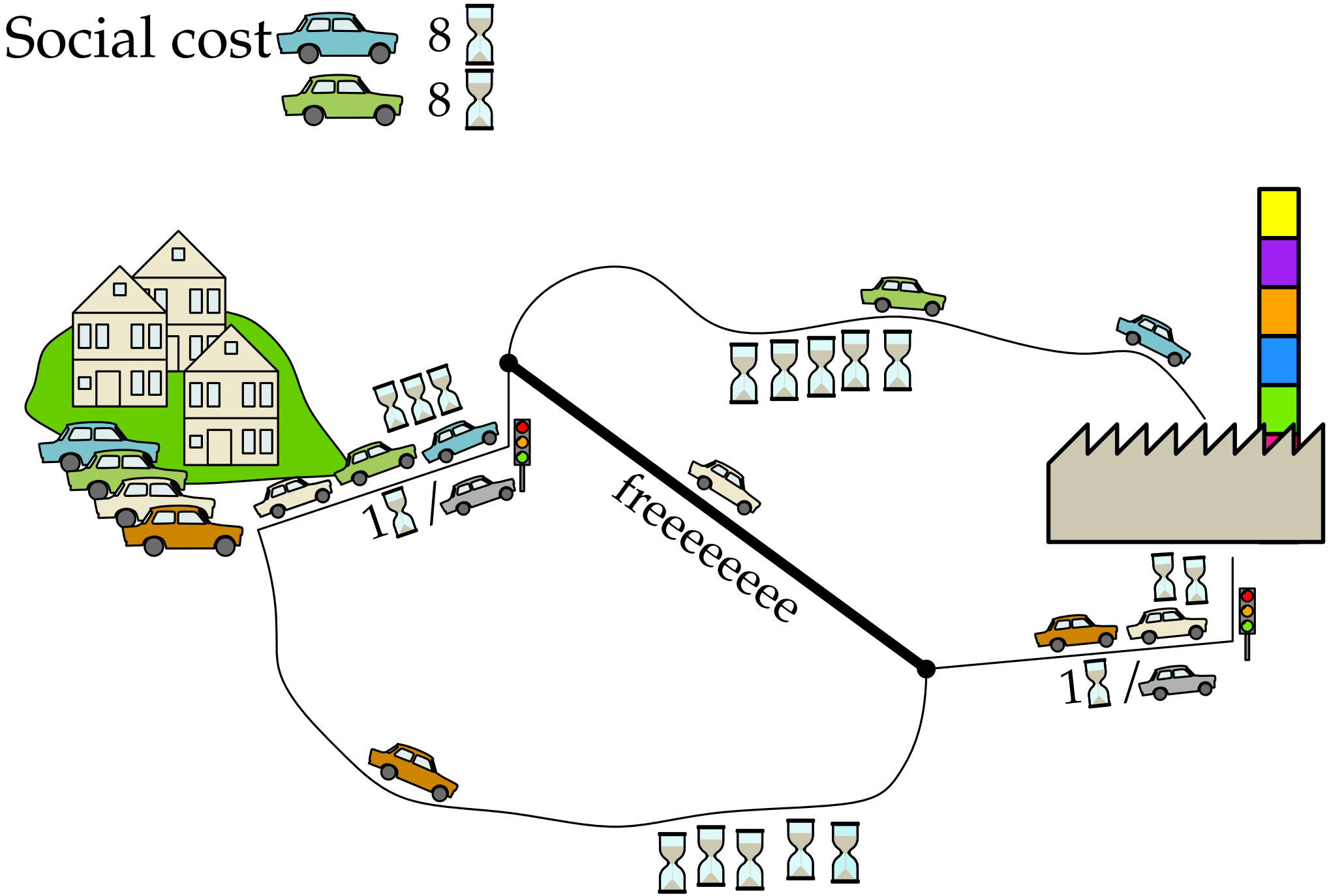
Social cost 8




Social cost
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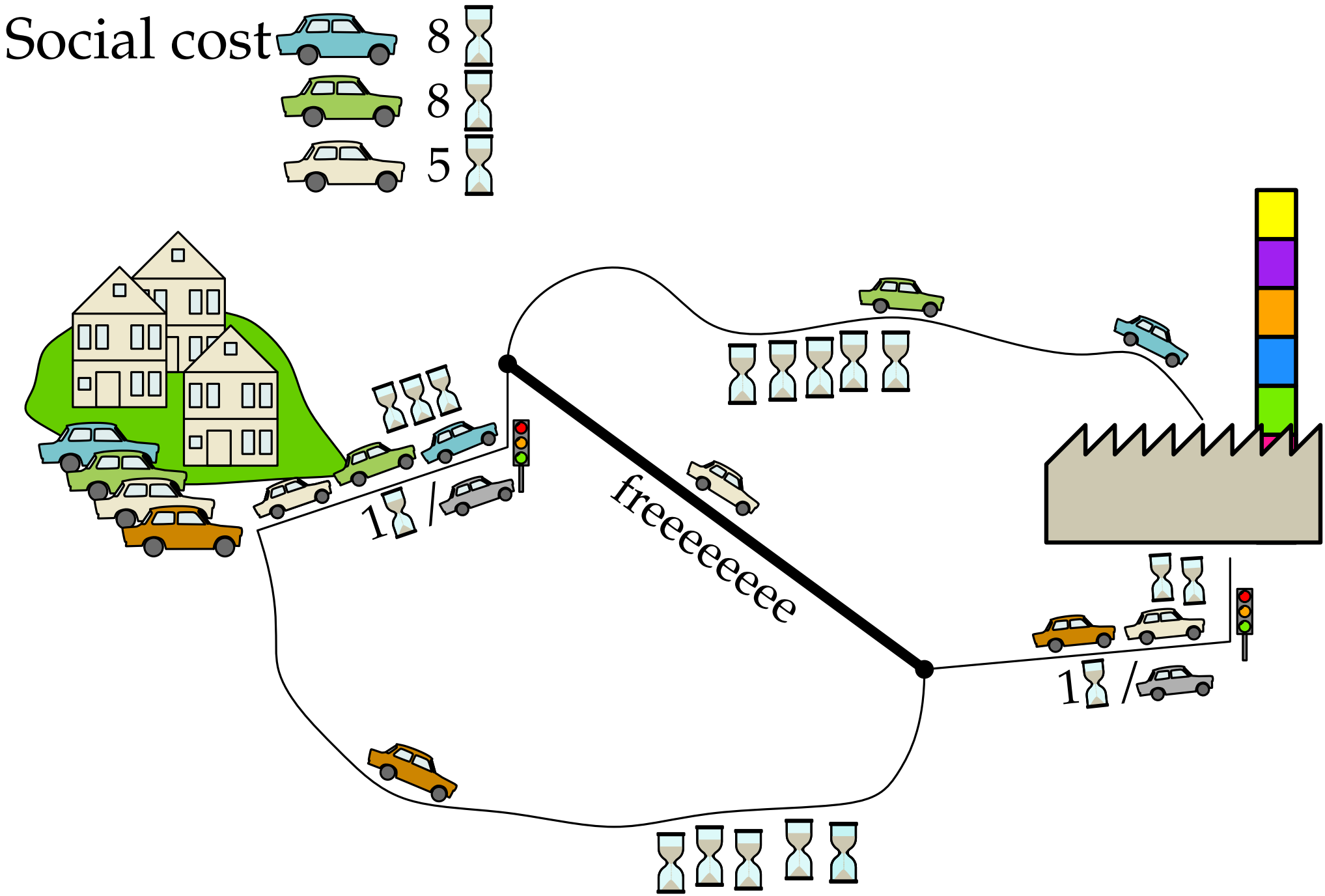
Social cost
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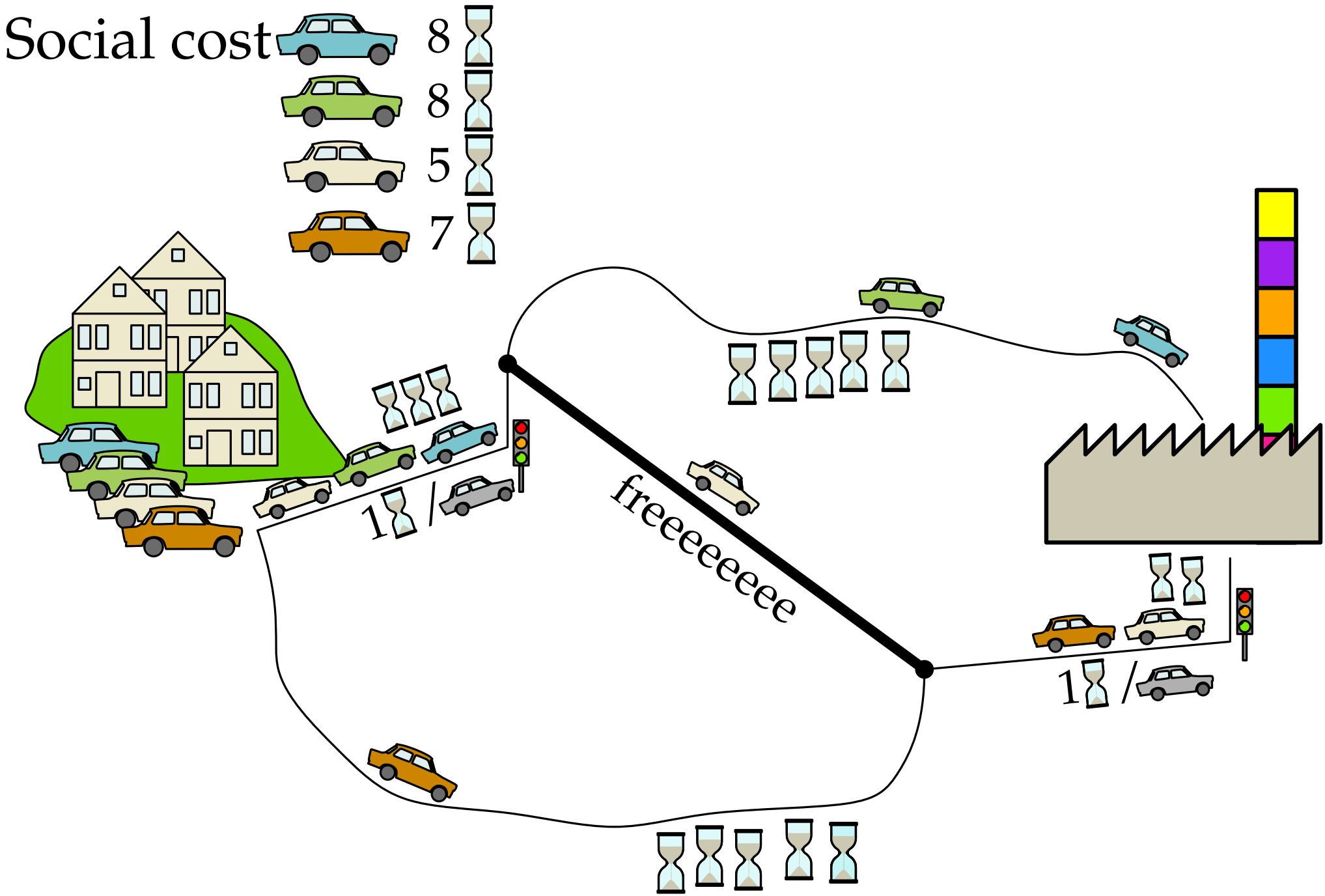
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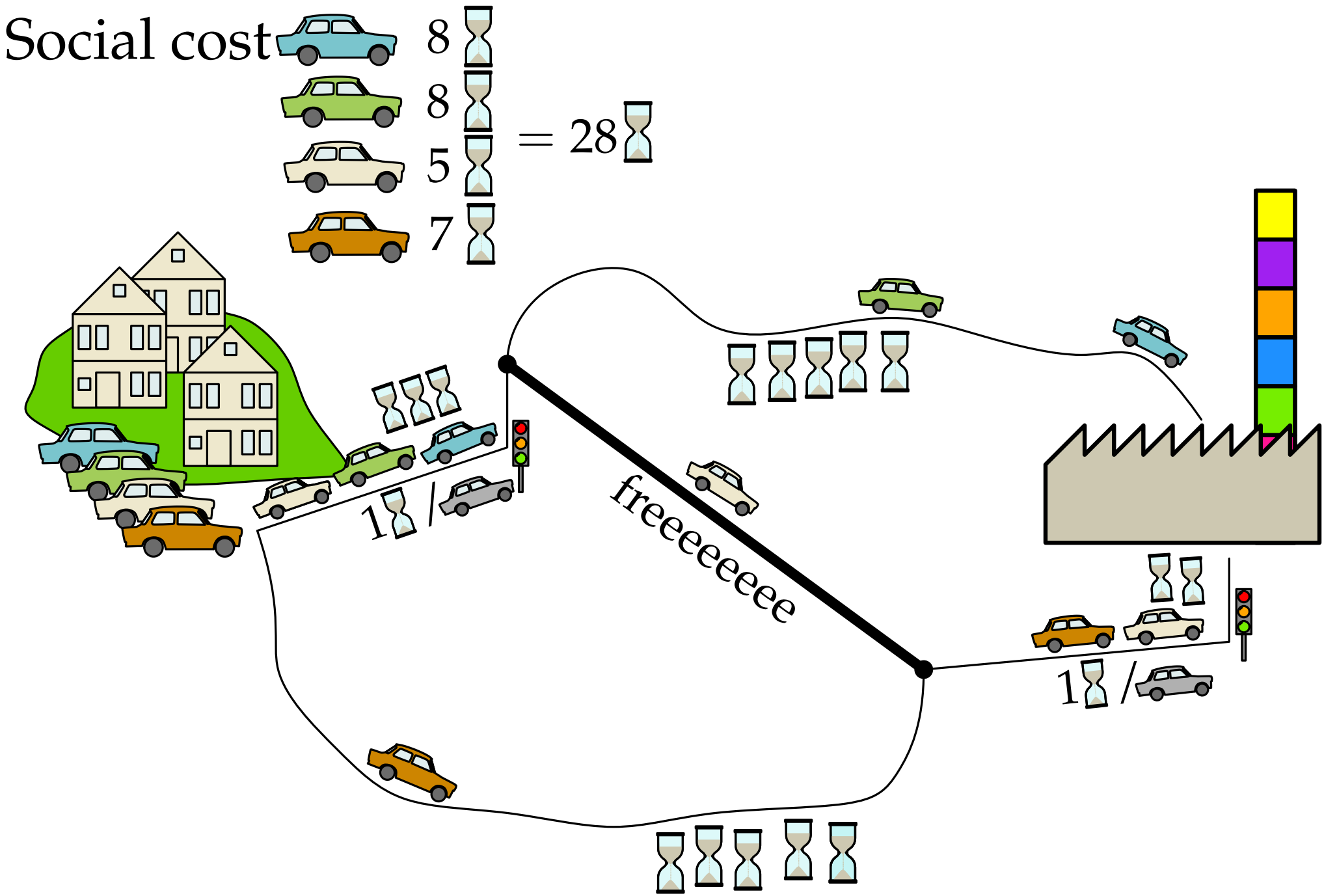
Social cost
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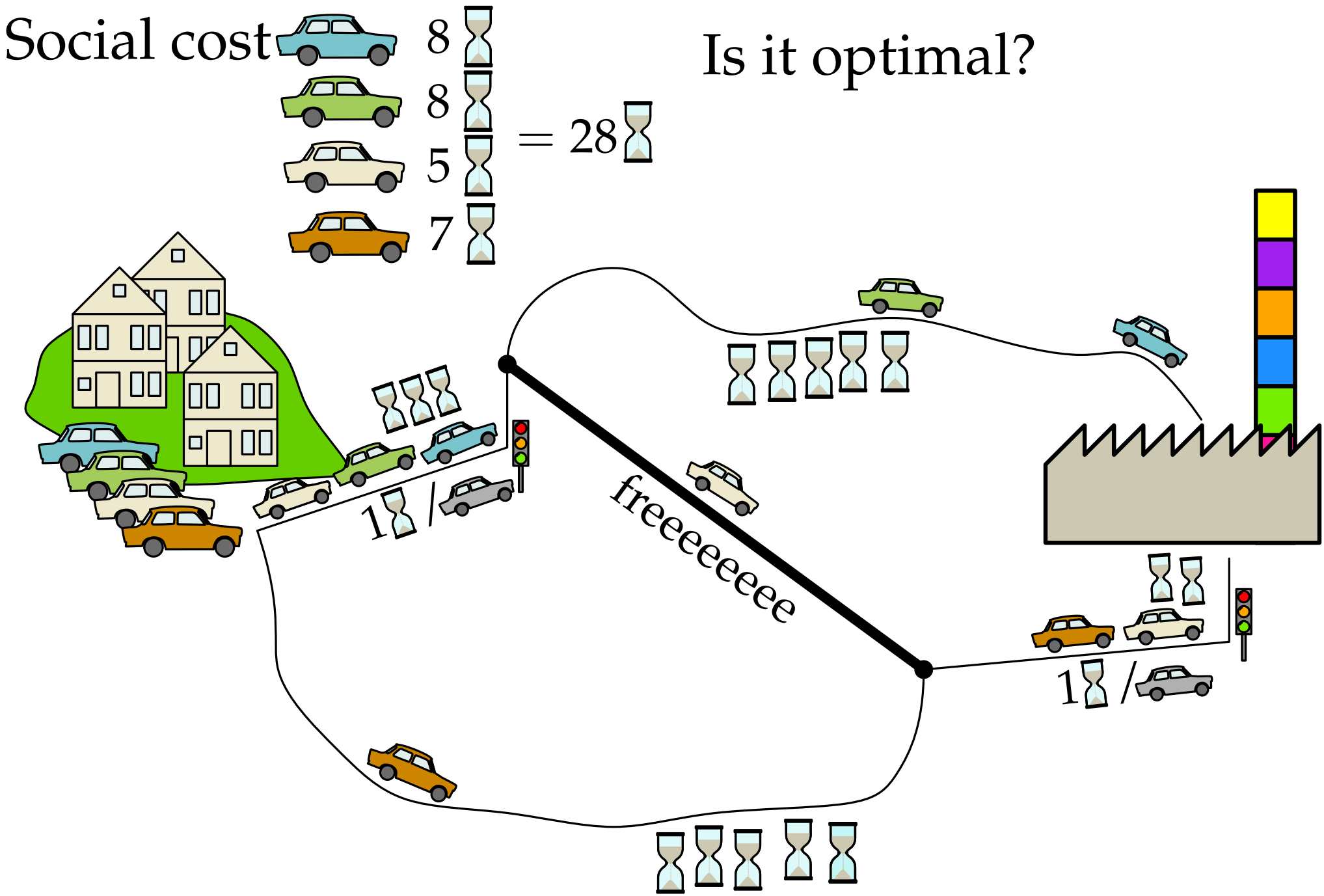
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Is it optimal?
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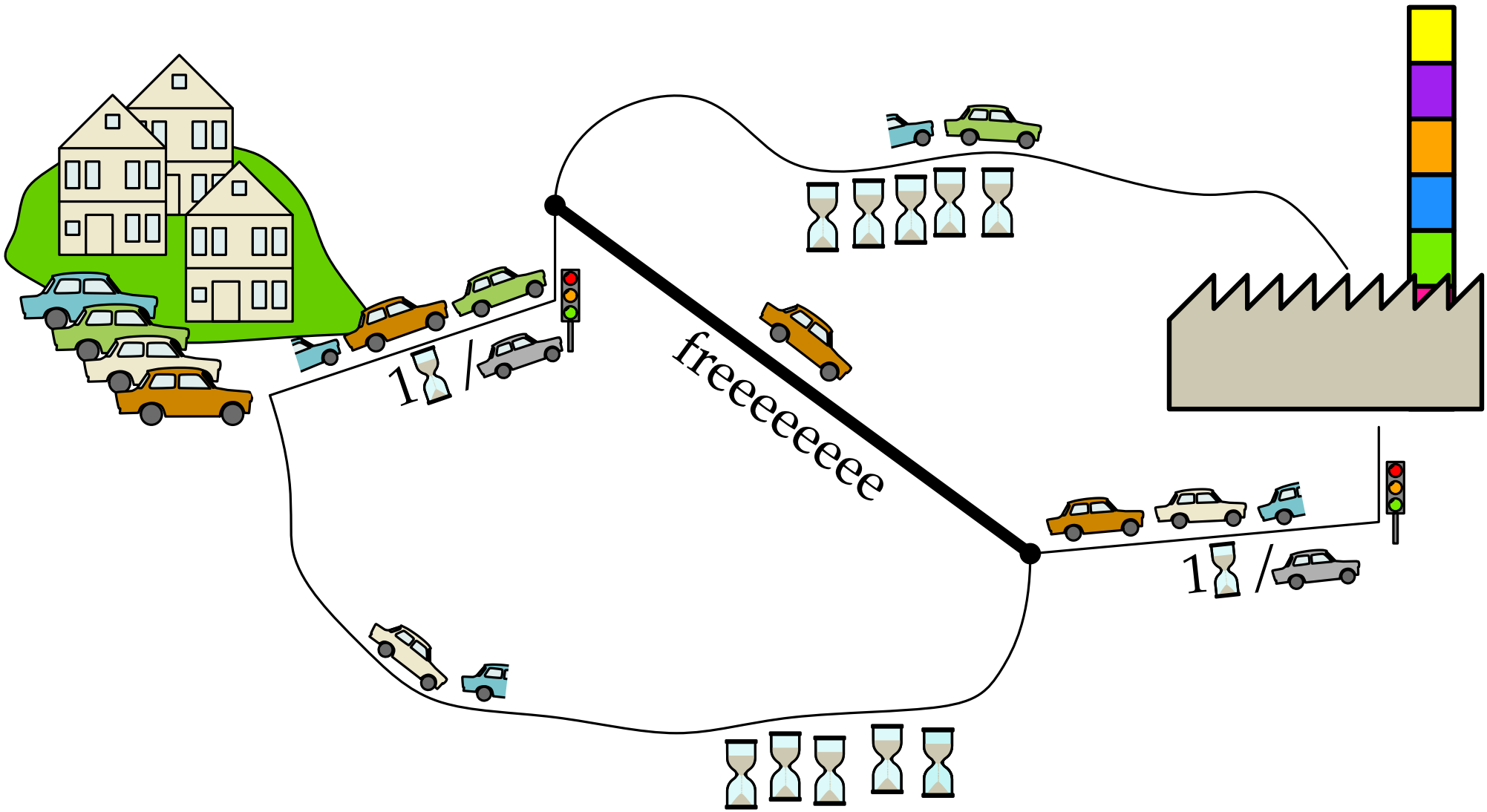
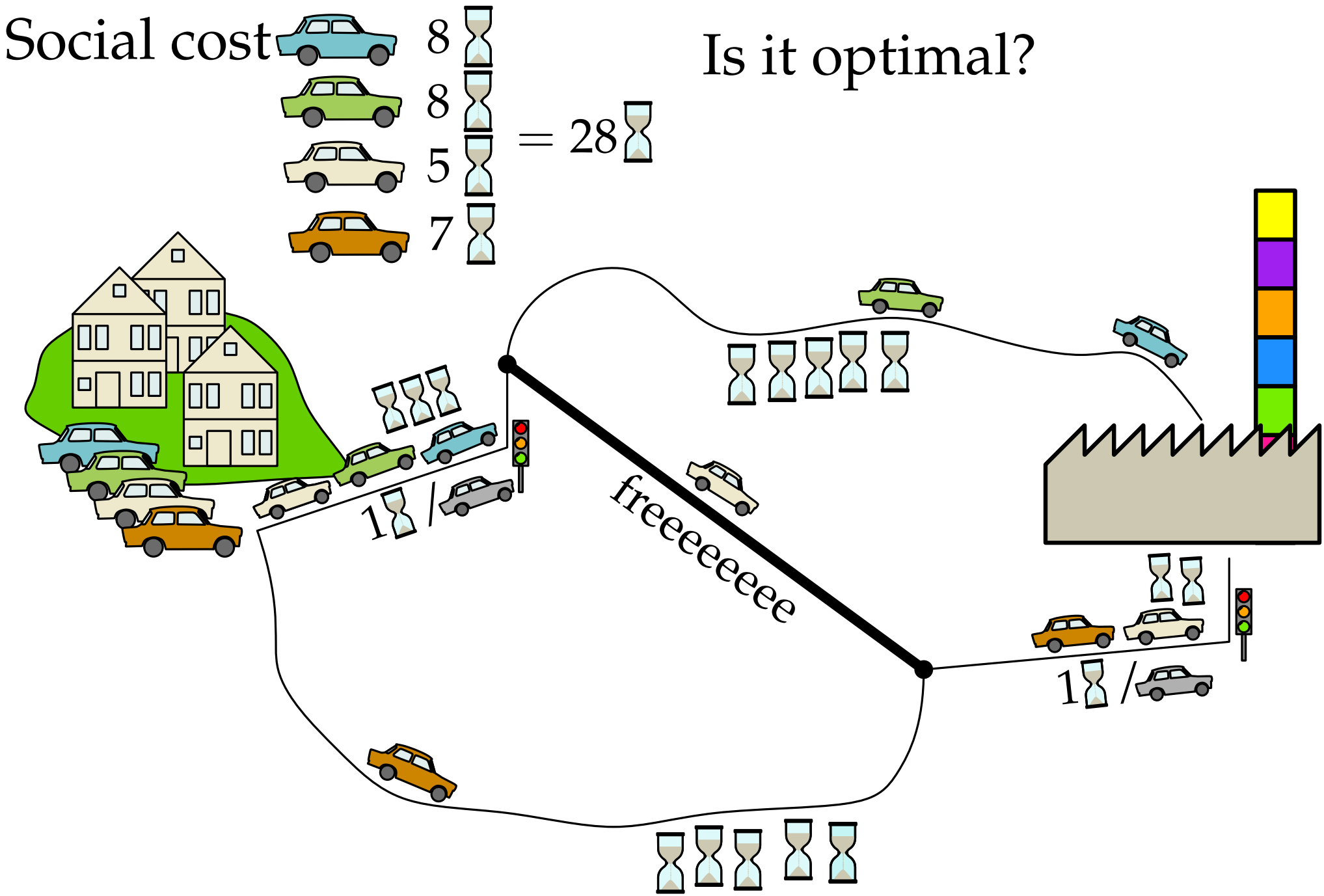
social cost = 27.5

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General congestions games

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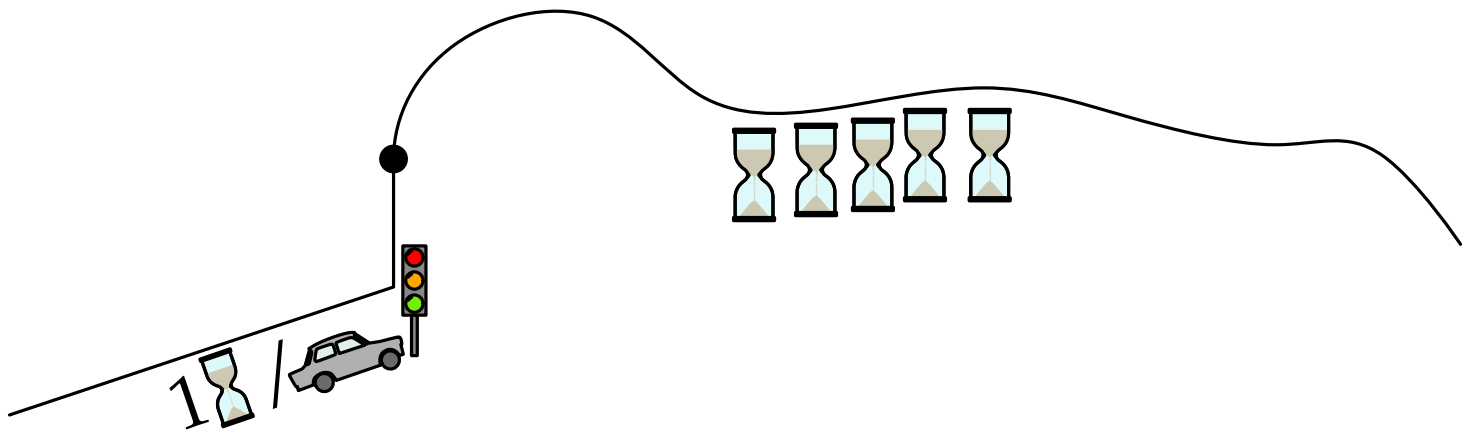

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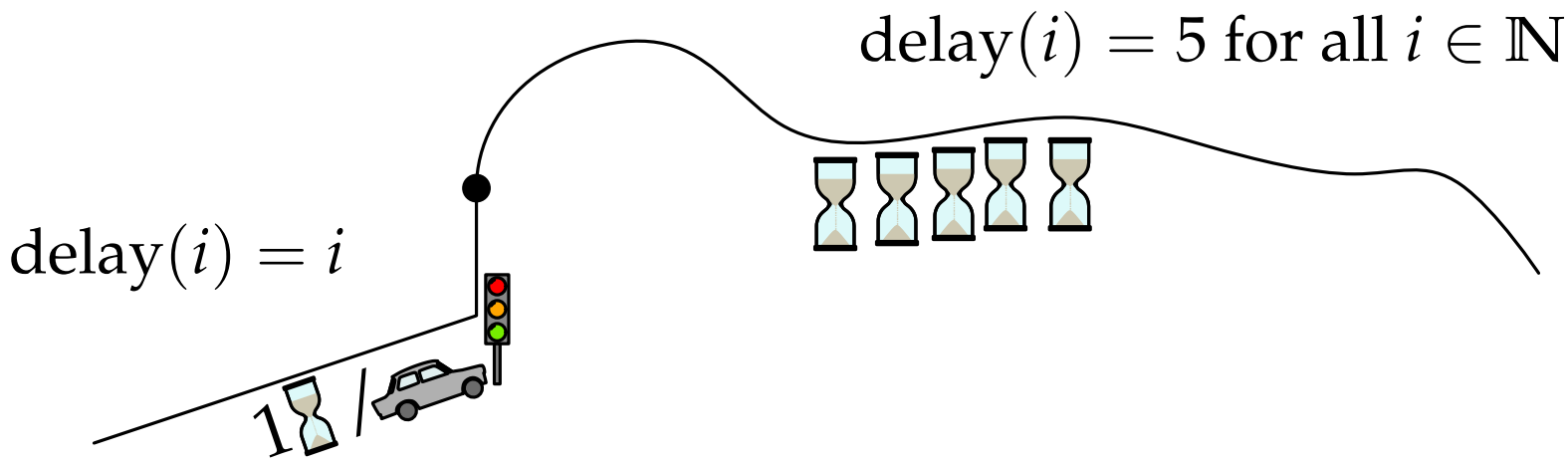

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$$\text{The cost for } p \text{ is } \text{cost}(p, \vec{s}) = \sum_{r\in s_p} \text{delay}_r(\text{load}(r, \vec{s}))$$

Yet Another Optimization Problem

Problem. Given a congestion game $(\mathcal{P}, \mathcal{R}, (S_p)_{p\in\mathcal{P}}, (\text{delay}_r)_{r\in\mathcal{R}})$, determine a strategy profile $\vec{s}$ that achieves the social optimum, i.e., that minimizes

$$\text{socialcost}(\vec{s}) := \sum_{p\in\mathcal{P}} \text{cost}(p, \vec{s}) \ .$$

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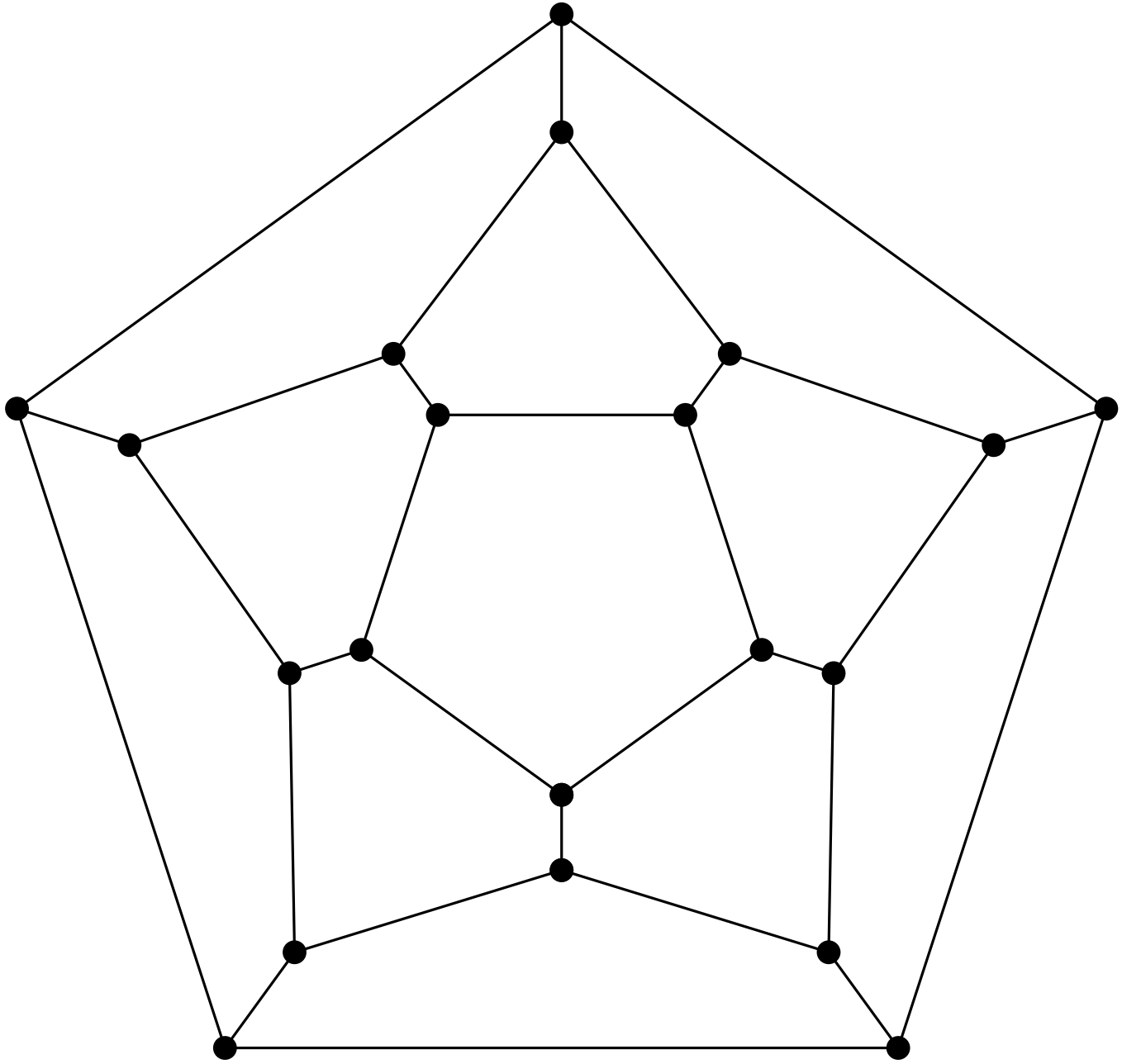
Theorem. This is NP-hard because we can use it to find a maximum independent set.

The Maximum Independent Set Problem



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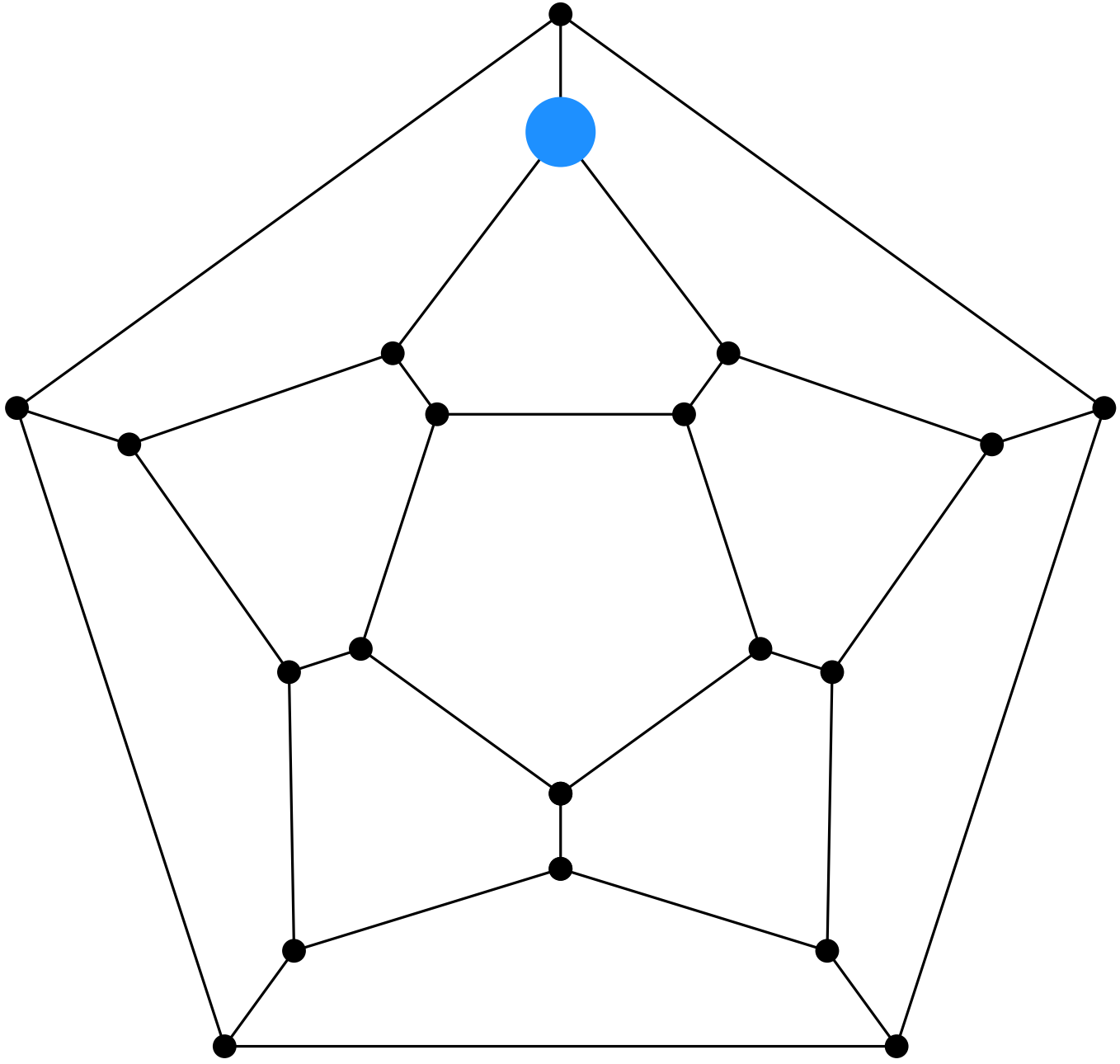

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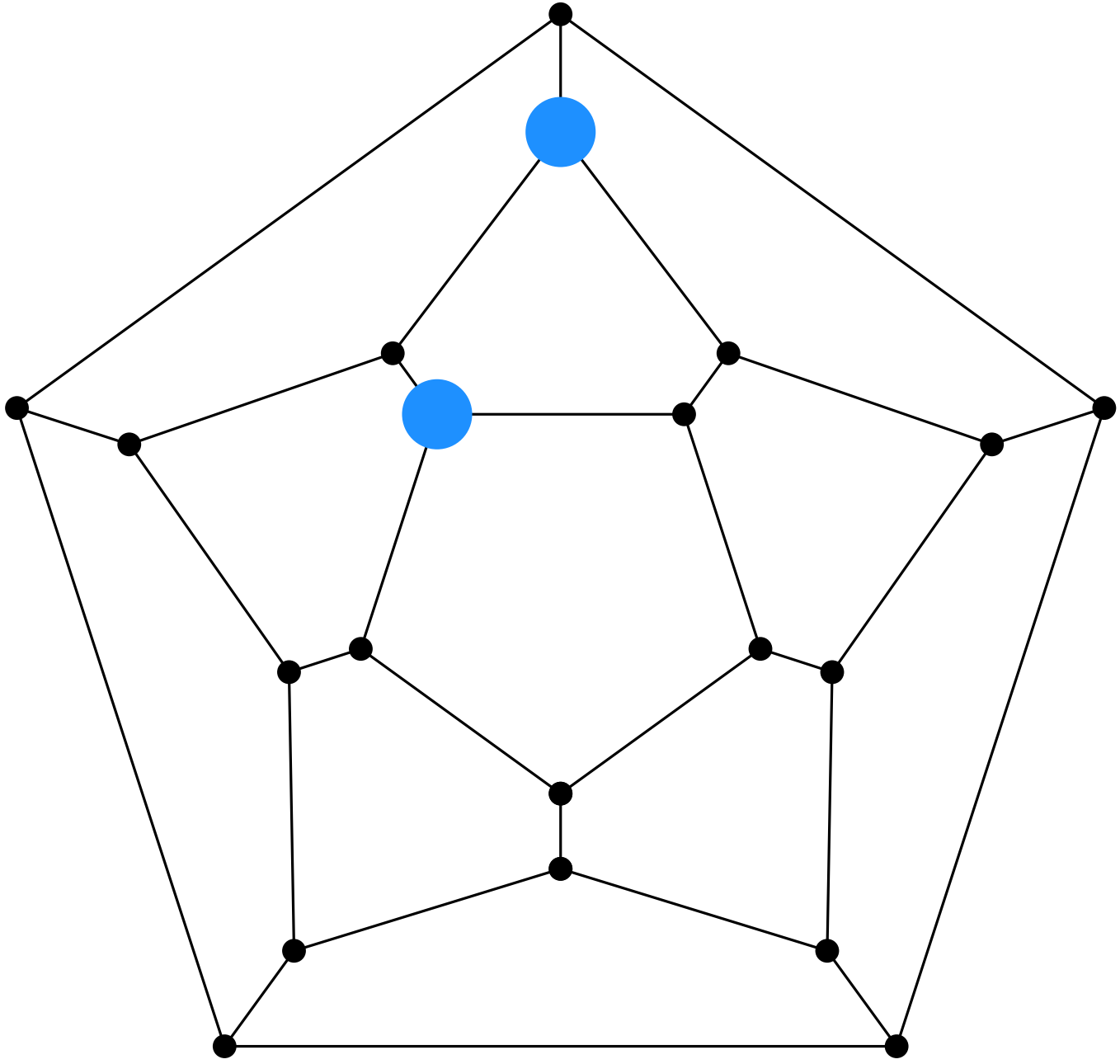

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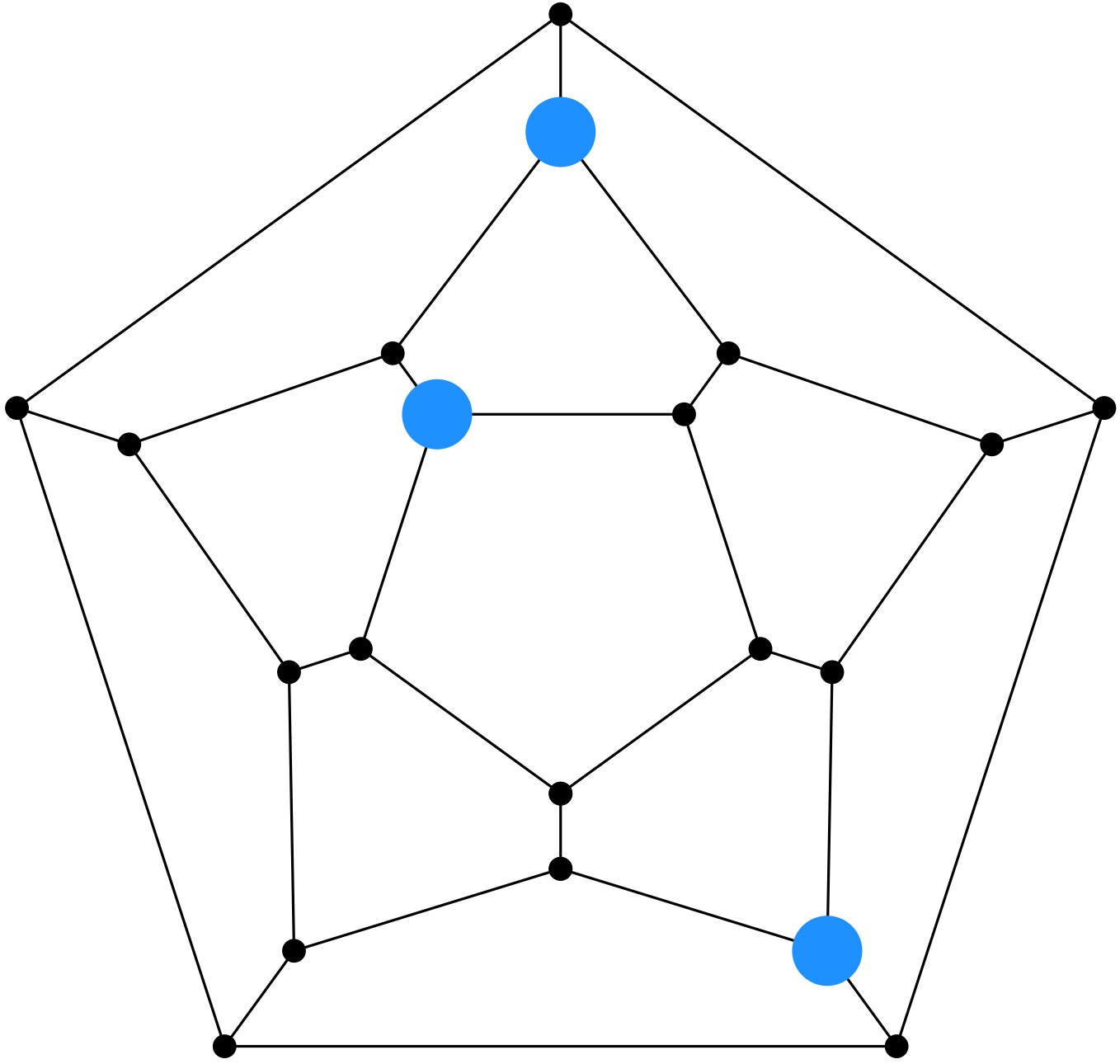

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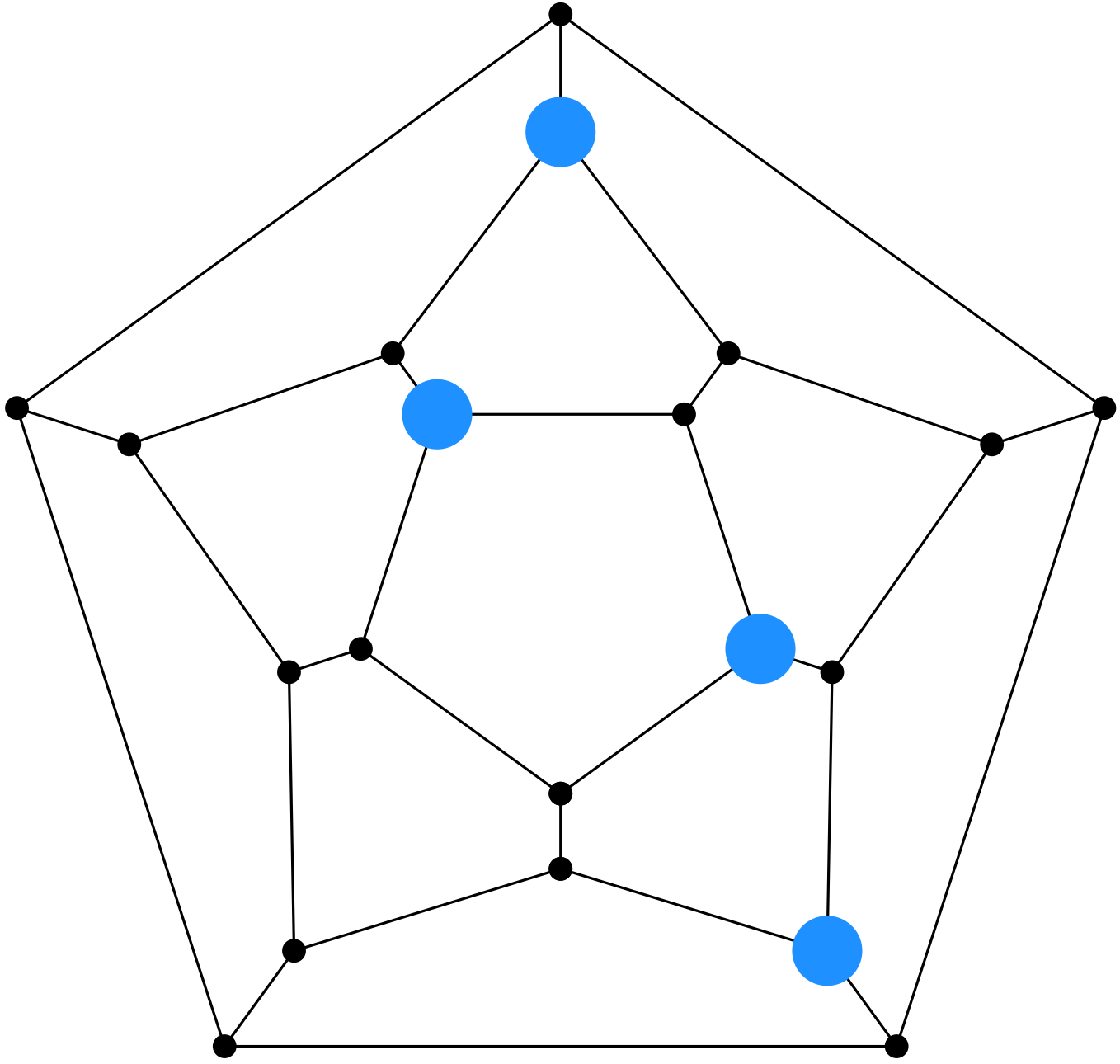

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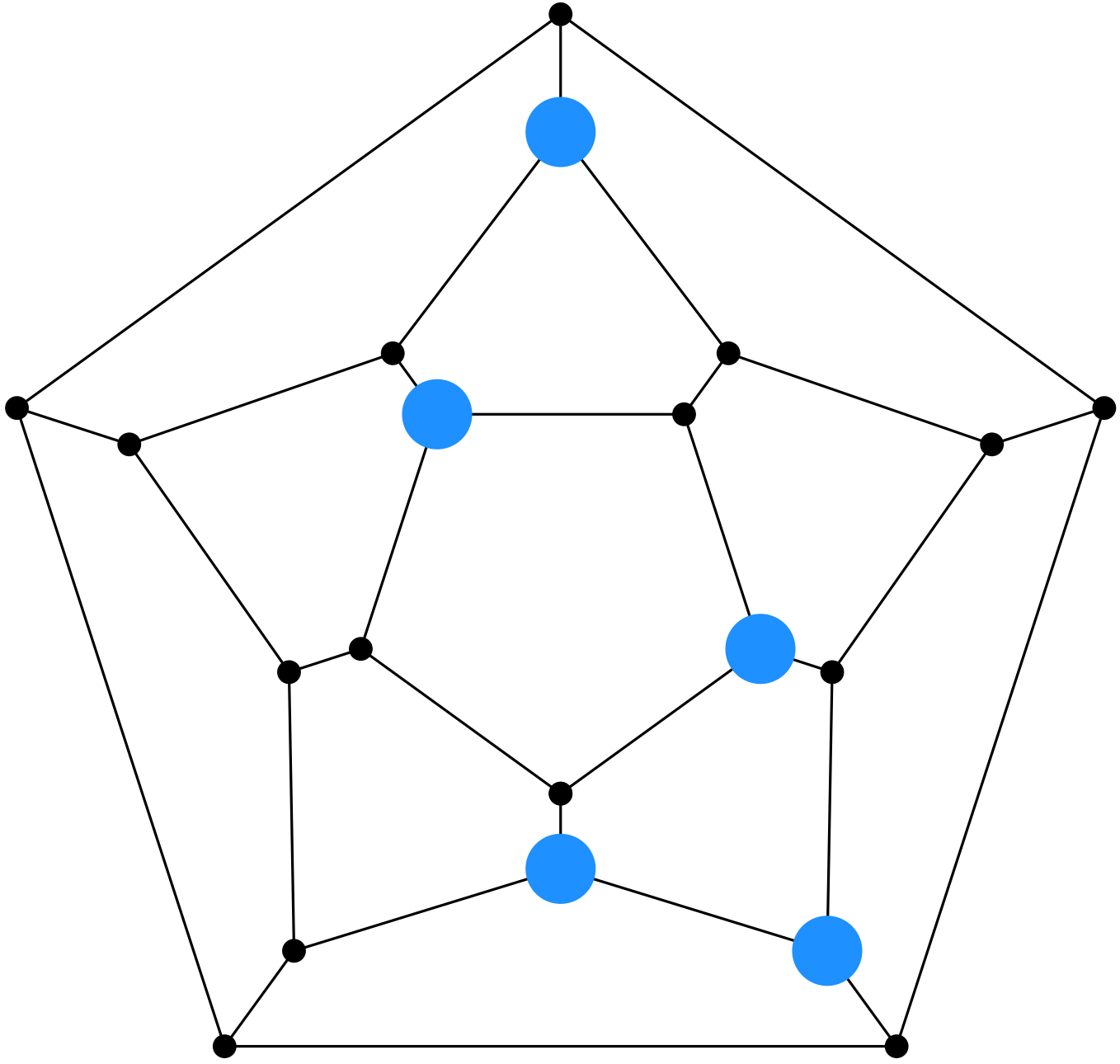

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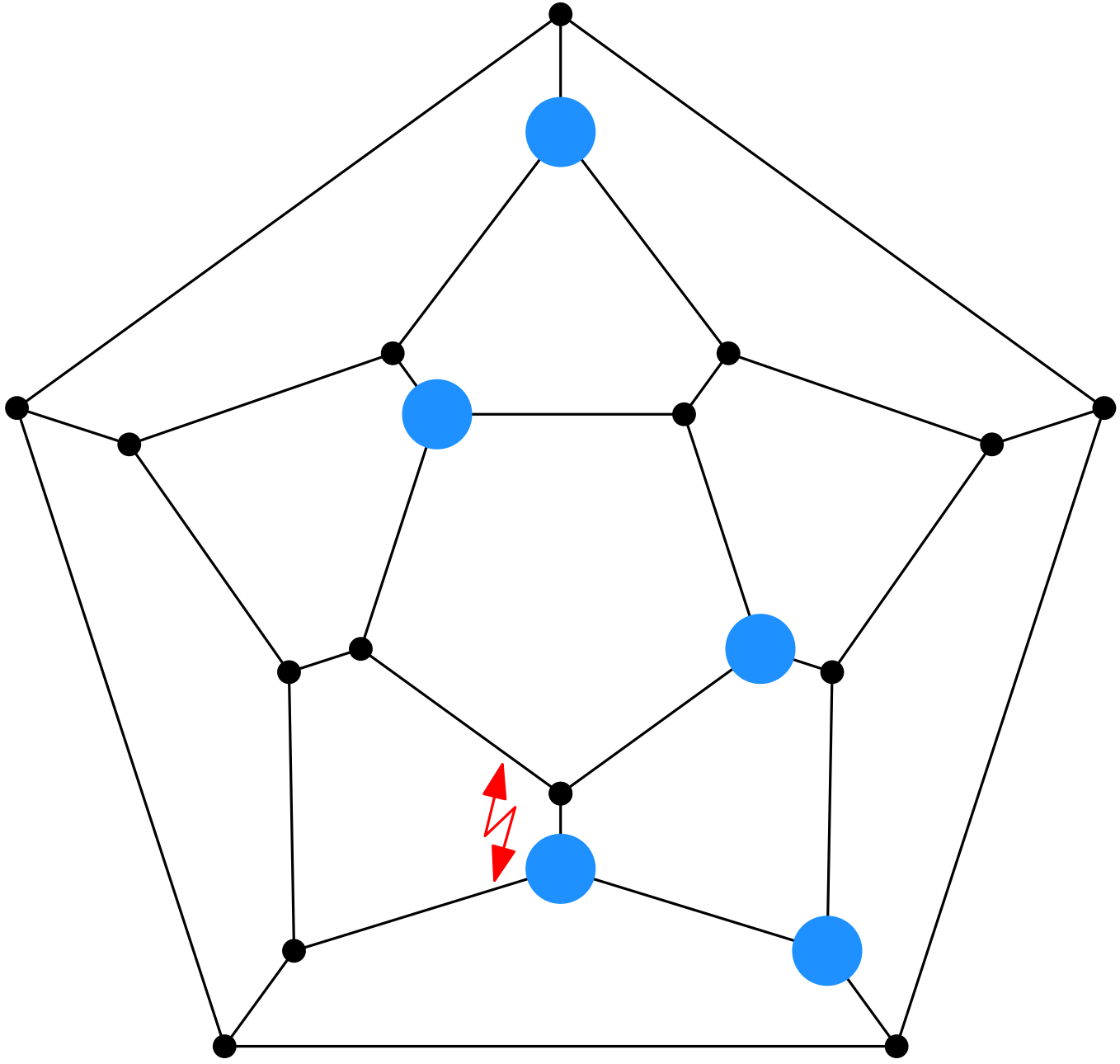

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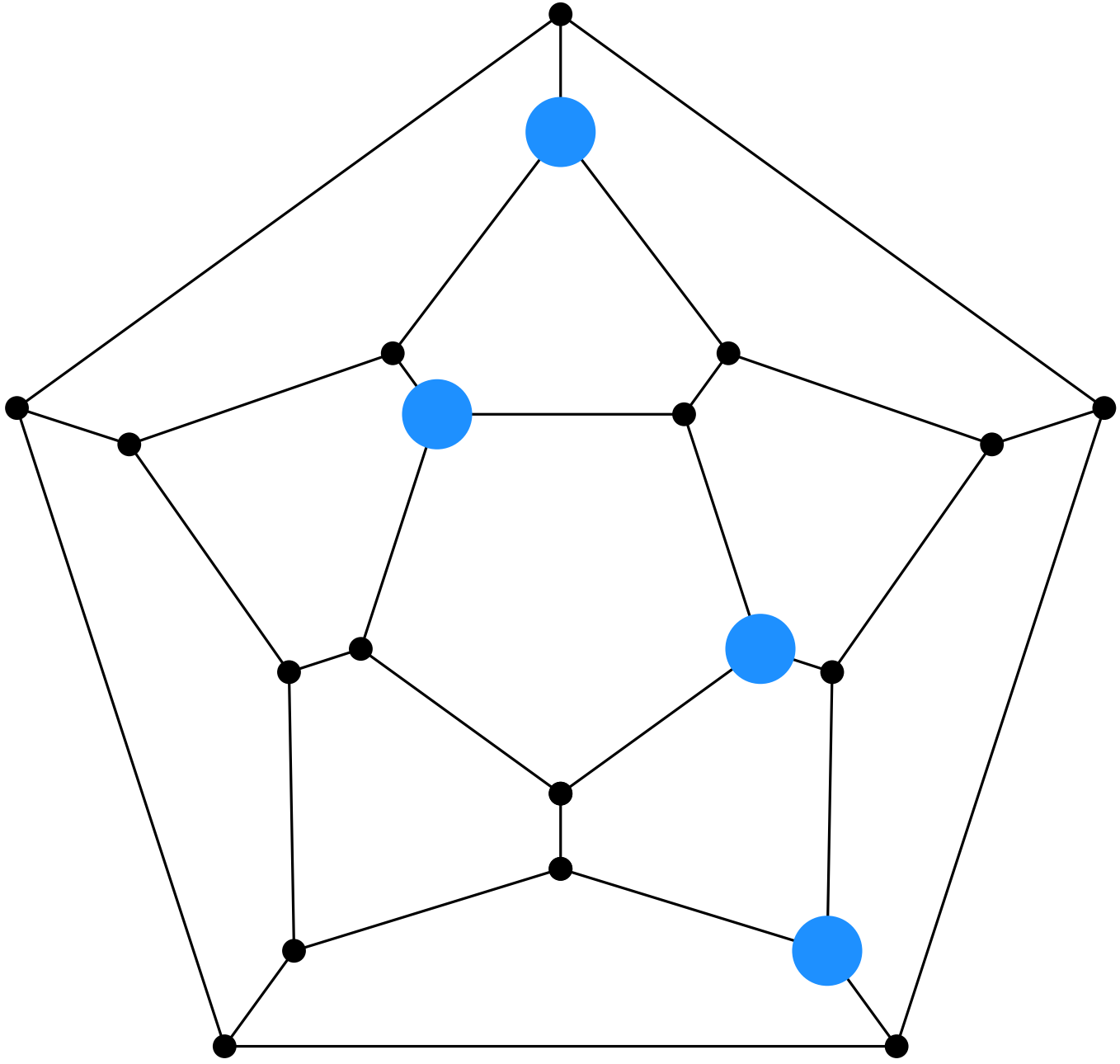

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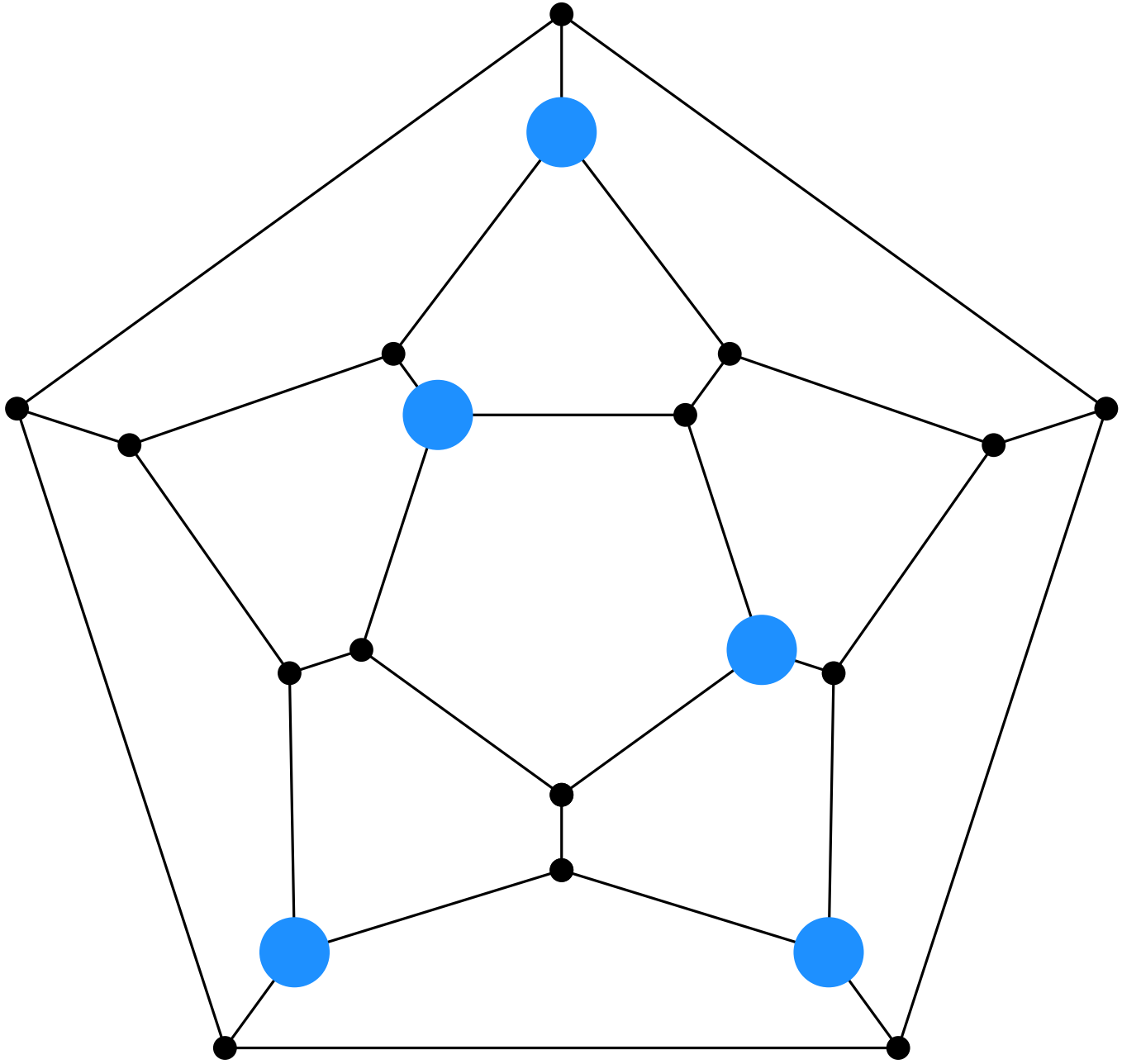

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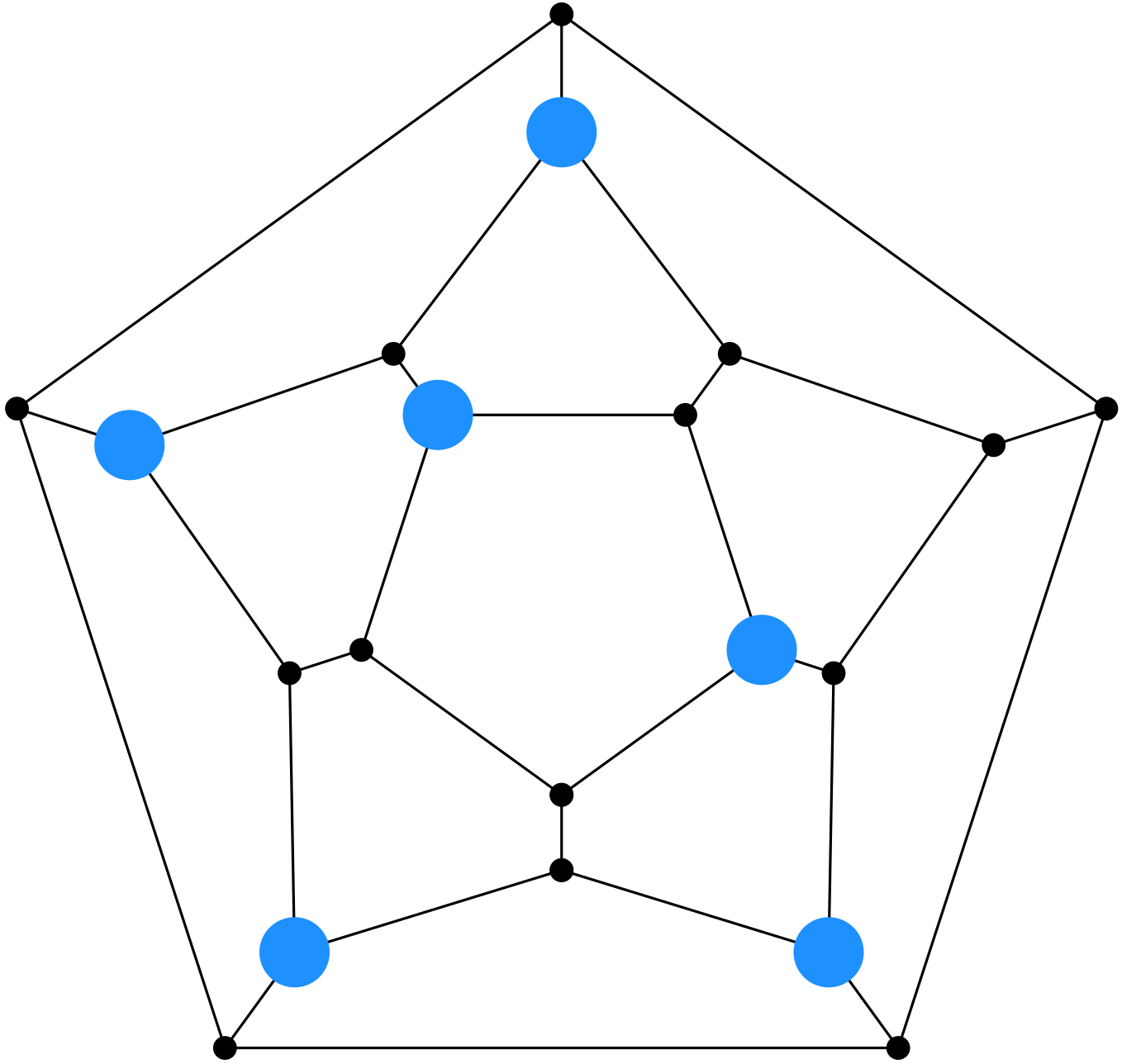

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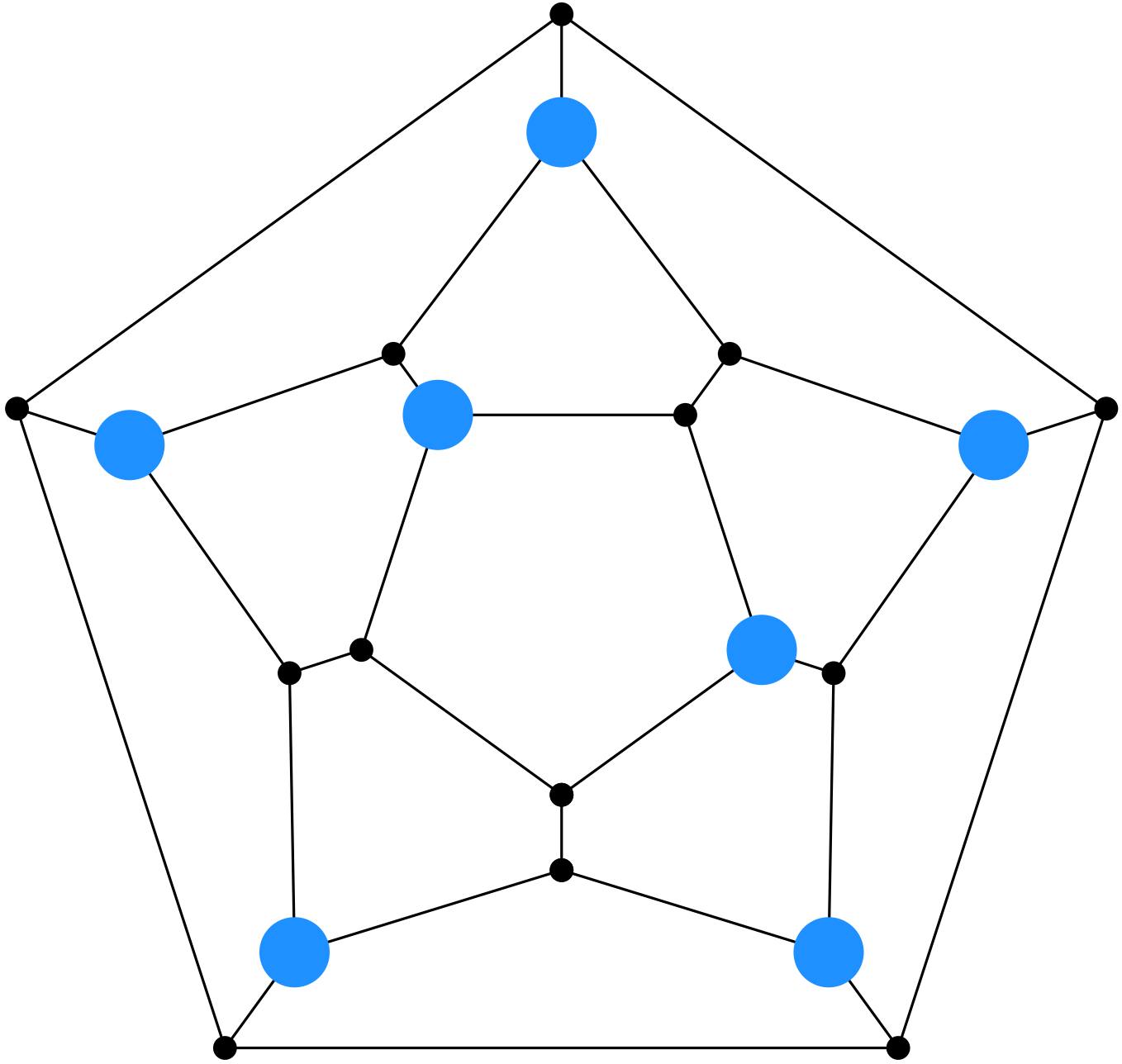

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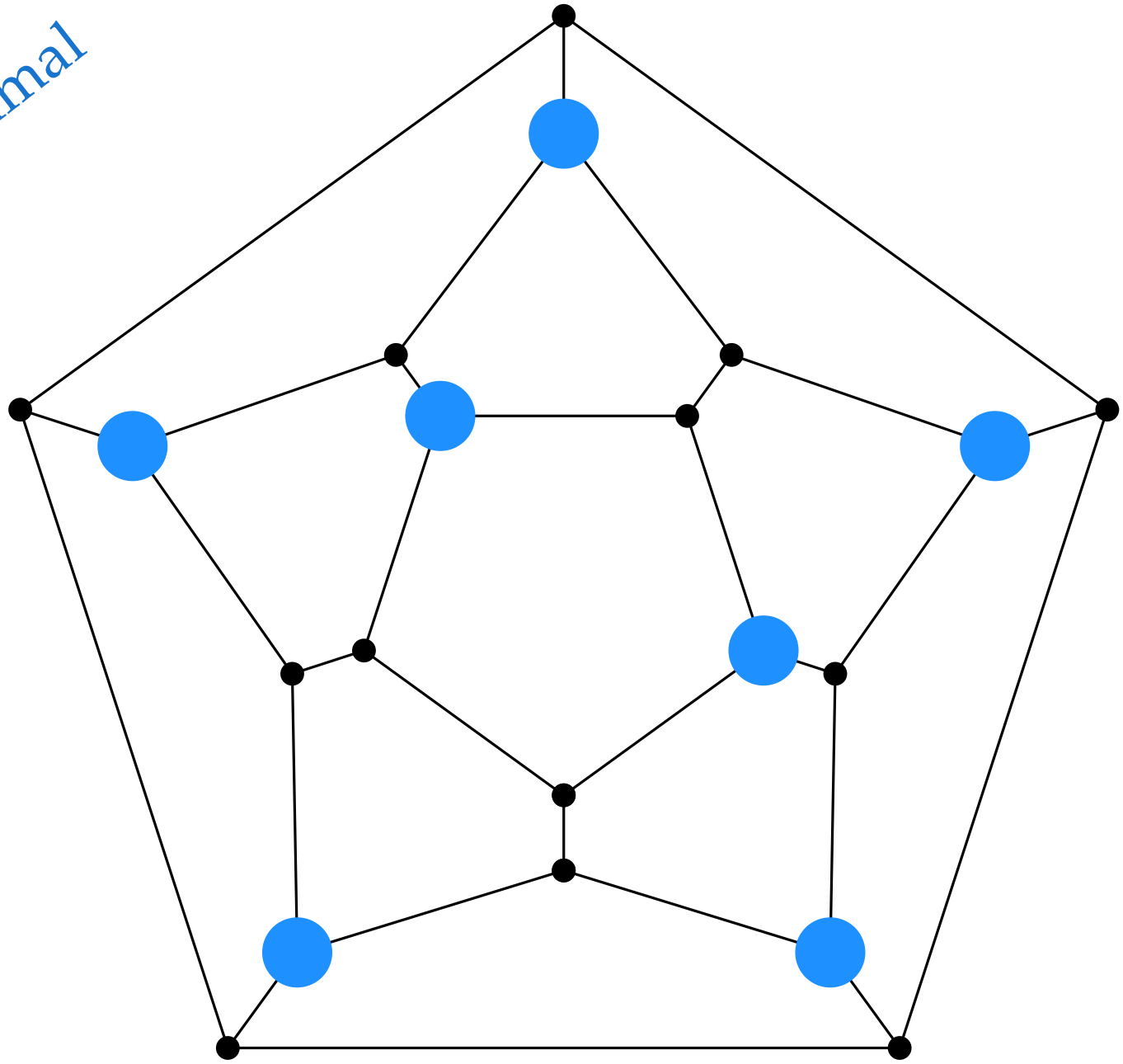

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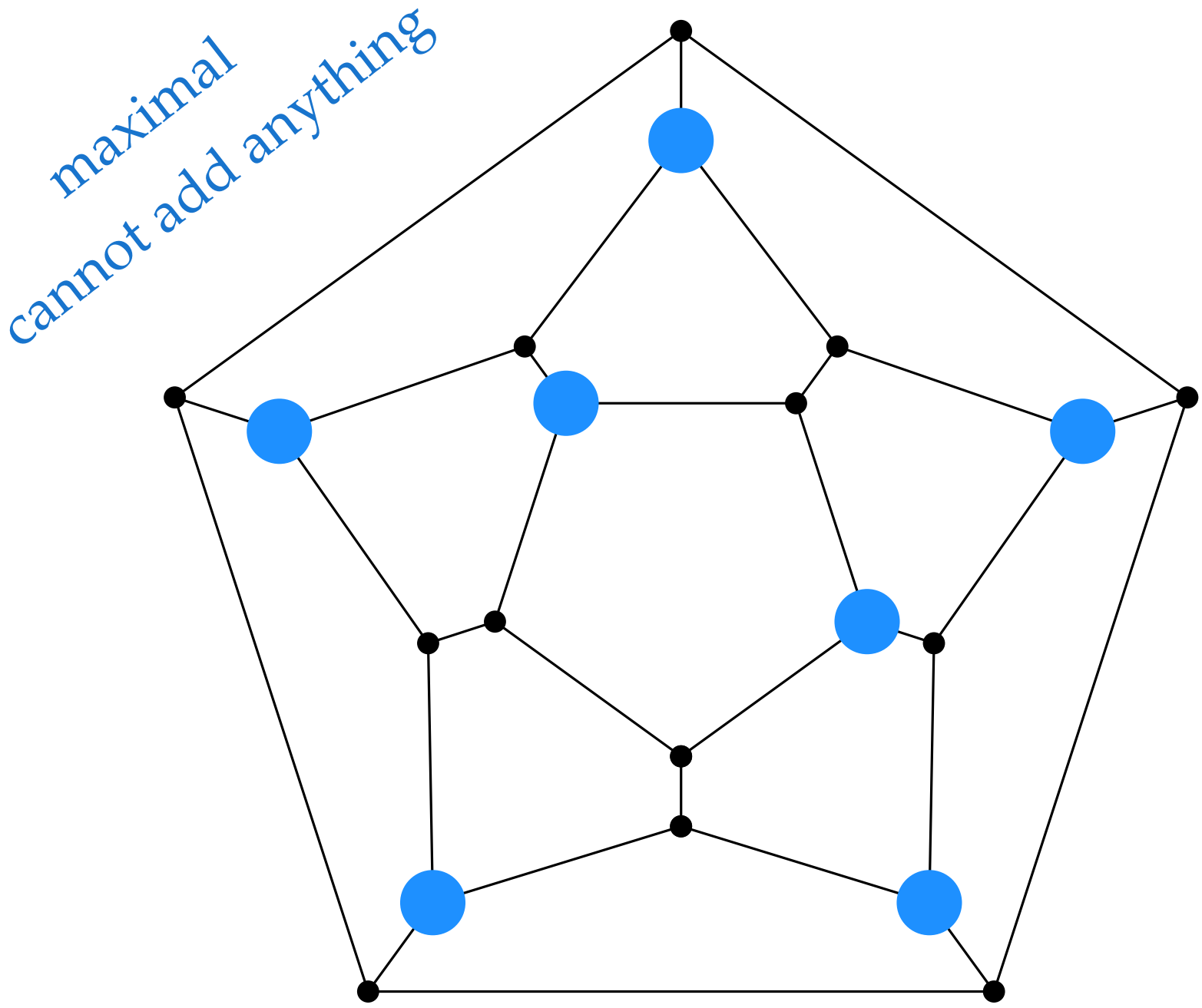
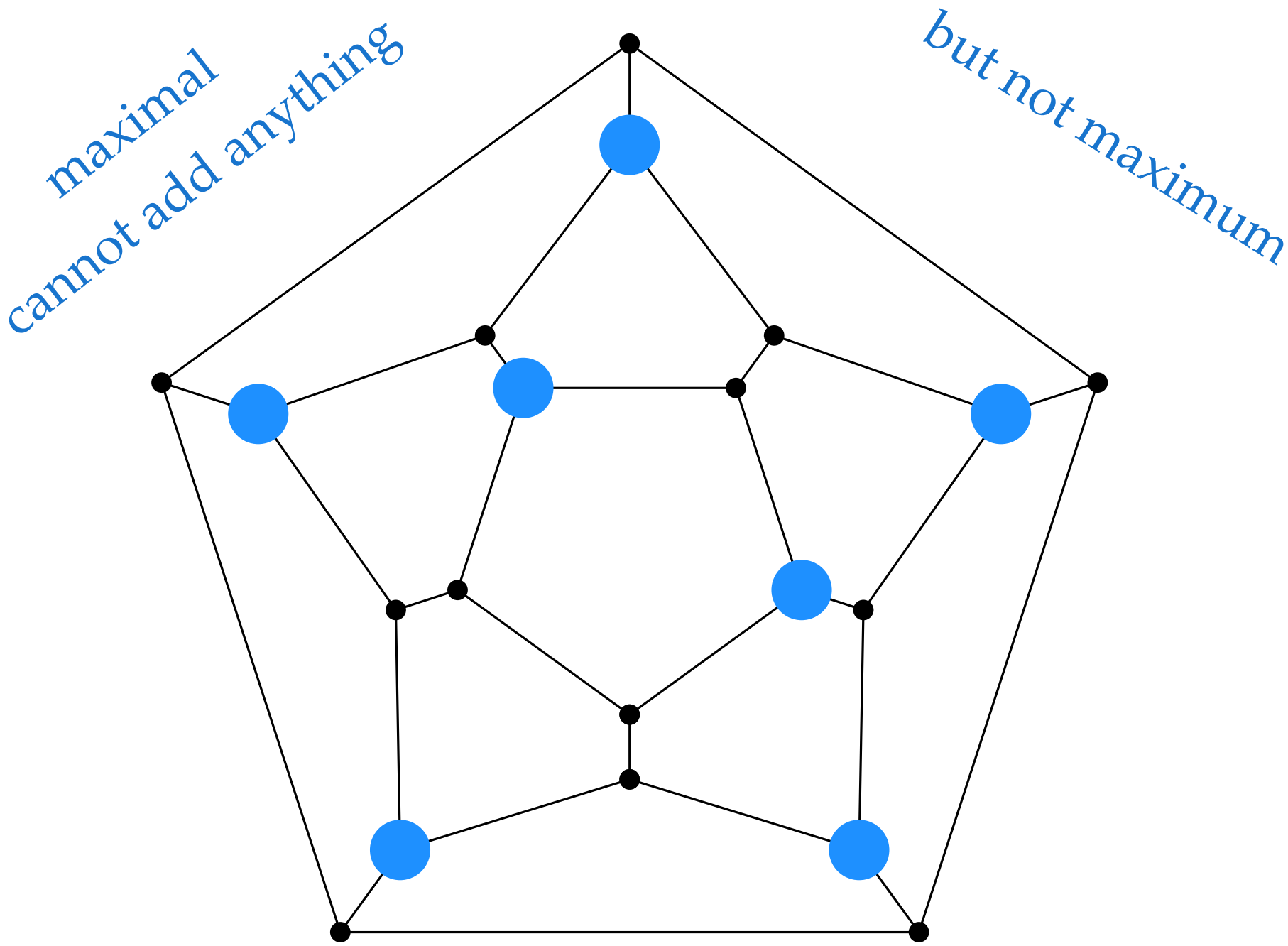


The Maximum Independent Set Problem



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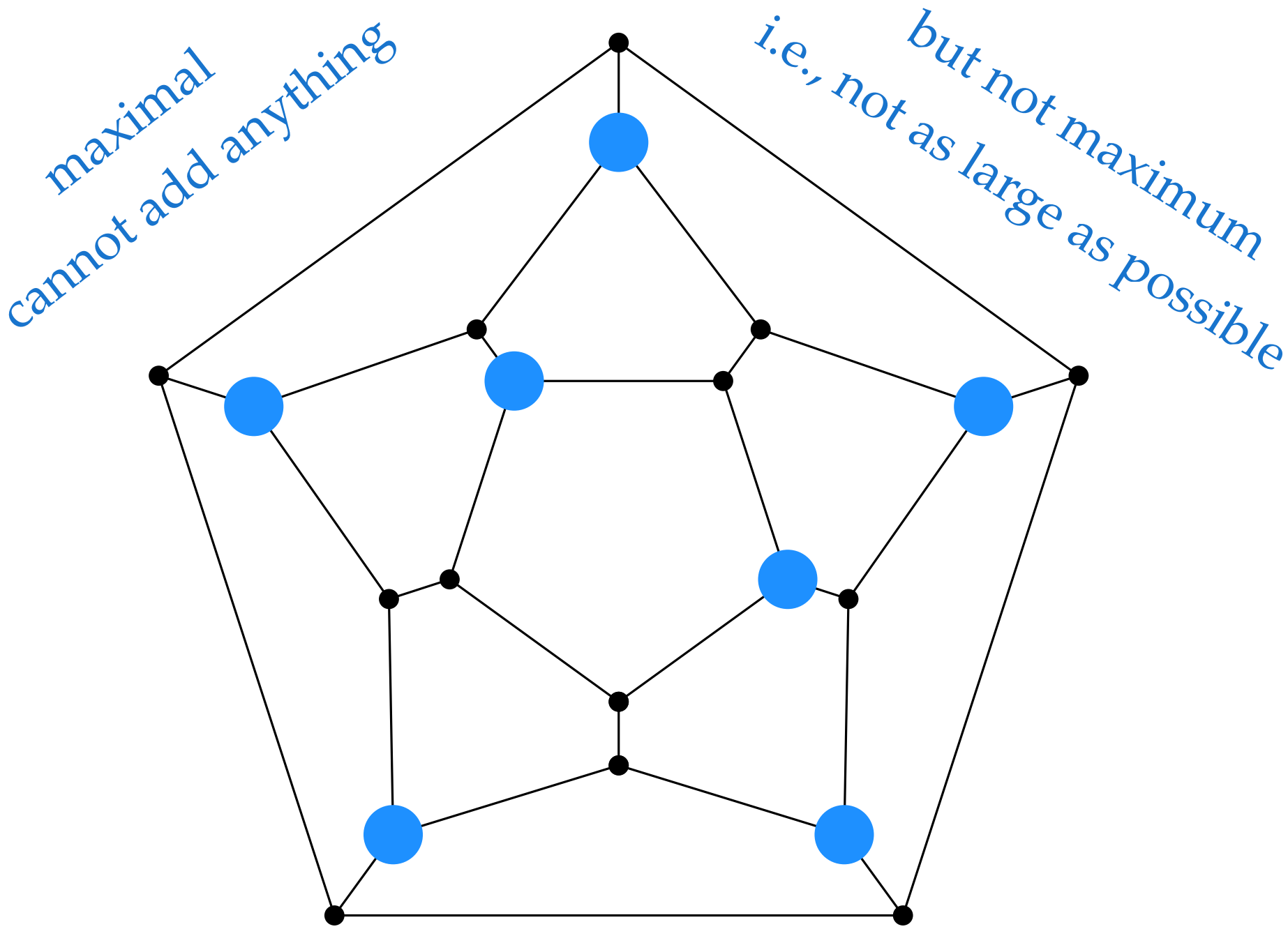

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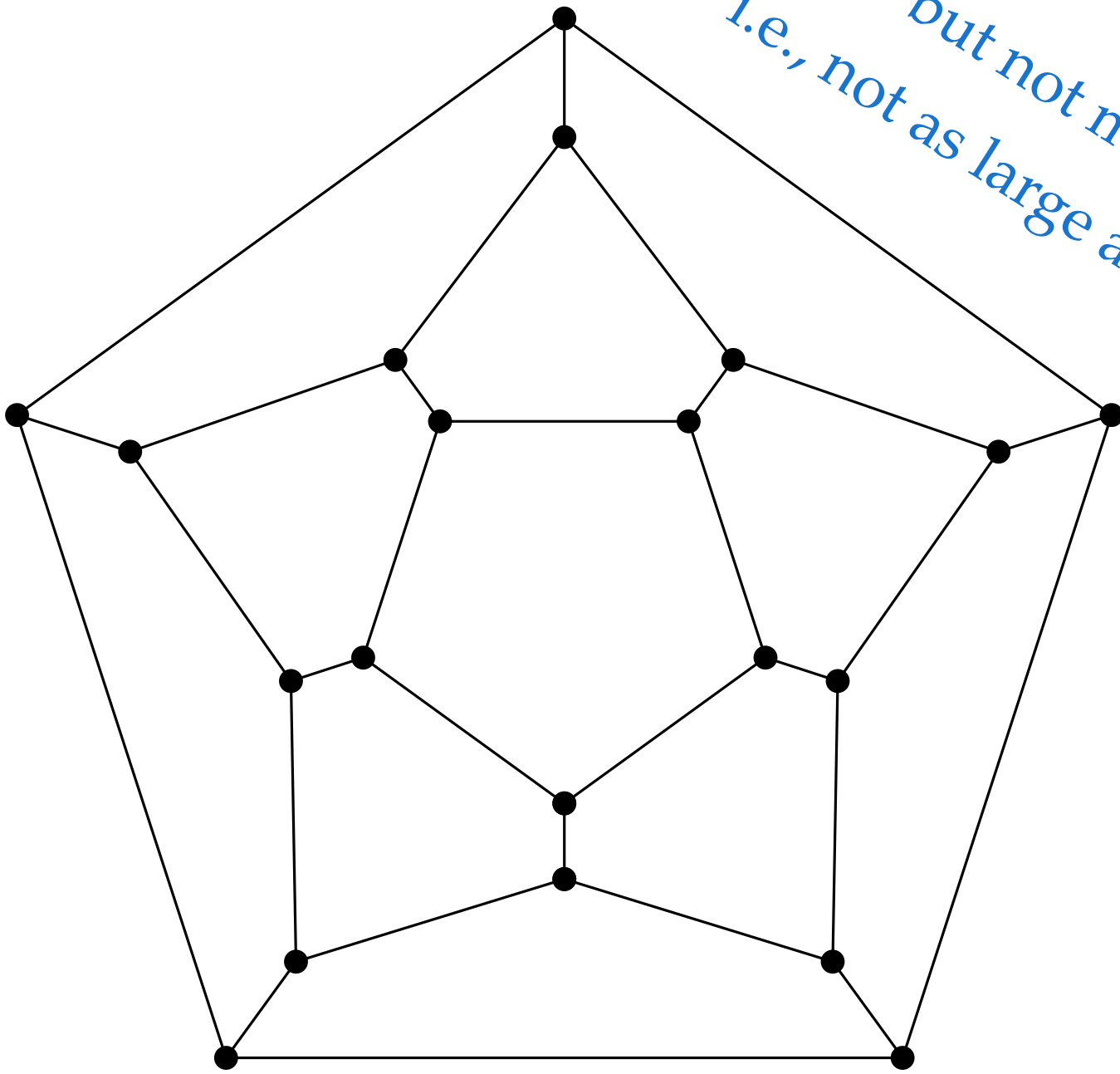
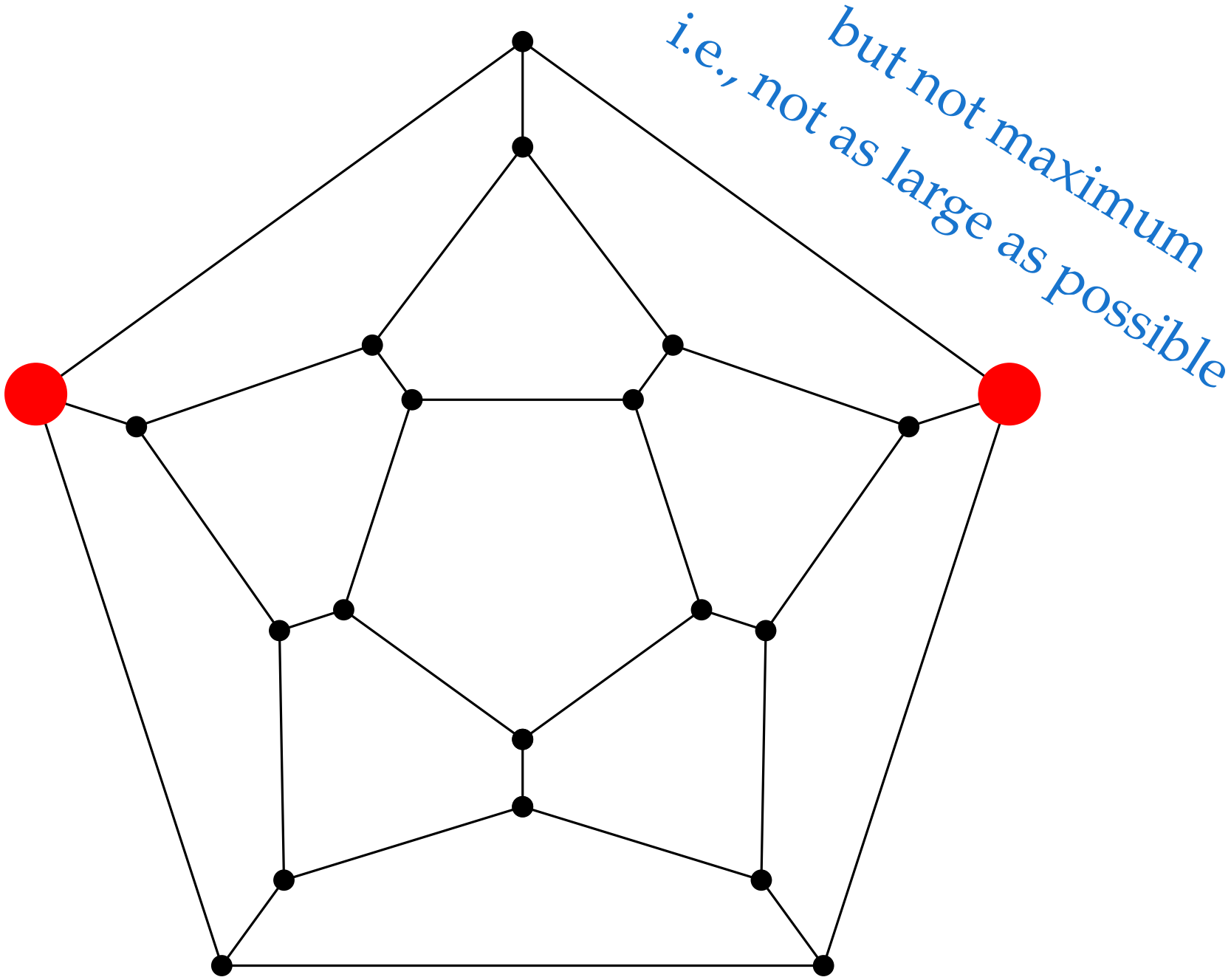


but not maximum

i.e., not as large as possible

The Maximum Independent Set Problem



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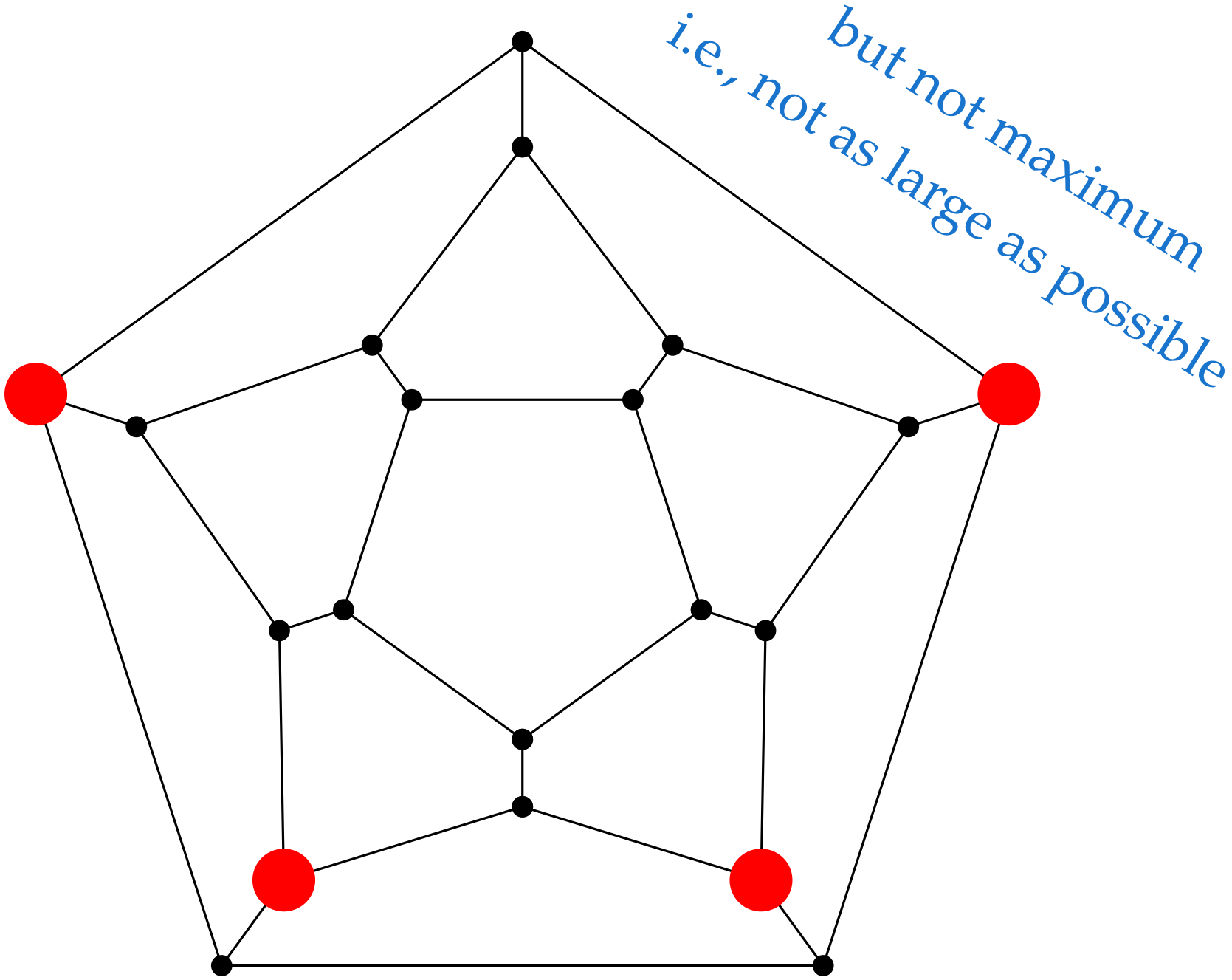

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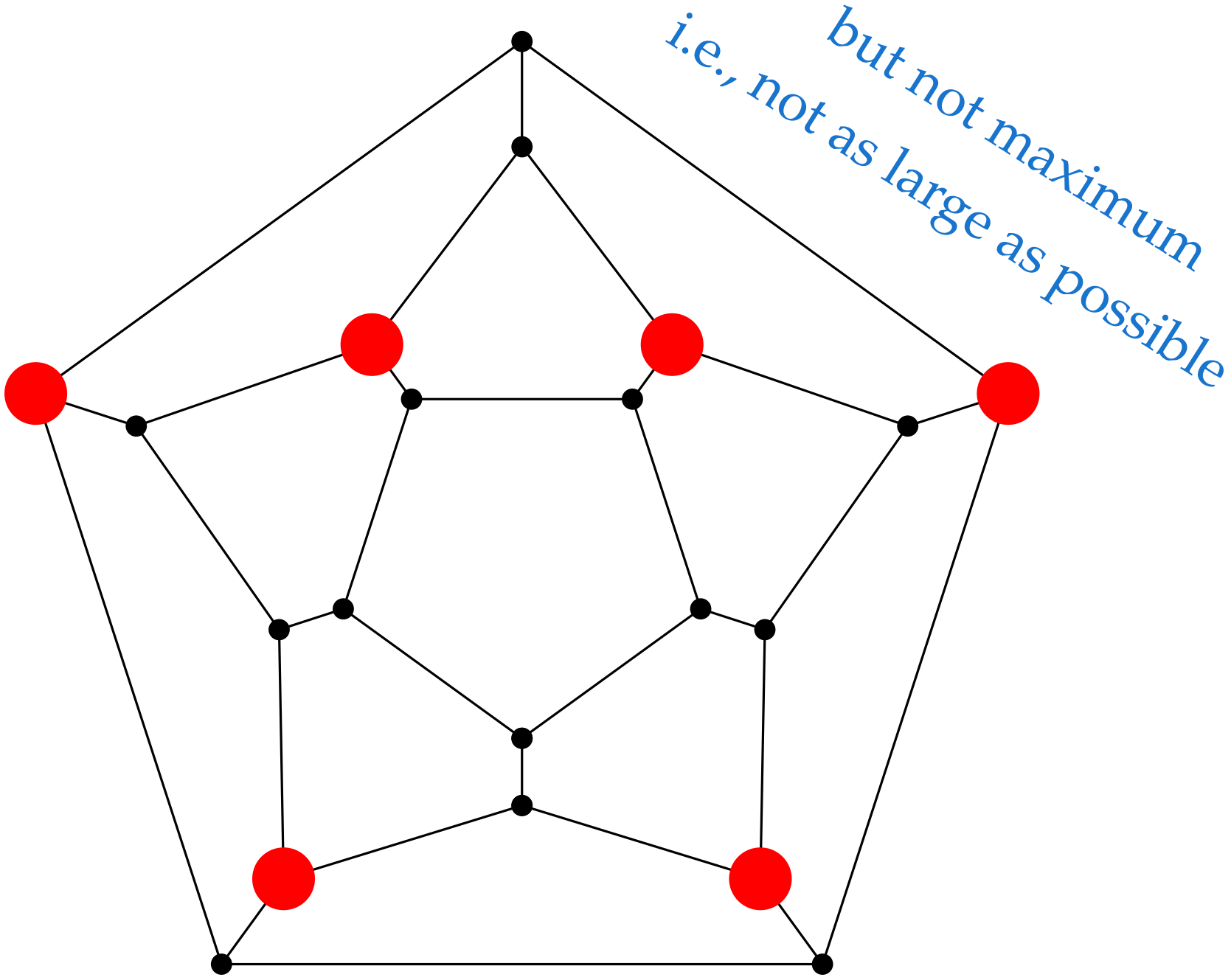
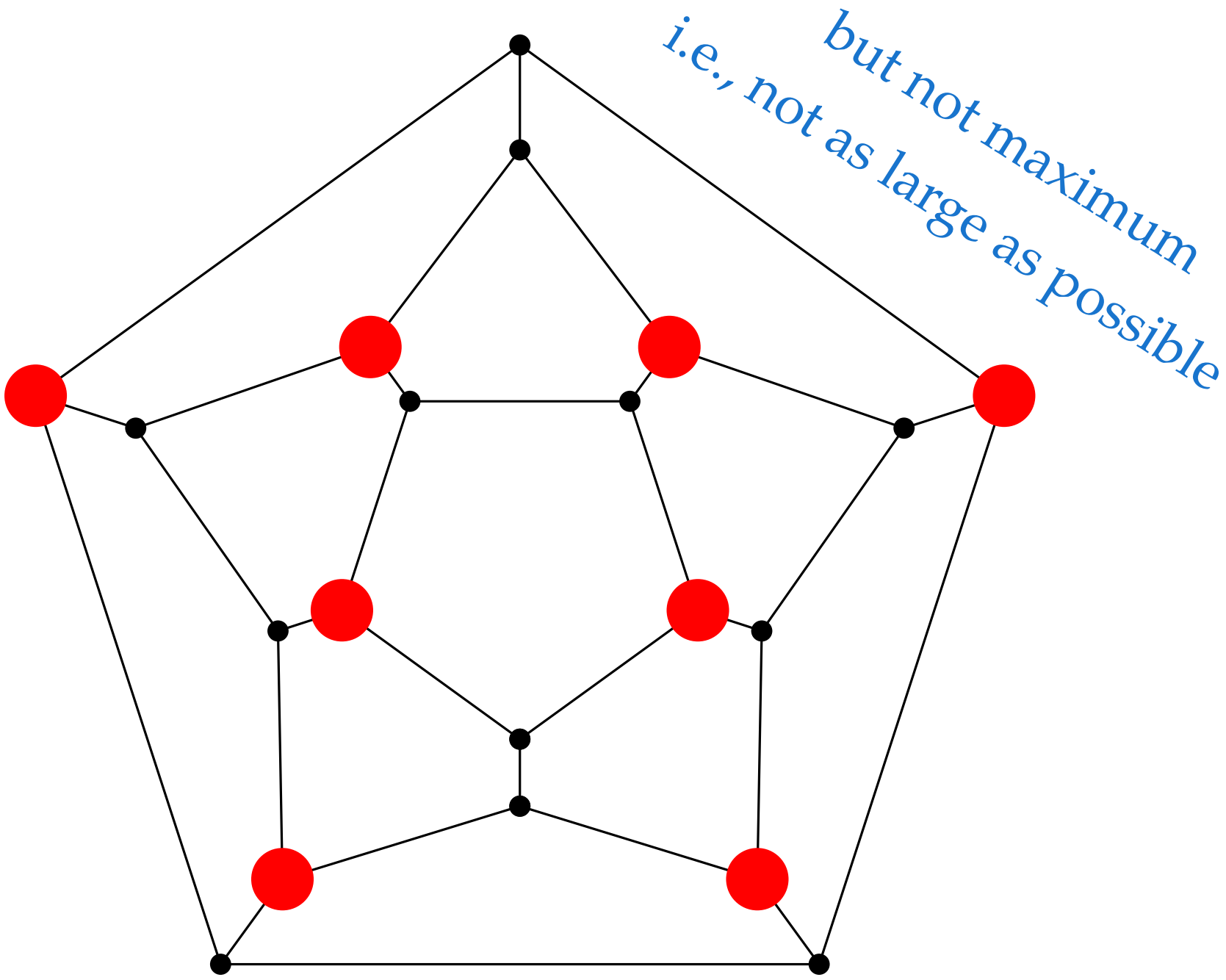


The Maximum Independent Set Problem



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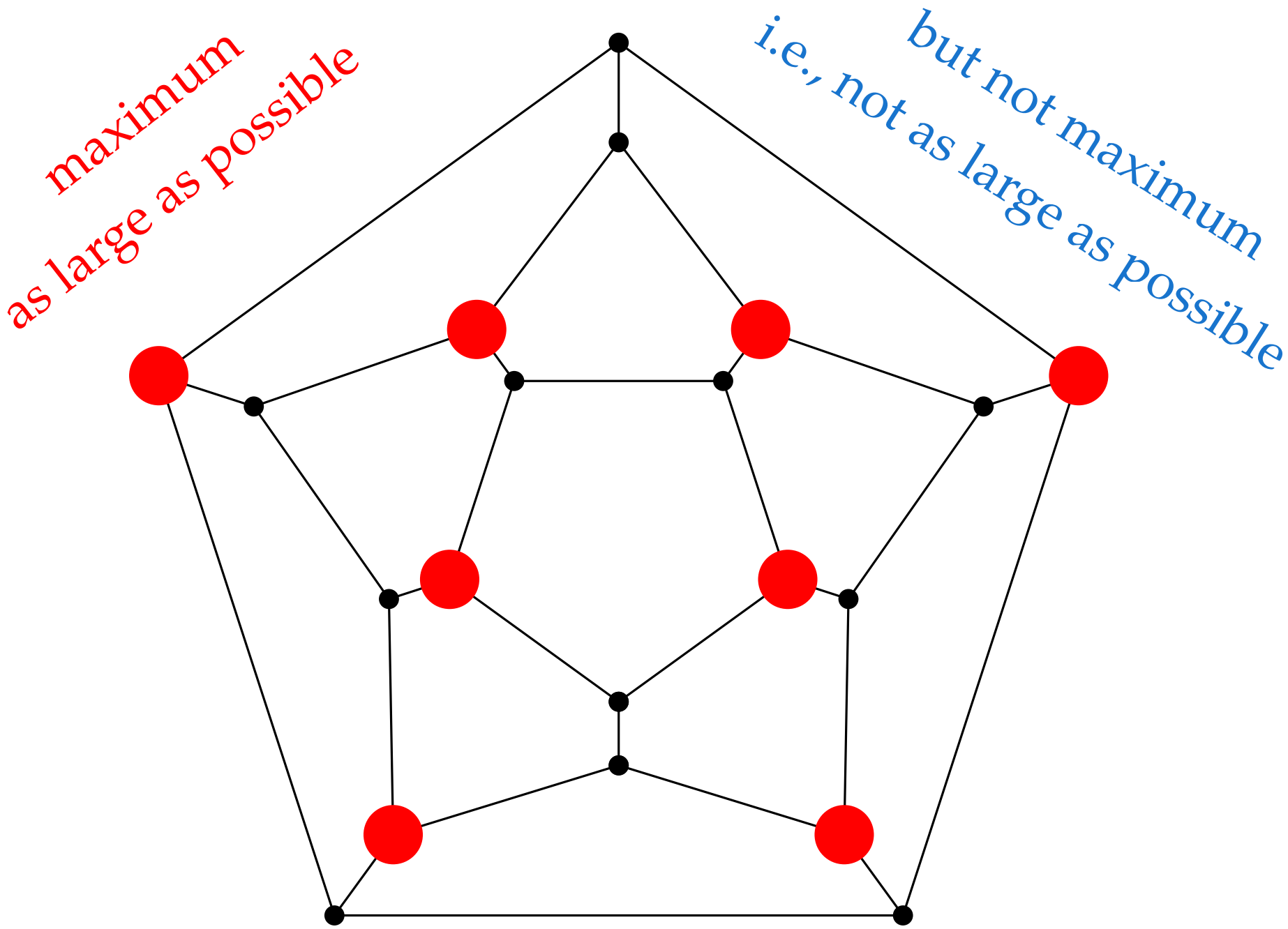
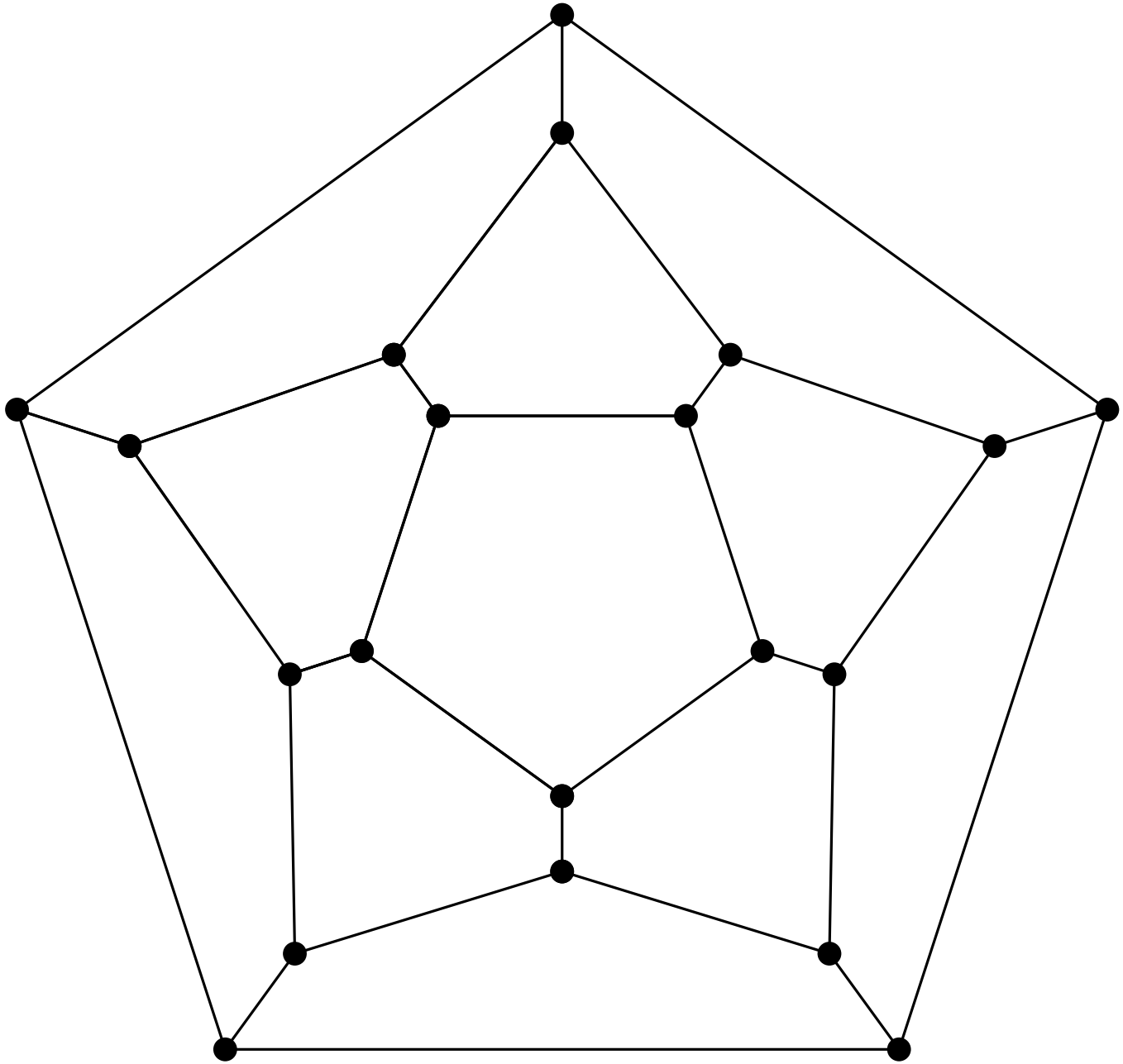


Using Congestion Games to Solve Max-IS



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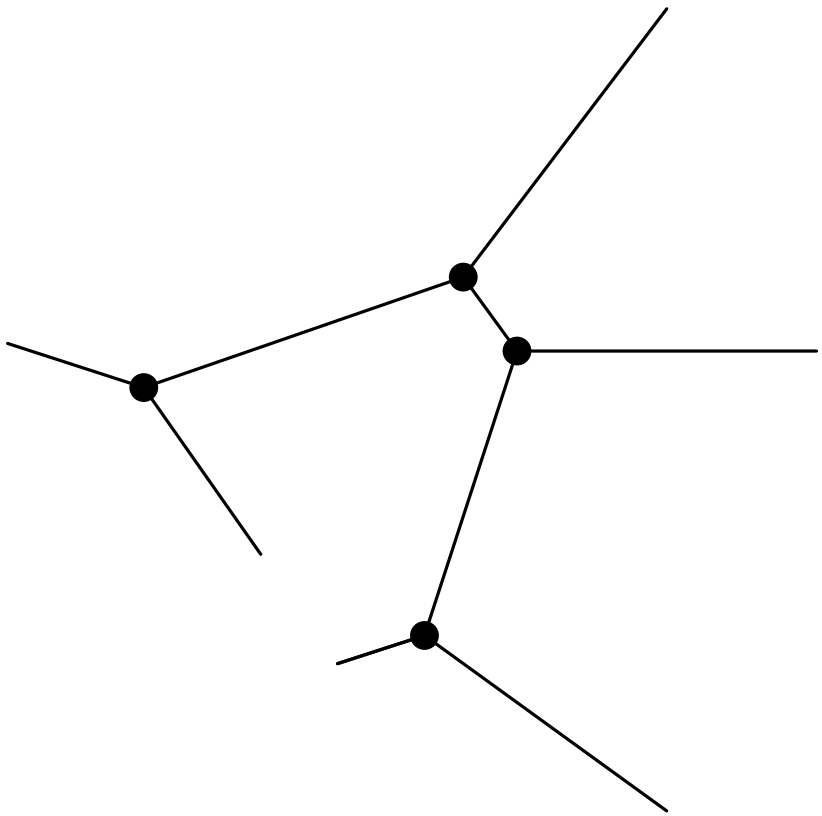
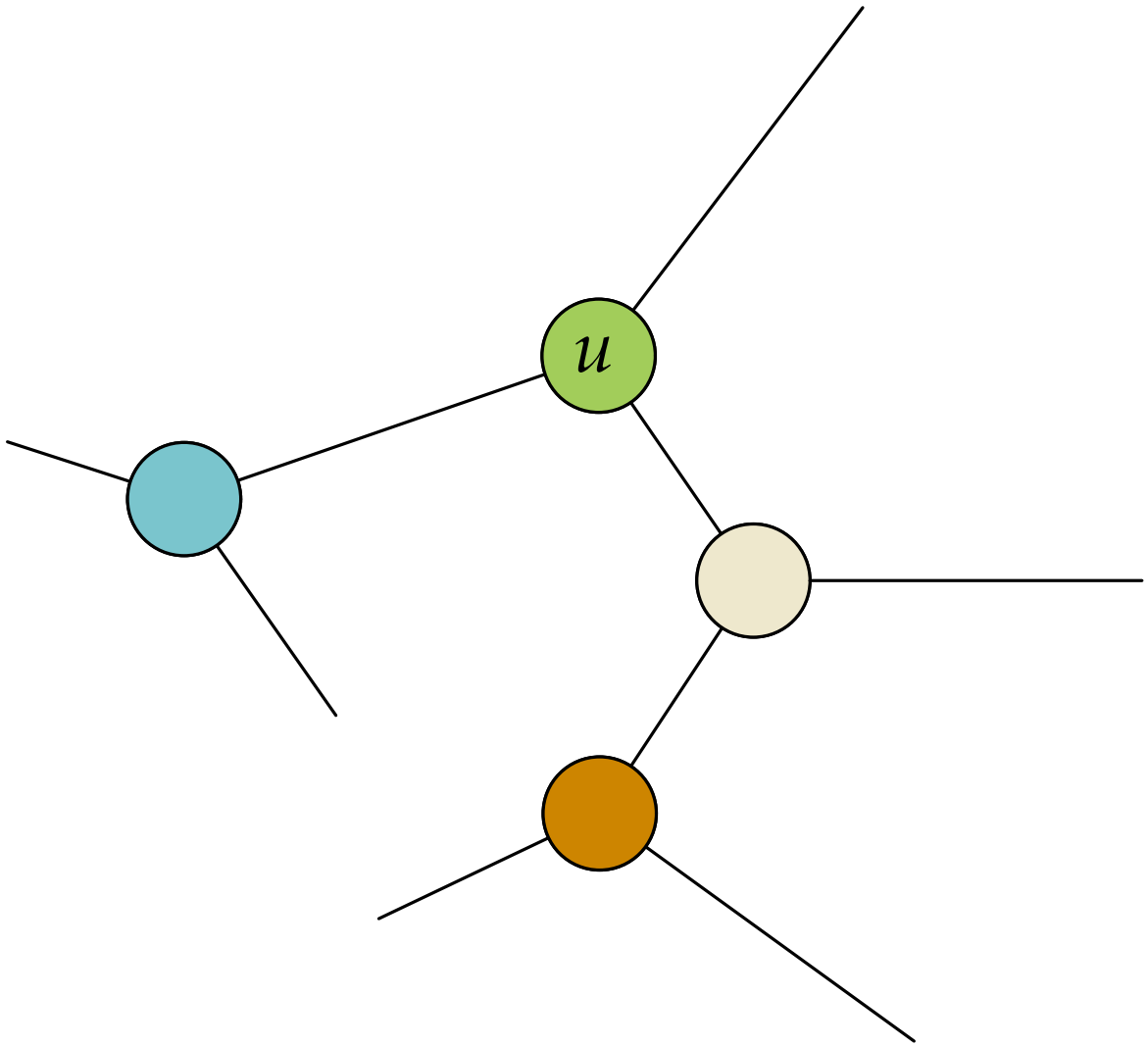


Using Congestion Games to Solve Max-IS

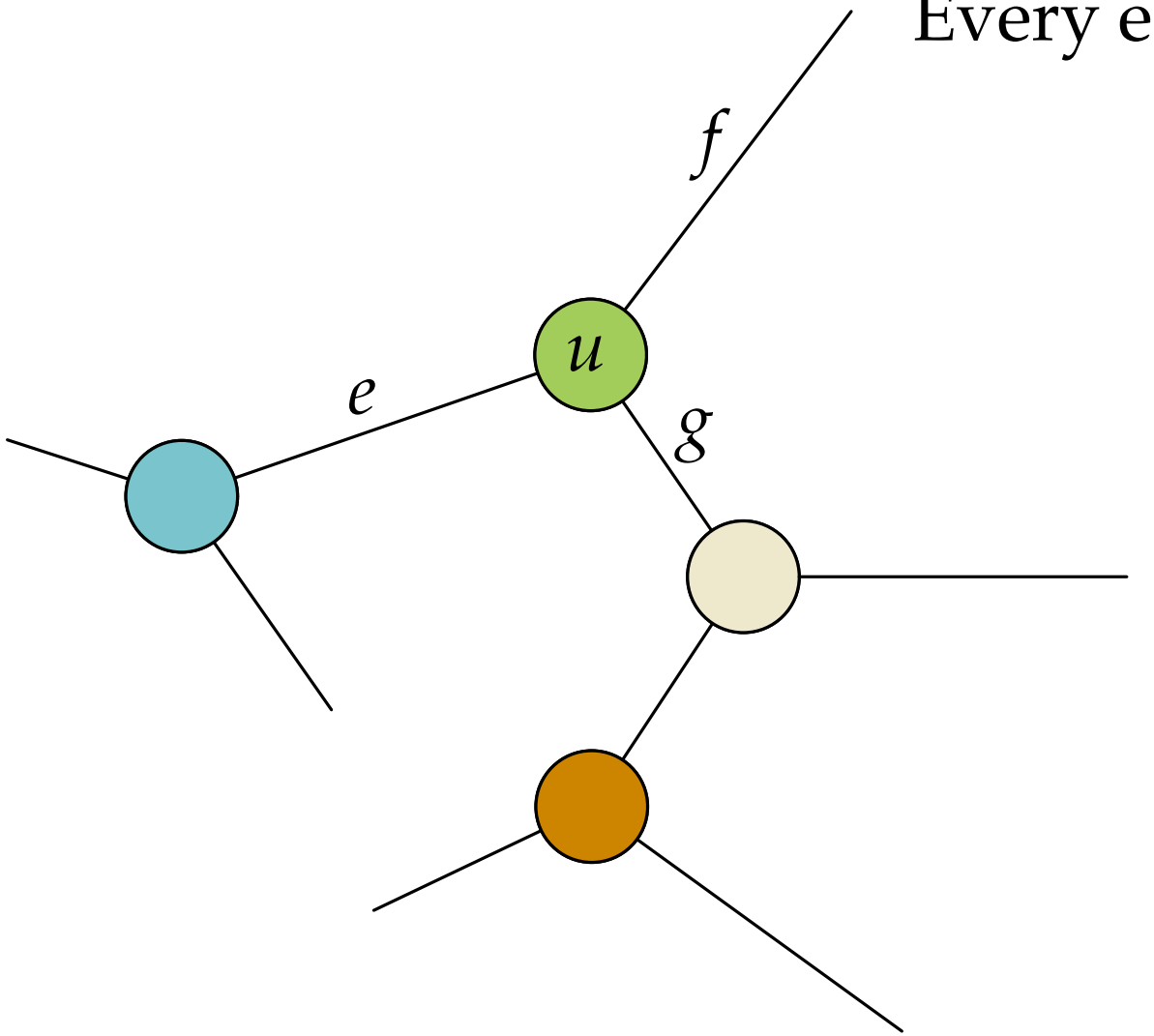
Every vertex is a player



Using Congestion Games to Solve Max-IS

Every vertex is a player

Every edge is a resource

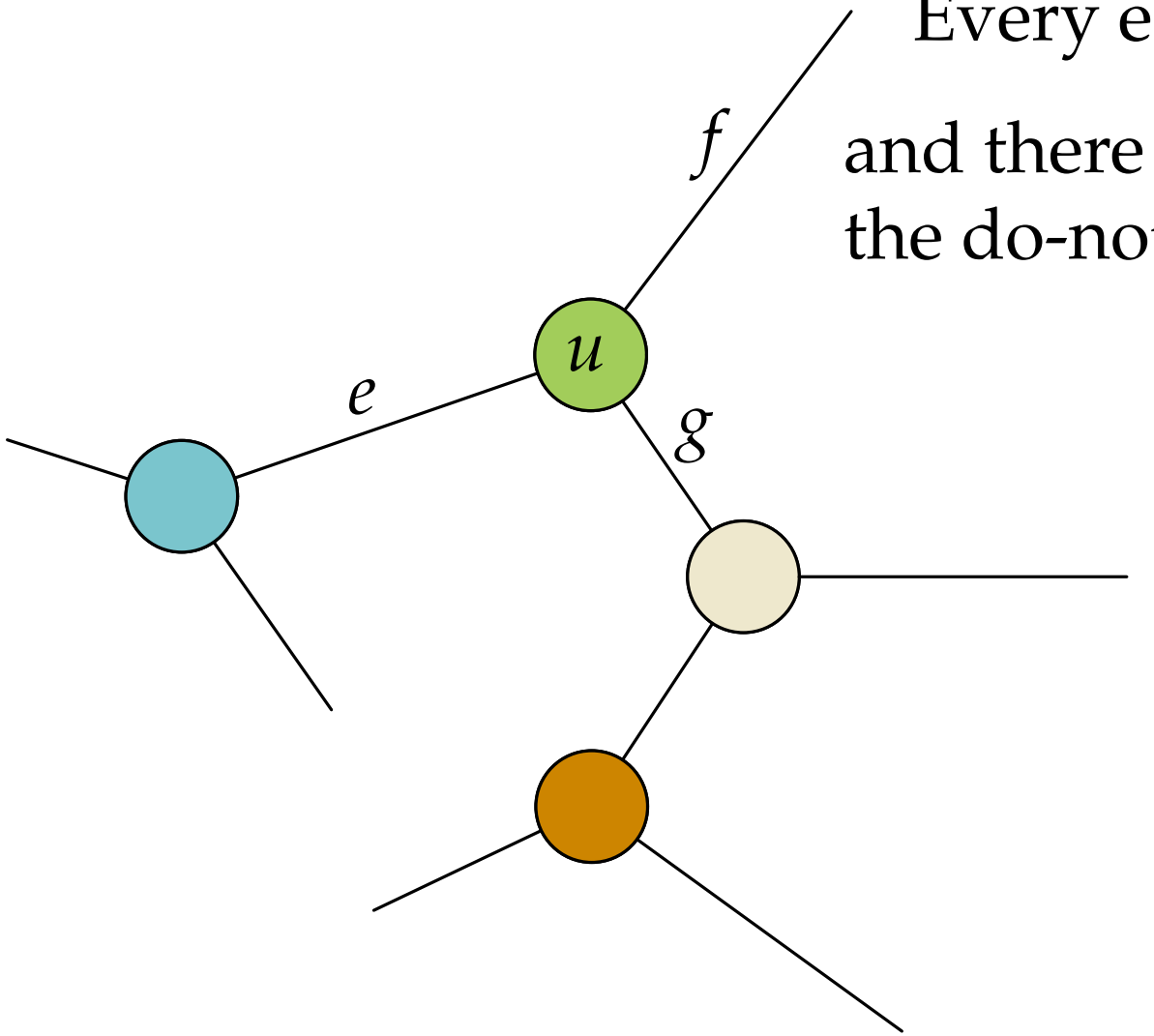


Using Congestion Games to Solve Max-IS

Every vertex is a player

Every edge is a resource

and there is another resource, the do-nothing-box.



$\text{delay}(\text{DNB}, i) = 1$

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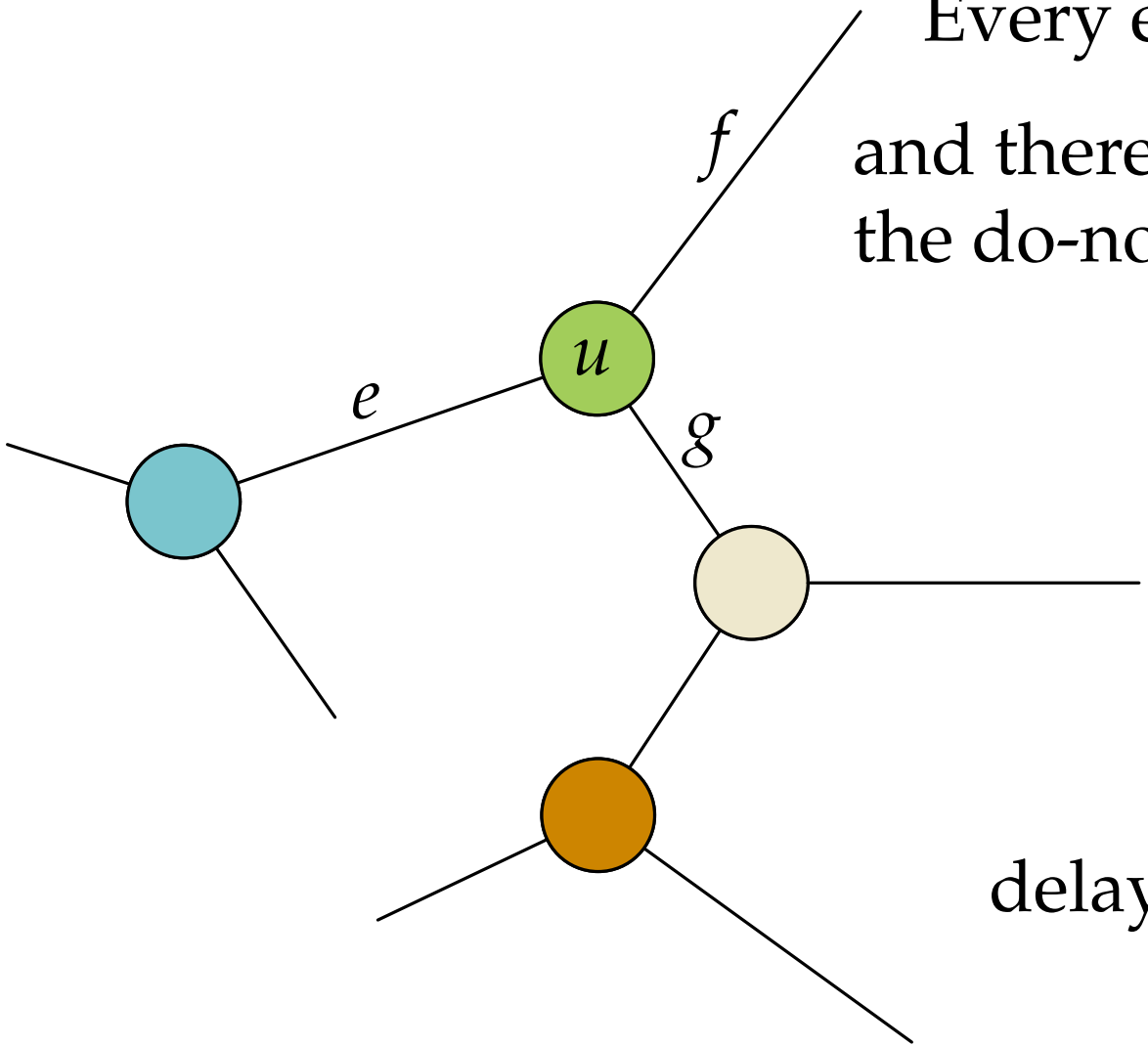

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$$\text{delay}(e, i) = \begin{cases} 0 & \text{if } i \leq 1 \\ \infty & \text{if } i \geq 2\,. \end{cases}$$

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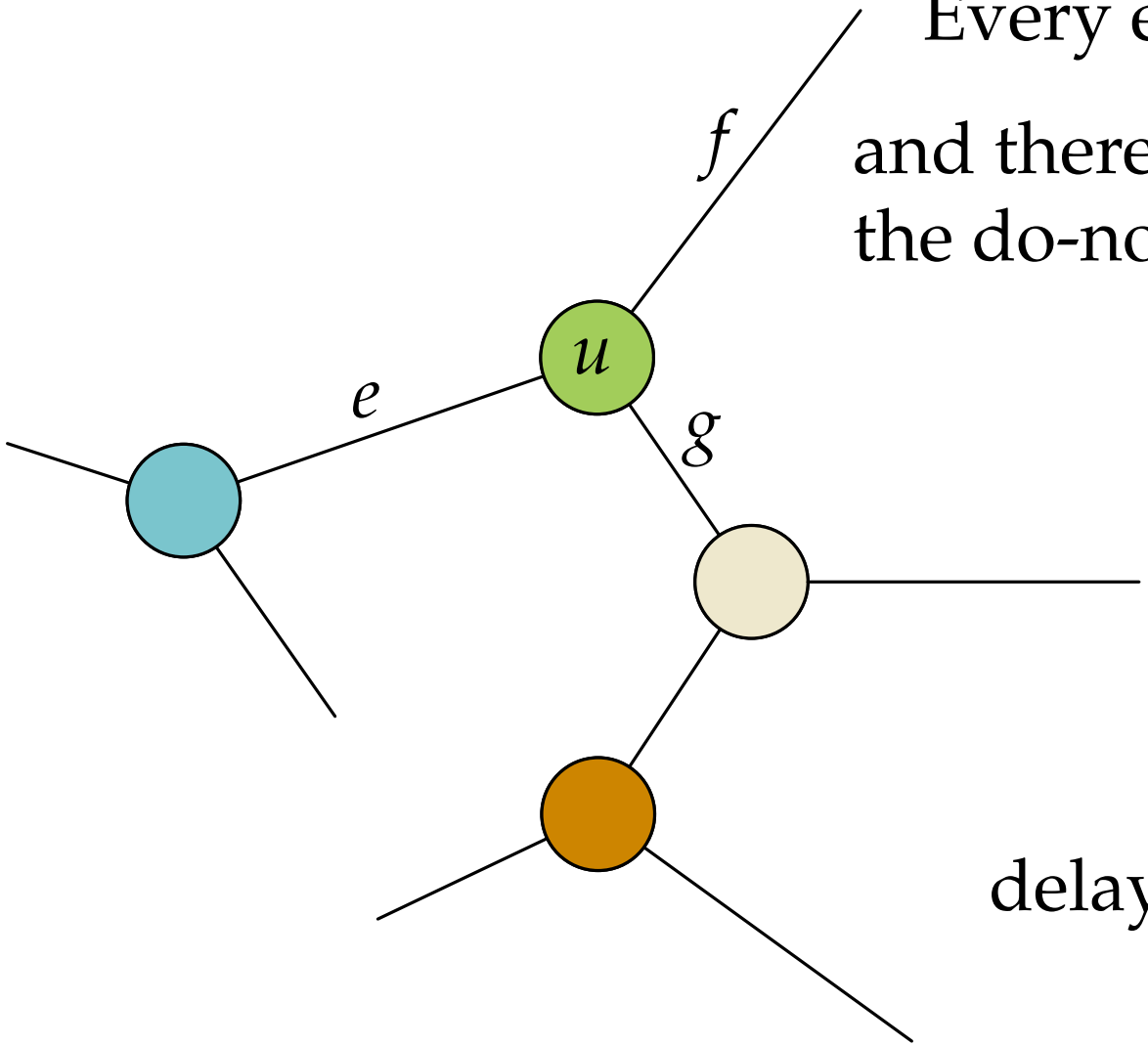

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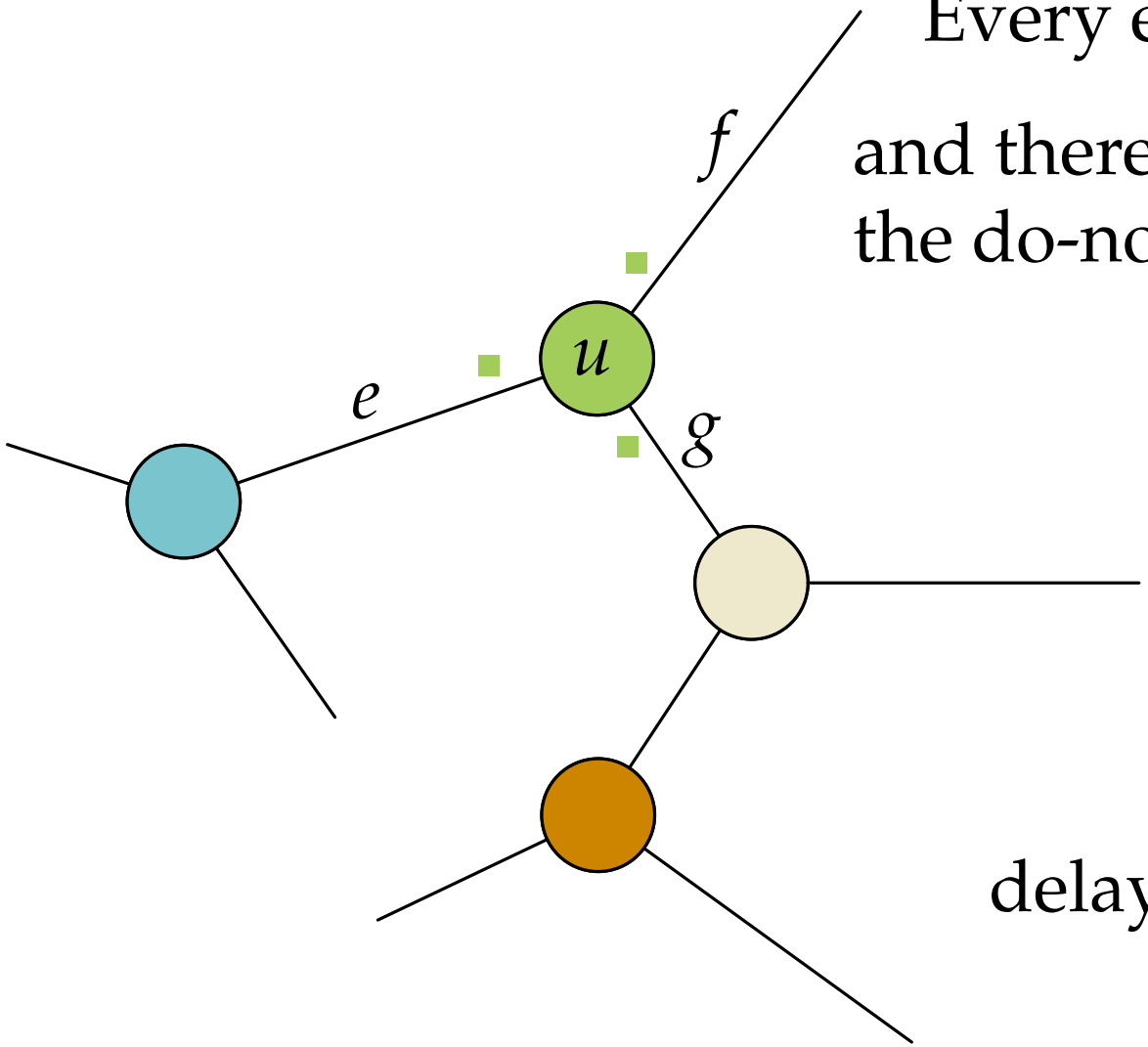

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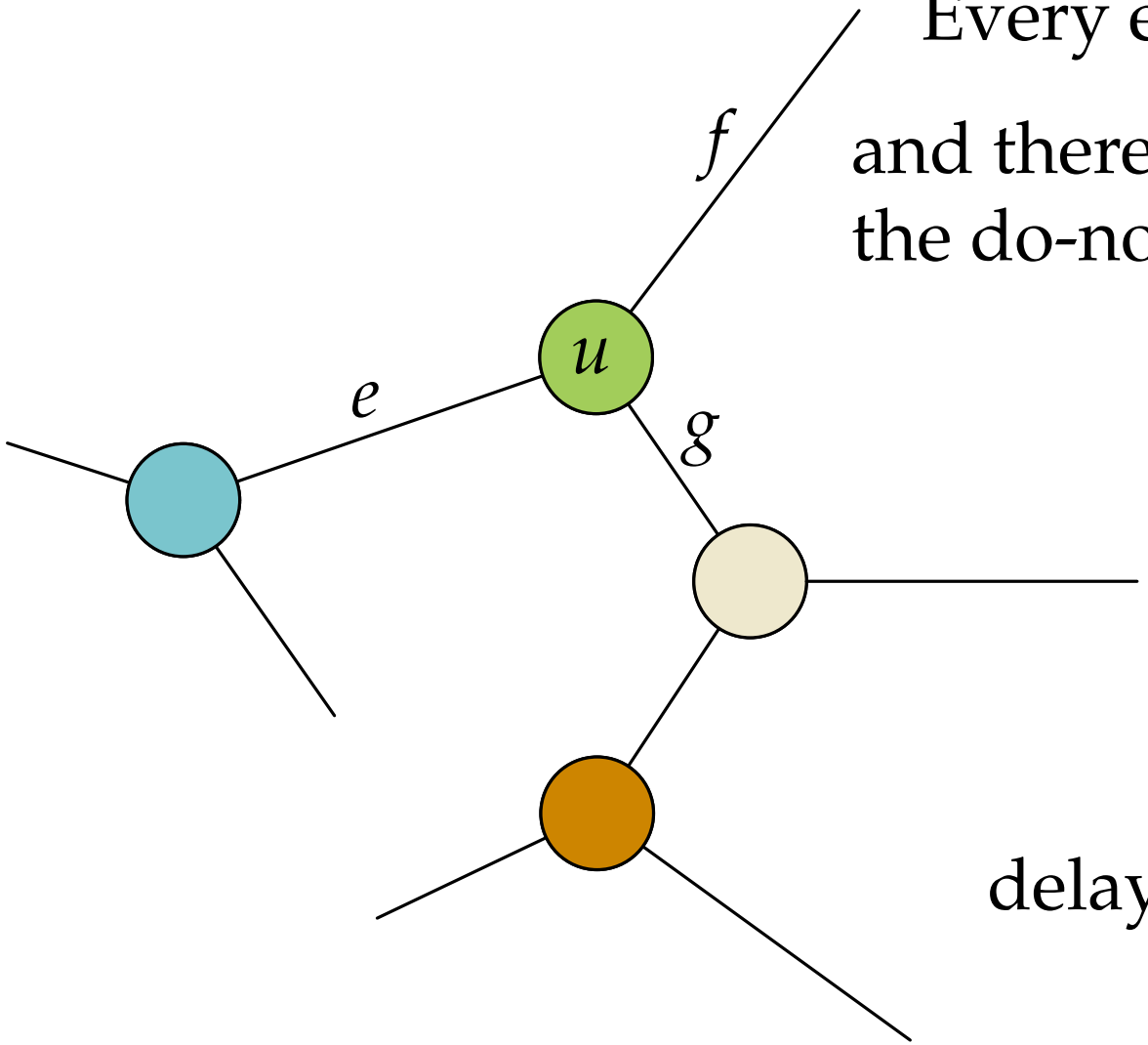

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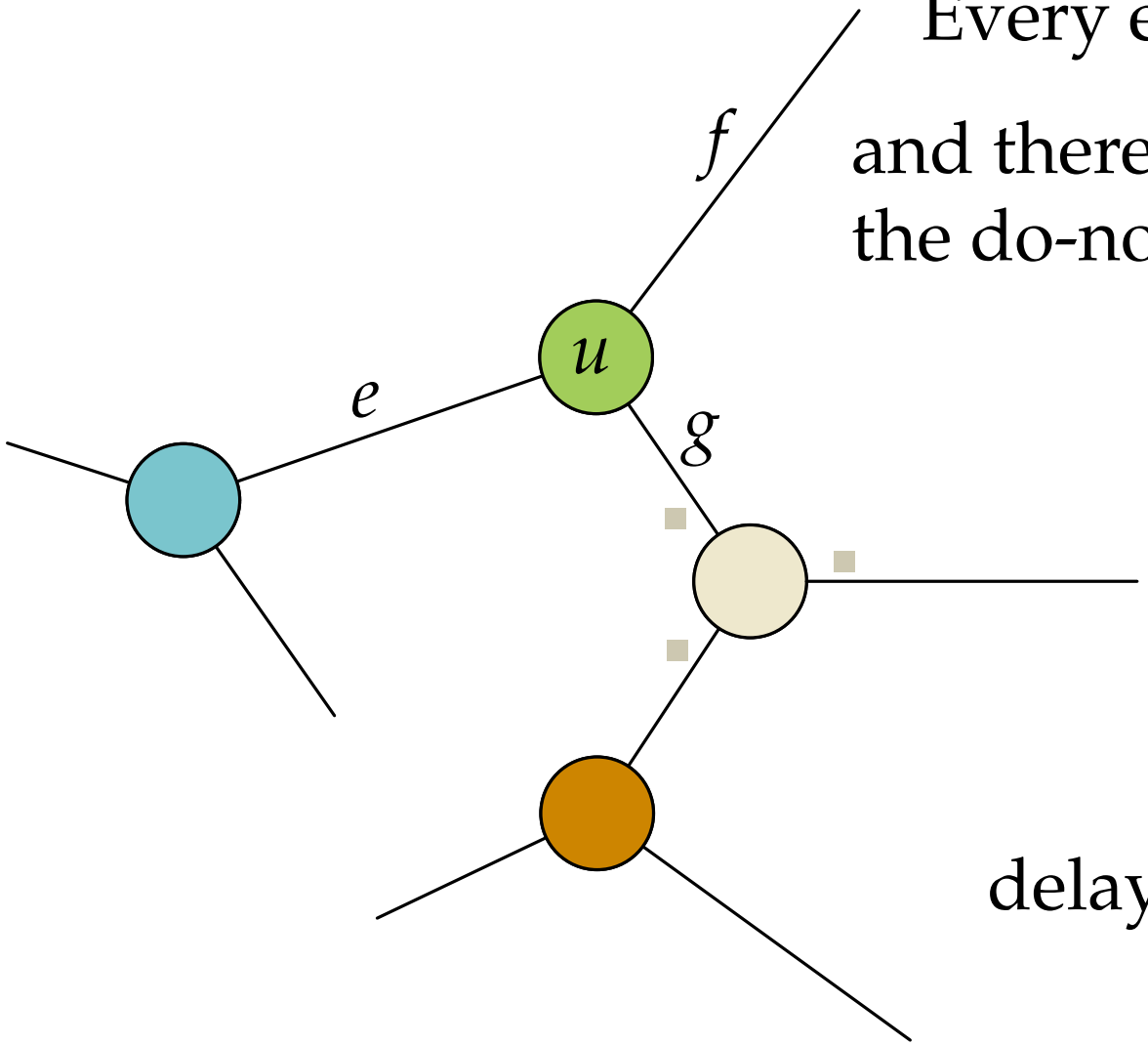

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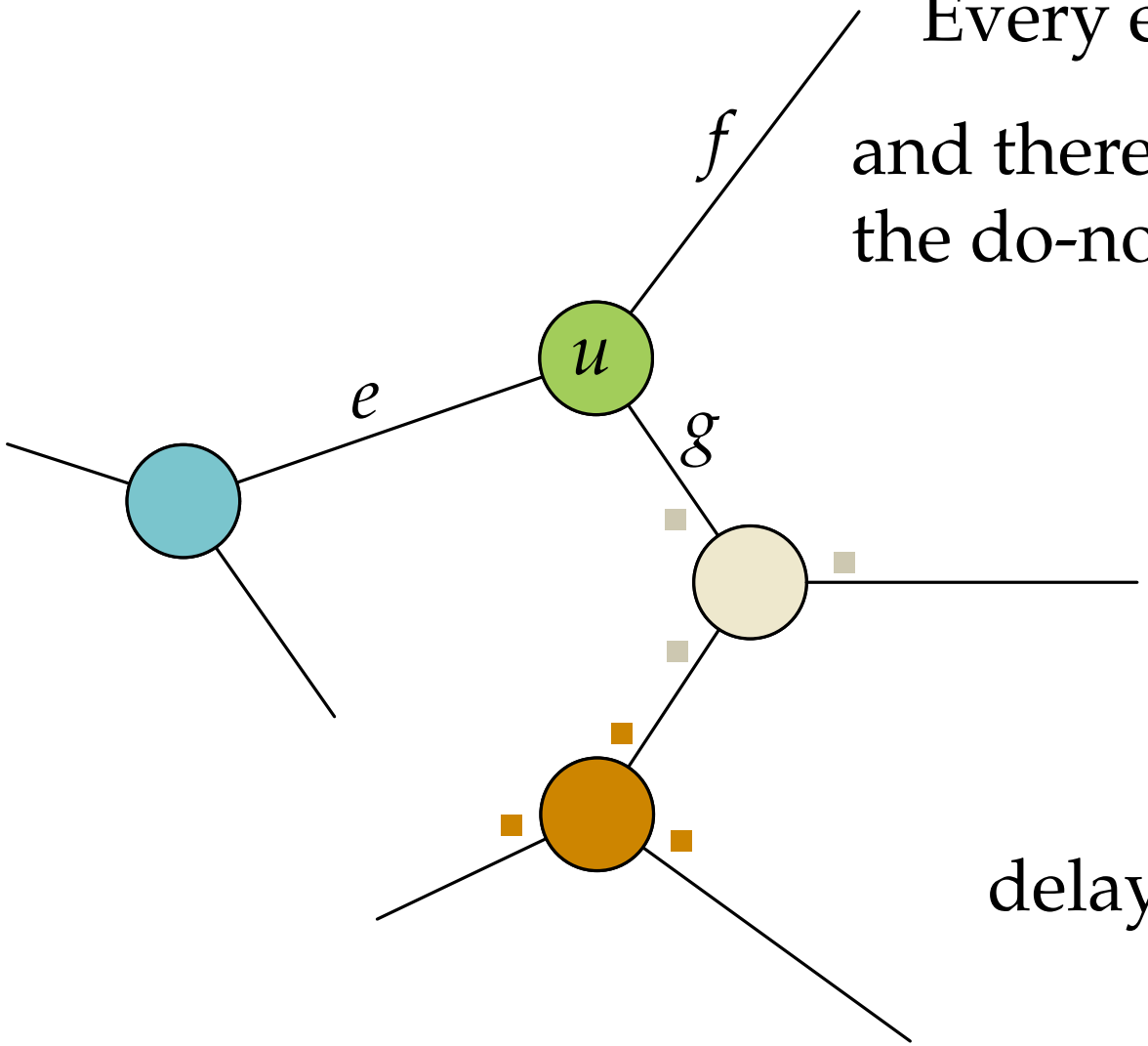

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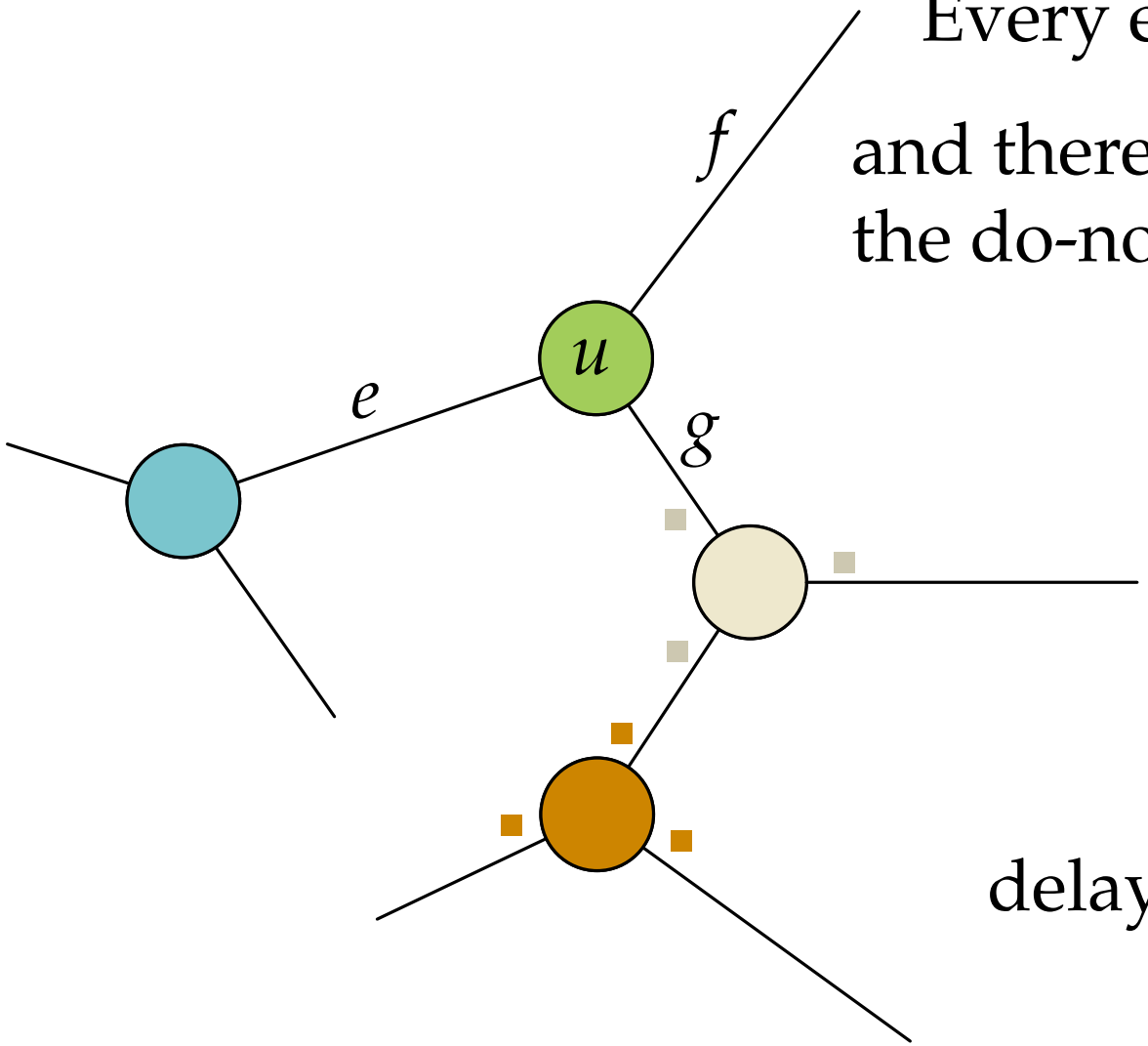

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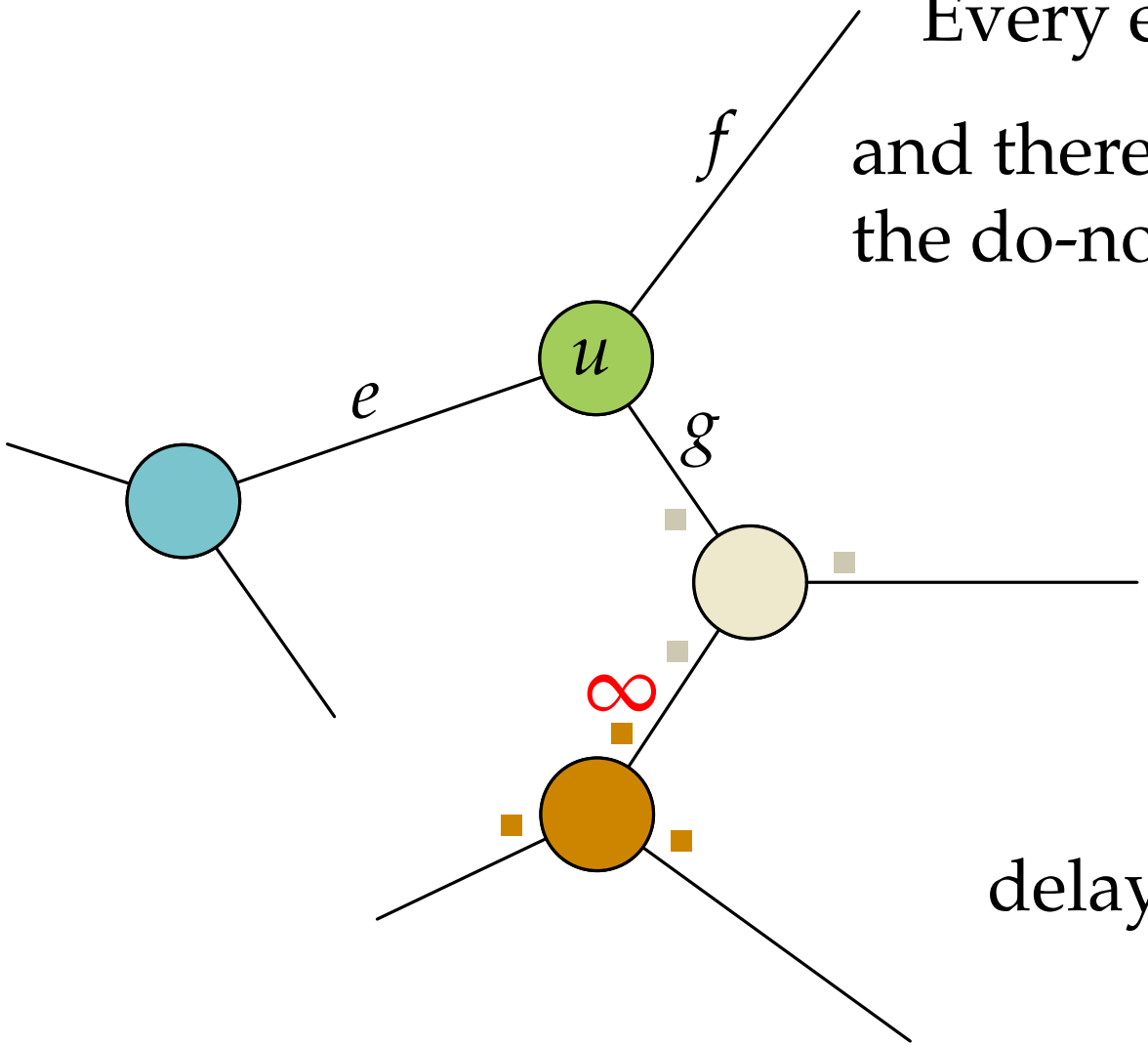

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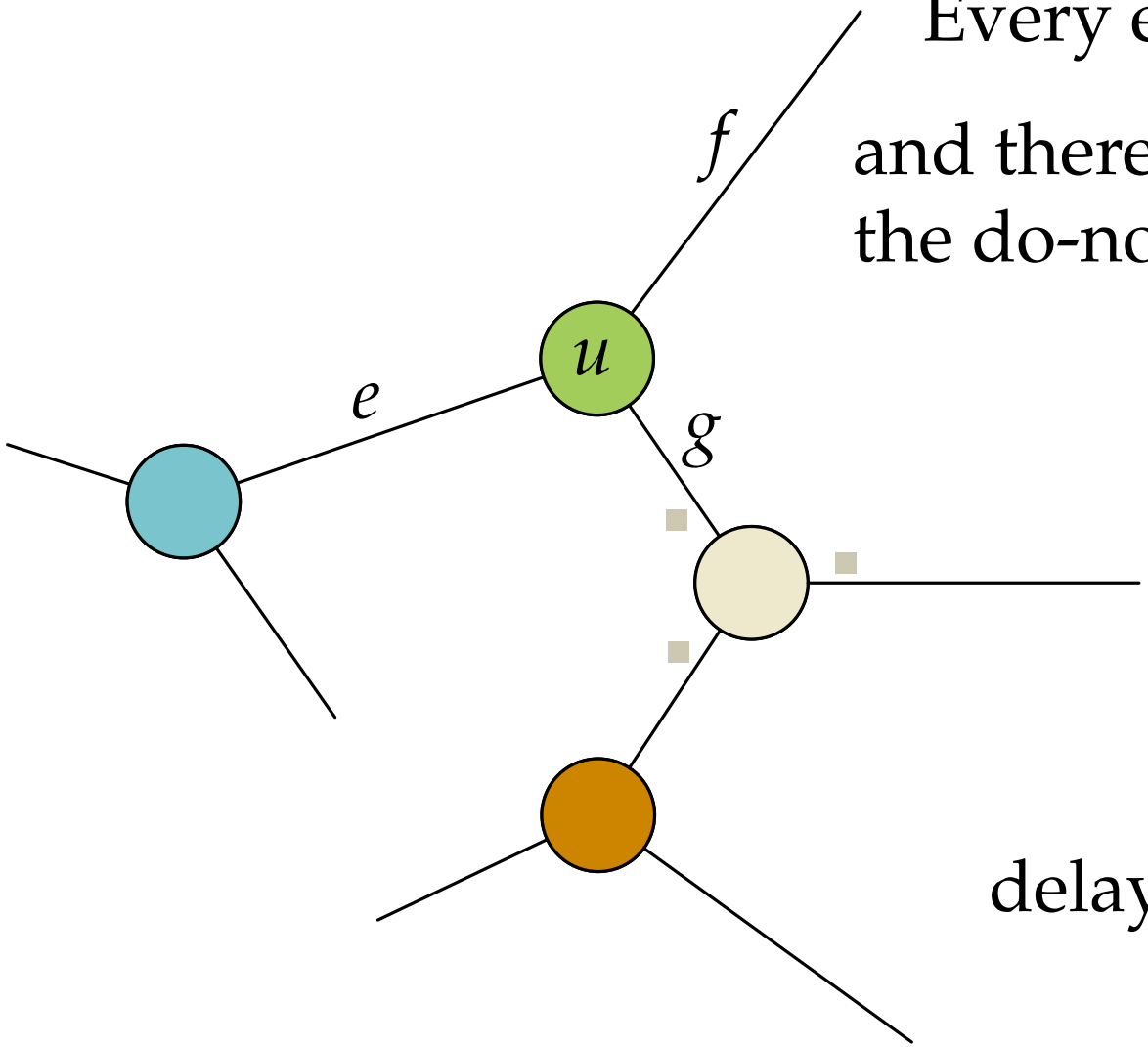

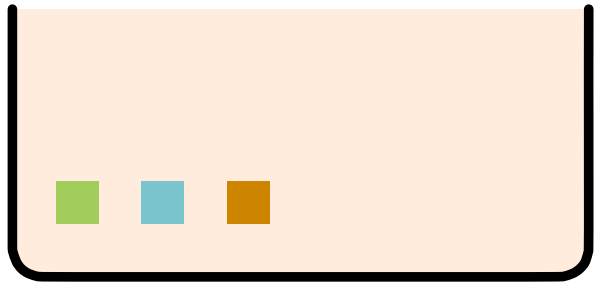

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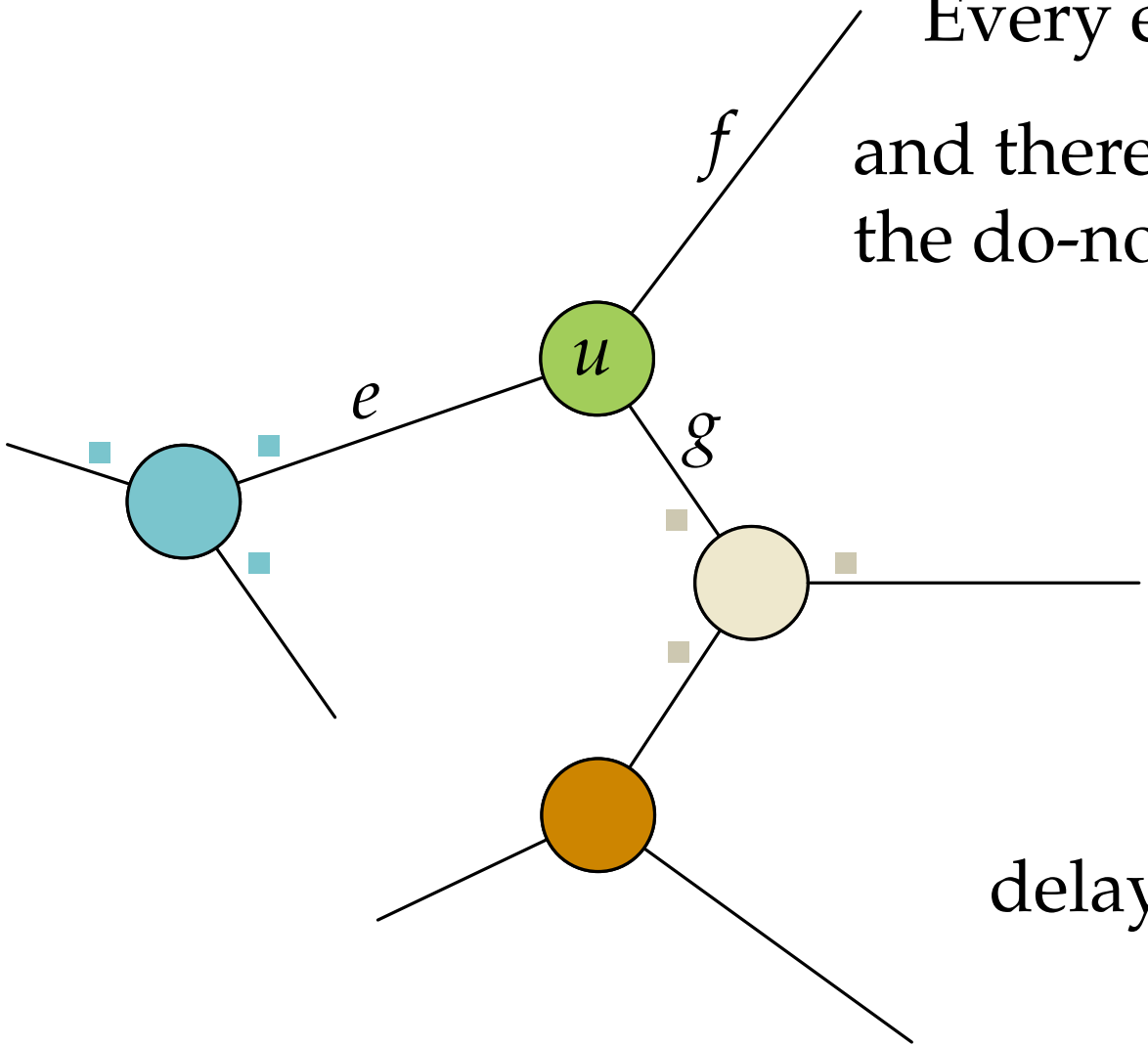

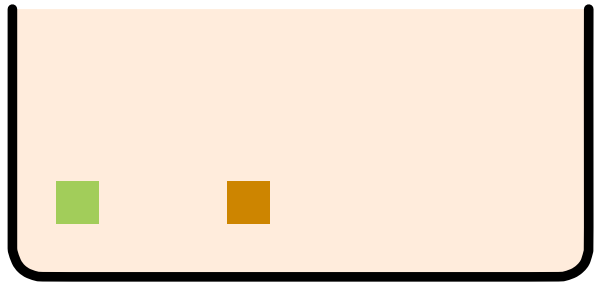

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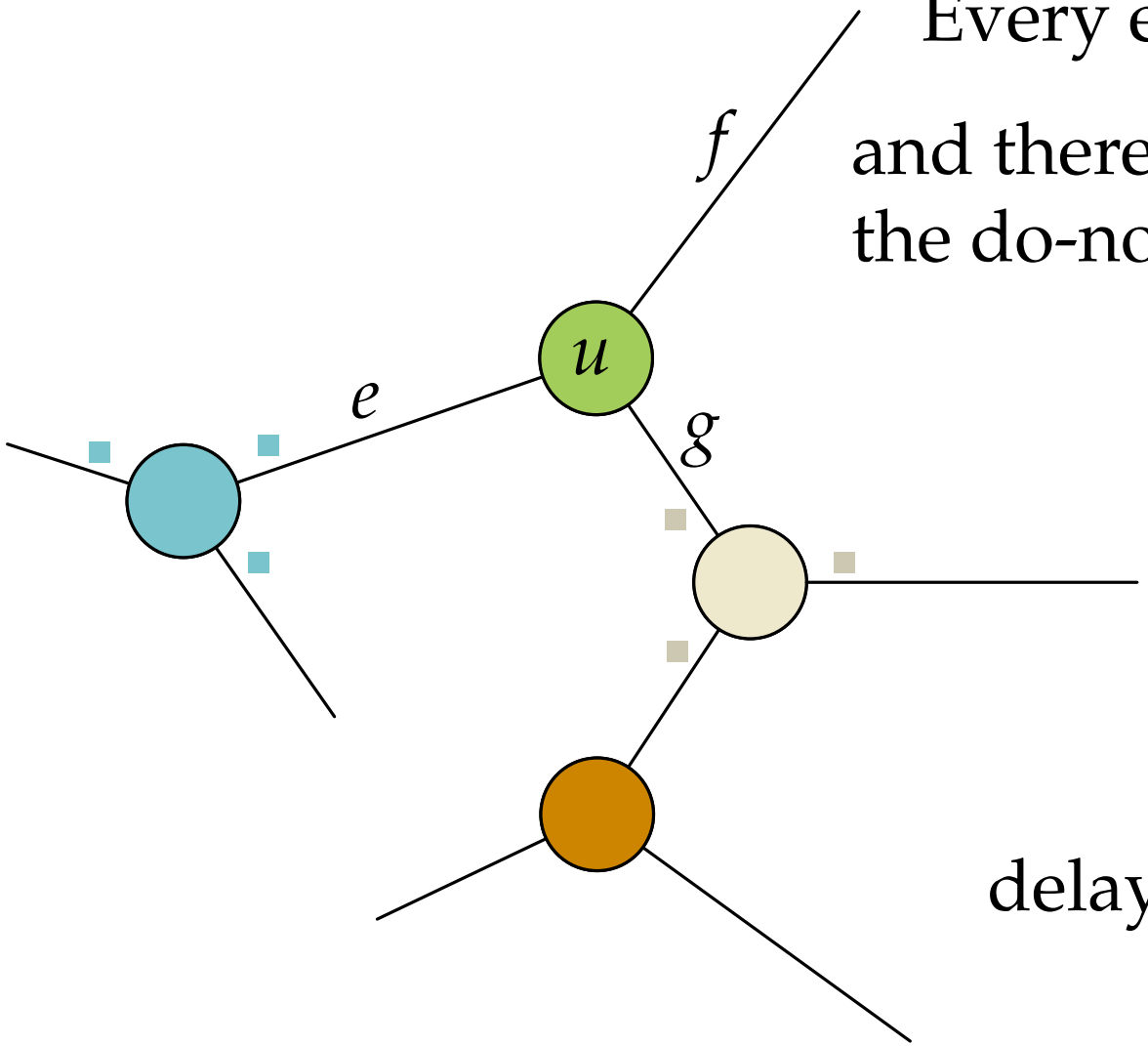

social cost = 2

$\text{delay}(\text{DNB}, i) = 1$

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$S_u = \{\{e, f, g\}, \{\text{DNB}\}\}$

Using Congestion Games to Solve Max-IS

$$S_u := \{\{e \in E \mid u \in e\}, \{\text{DNB}\}\}$$

Observation. The social optimum is $n - |\text{Max-IS}|$.

$$\text{delay}(\text{DNB}, i) = 1$$

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Observation. The social optimum is $n - |\text{Max-IS}|$.

Corollary. Computing the social optimum is NP-hard.

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Using Congestion Games to Solve Max-IS

$$S_u := \{\{e \in E \mid u \in e\}, \{\text{DNB}\}\}$$

Observation. The social optimum is $n - |\text{Max-IS}|$.

Corollary. Computing the social optimum is NP-hard.

But. It's usually not the social optimum we are looking for.

$$\text{delay}(\text{DNB}, i) = 1$$

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Social optimum in an ideal world

Social optimum in an ideal world

Collective farm holiday, A. A. Plastov, 1937

Image source: `https://rusmuseumvrm.ru`

Social optimum in an ideal world

Collective farm holiday, A. A. Plastov, 1937

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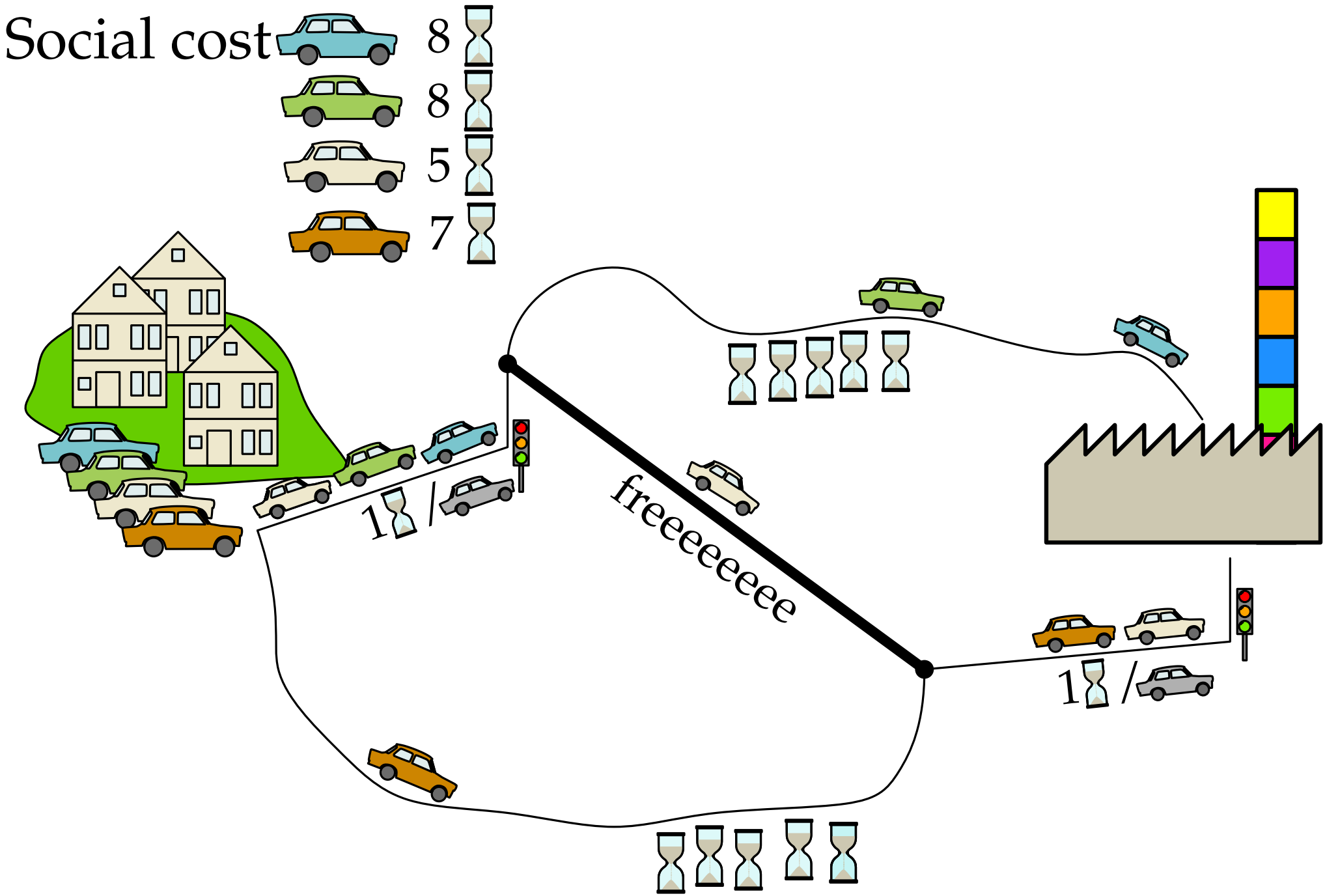
Outcome in the real world

Unemployed men queued outside a depression soup kitchen opened in Chicago by Al Capone

Image source: wikimedia


Social cost
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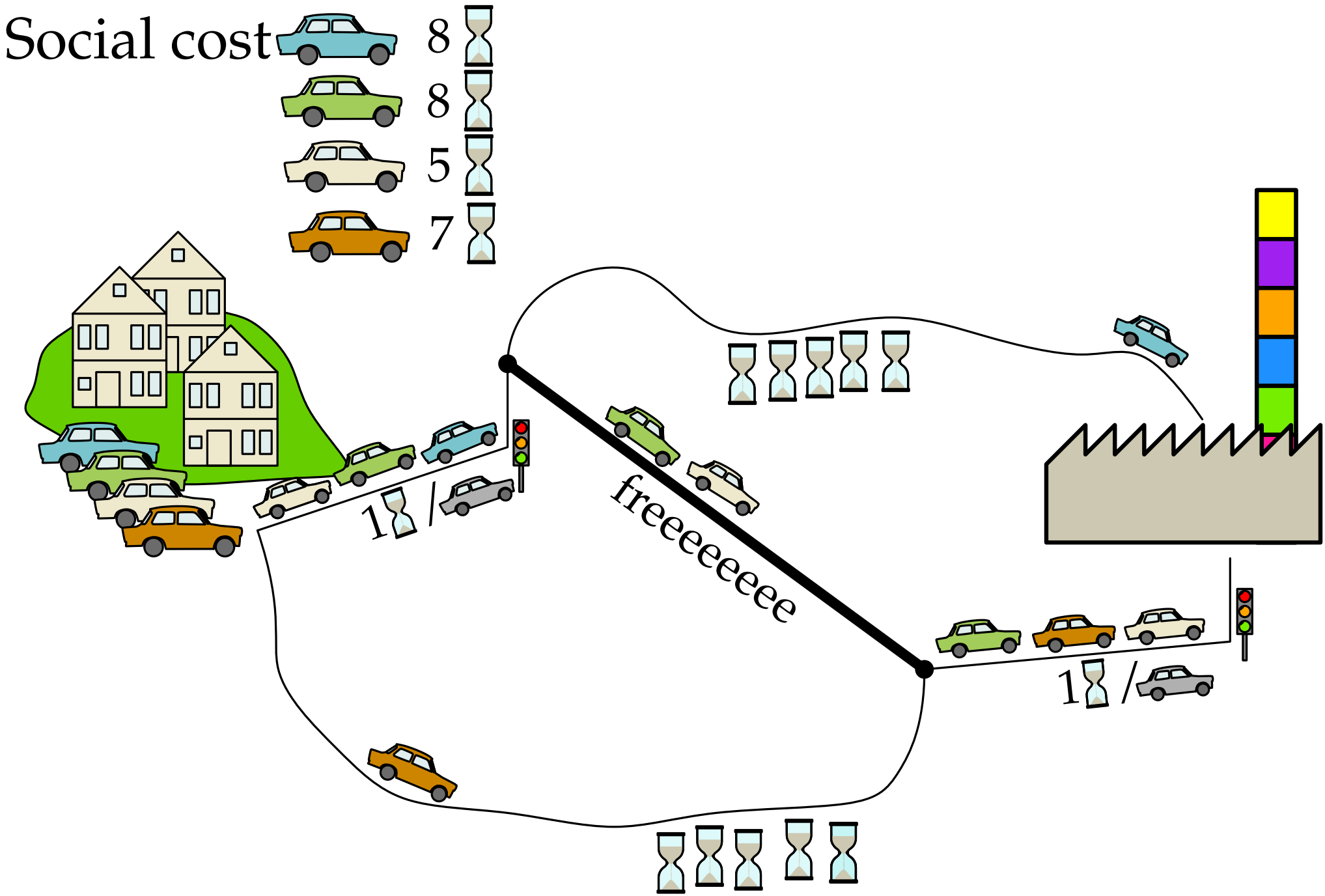
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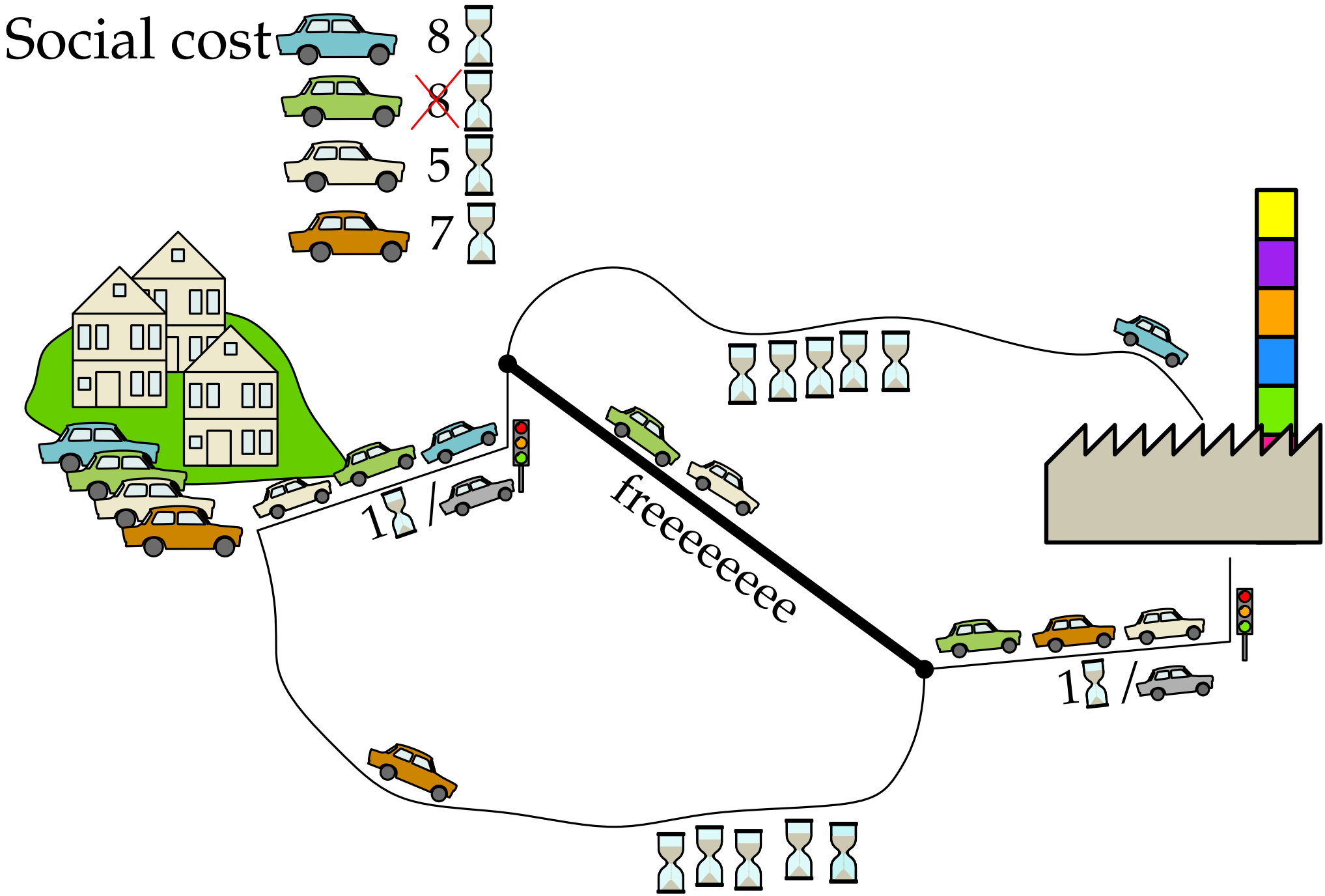
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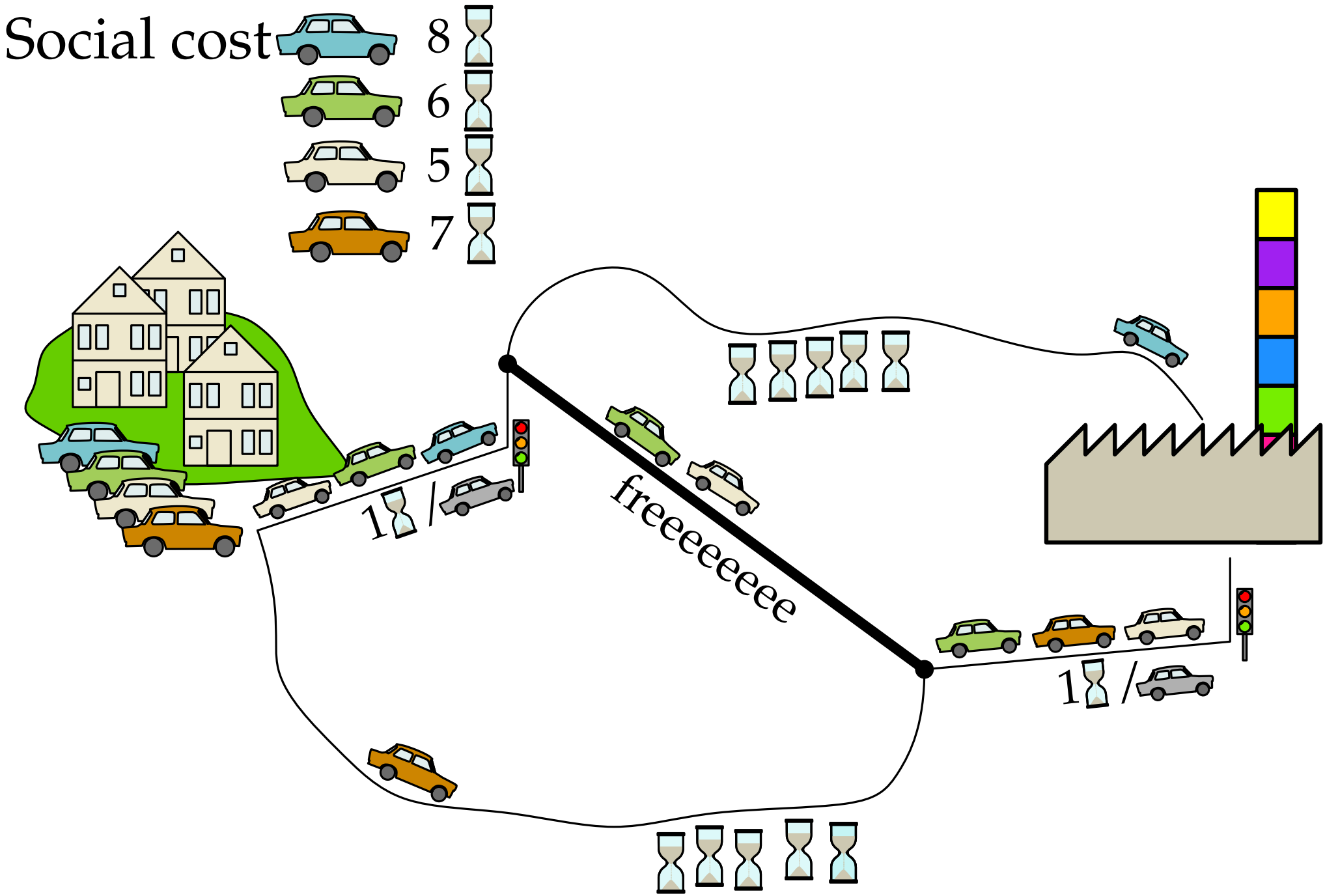
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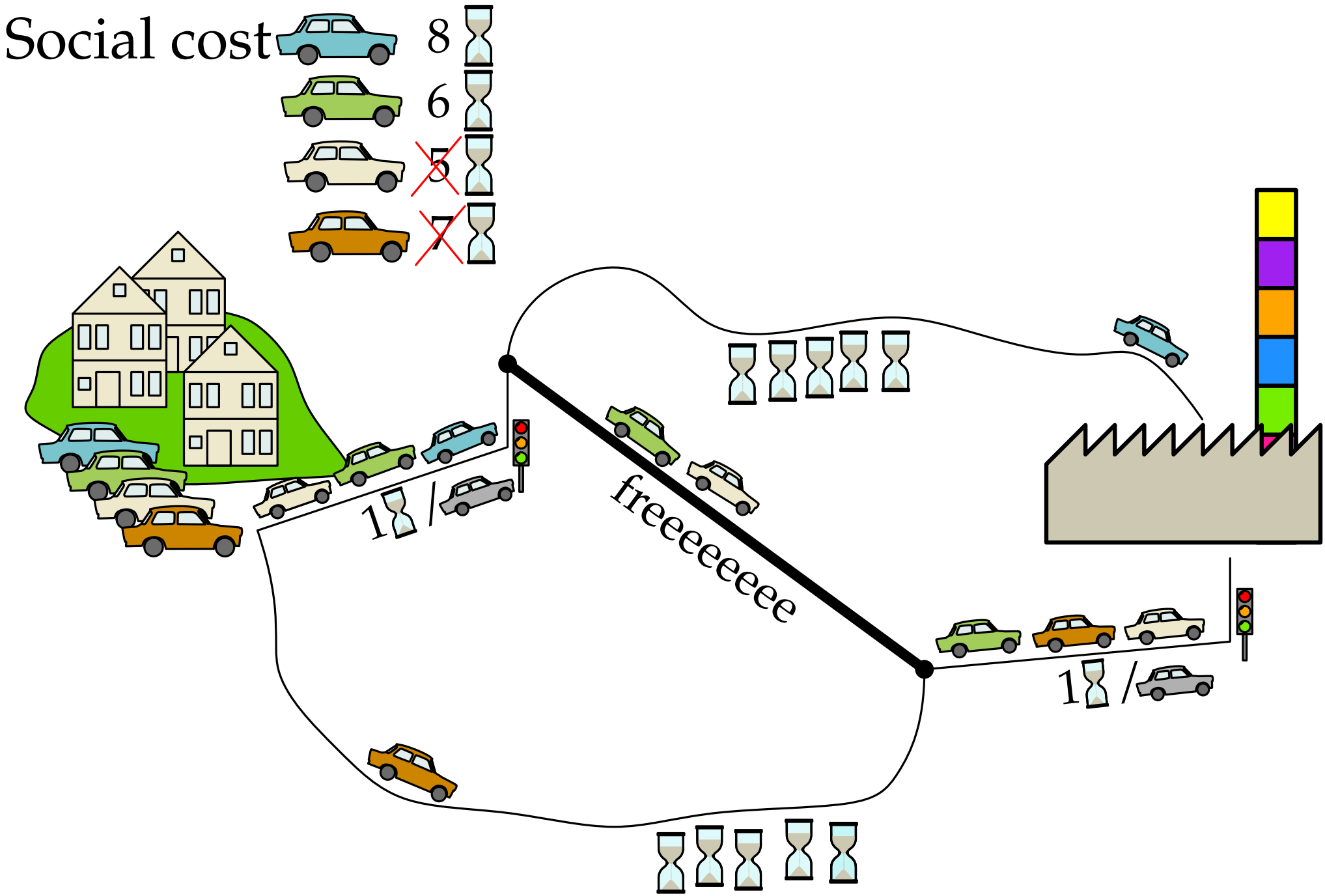
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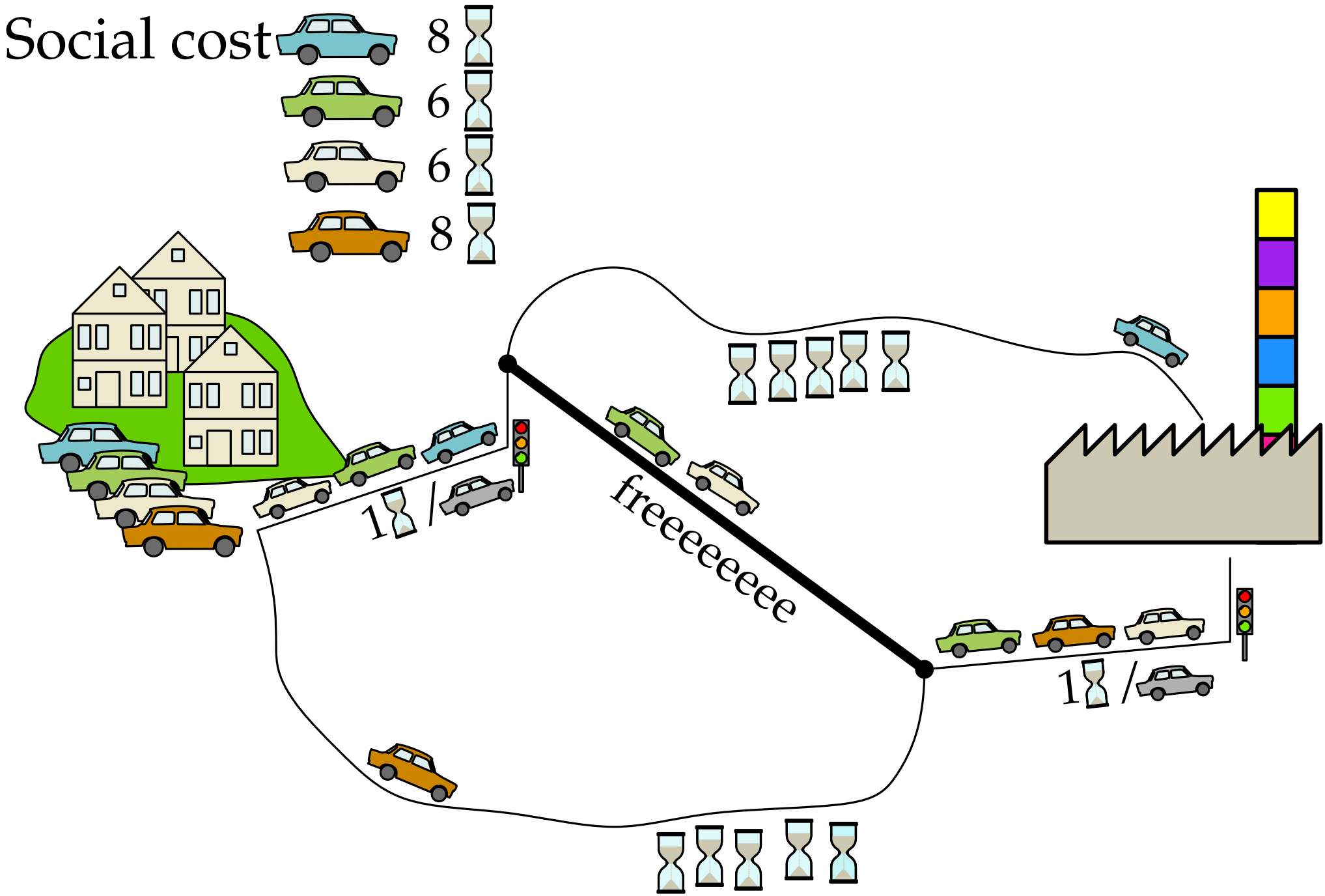
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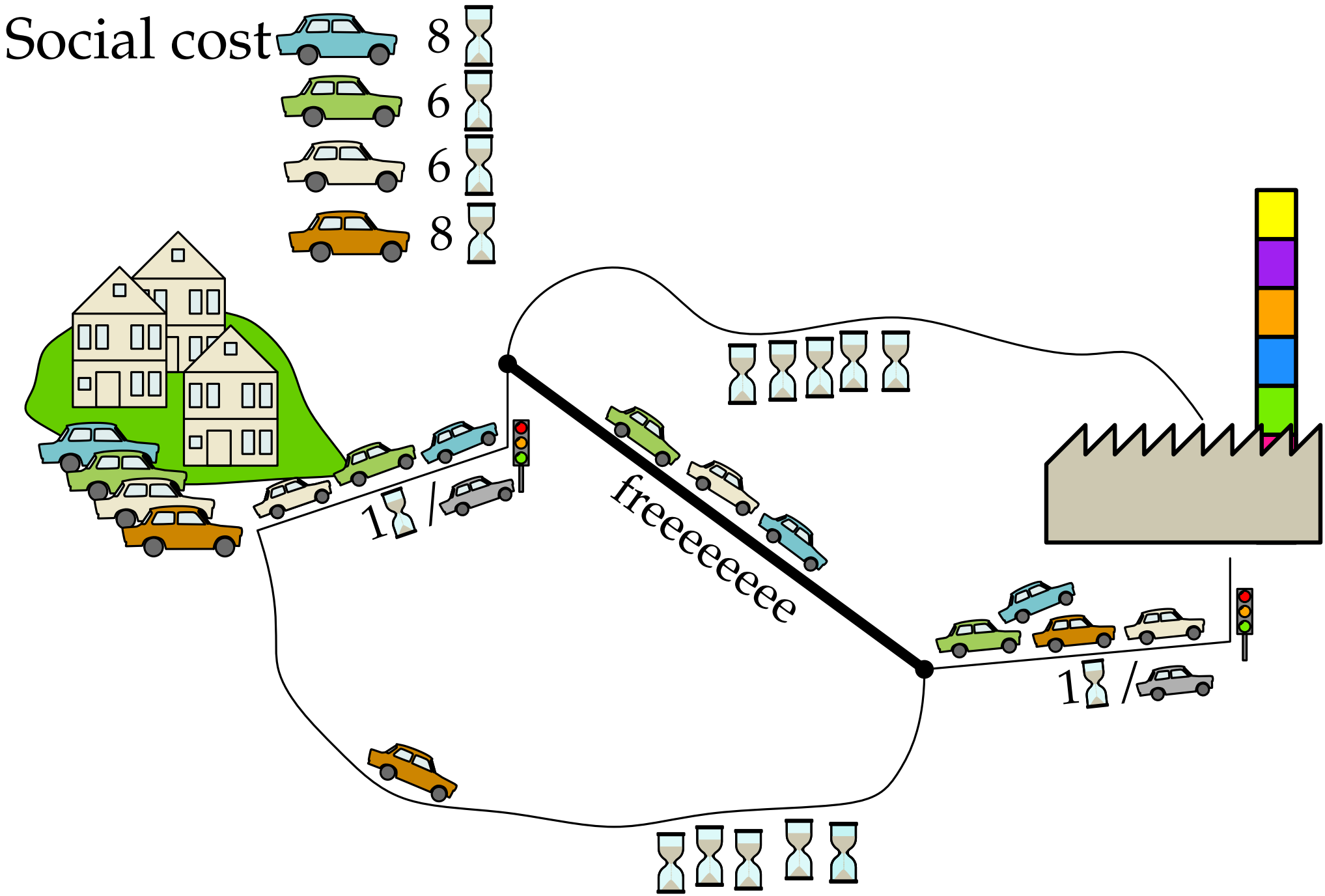
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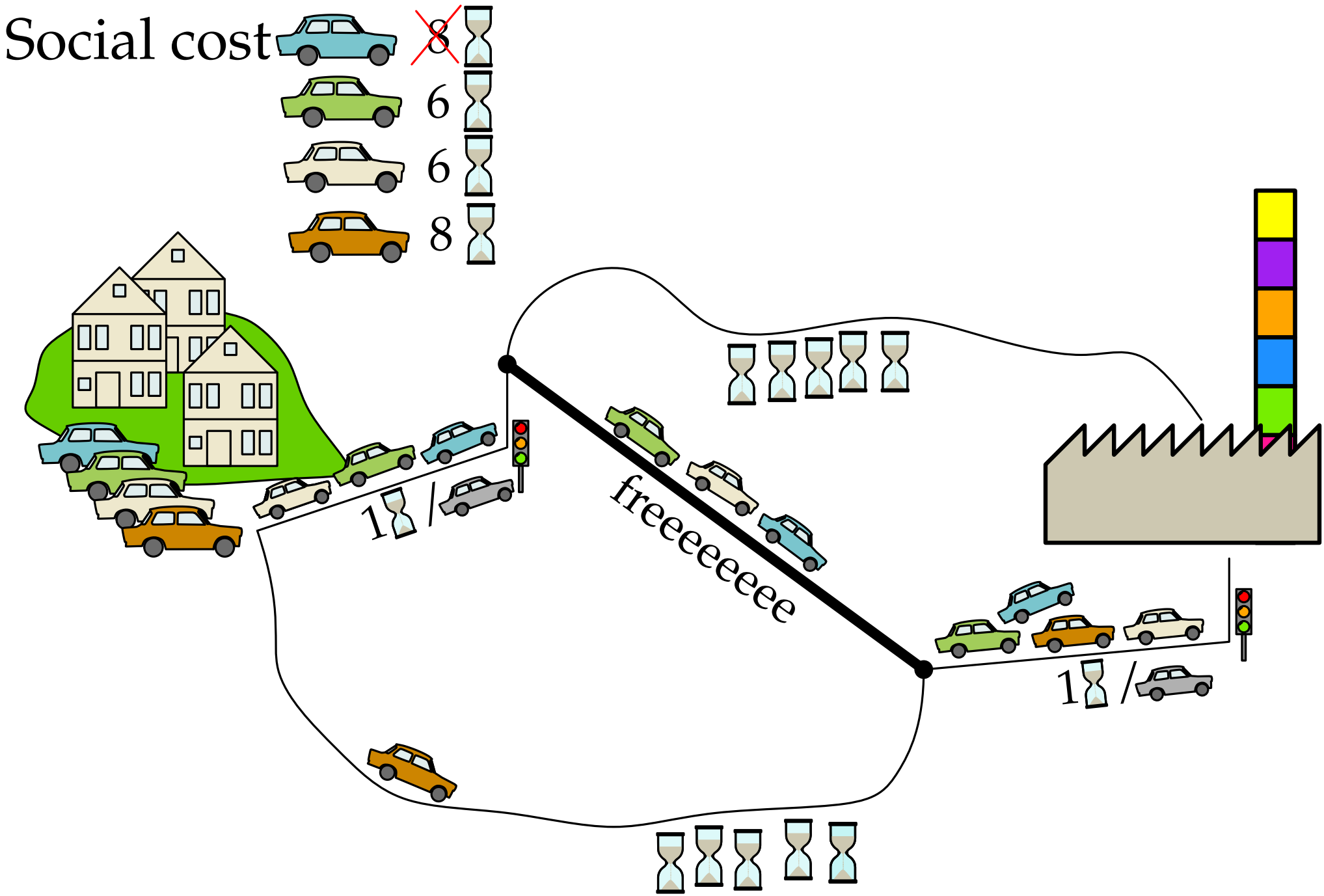
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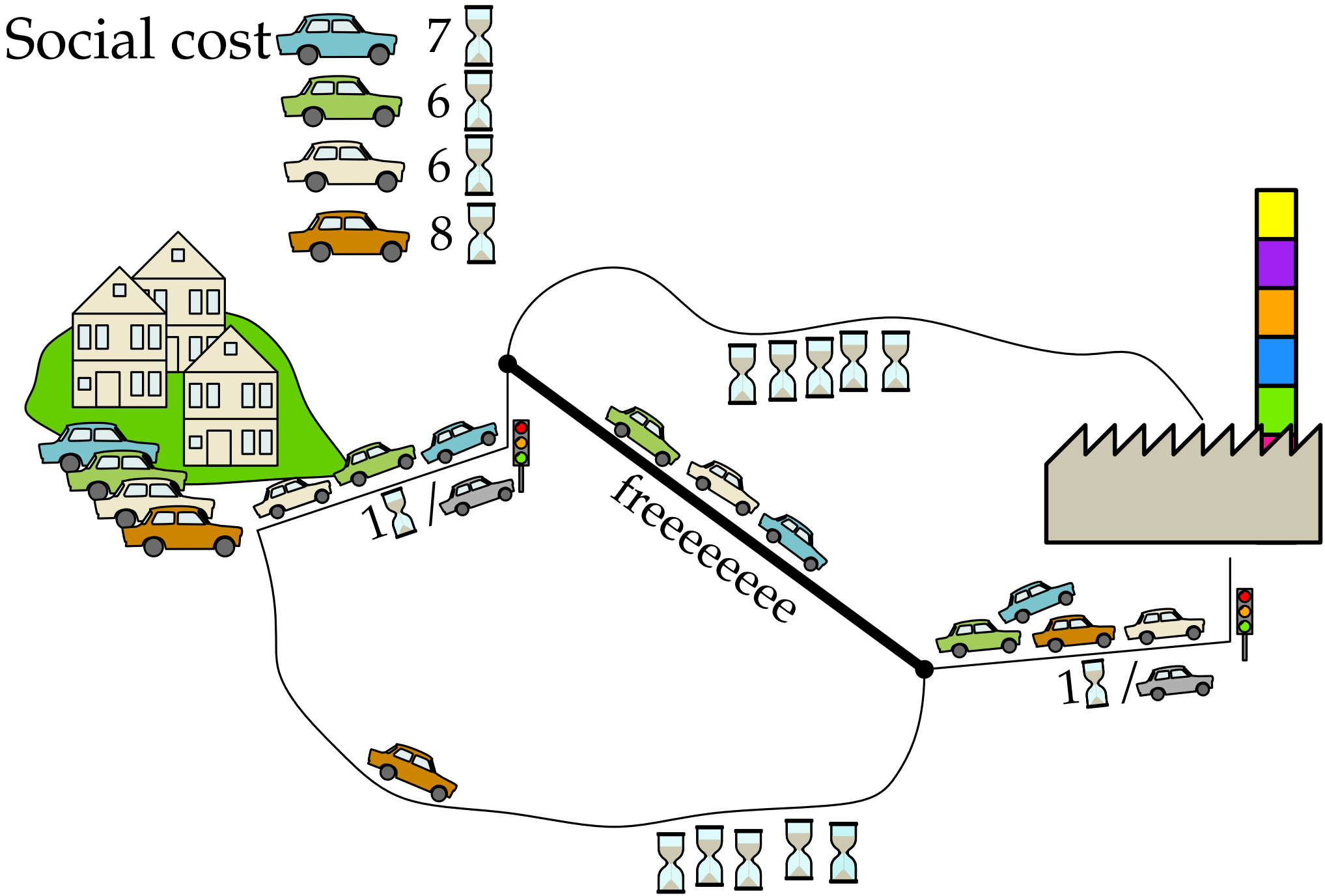
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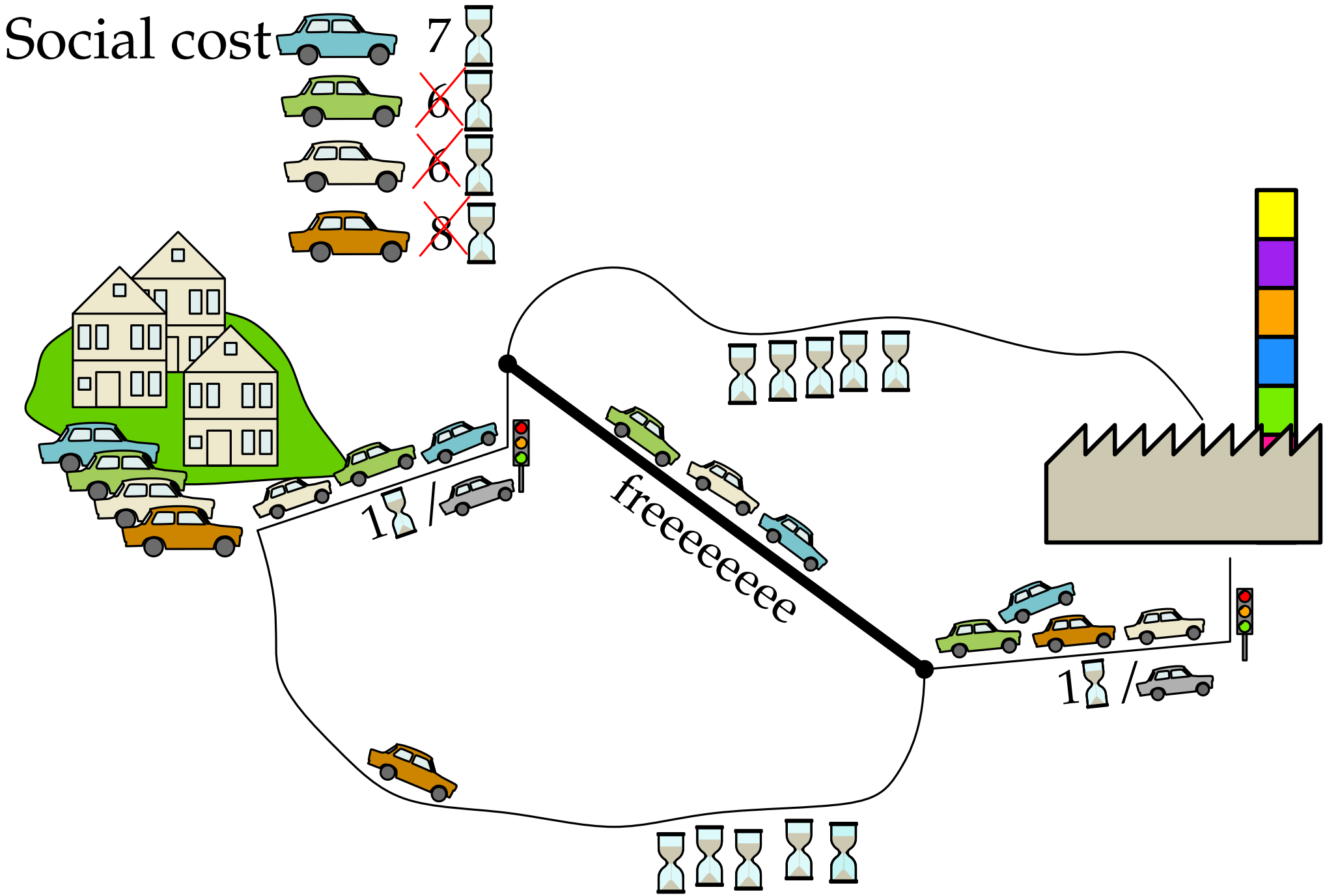
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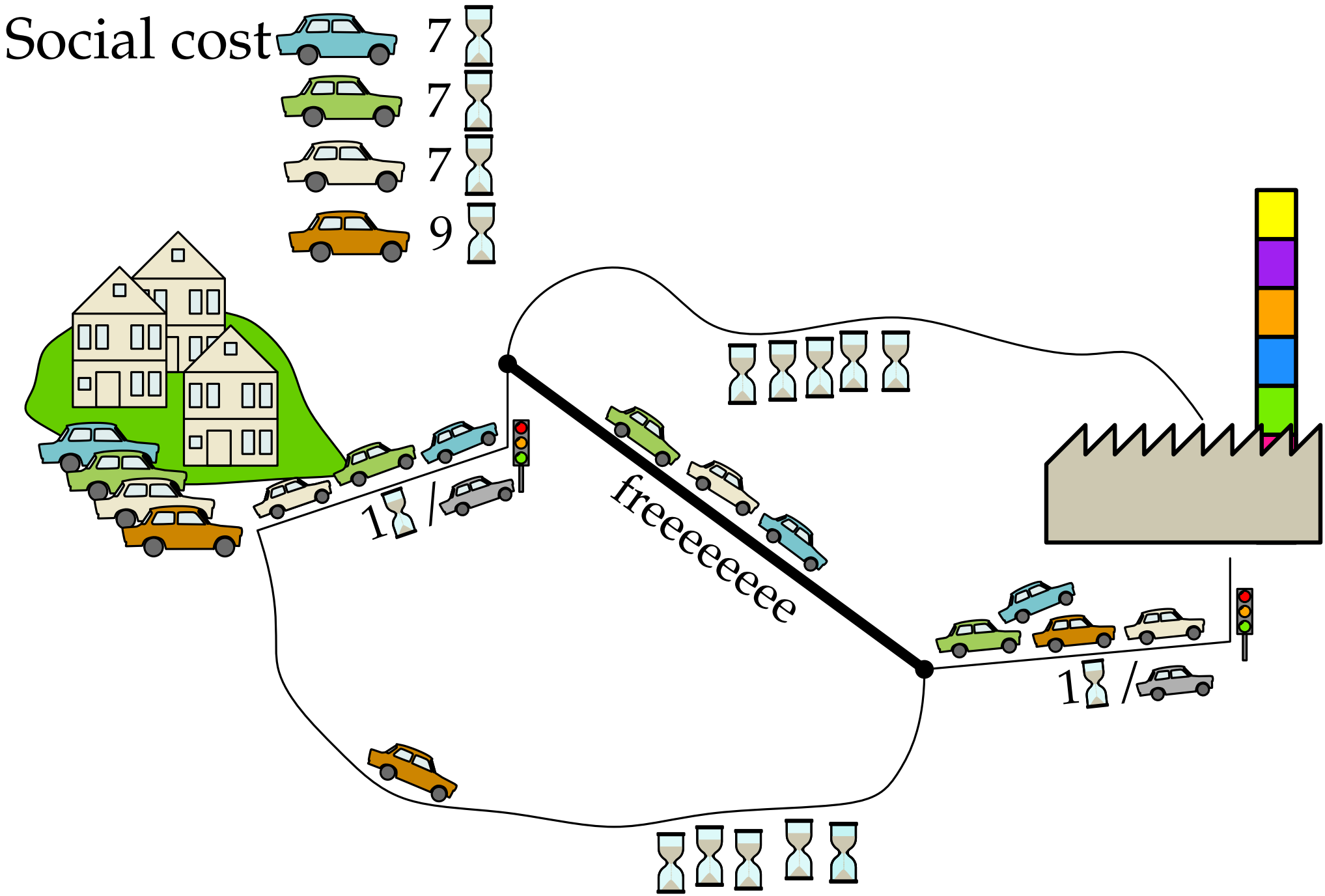
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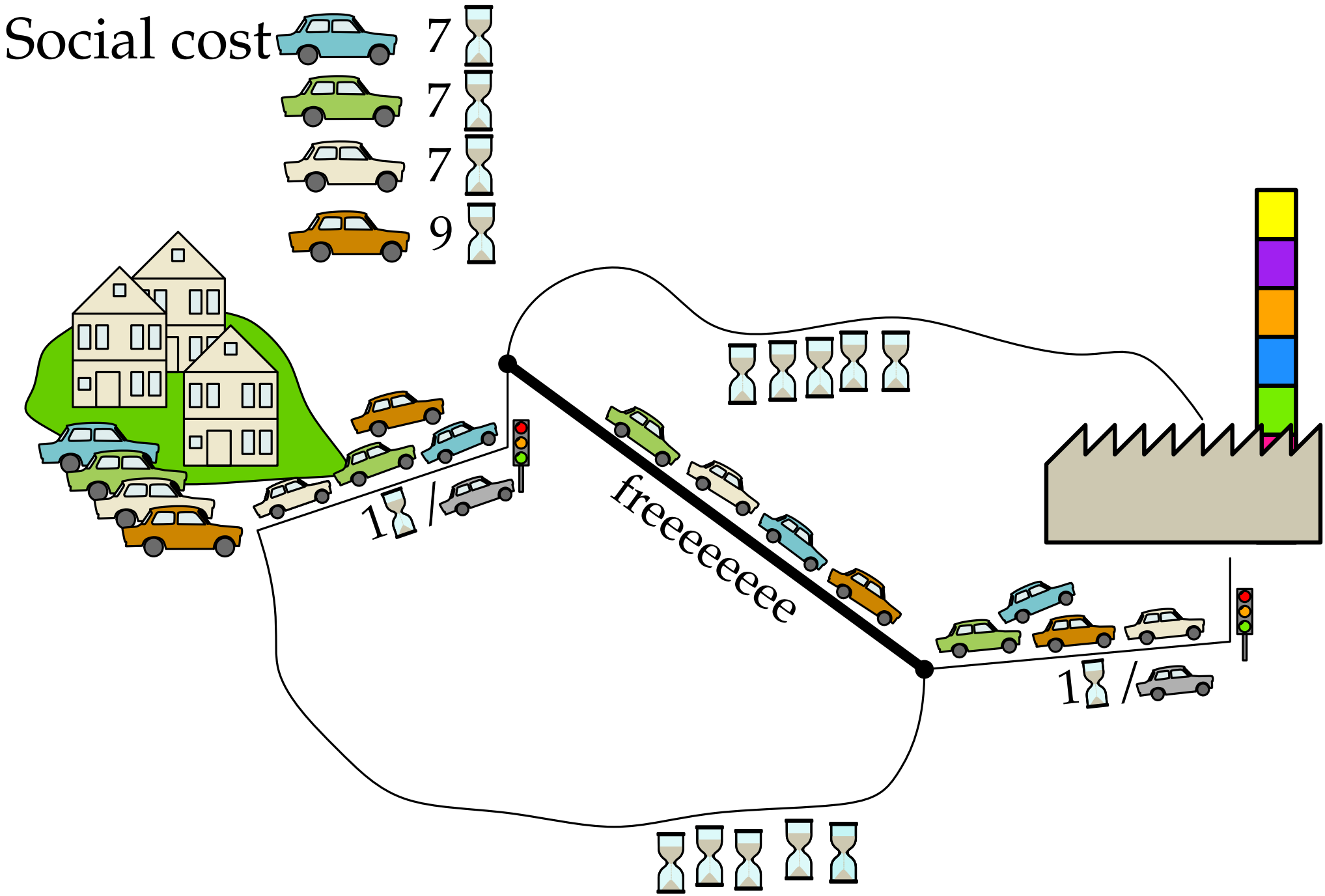
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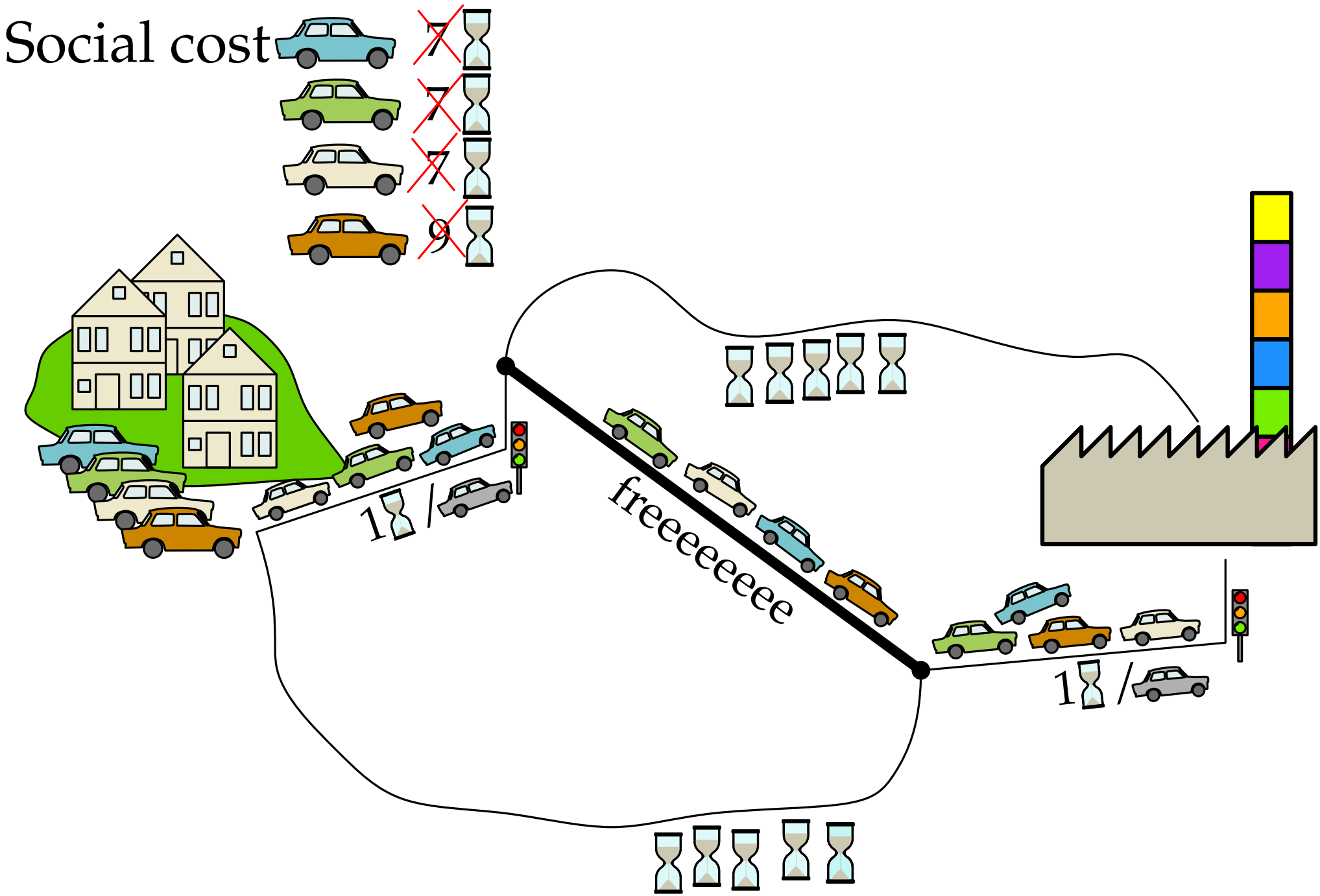
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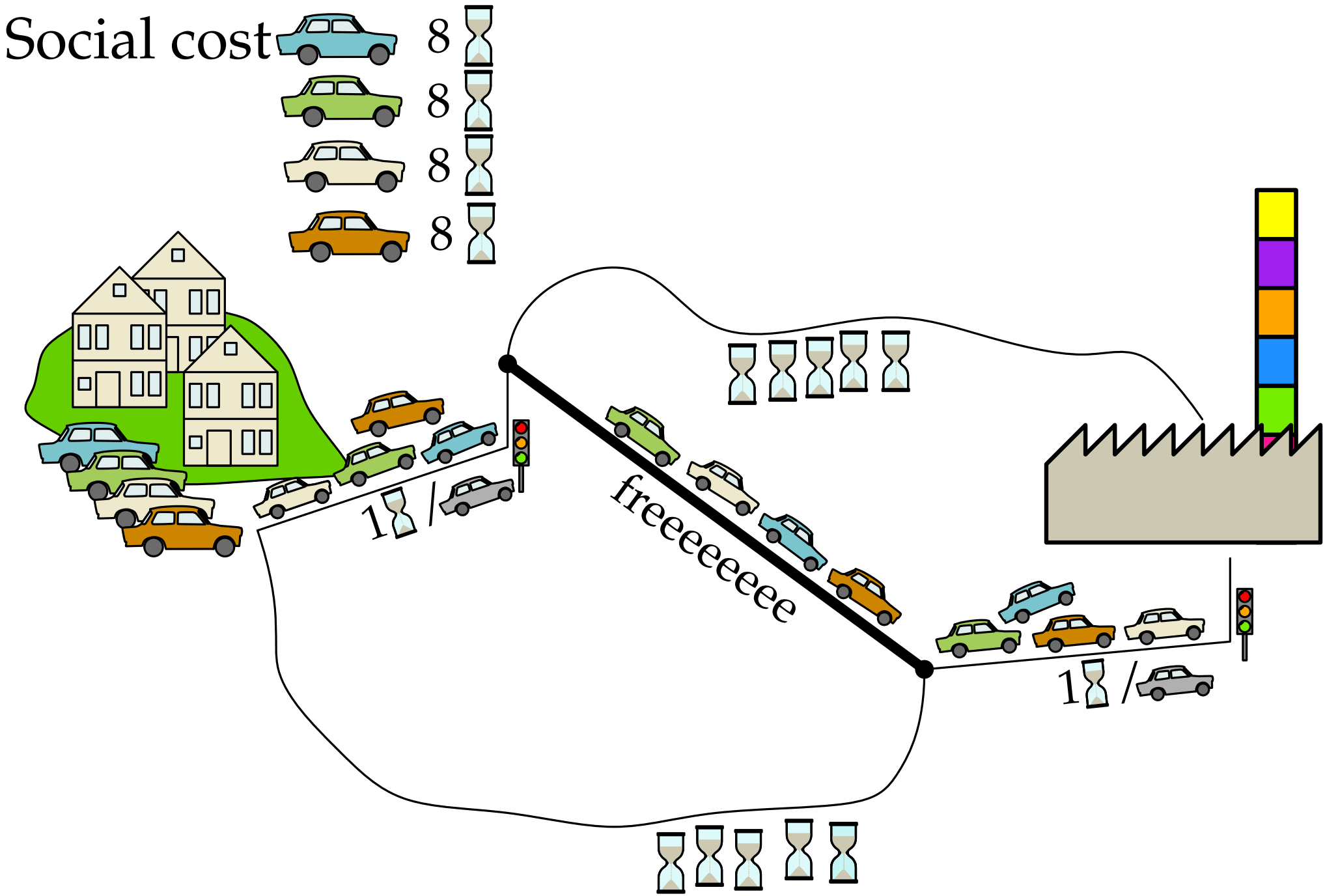
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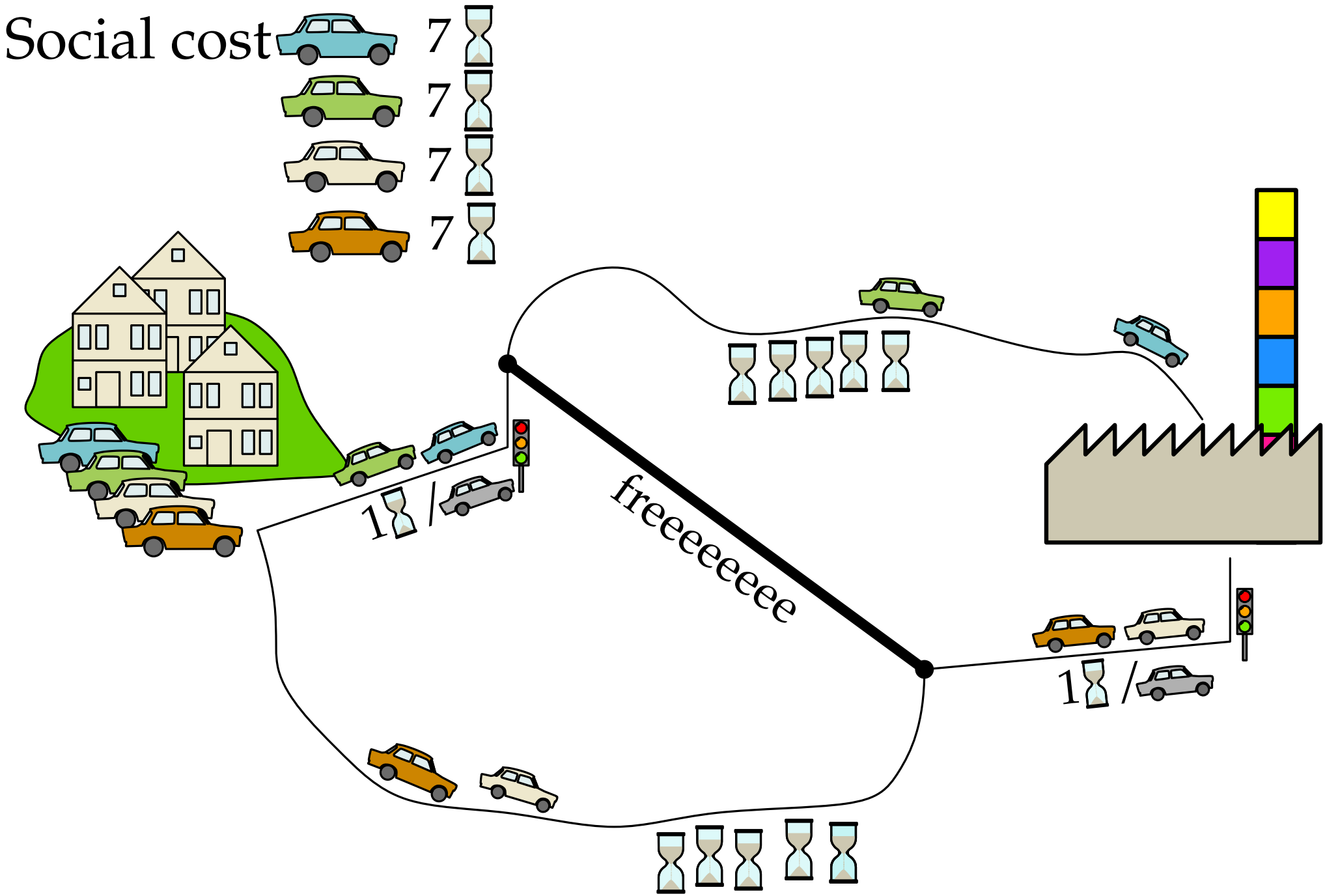
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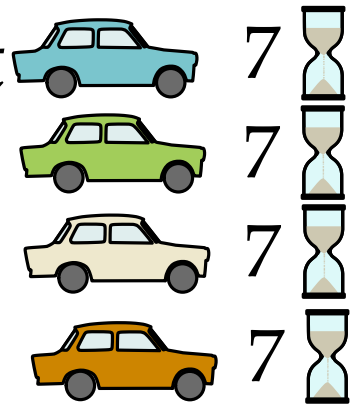
Social cost  This is optimal.

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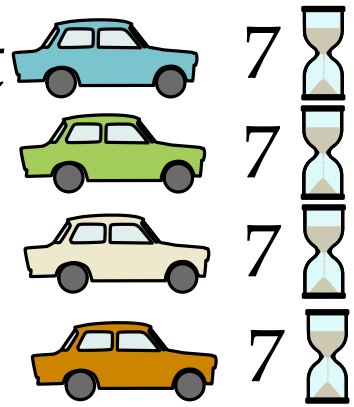

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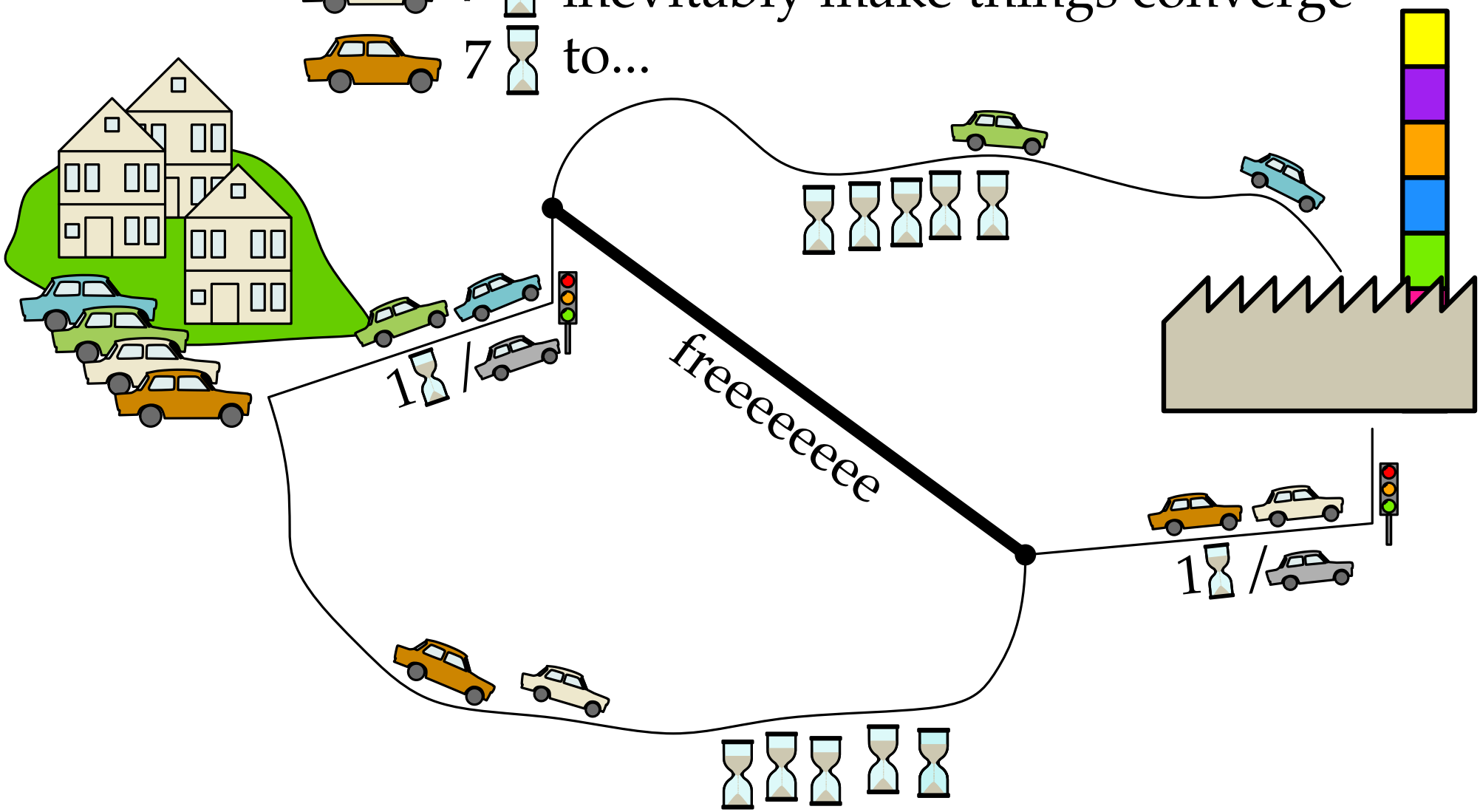
But individual decisions will inevitably make things converge to...



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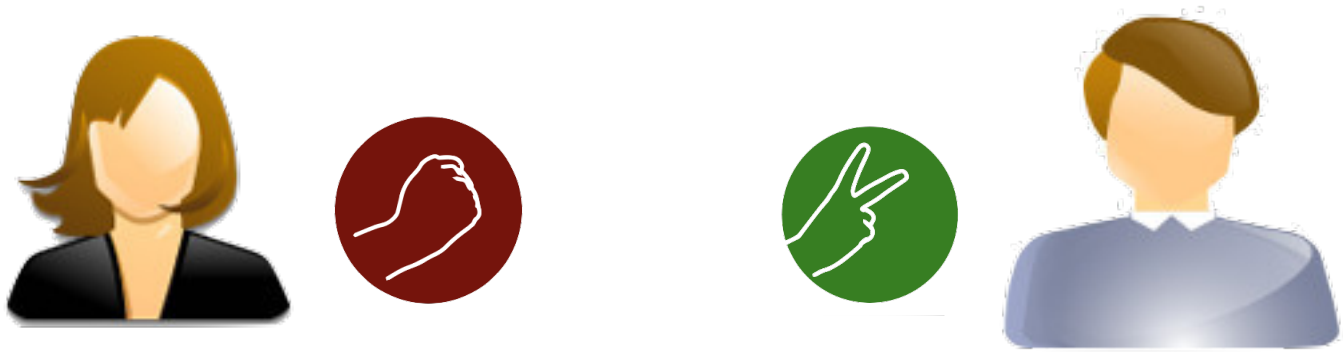

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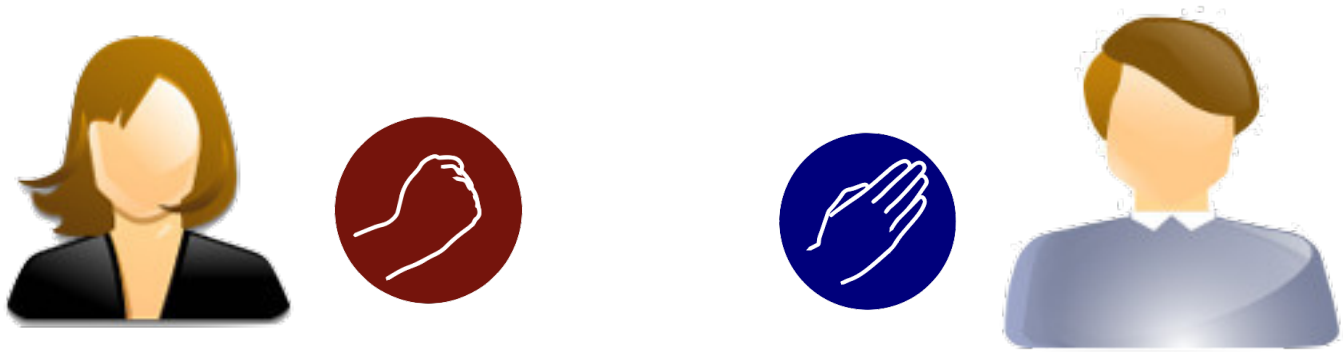

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Yes, there is the concept of a *mixed Nash equilibrium* using randomness, but we will not consider this in this talk.

image source: wikimedia

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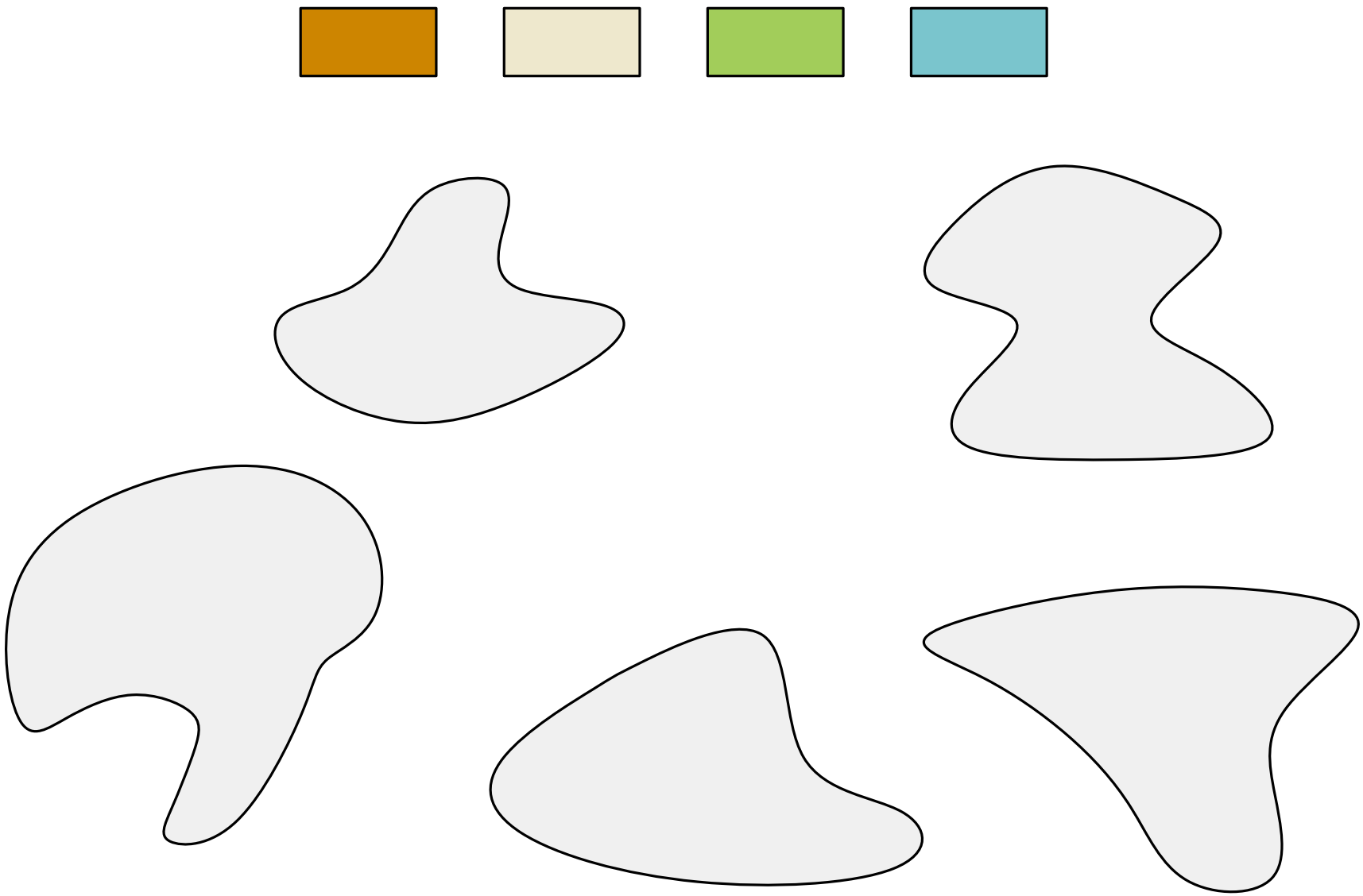

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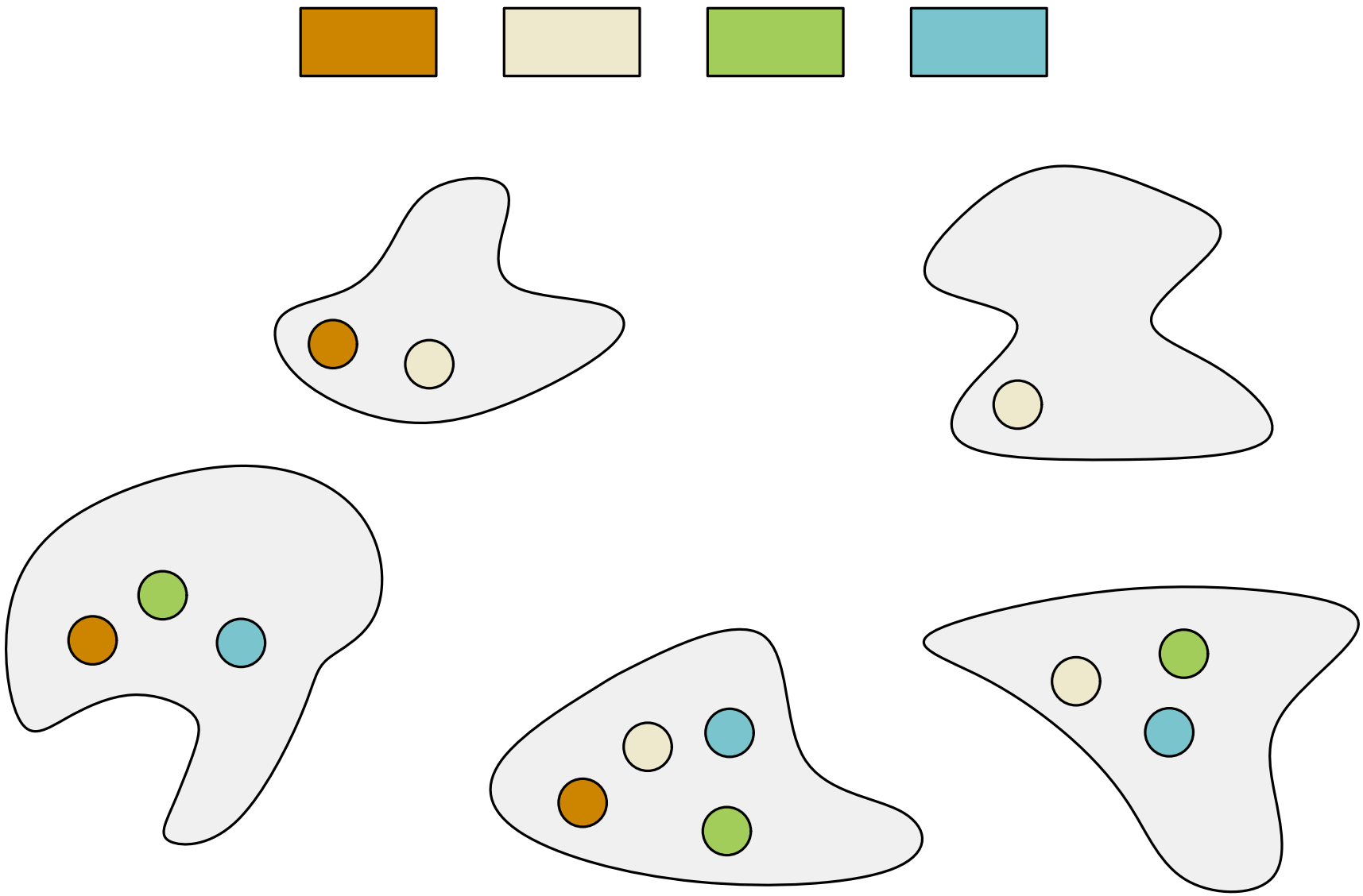

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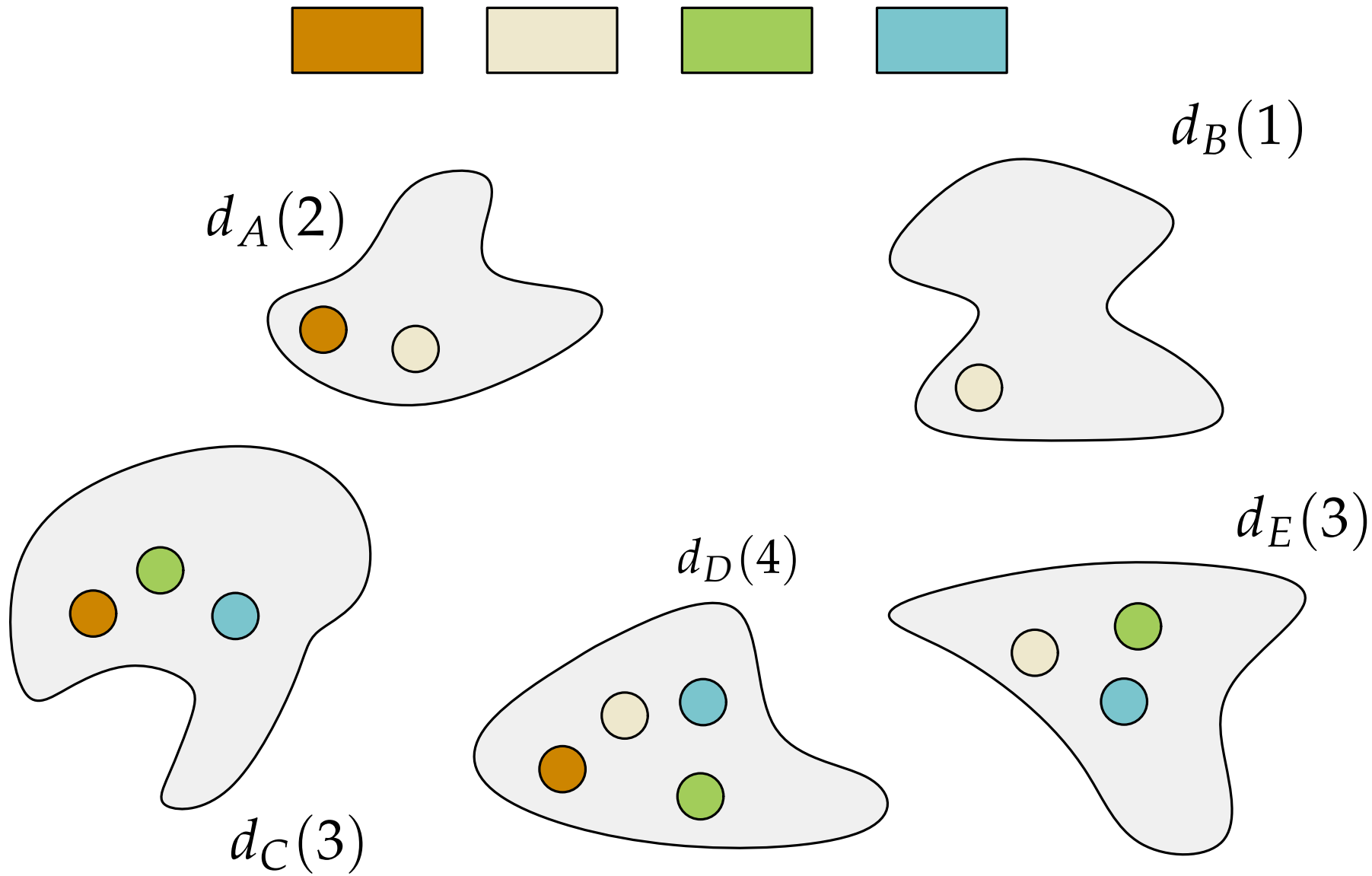

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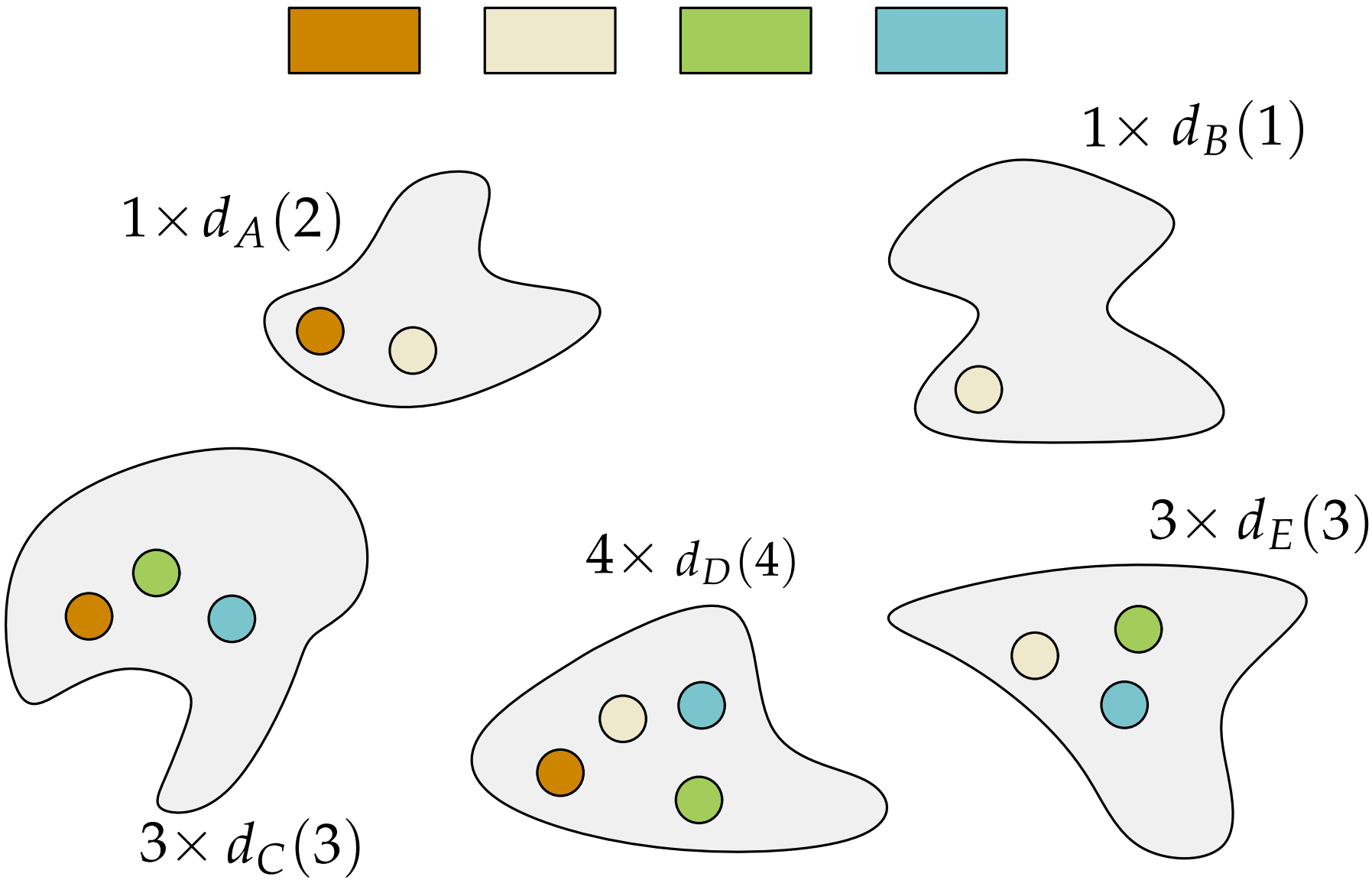

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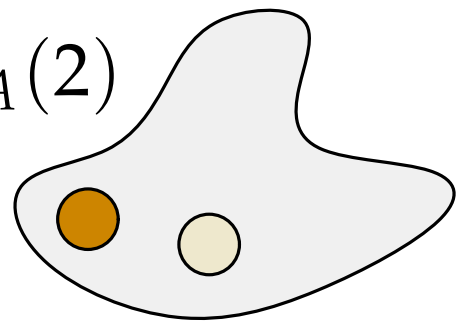
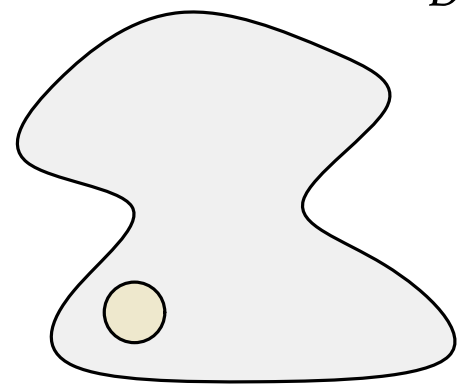
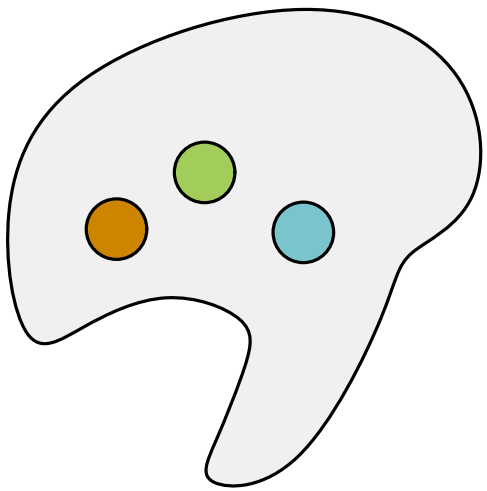
$d_A(1)+d_A(2)$

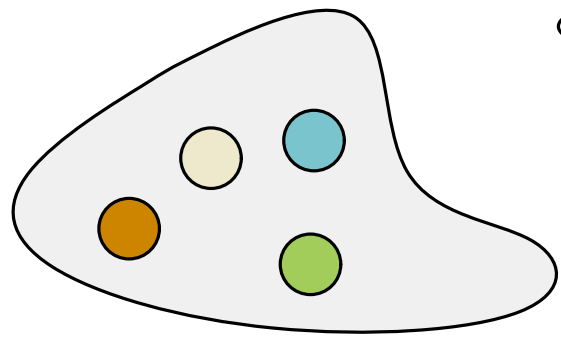


$d_B(1)$





$d_D(1)+\cdots+d_D(4)$



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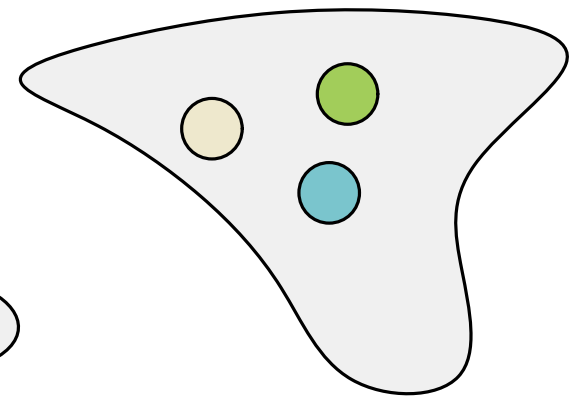

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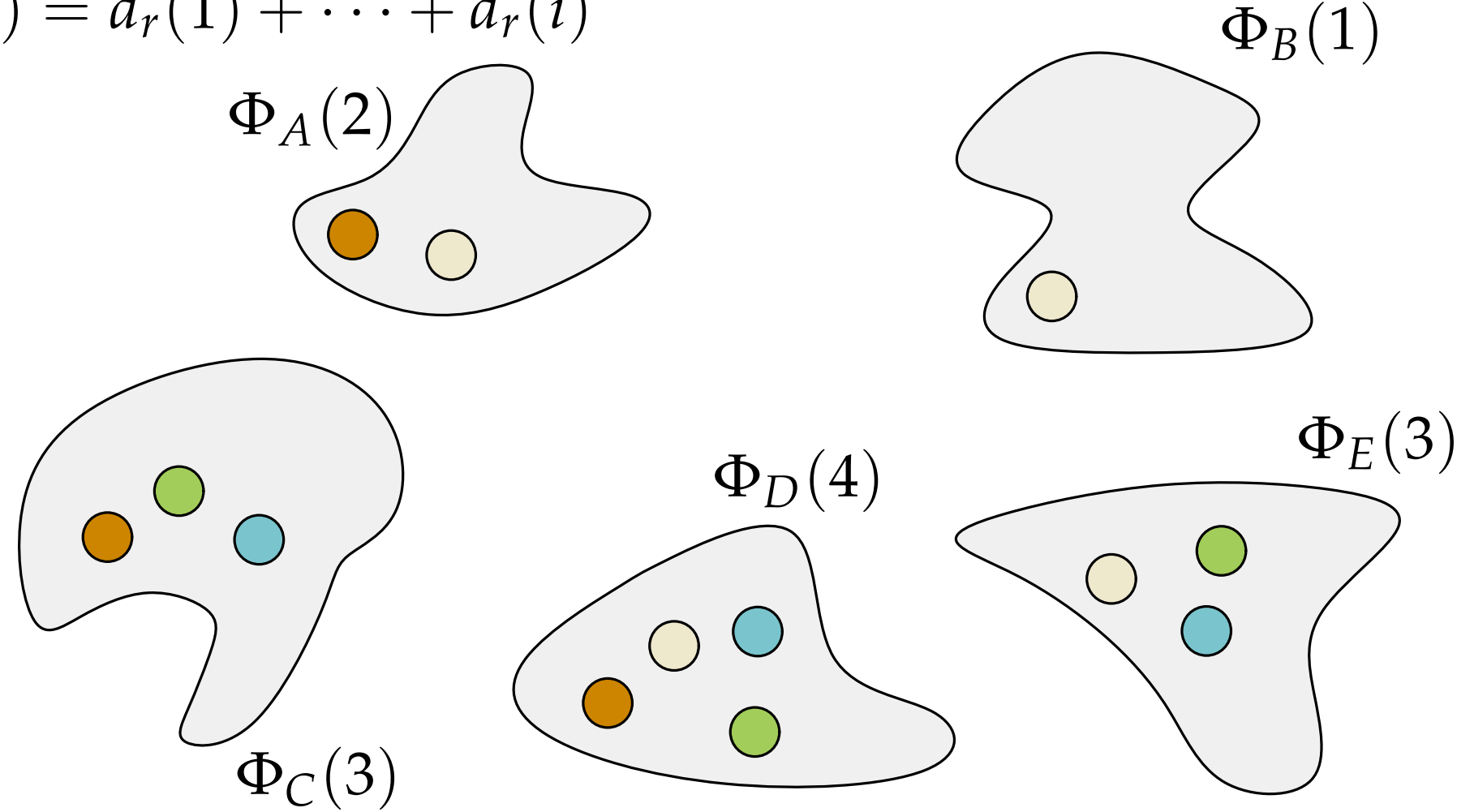

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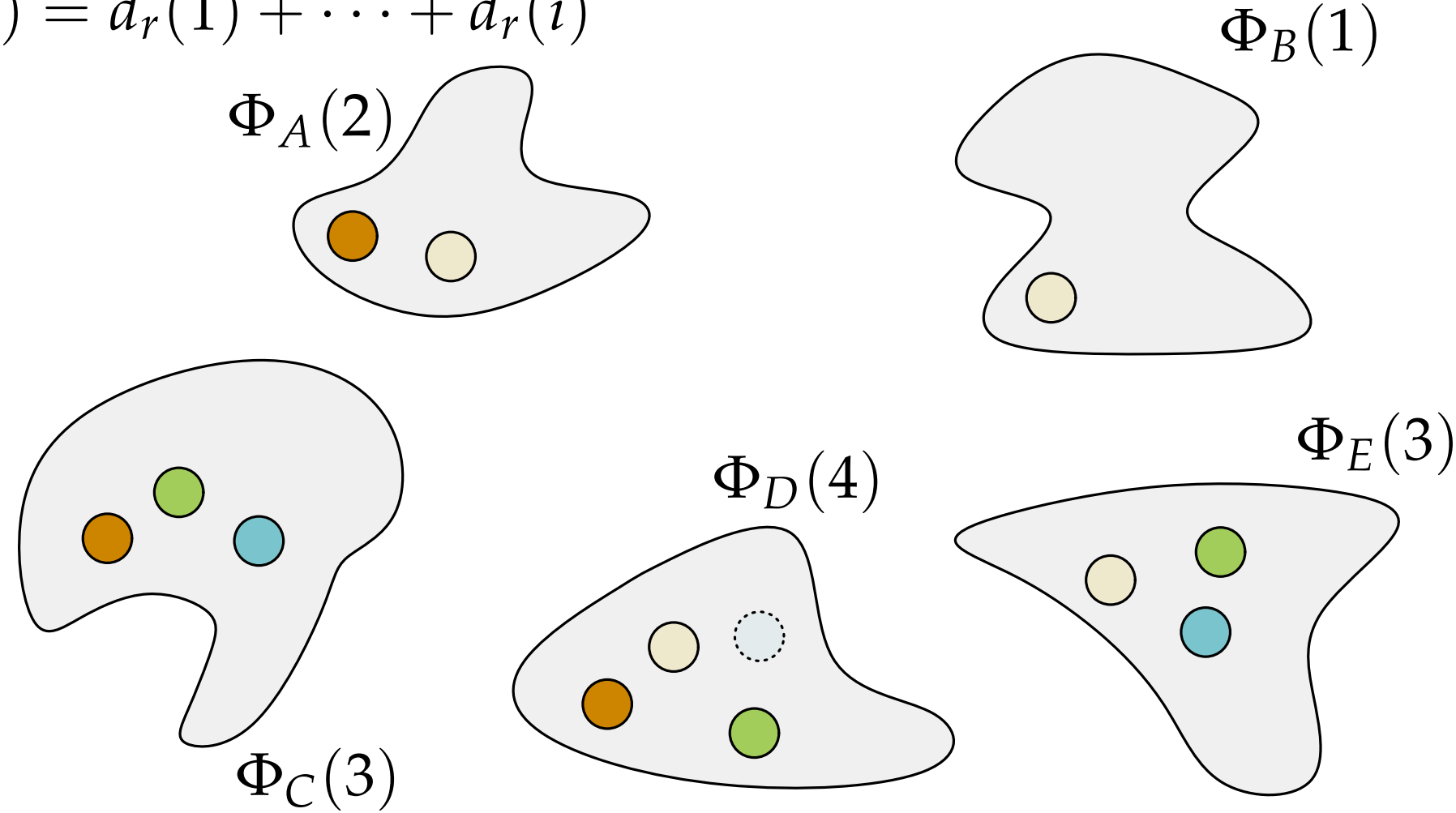

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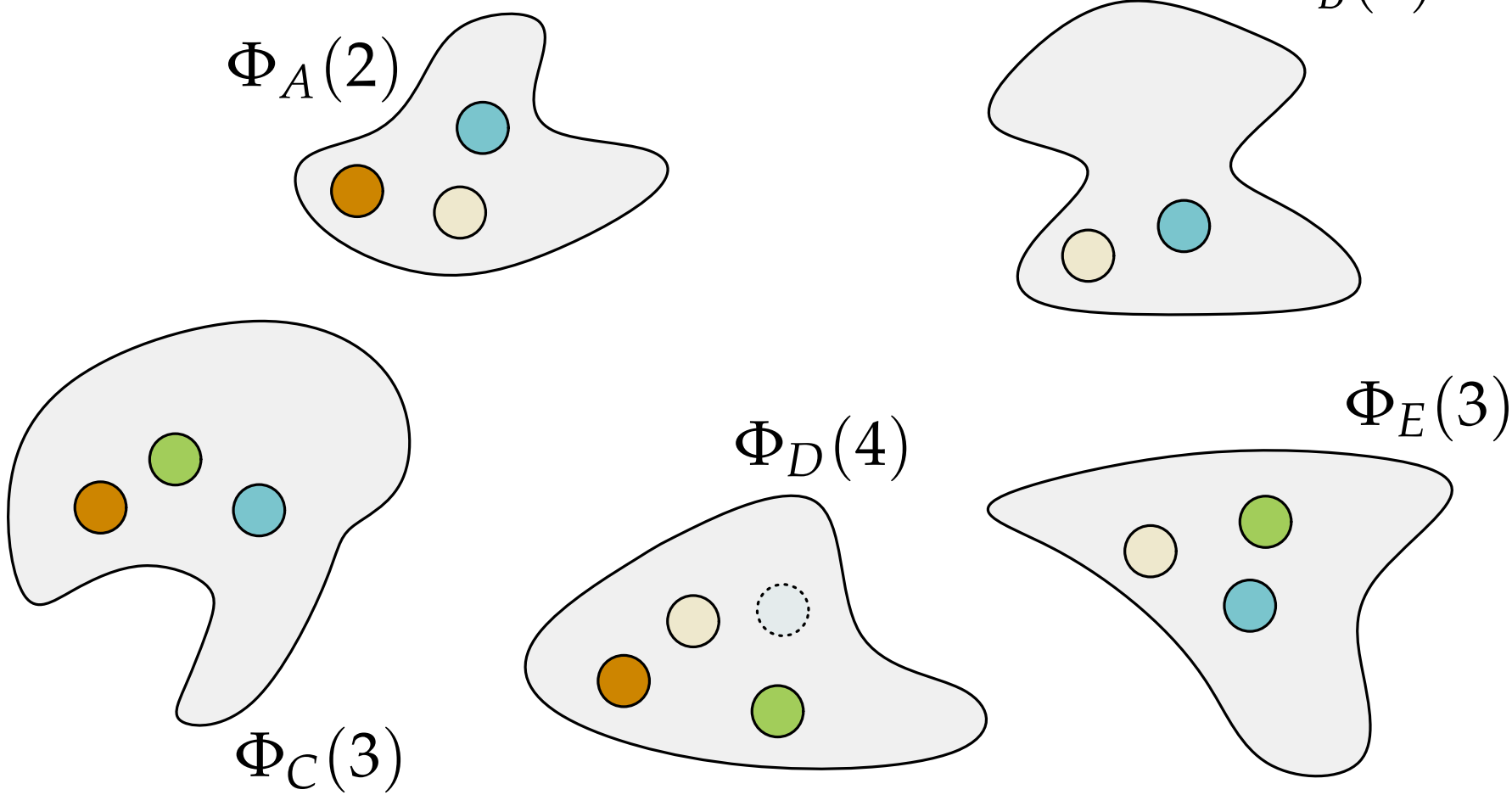

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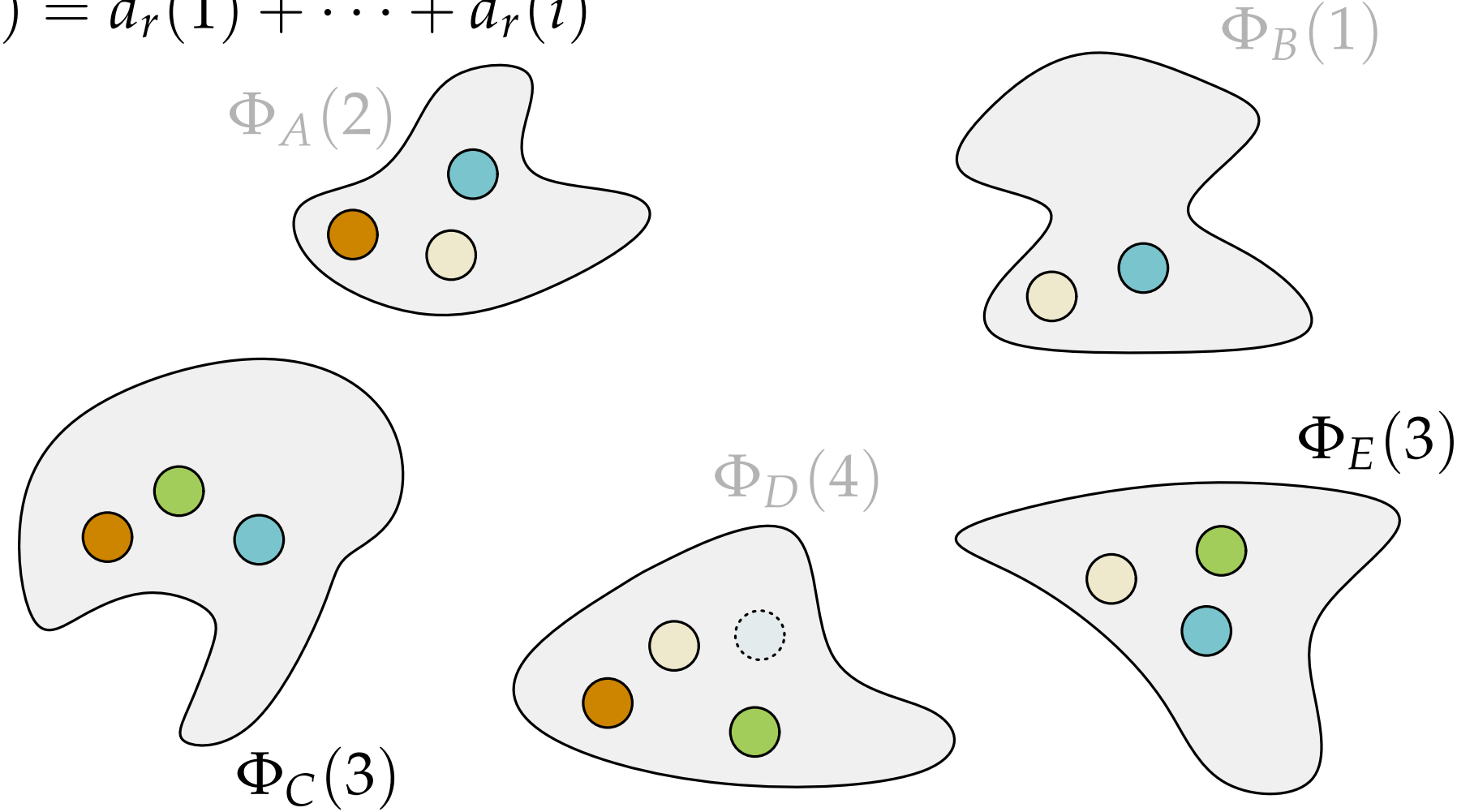

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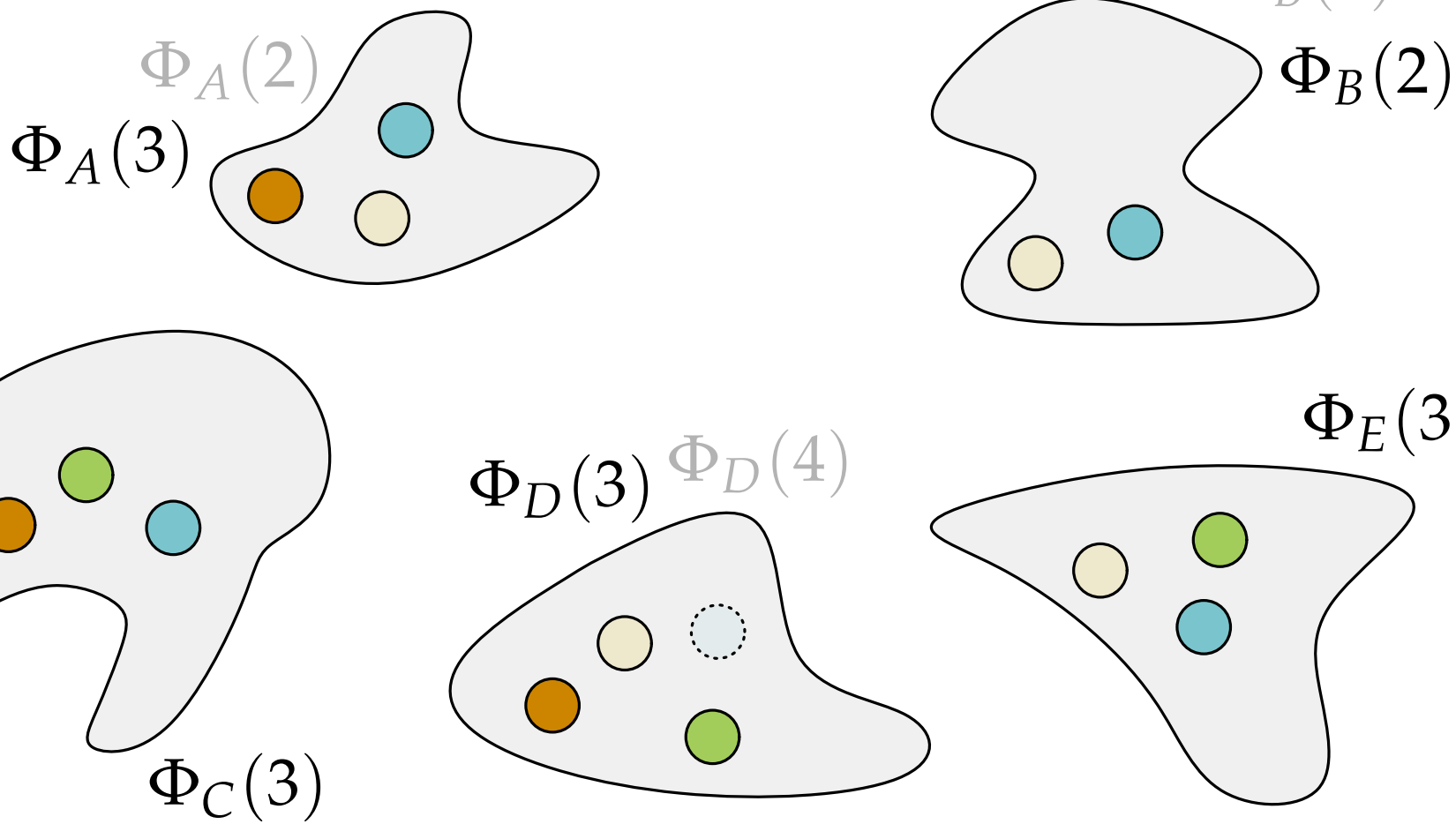

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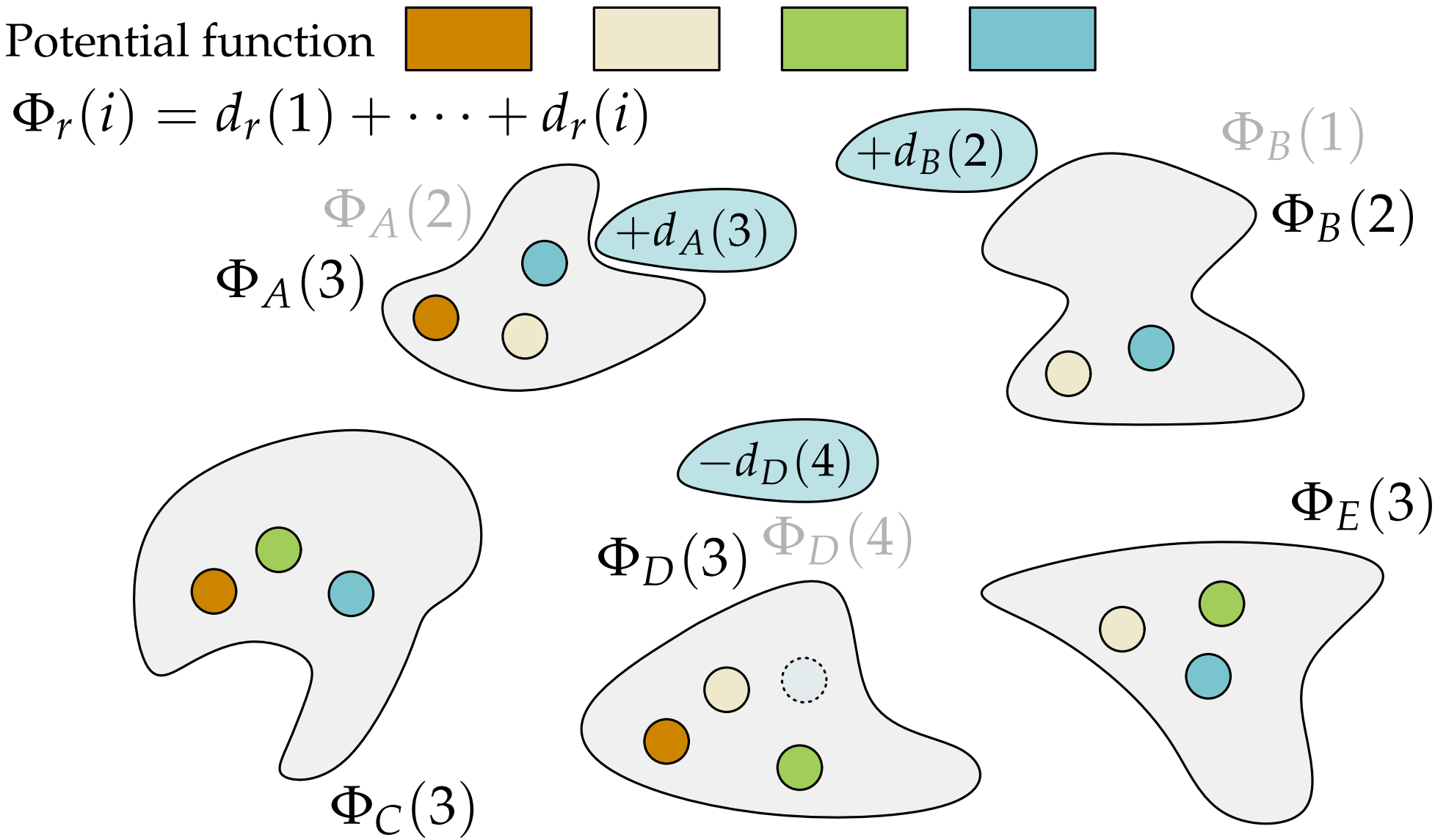

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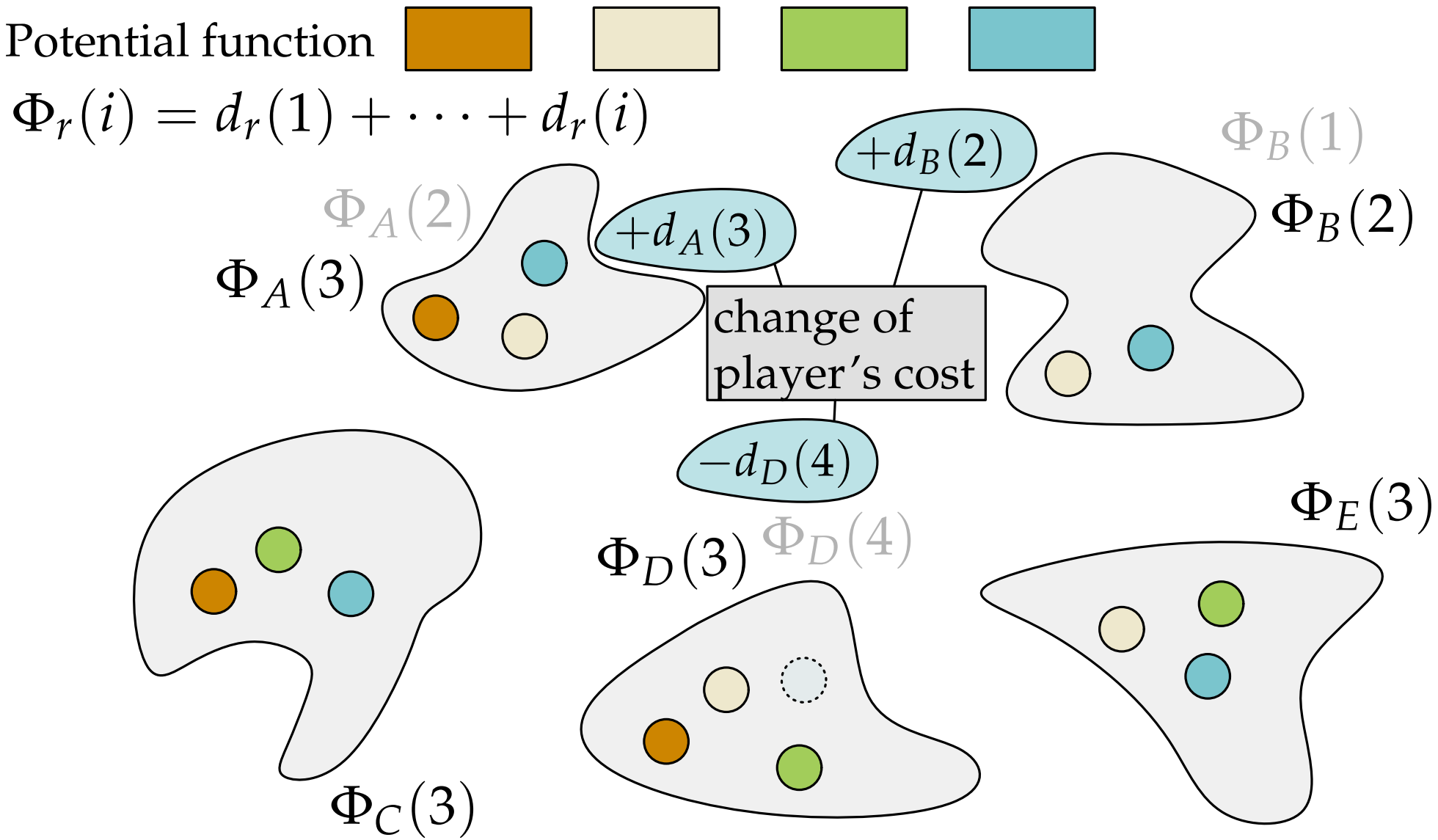

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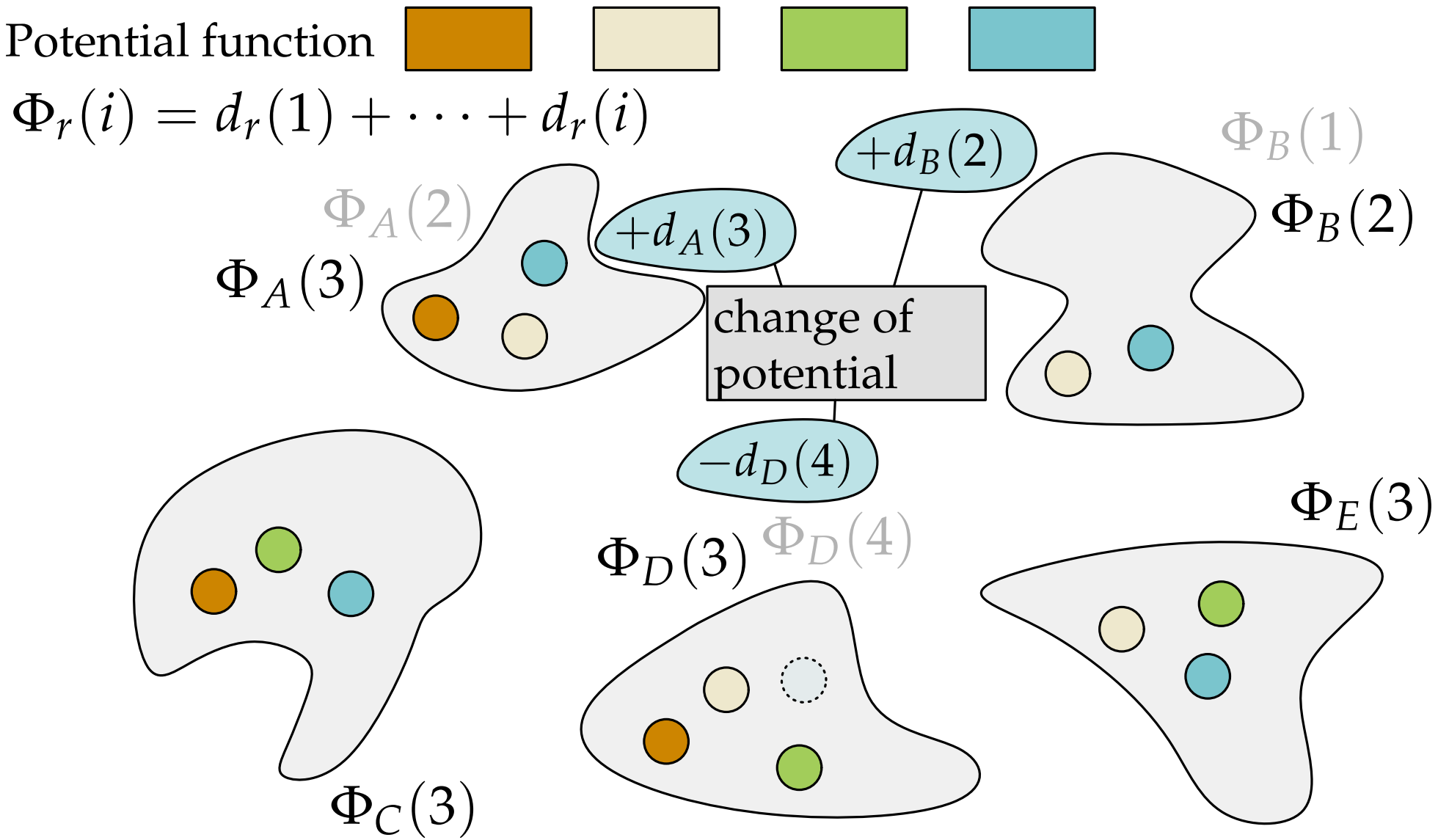

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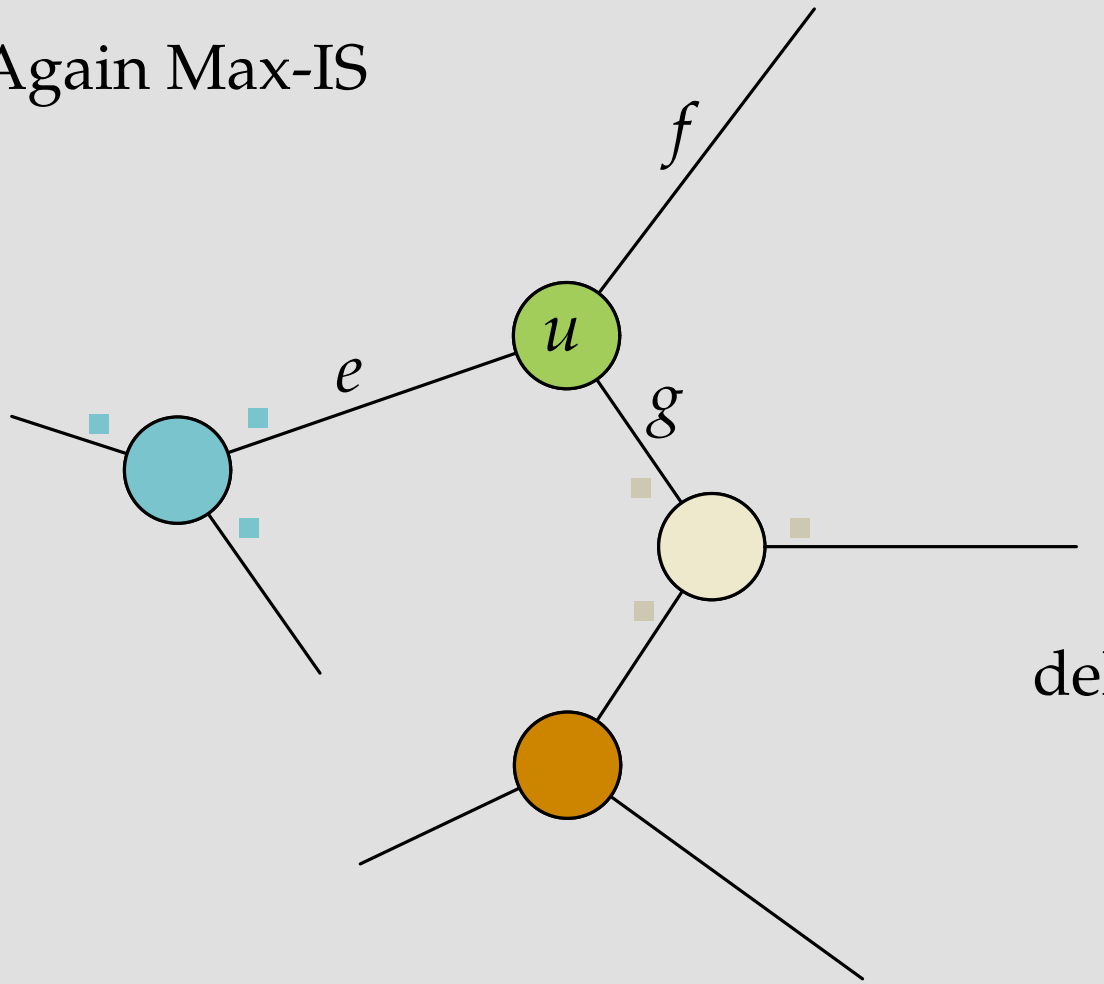

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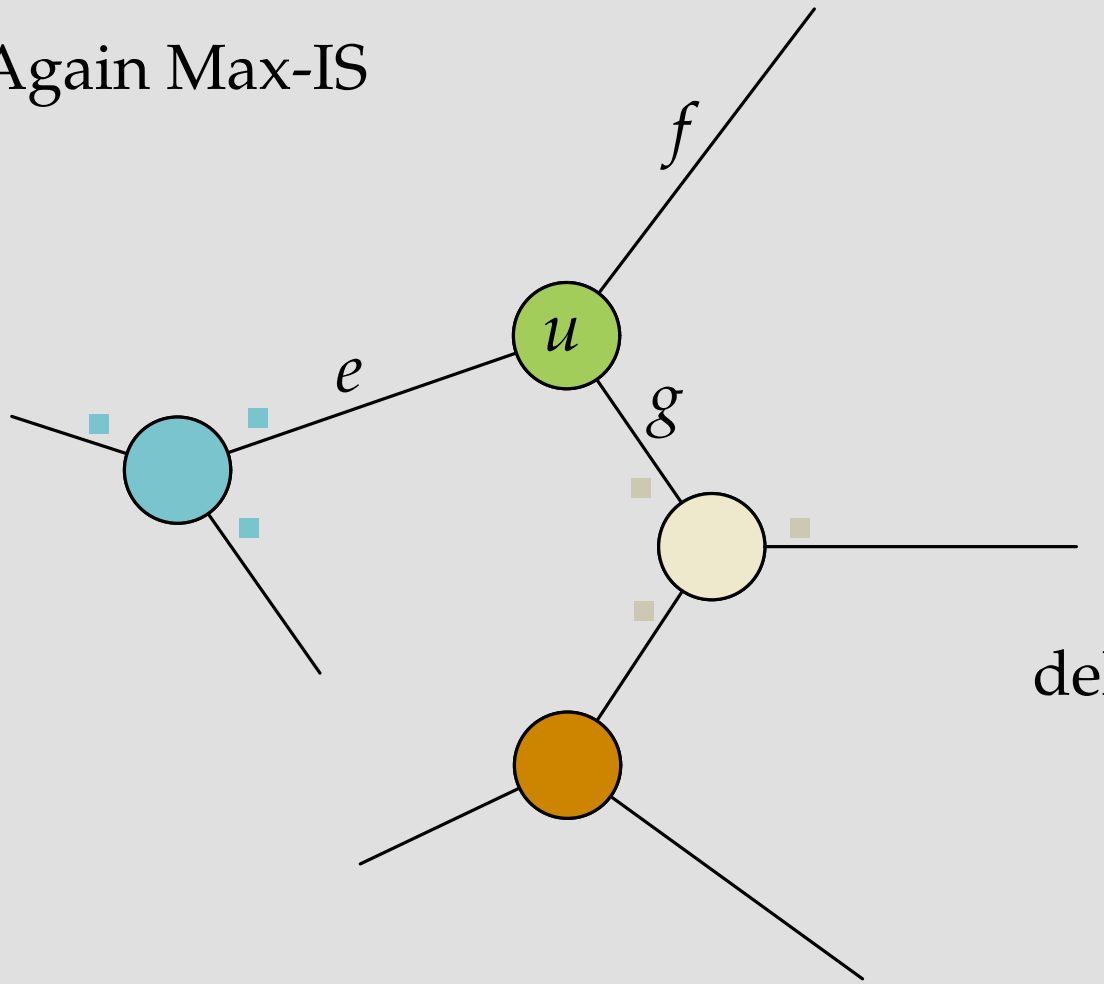

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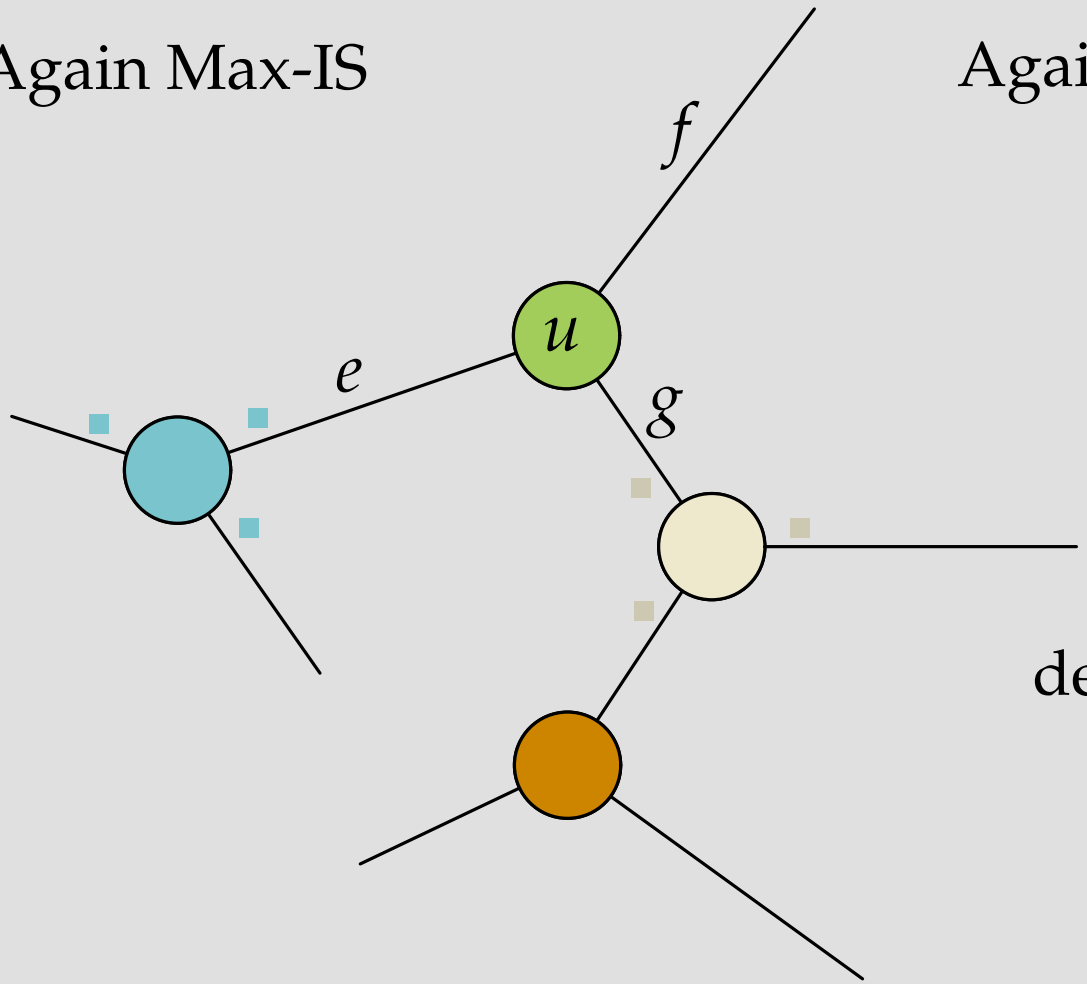

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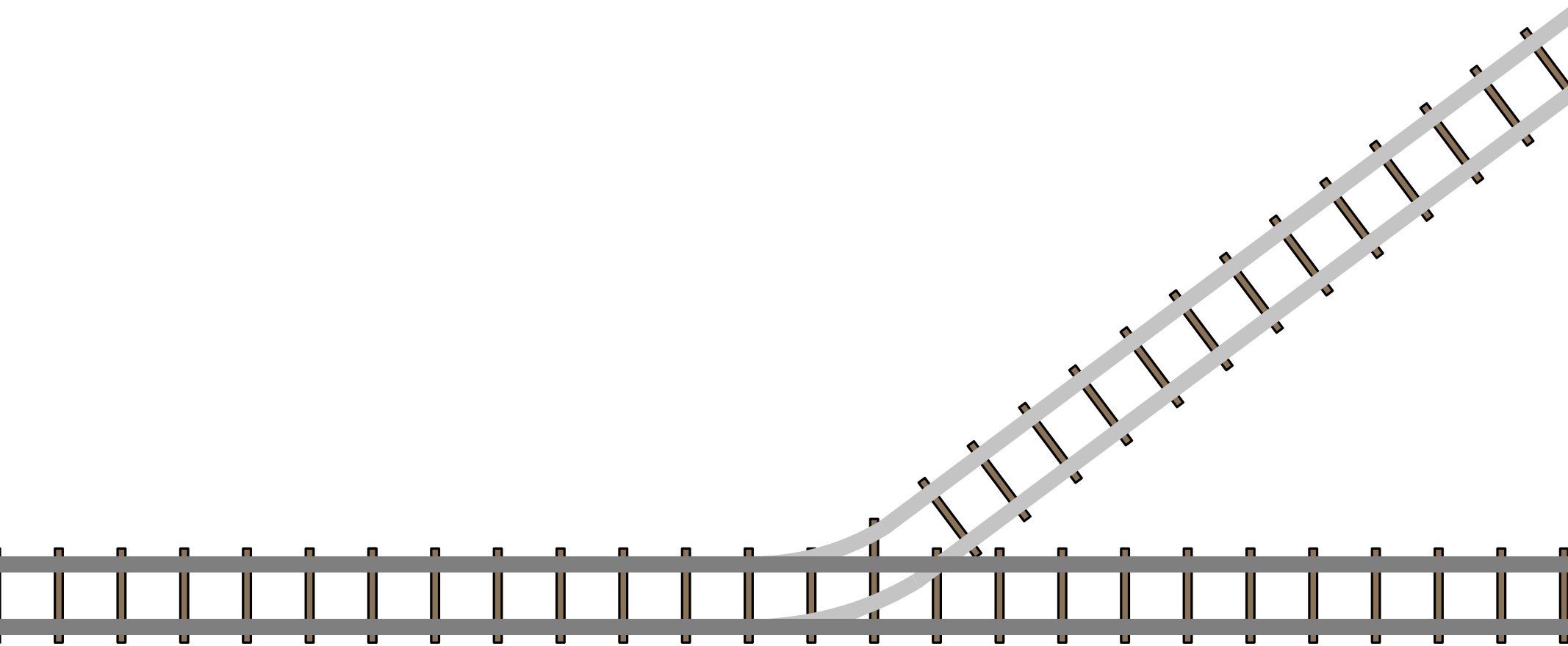
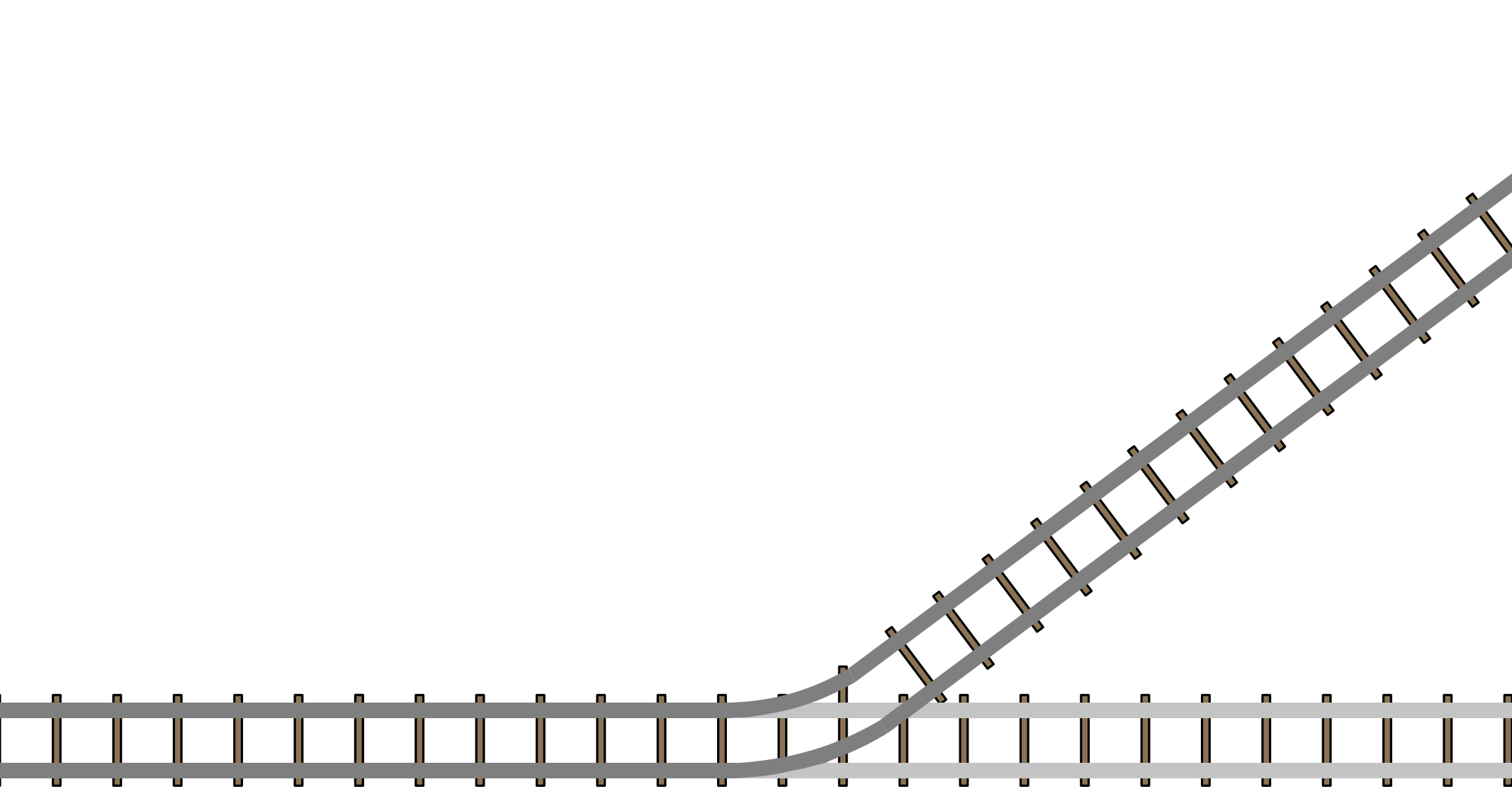
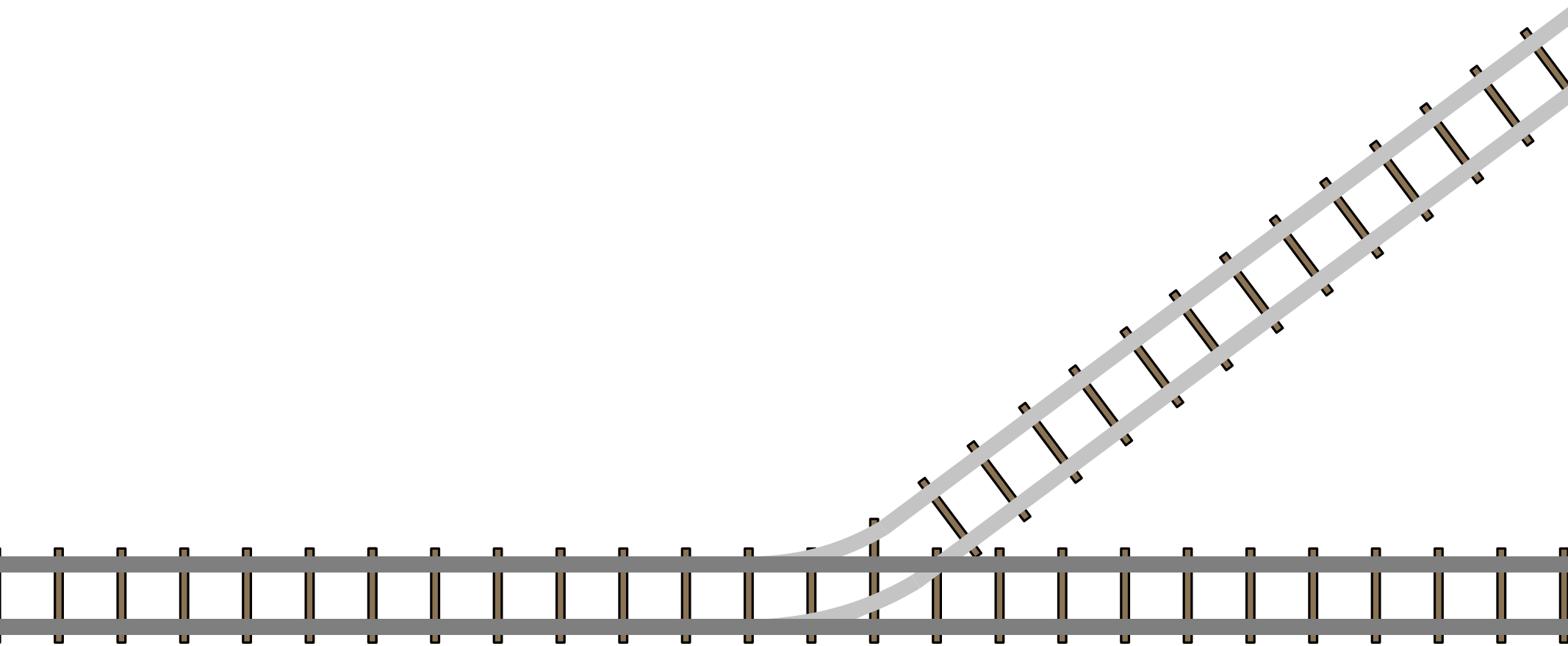
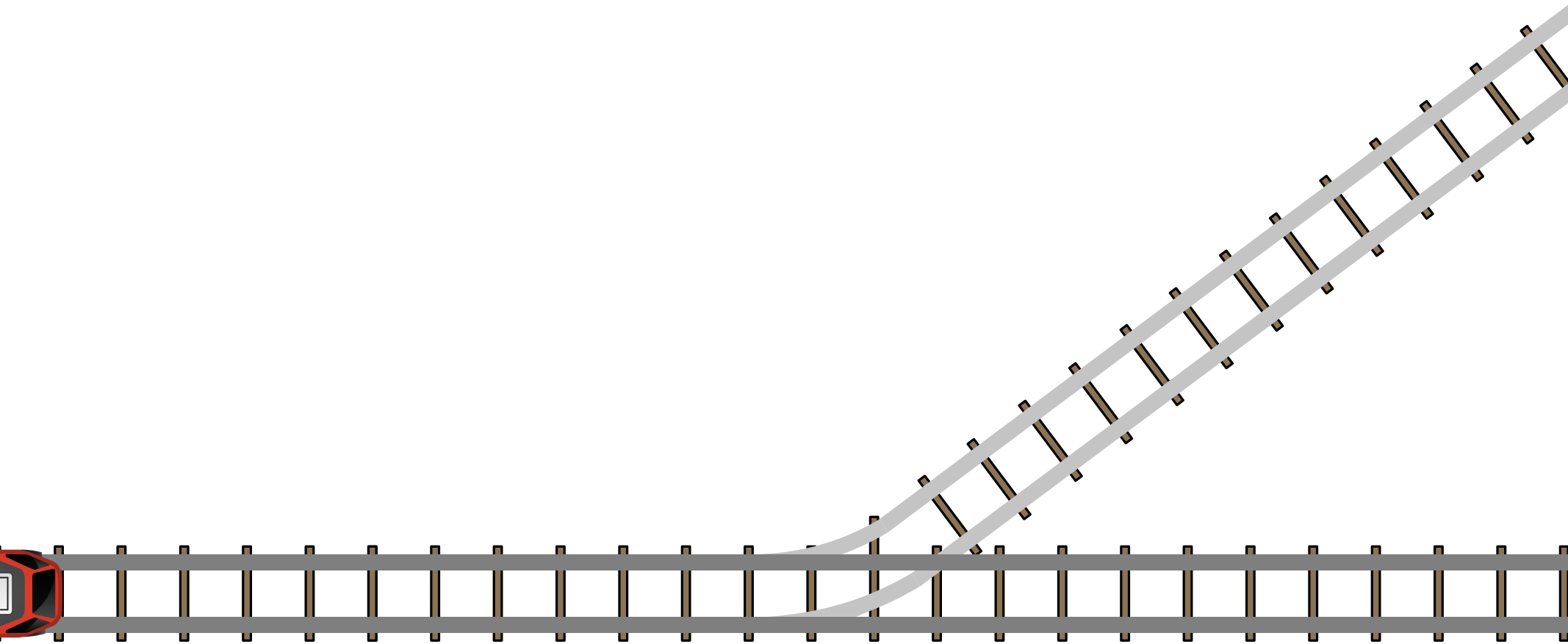
Arrival

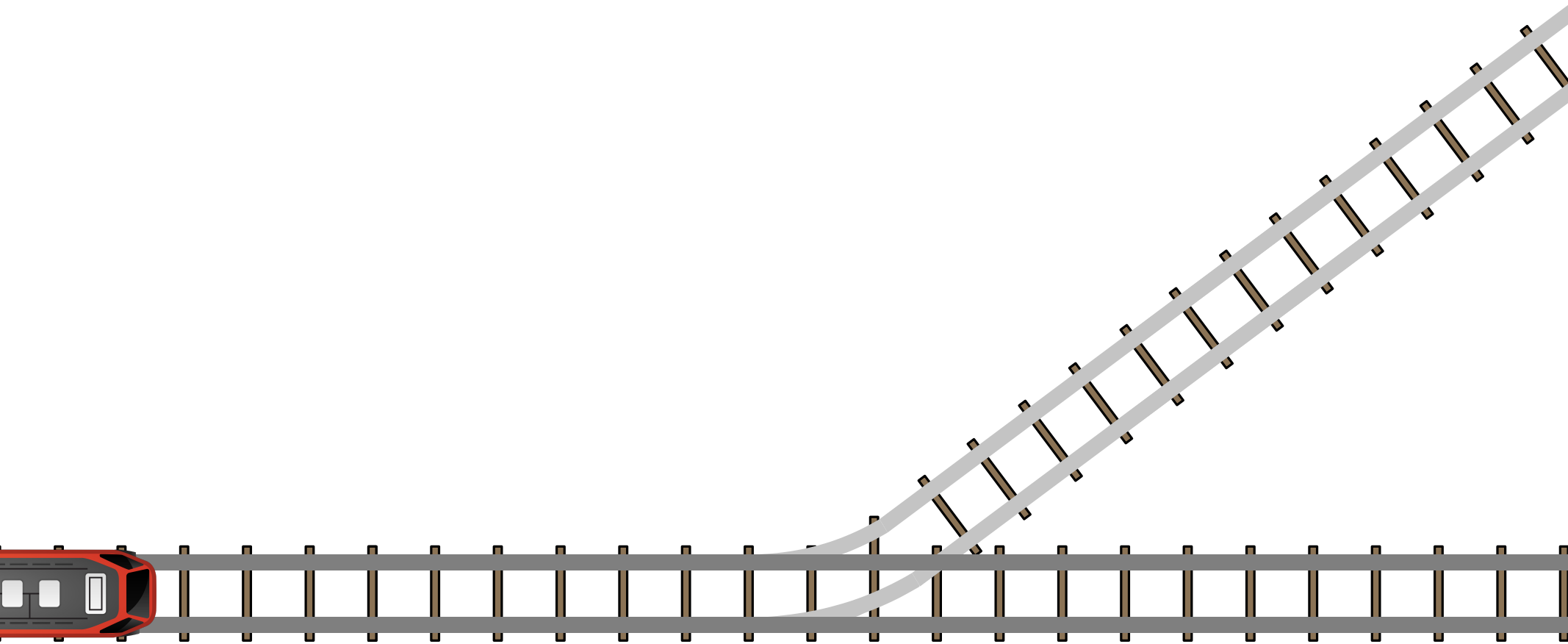
Dohrau, Gärtner, Kohler, Matoušek, Welzl (2017)

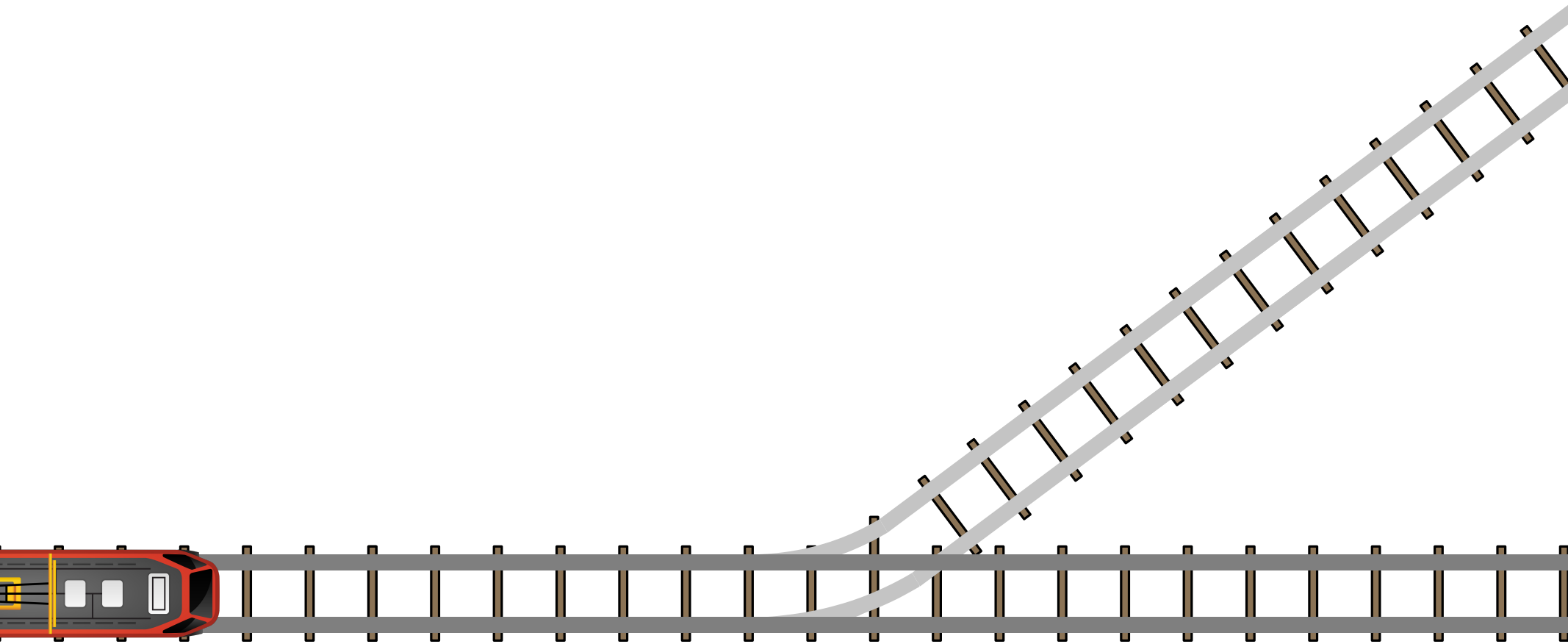


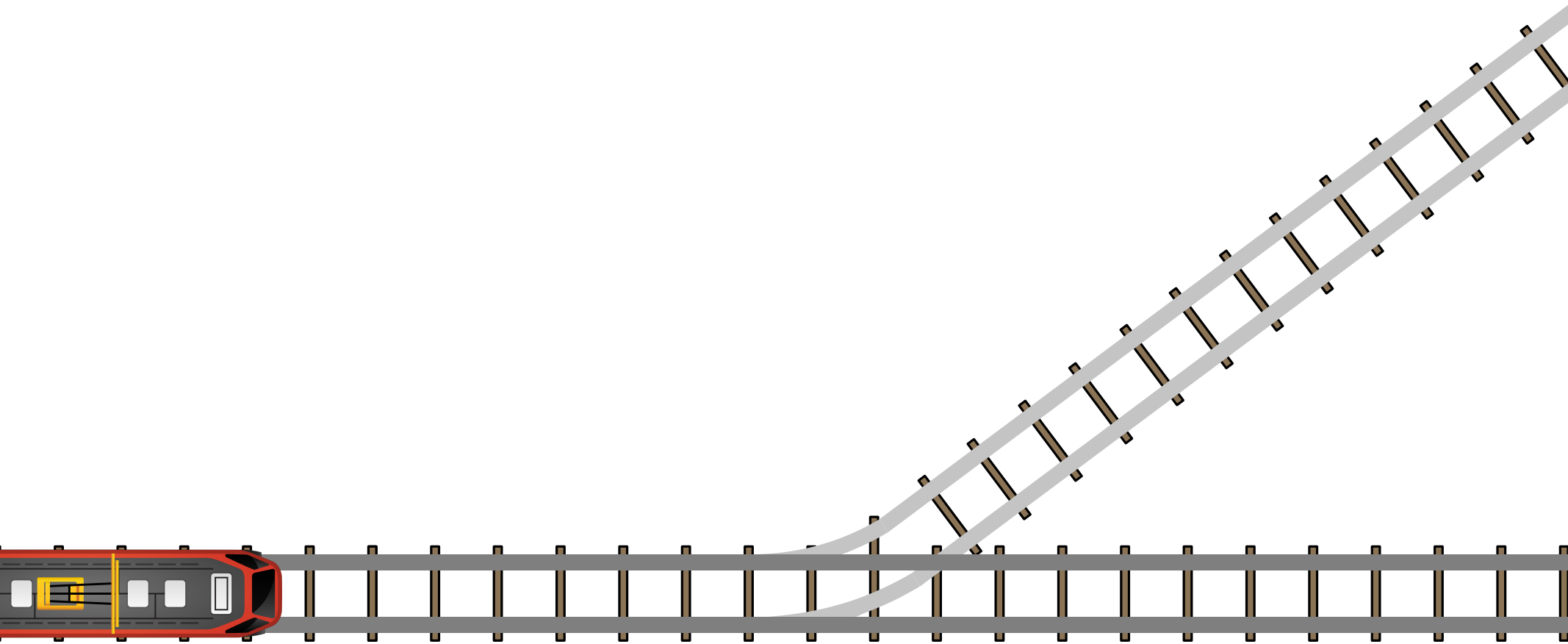


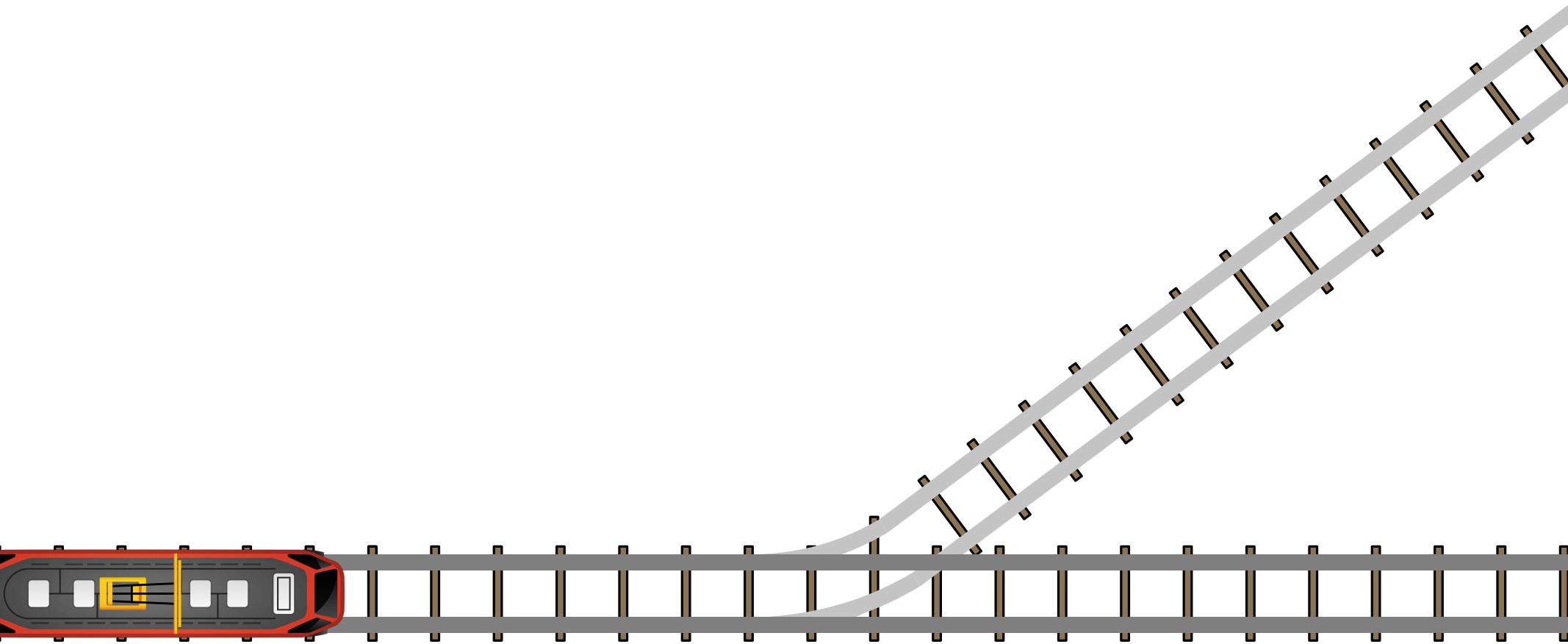


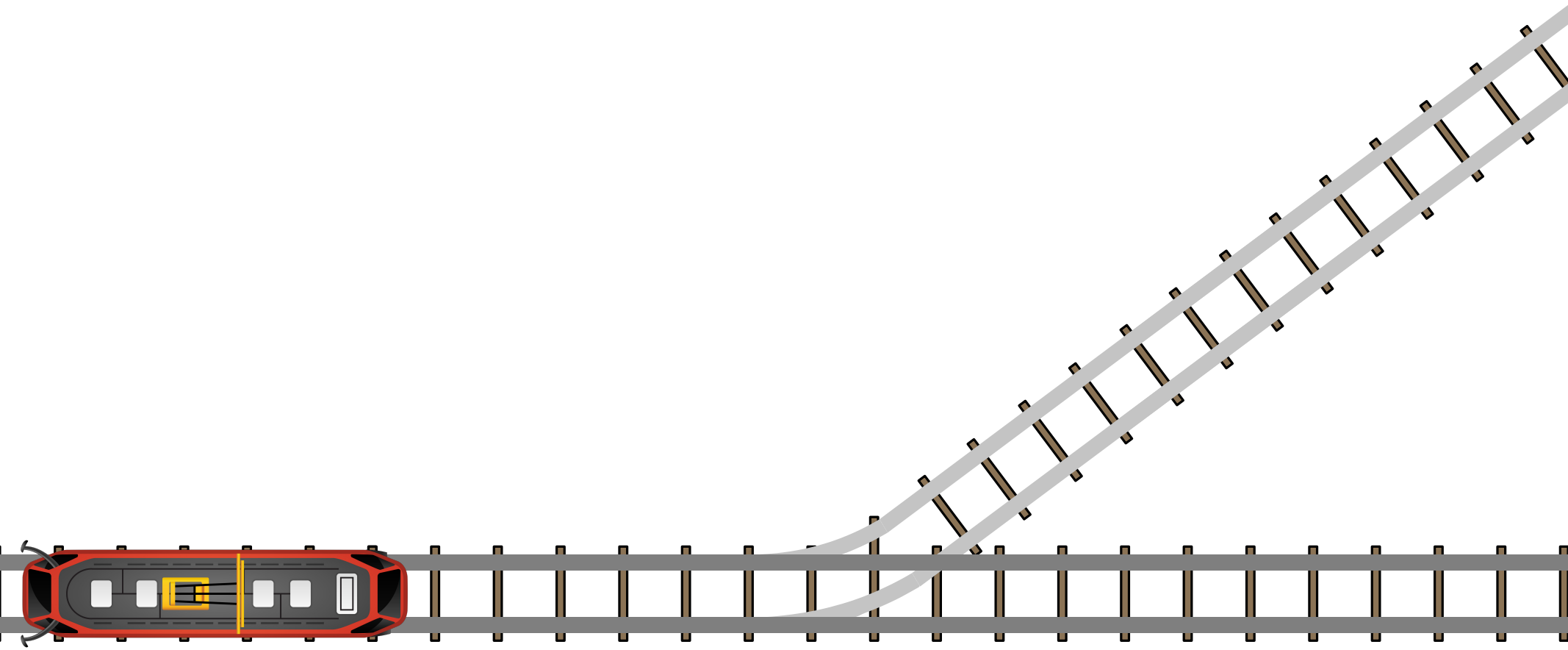


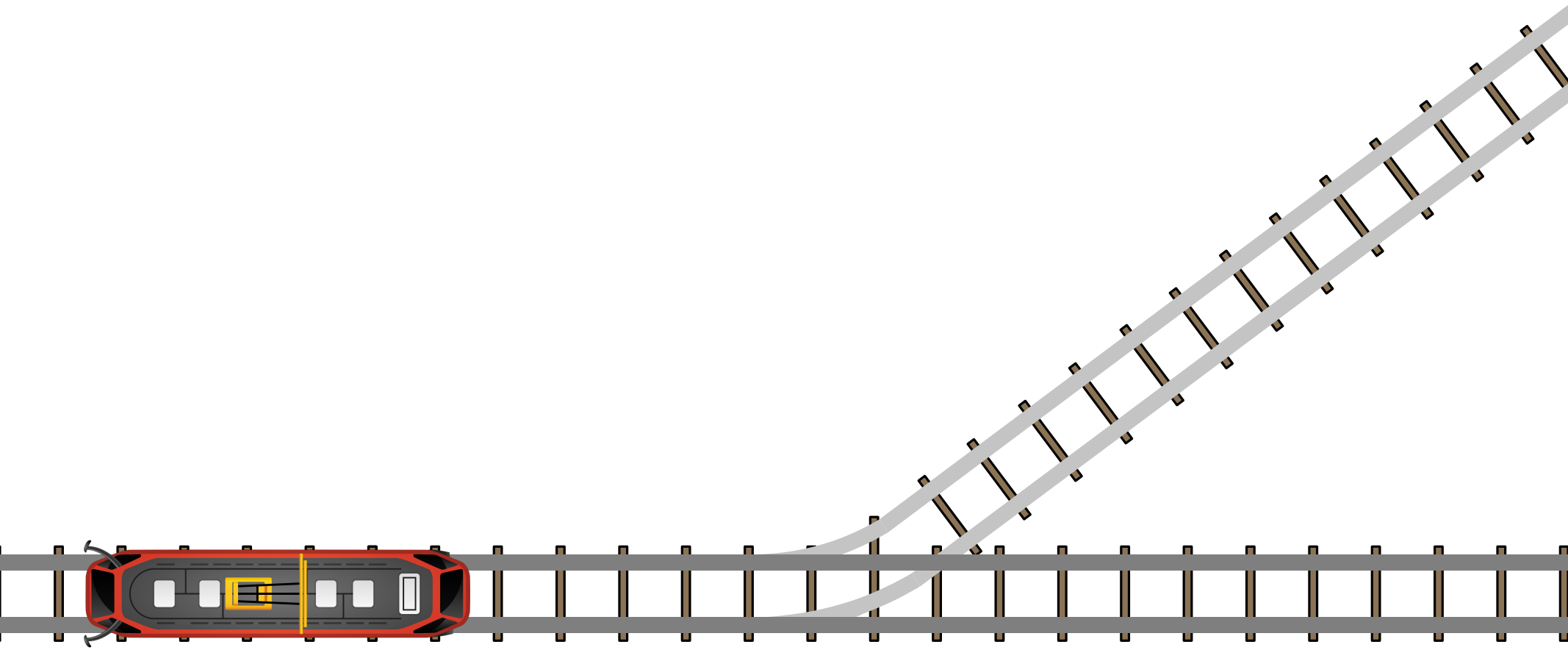


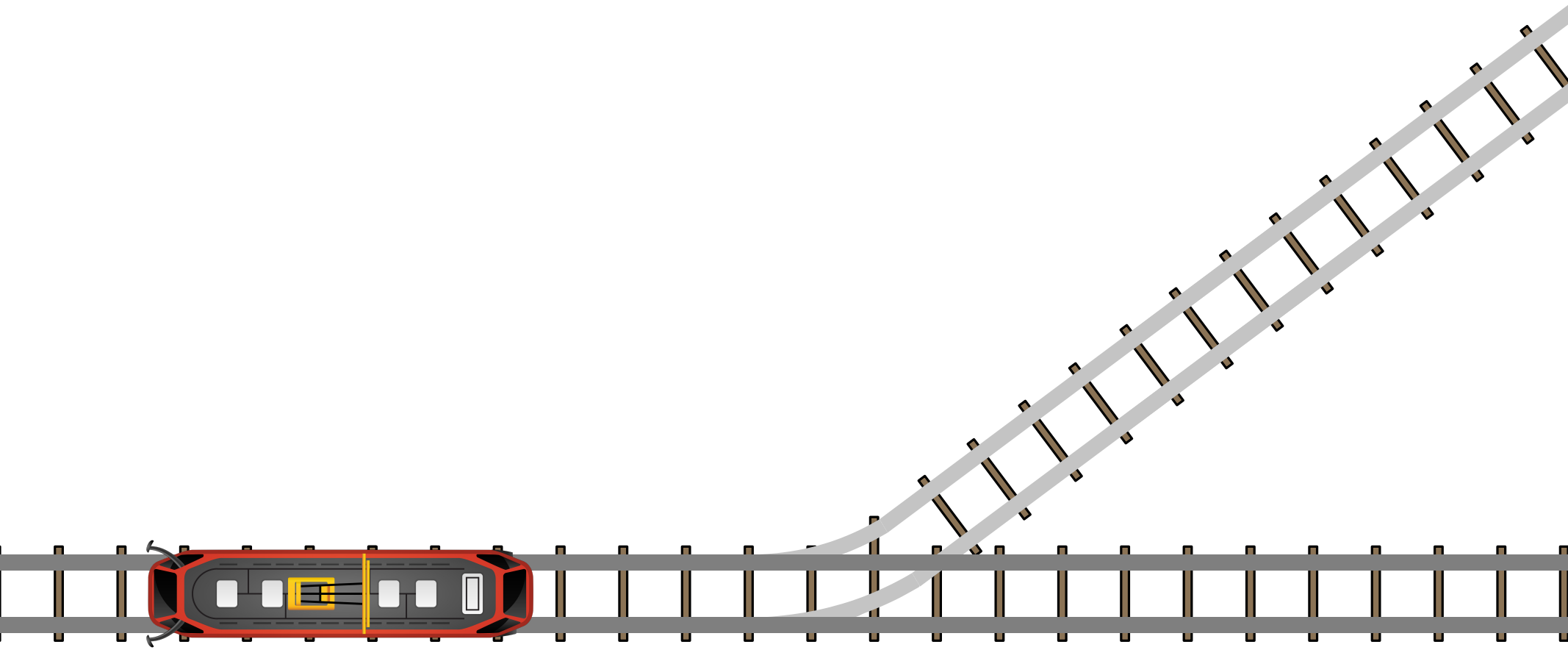












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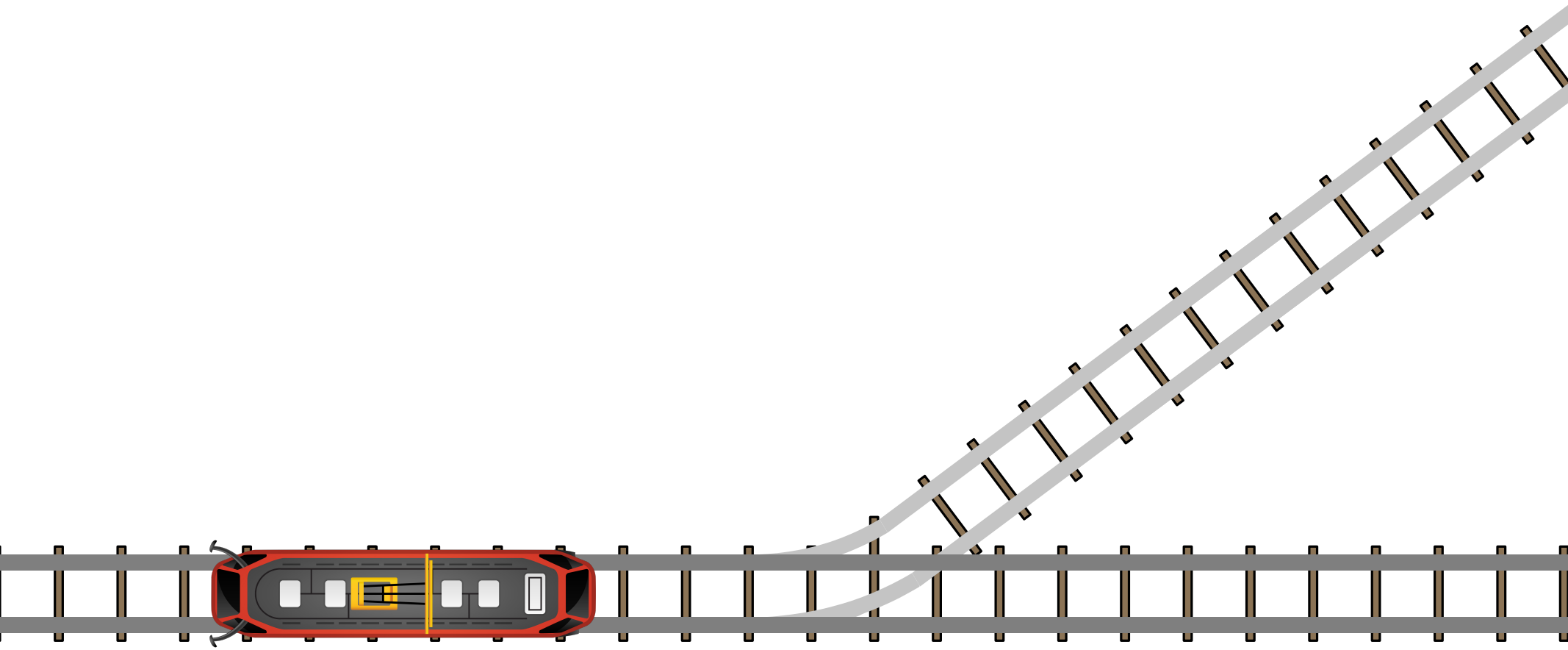

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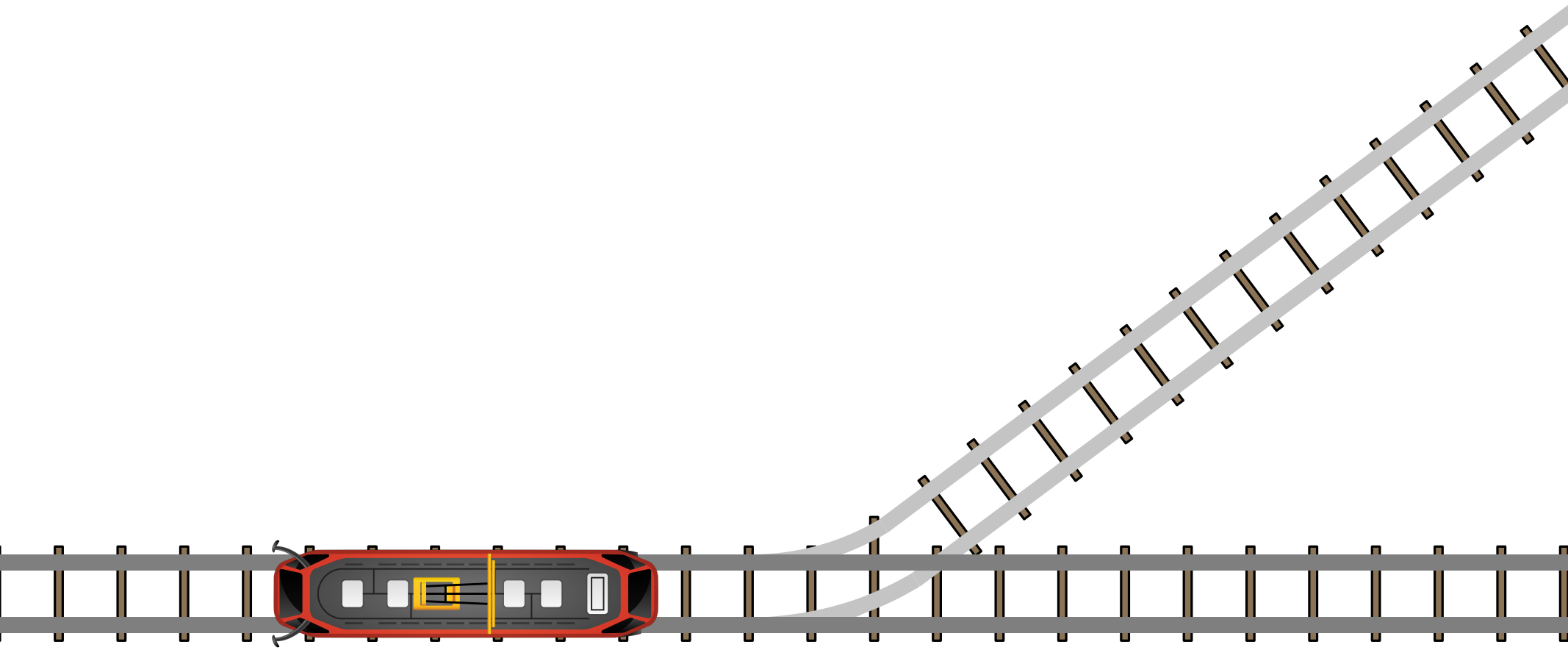

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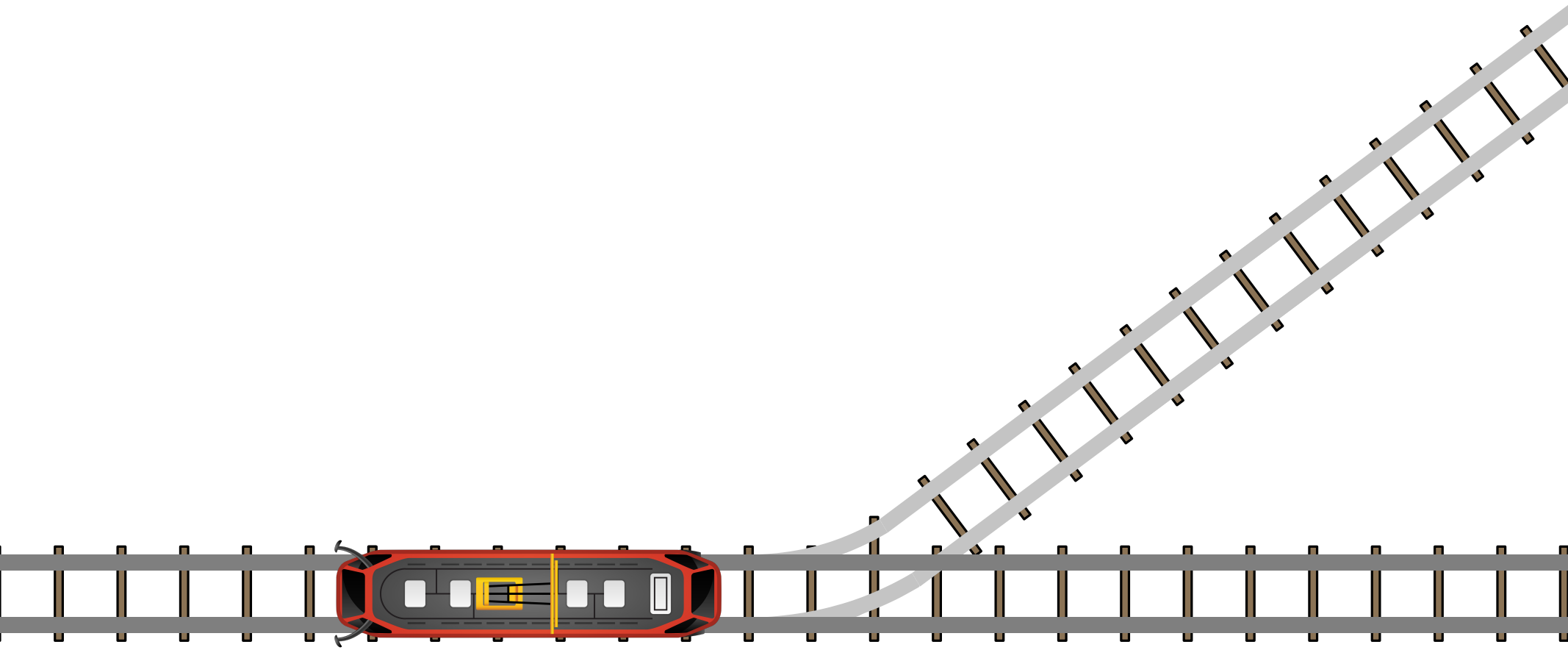

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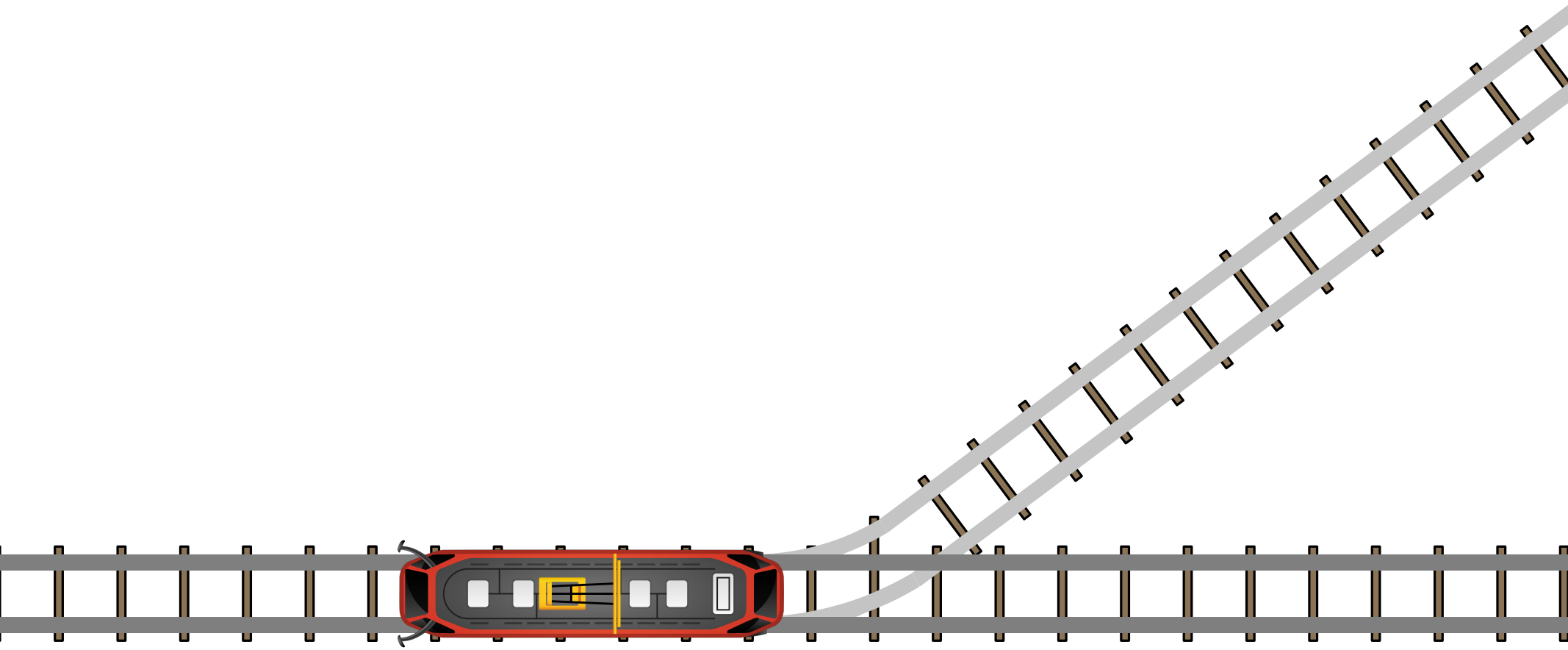

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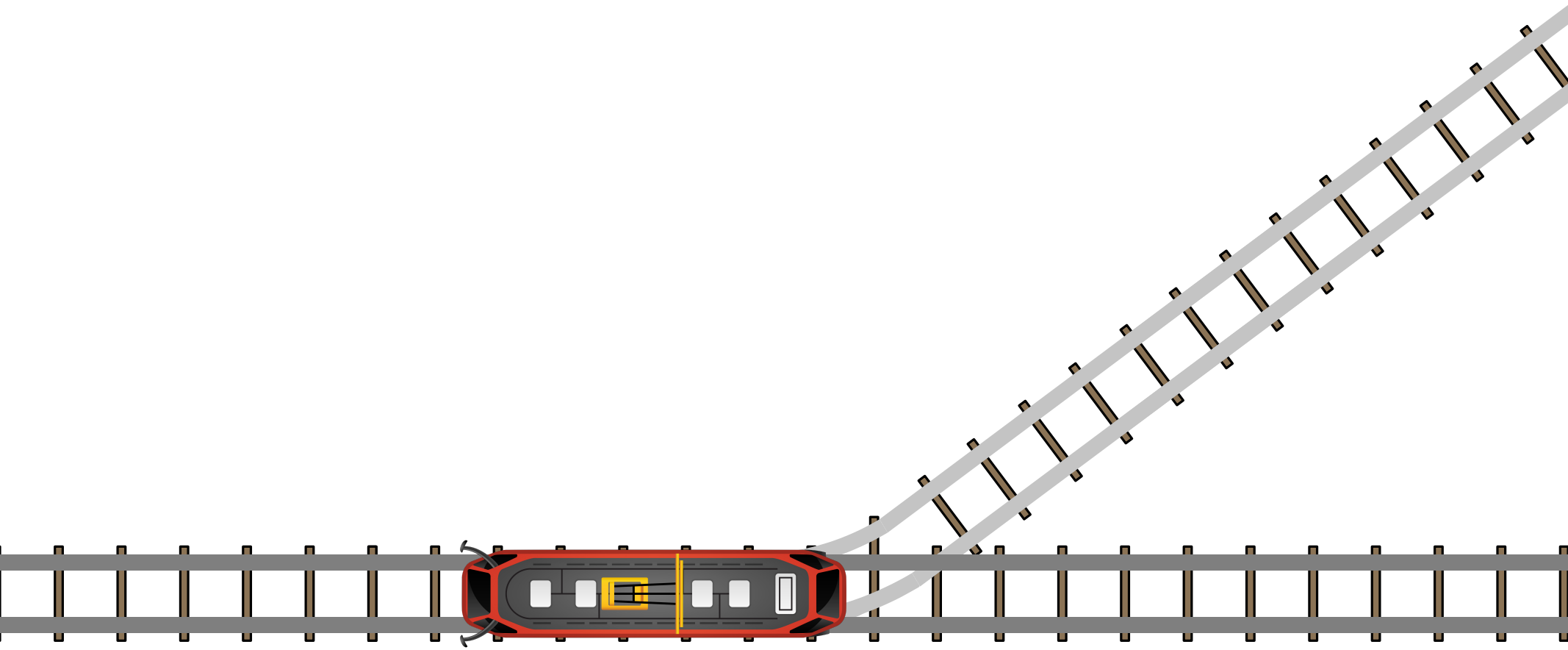

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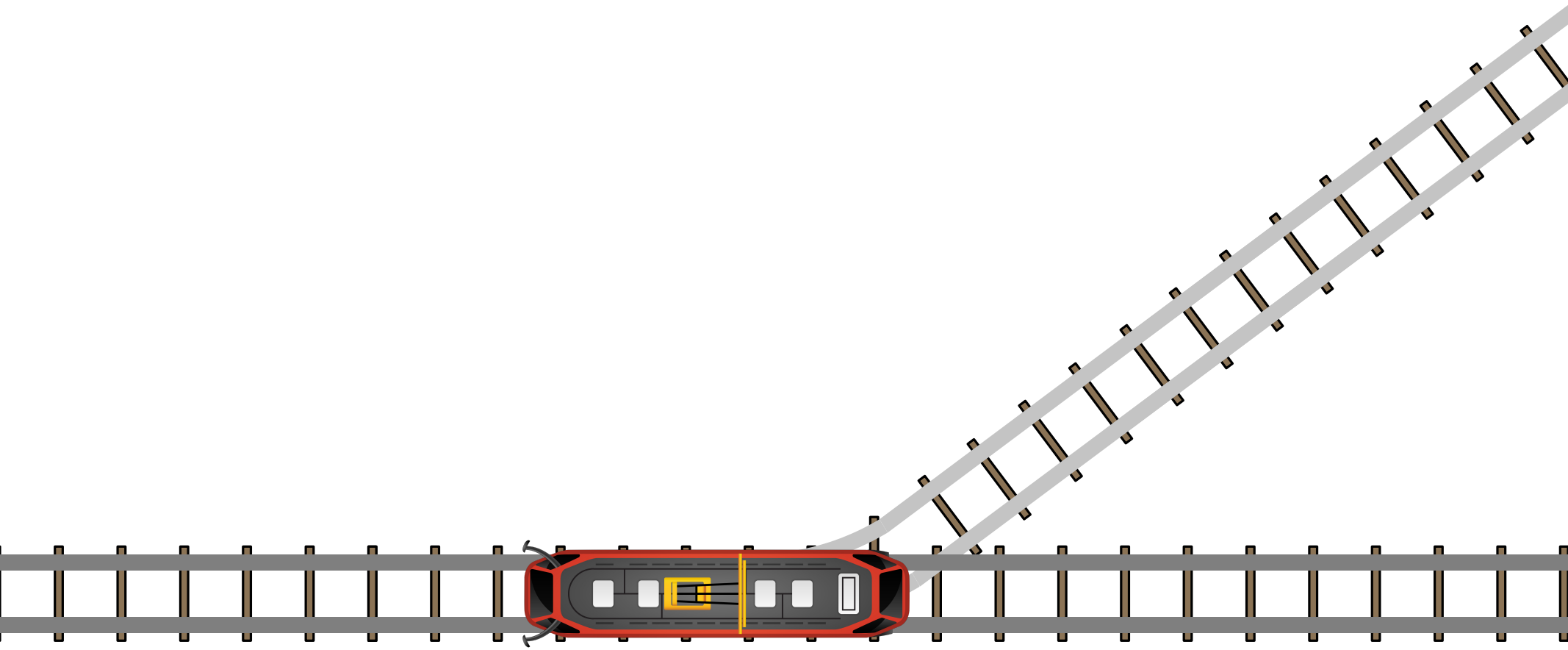

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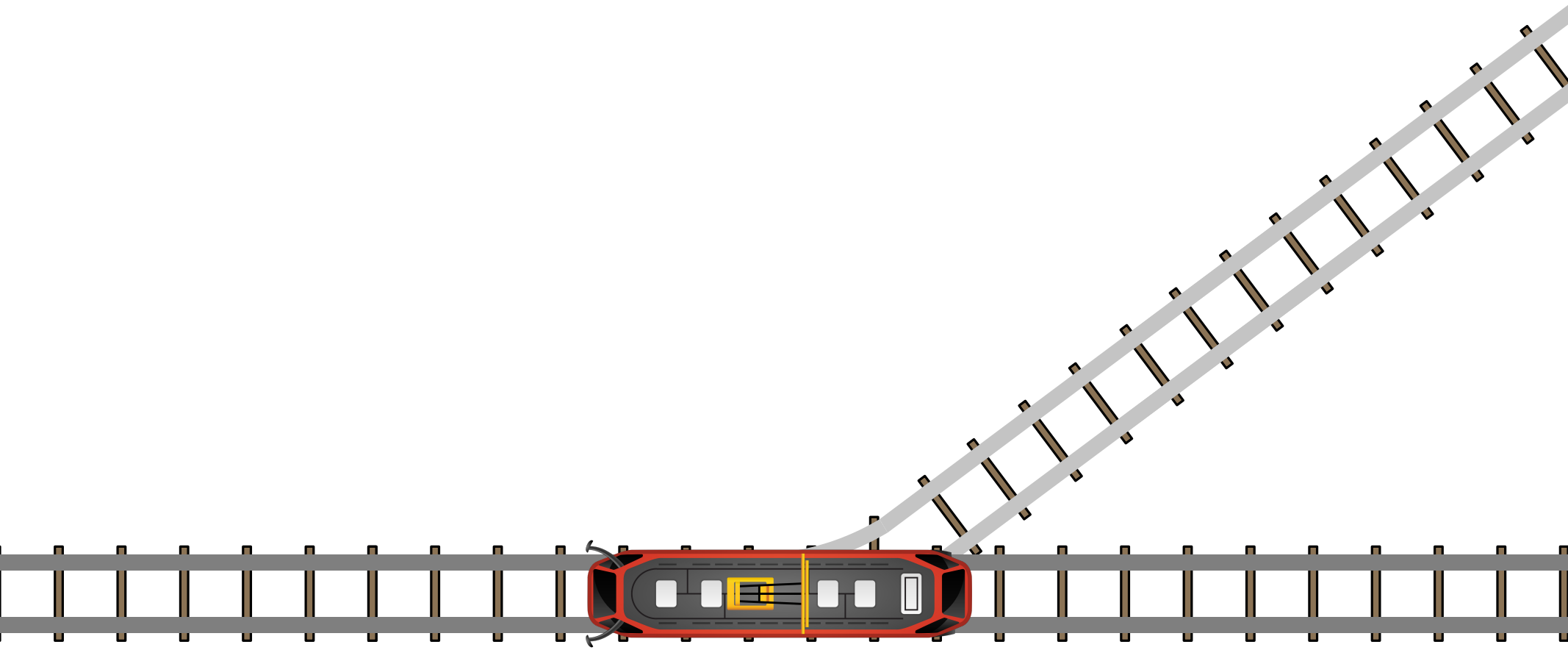

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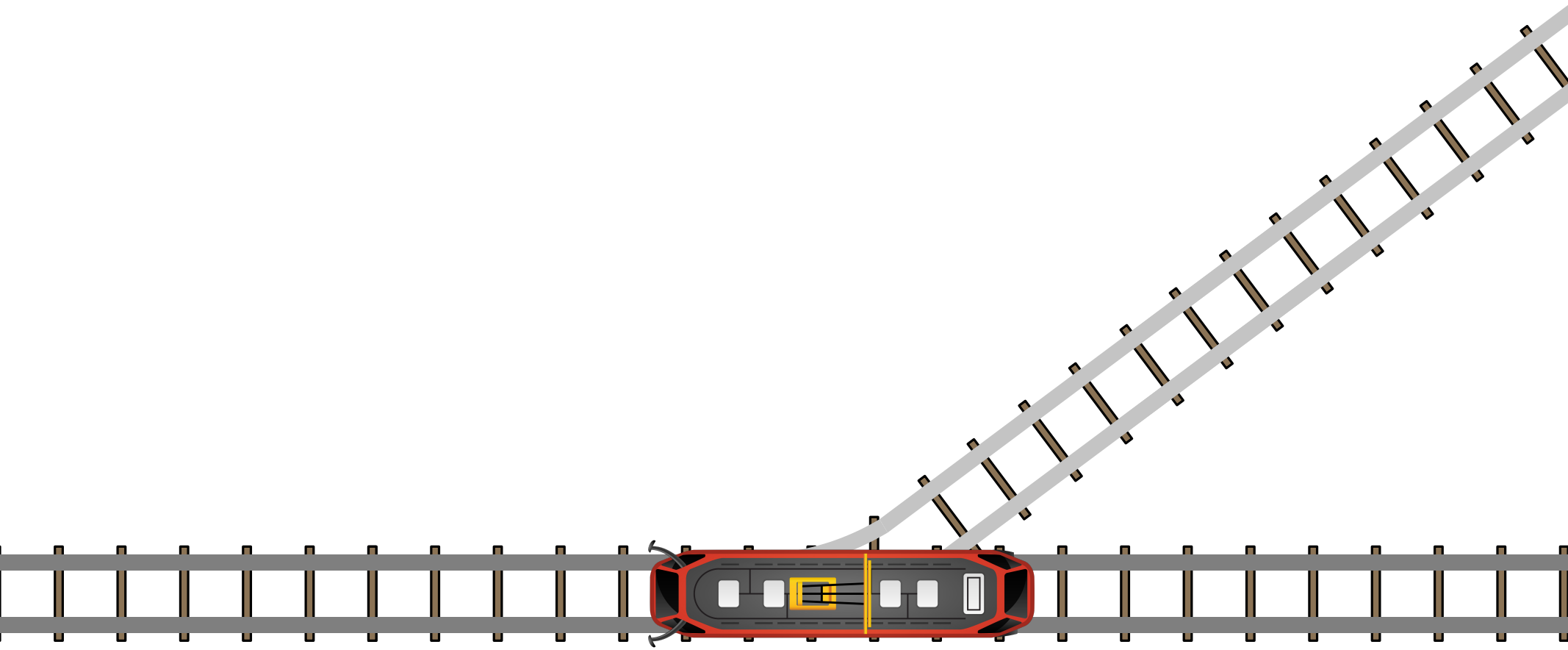

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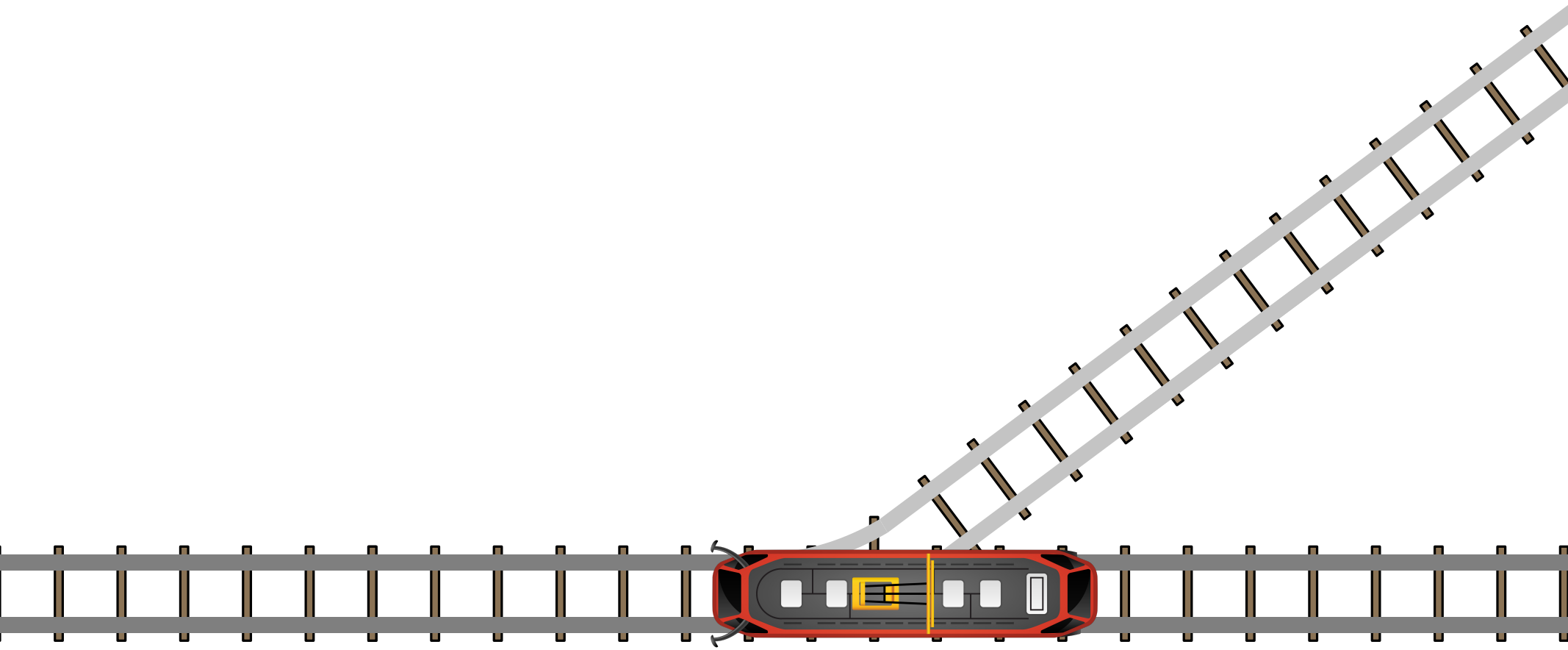

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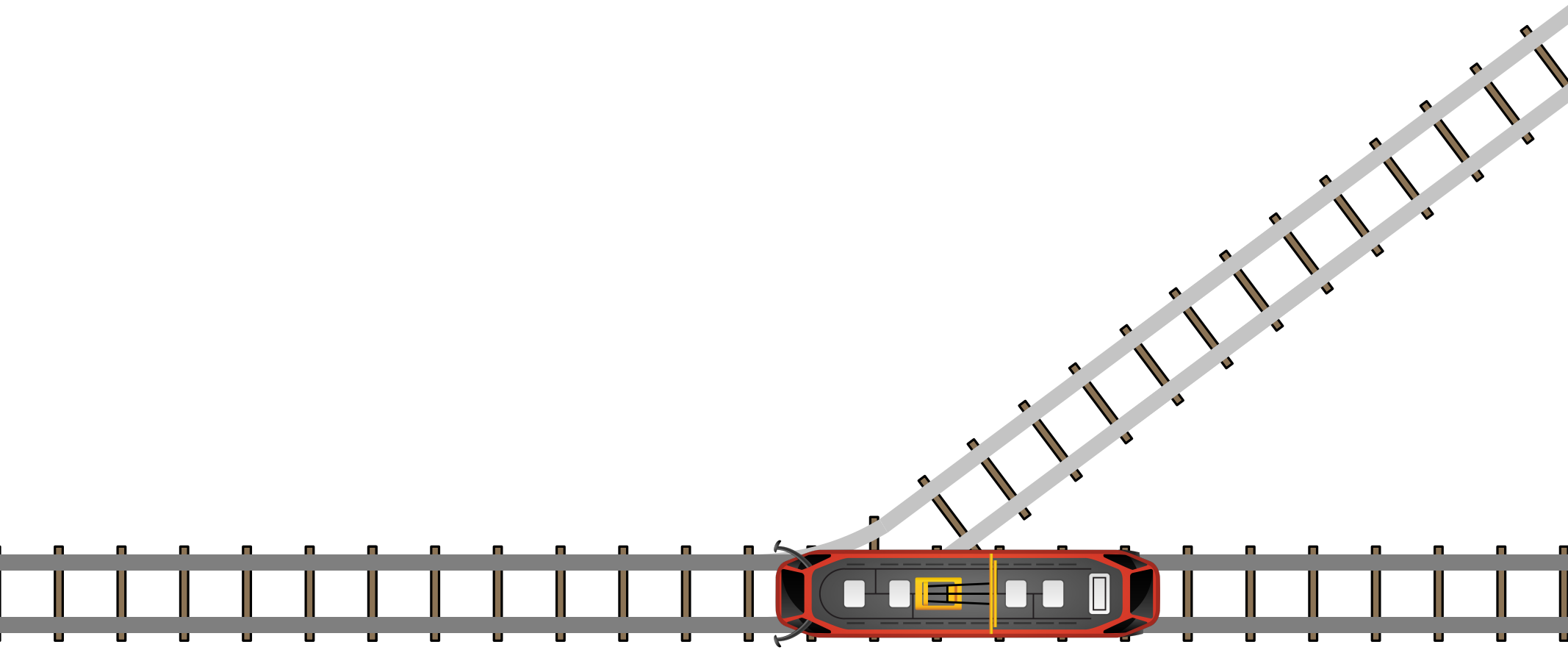

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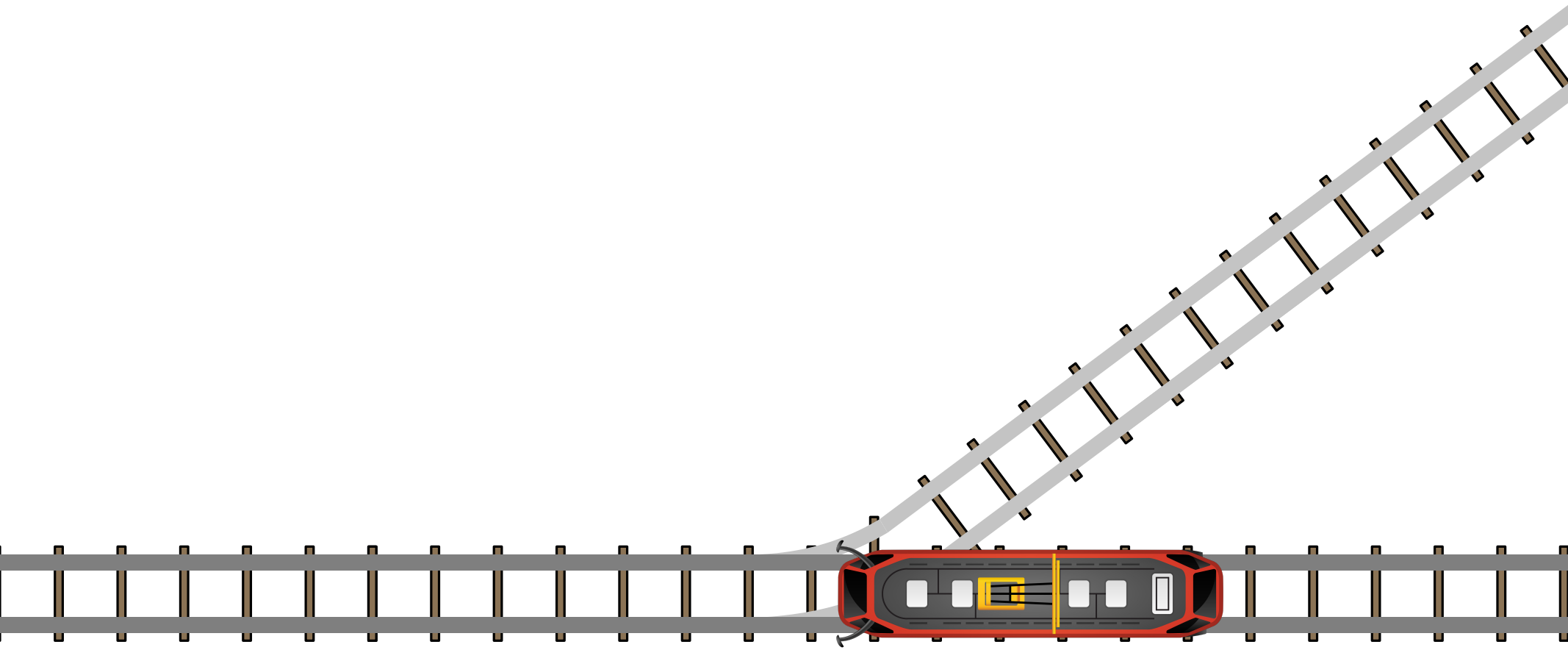

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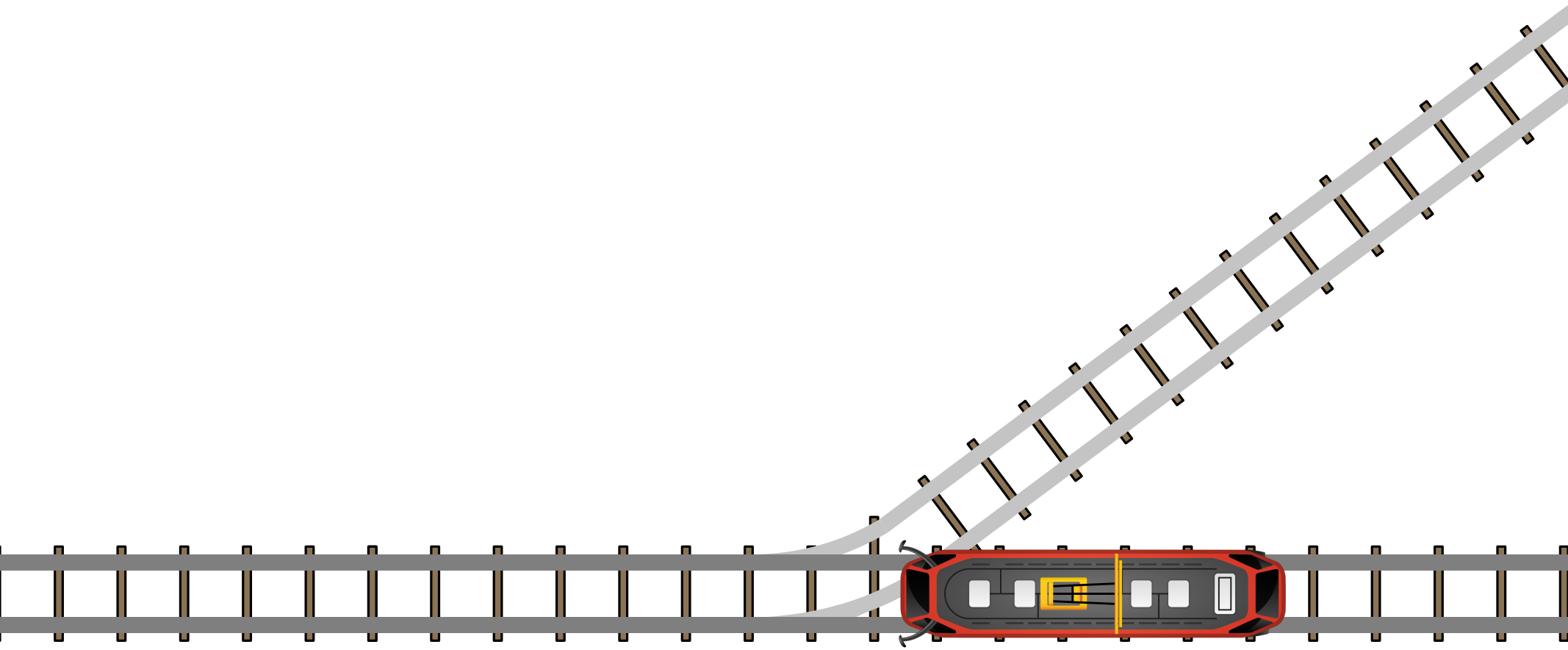

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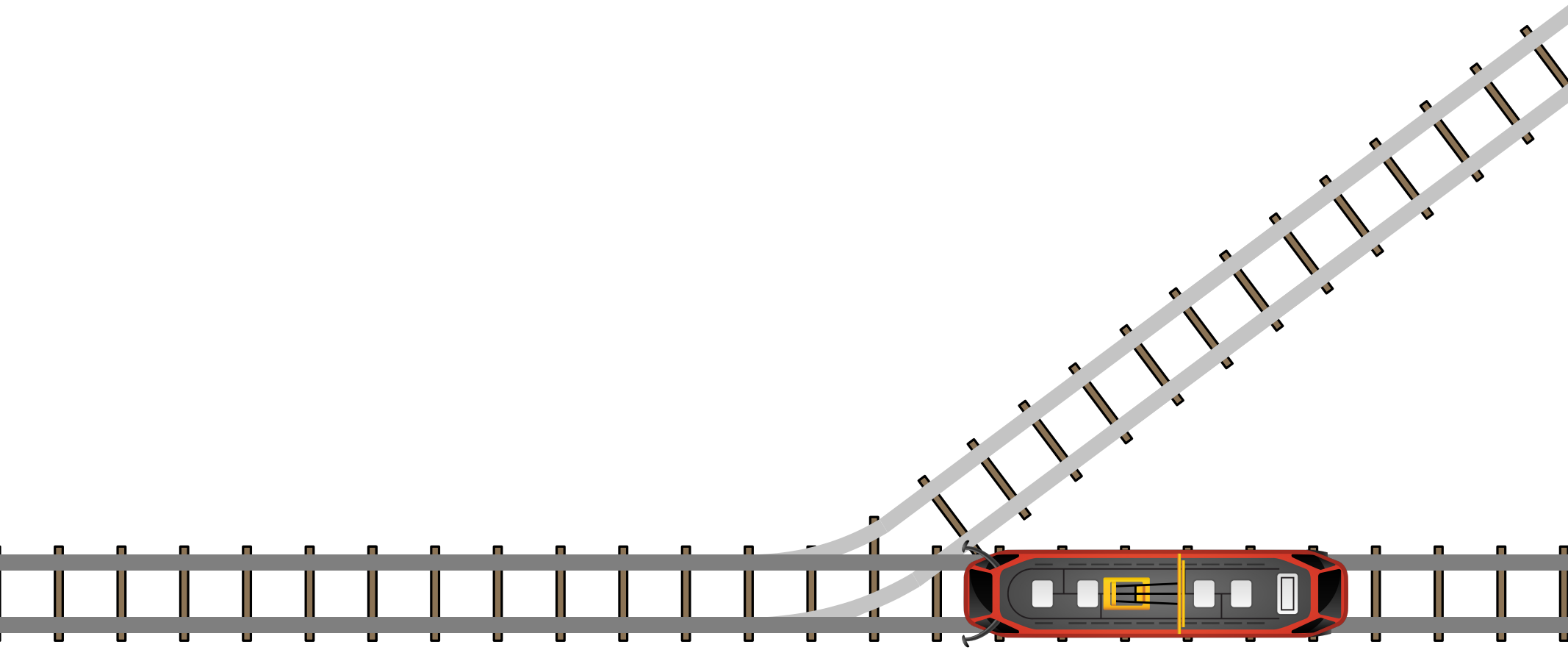

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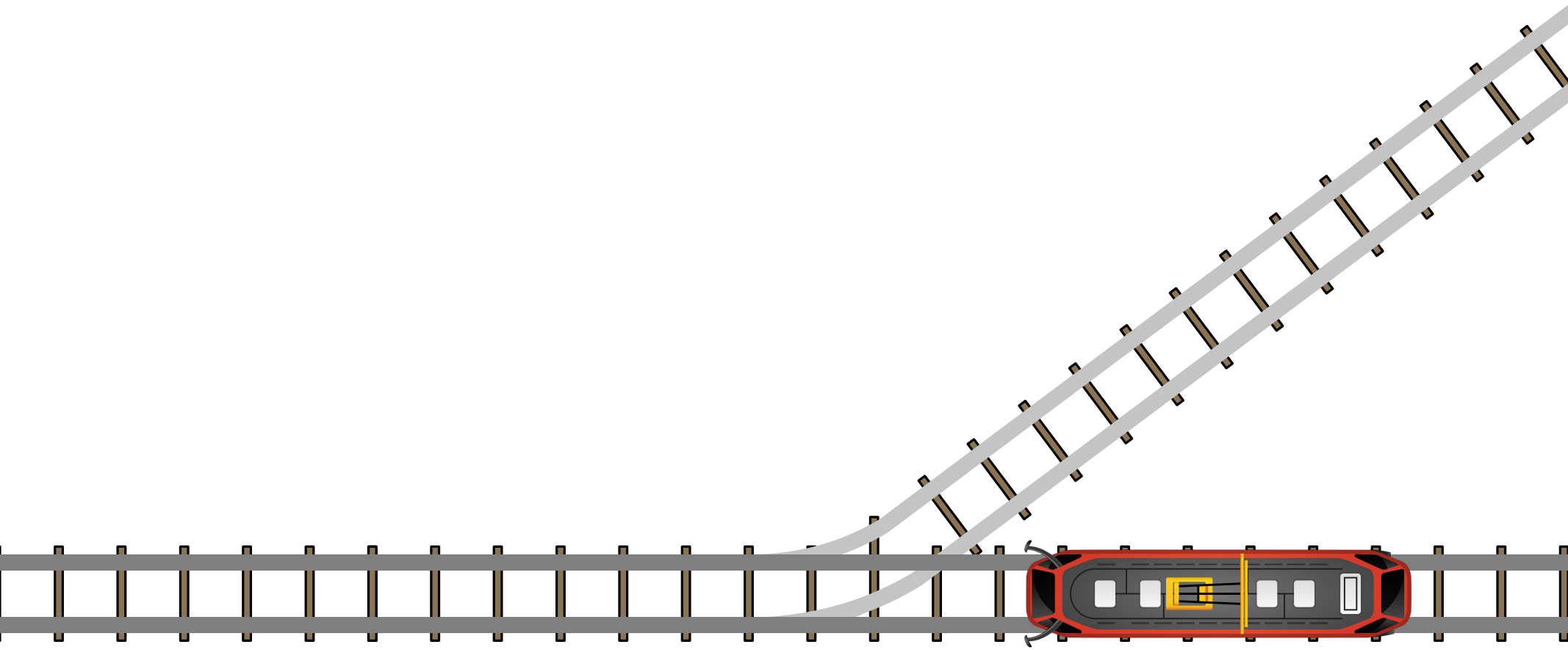

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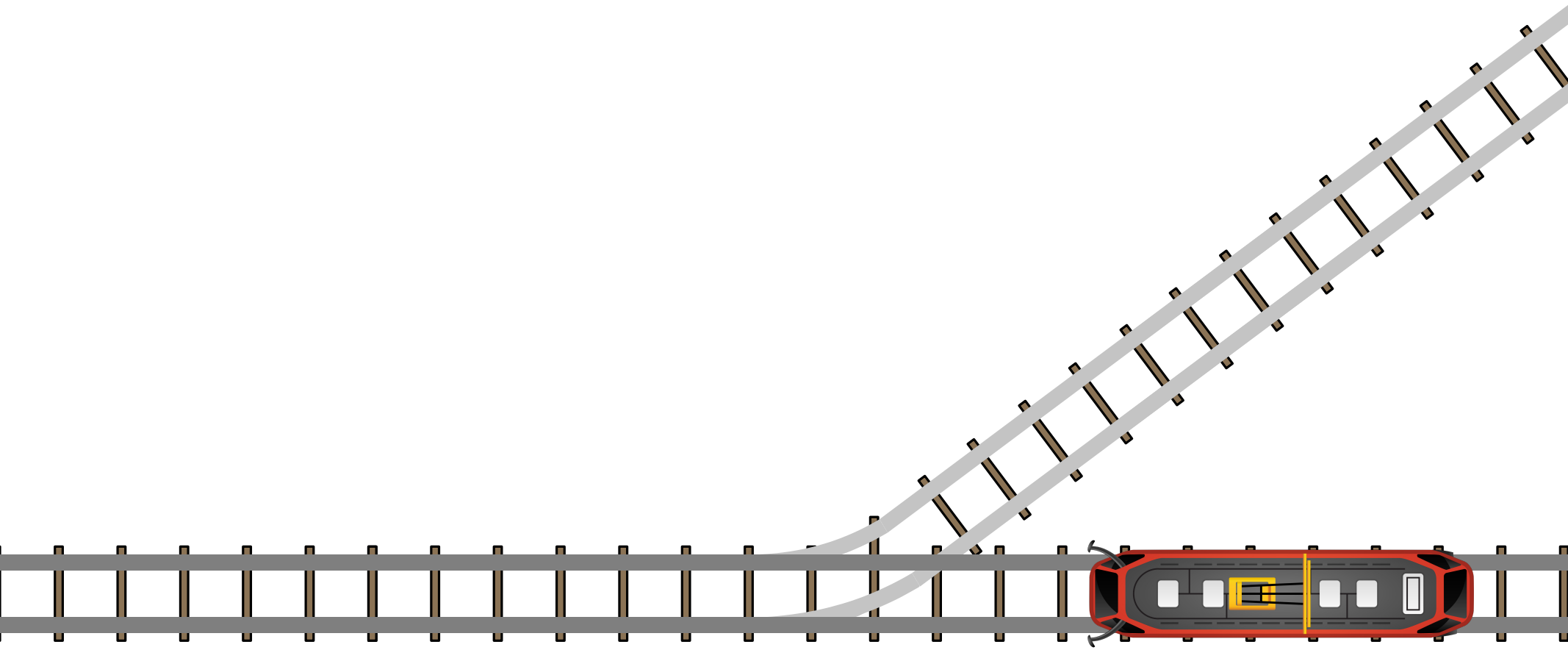

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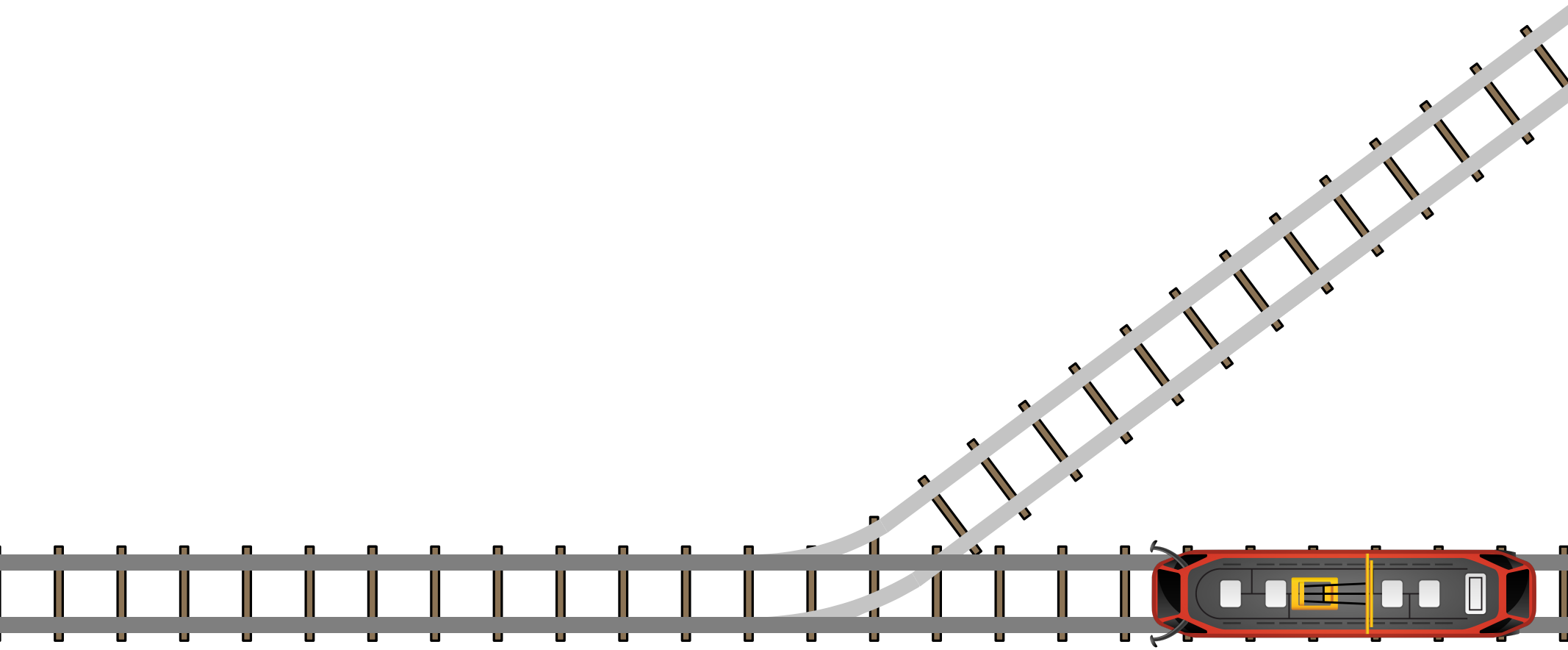

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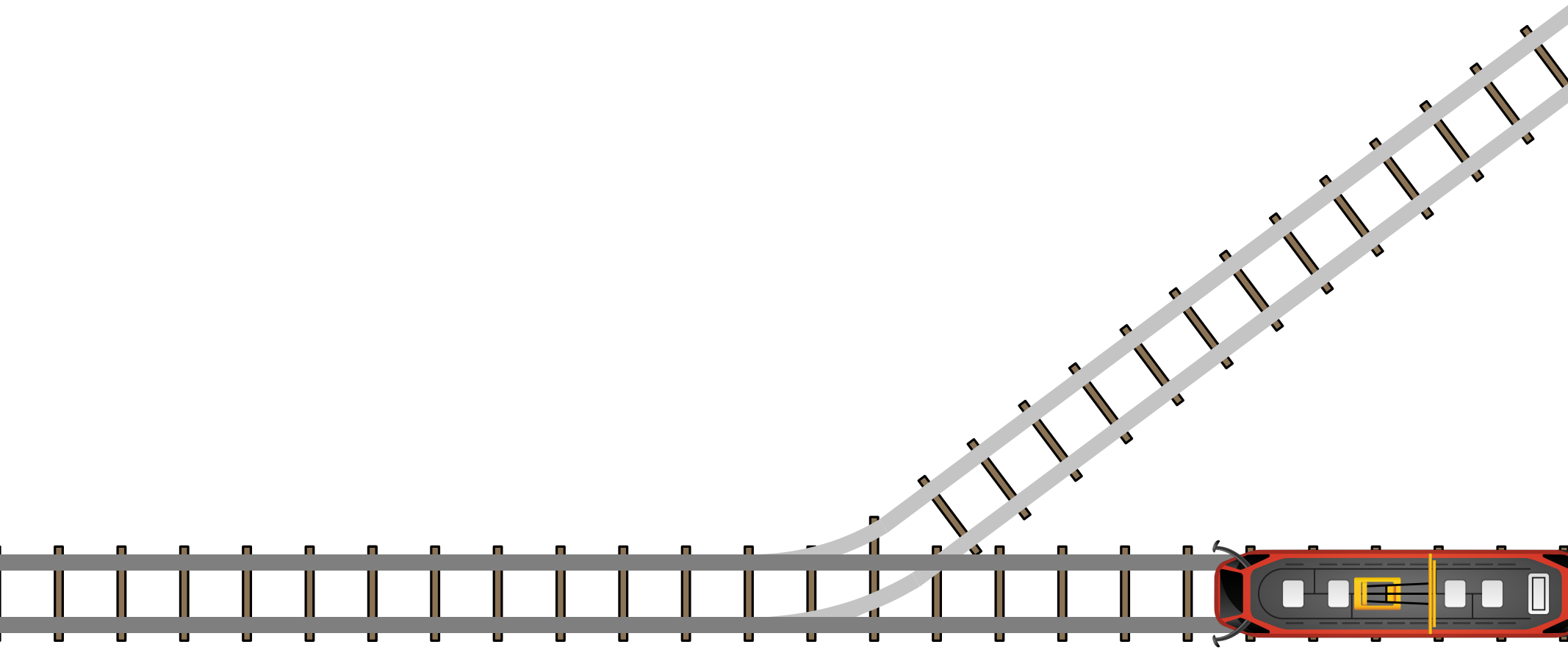

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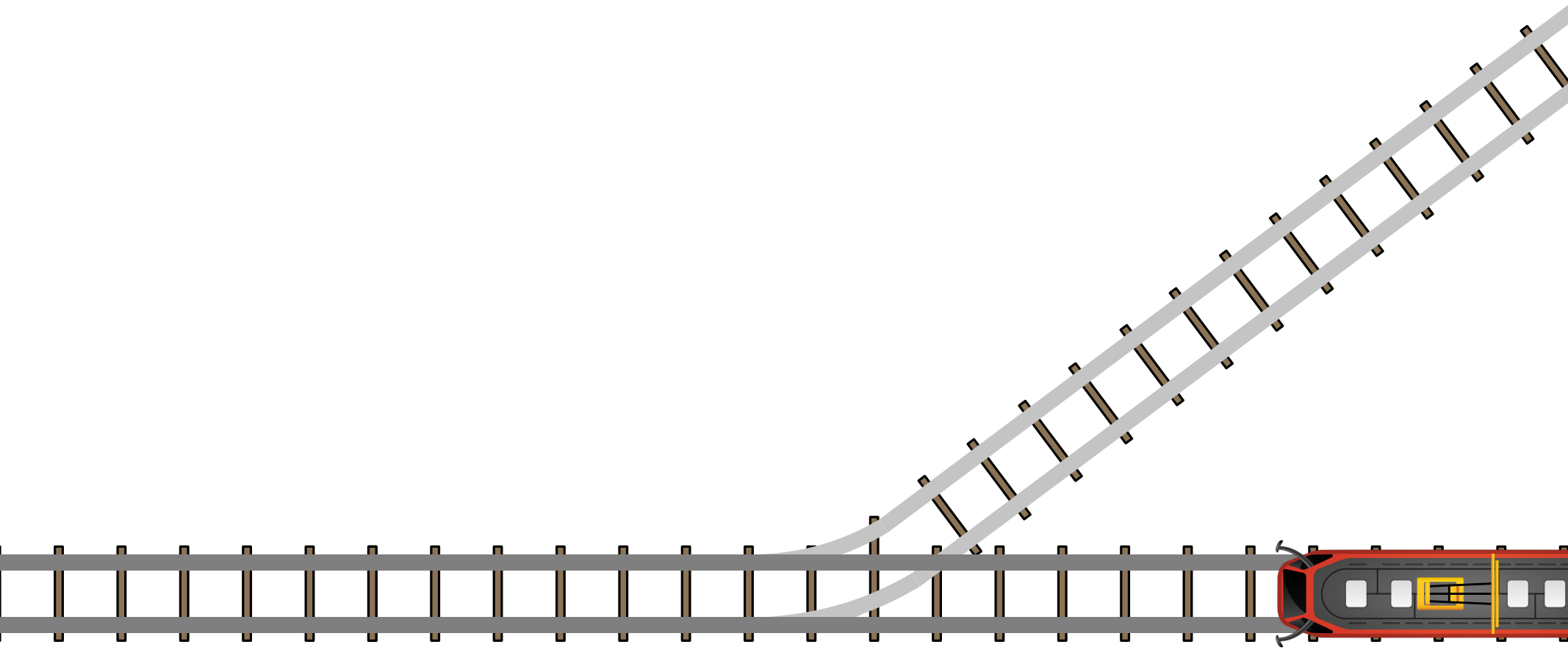

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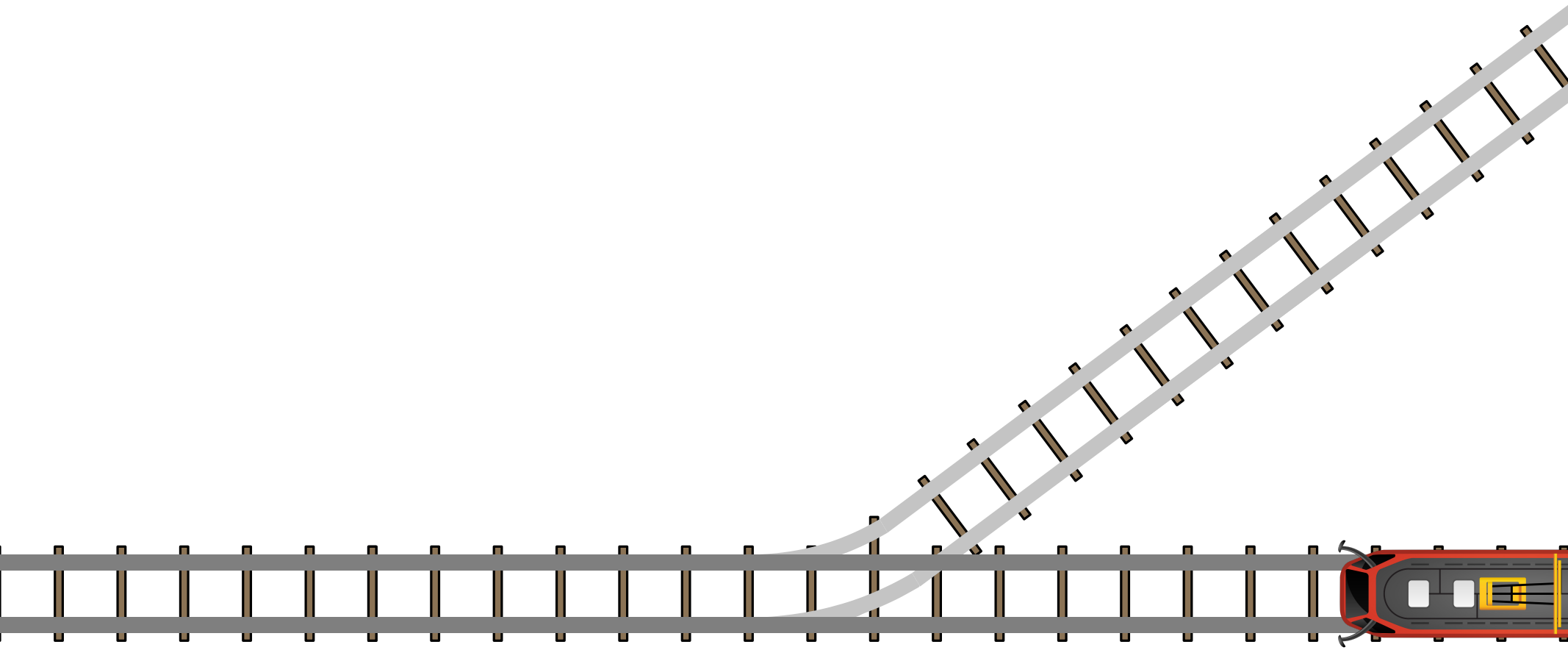

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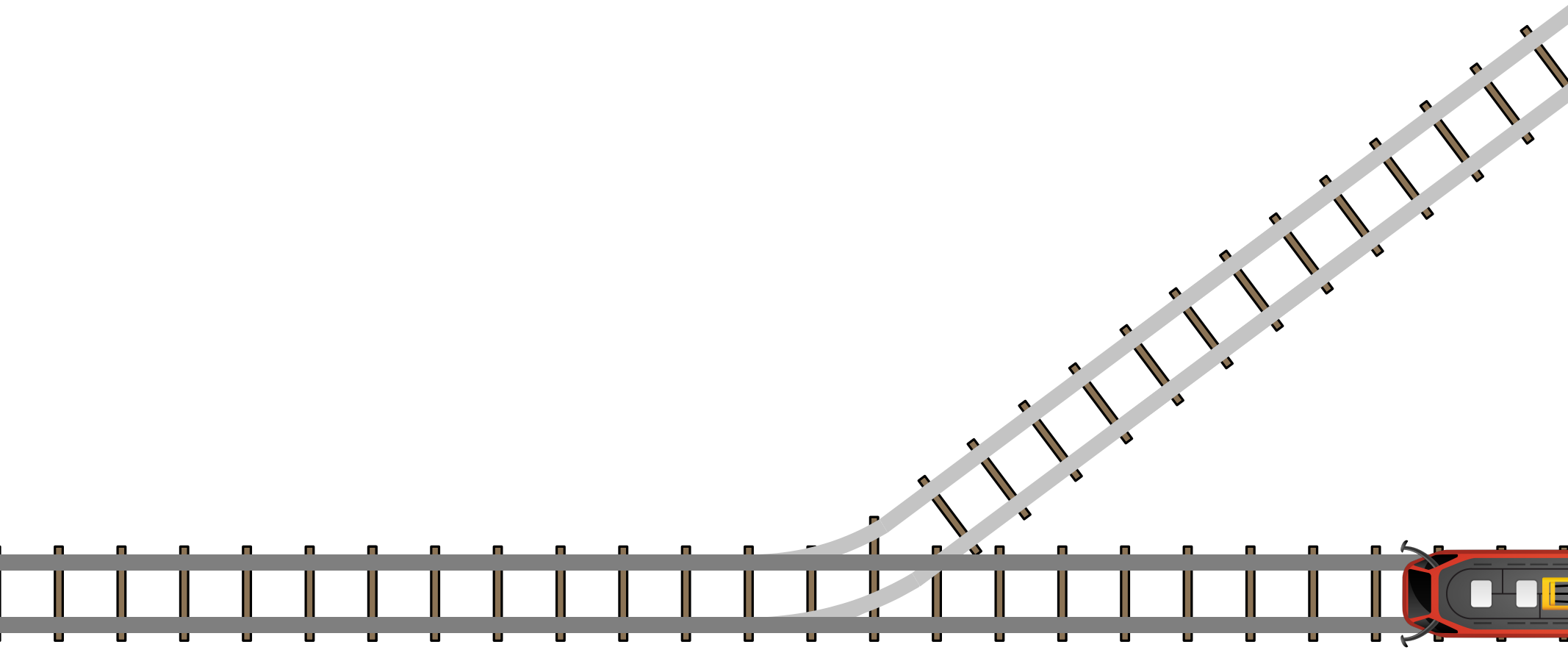

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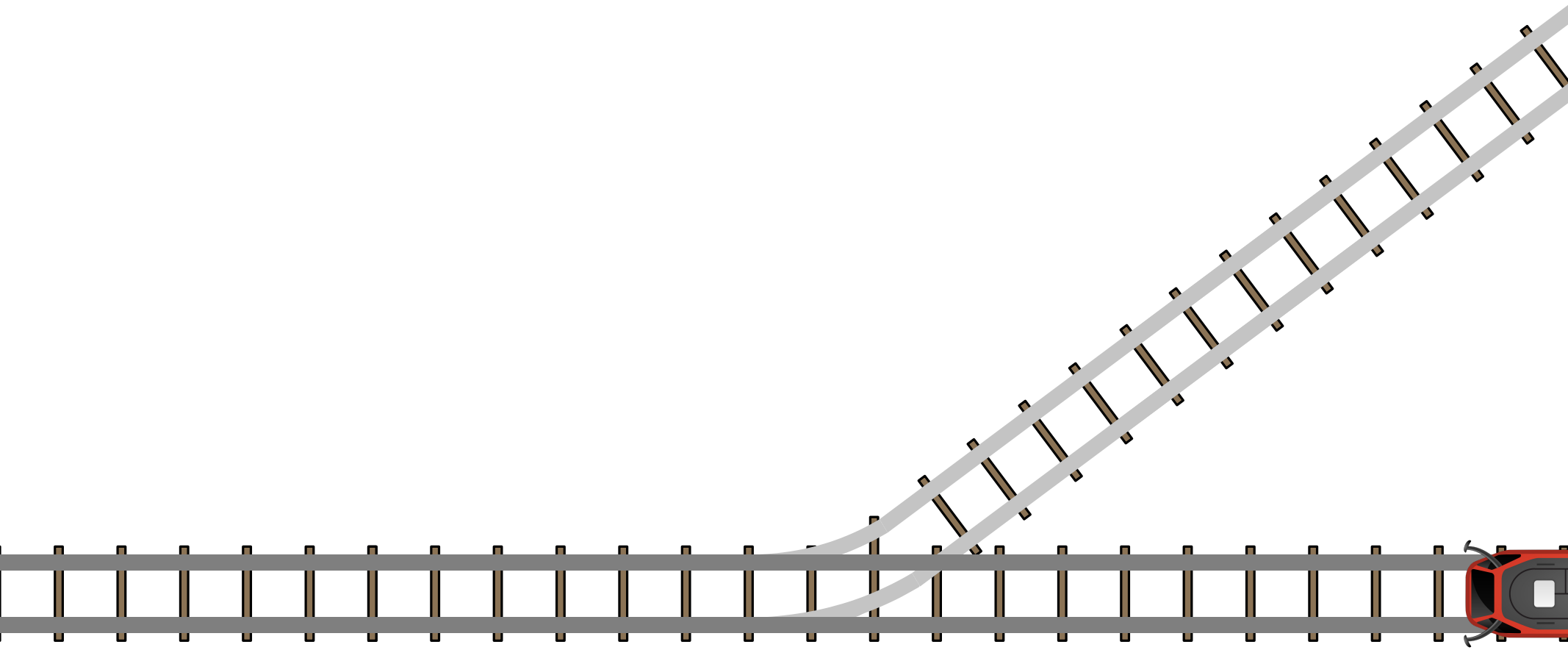

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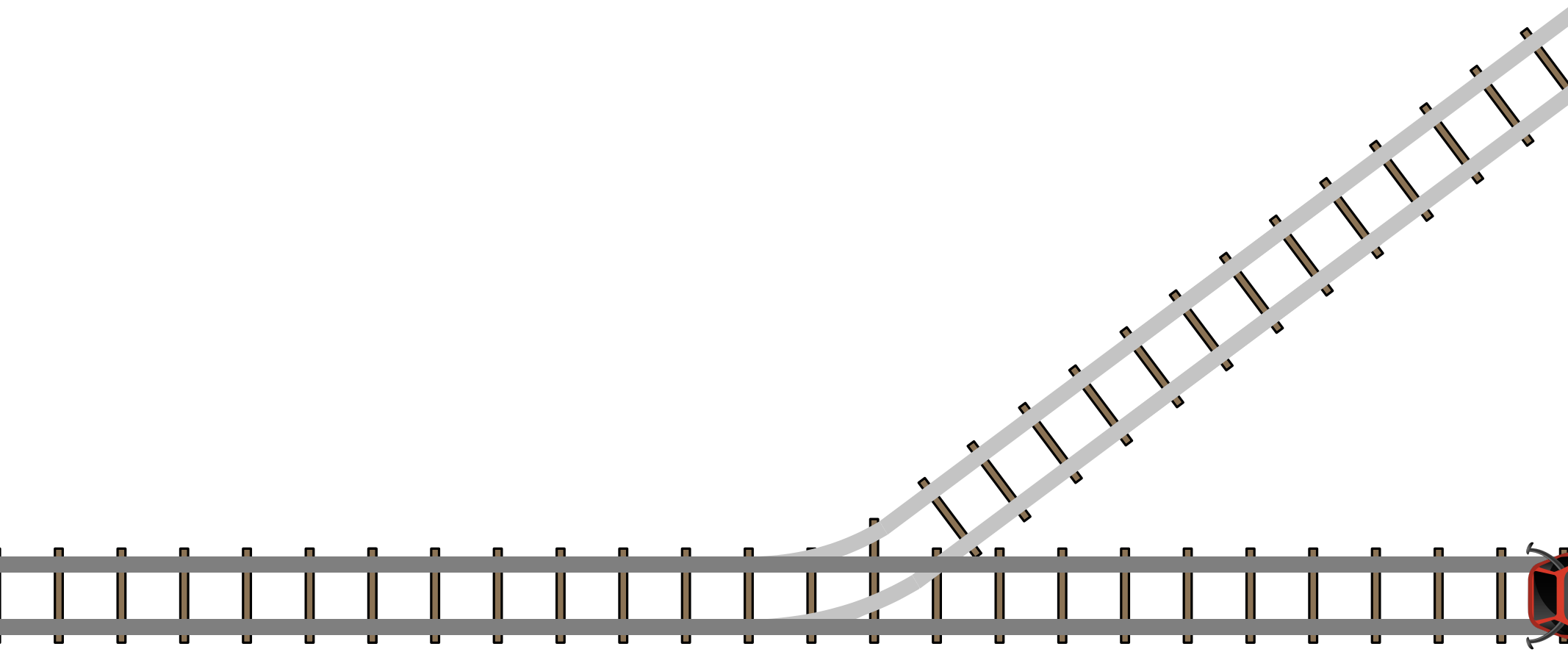

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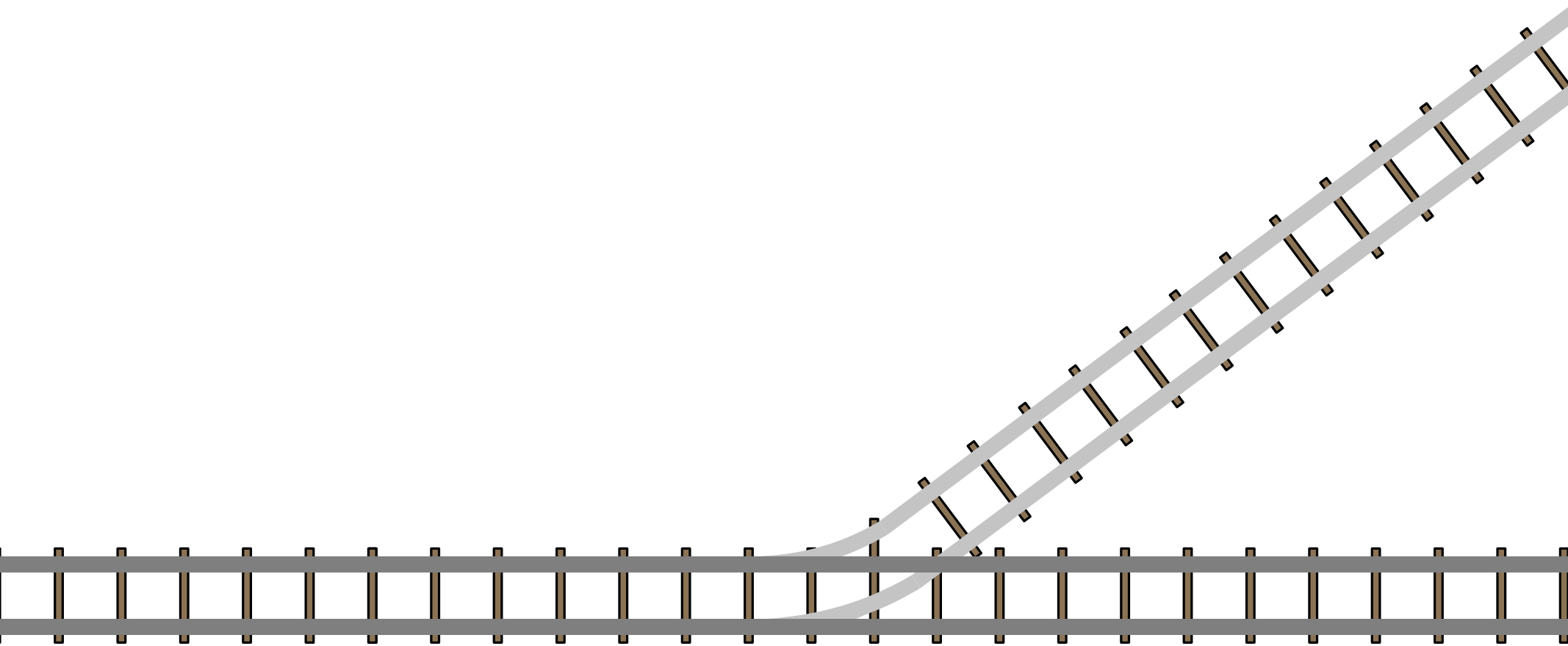

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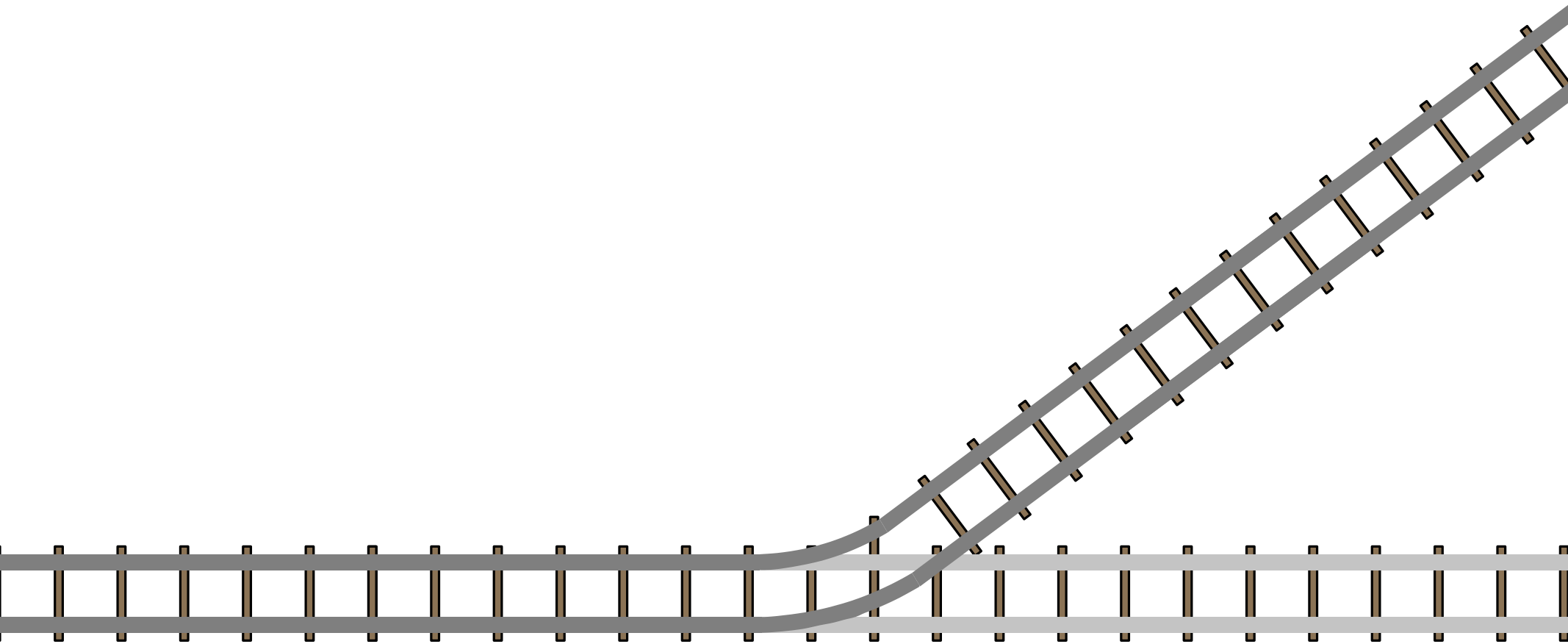

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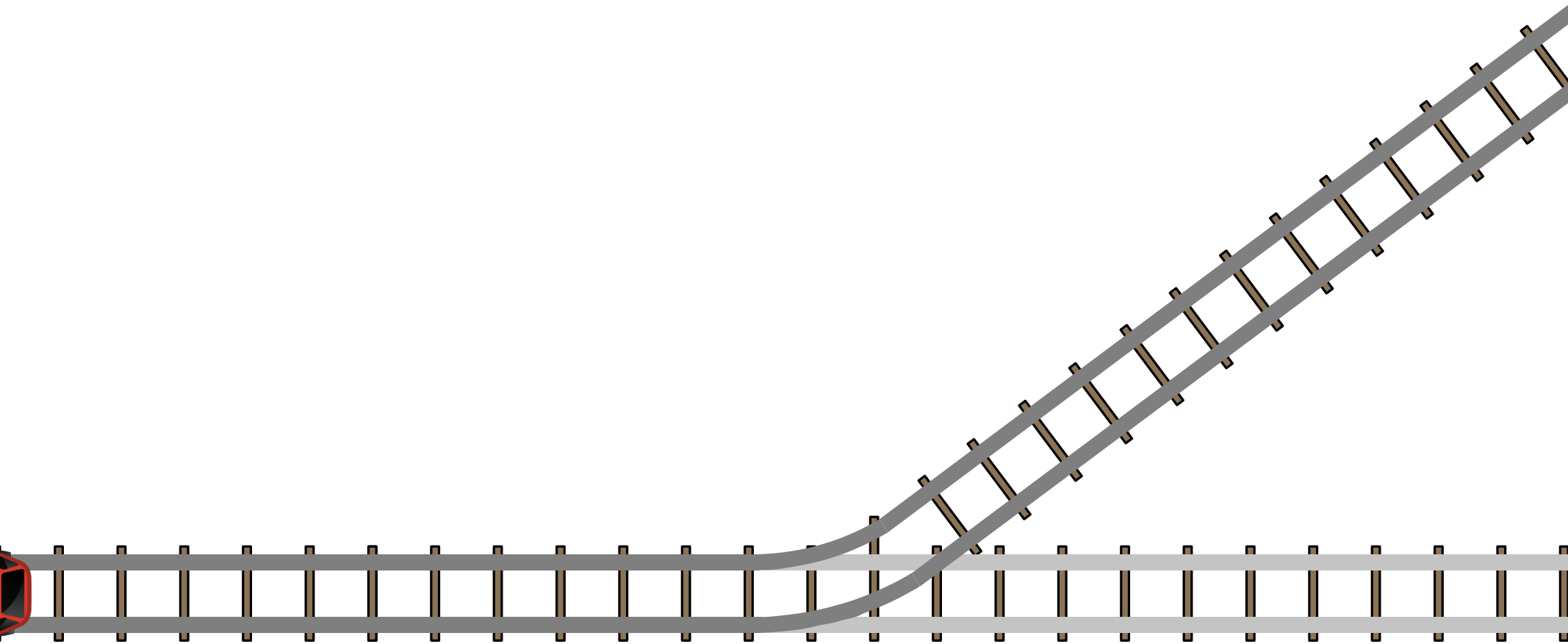

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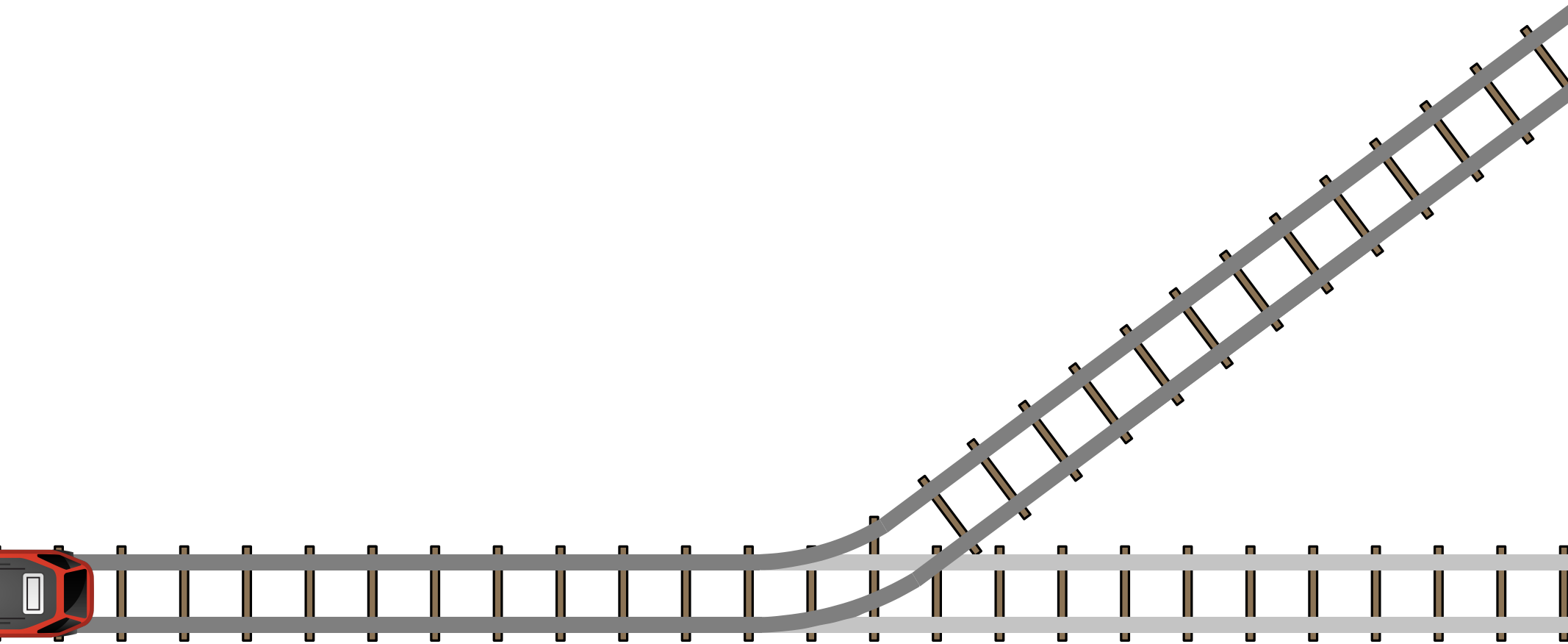

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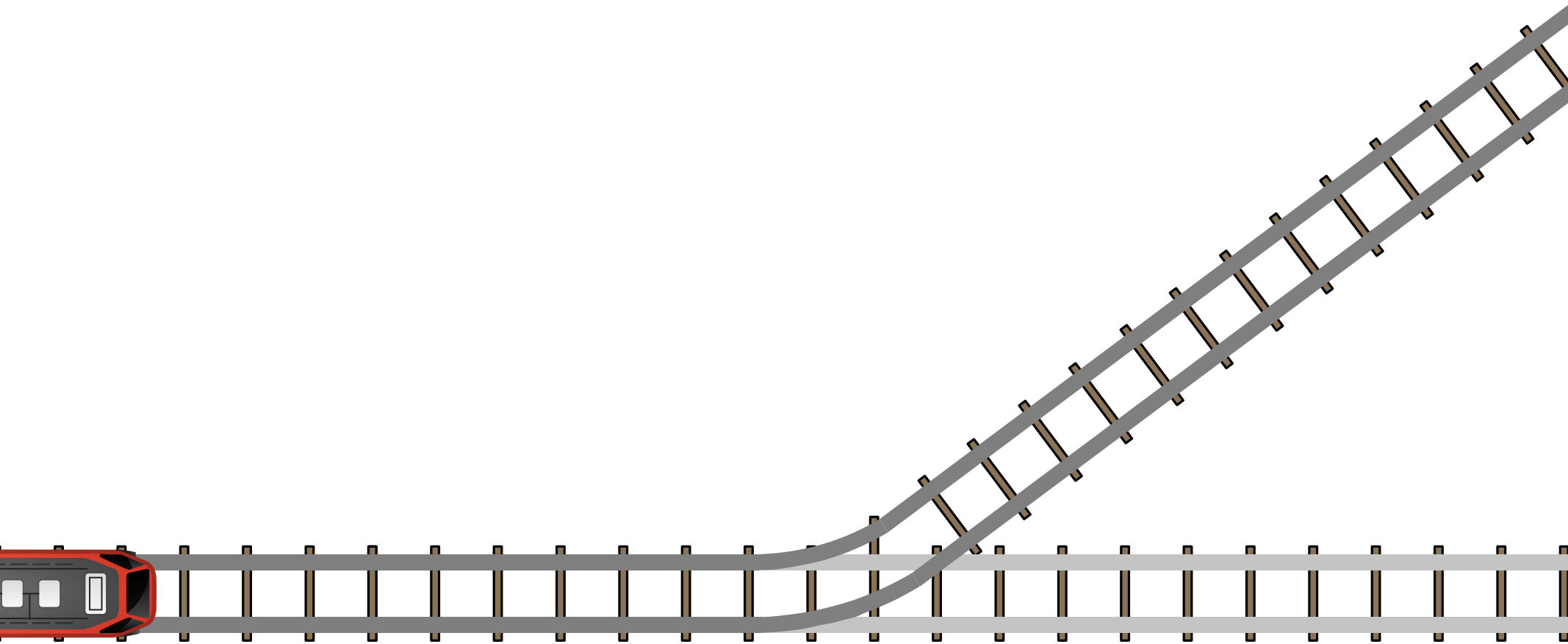

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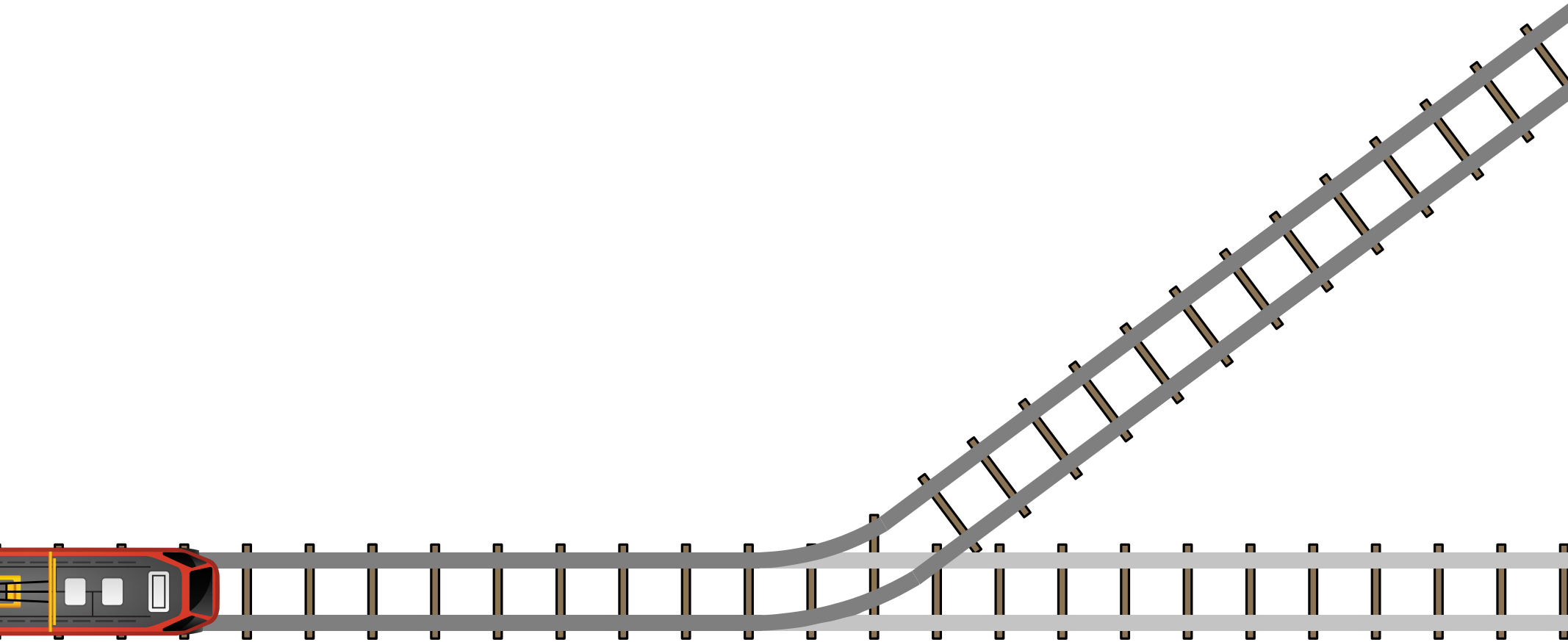

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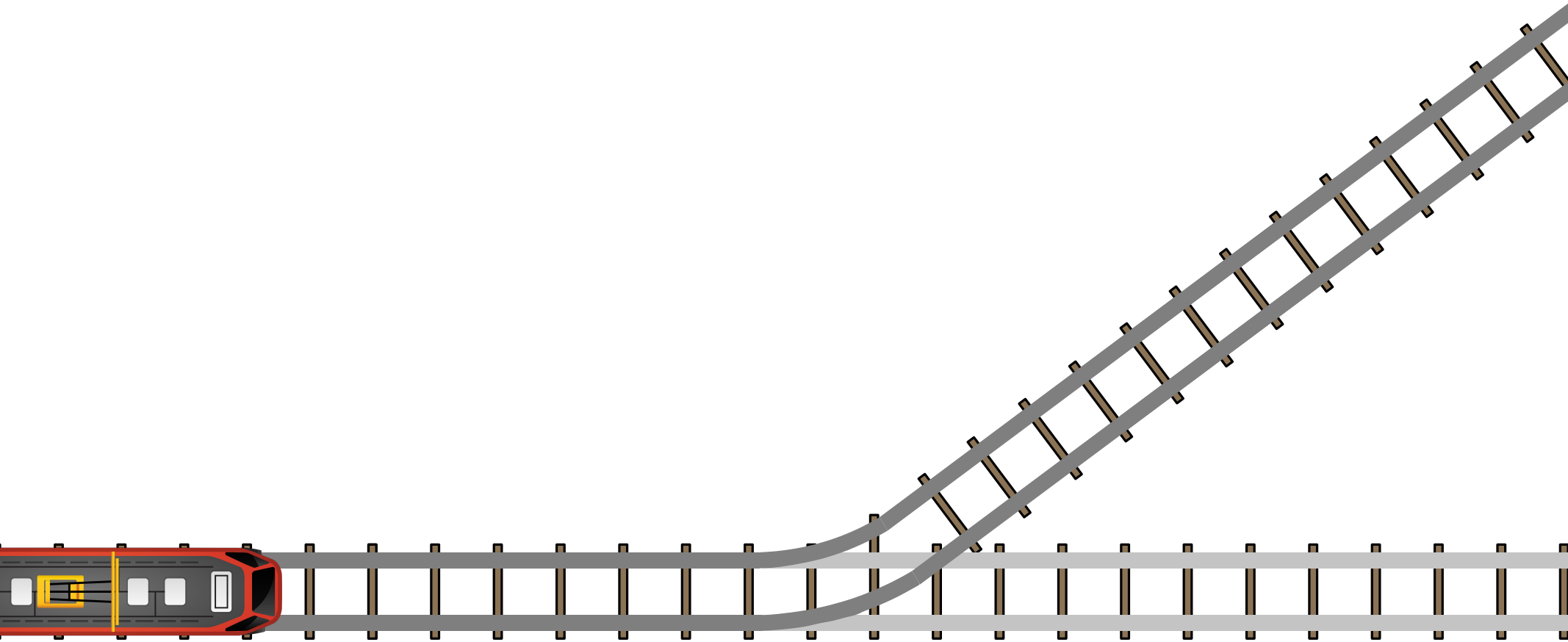

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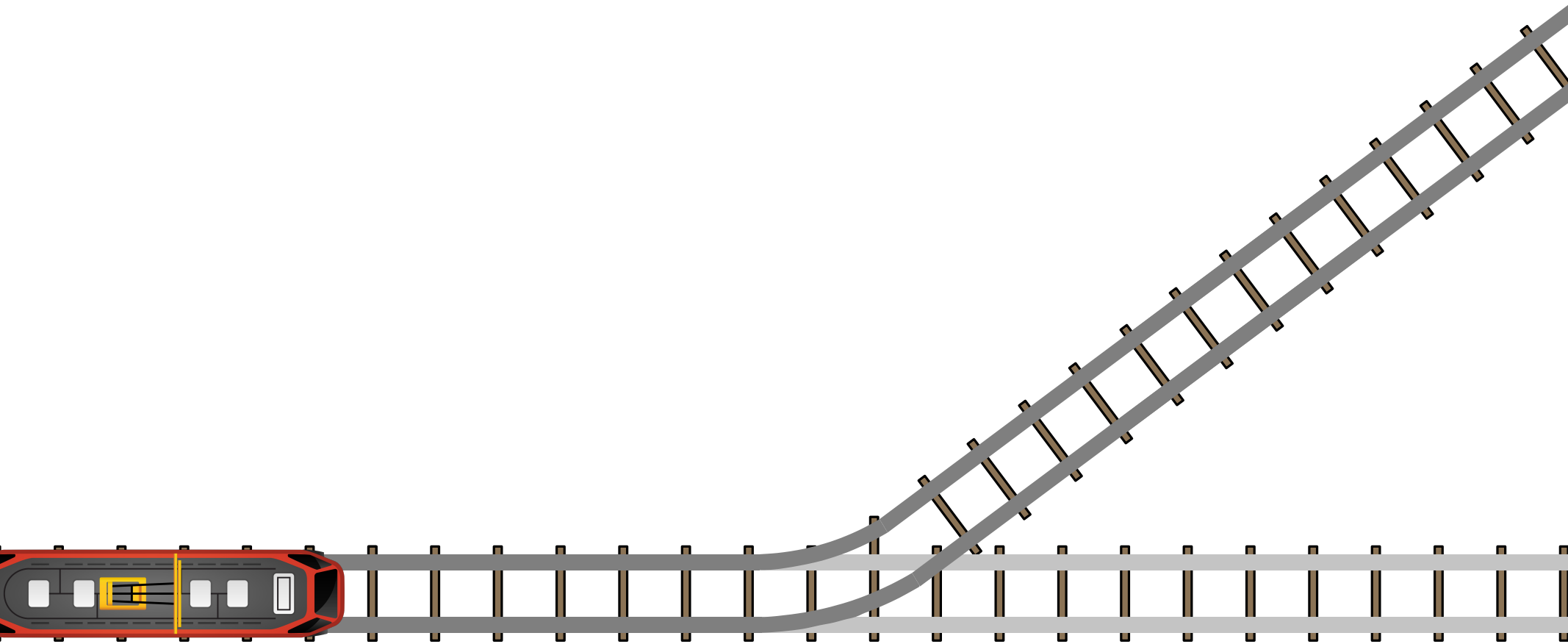

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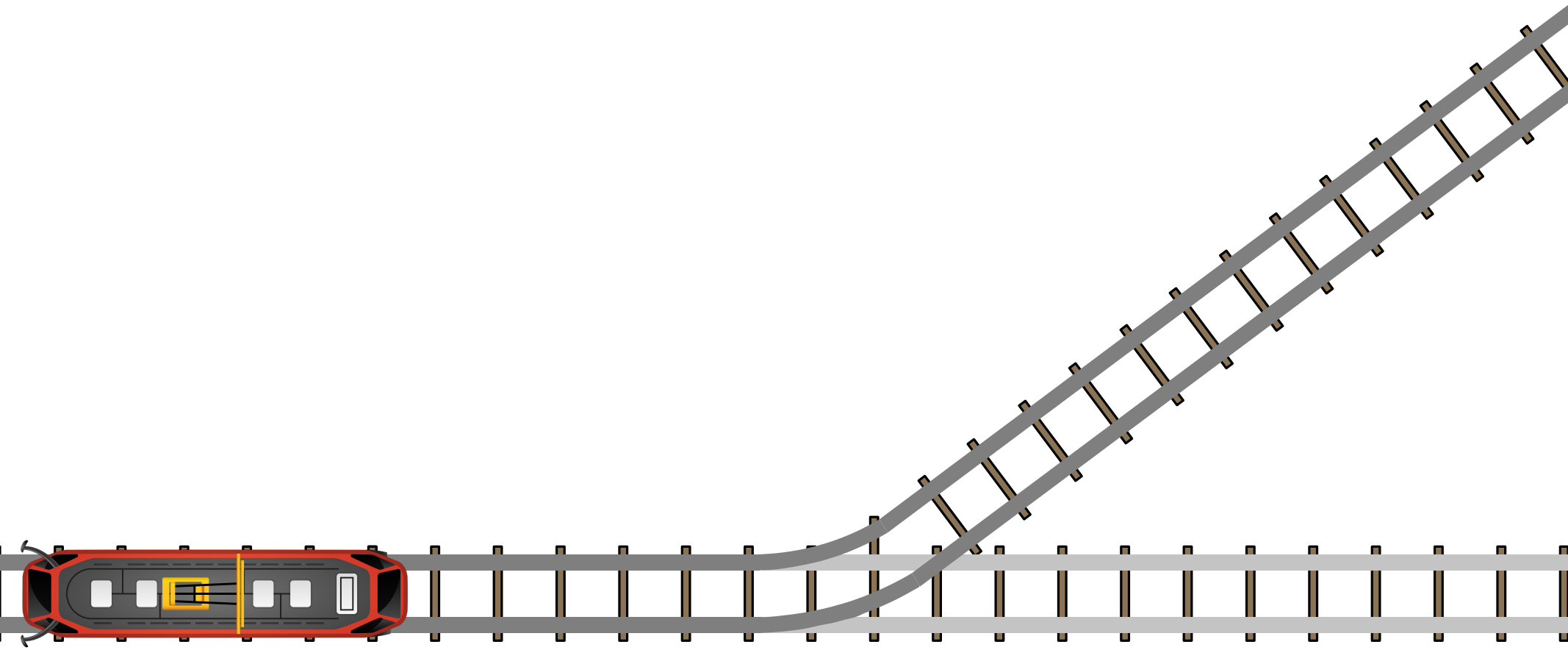

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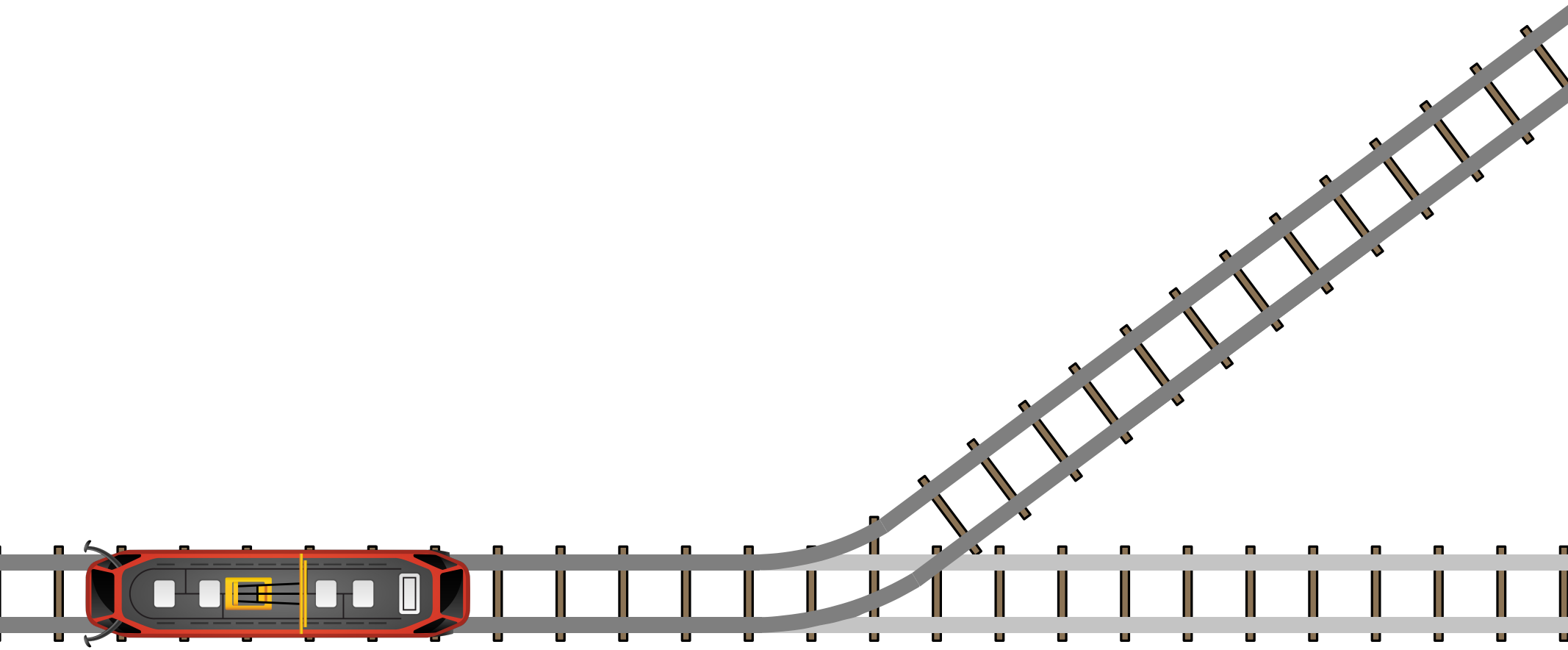

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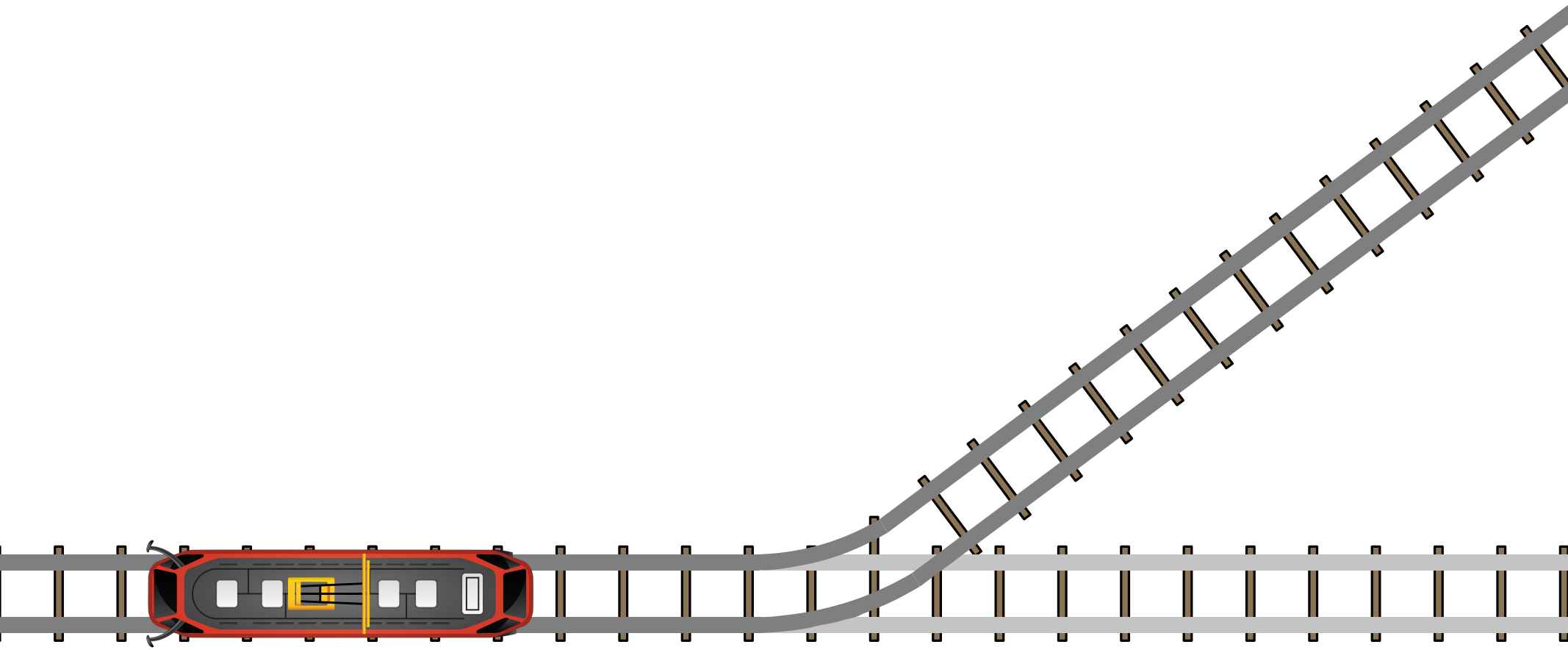

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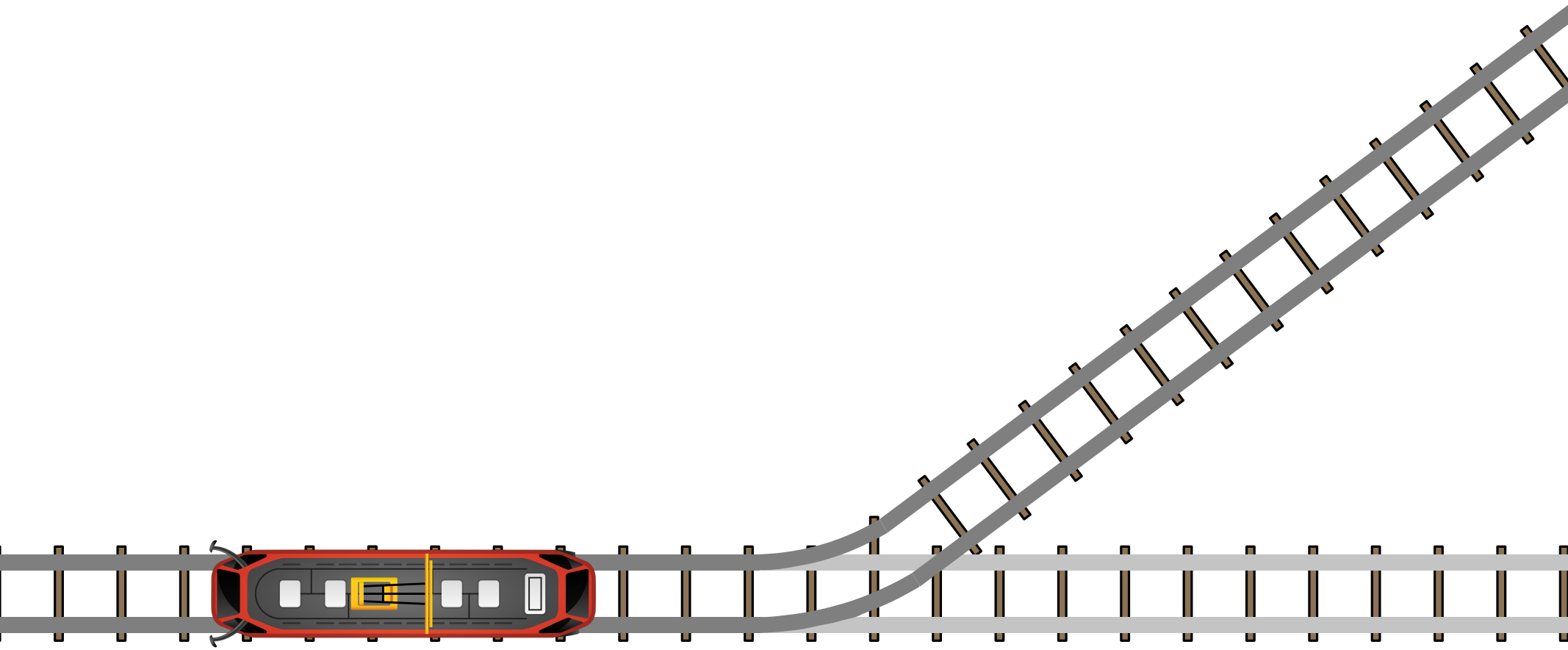

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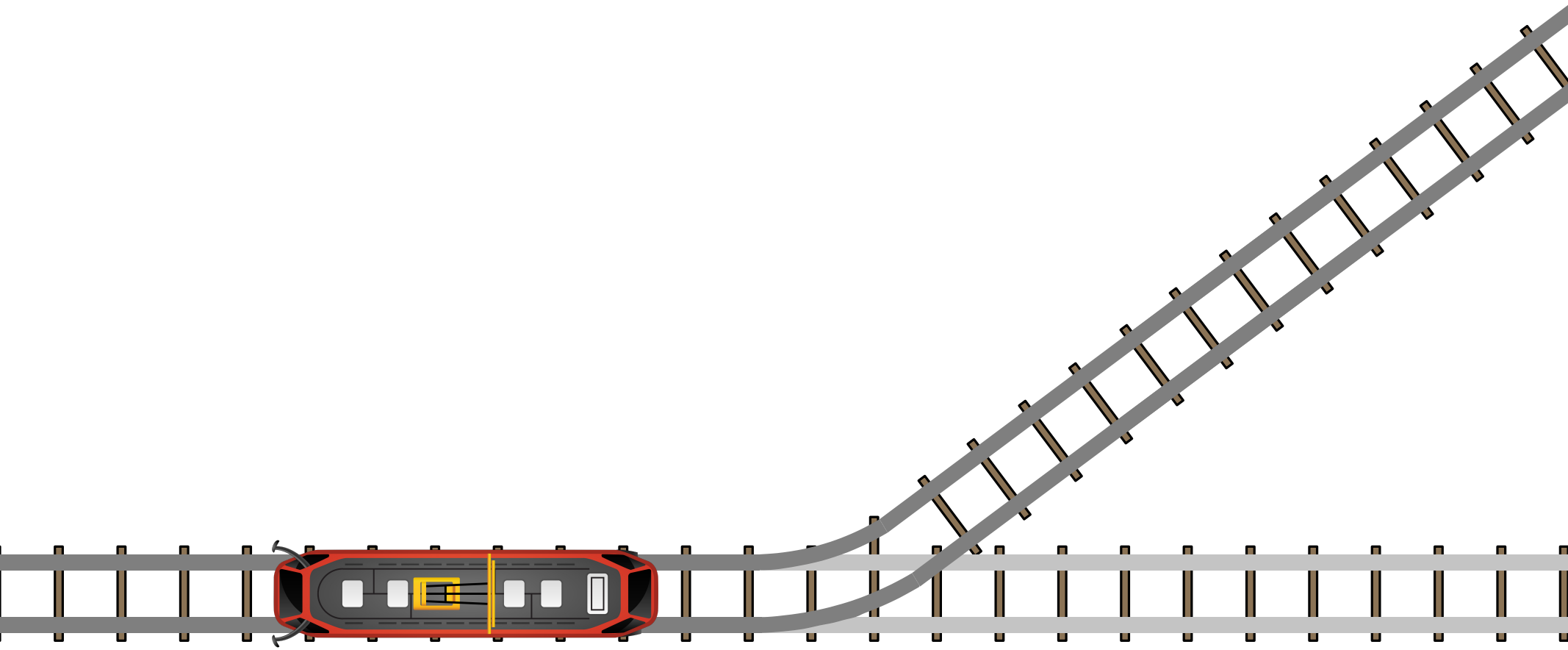

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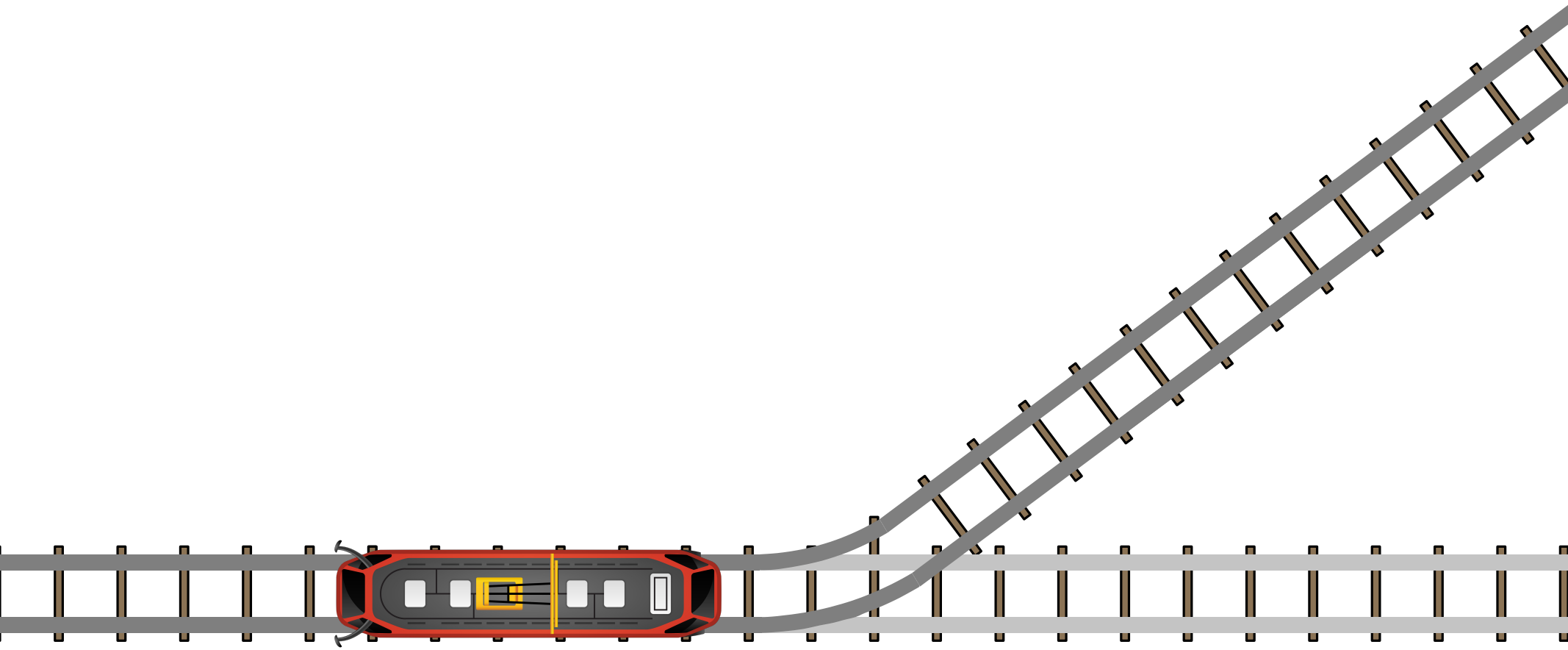

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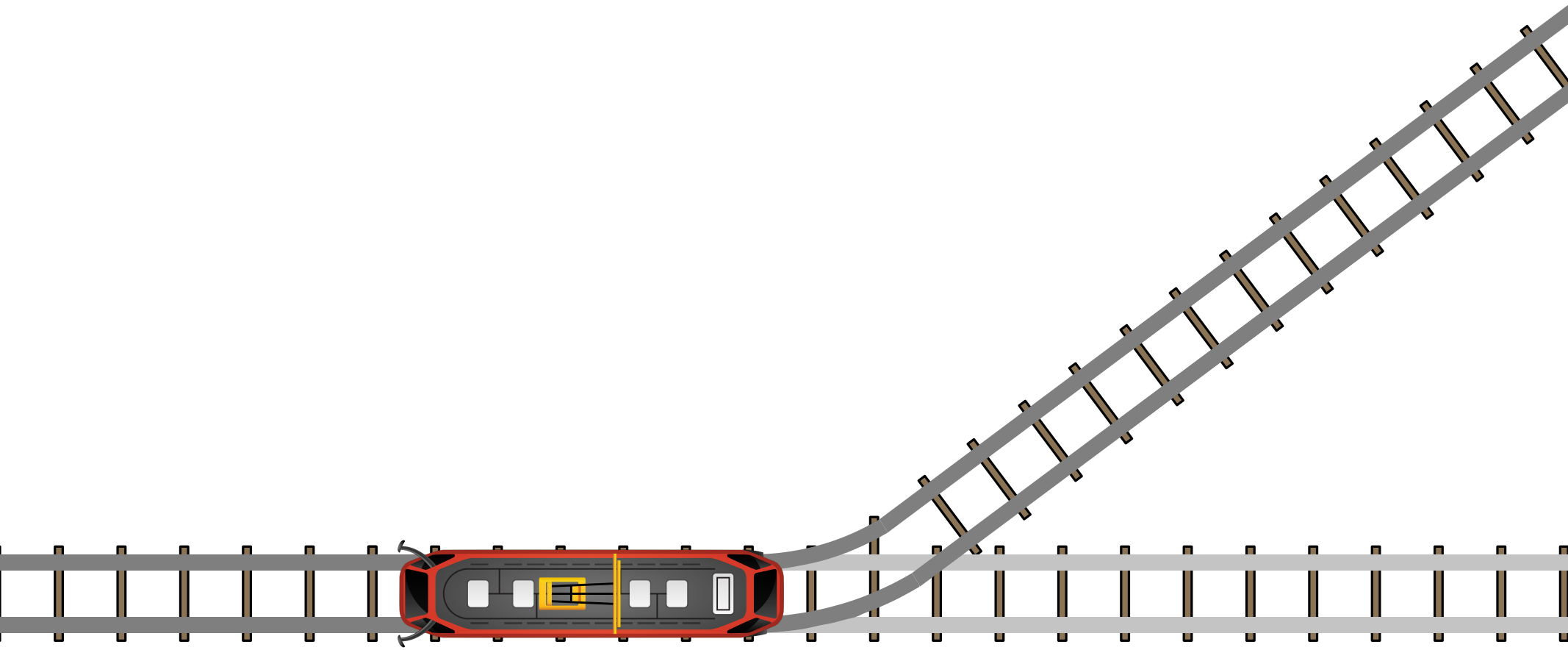

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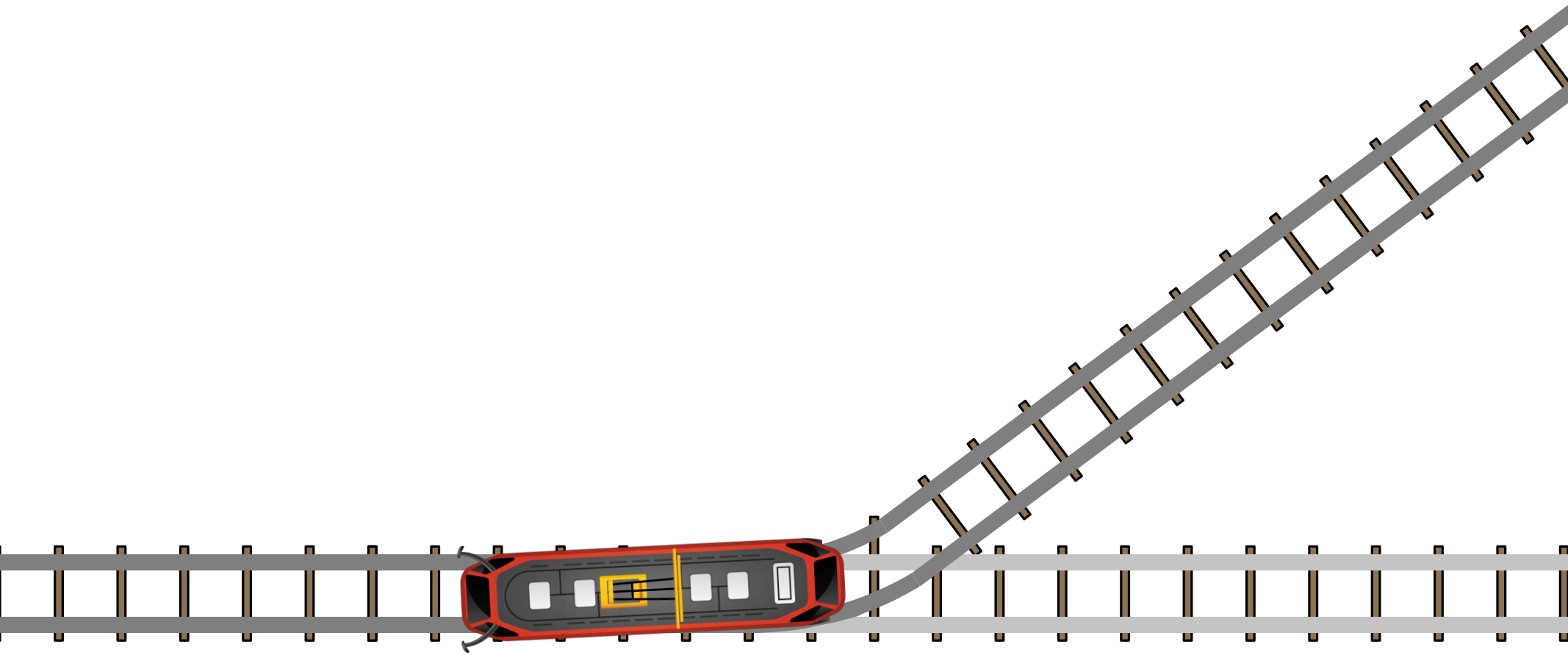

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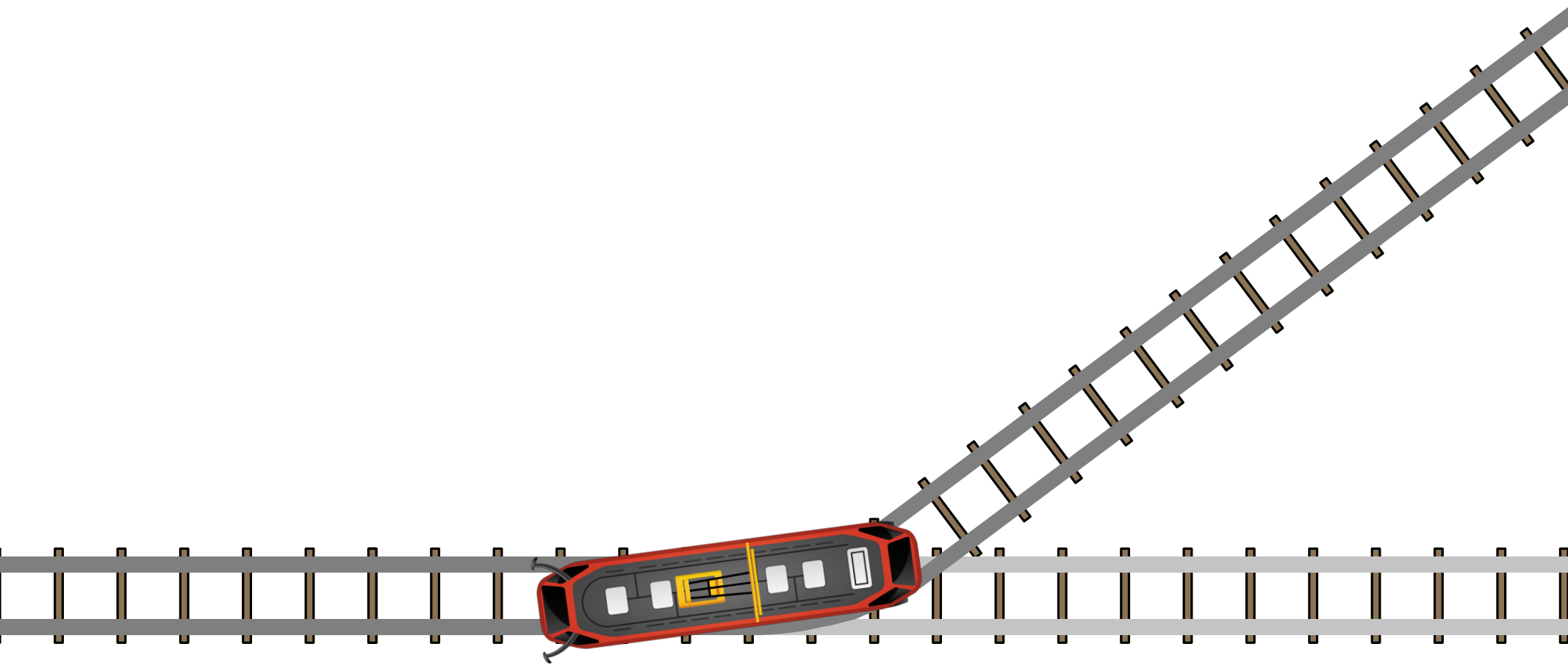

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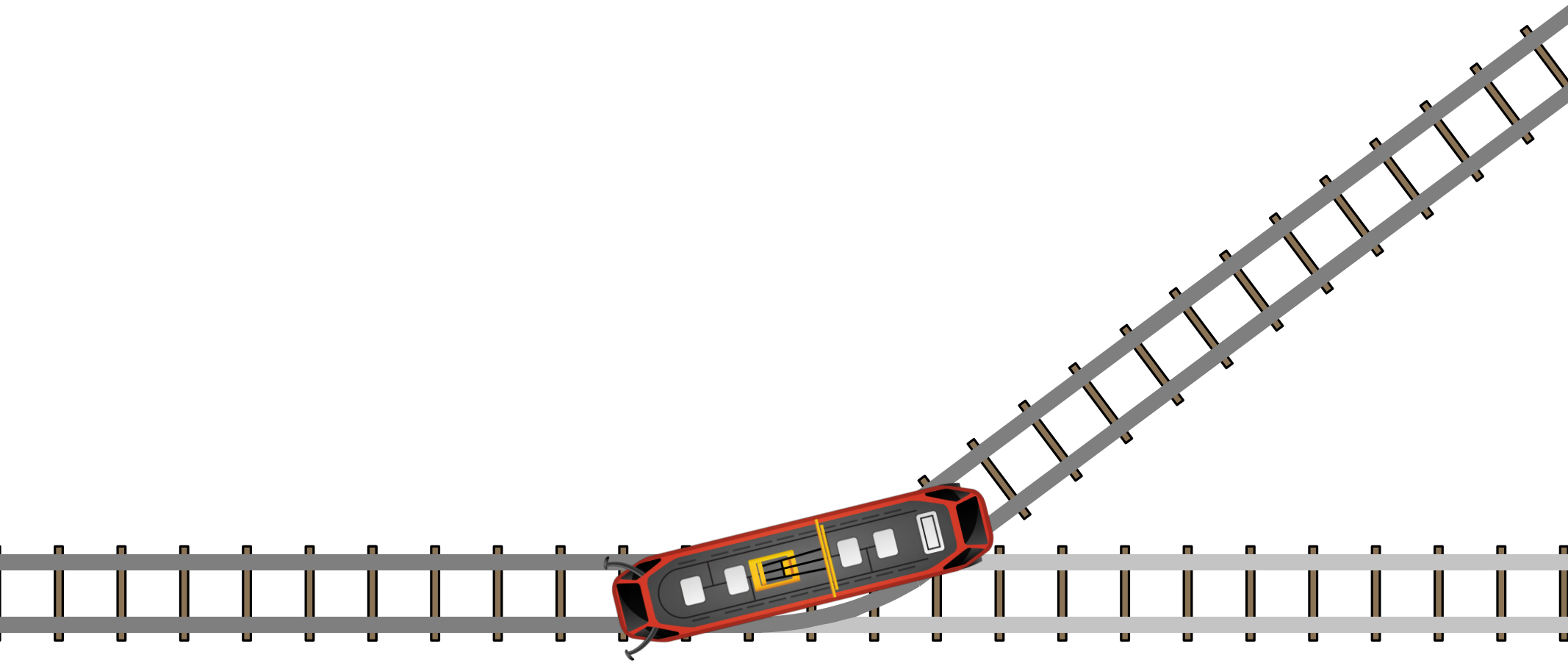

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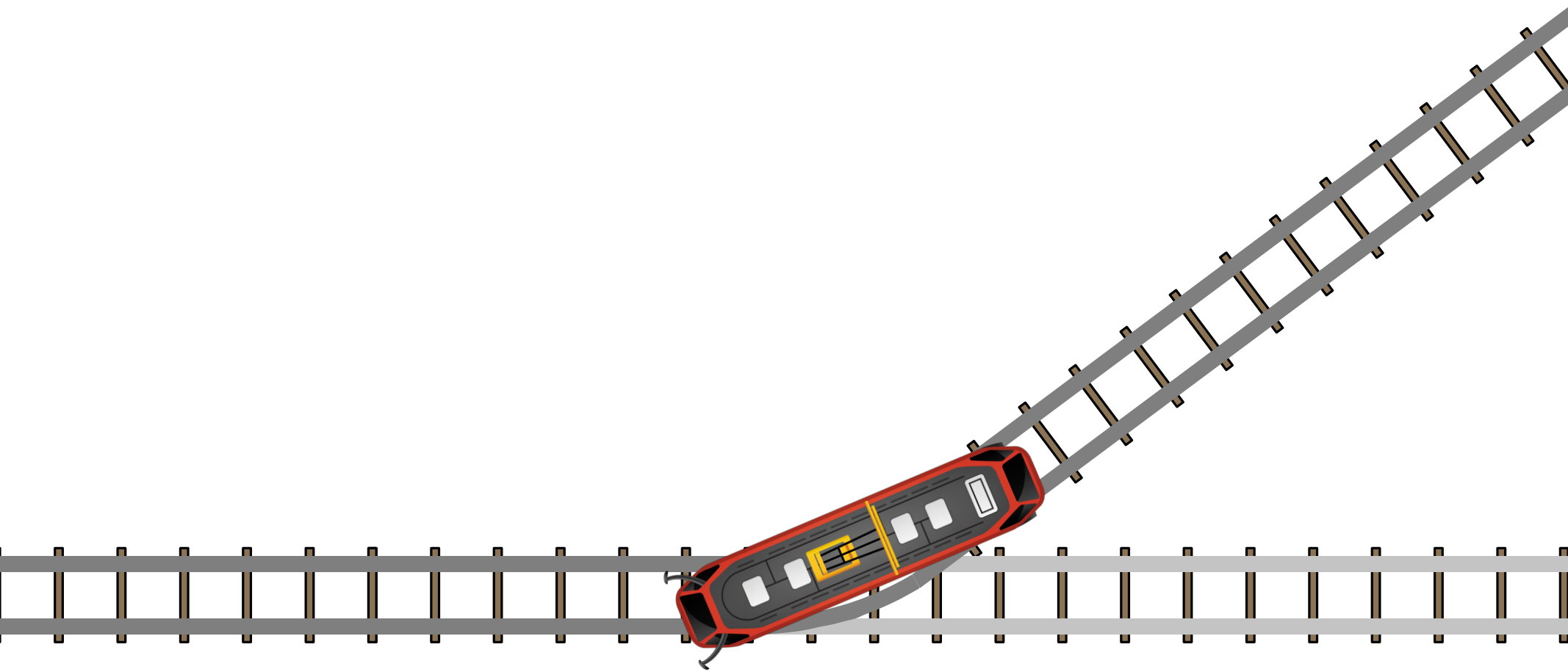

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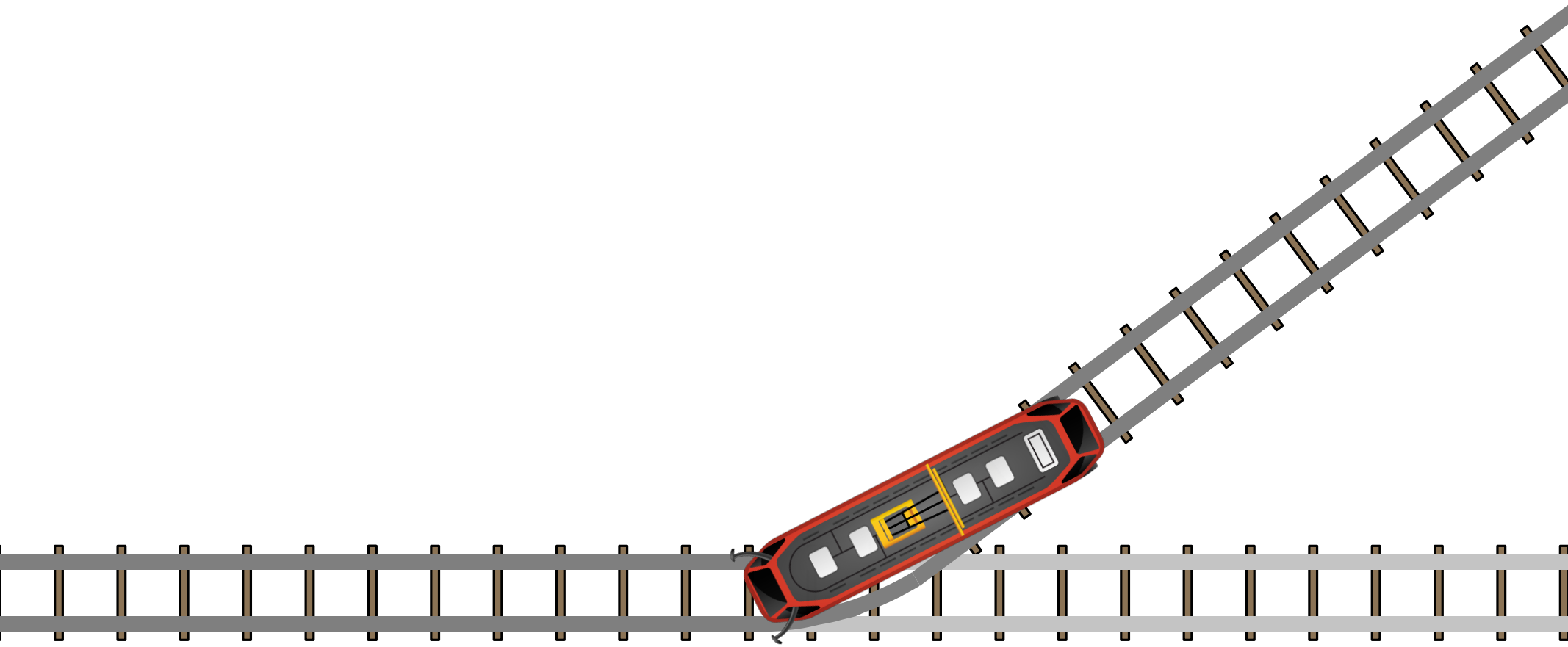

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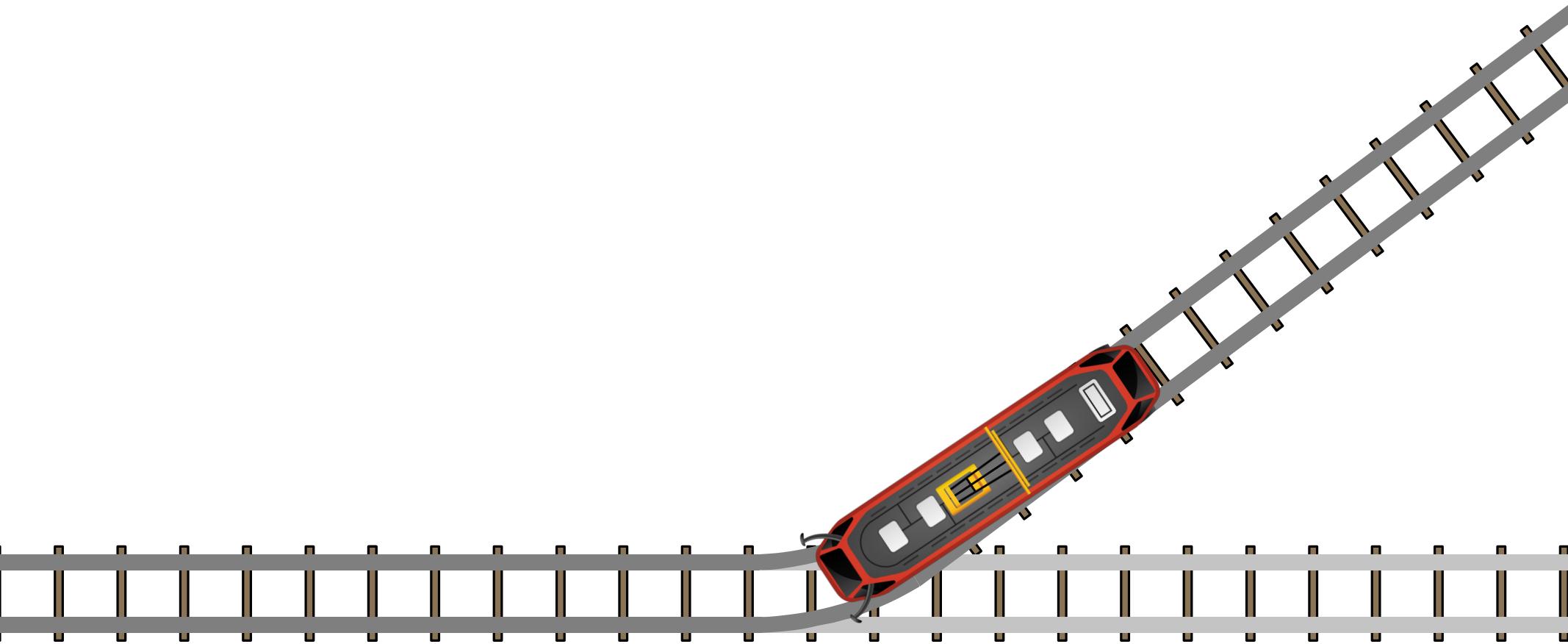
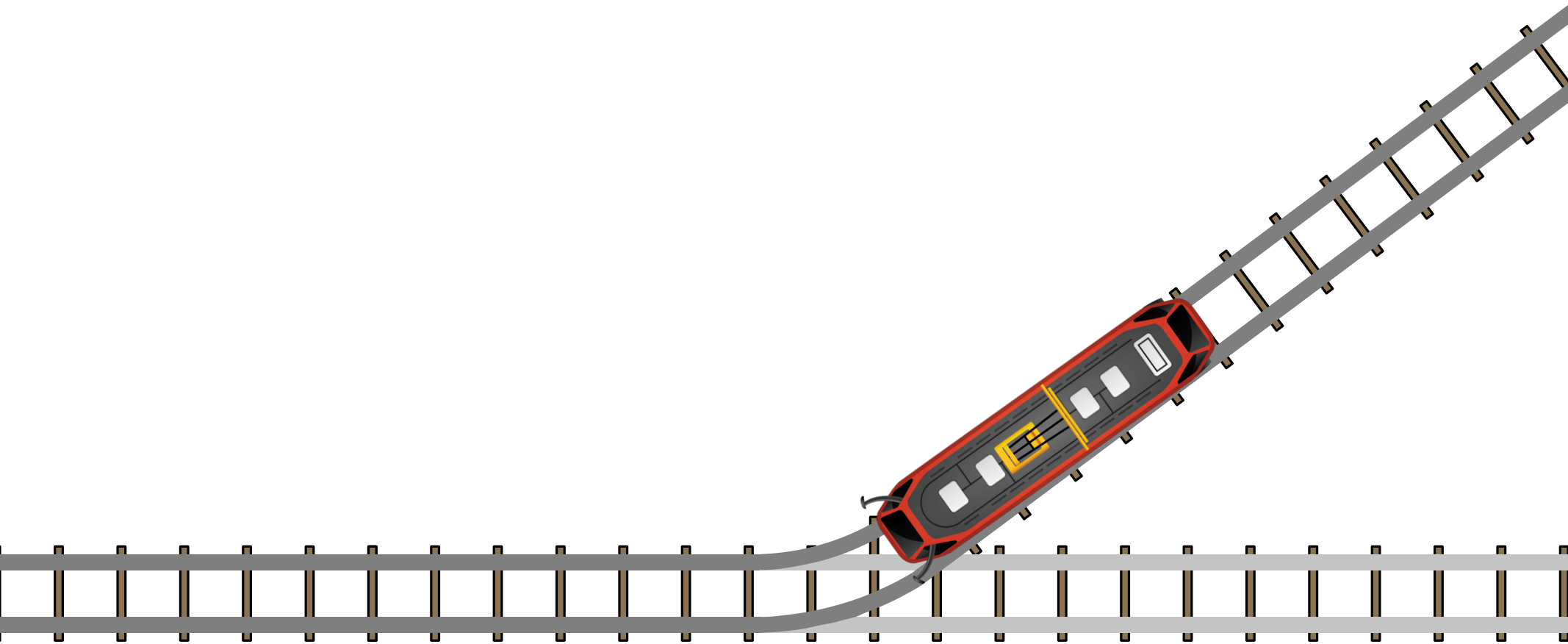
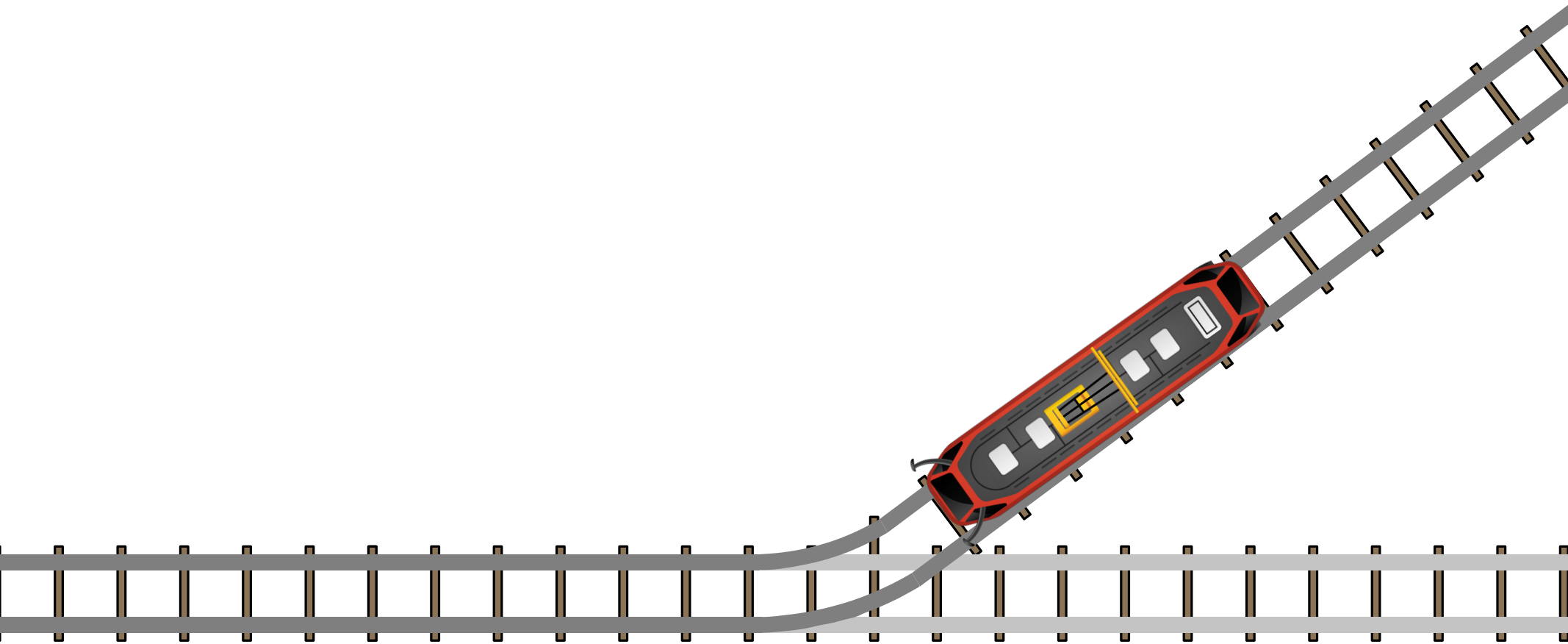
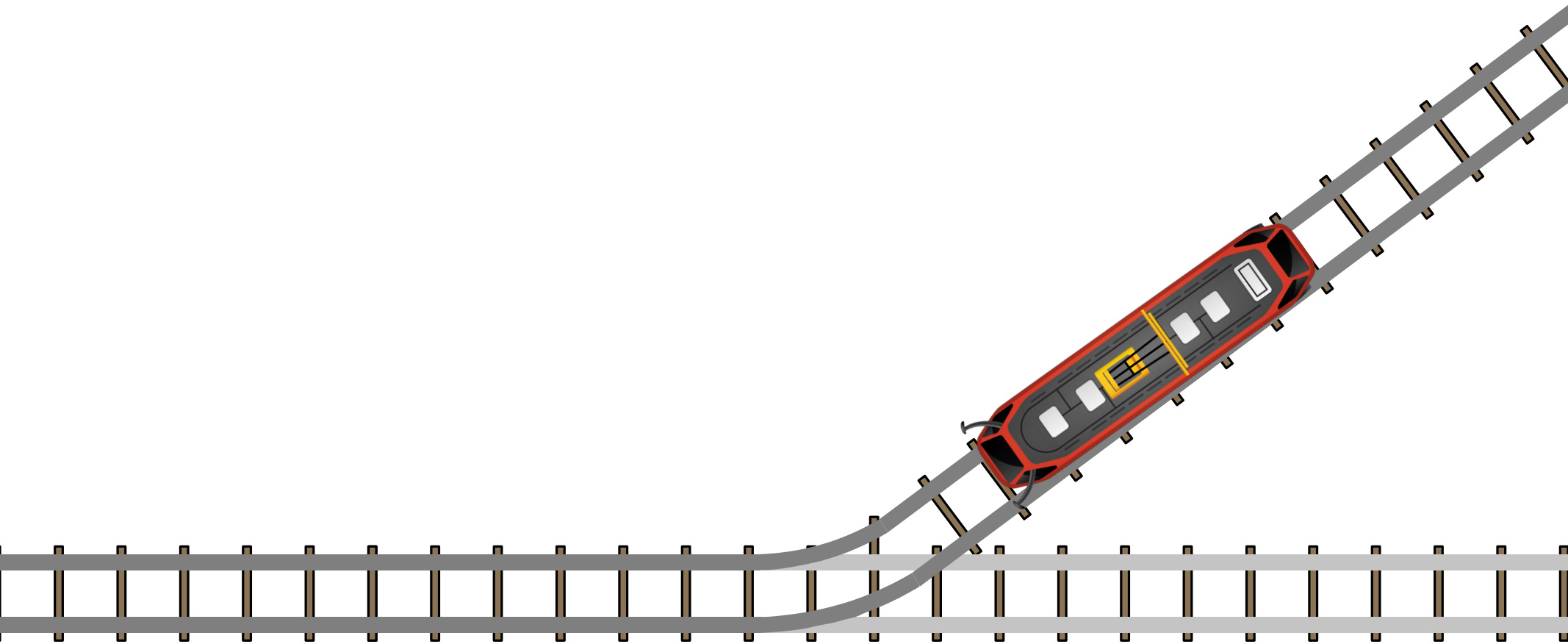
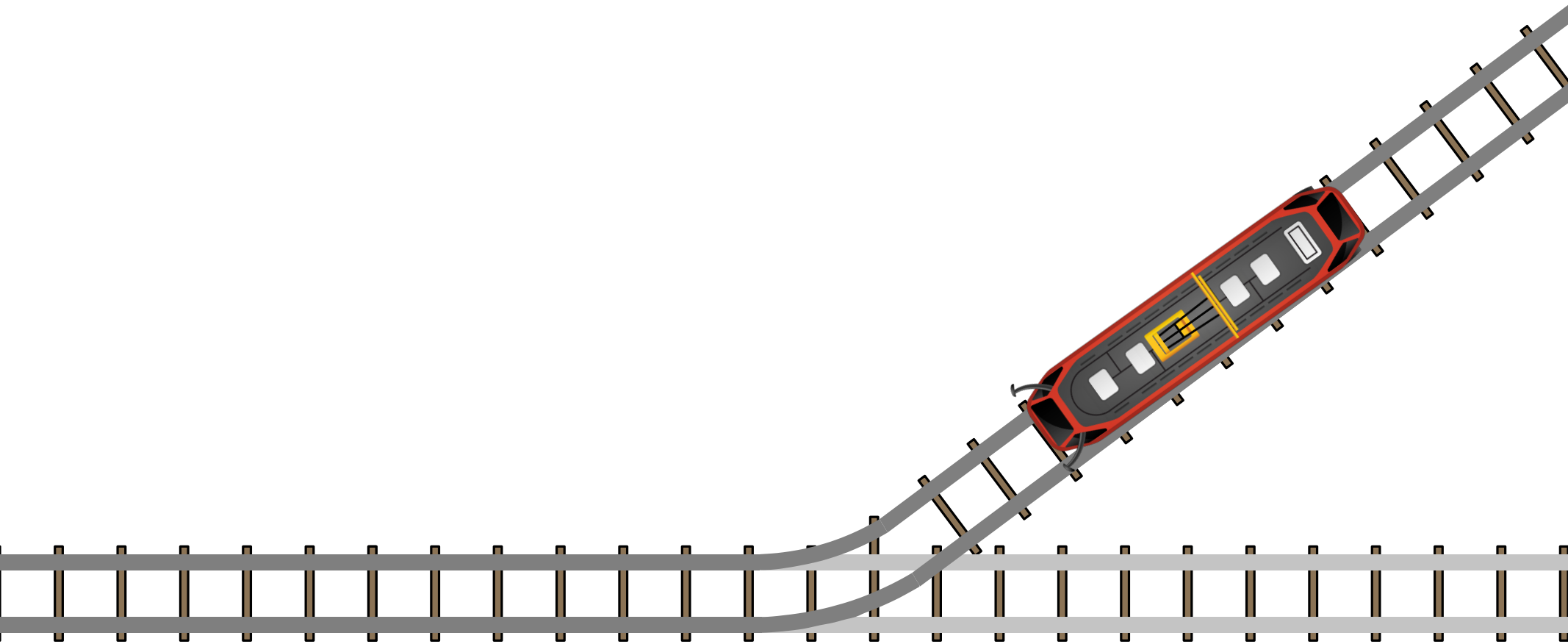
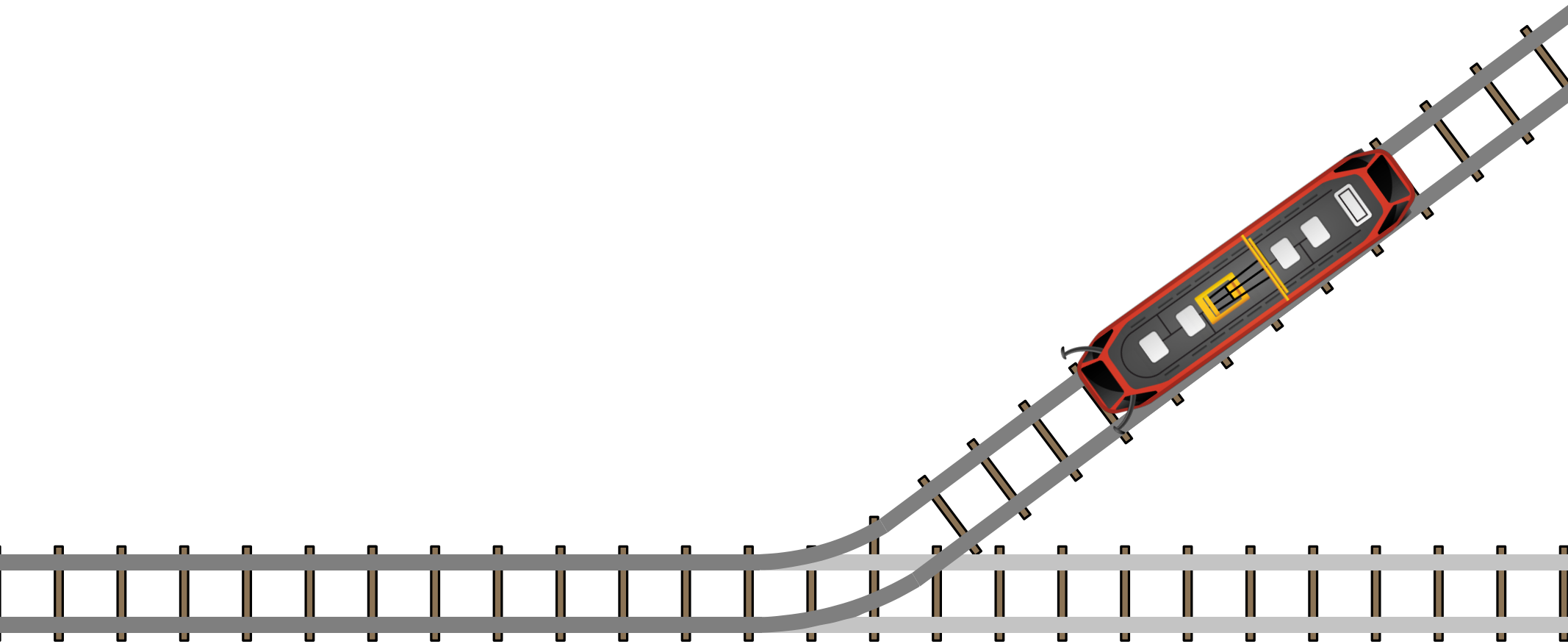


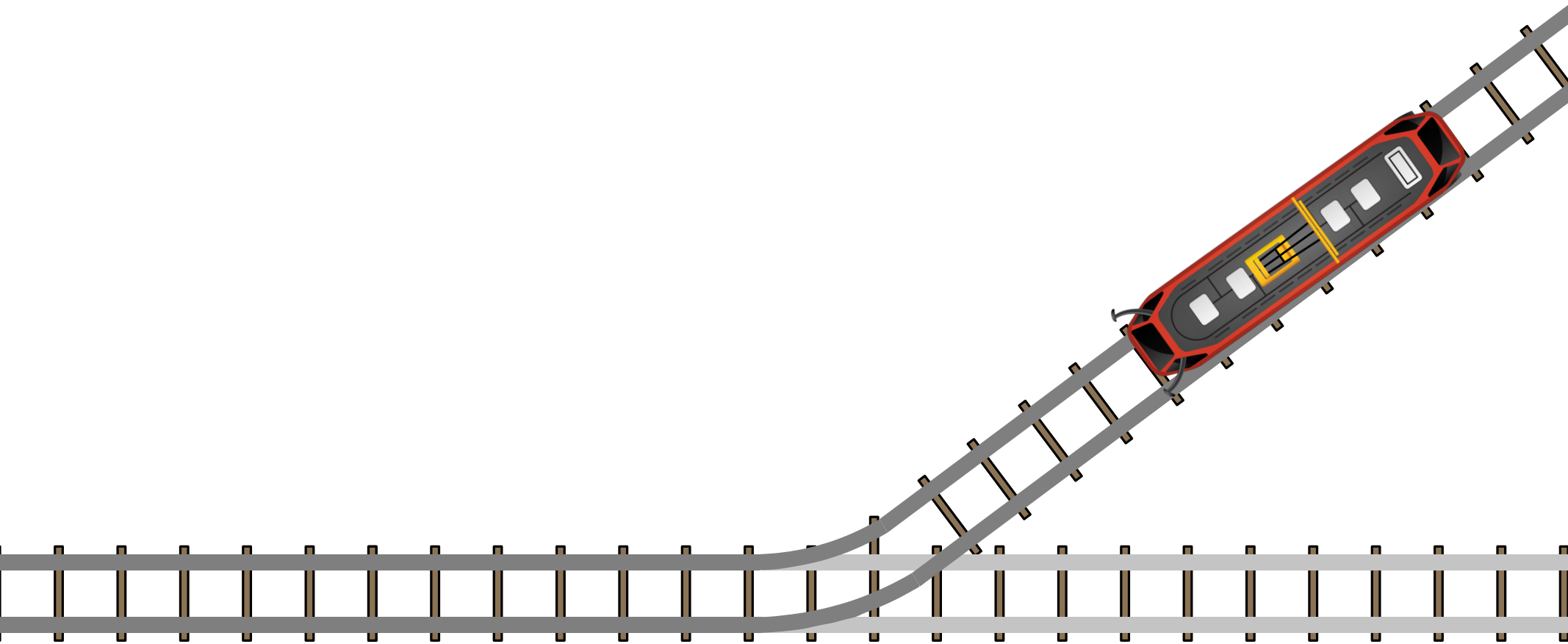


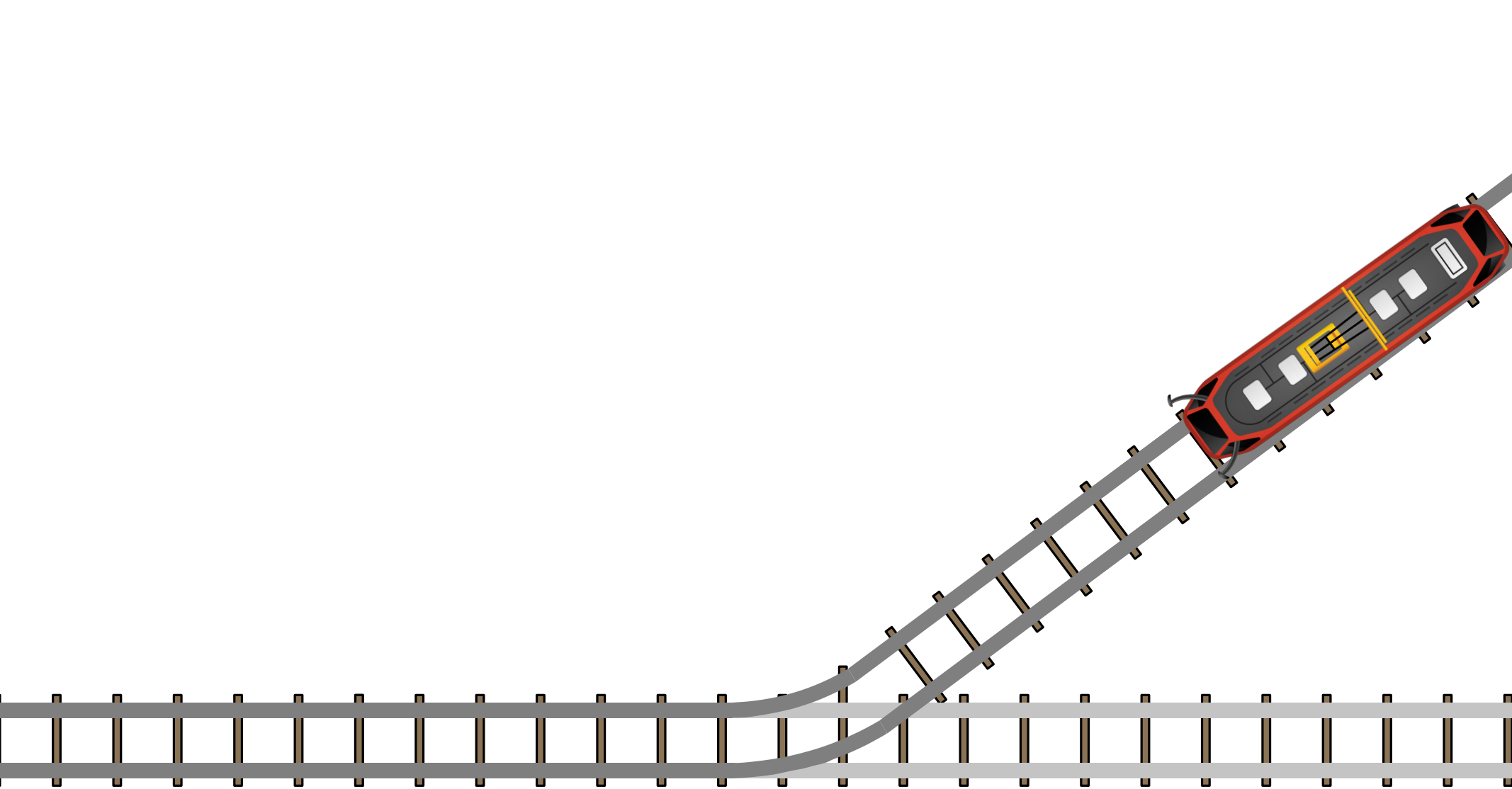


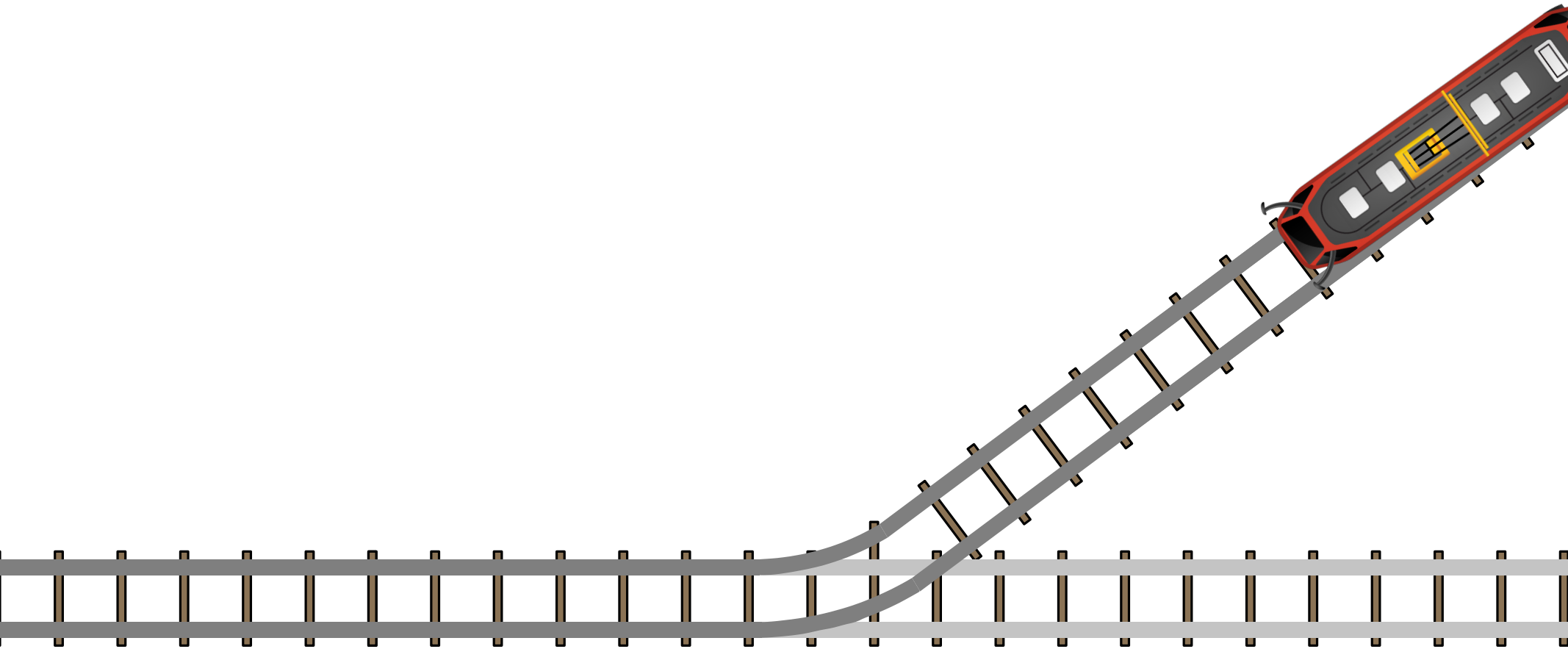


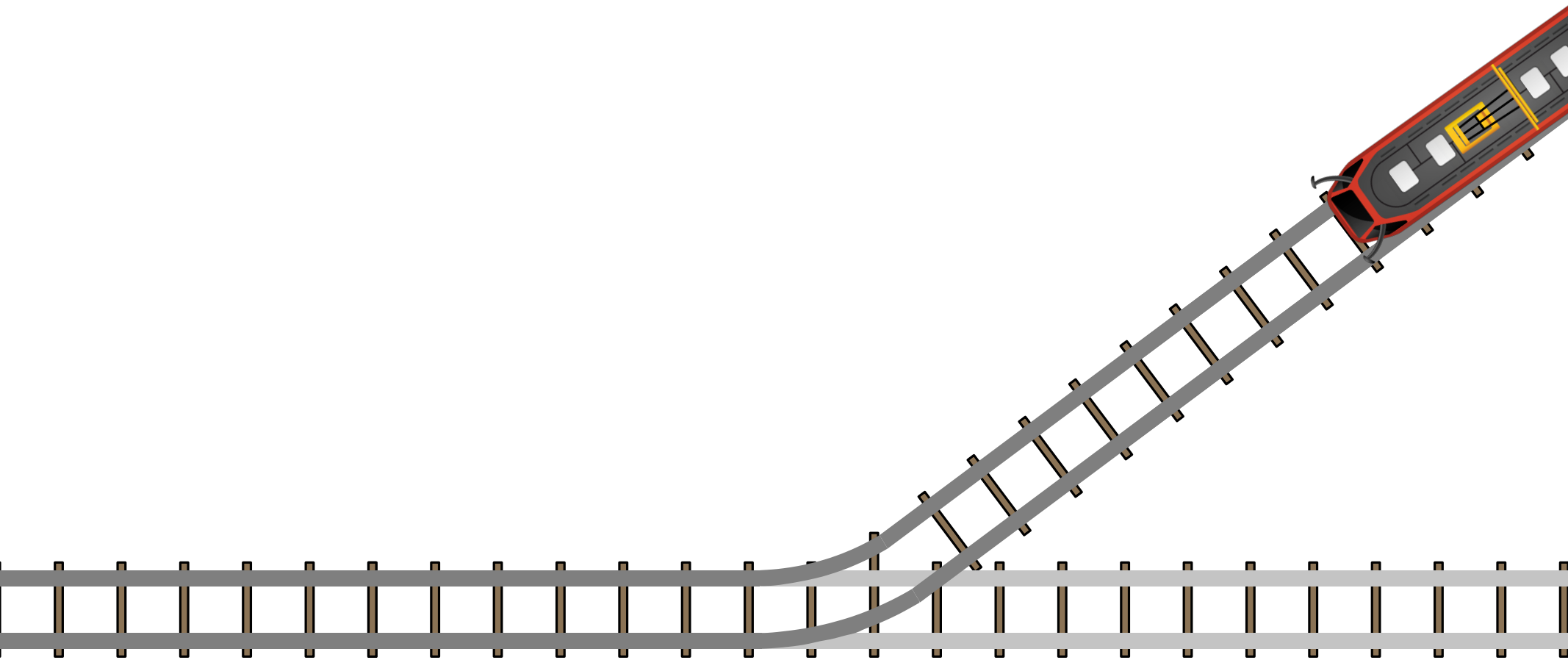


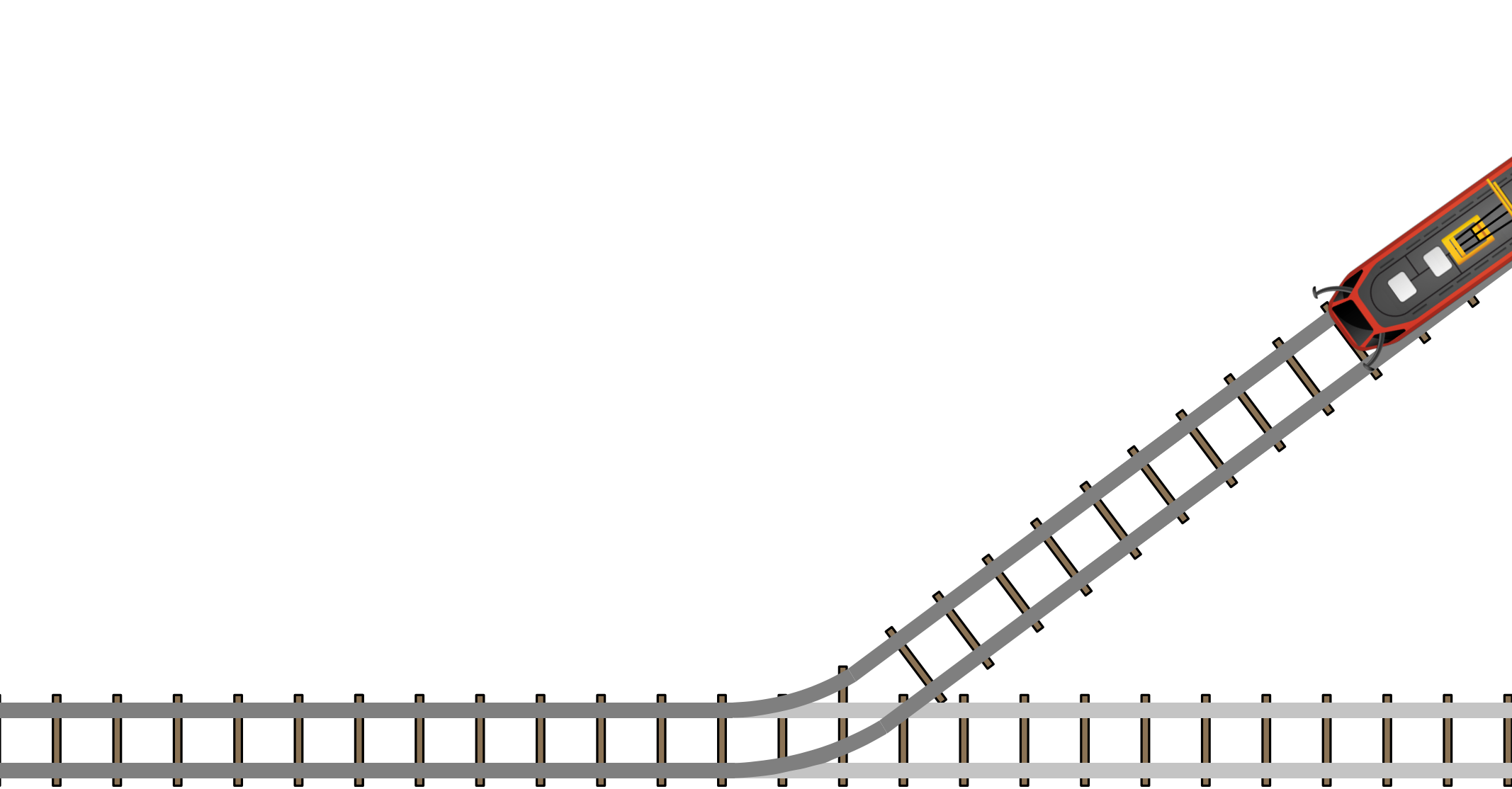


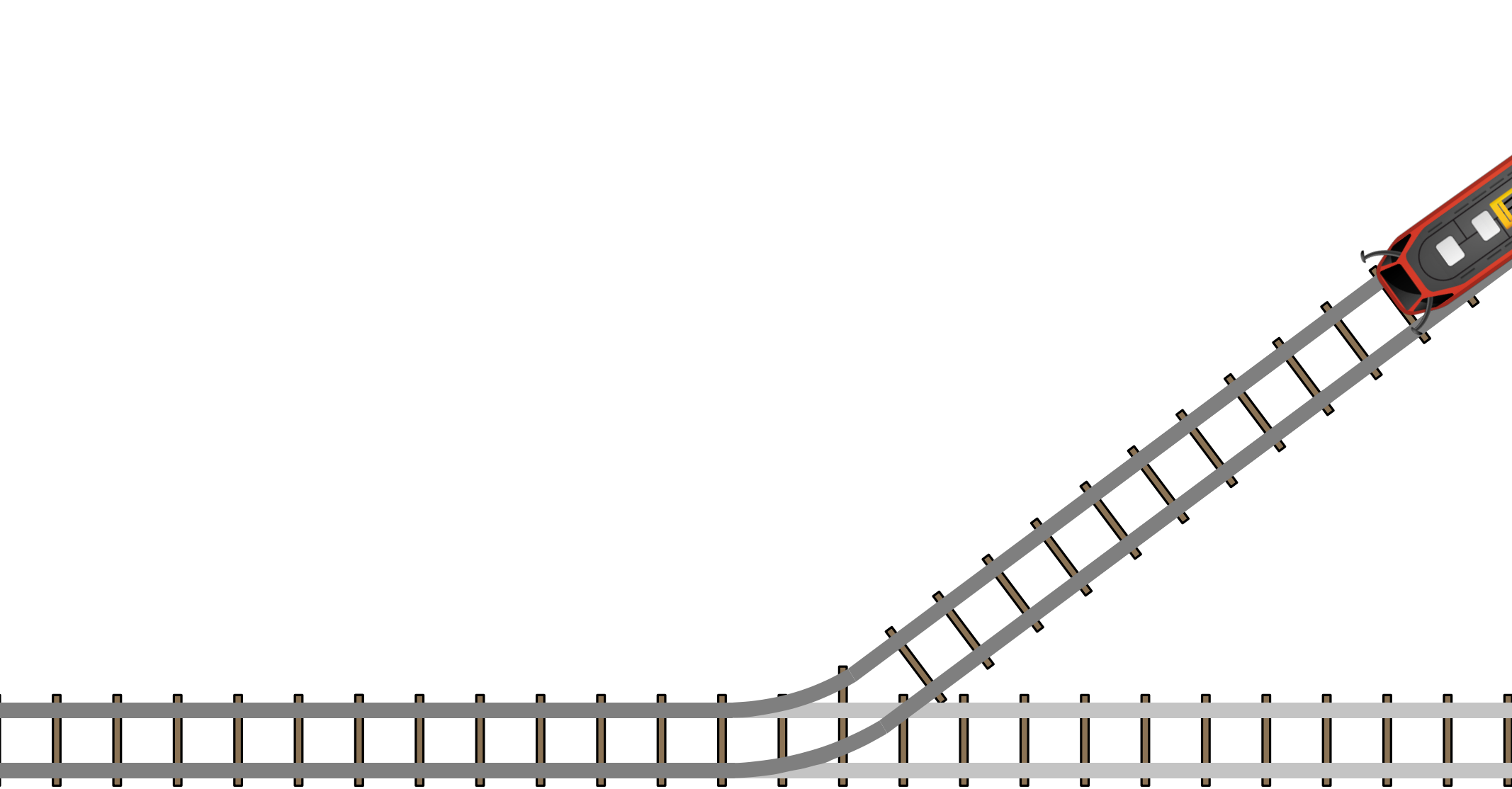


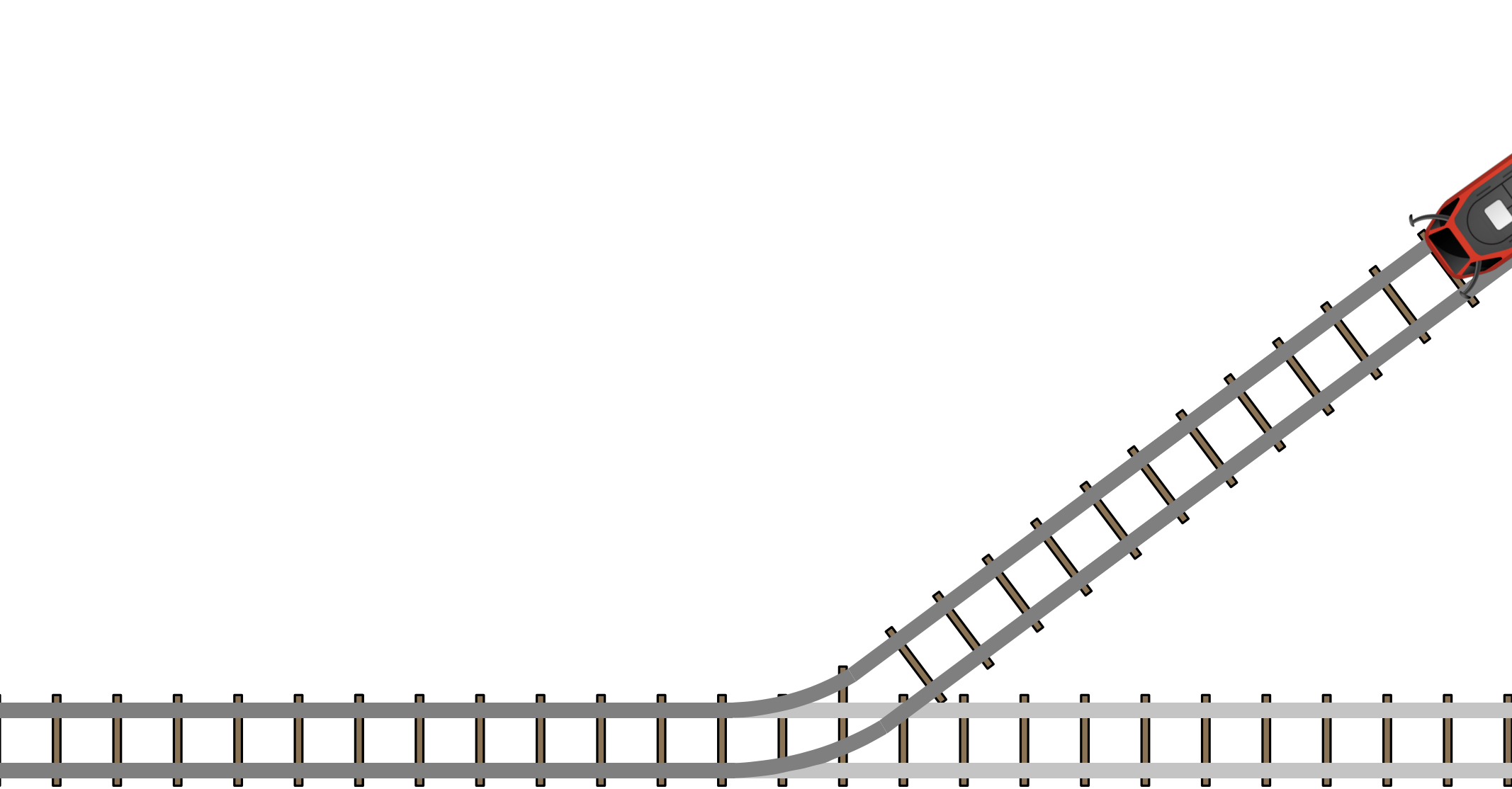


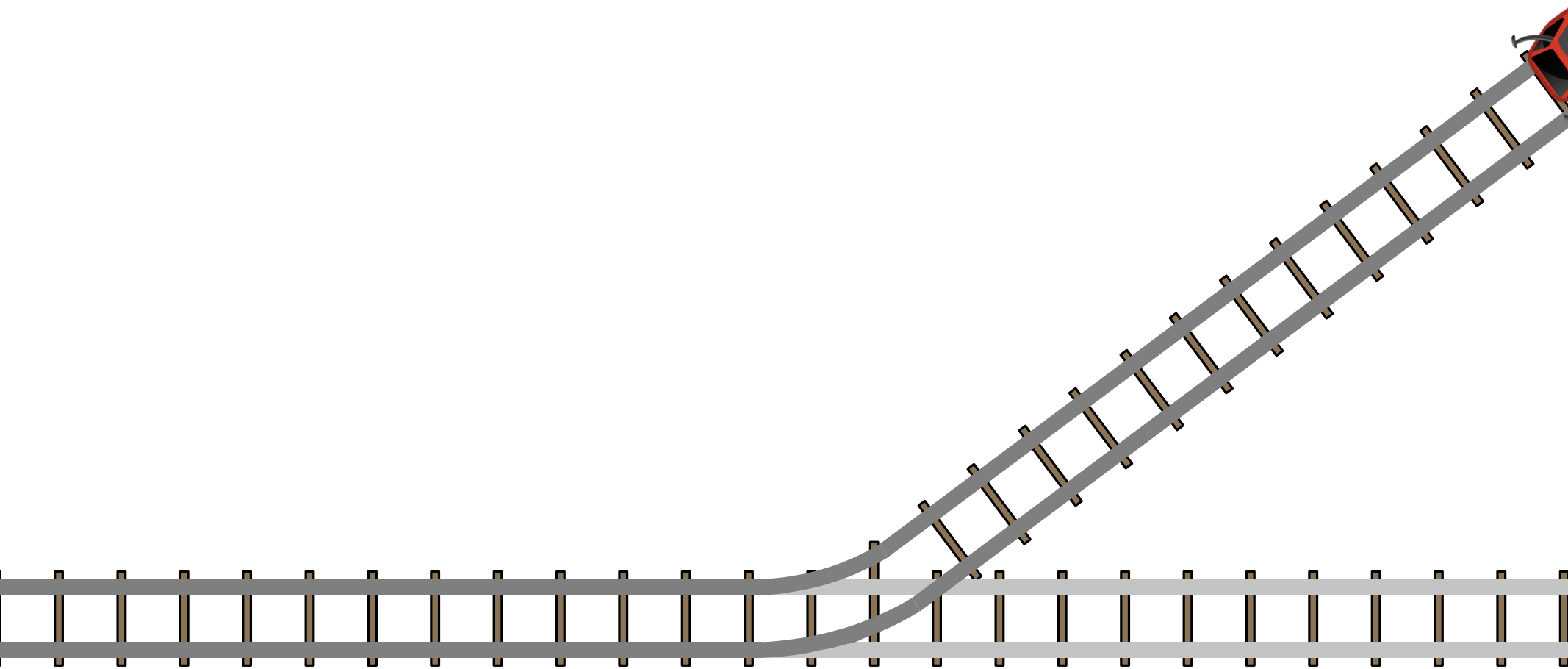


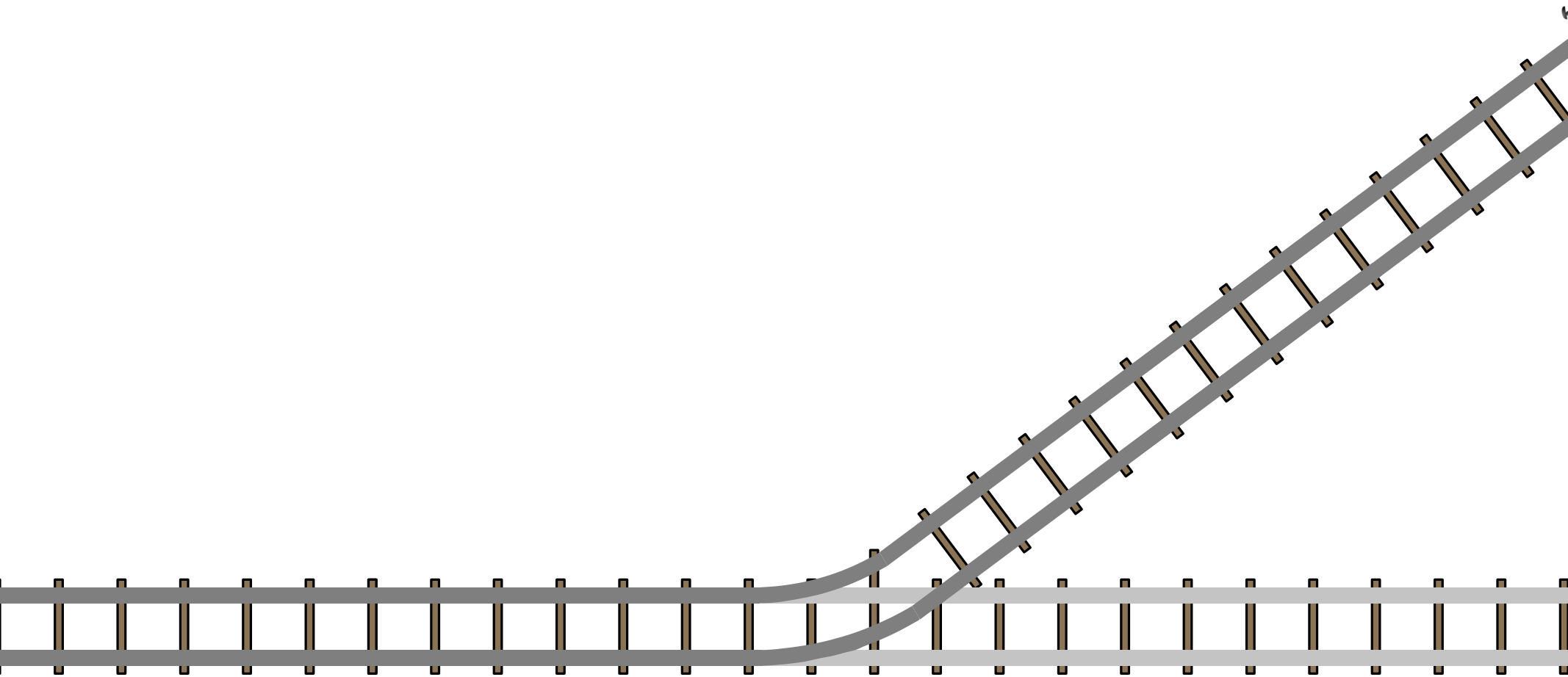


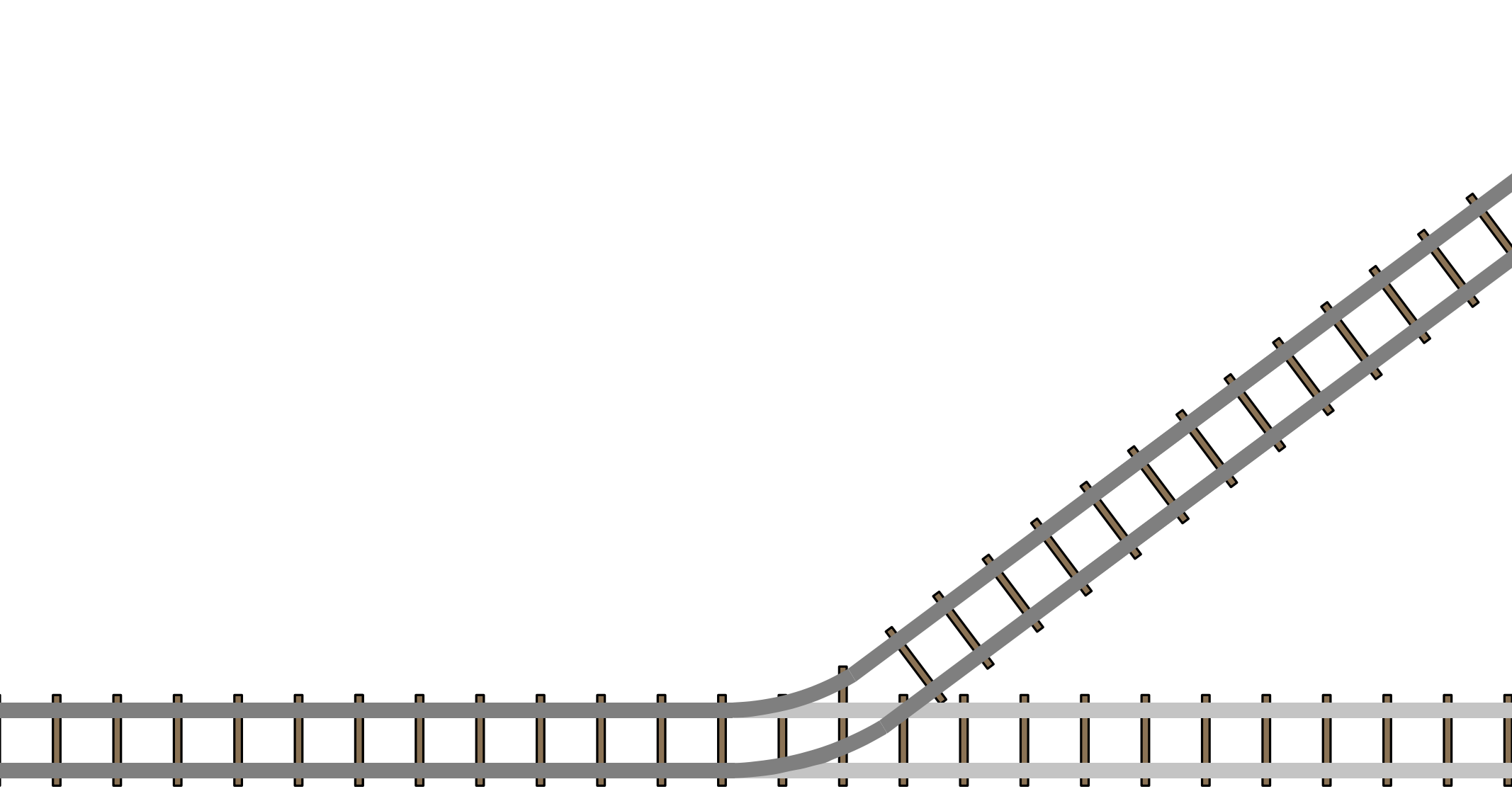


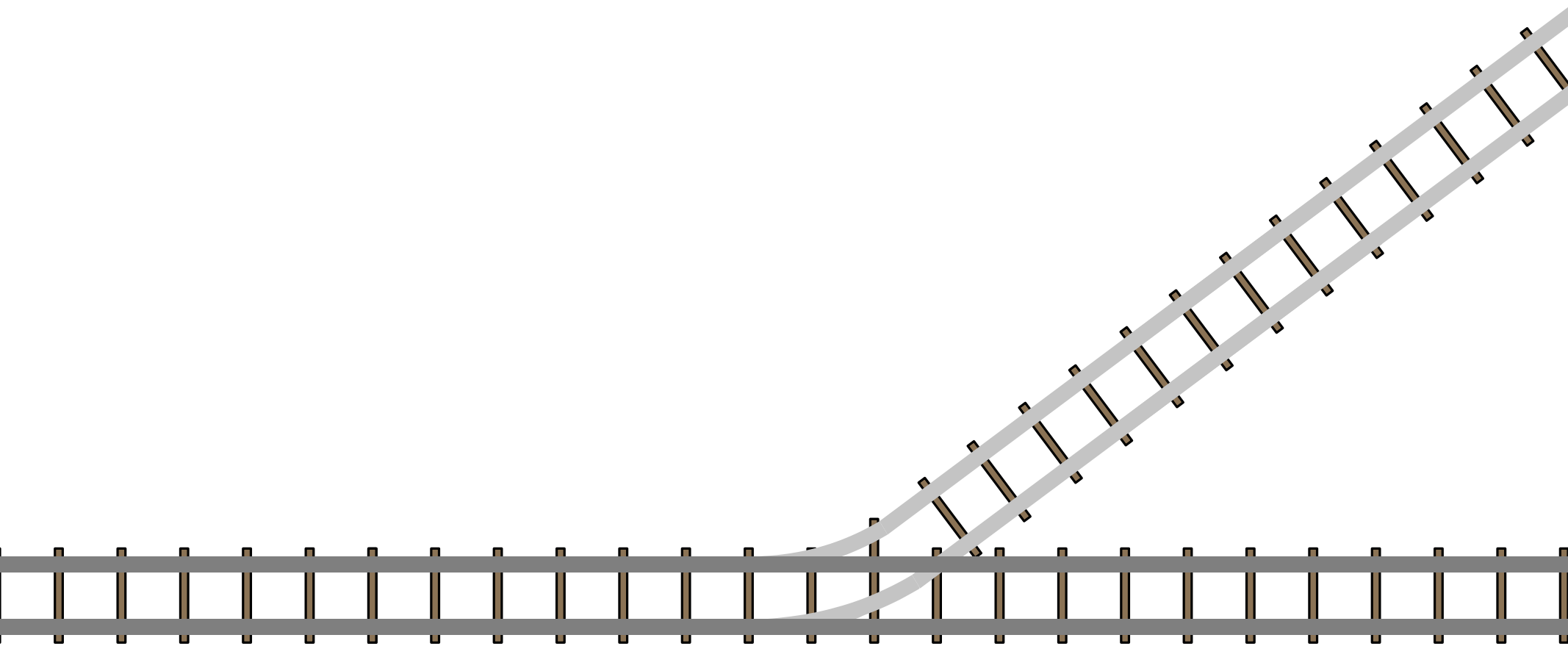






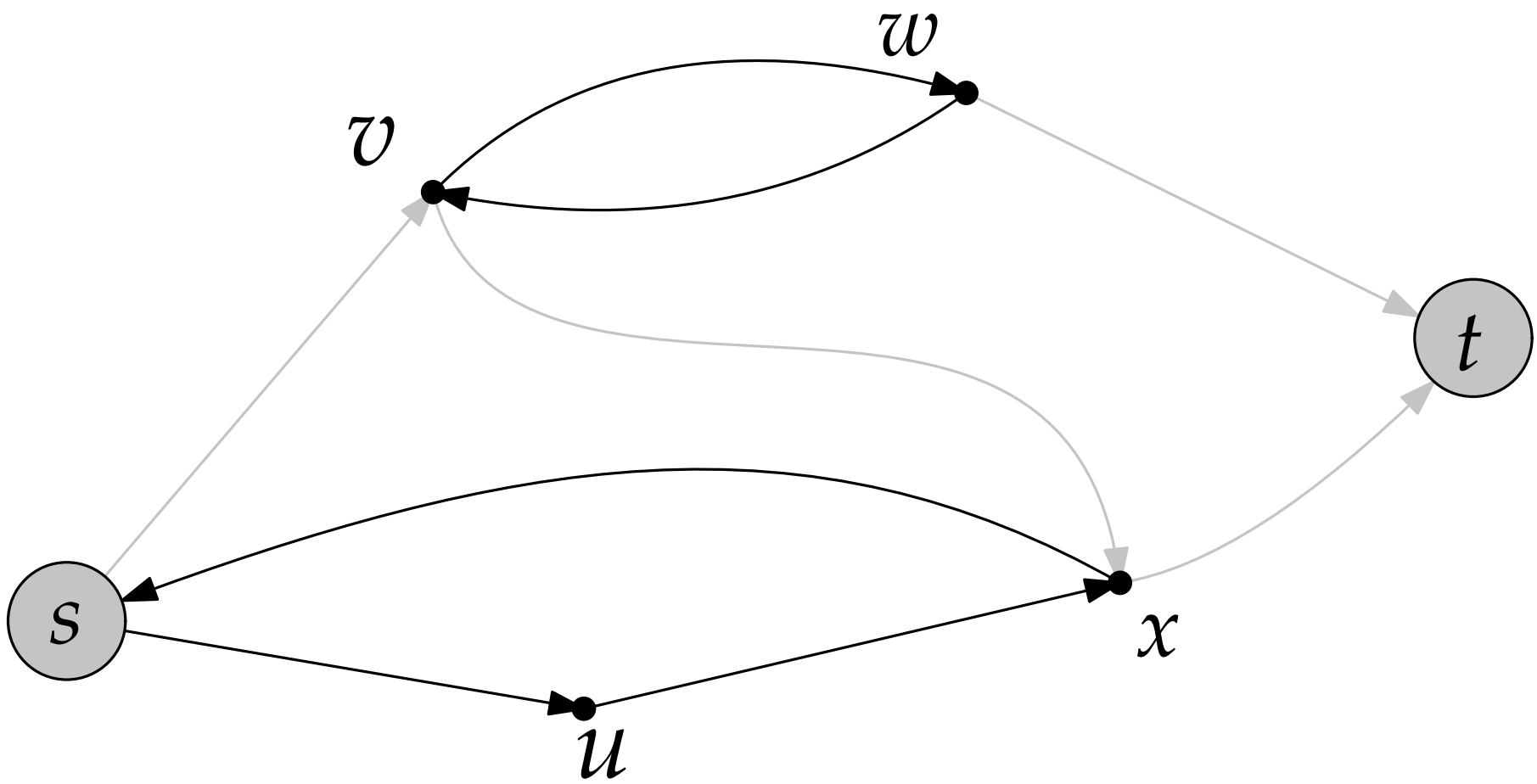






Arrival

Dohrau, Gärtner, Kohler, Matoušek, Welzl (2017)



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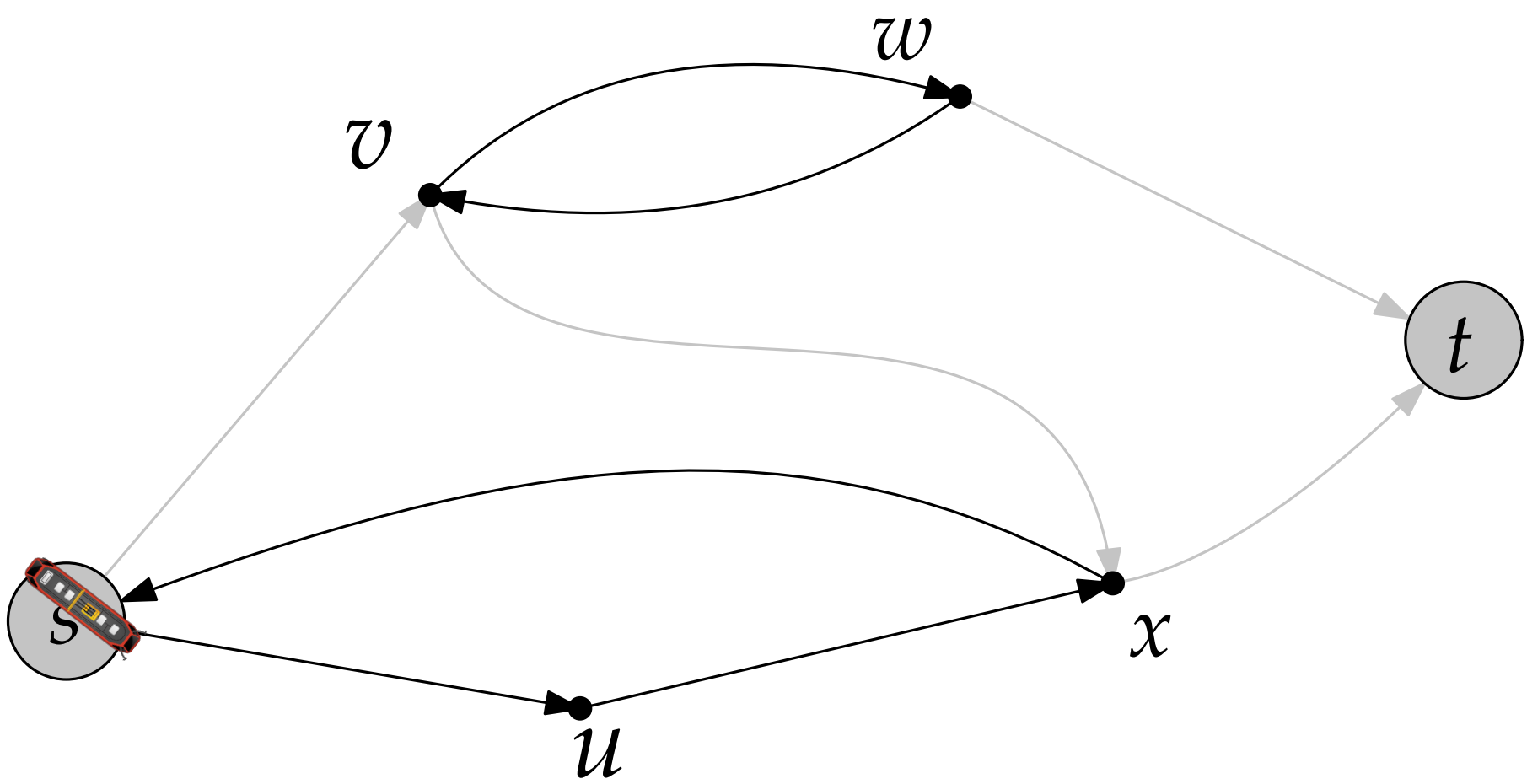
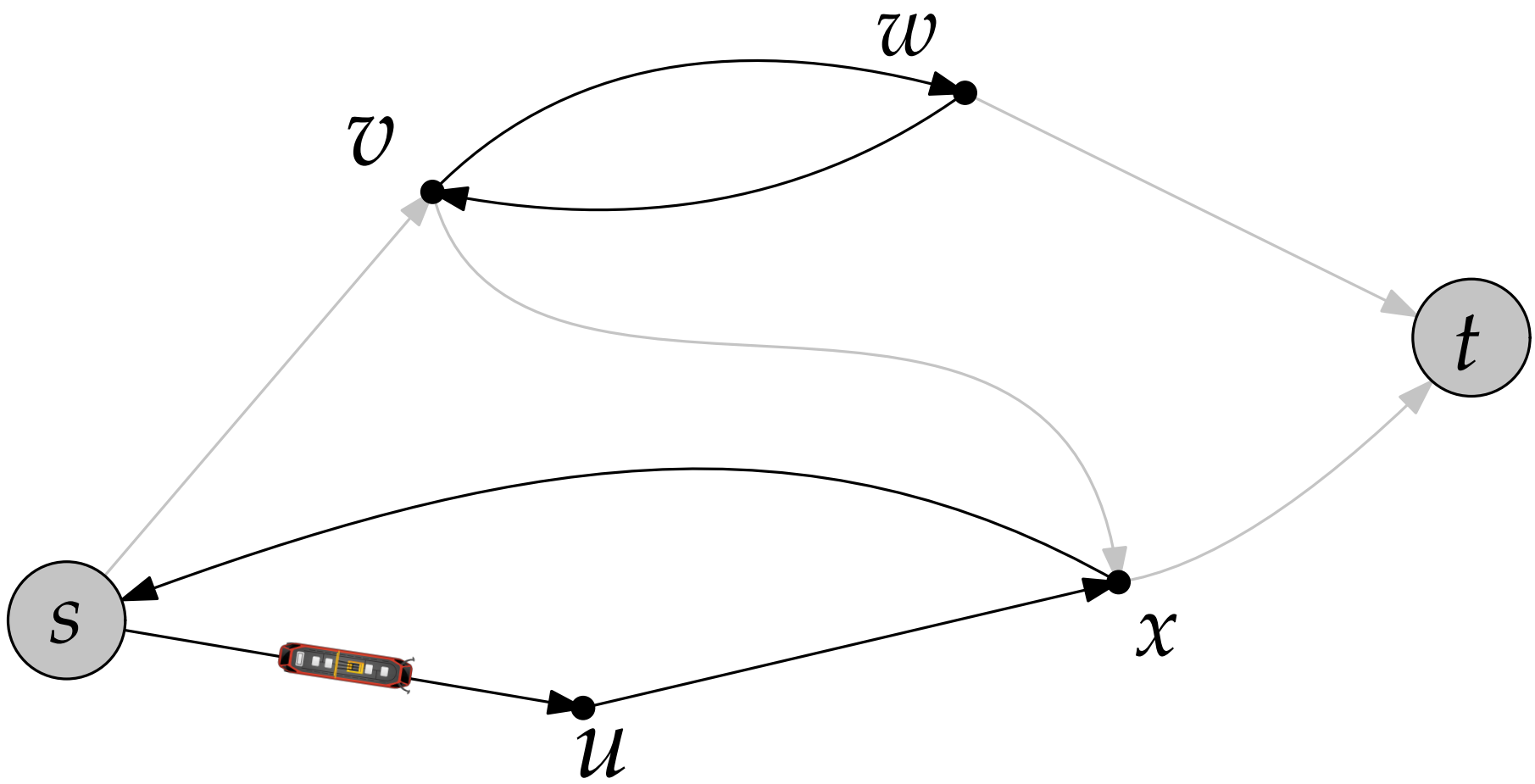
Dohrau, Gärtner, Kohler, Matoušek, Welzl (2017)



Will the train ever reach the station?

Arrival

Dohrau, Gärtner, Kohler, Matoušek, Welzl (2017)



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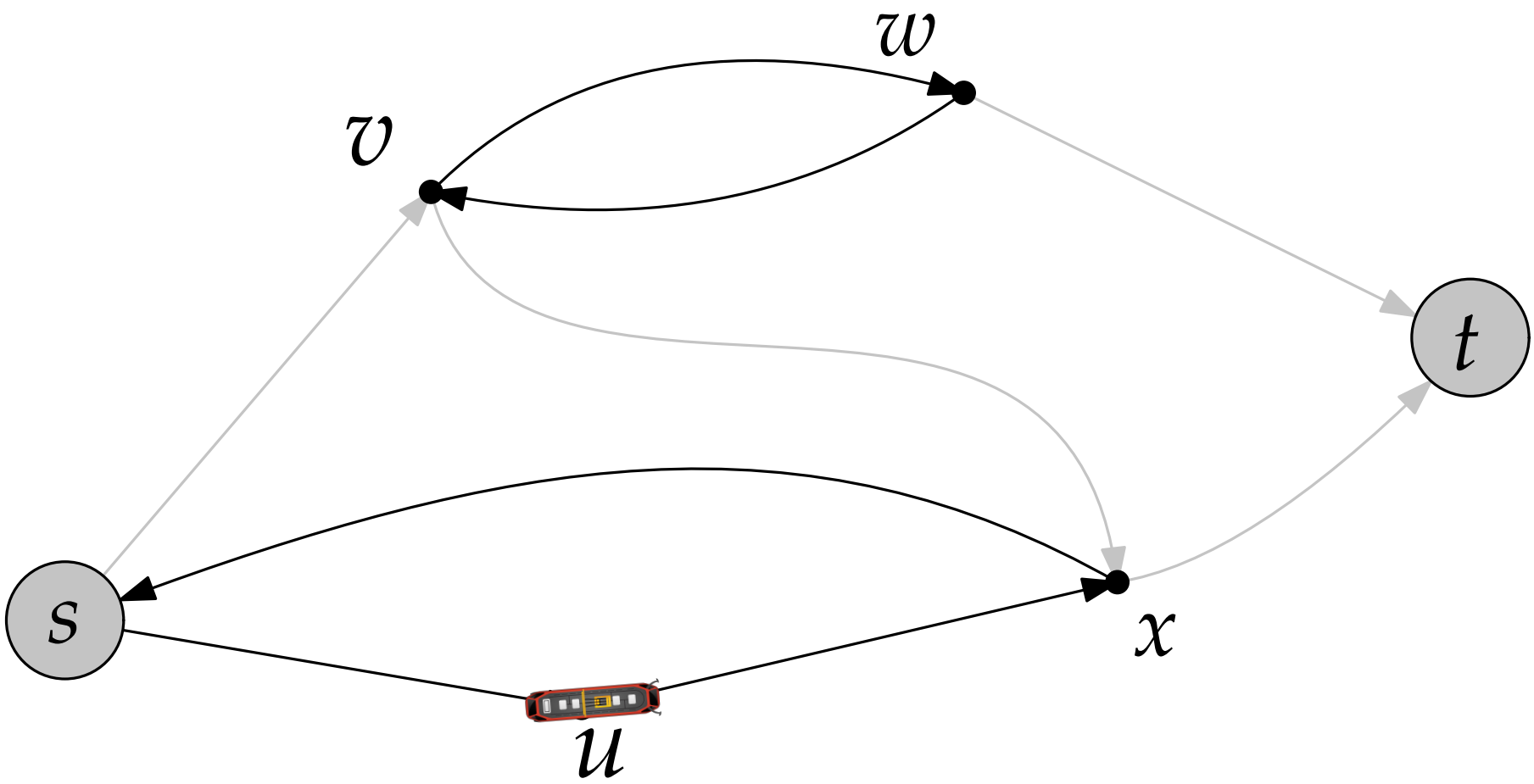

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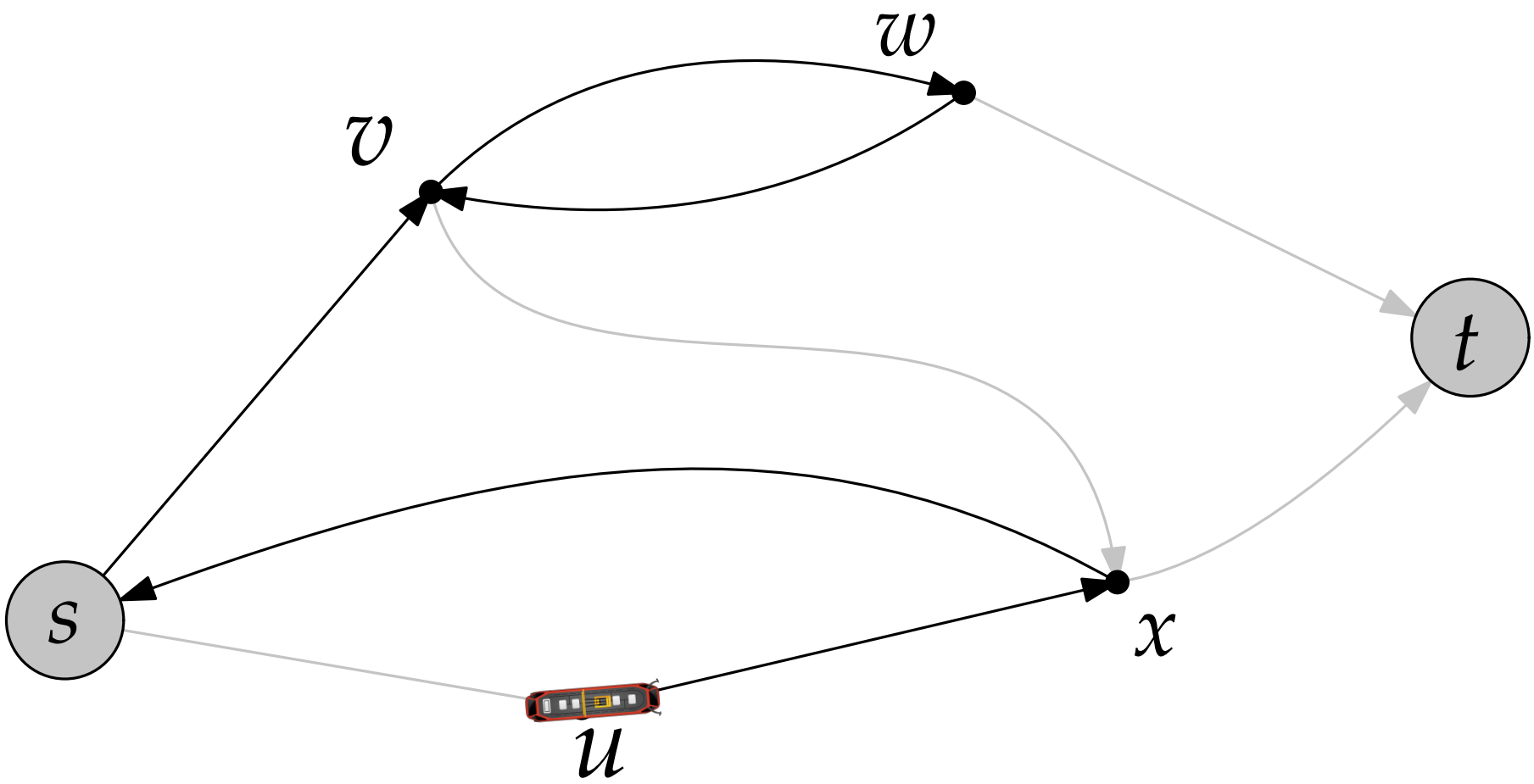

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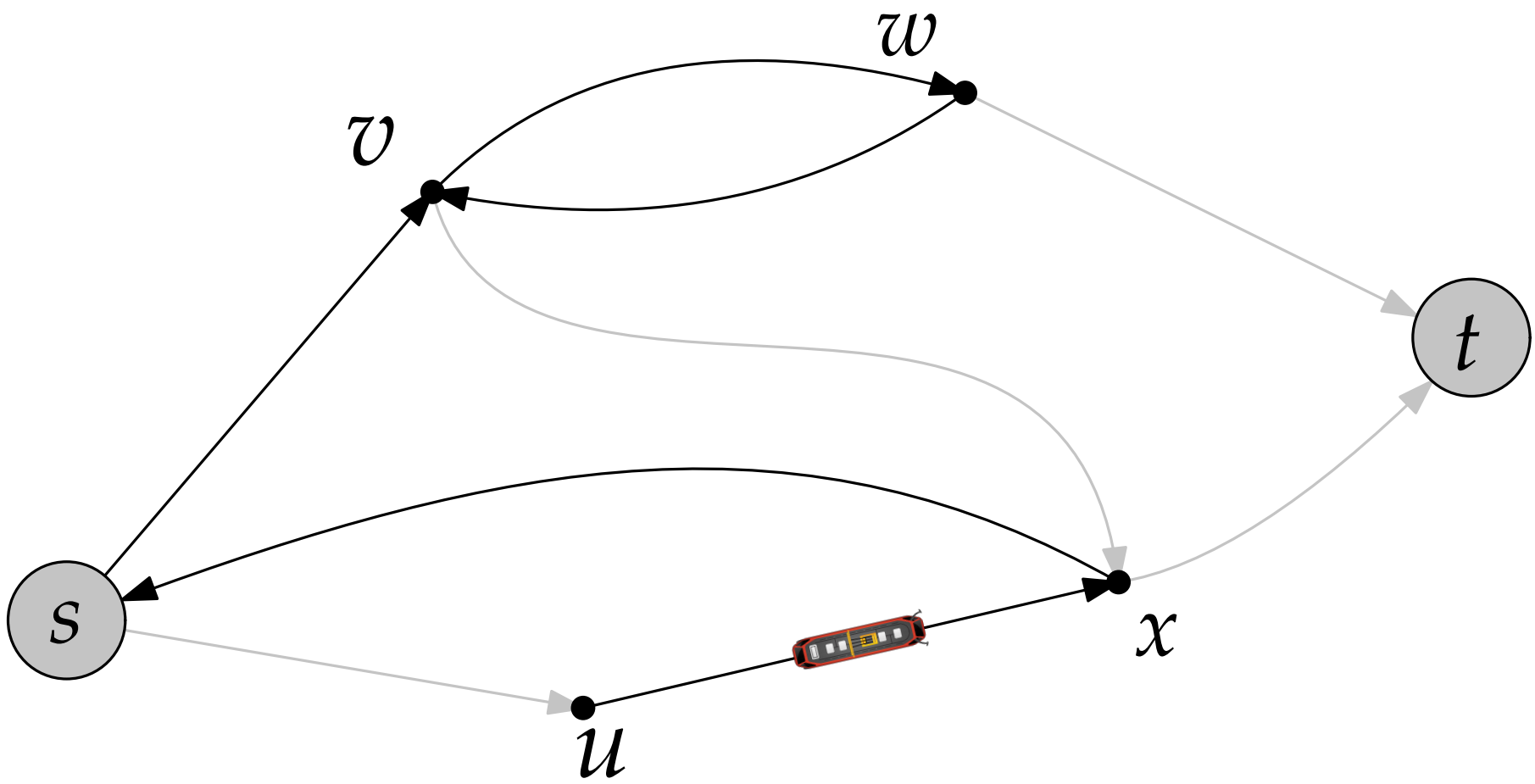

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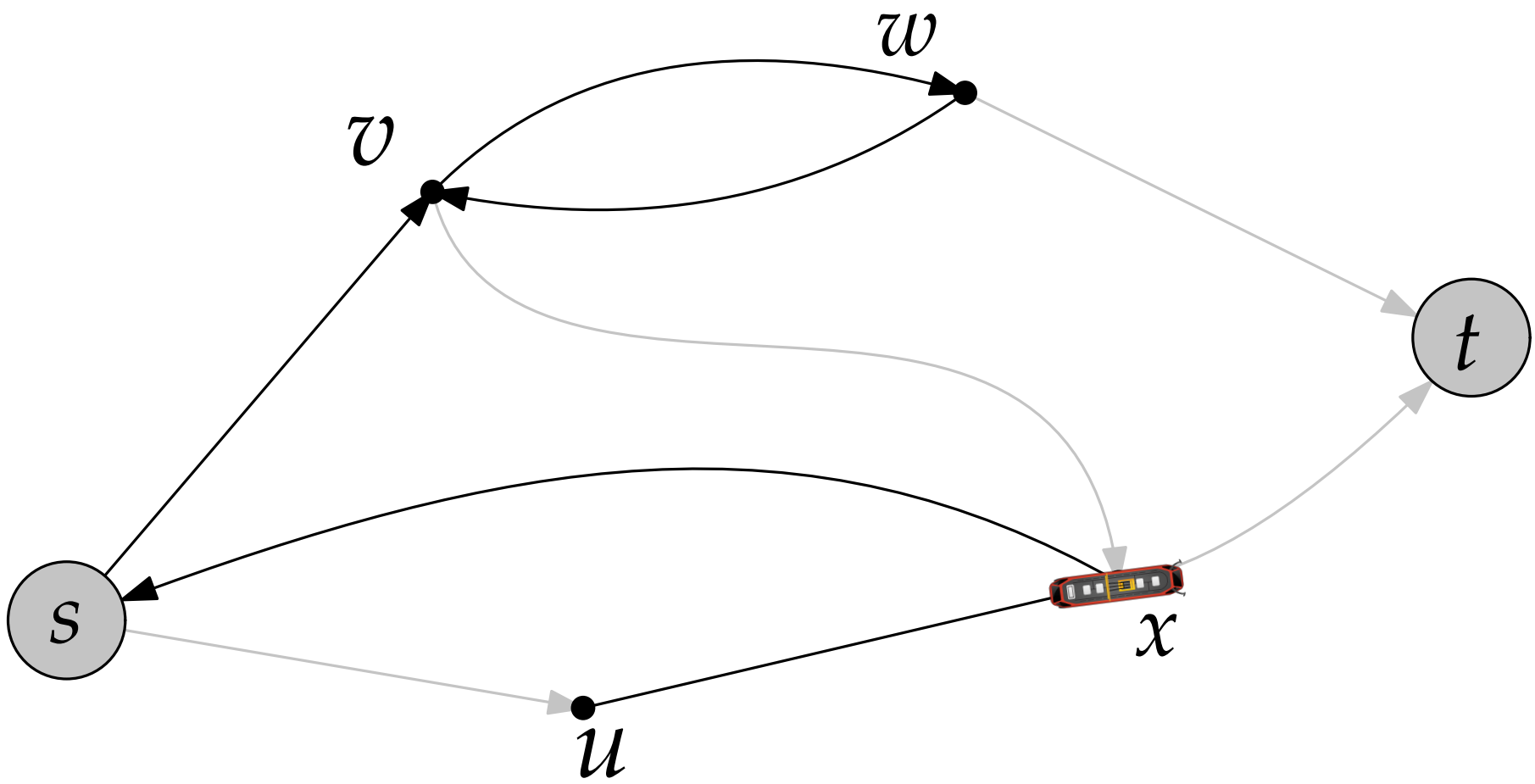

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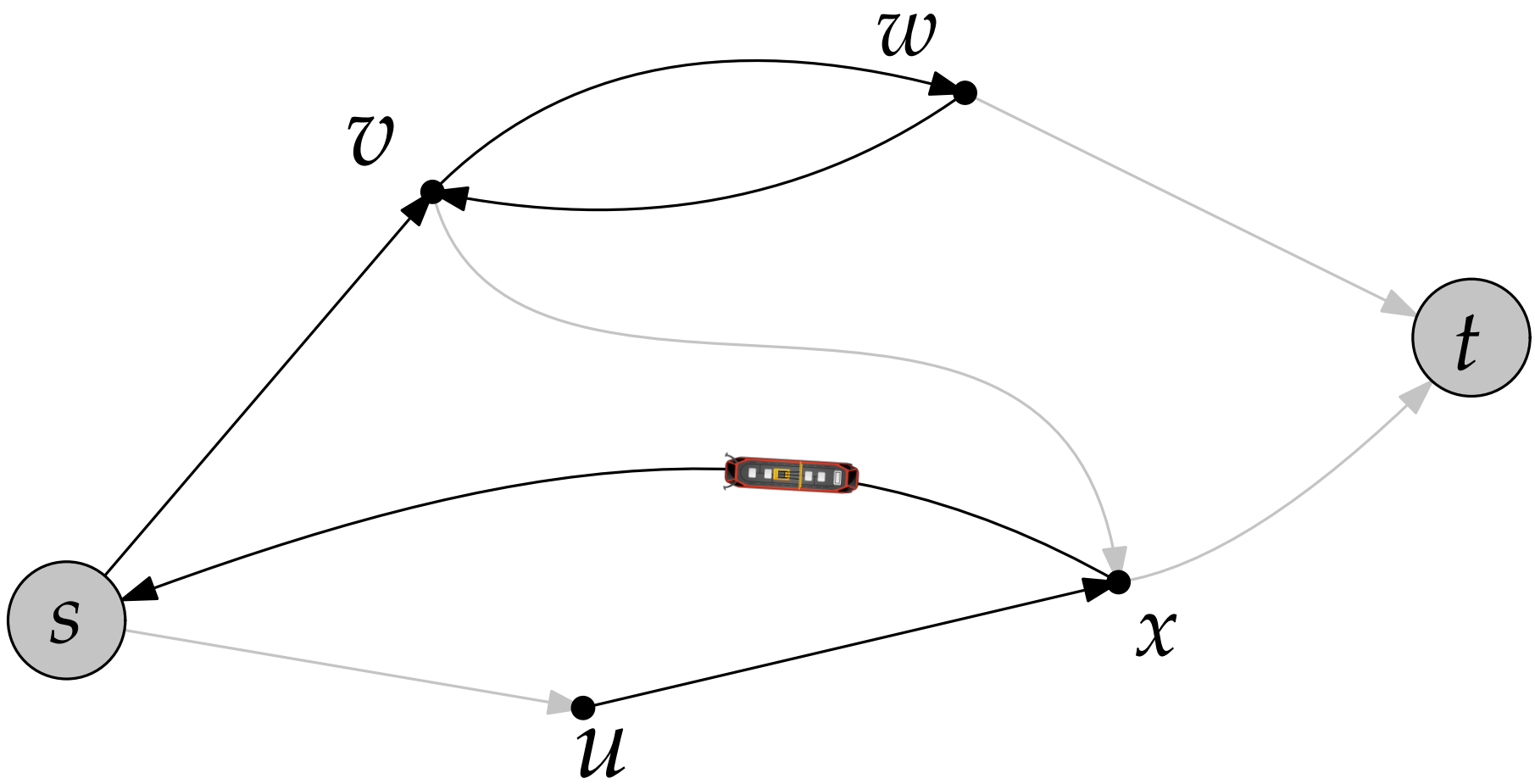

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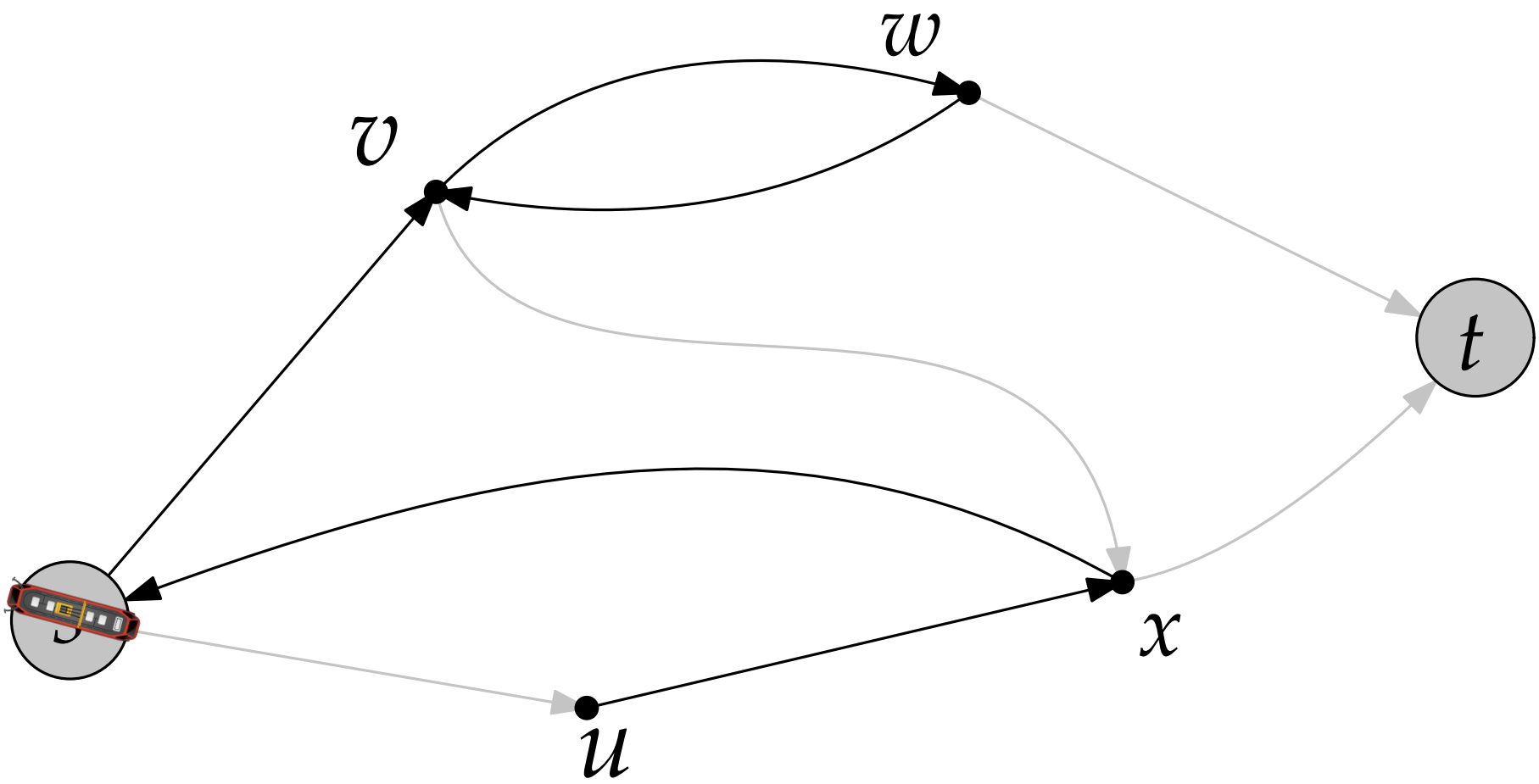

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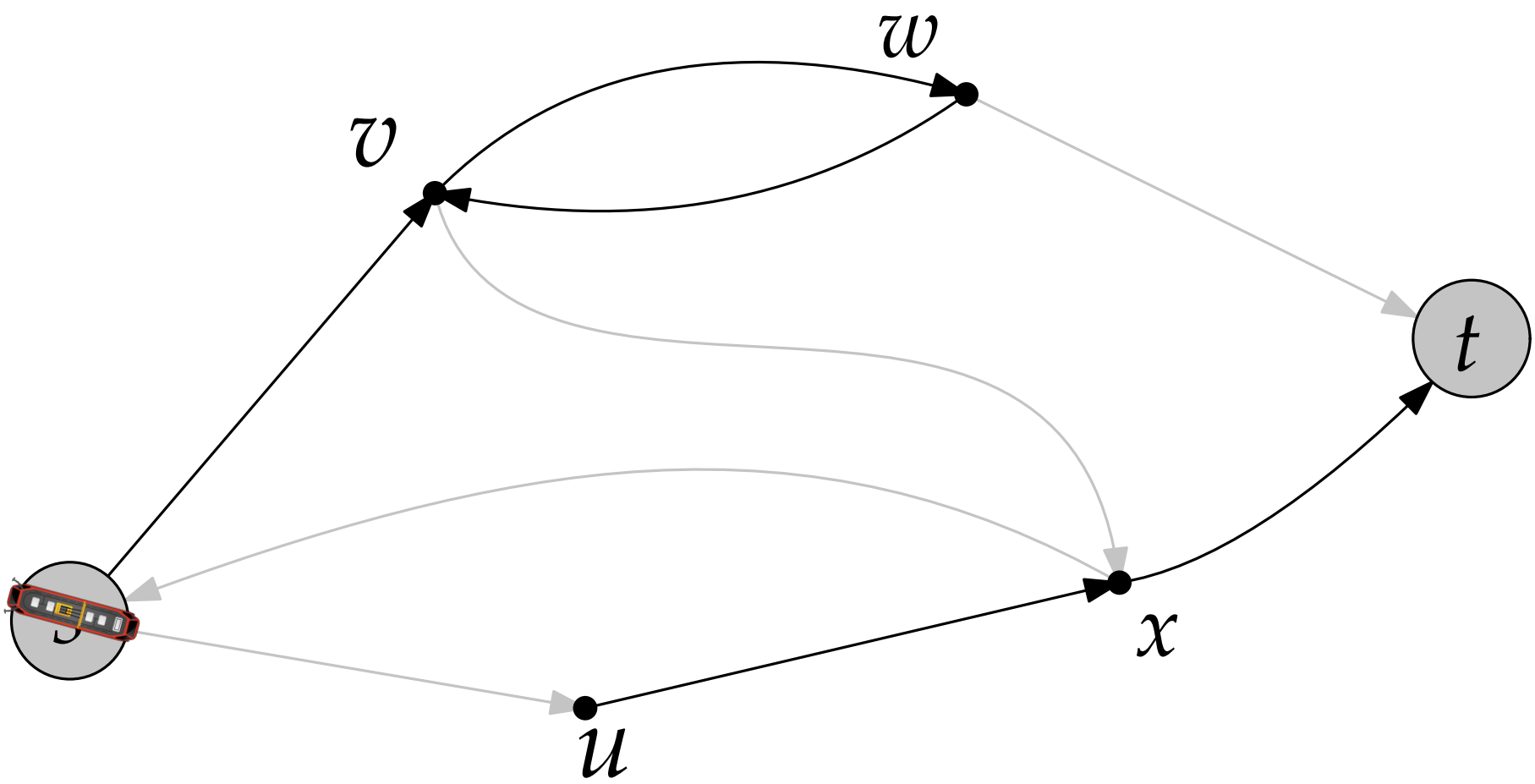

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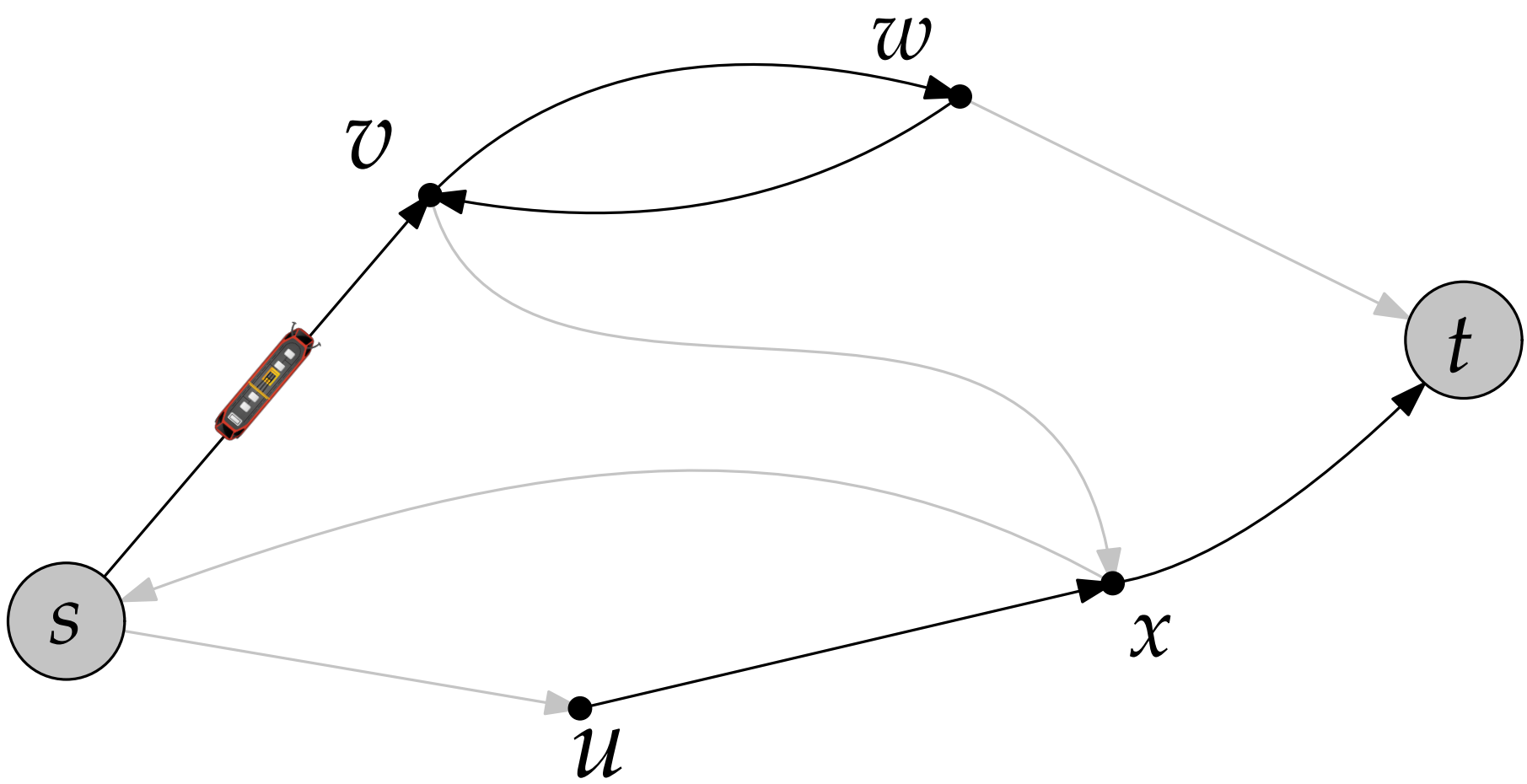

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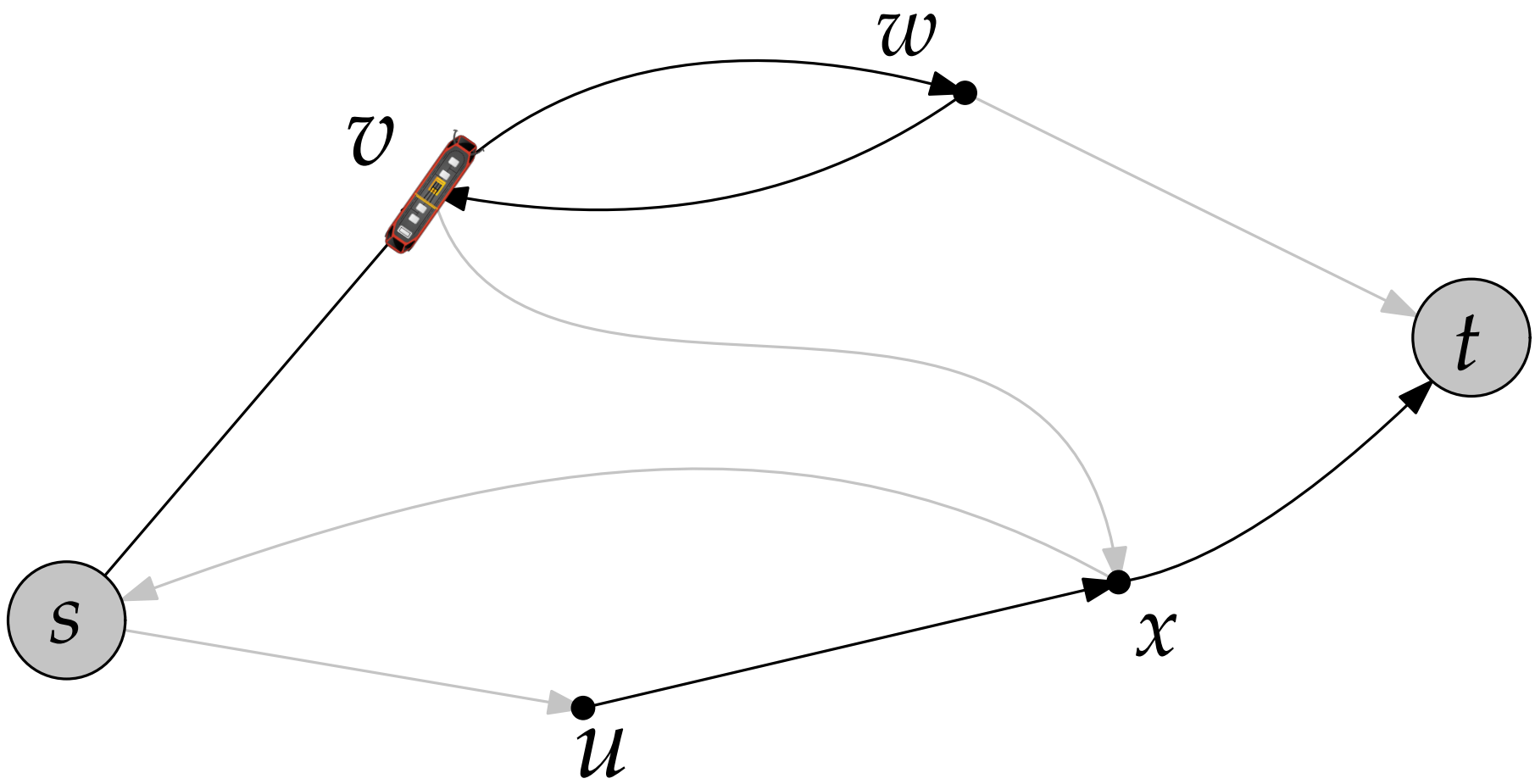

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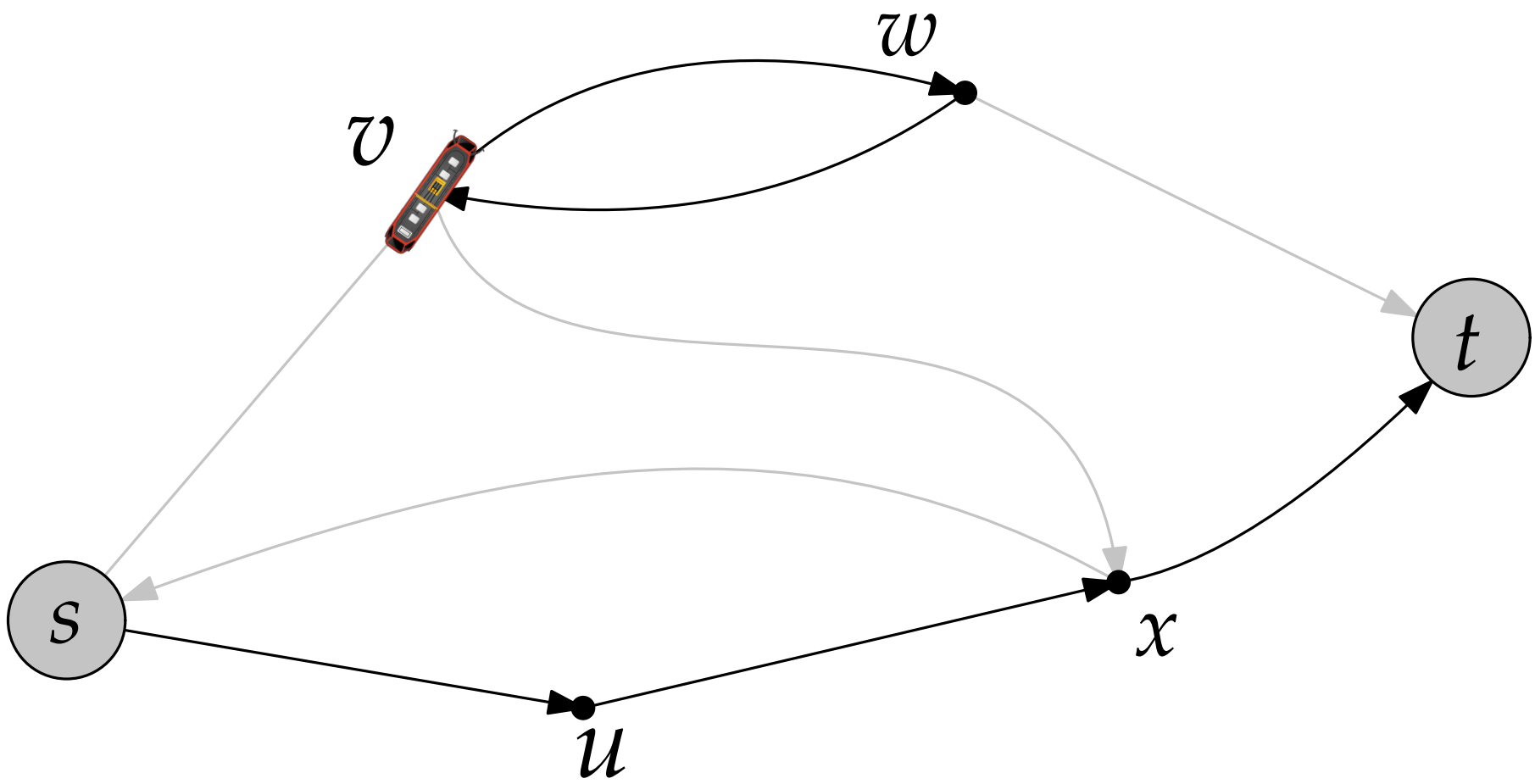

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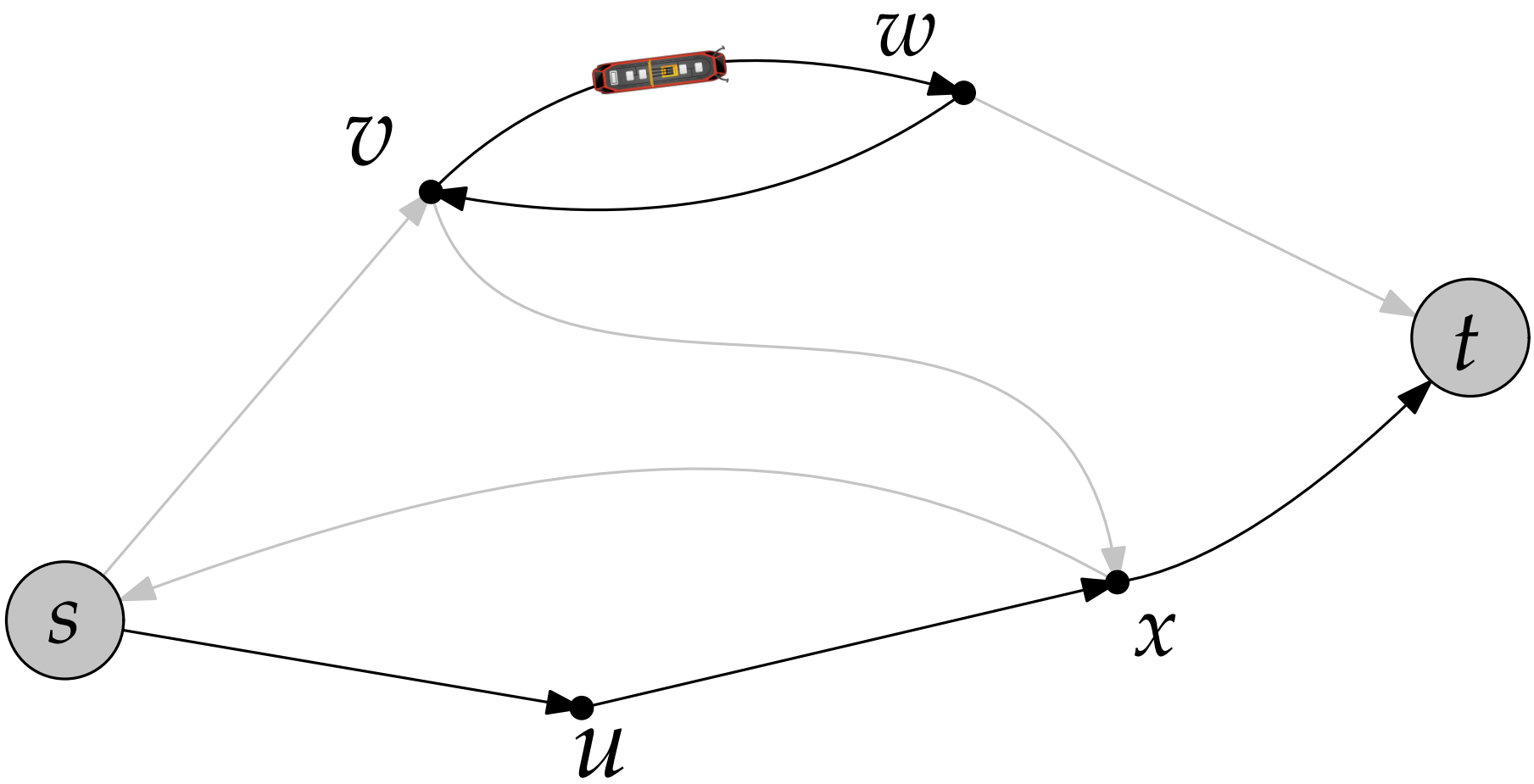

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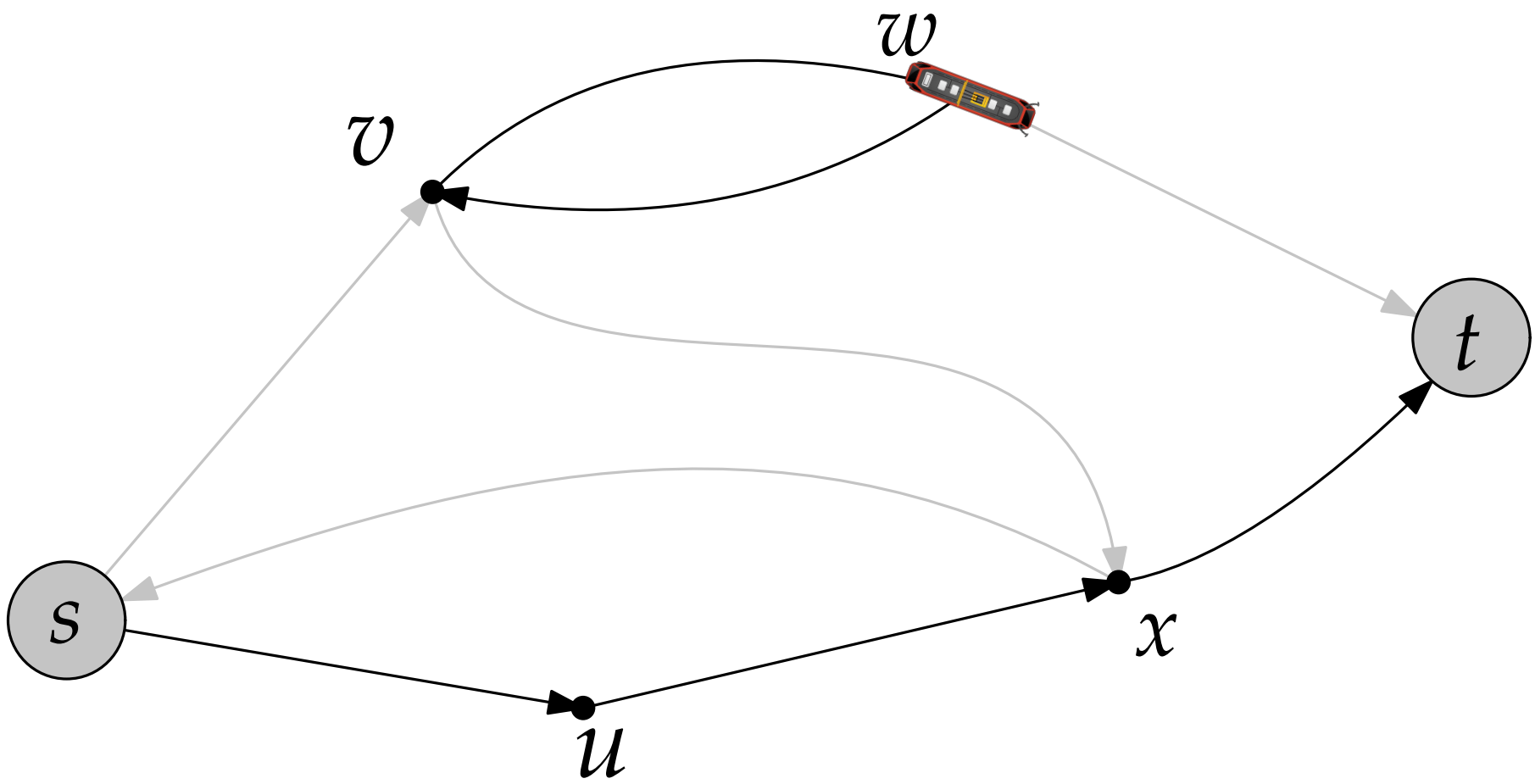

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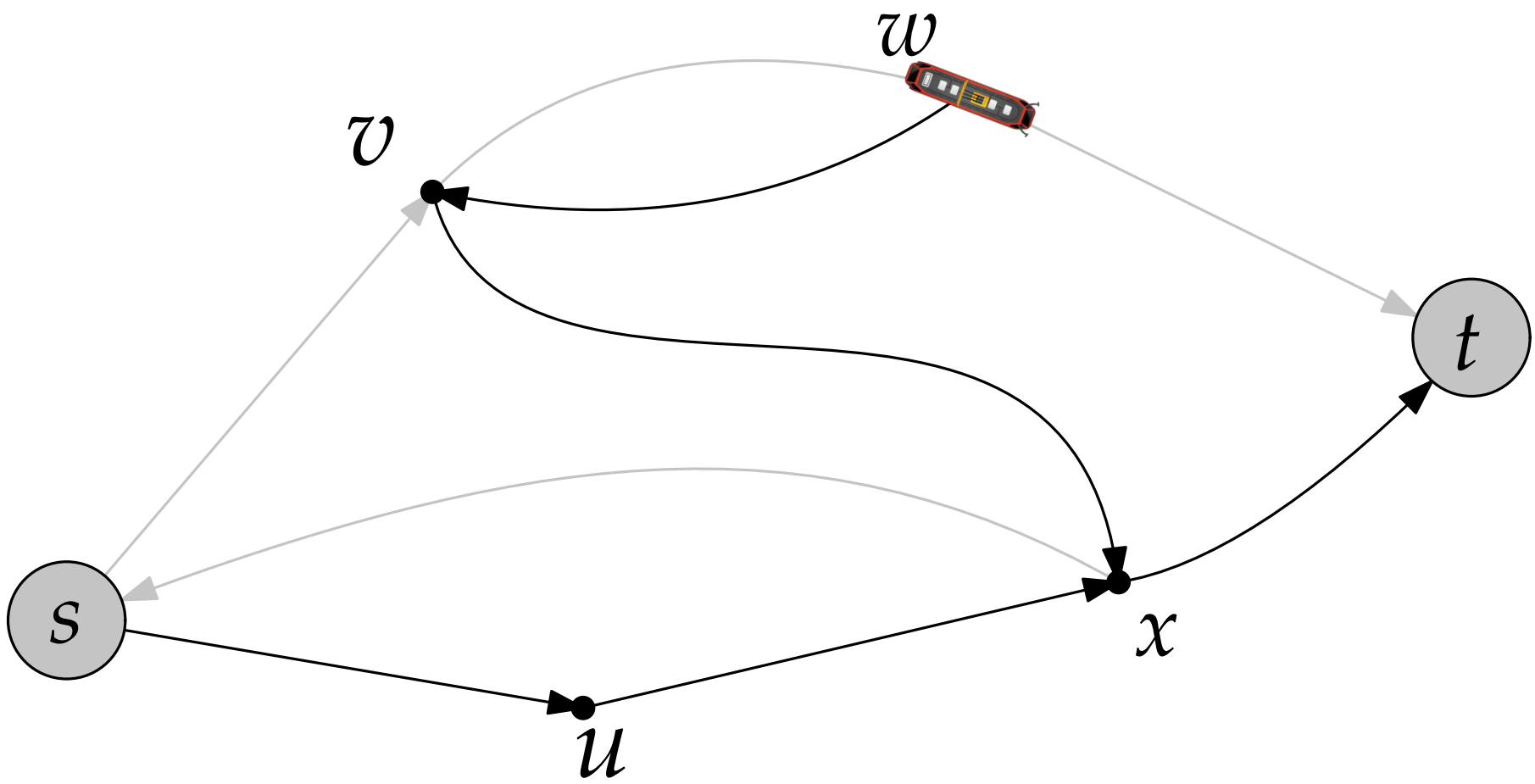

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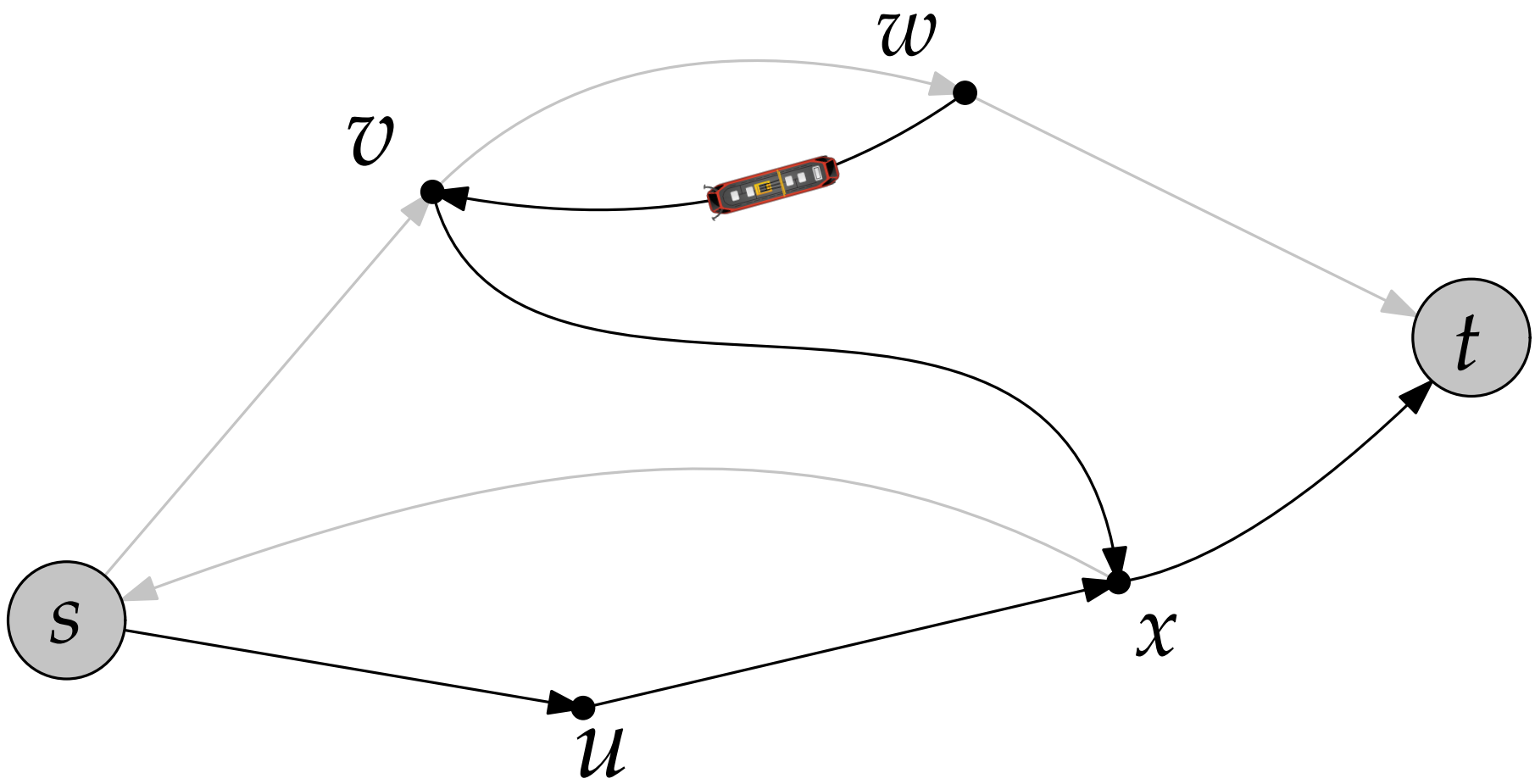

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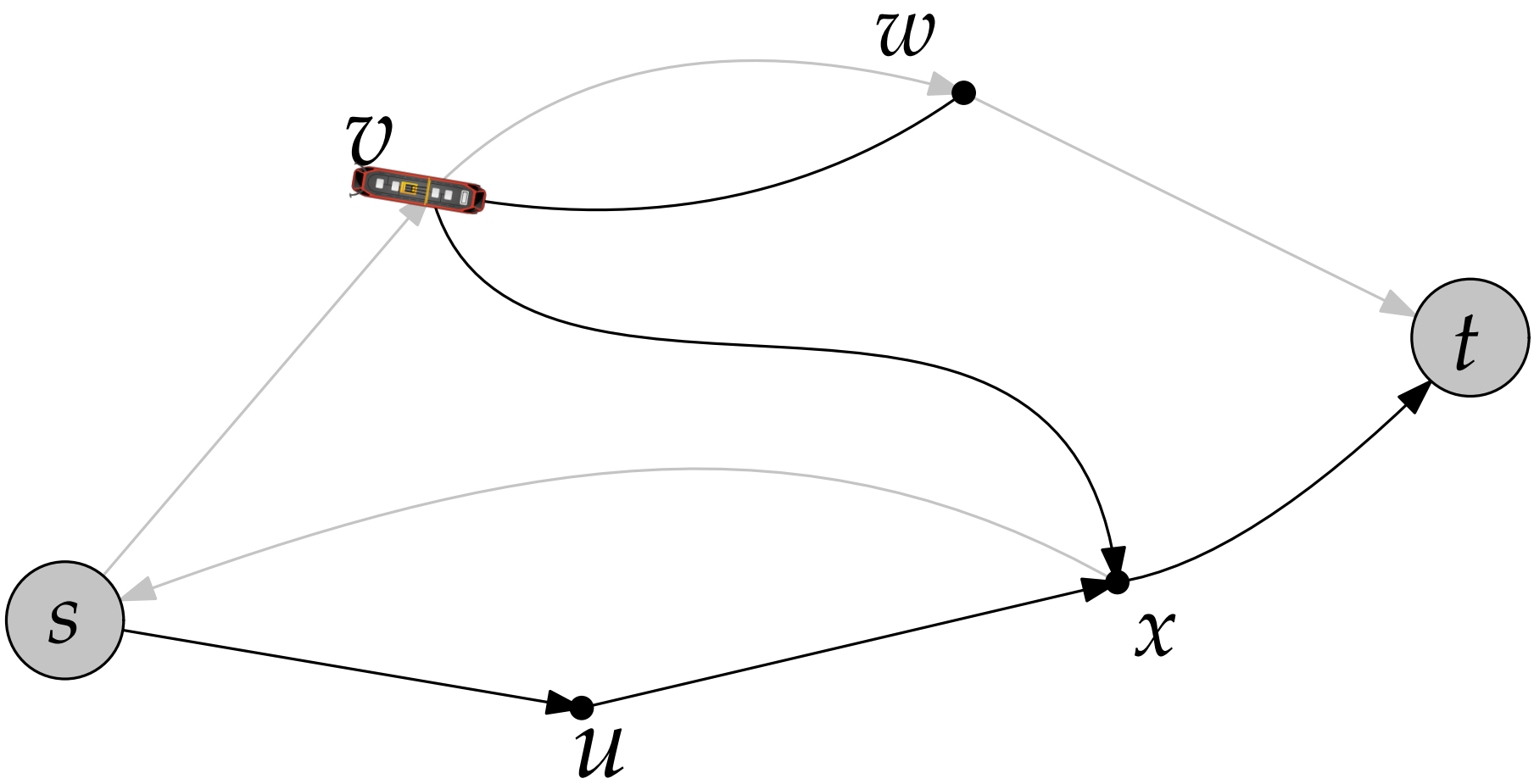

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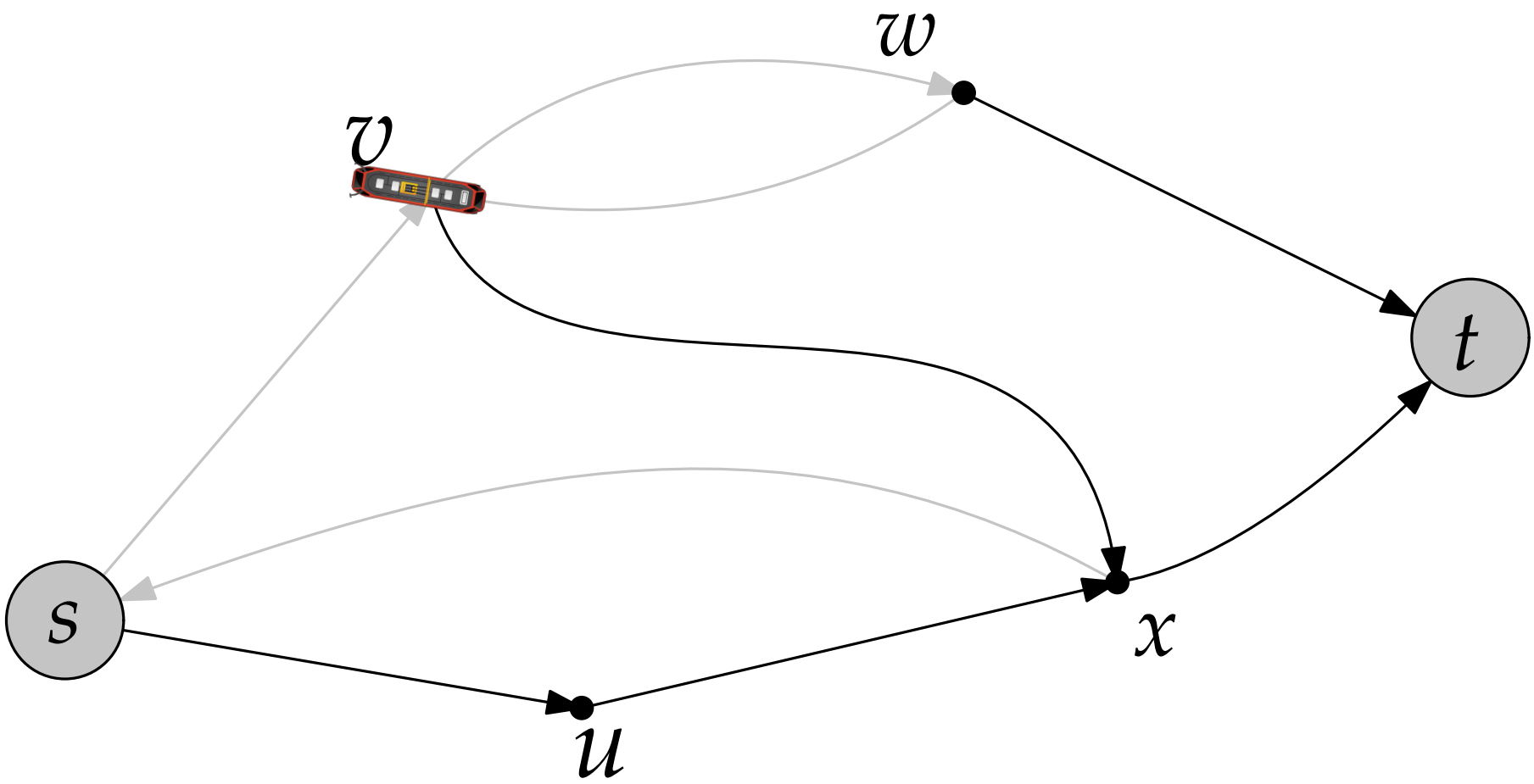

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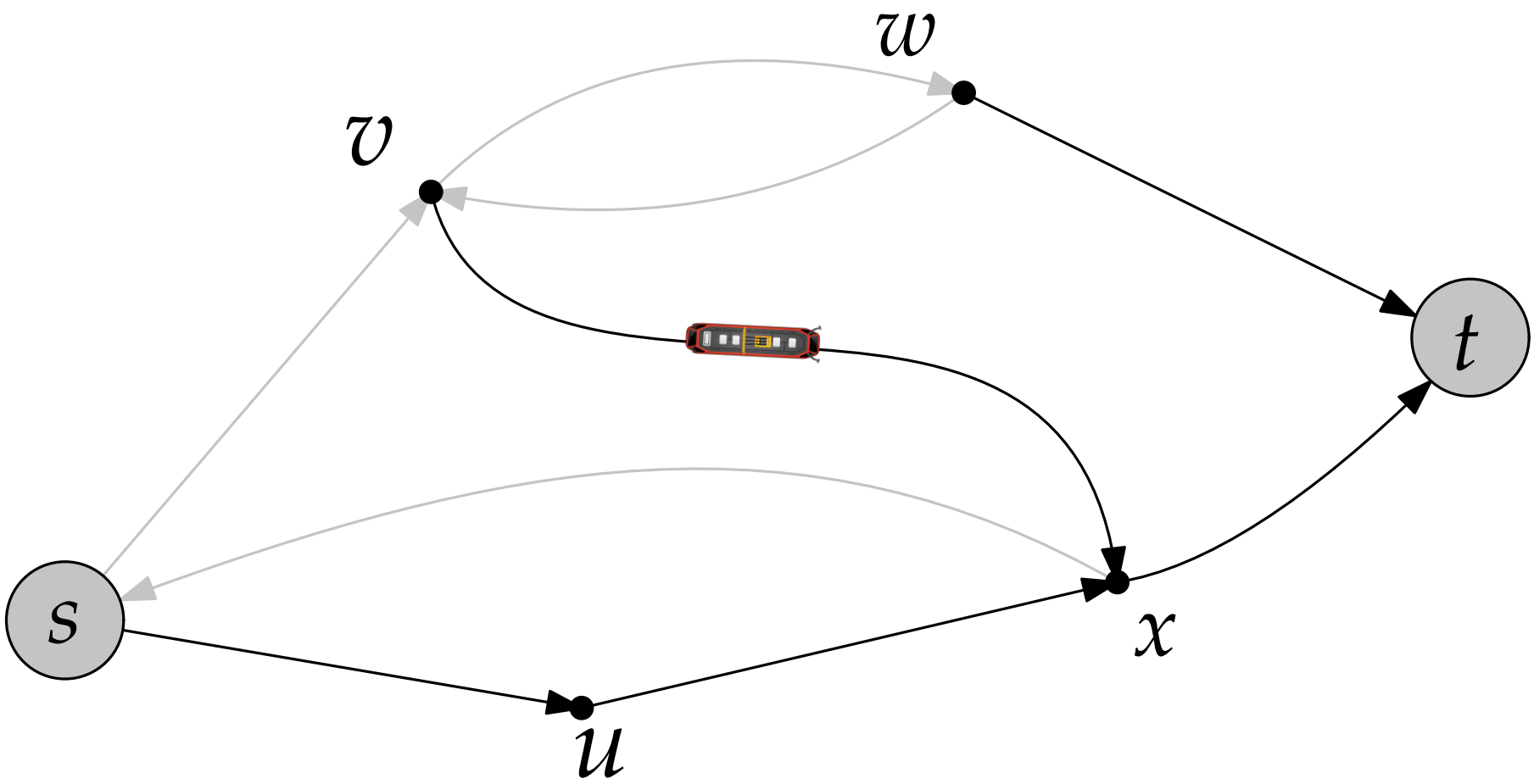

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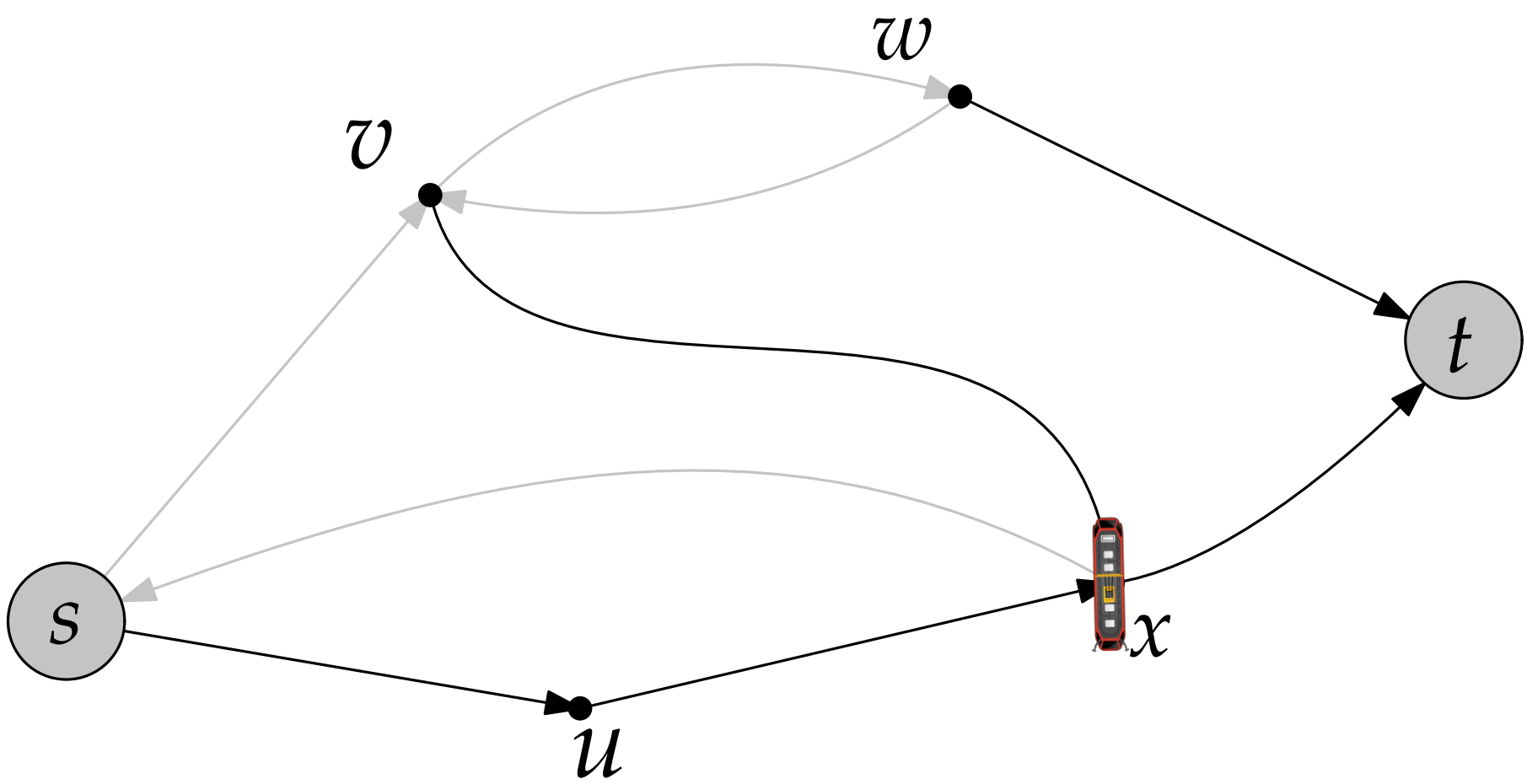

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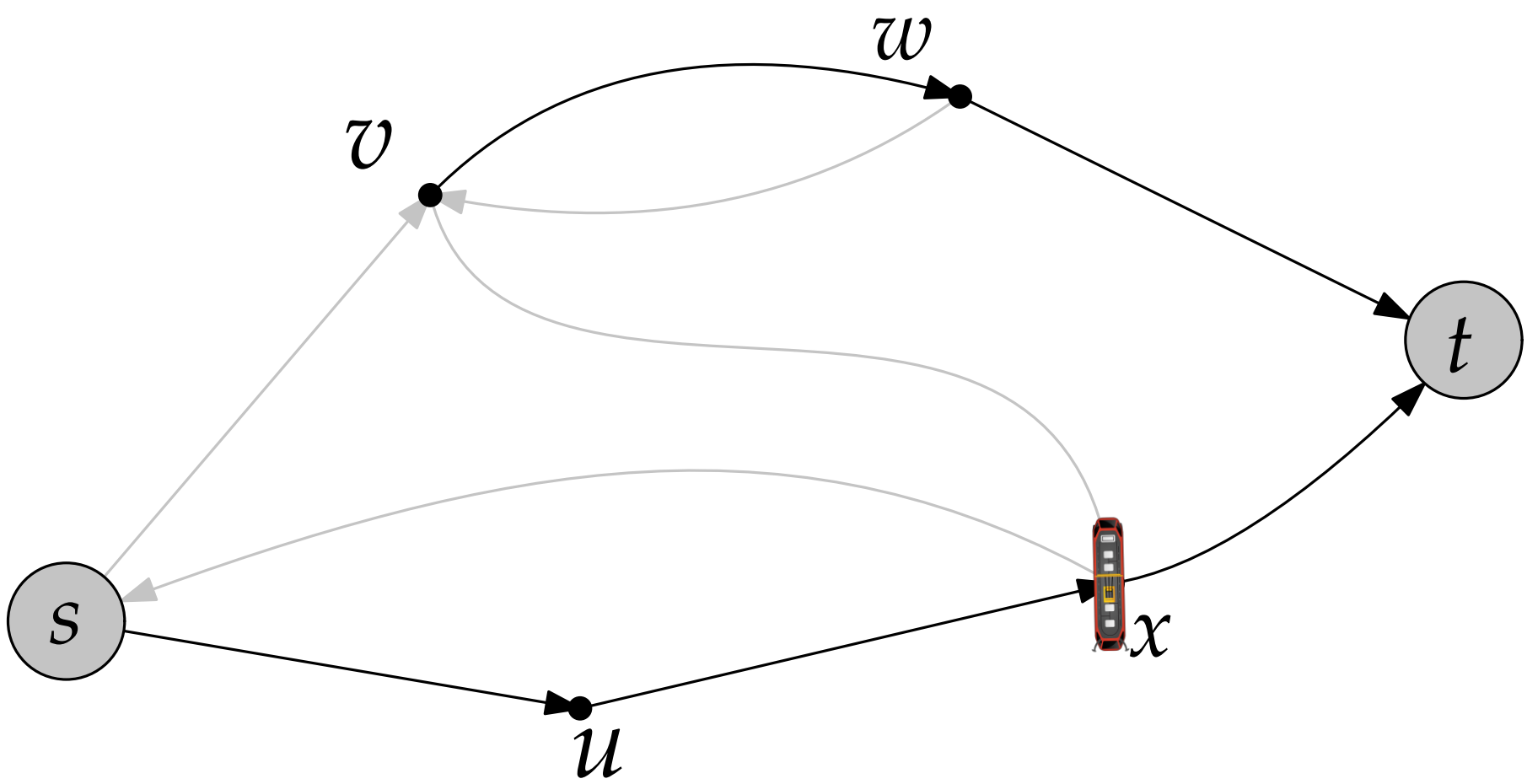

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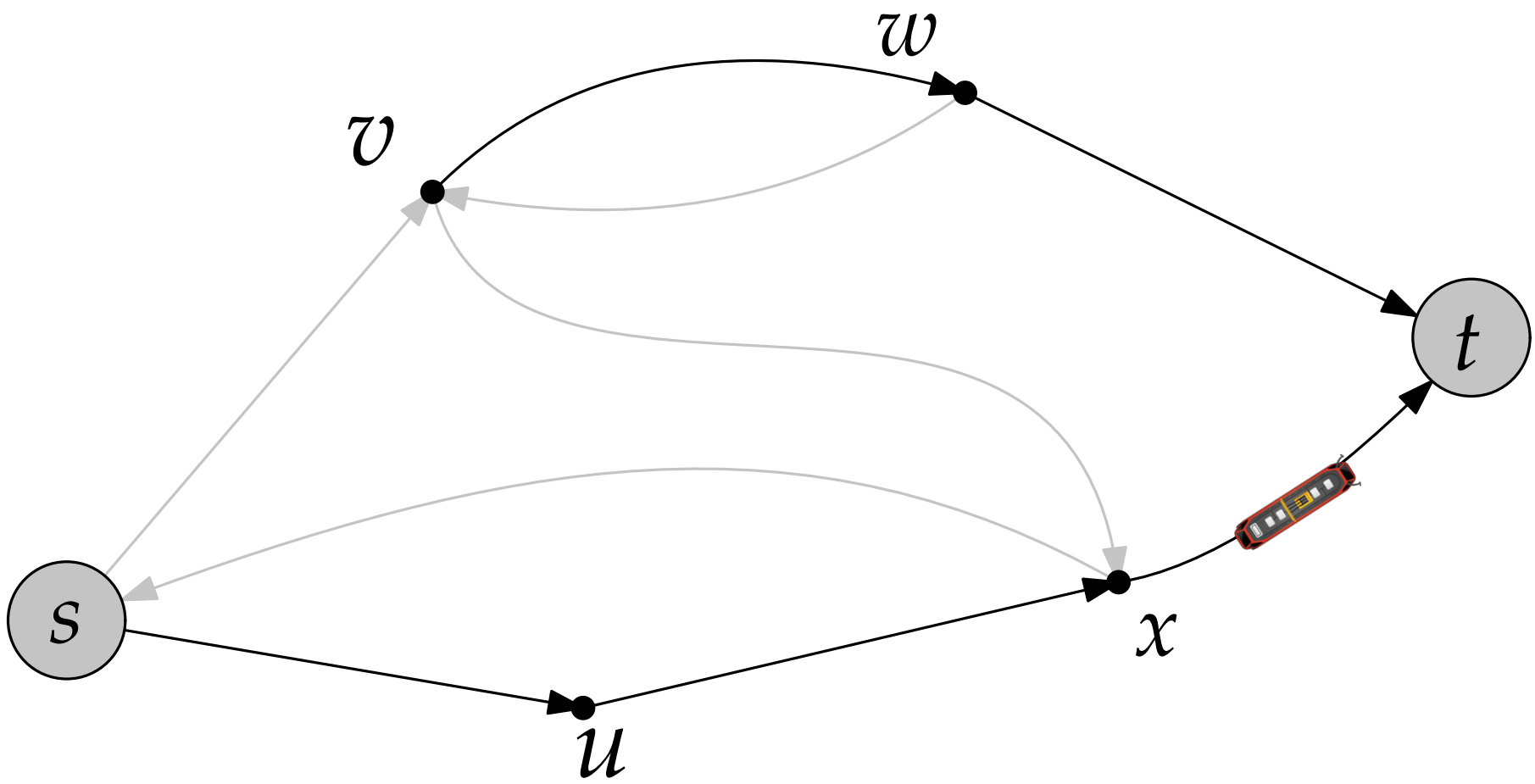
Arrival

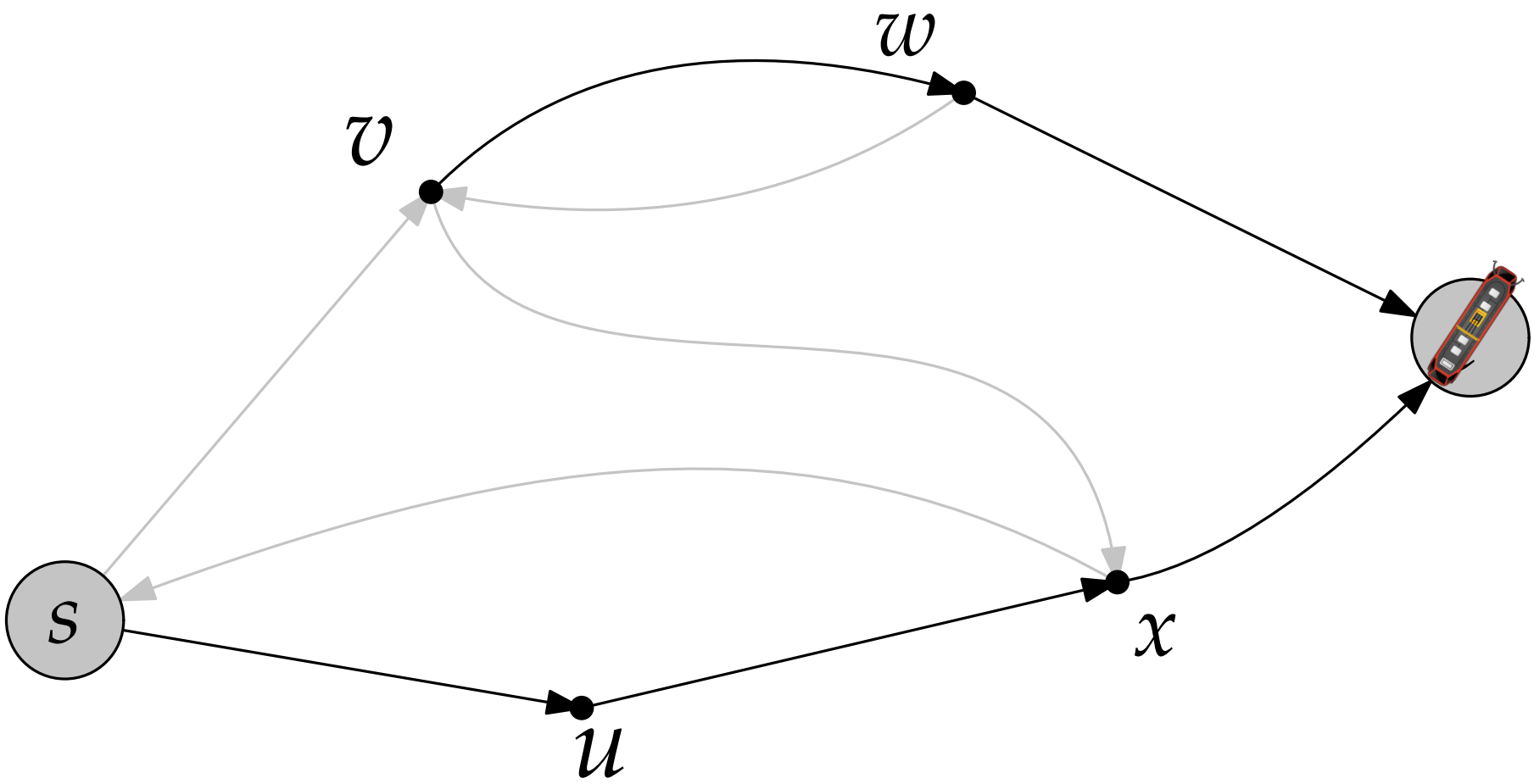
Dohrau, Gärtner, Kohler, Matoušek, Welzl (2017)



Will the train ever reach the station?

Arrival

Dohrau, Gärtner, Kohler, Matoušek, Welzl (2017)



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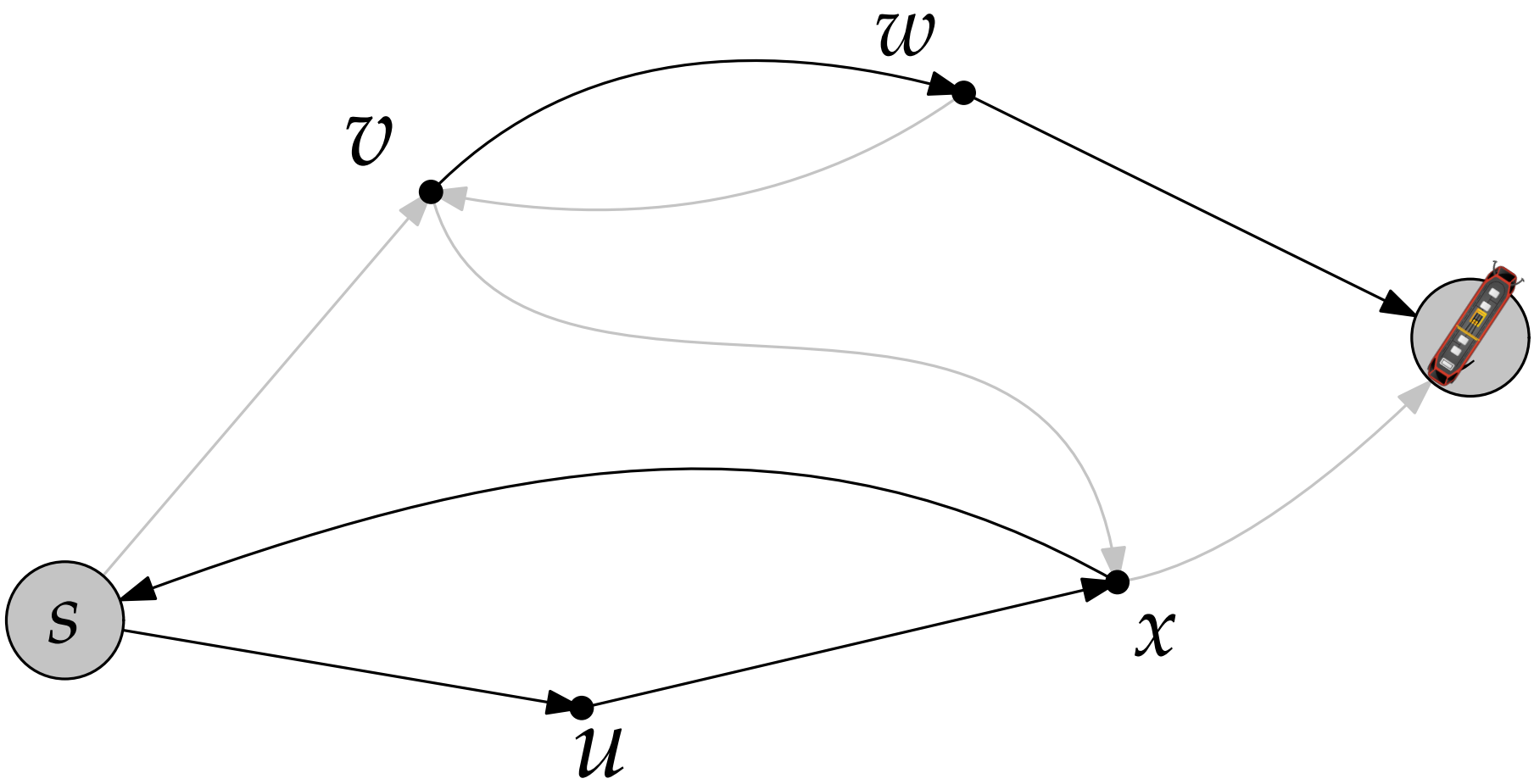
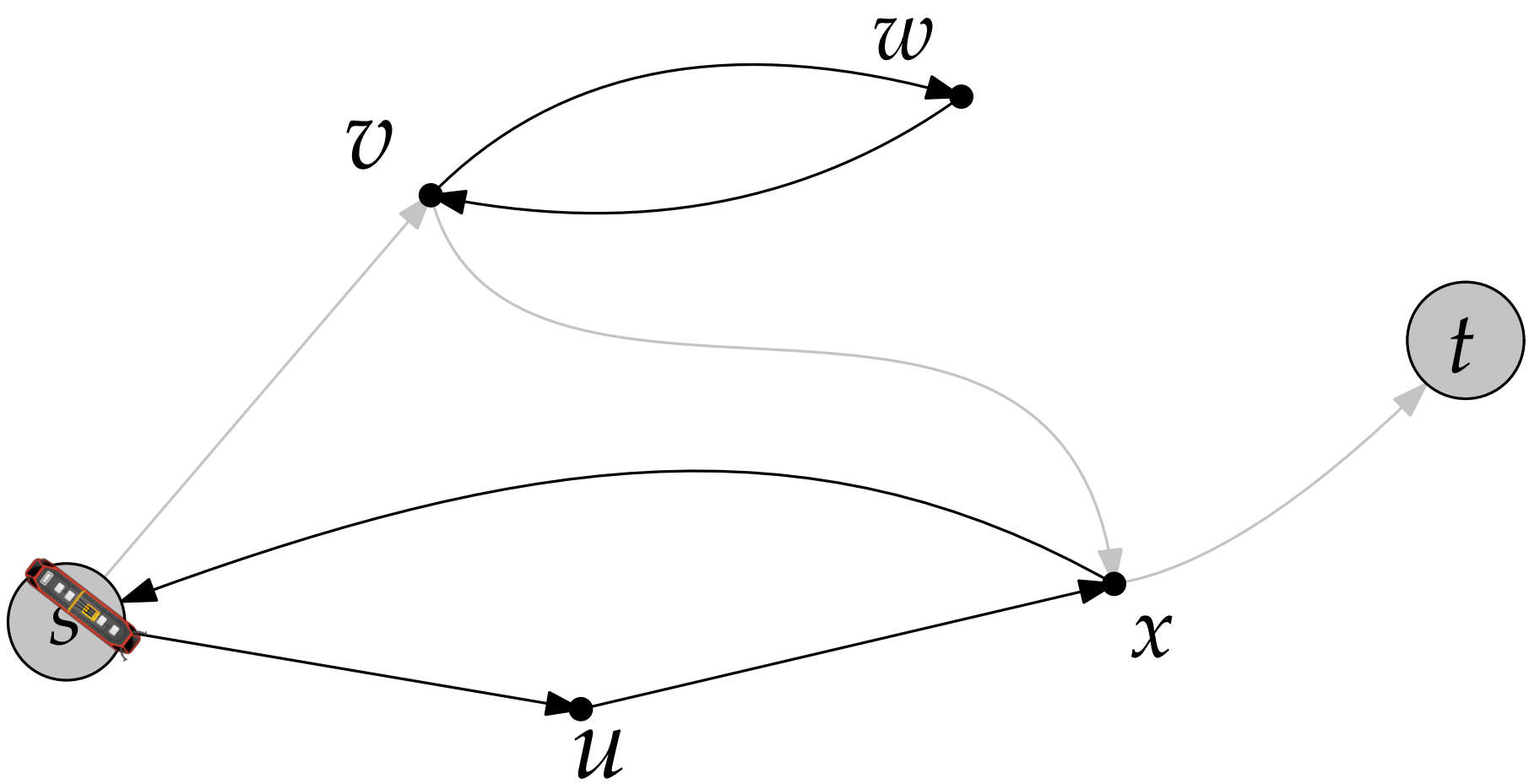


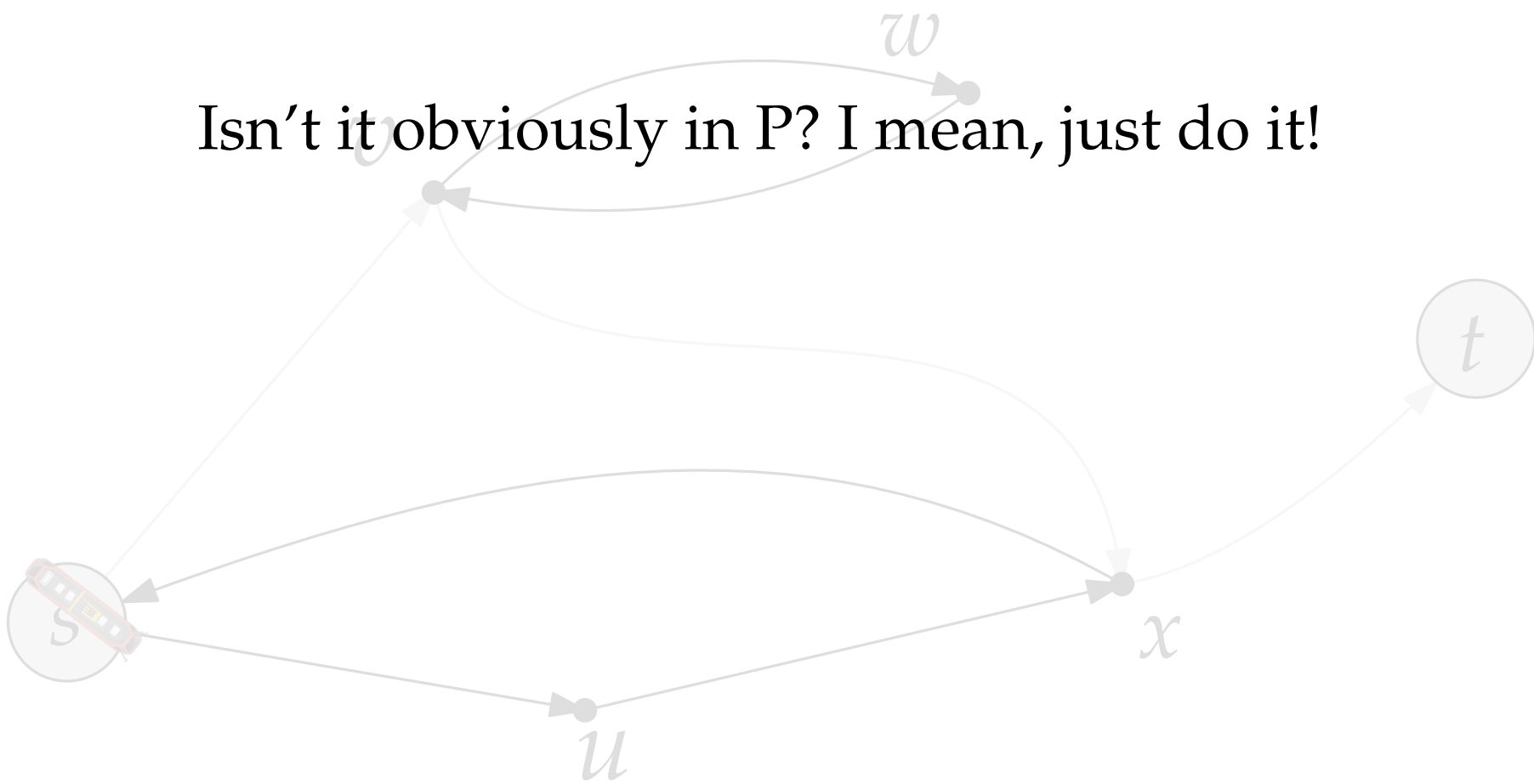
Will the train ever reach the station?

Arrival



Arrival

Isn't it obviously in P? I mean, just do it!



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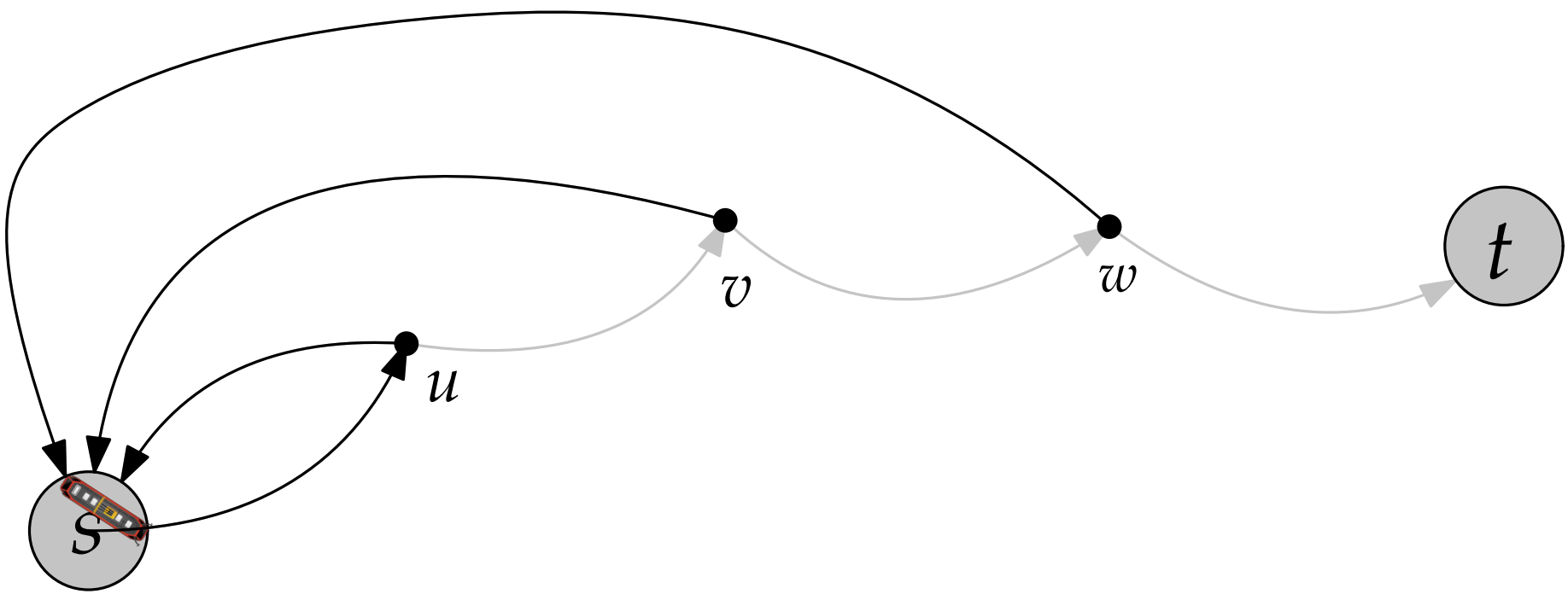
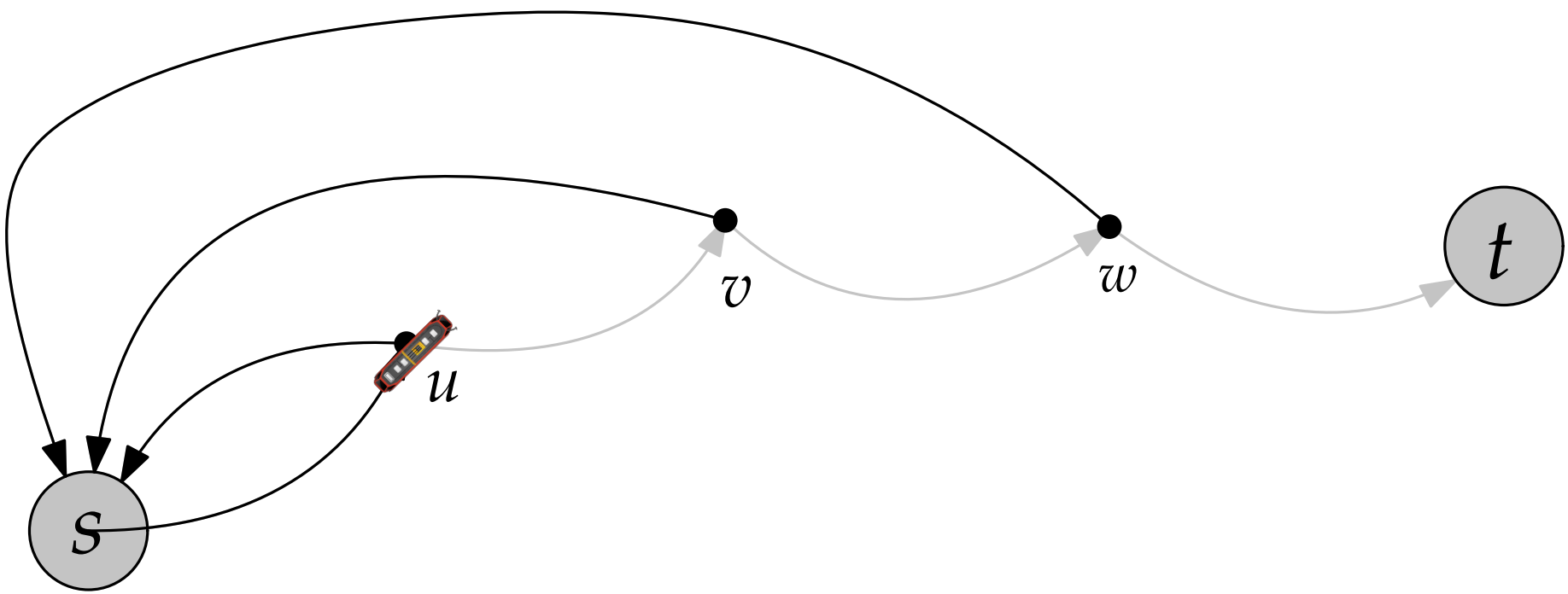


Arrival



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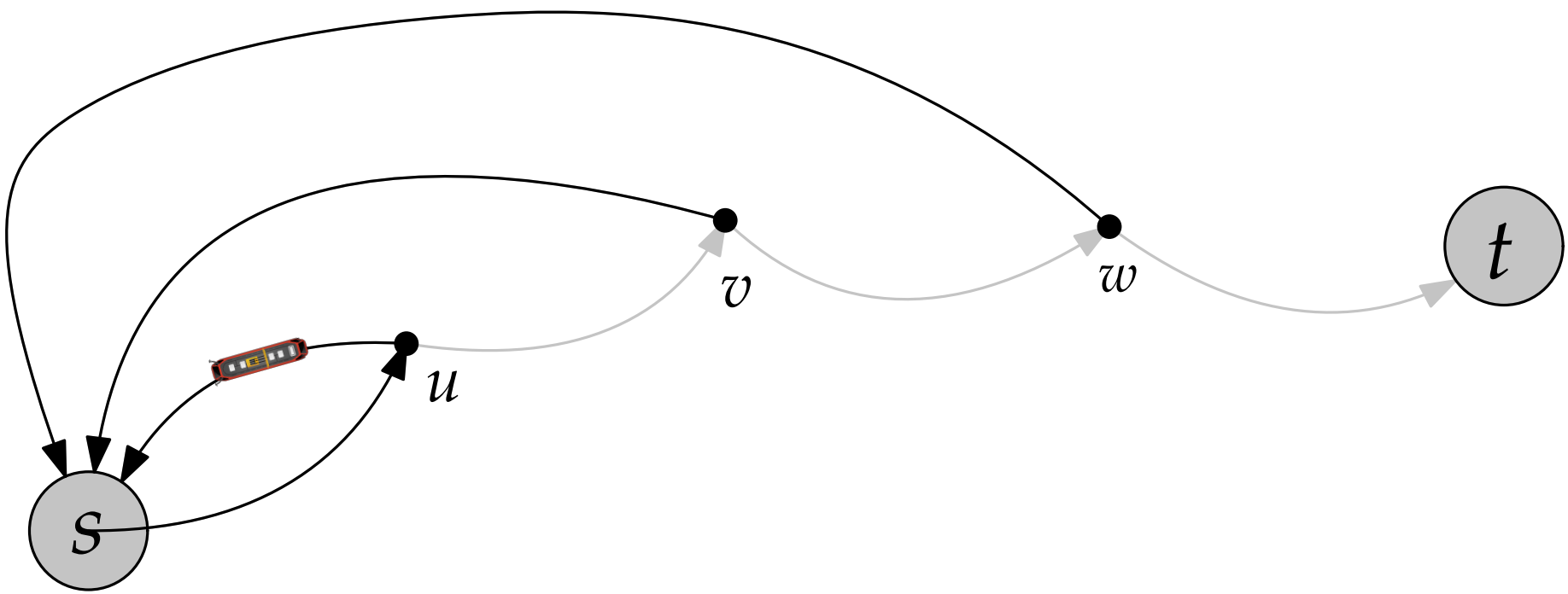

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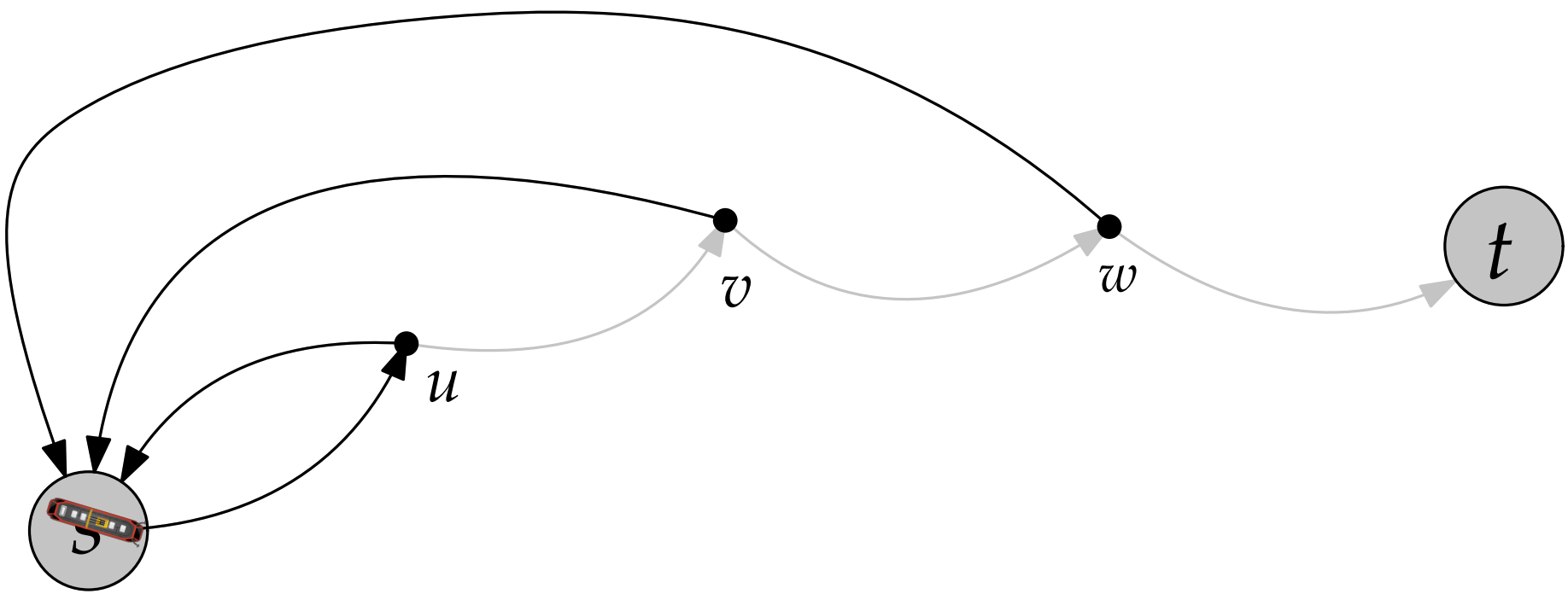
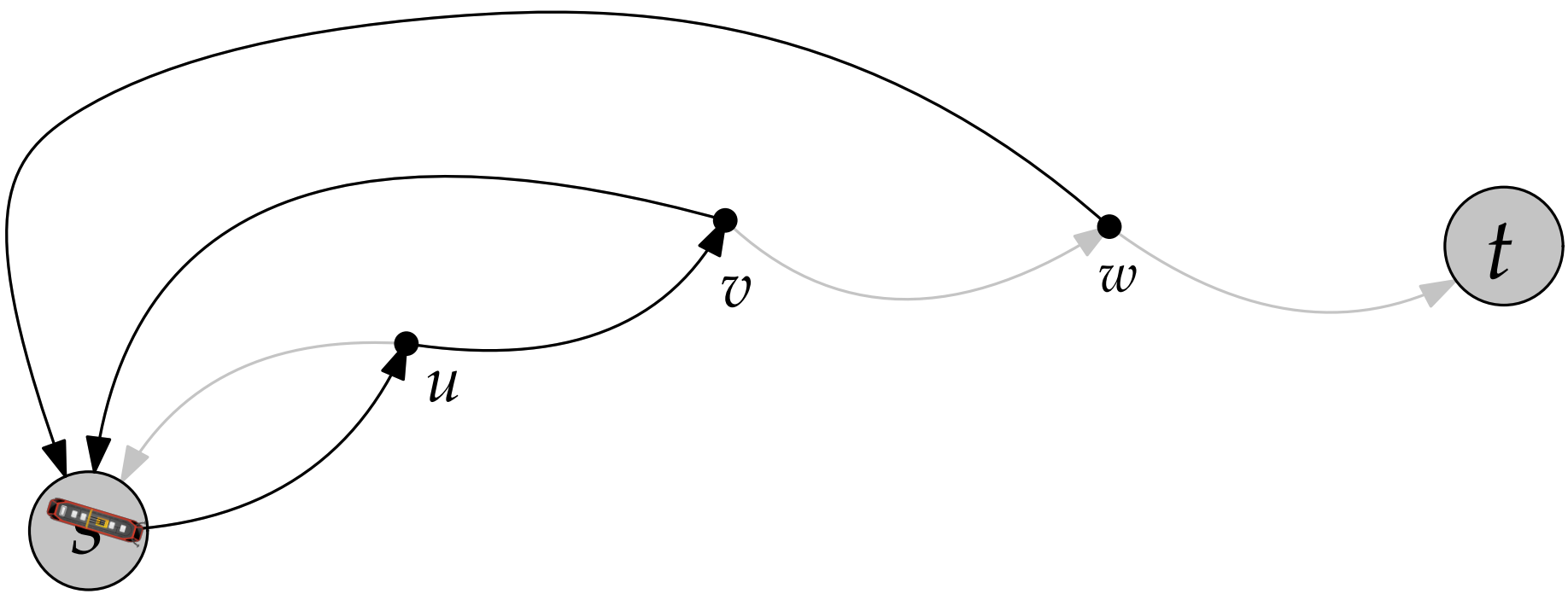


Arrival



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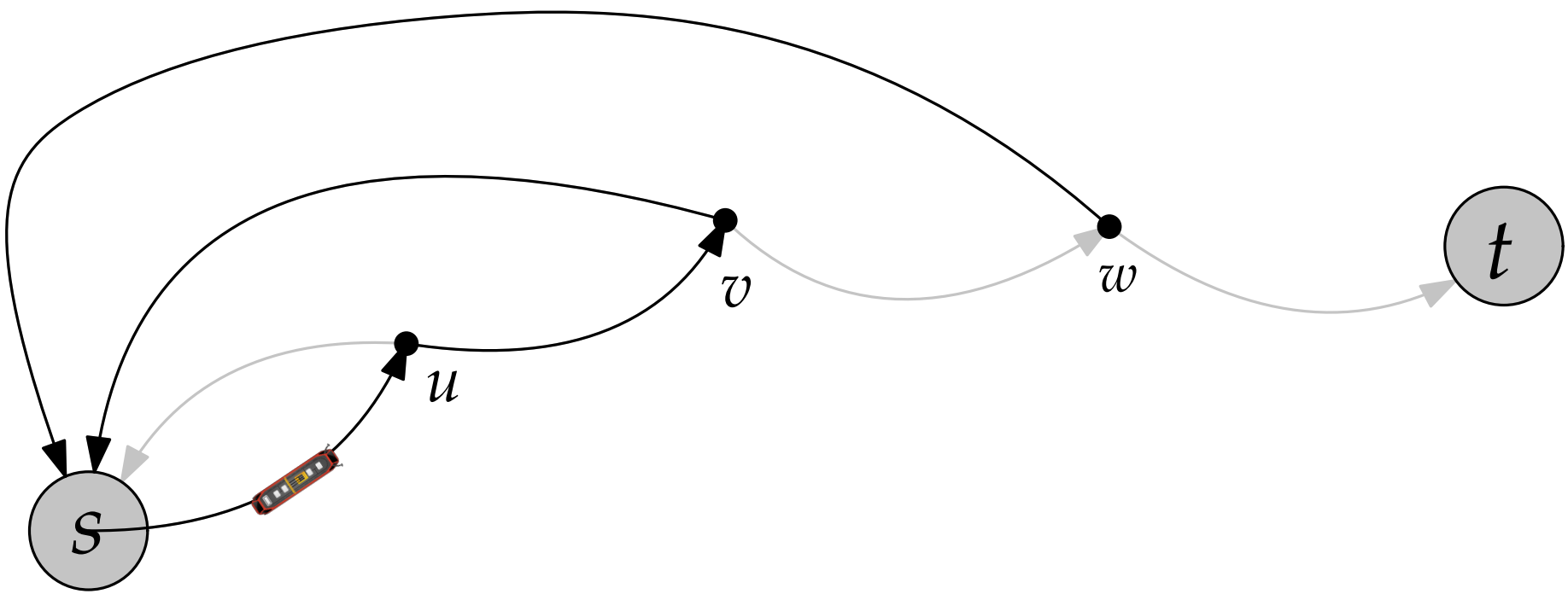

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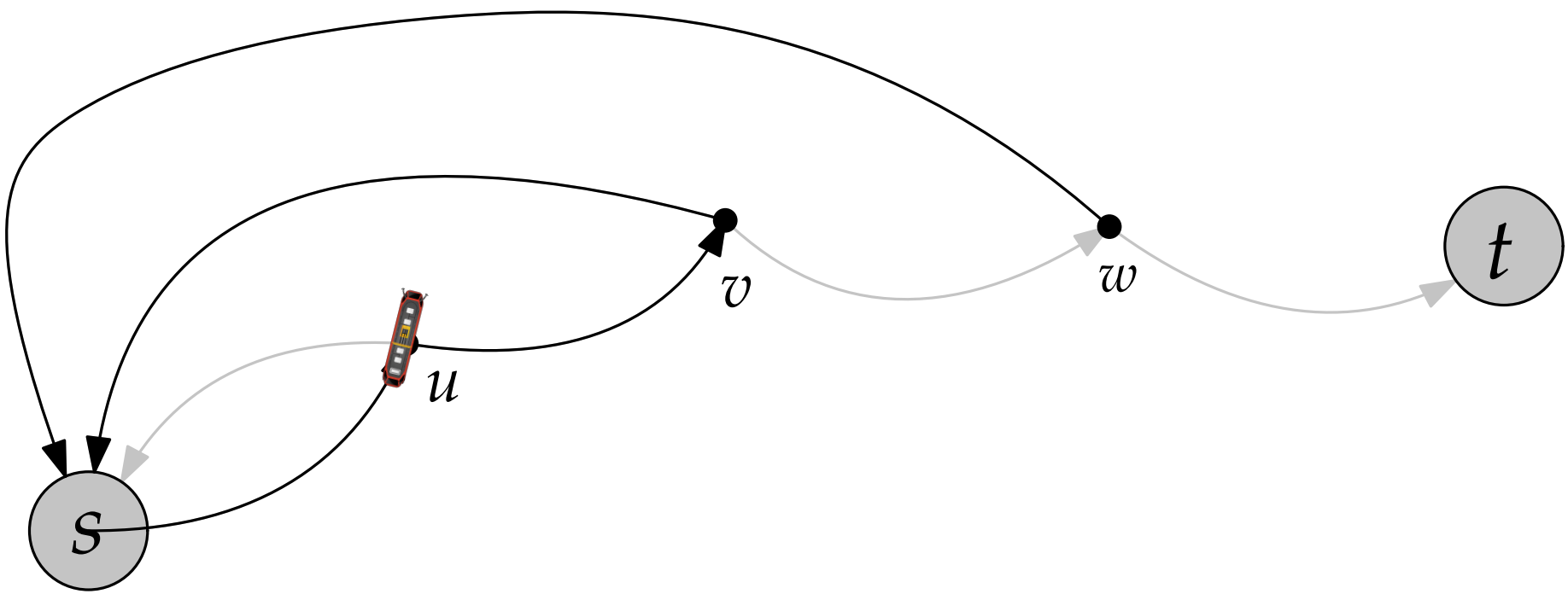

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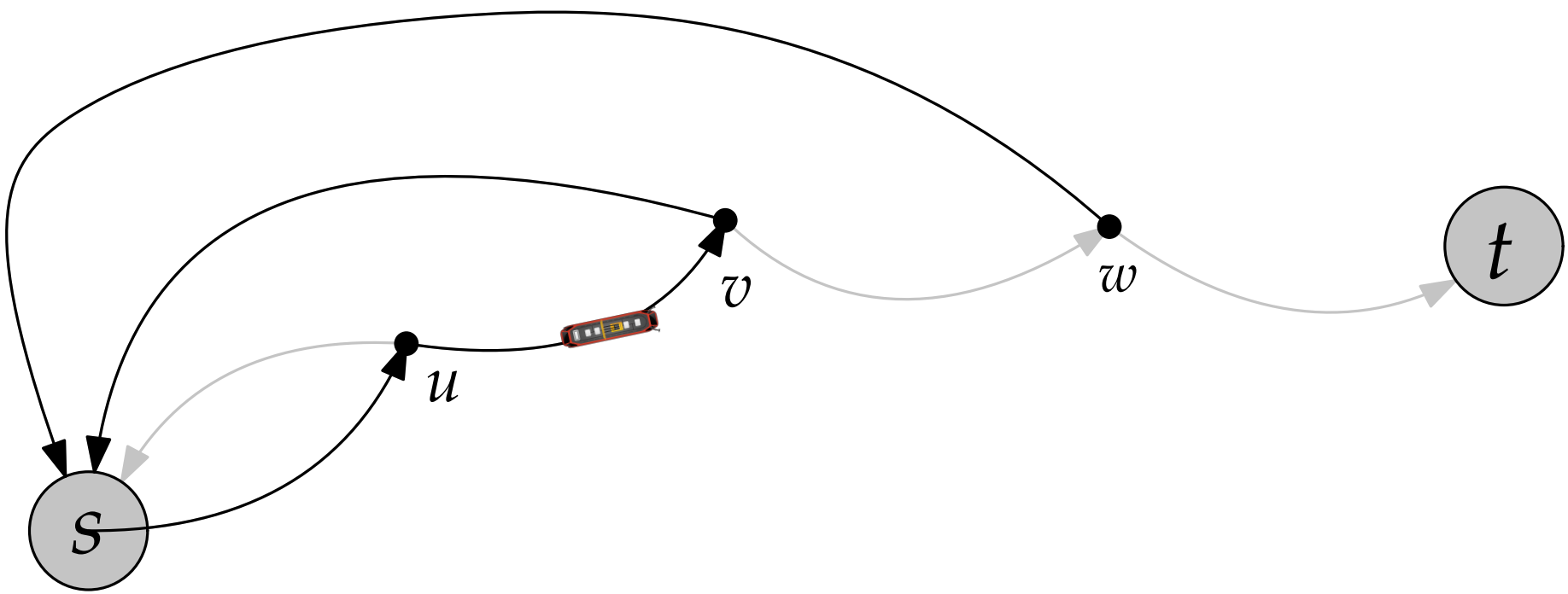

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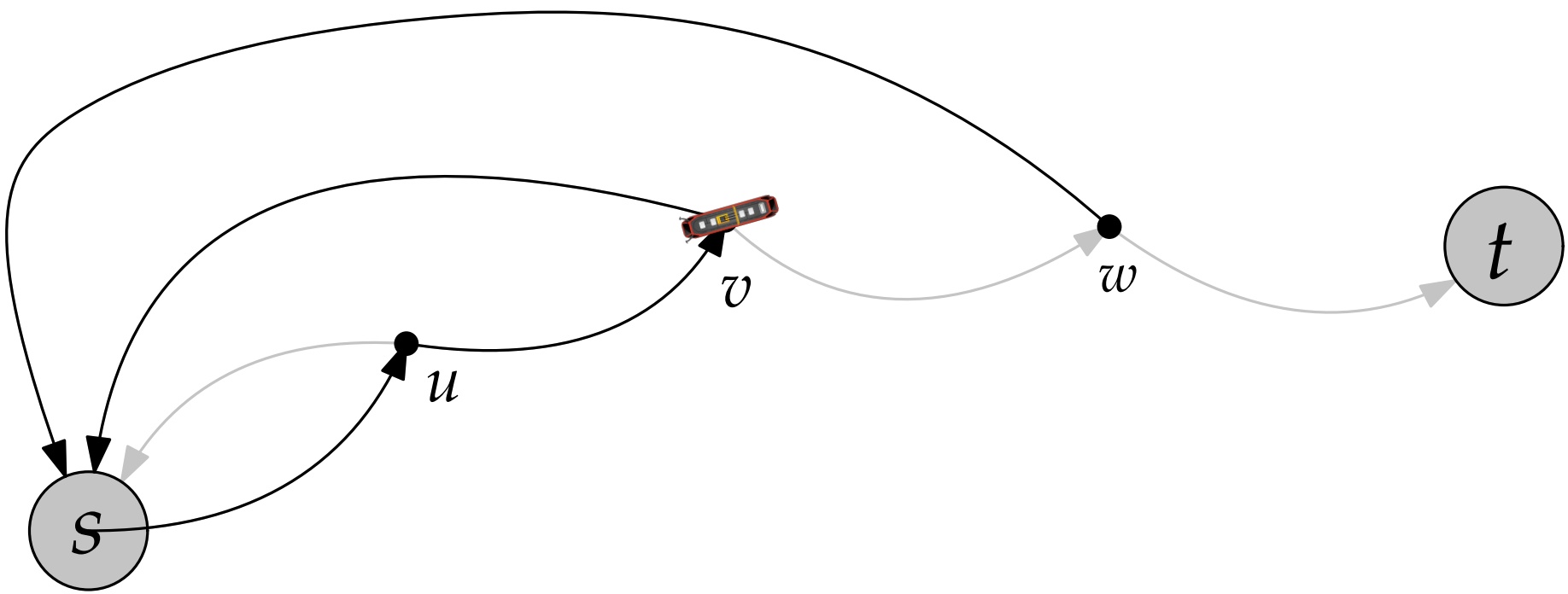

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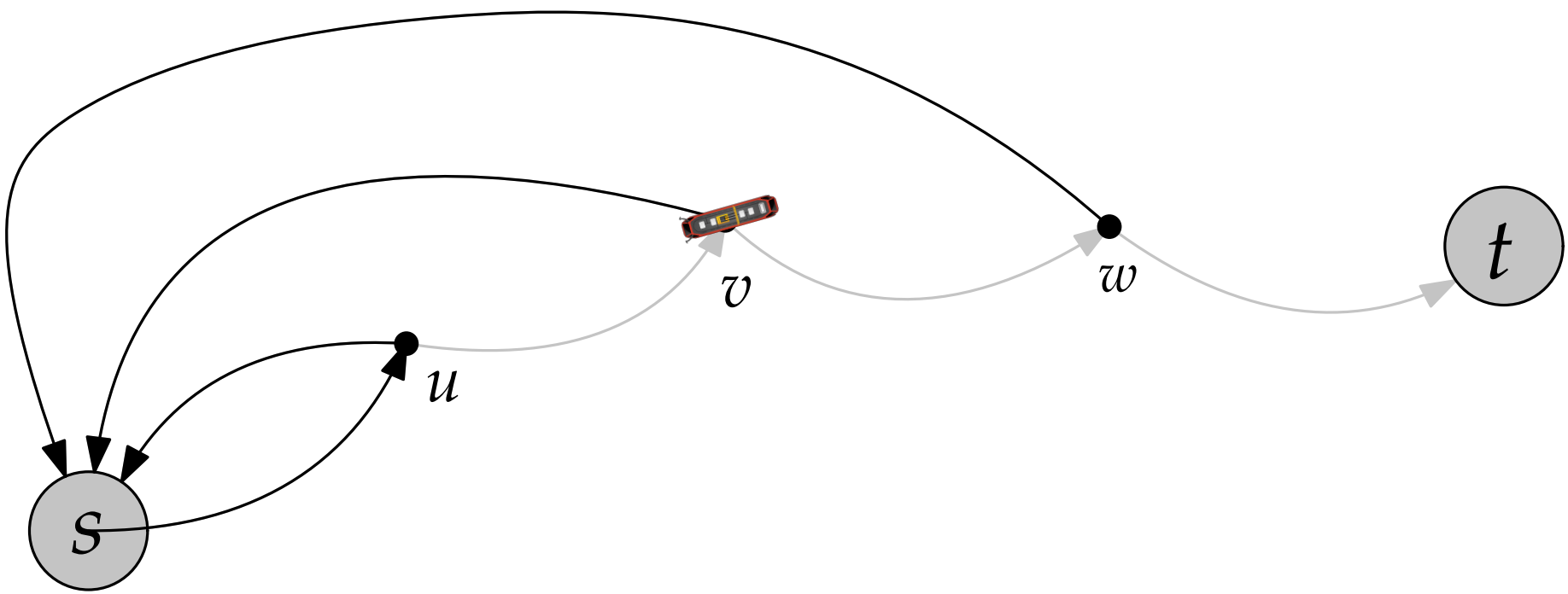

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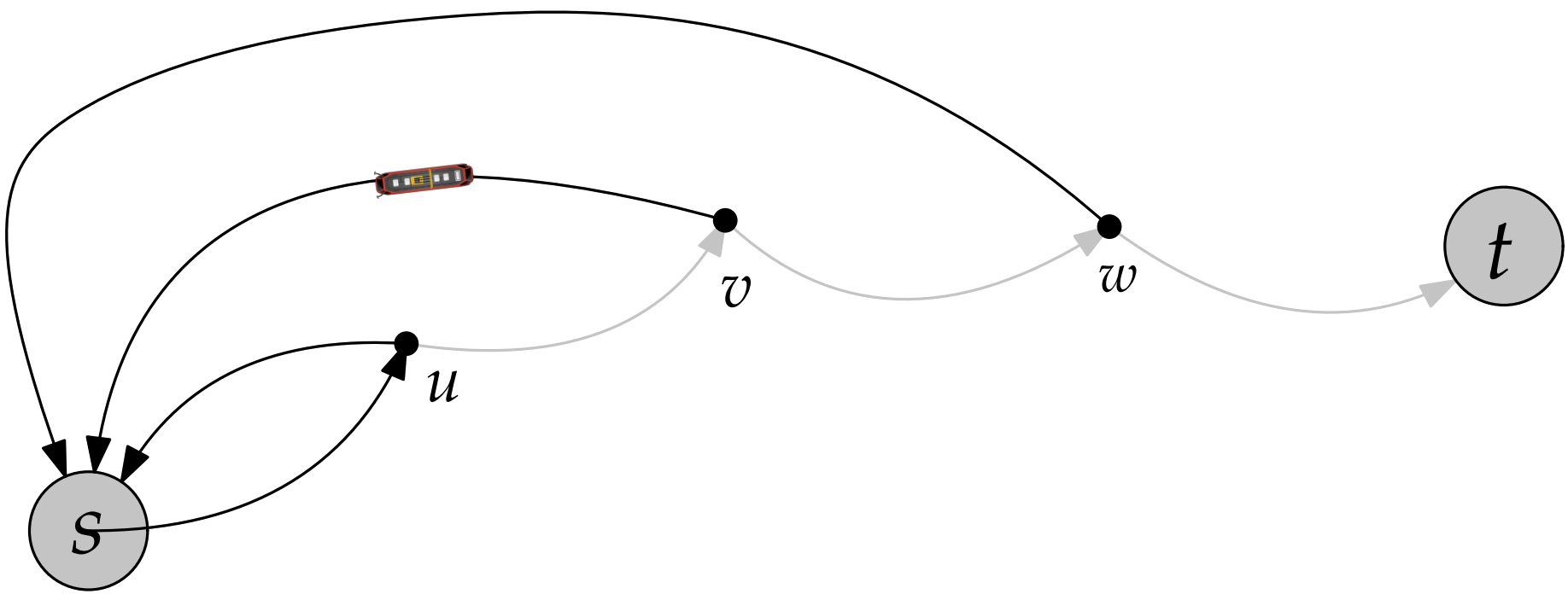

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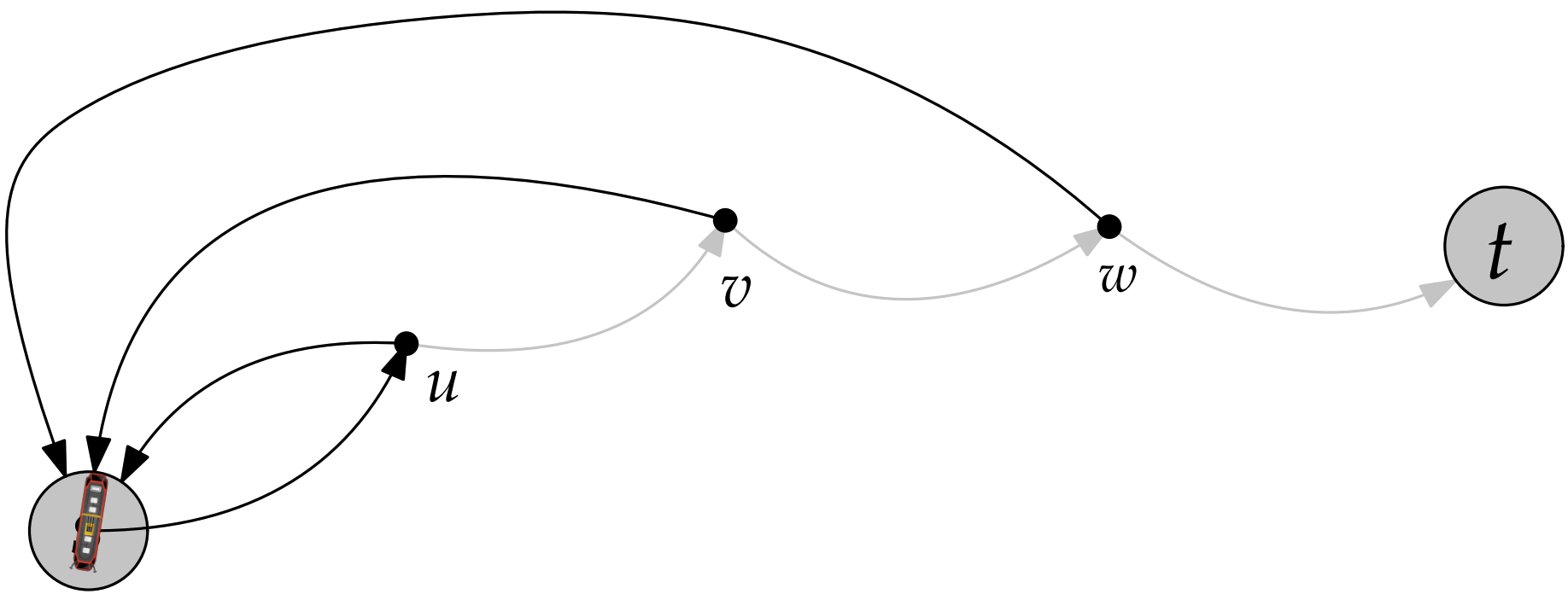

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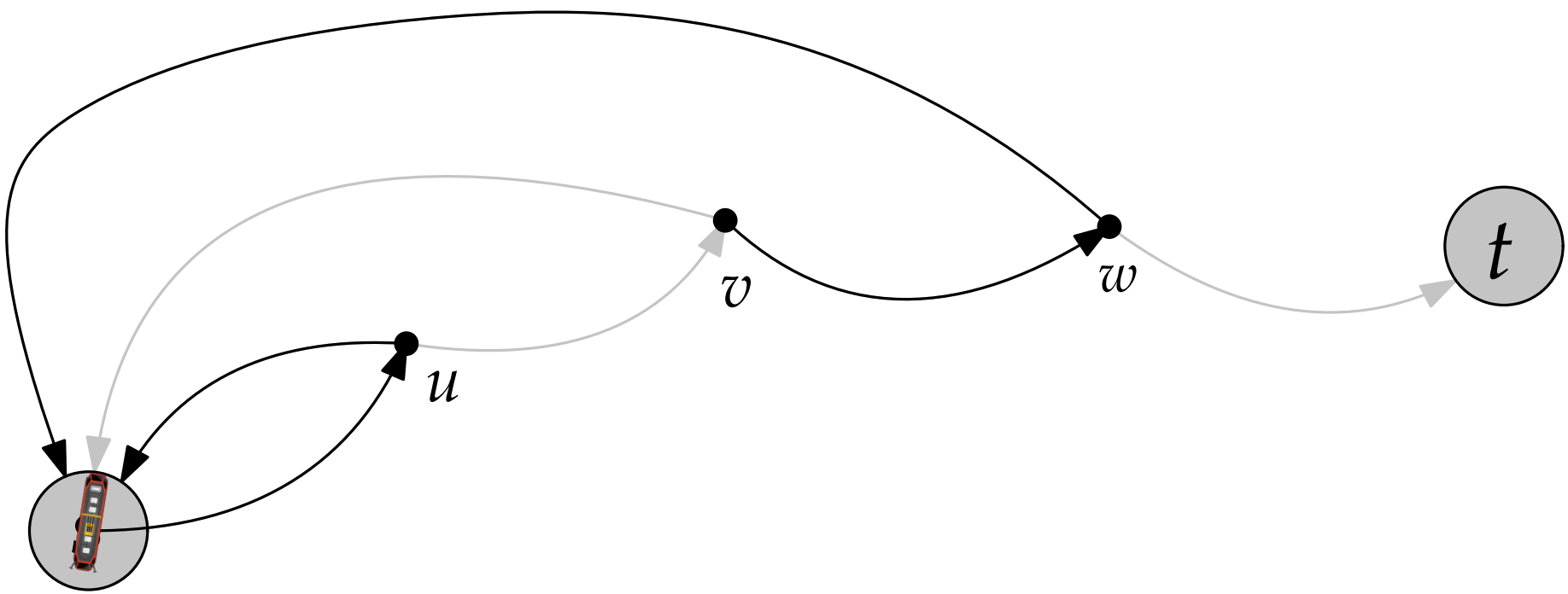

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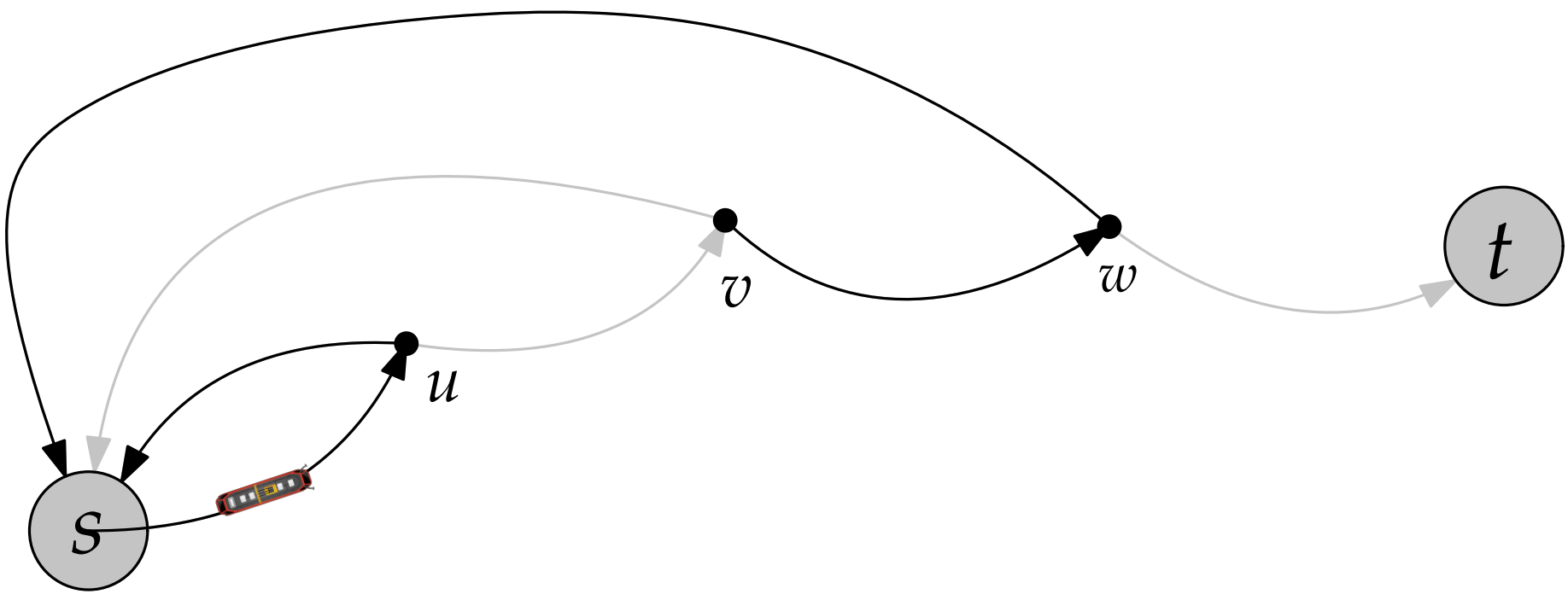

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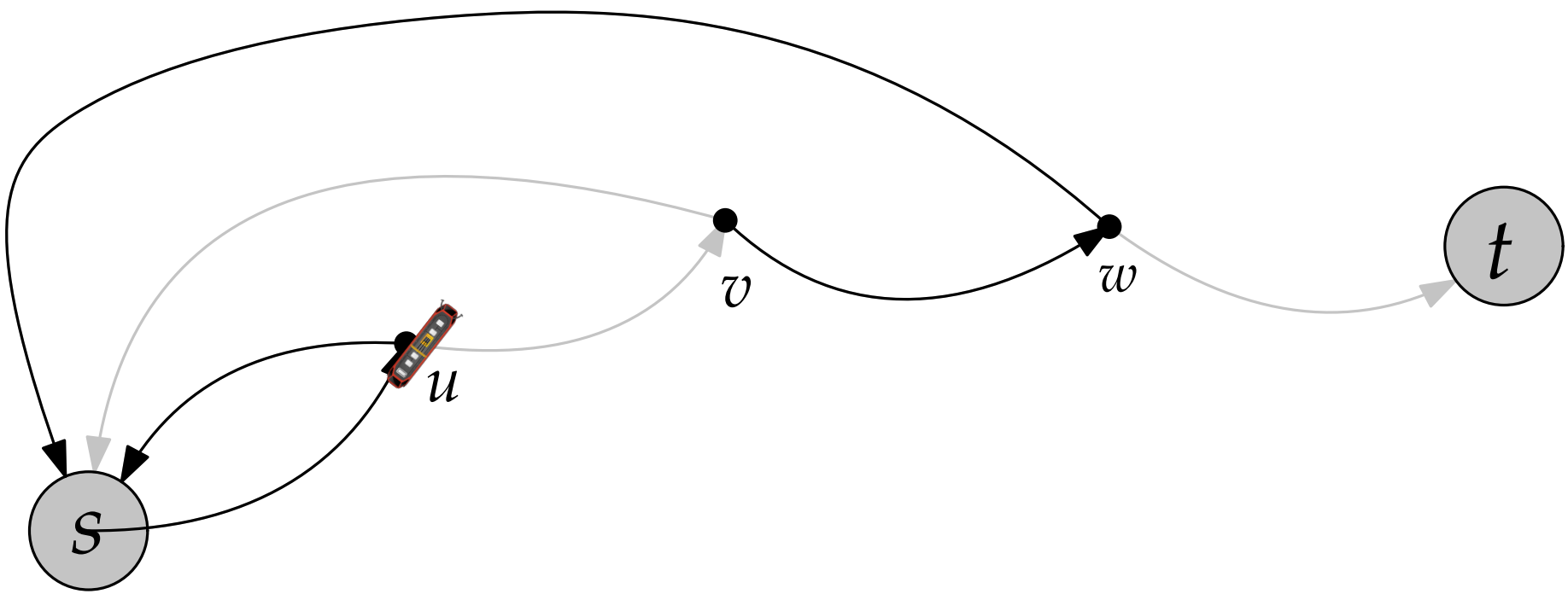

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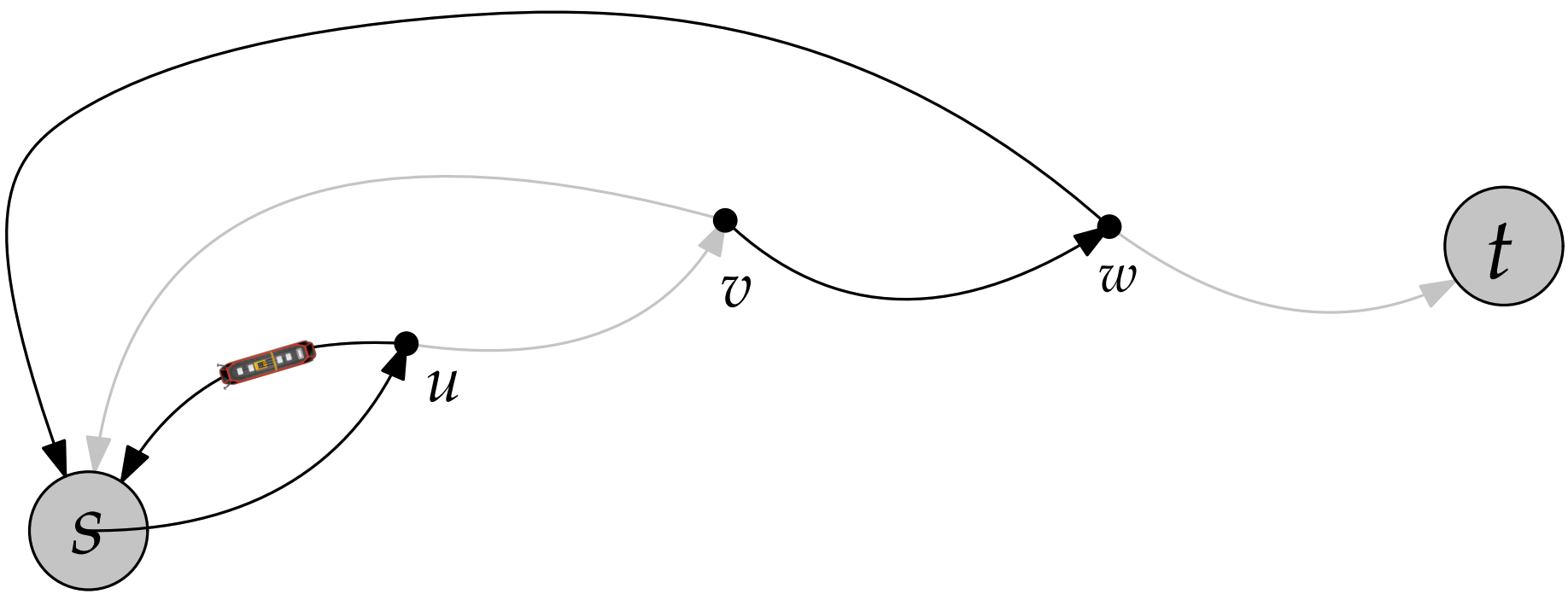

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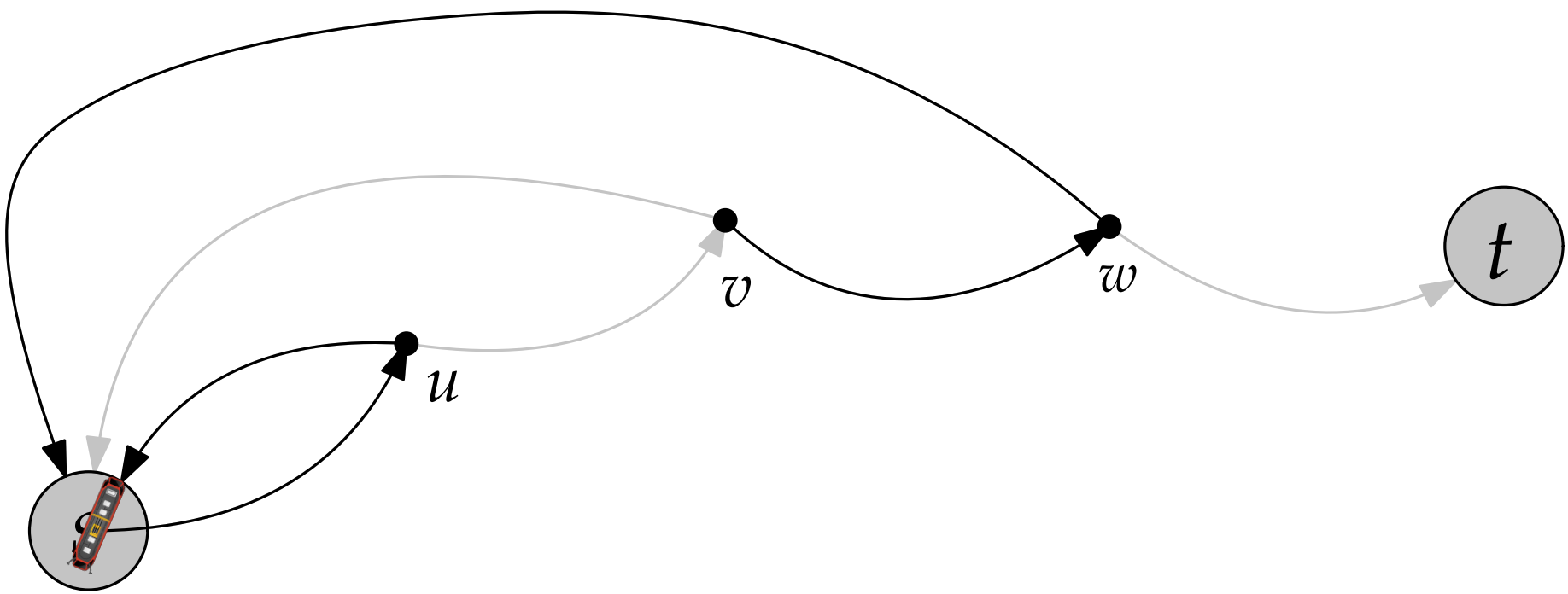

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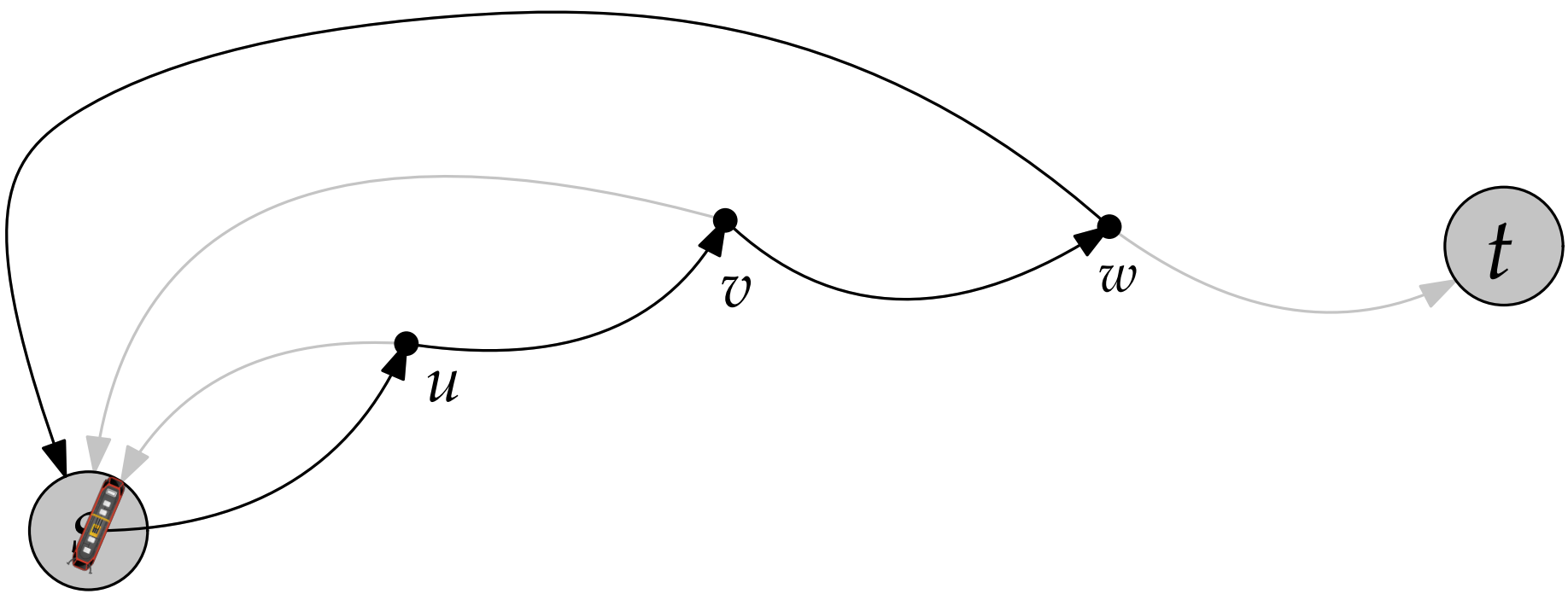

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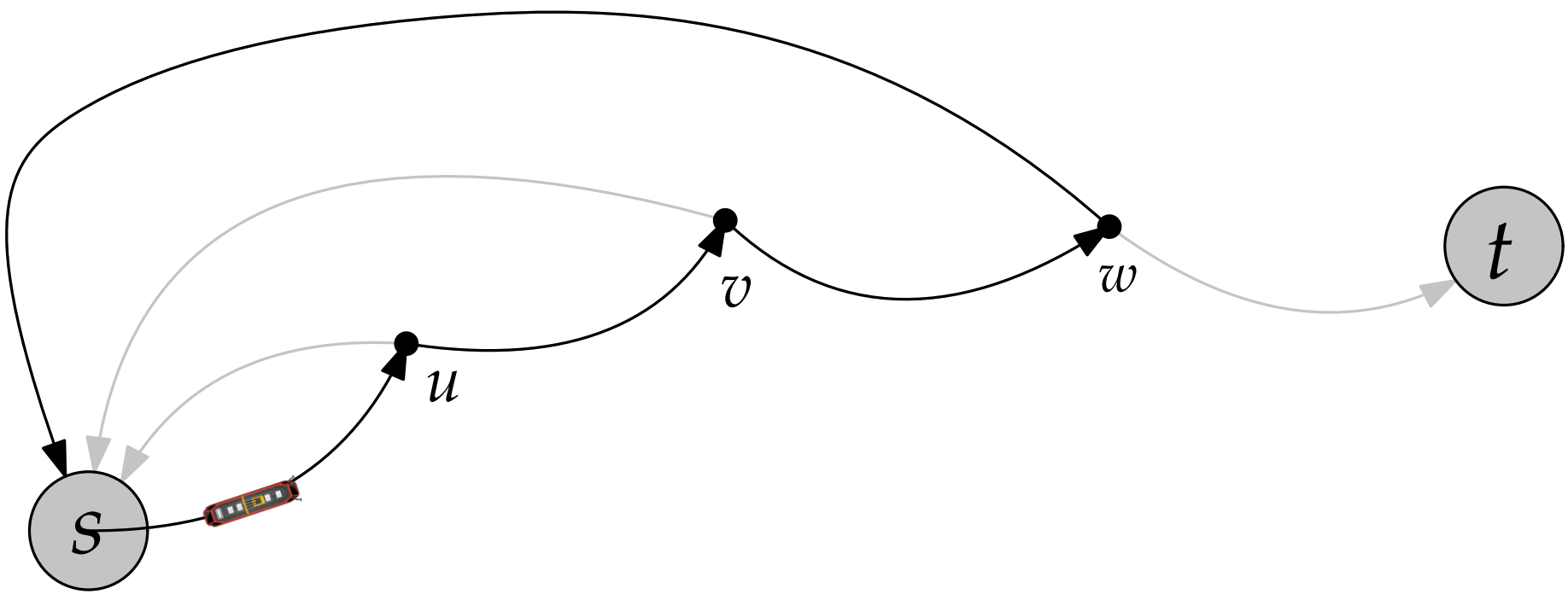

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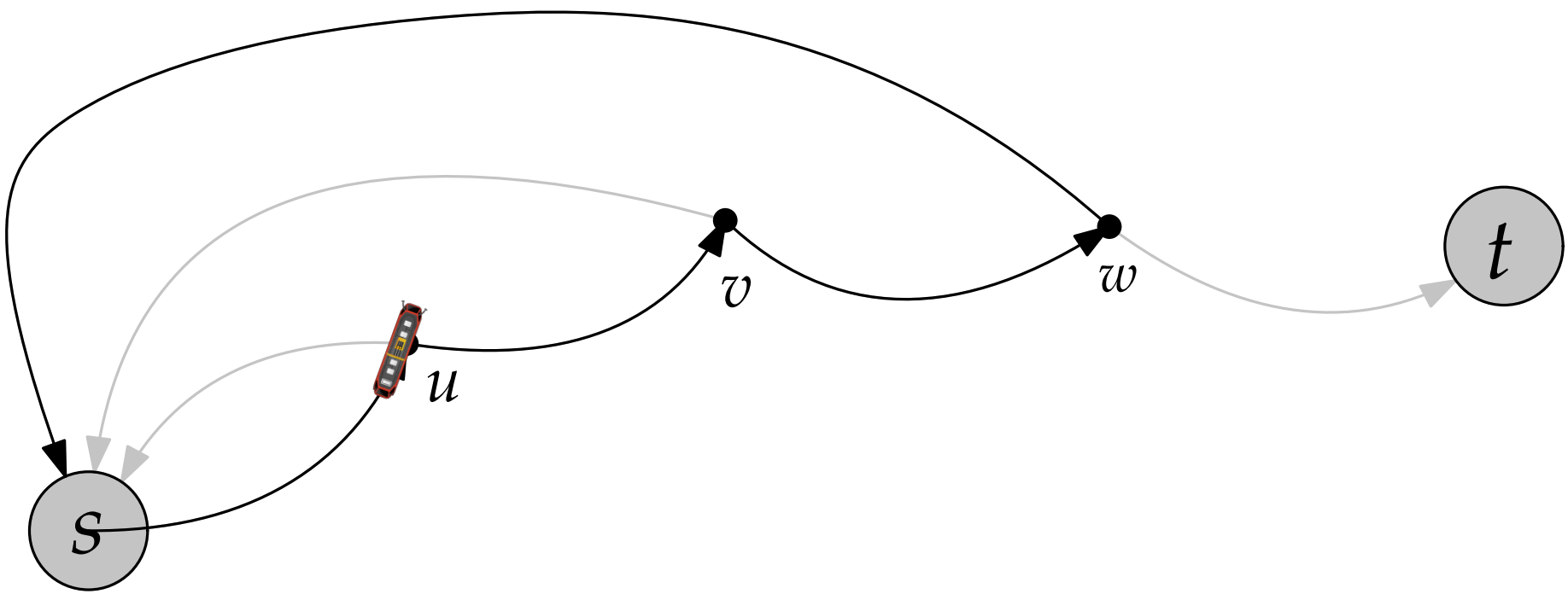

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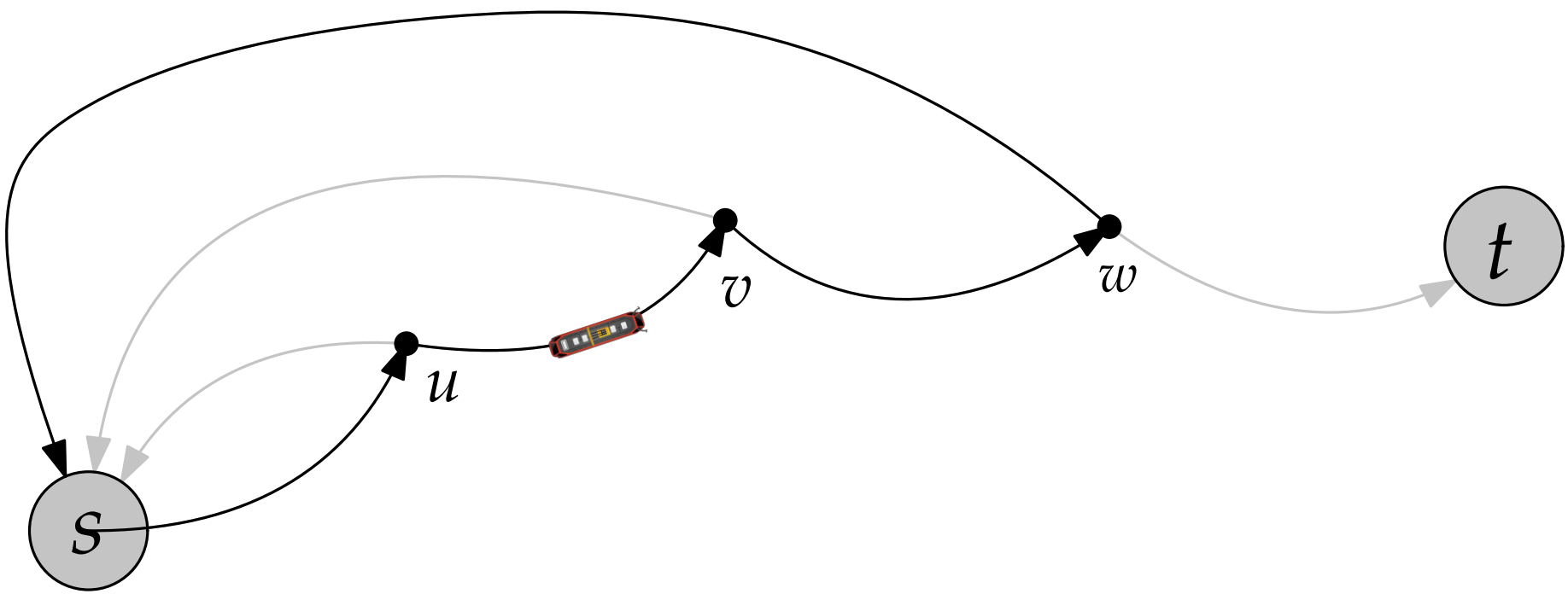

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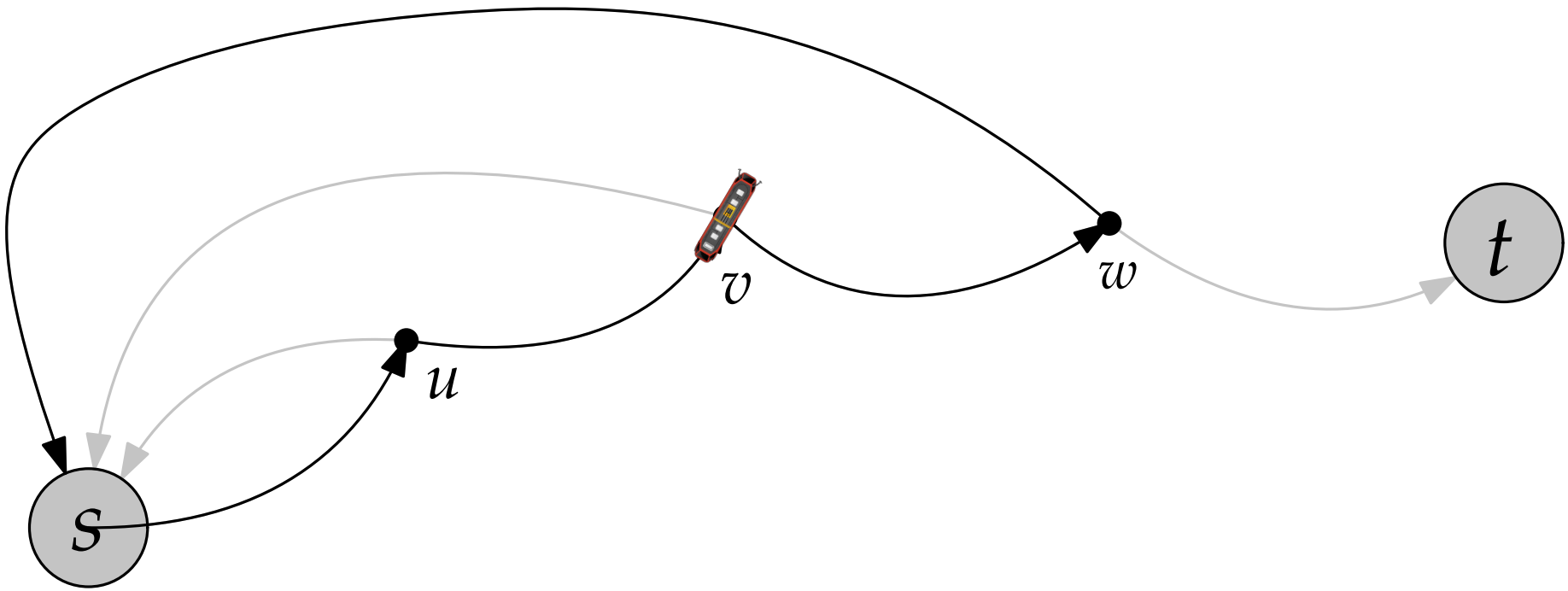

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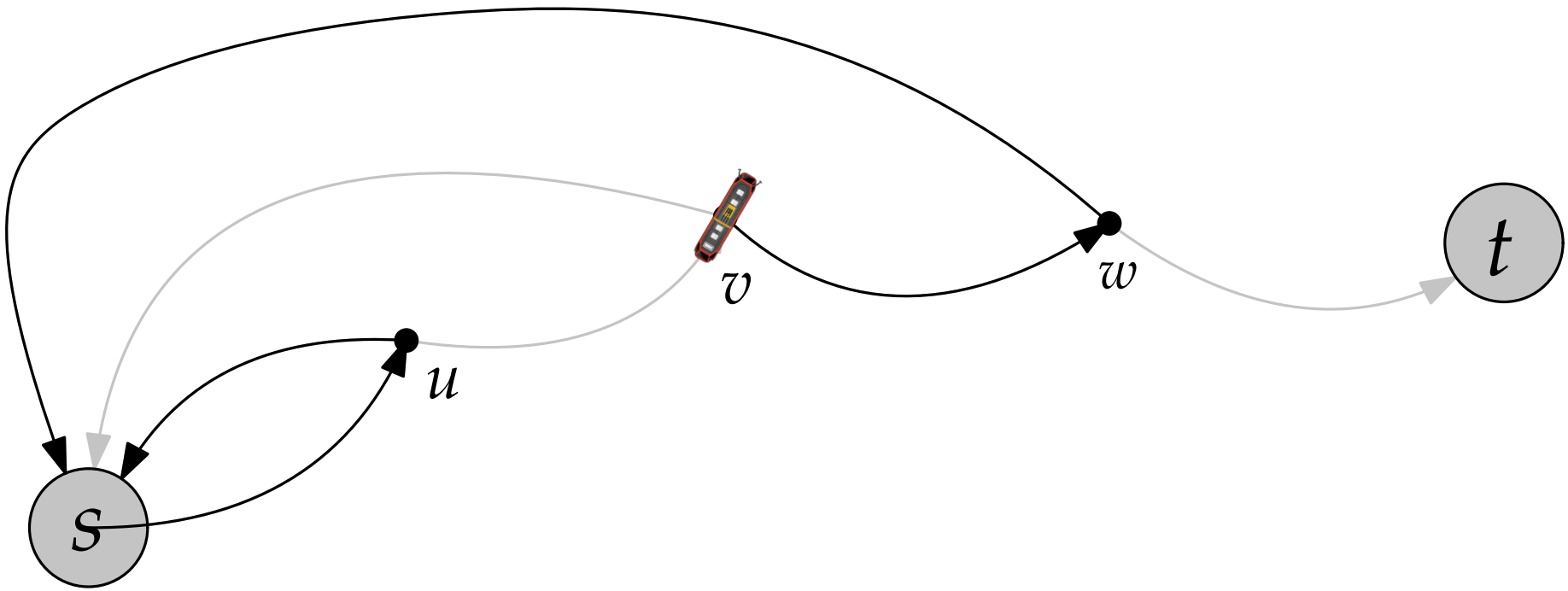

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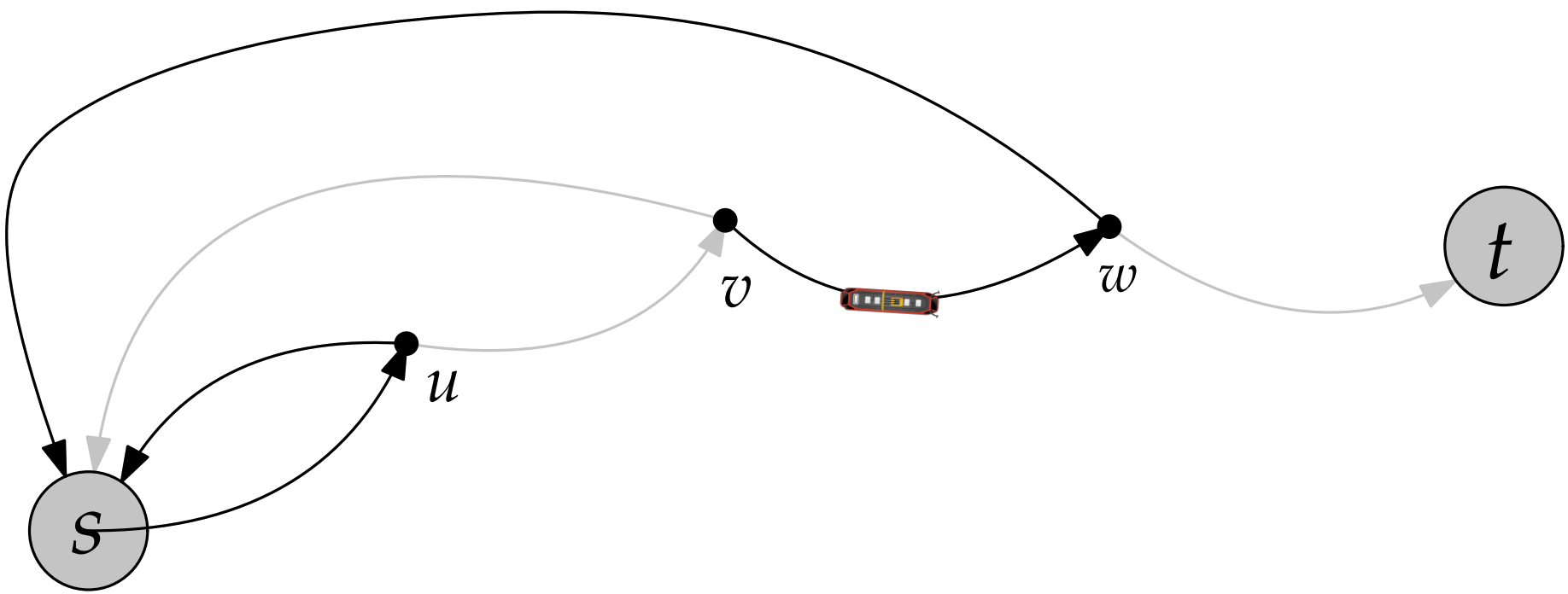

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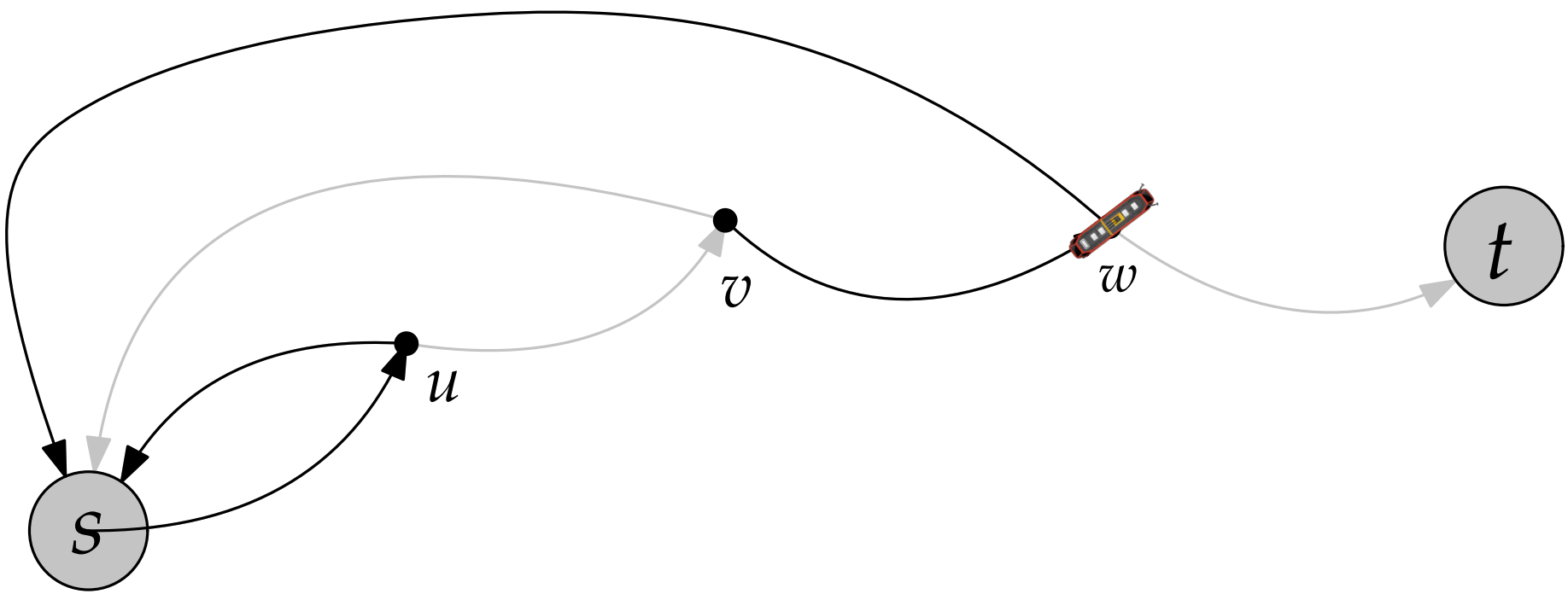

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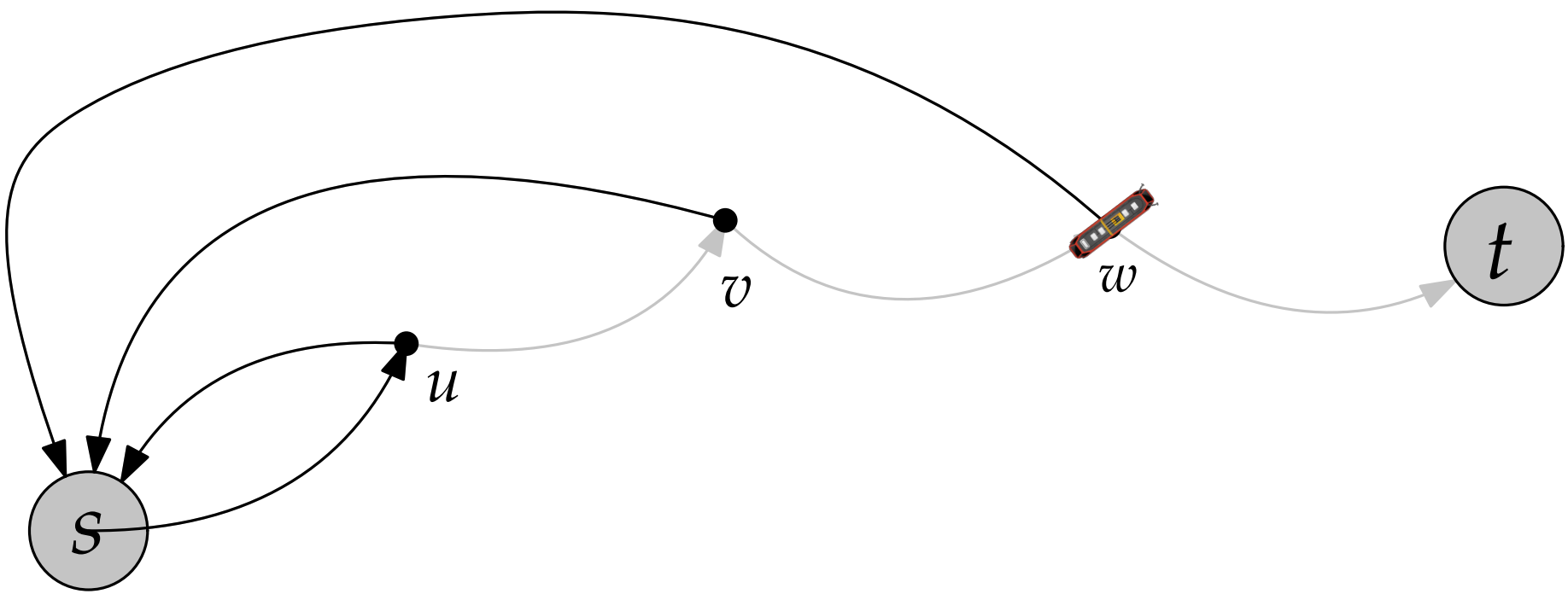

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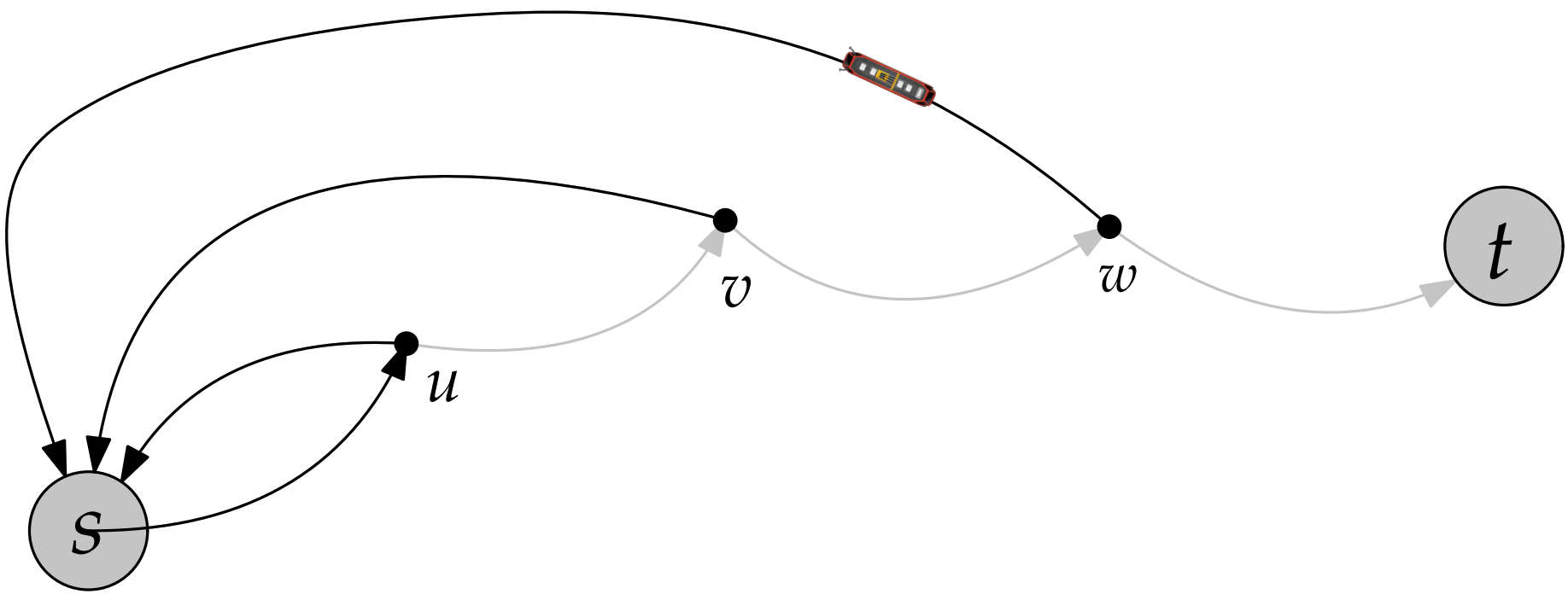

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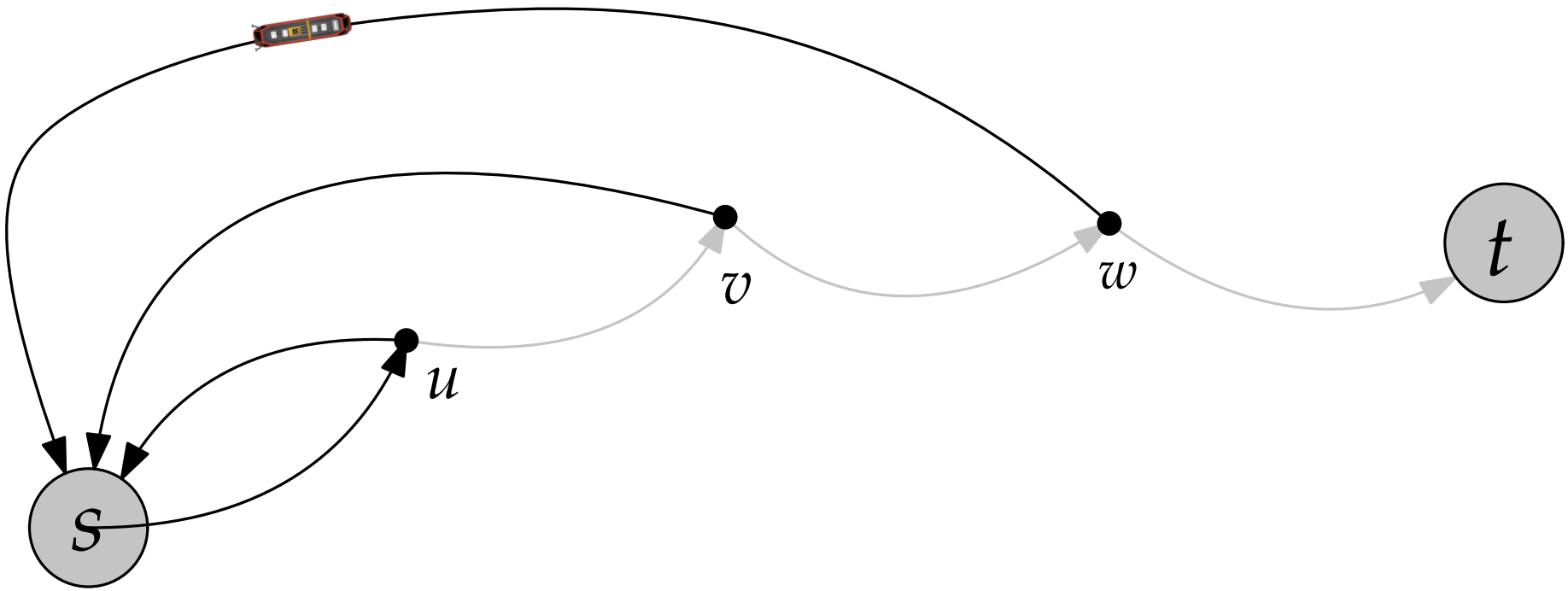

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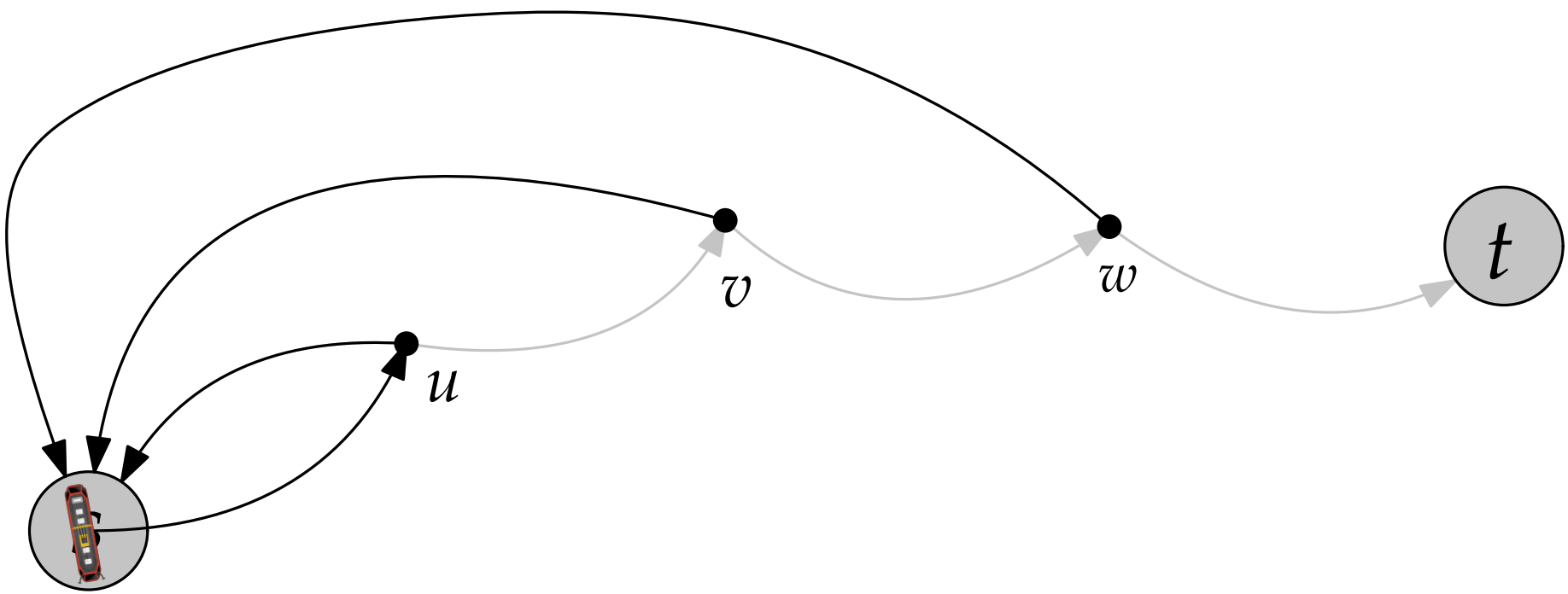

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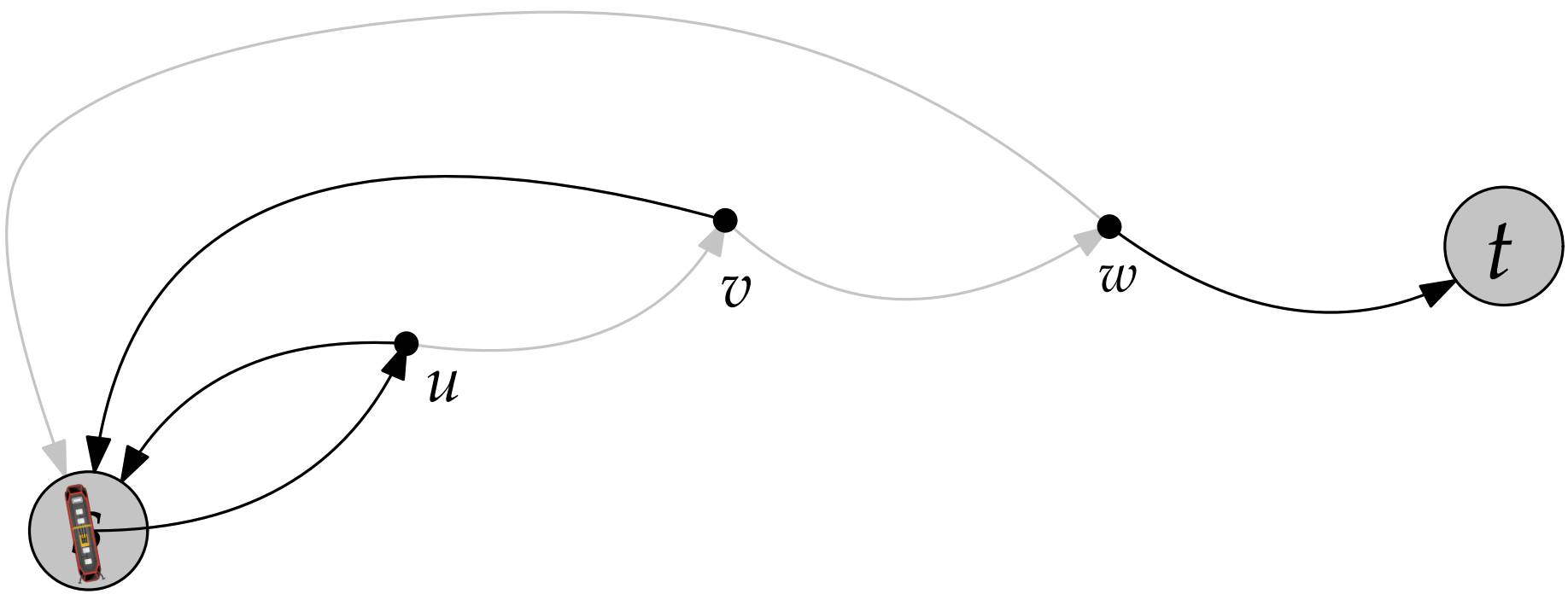

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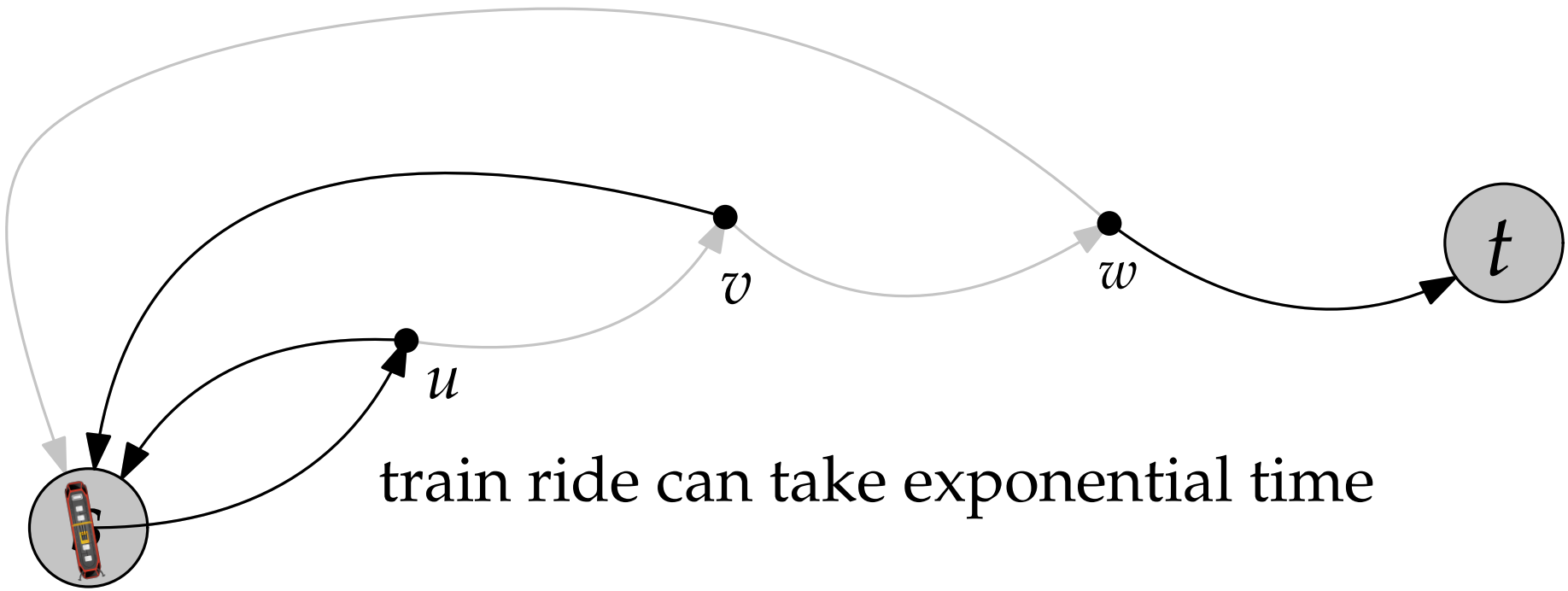
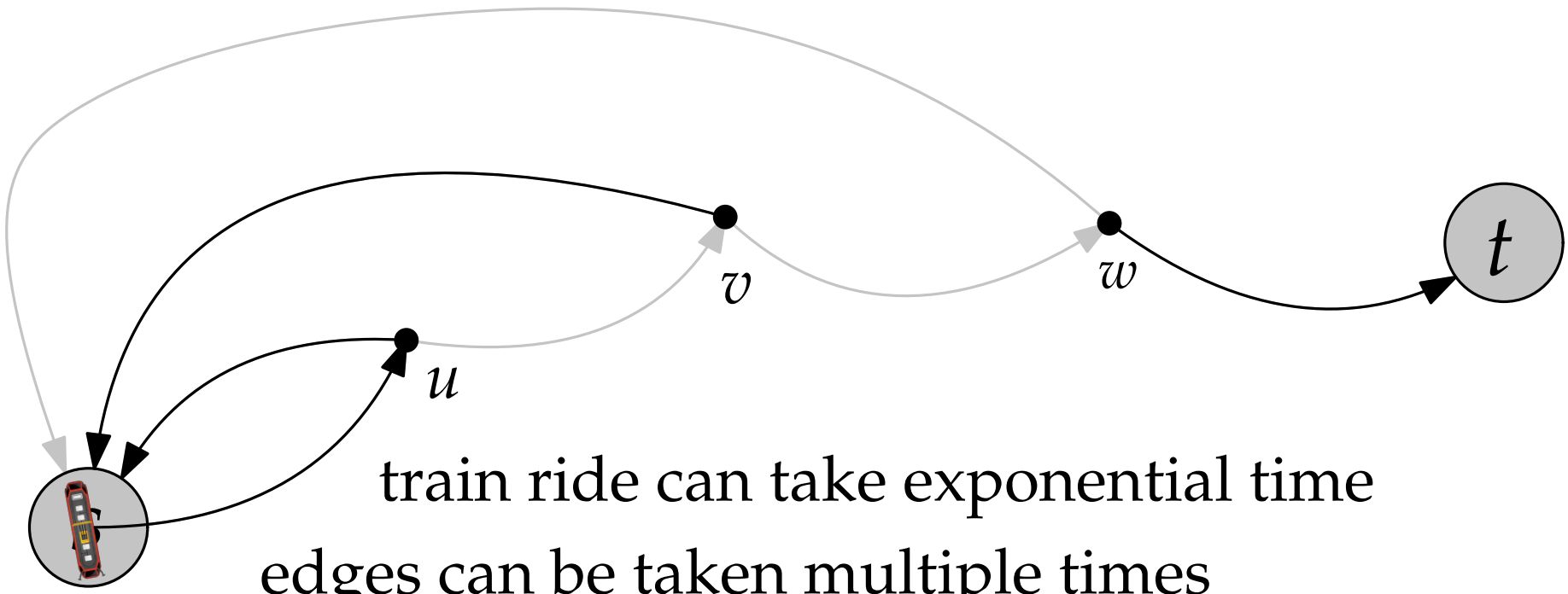


train ride can take exponential time

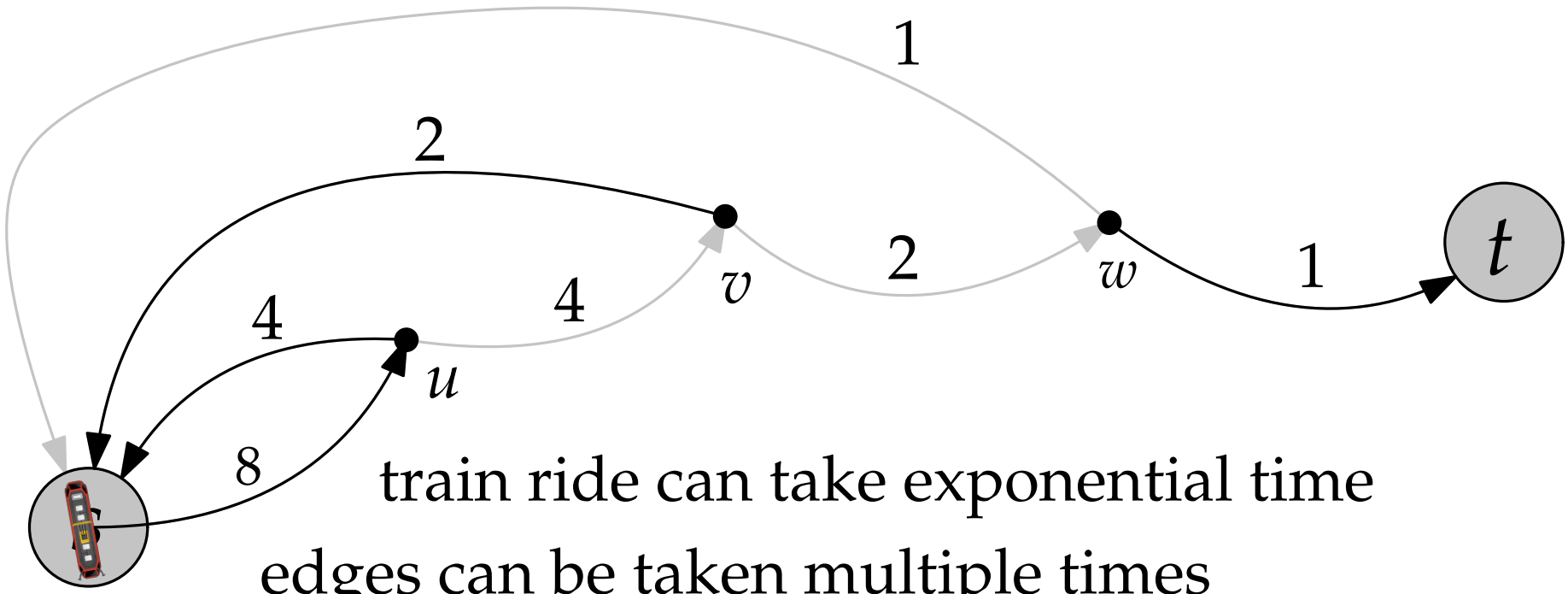
Arrival



train ride can take exponential time

edges can be taken multiple times

Arrival



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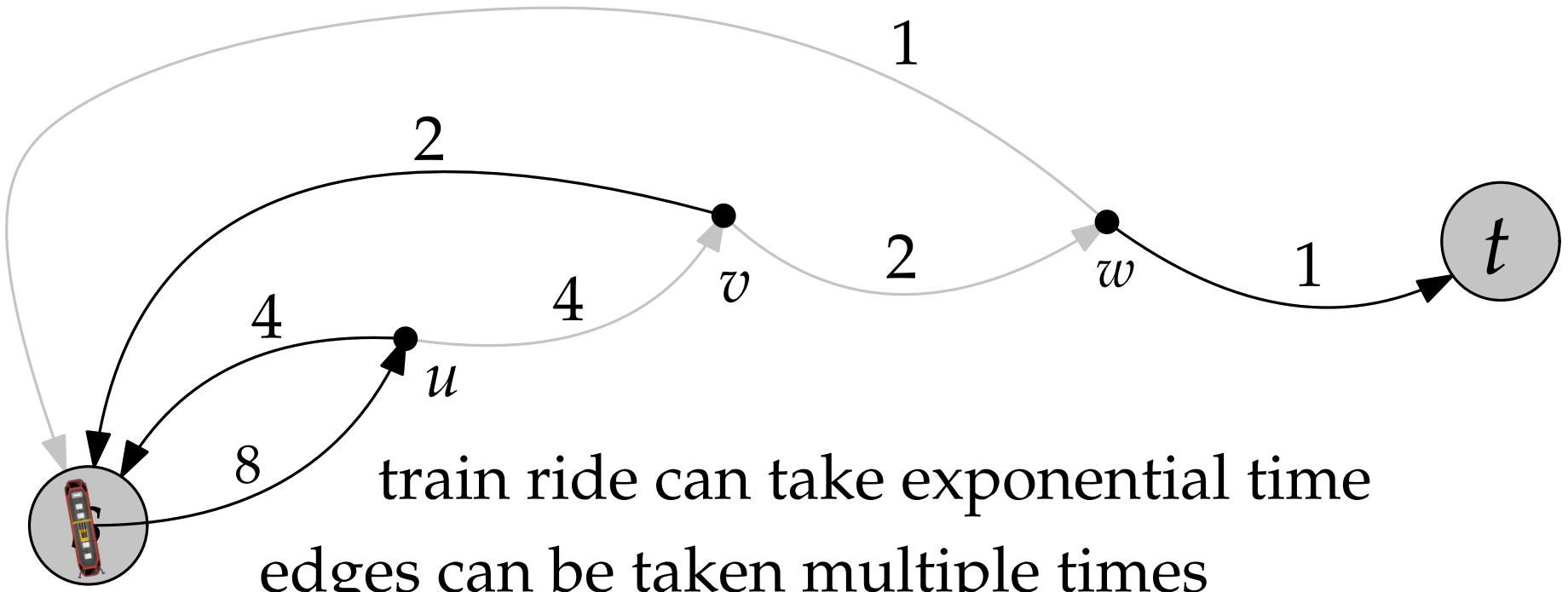
ARRIVAL

Theorem (Dohrau, Gärtner, Kohler, Matoušek, Welzl).
ARRIVAL is in NP.



train ride can take exponential time

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We call this the *profile* of the train ride from s to t.

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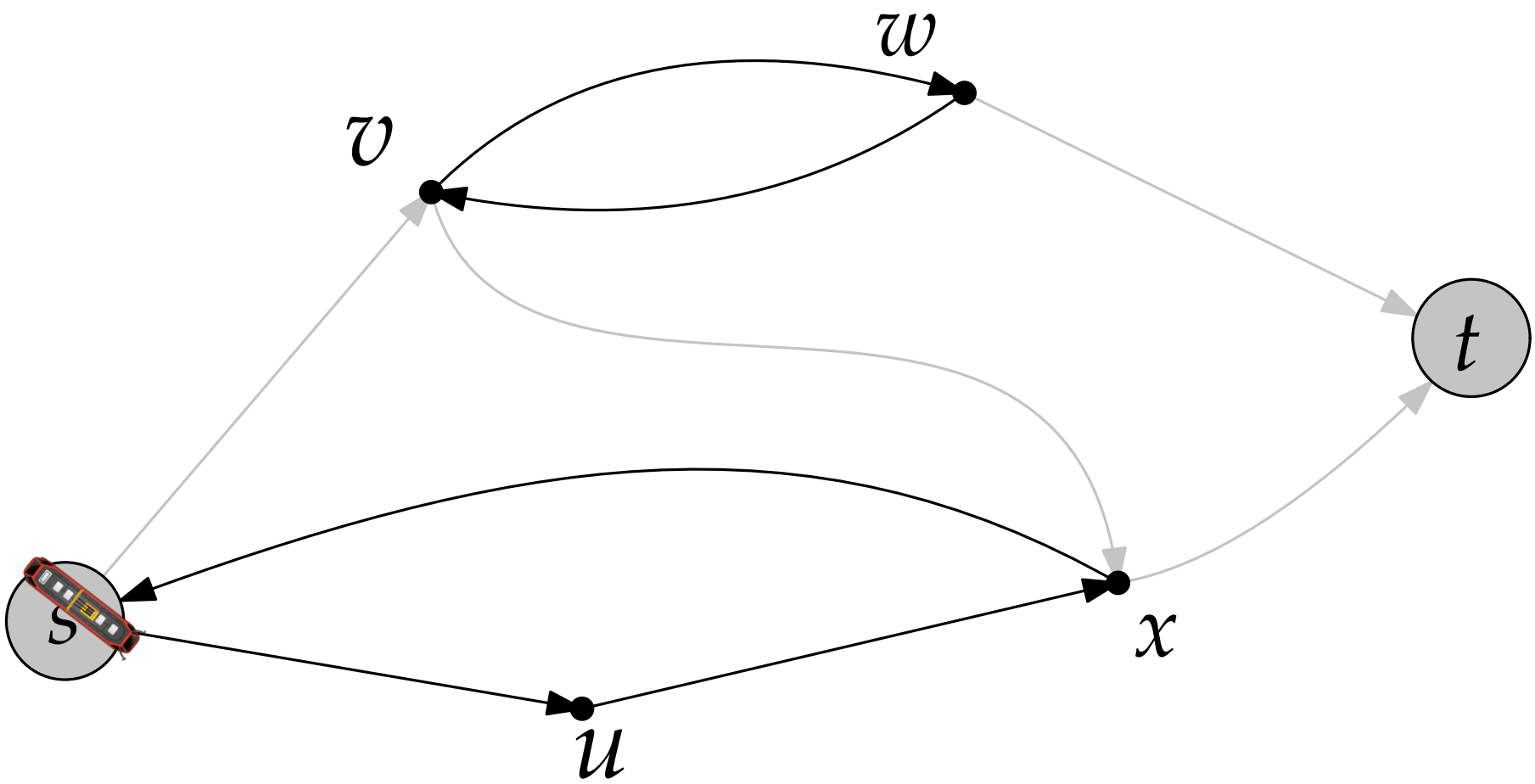

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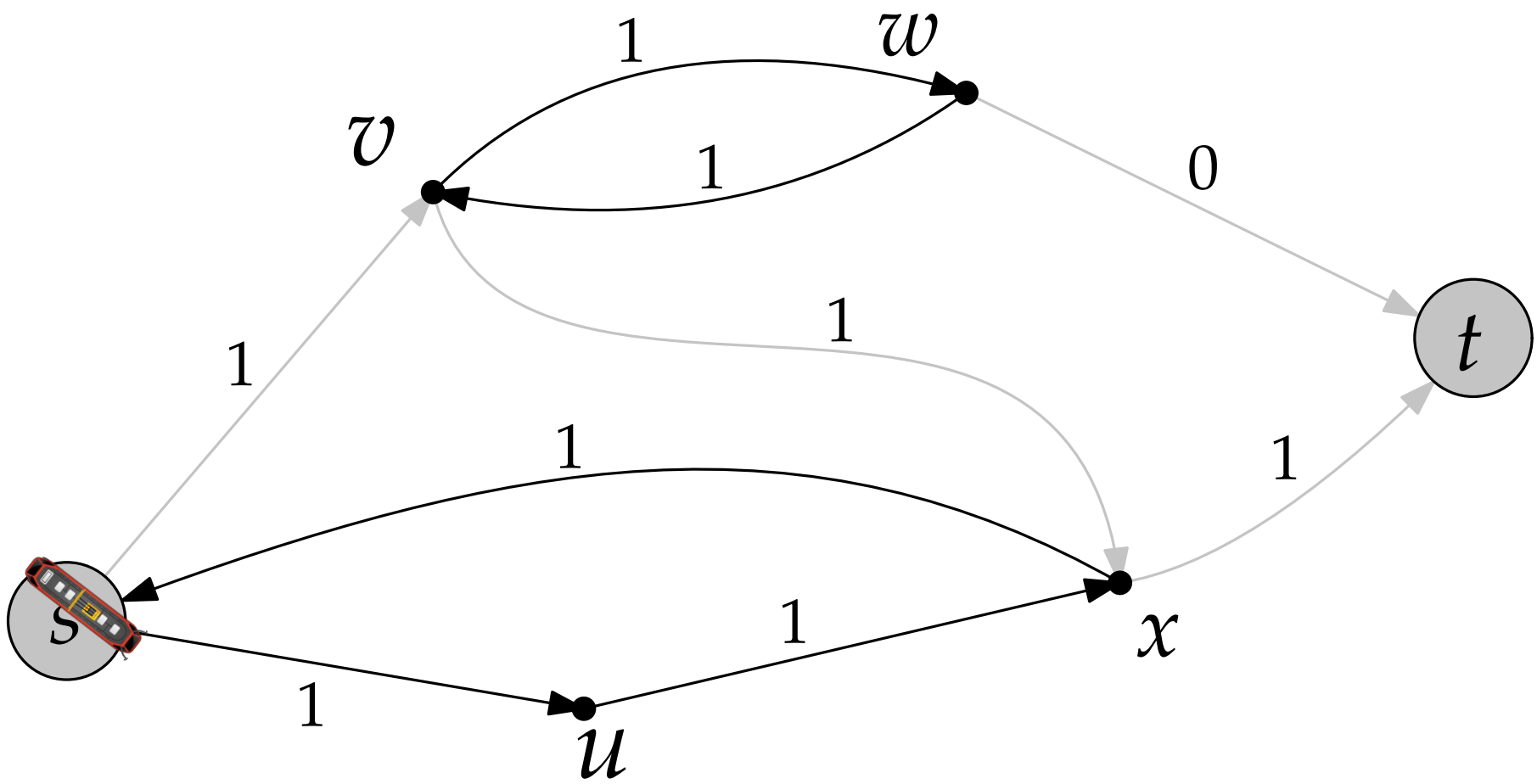

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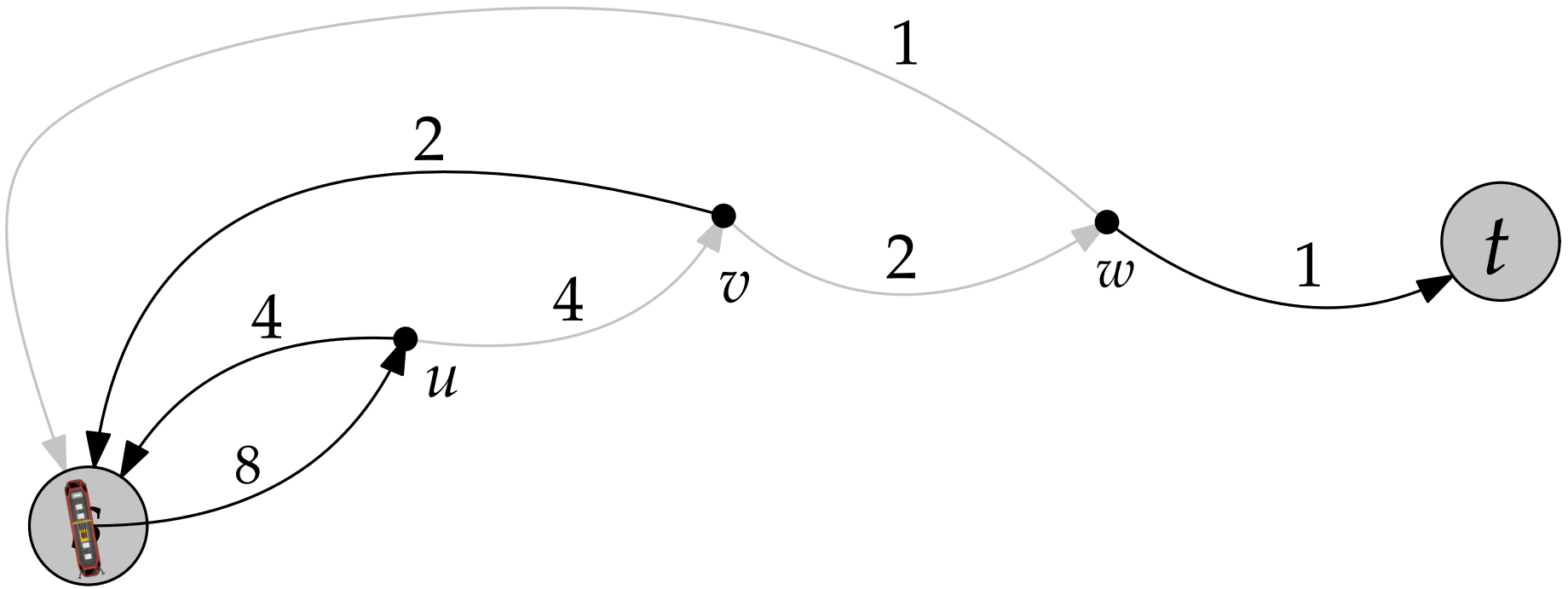

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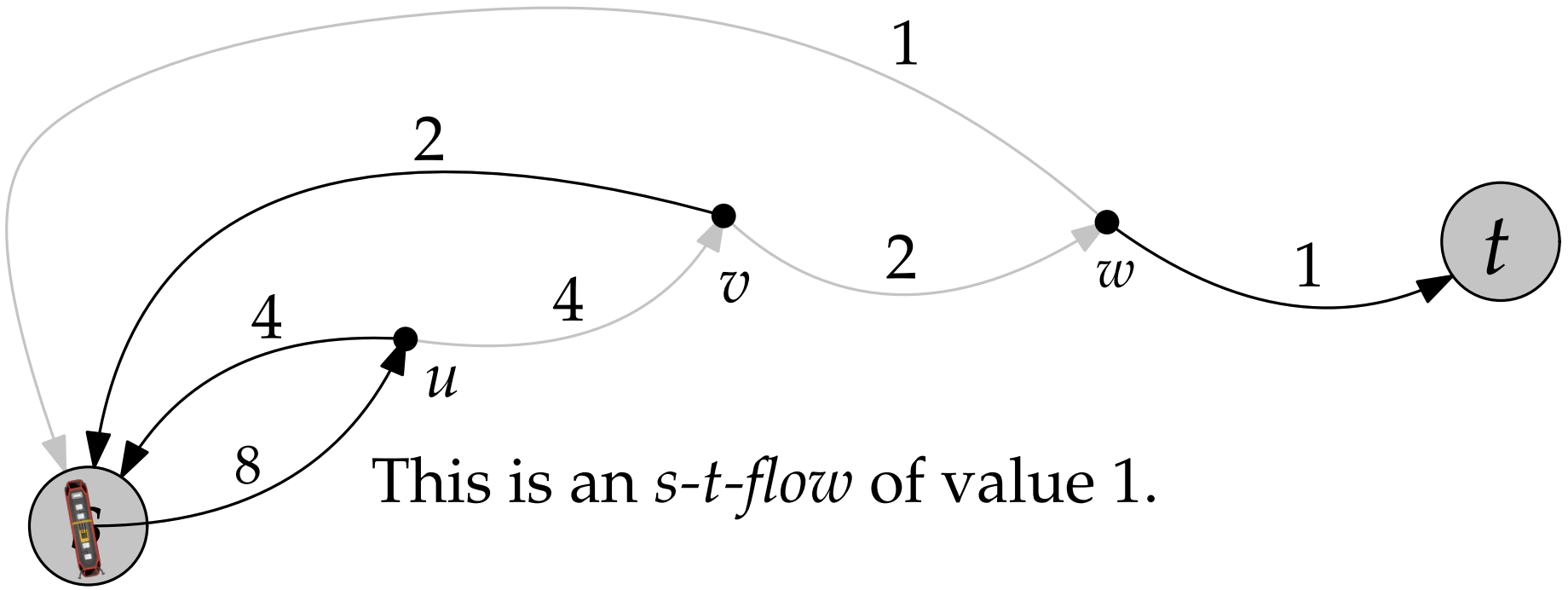

This is an *s-t-flow* of value 1.

ARRIVAL

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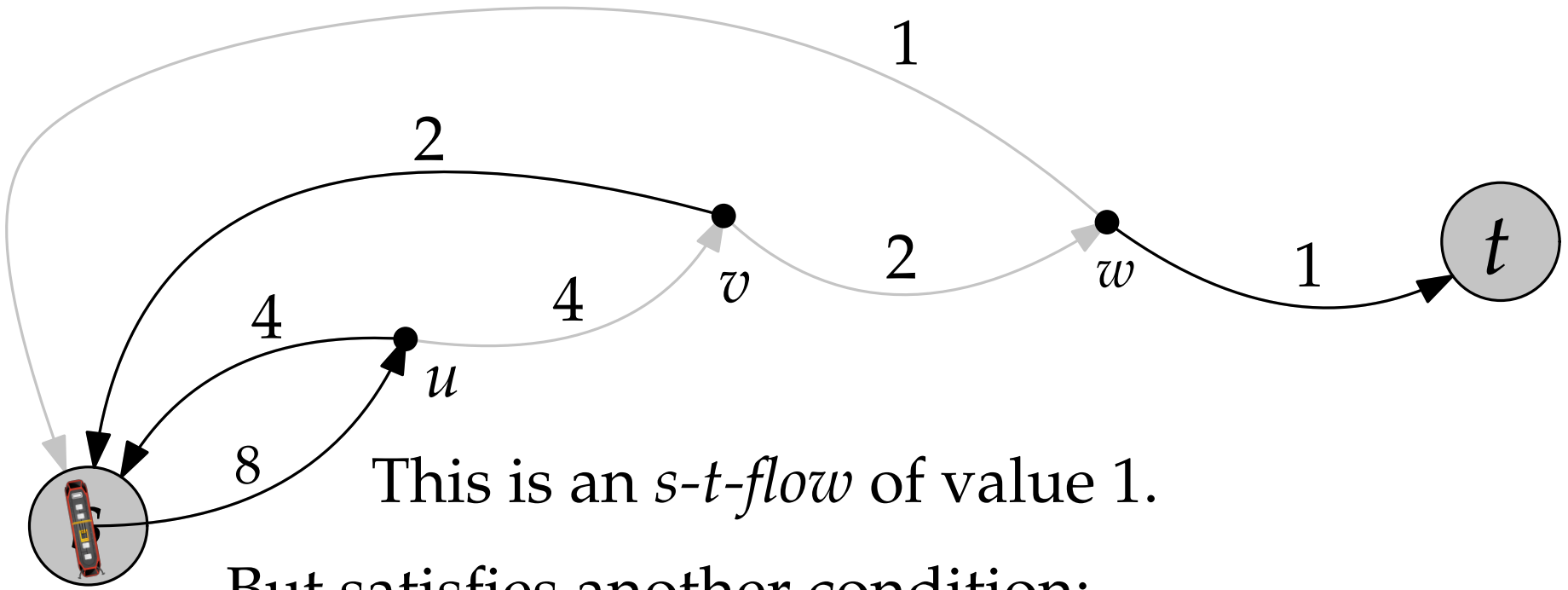
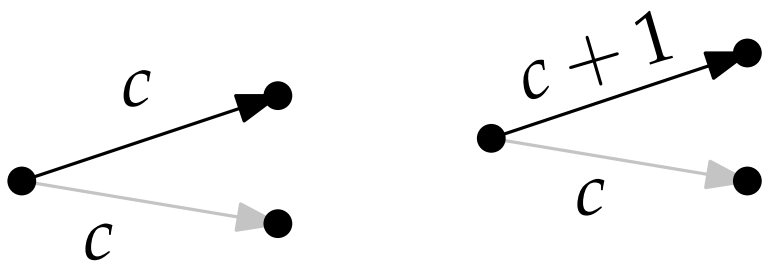


This is an *s-t-flow* of value 1.

But satisfies another condition:



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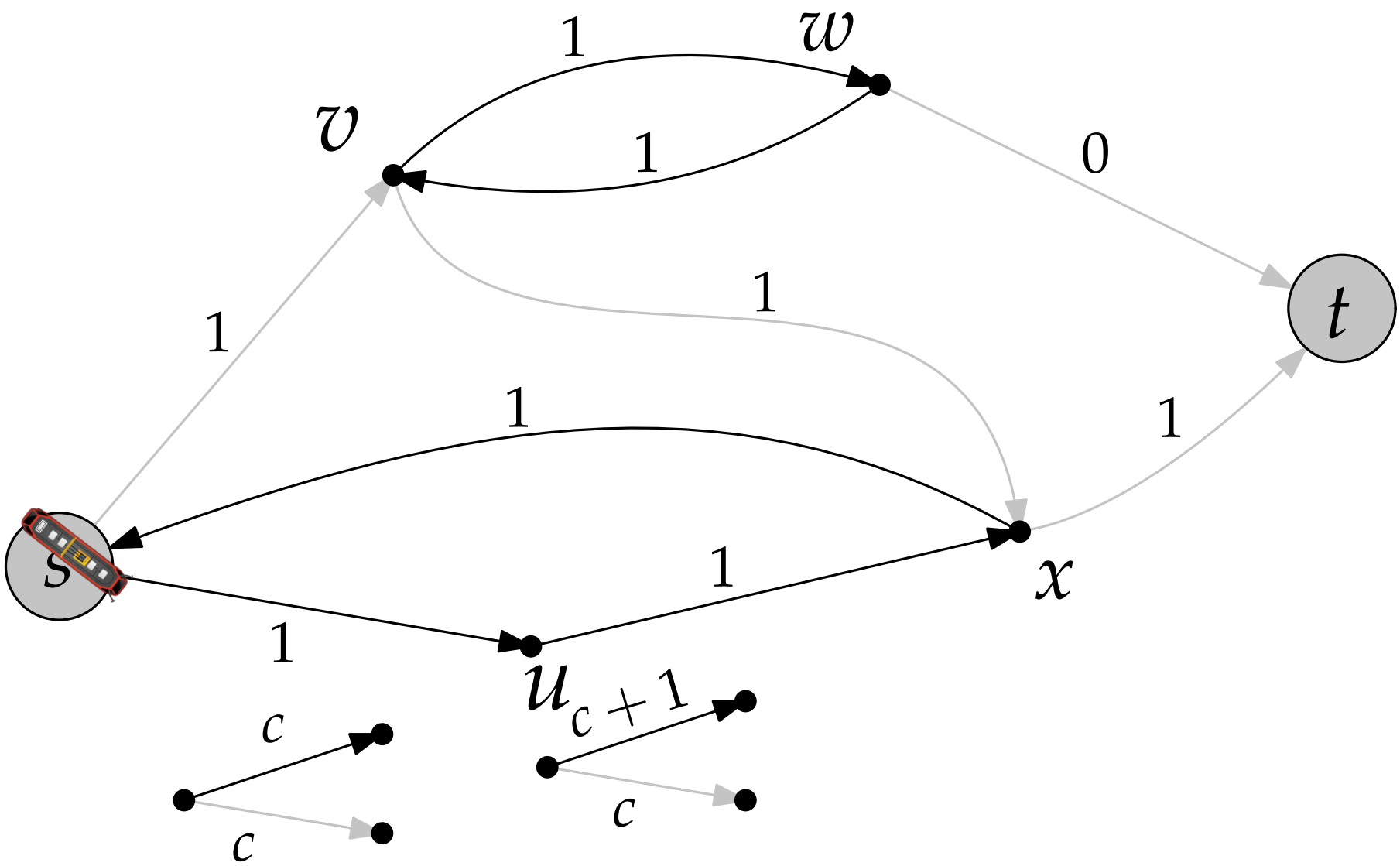


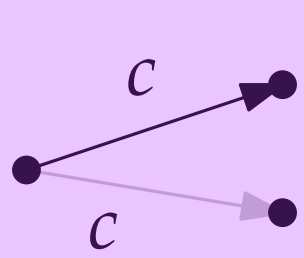
ARRIVAL

Theorem (Dohrau, Gärtner, Kohler, Matoušek, Welzl).
ARRIVAL is in NP.

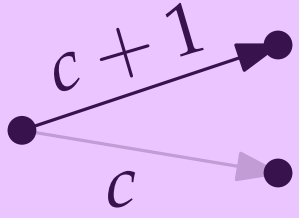
Definition. A function $f : E \to \mathbb{N}$ is a *train flow* from s to t if

$$\sum_w f(v,w) - \sum_u f(u,v) = \begin{cases} 1 & \text{if } v = s \\ -1 & \text{if } v = t \\ 0 & \text{else.} \end{cases}$$

and



or



at all vertices.

Arrival

Definition. A function $f : E \to \mathbb{N}$ is a *train flow* from s to t if

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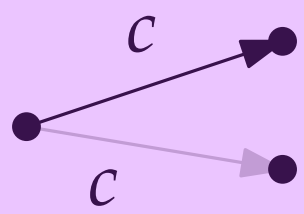

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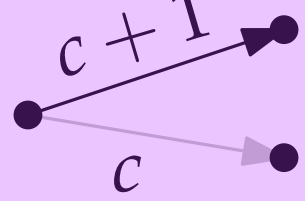

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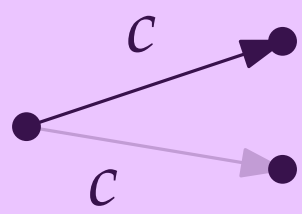

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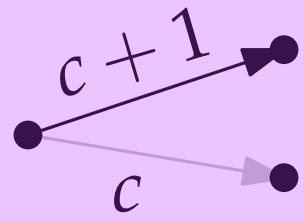

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Observation. Every train ride profile is a train flow.

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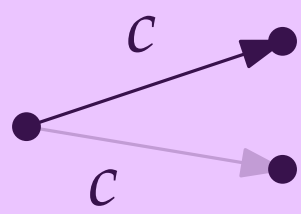

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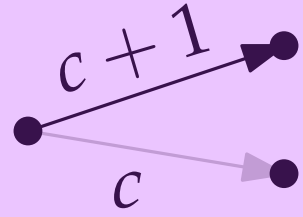

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Question. Is every train flow a train ride profile?

Arrival

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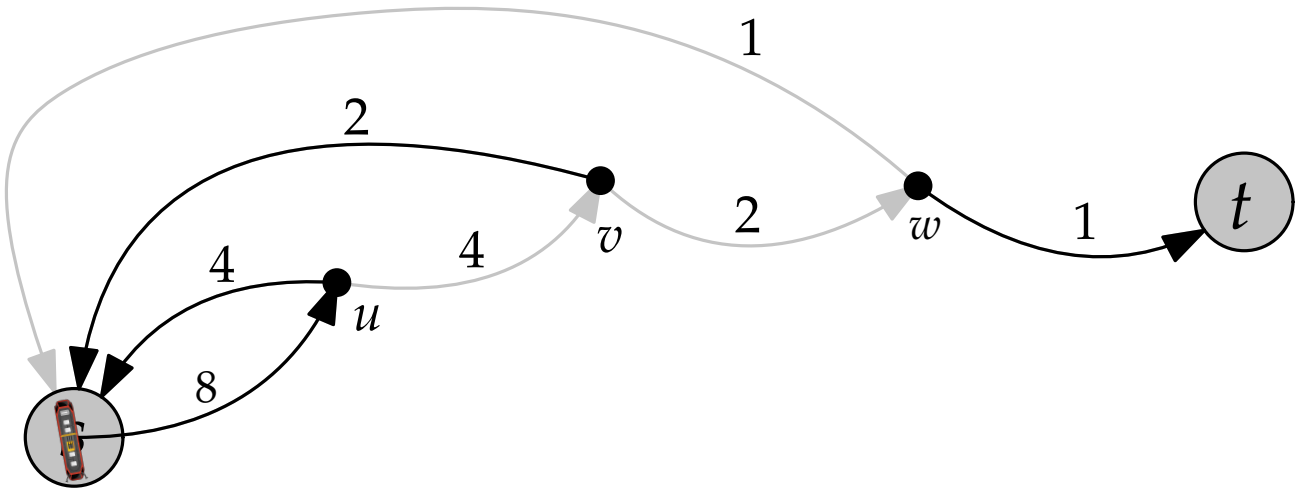

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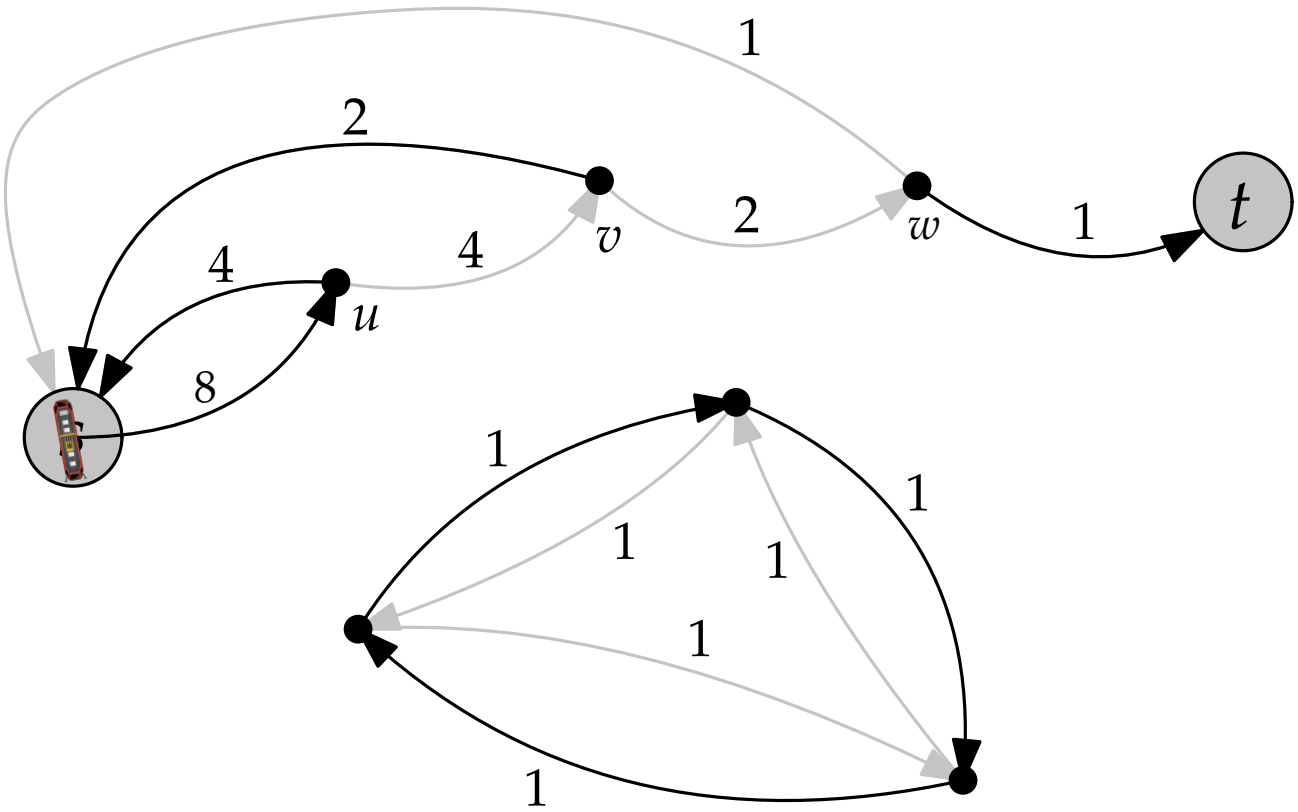

ARRIVAL

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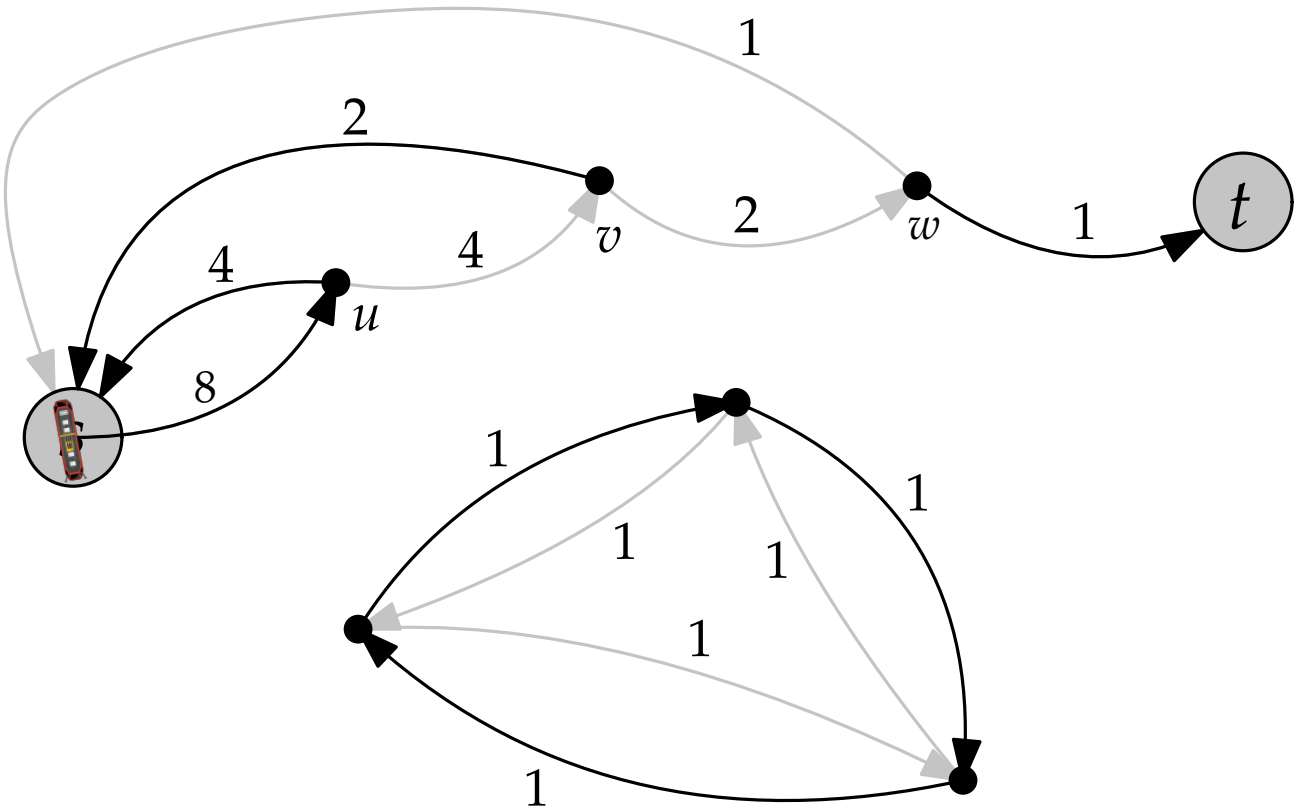

...this is a train flow but not a train ride profile...

ARRIVAL

Lemma (Dohrau, Gärtner, Kohler, Matoušek, Welzl). If a train flow f exists then the train reaches the destination t.

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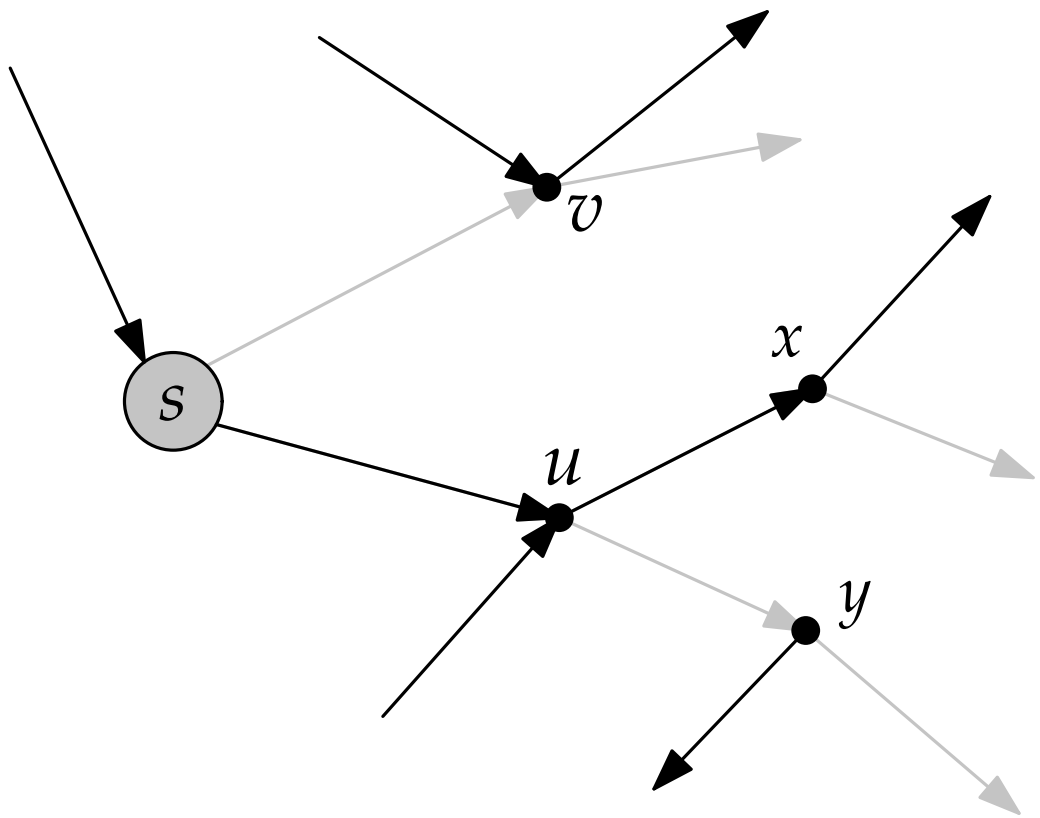

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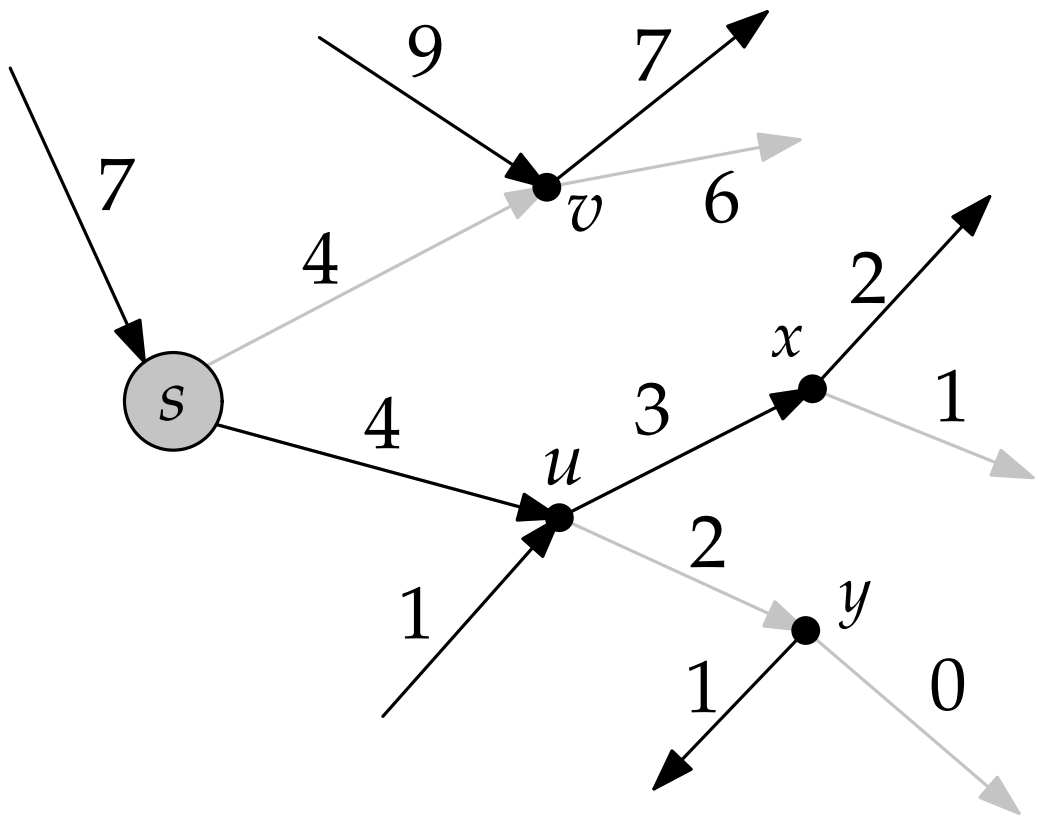

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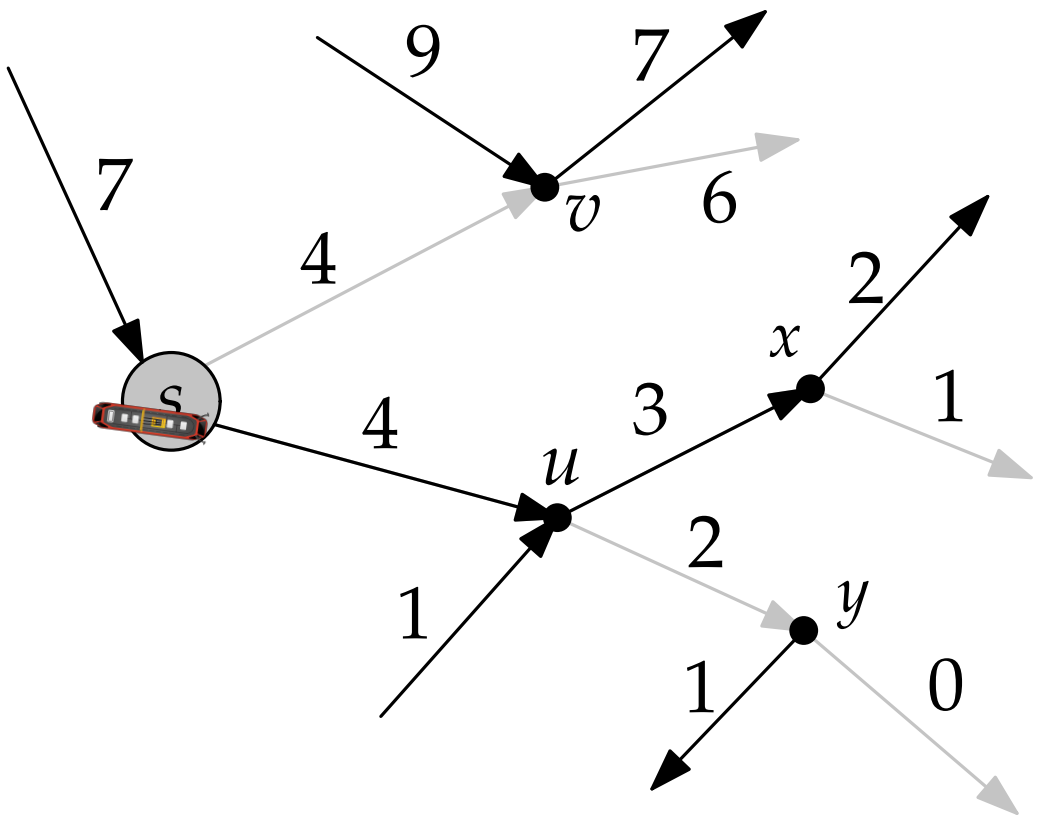

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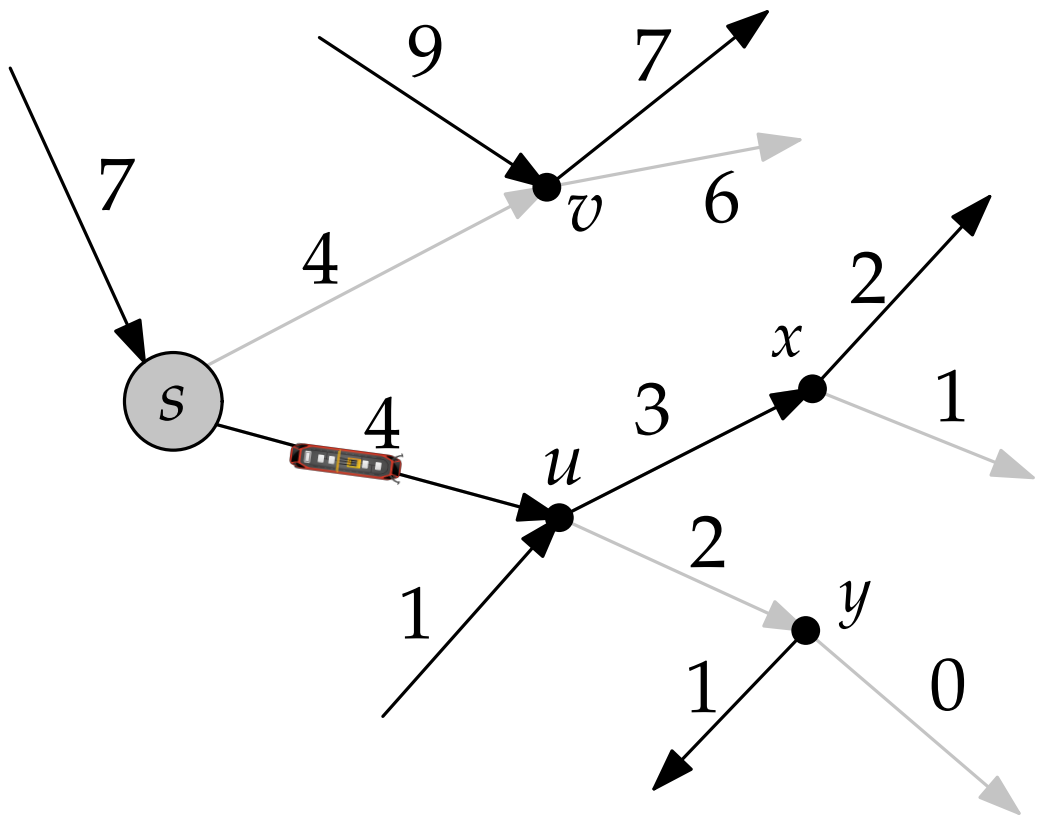

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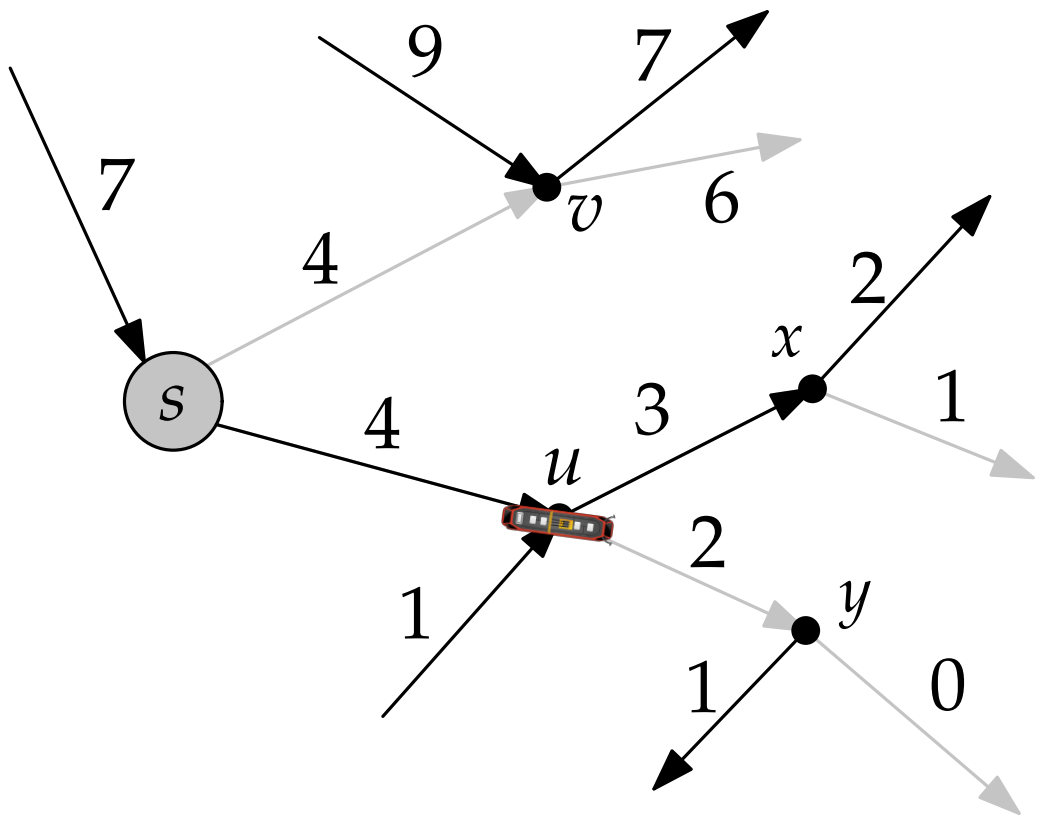

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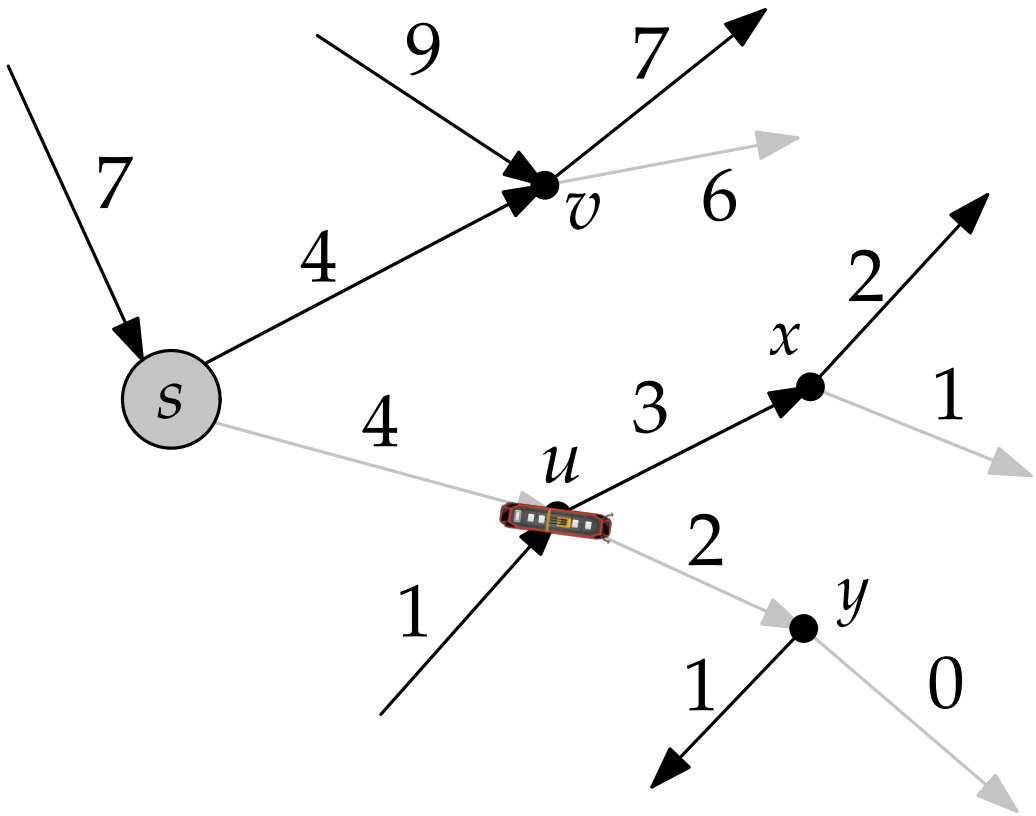

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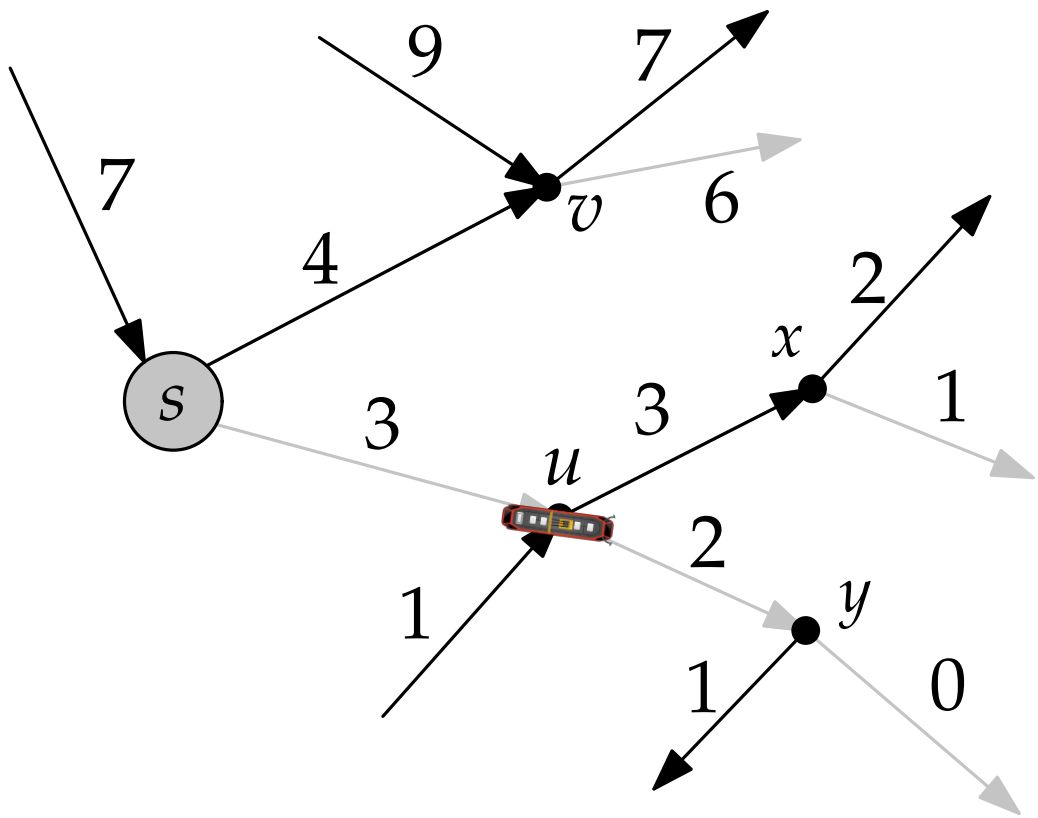

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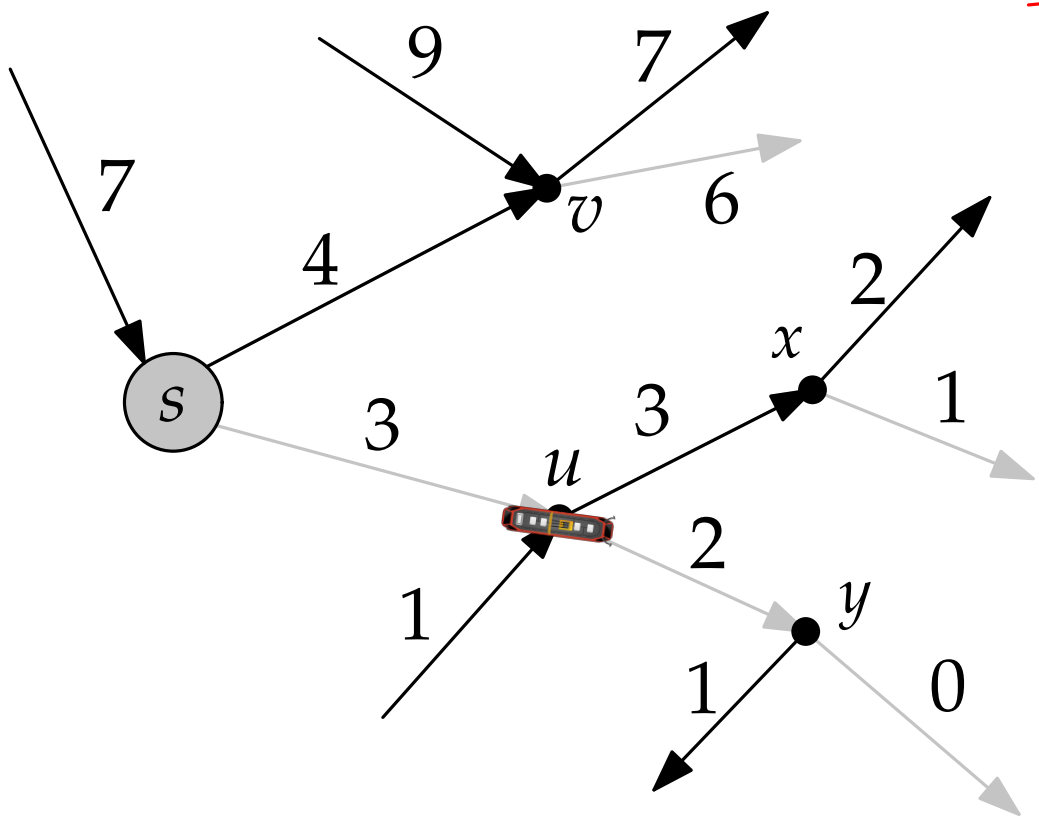

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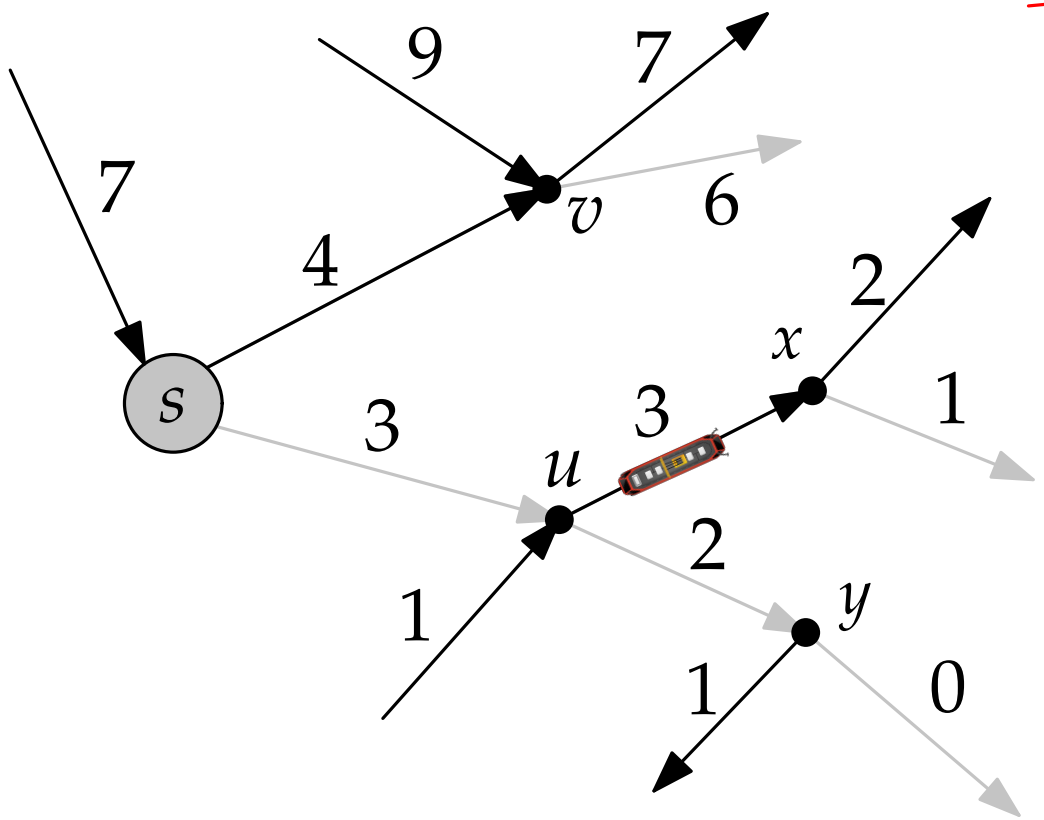

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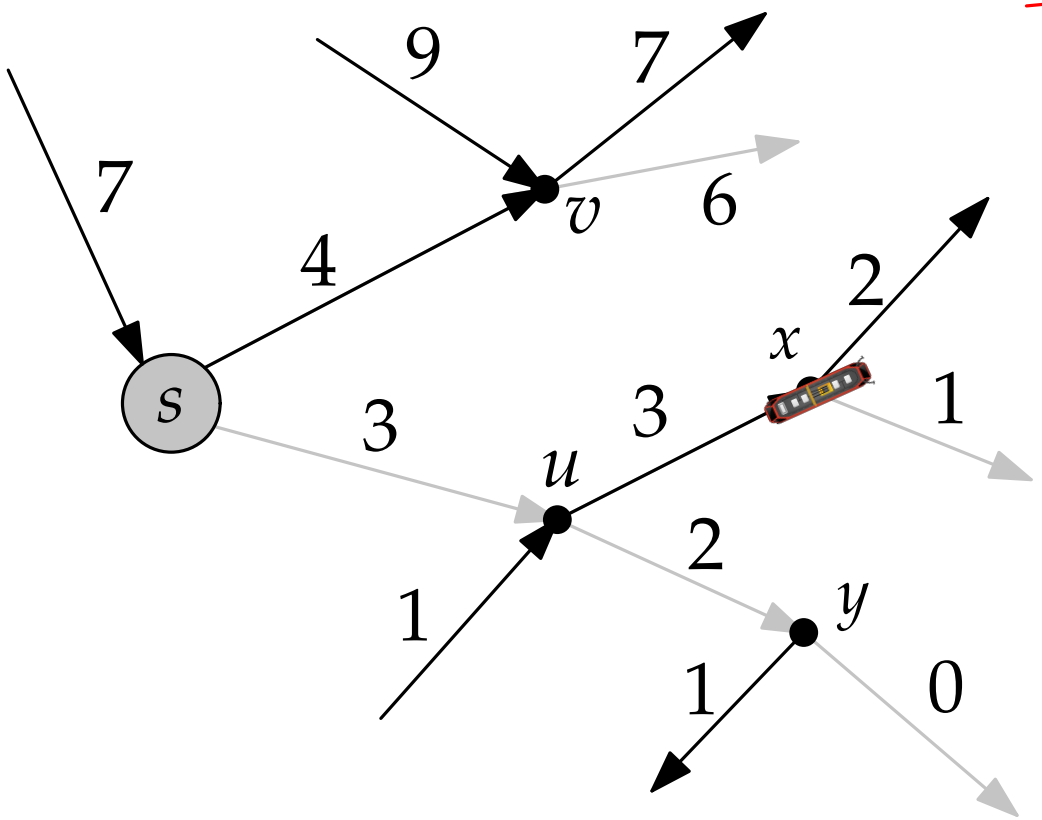

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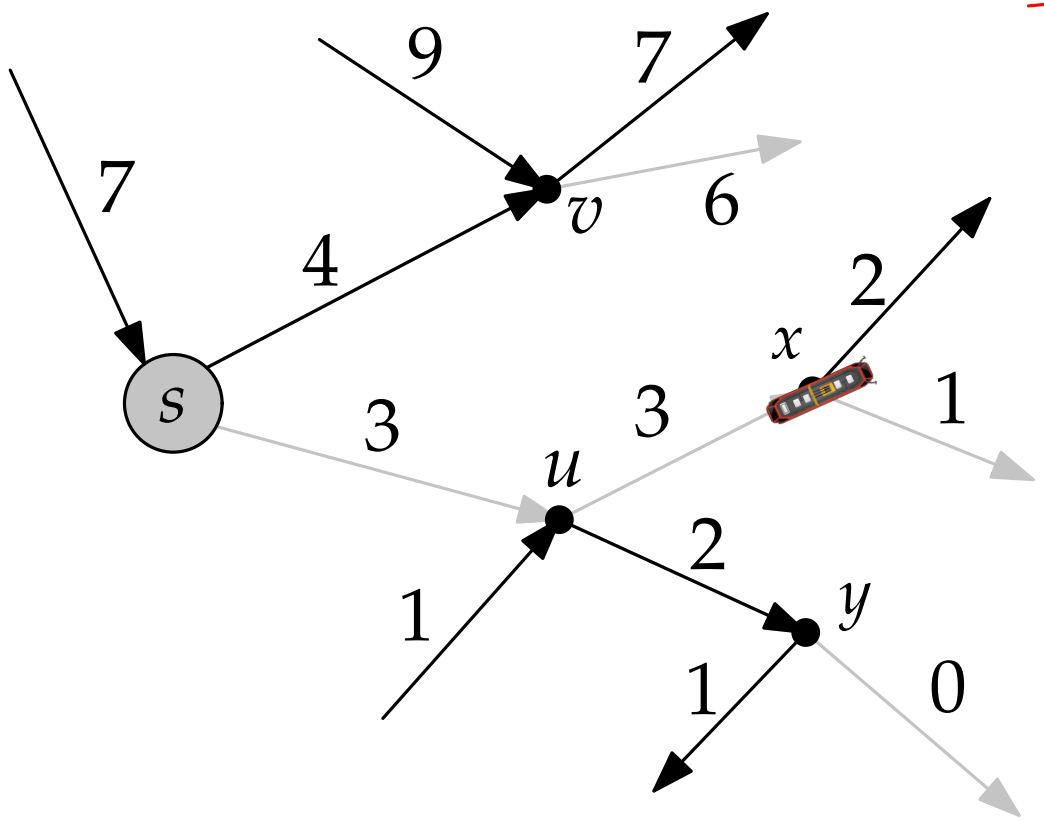

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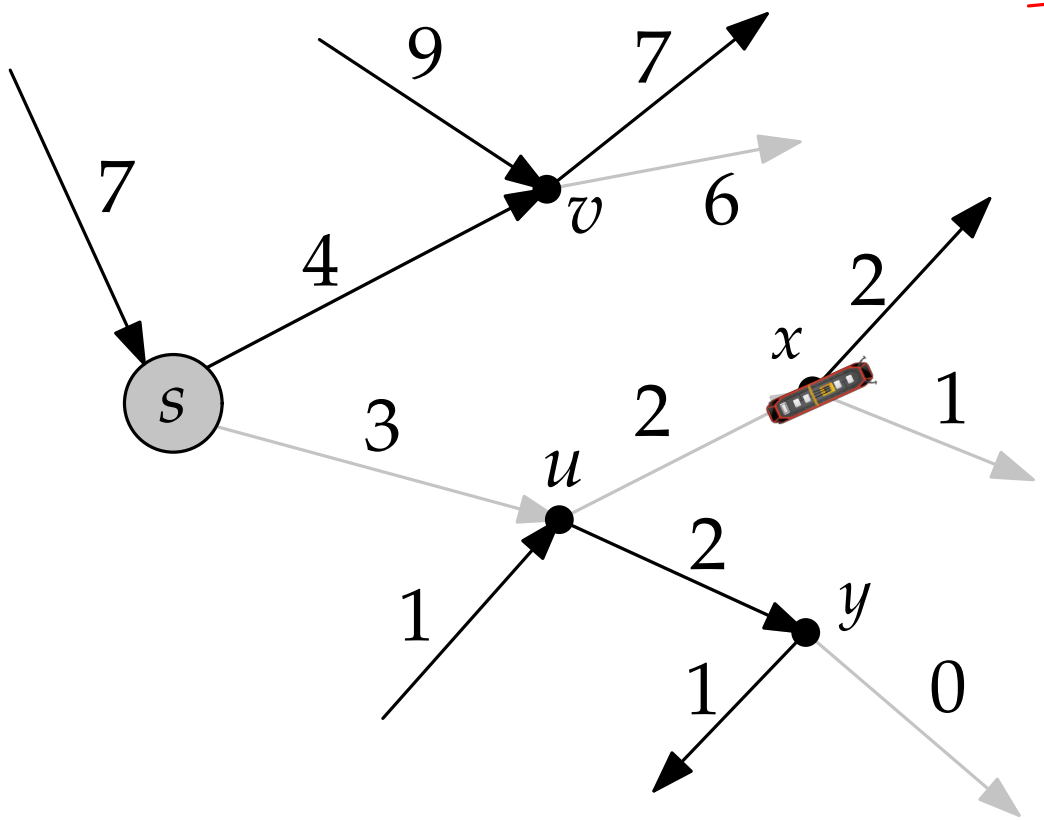

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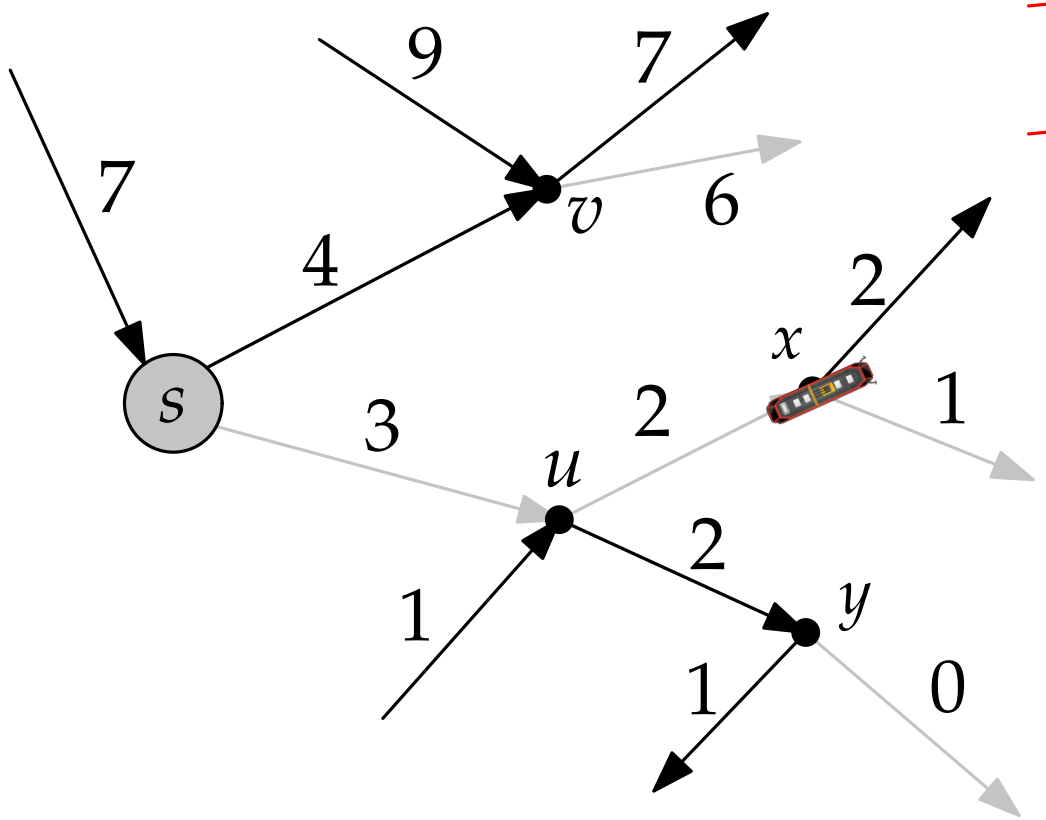

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~~train flow from u to t~~

train flow from x to t

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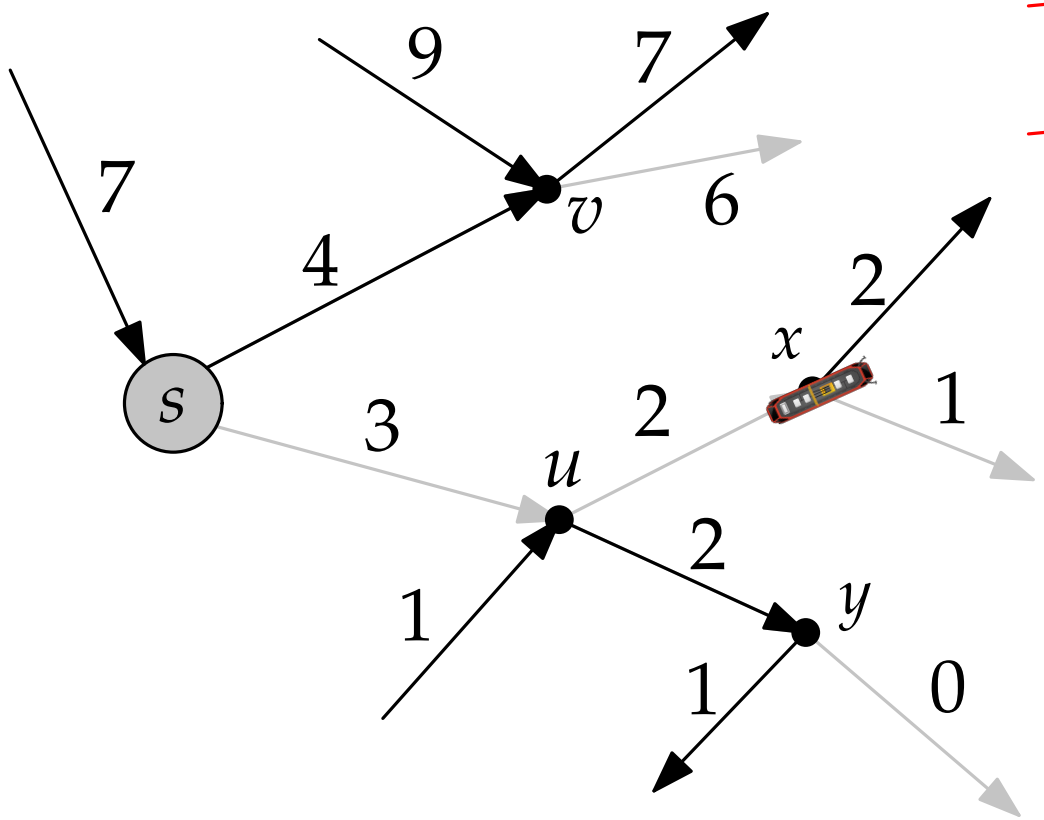

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numbers decrease

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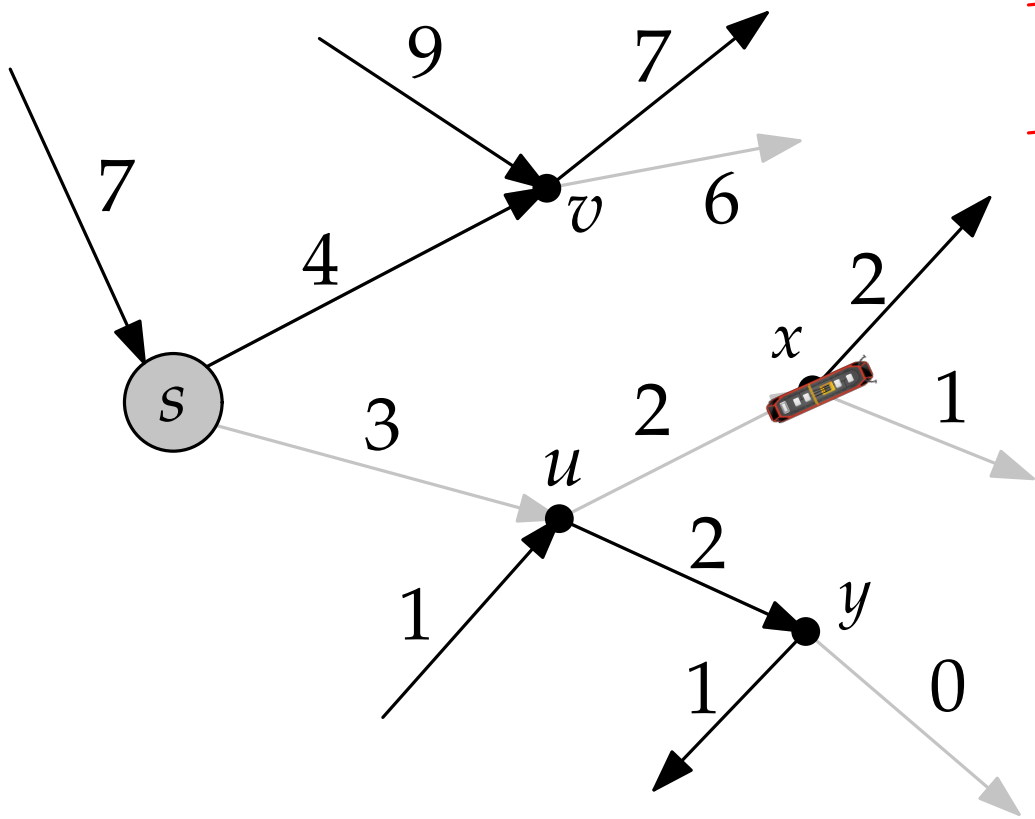

~~train flow from s to t~~

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train flow from x to t

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never become negative

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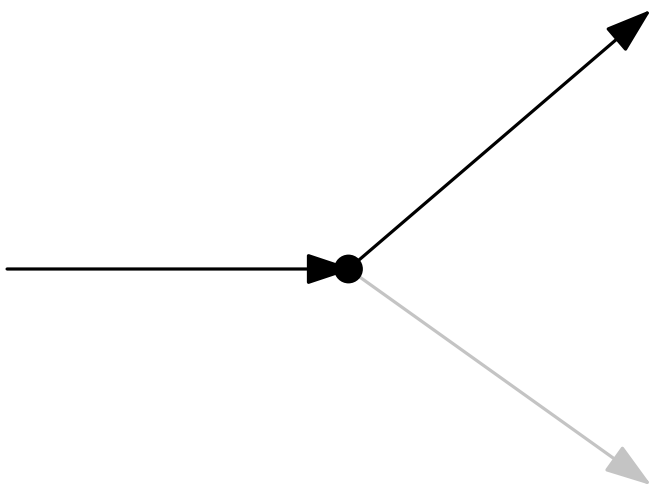

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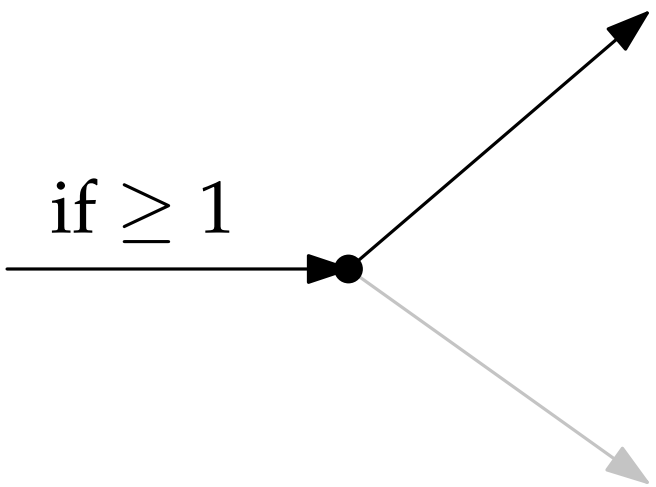

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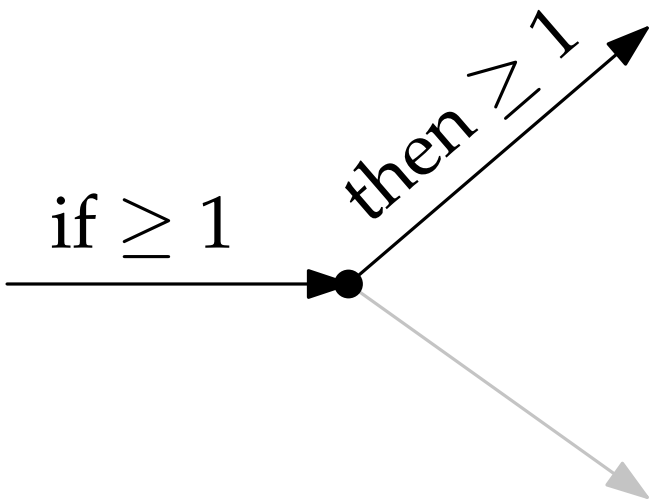

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Theorem (Dohrau, Gärtner, Kohler, Matoušek, Welzl). ARRIVAL is in NP.

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Theorem (Dohrau, Gärtner, Kohler, Matoušek, Welzl). ARRIVAL is in NP.

Proof. Given $f : E \to \mathbb{N}$, verify that it is a train flow from s to t.
If yes, then f is a certificate that the train reaches the station.
Thus, ARRIVAL is in NP.

ARRIVAL

Theorem (Dohrau, Gärtner, Kohler, Matoušek, Welzl).
ARRIVAL is in co-NP.

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Observation. If train reaches a trap, it will never reach the destination t.

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Less obvious Observation. If the train does not eventually reach t, then it will reach a trap.

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A train flow from s to a trap is a certificate that the train does not reach destination t.

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So ARRIVAL is in co-NP.

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Theorem (Karthik C.S.). ARRIVAL is in PLS.

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$\text{val} : \mathcal{F} \to \mathbb{N}, \sum_{e \in E} f(e)$.

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Goal. Find a *local maximum*. That is, a configuration $f \in \mathcal{F}$ with $\text{val}(g) \leq \text{val}(f)$ for all successors g of f.

ARRIVAL

Theorem (Karthik C.S.). ARRIVAL is in PLS. That is, computing where the train ends up (destination t or some trap) can be phrased as a quest for finding a local ~~minimum~~ maximum of a certain function.

$\mathcal{F}$: set of configurations, all train flows from s to some u

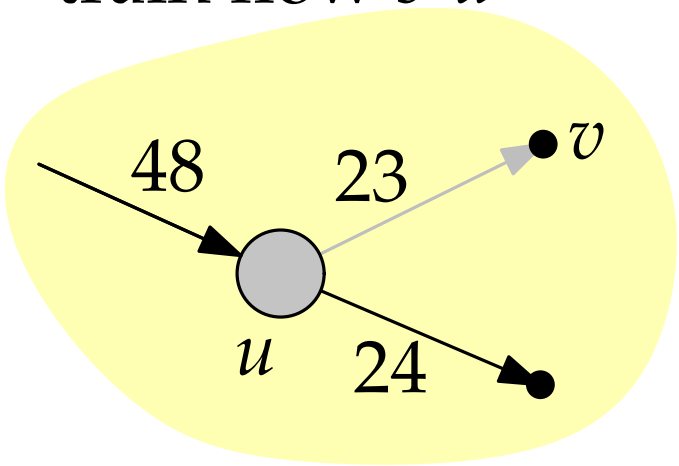
$\text{val} : \mathcal{F} \to \mathbb{N}$, $\sum_{e \in E} f(e)$.

$\text{succ} \subseteq \mathcal{F} \times \mathcal{F}$, what would happen next

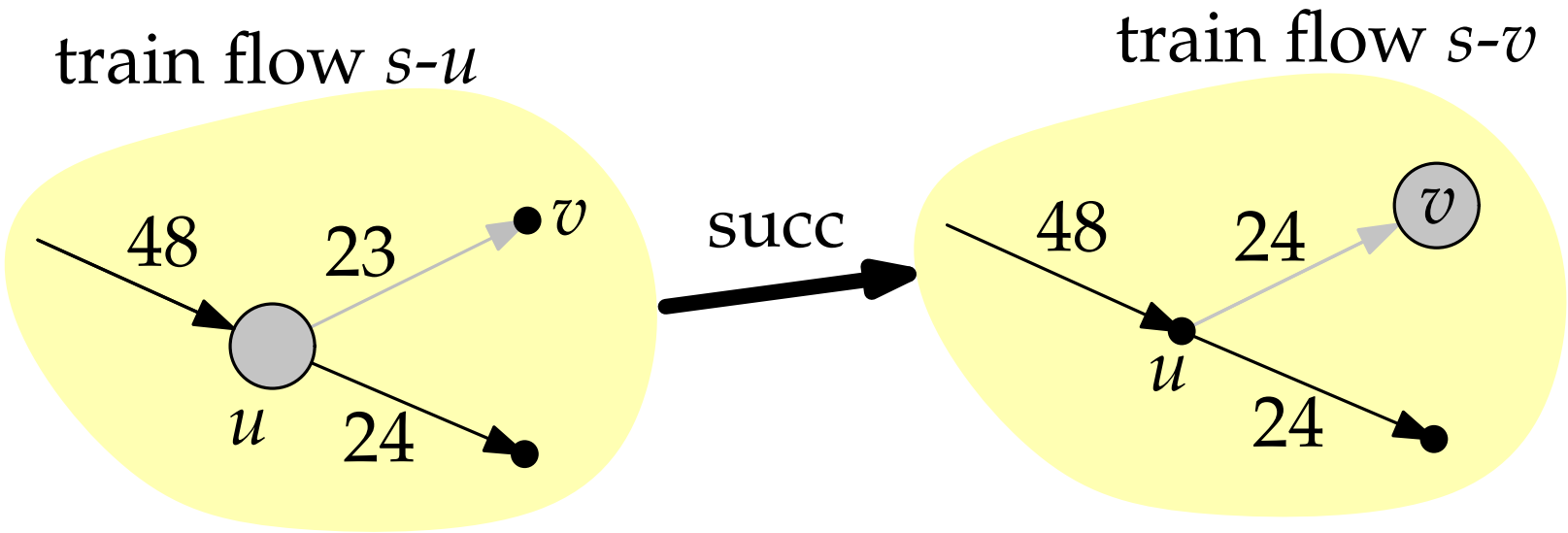
Goal. Find a *local maximum*. That is, a configuration $f \in \mathcal{F}$ with $\text{val}(g) \leq \text{val}(f)$ for all successors g of f.

Arrival

train flow s-u



Arrival



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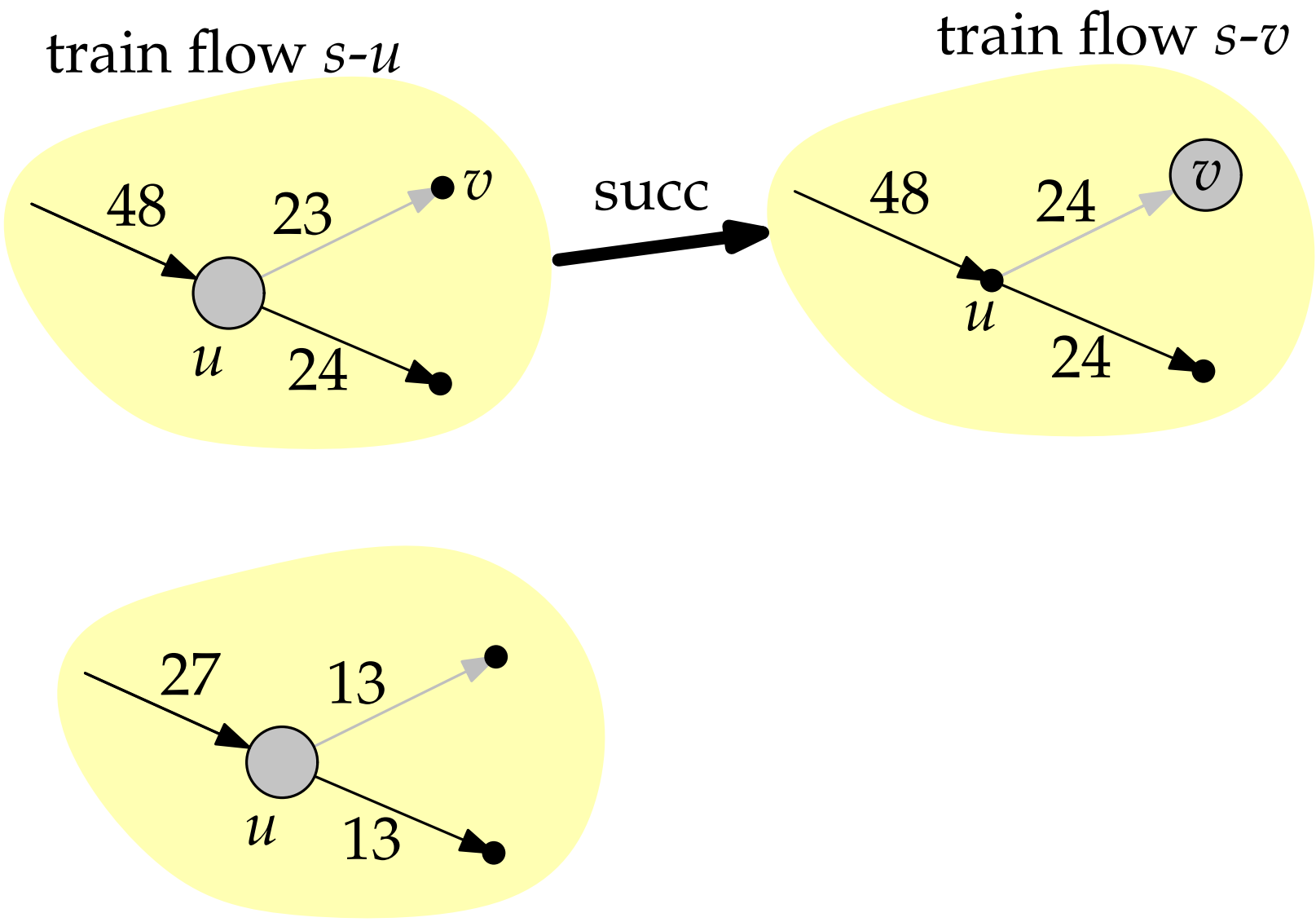

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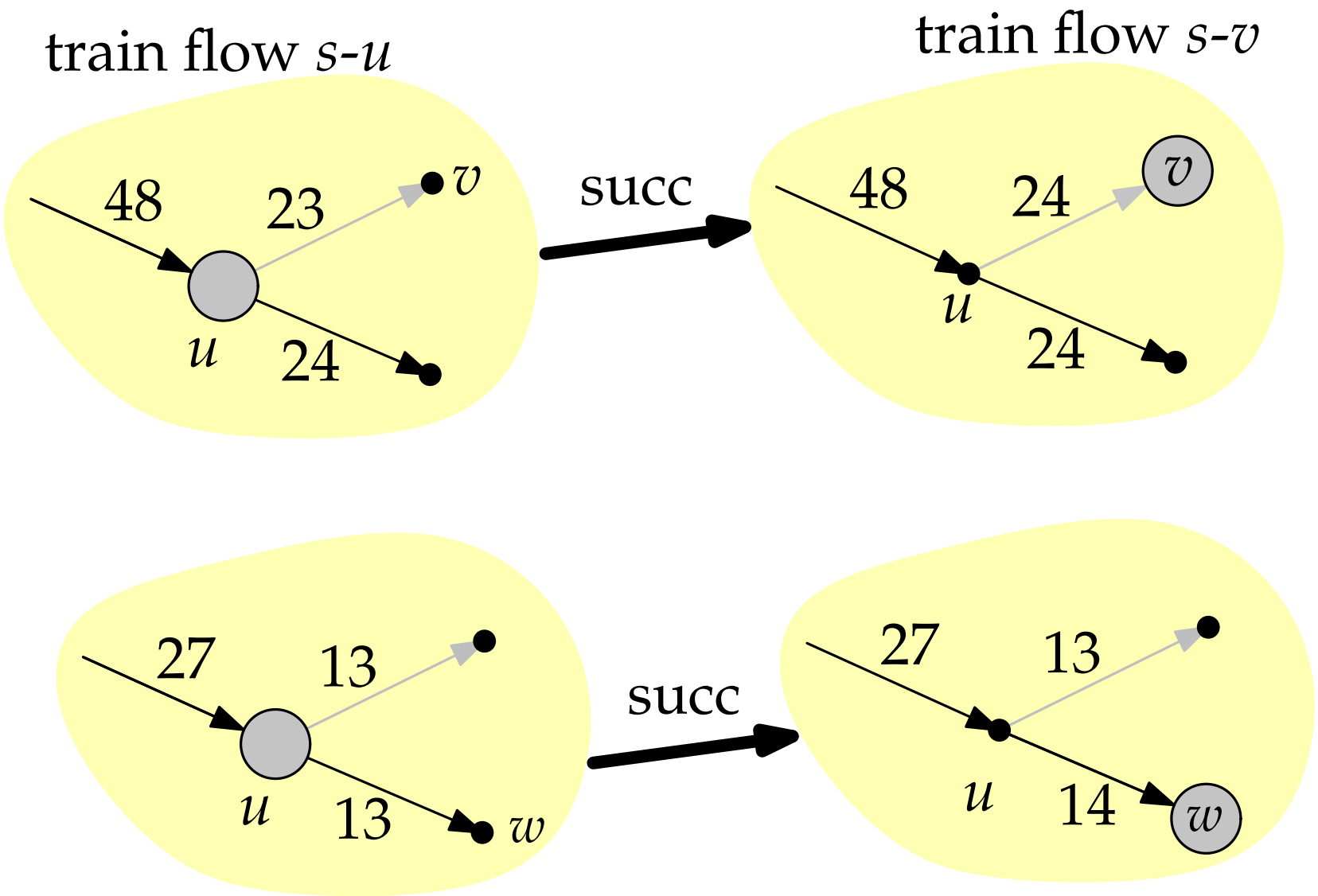
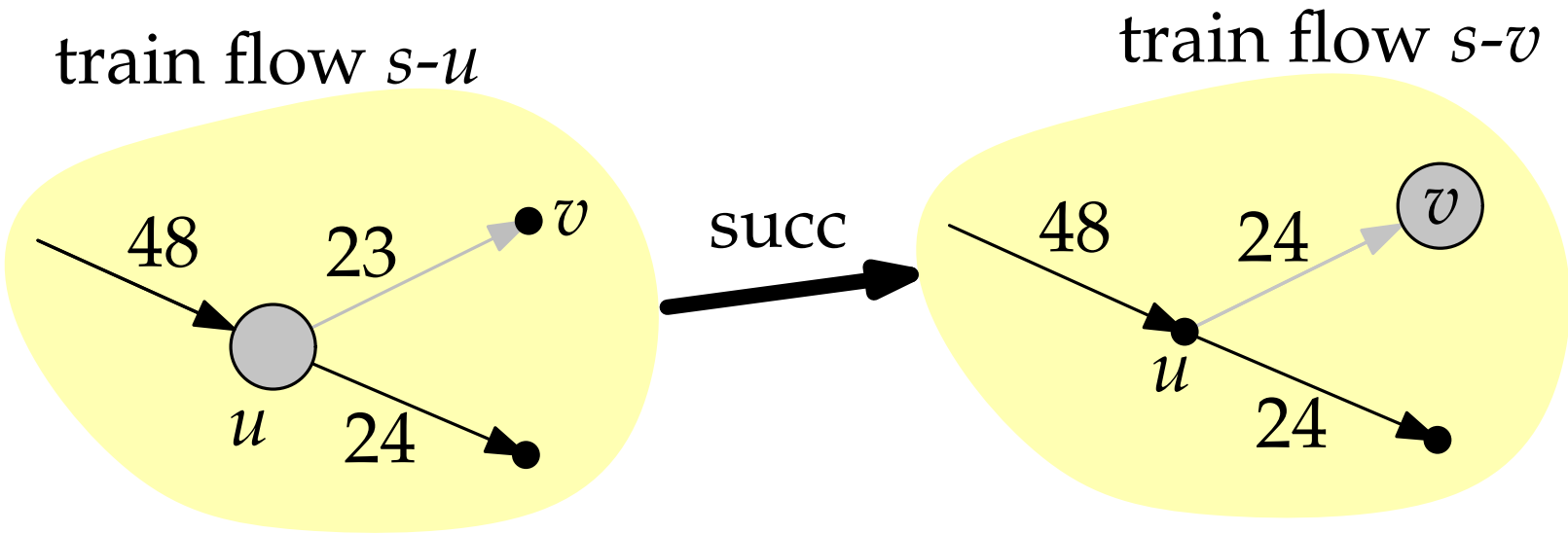
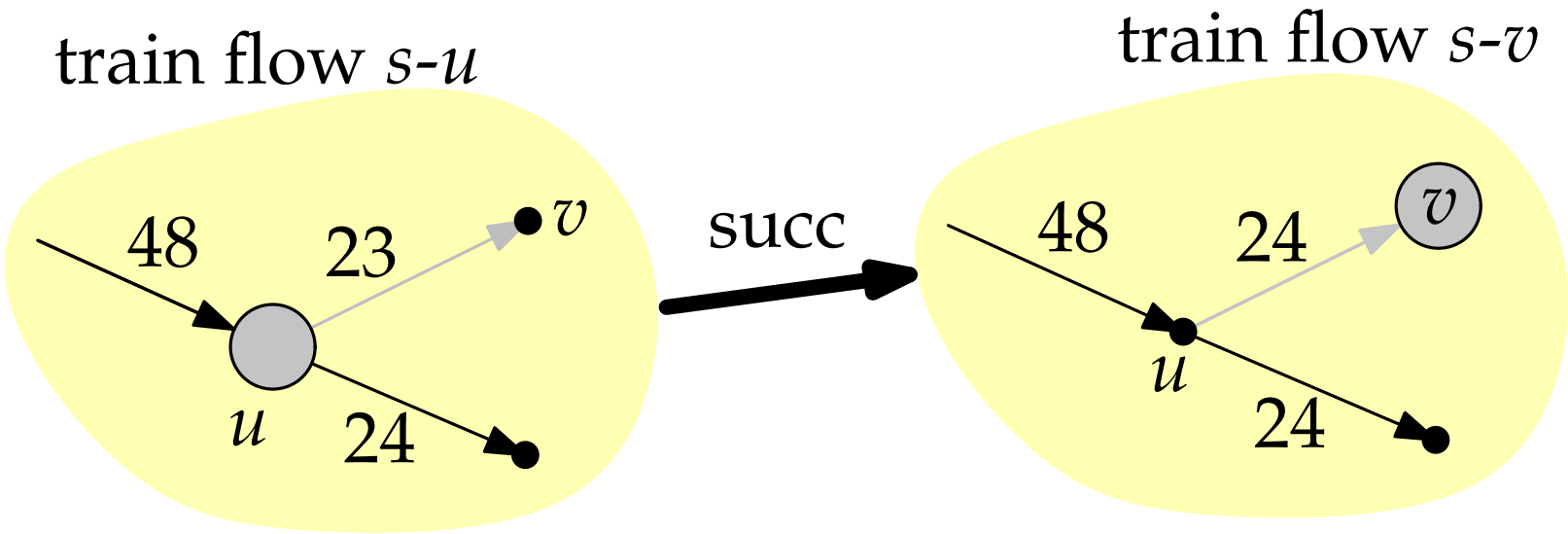

train flow s-u
48
23
v
u
24
succ
train flow s-v
48
24
v
u
24
27
13
u
13
w
succ
27
13
u
14
w

ARRIVAL



Exception: $\mathrm{succ}(f) := f$ when f is a train flow from s to t or to some trap.

ARRIVAL



Exception: $\mathrm{succ}(f) := f$ when f is a train flow from s to t or to some trap.

Theorem (Karthik C.S.). Deciding ARRIVAL is no harder than finding a local maximum of the succ function.
Thus, ARRIVAL is in PLS.

PLS—More Formal Definition

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Informal. A PLS-problem asks us to compute a *solution* that is *locally optimal* with respect to some *neighborhood relation* and a *valuation / cost / revenue* function.

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ARRIVAL

- Solutions: trains flows from s to some u
- Neighbor: successor flow
- Value: sum of numbers

PLS—More Formal Definition

"The" generic PLS problem:

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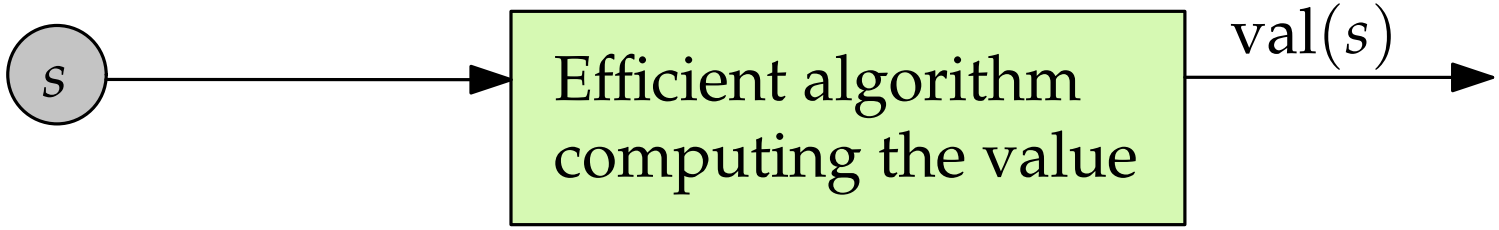

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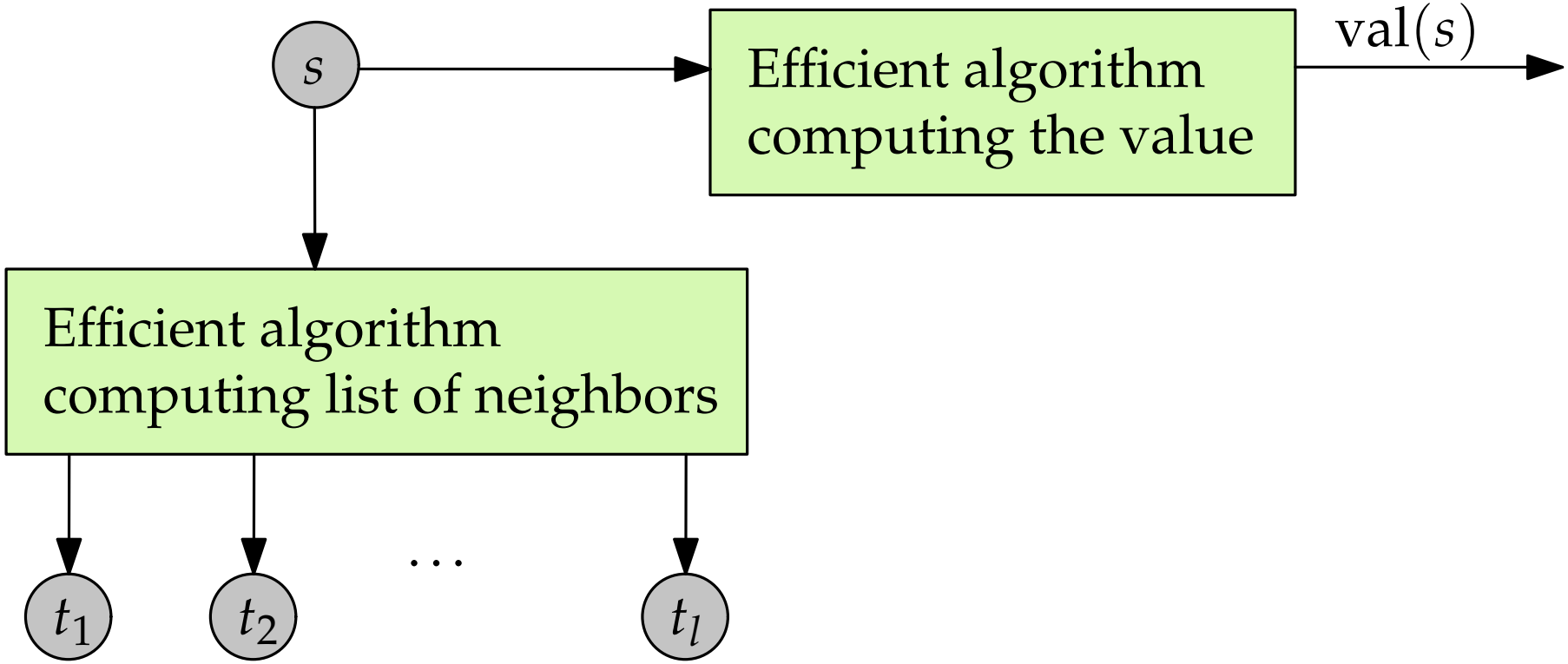

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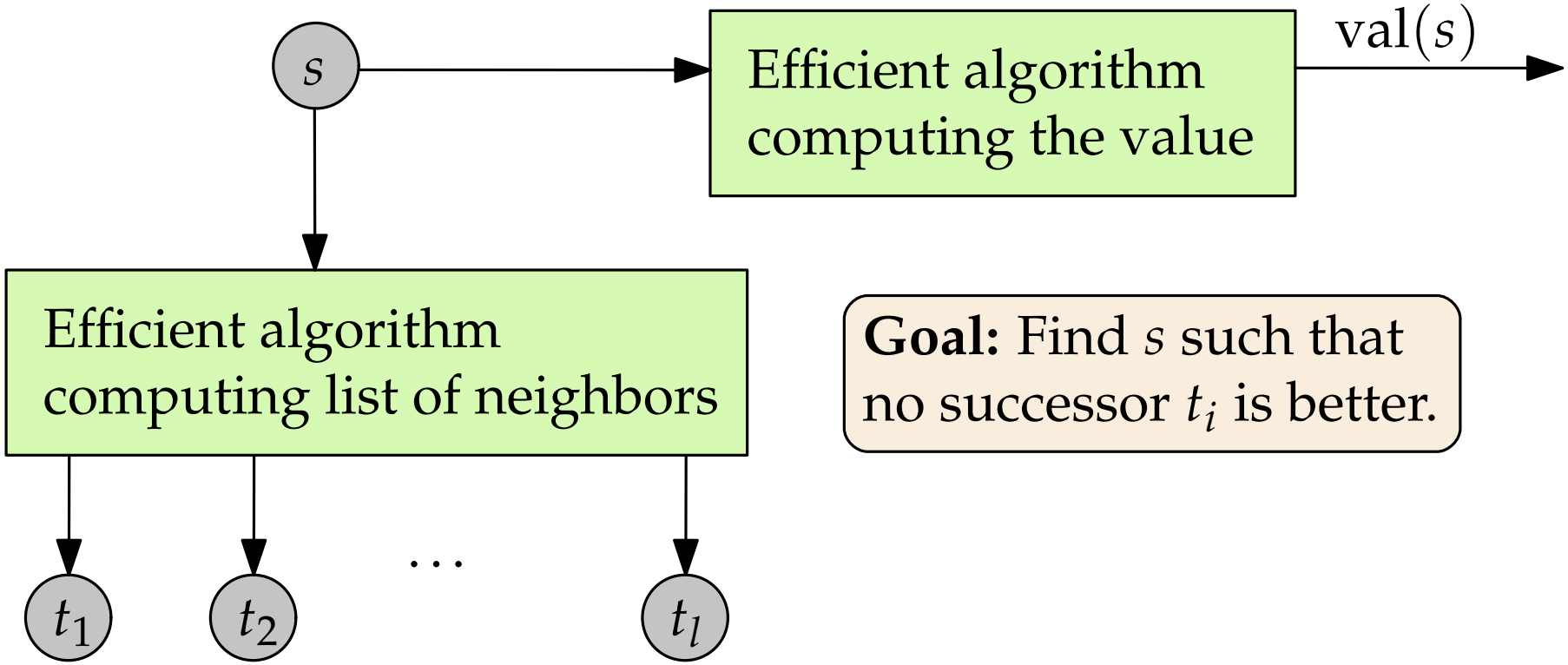

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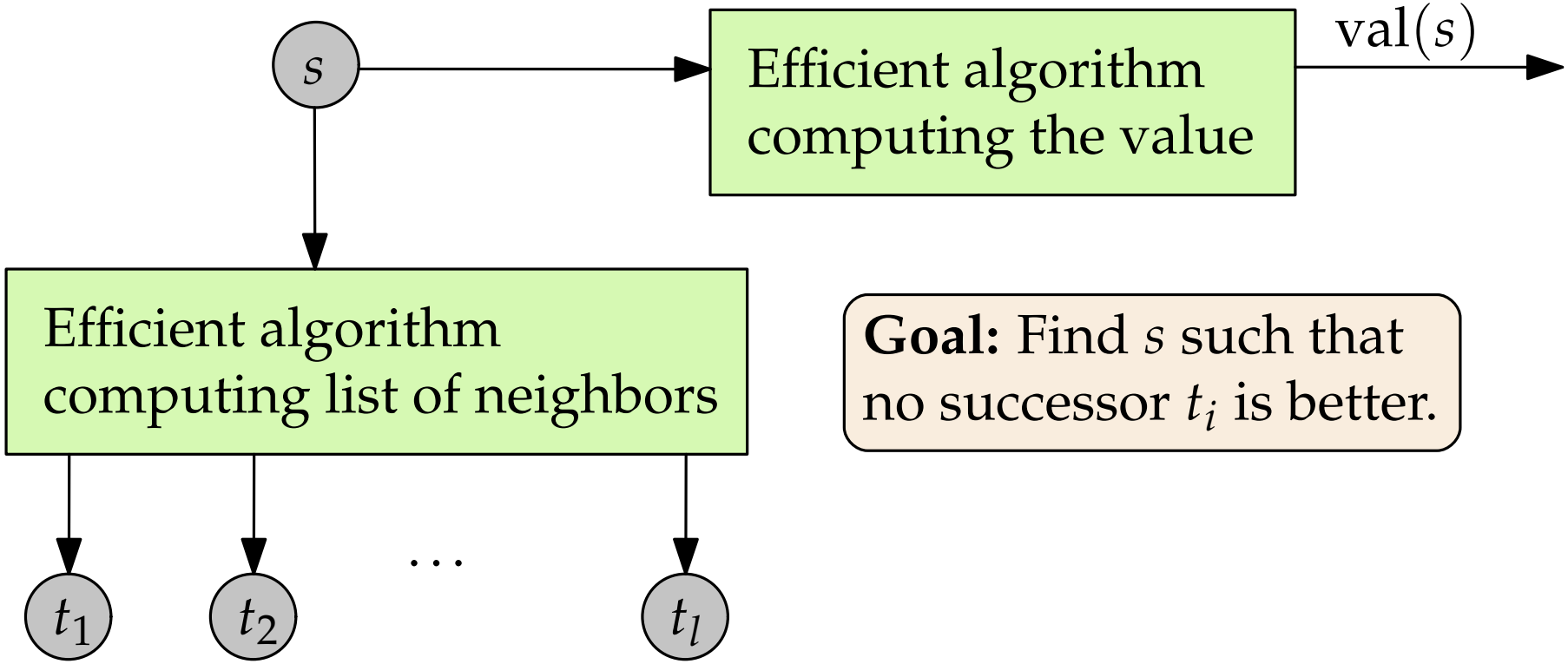

Goal: Find s such that no successor t_i is better.

l should be polynomial in $|s|$.

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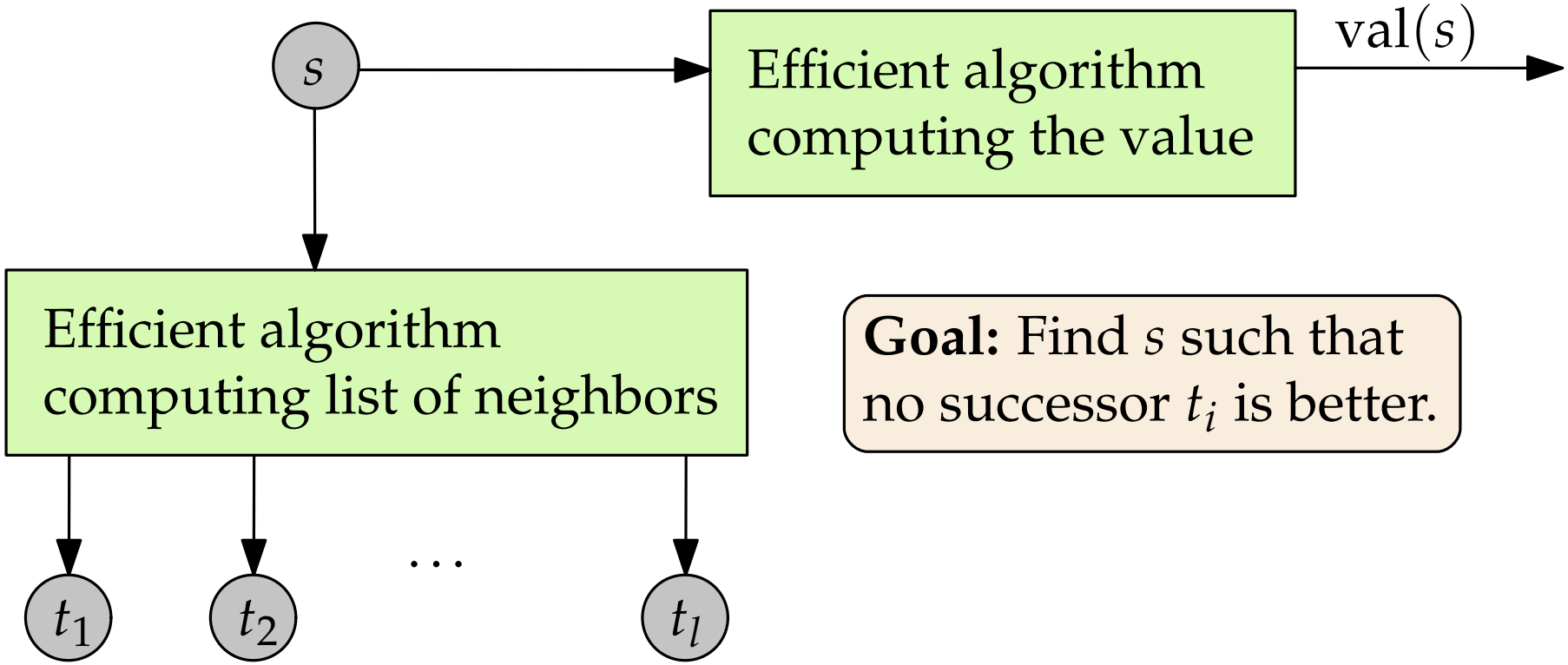

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$|t|$ polynomial in $|s|$ for every "reachable" t.

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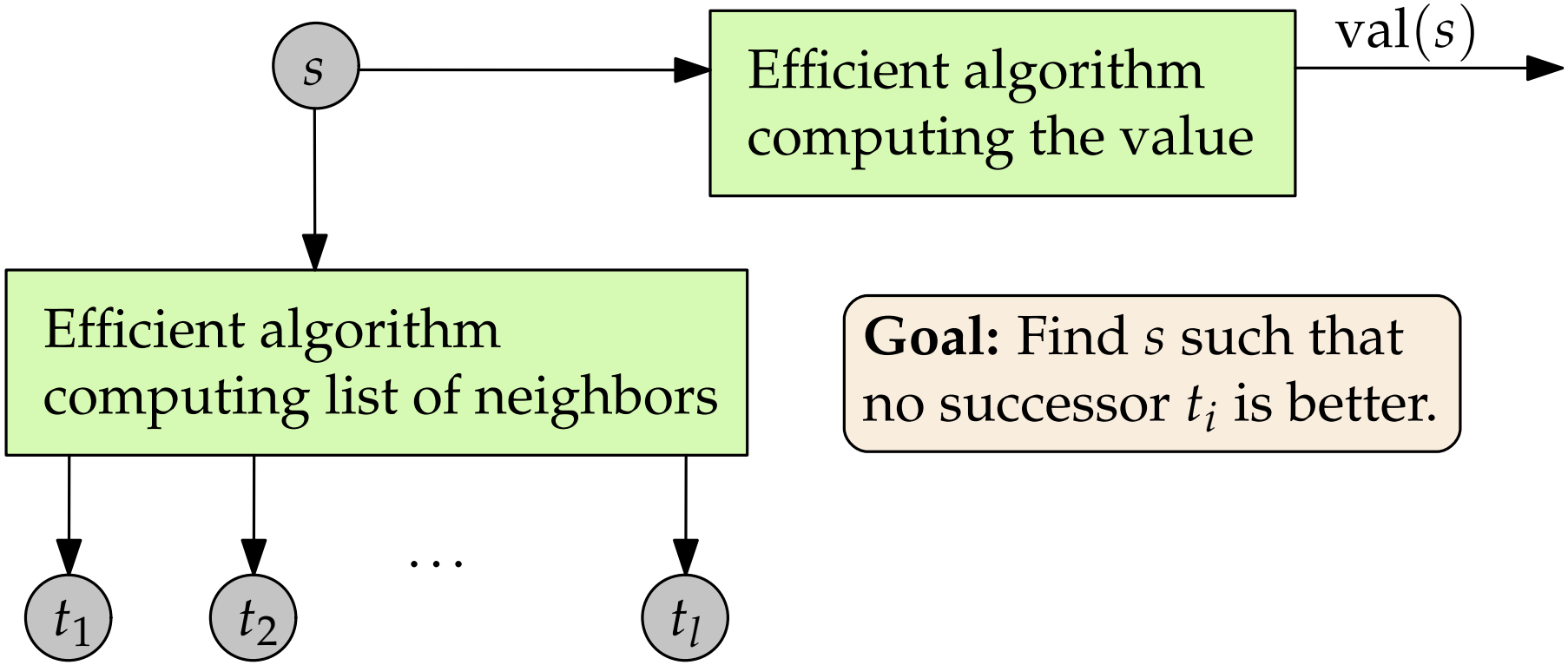

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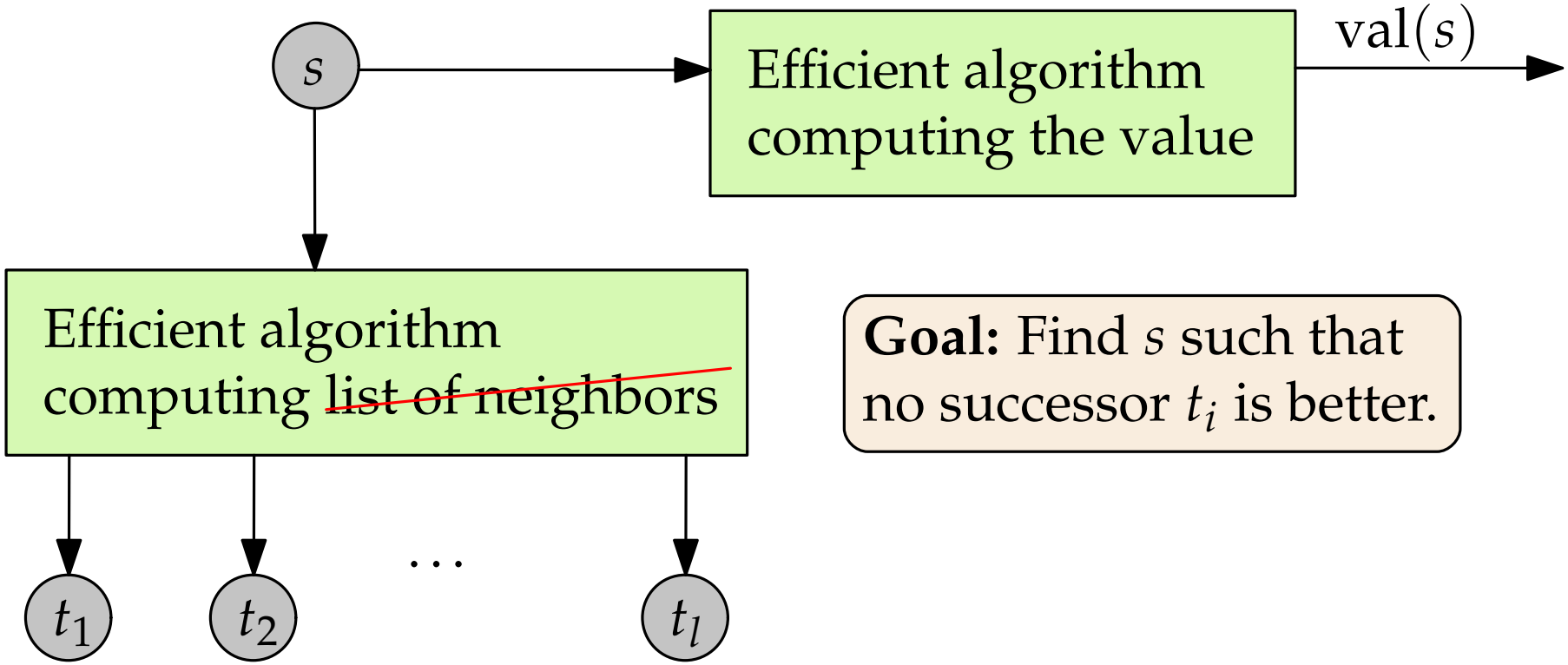

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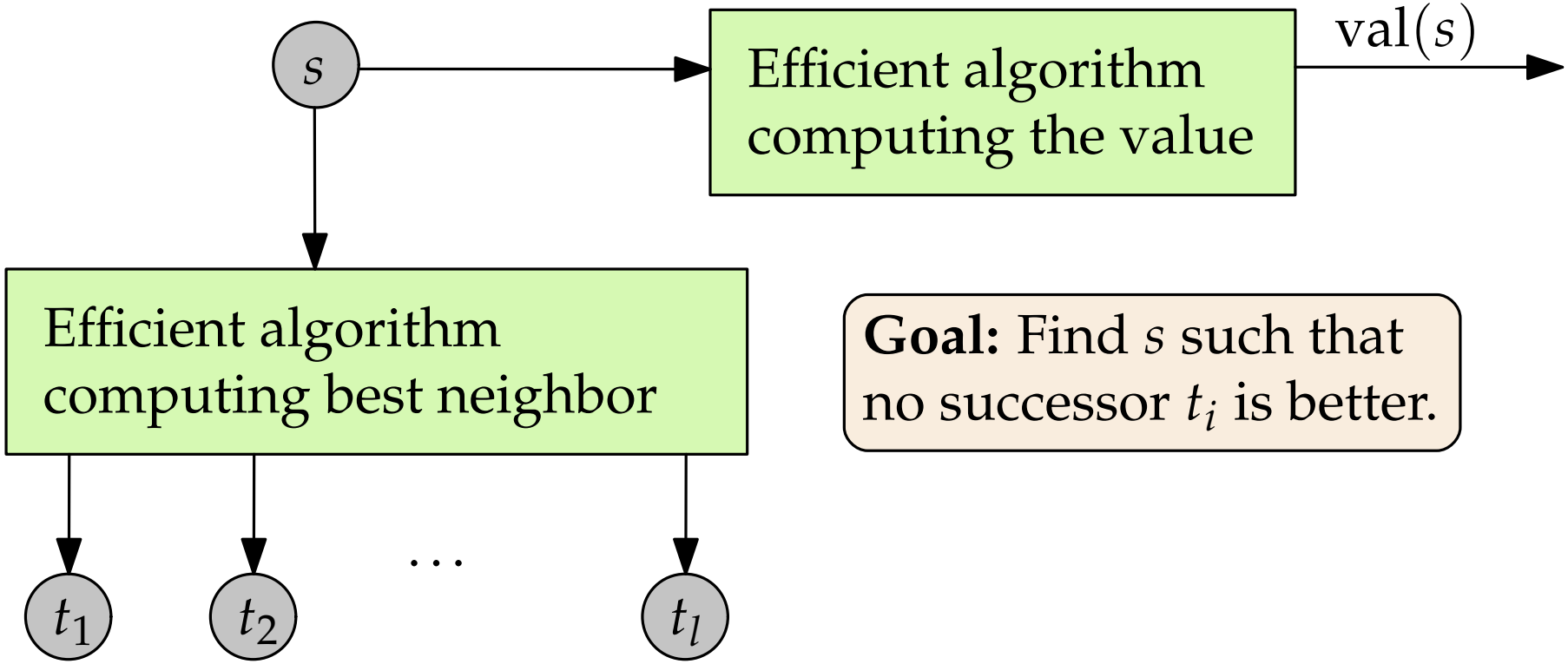

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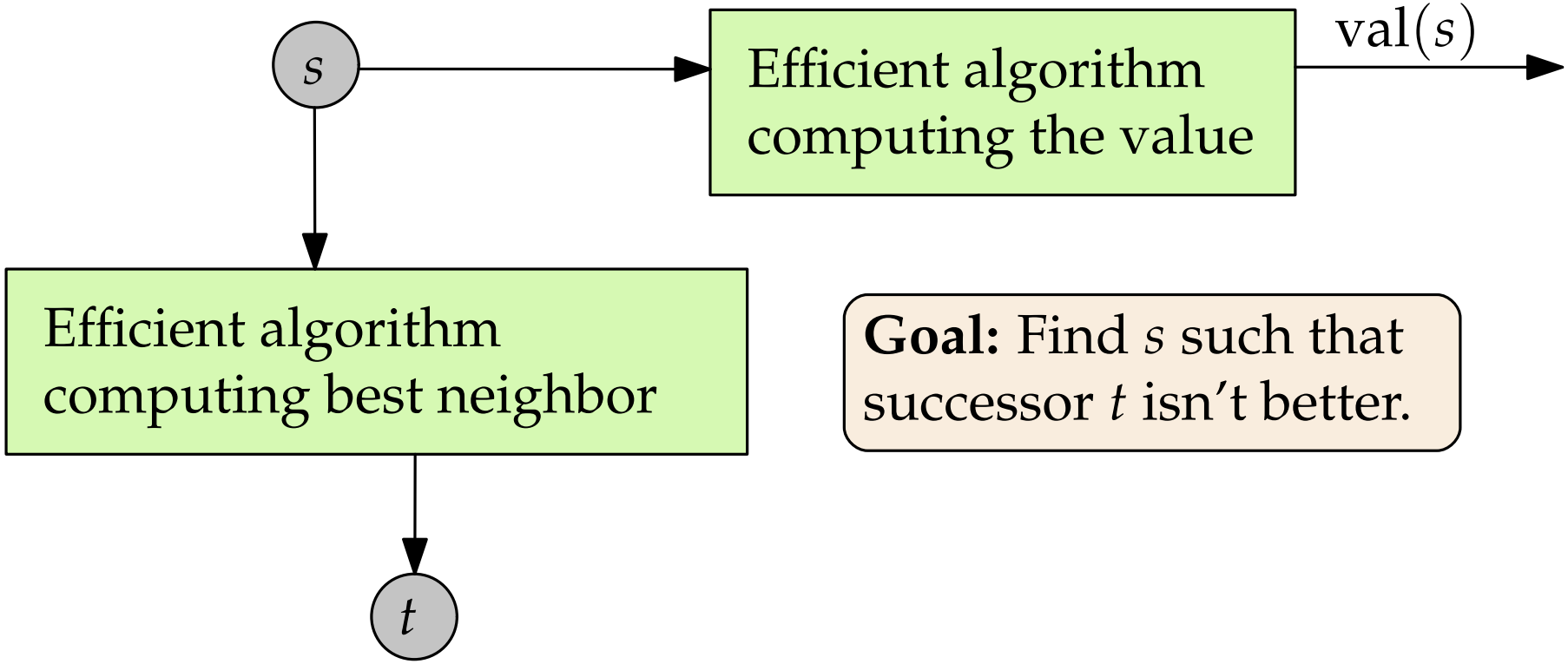

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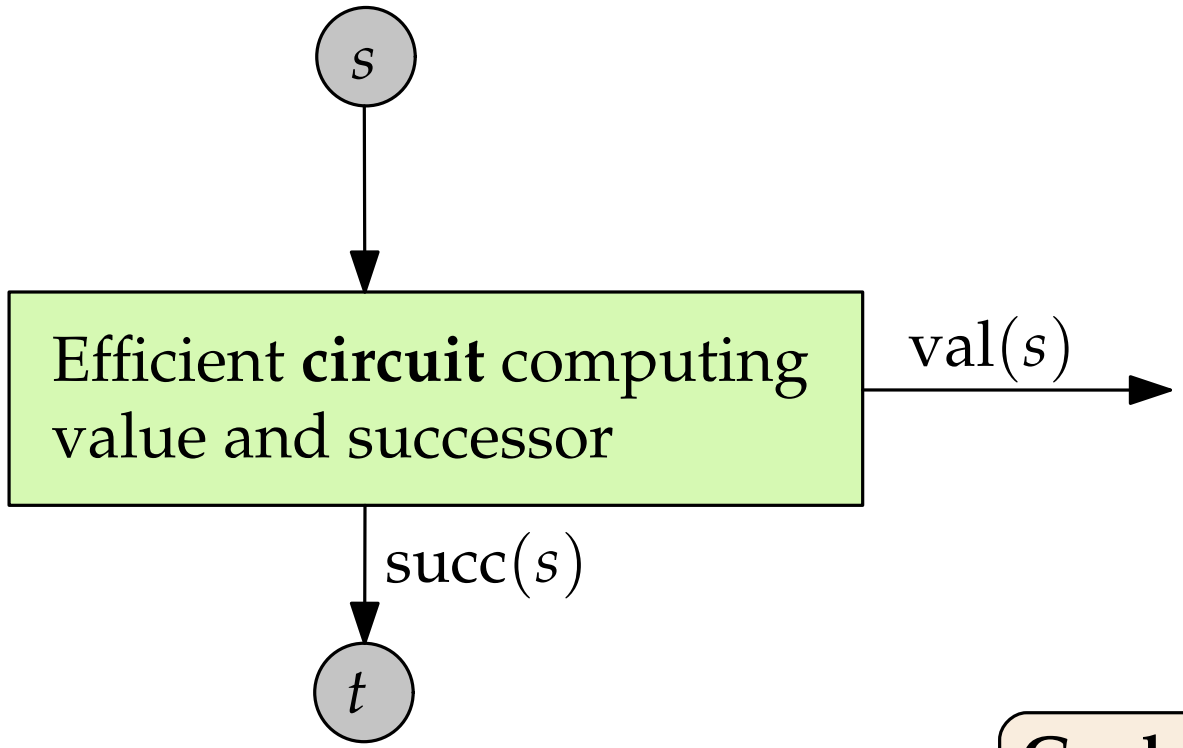
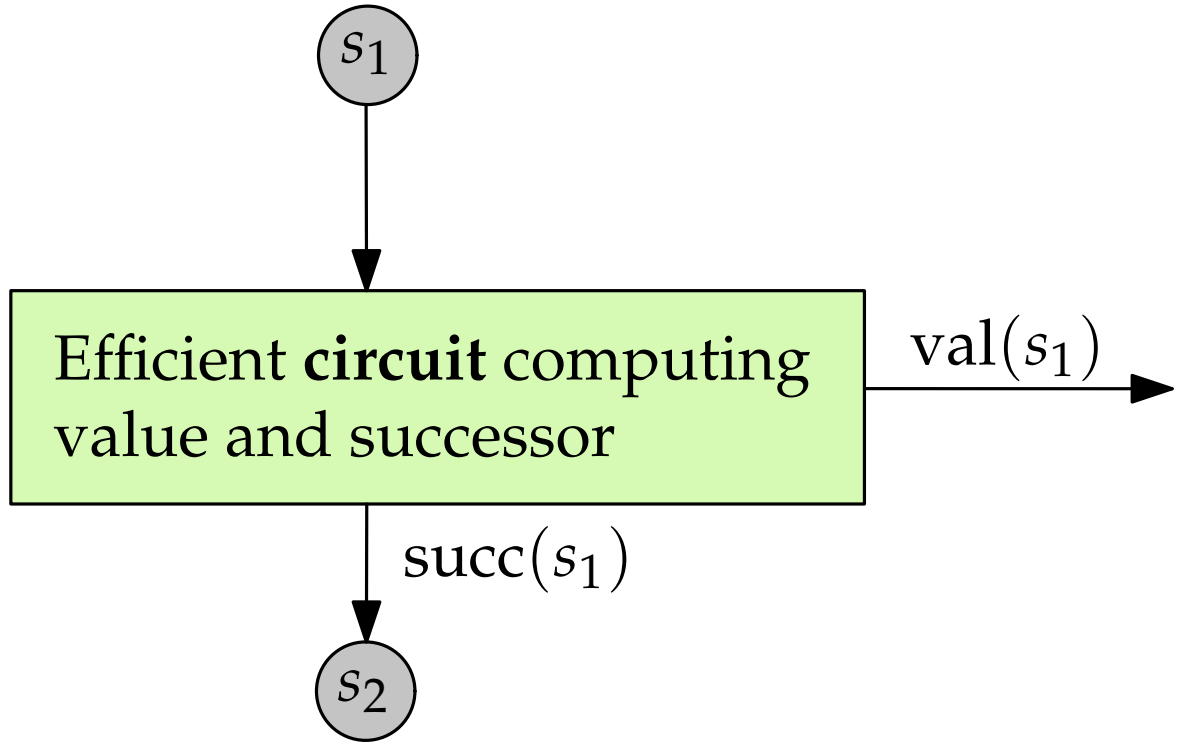


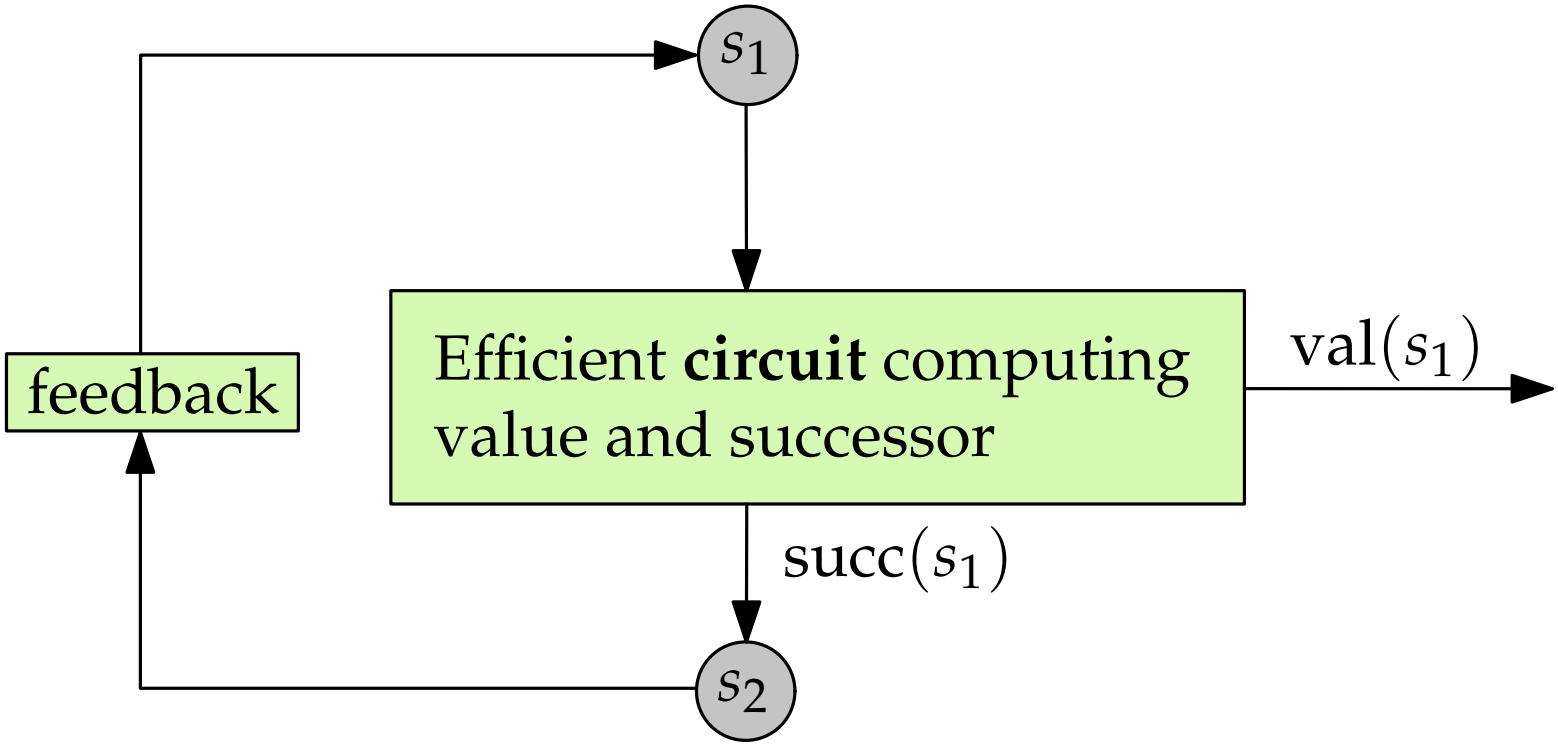
Goal: Find s such that successor t isn't better.

IterateCircuit



Goal: Find s such that $\mathrm{val}(s) \geq \mathrm{val}(\mathrm{succ}(s))$

IterateCircuit



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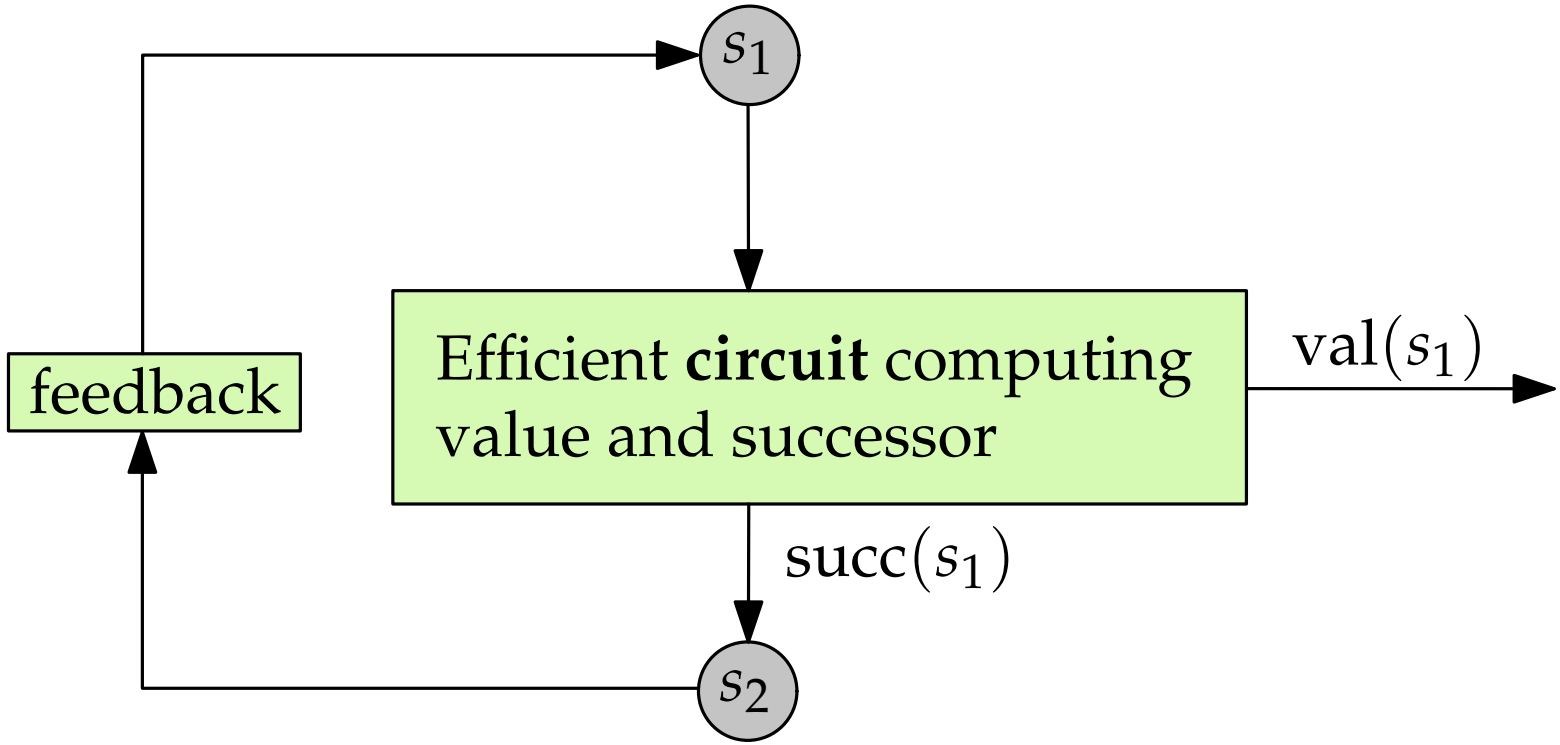

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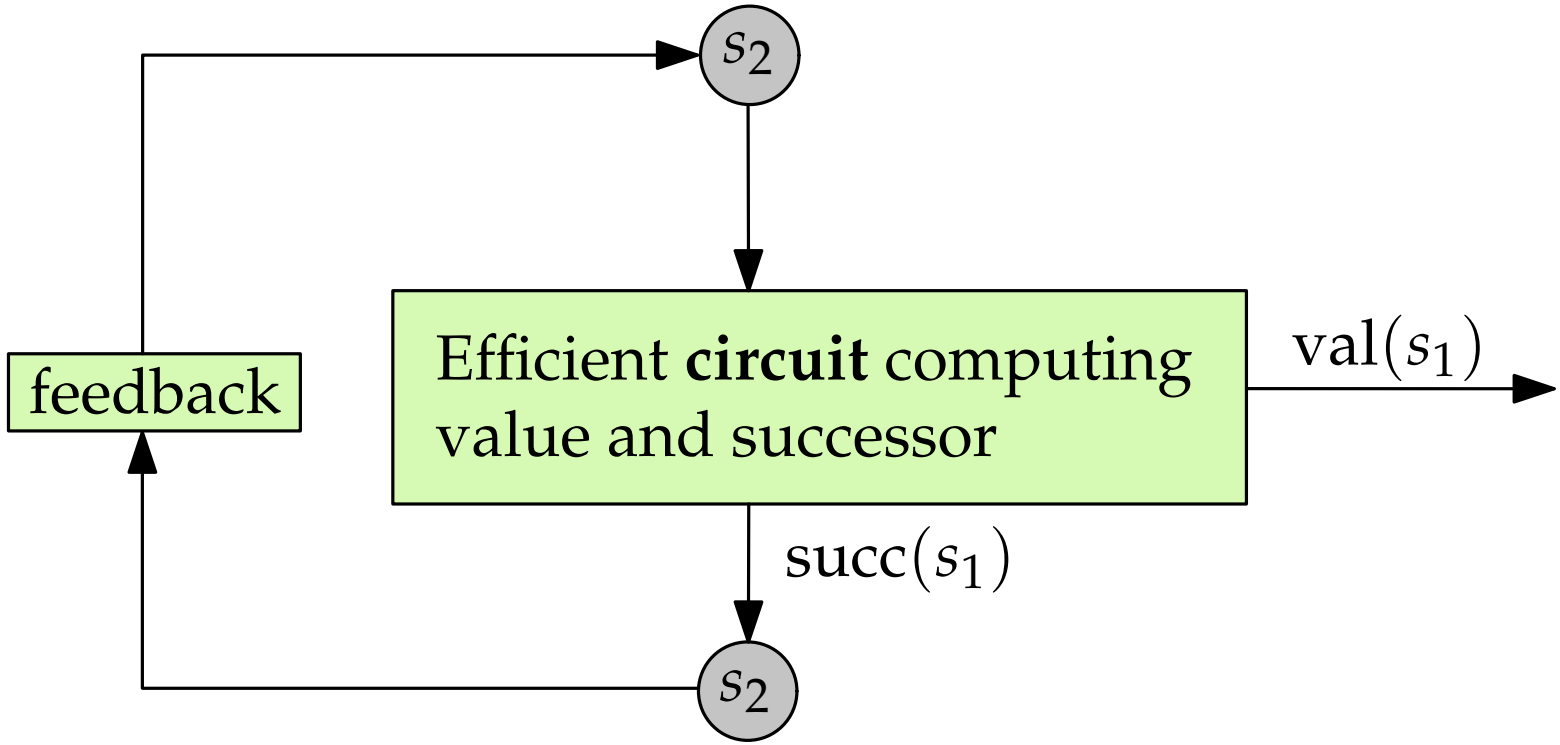

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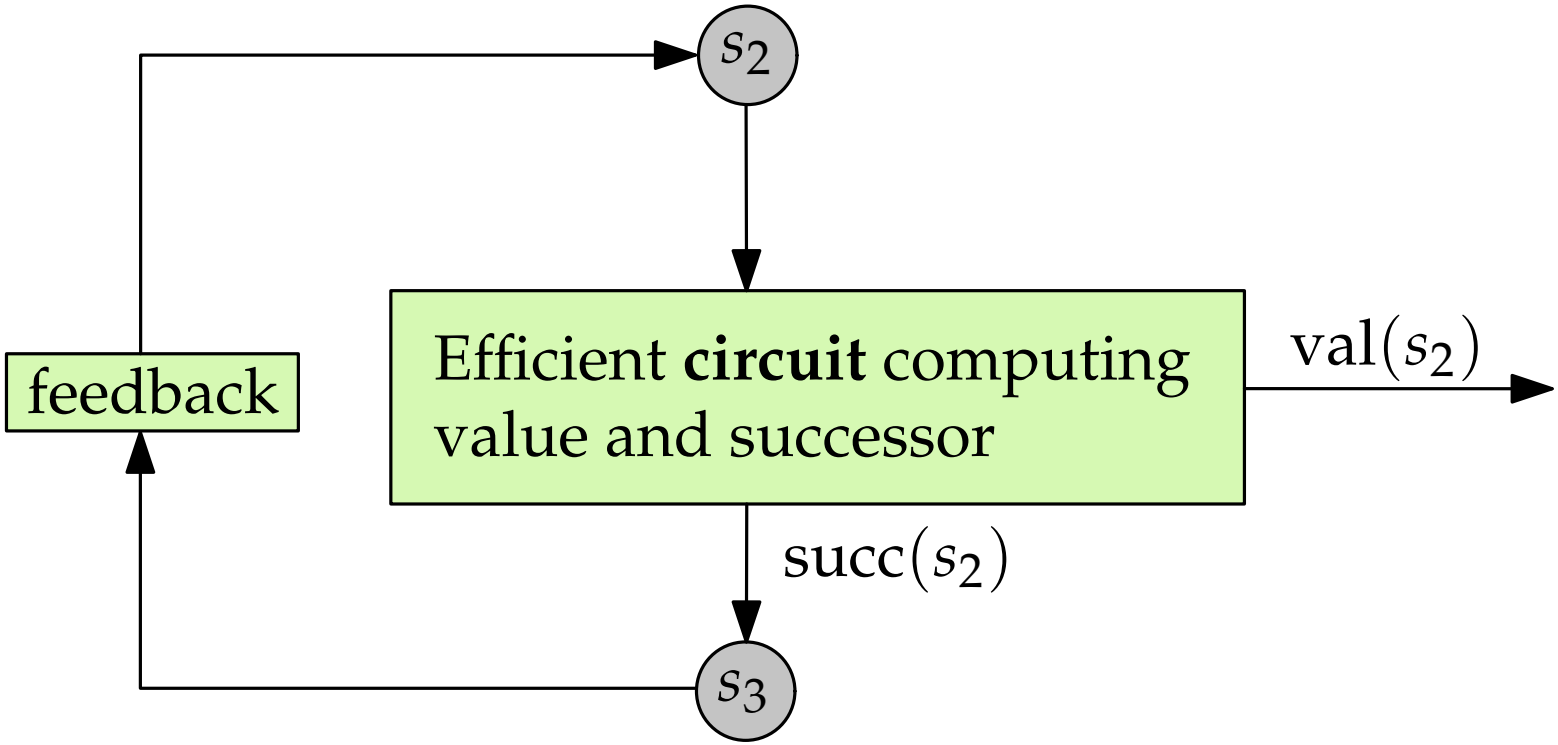

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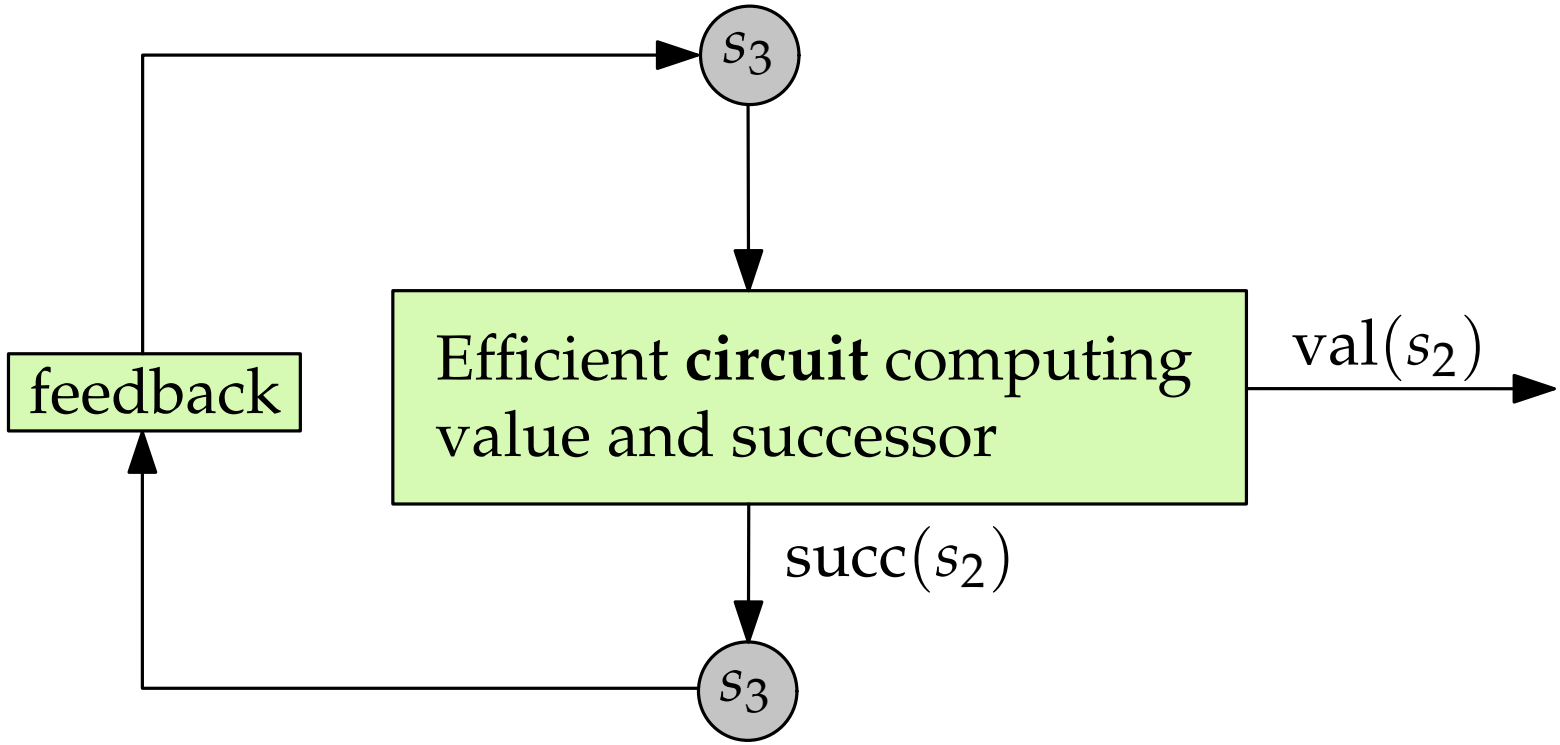

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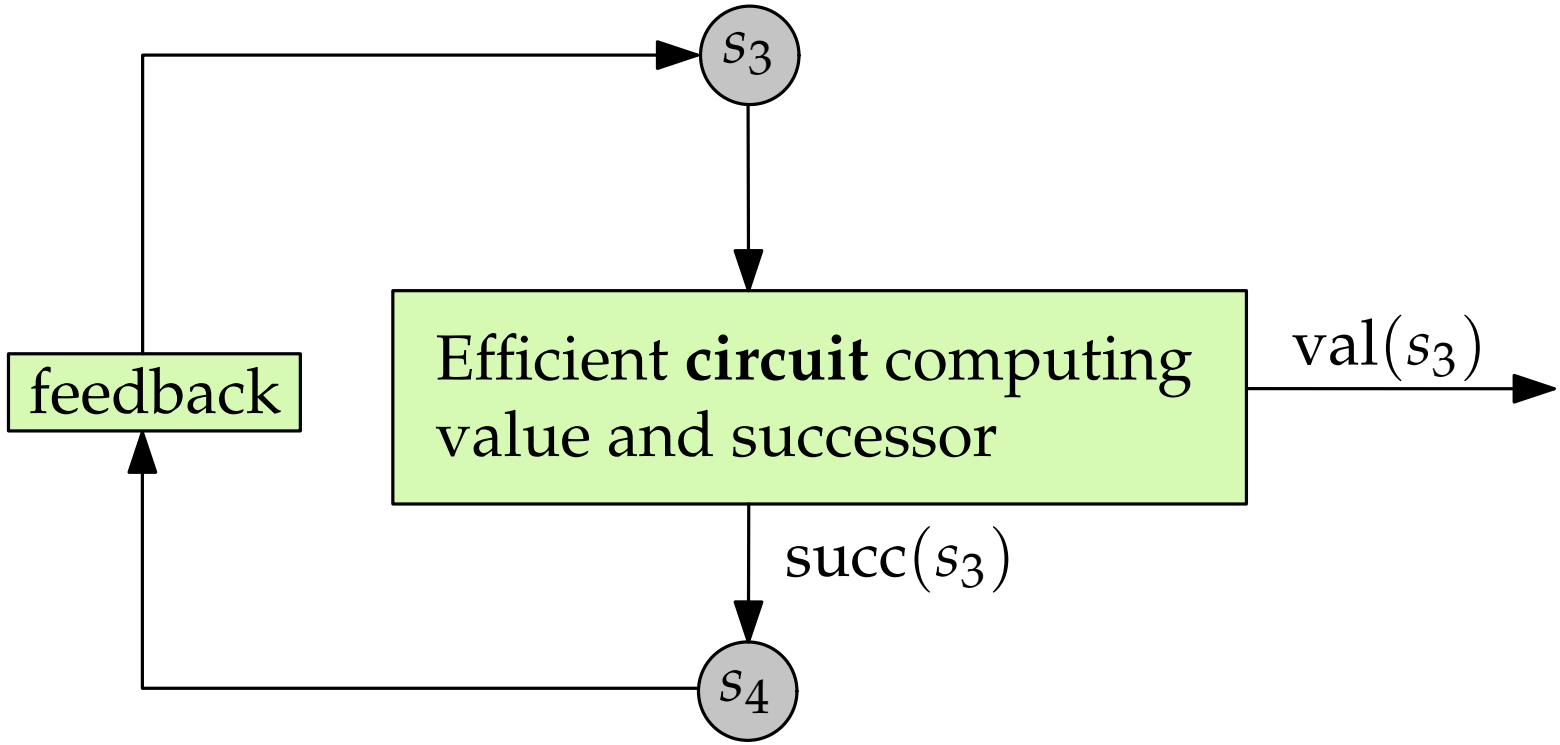

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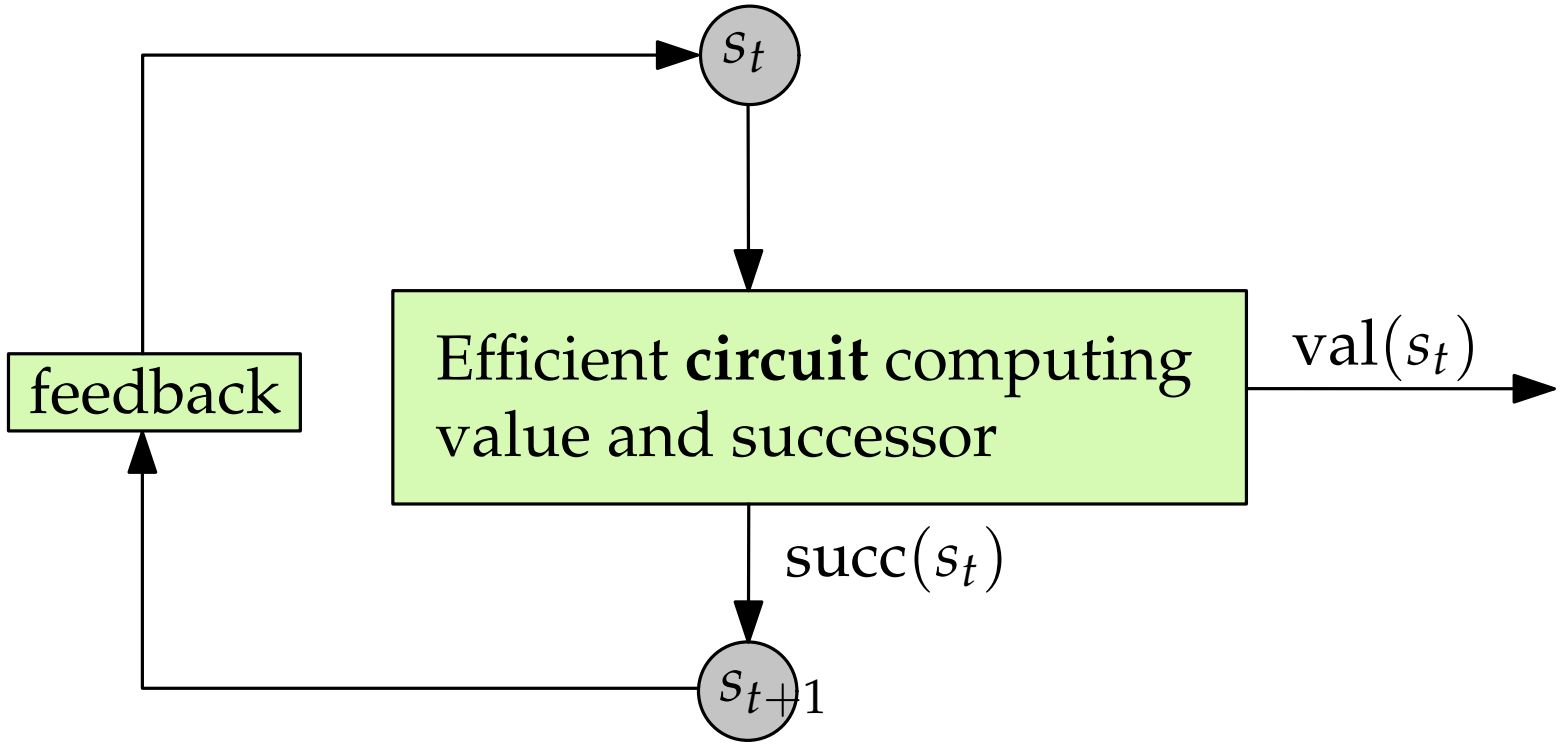

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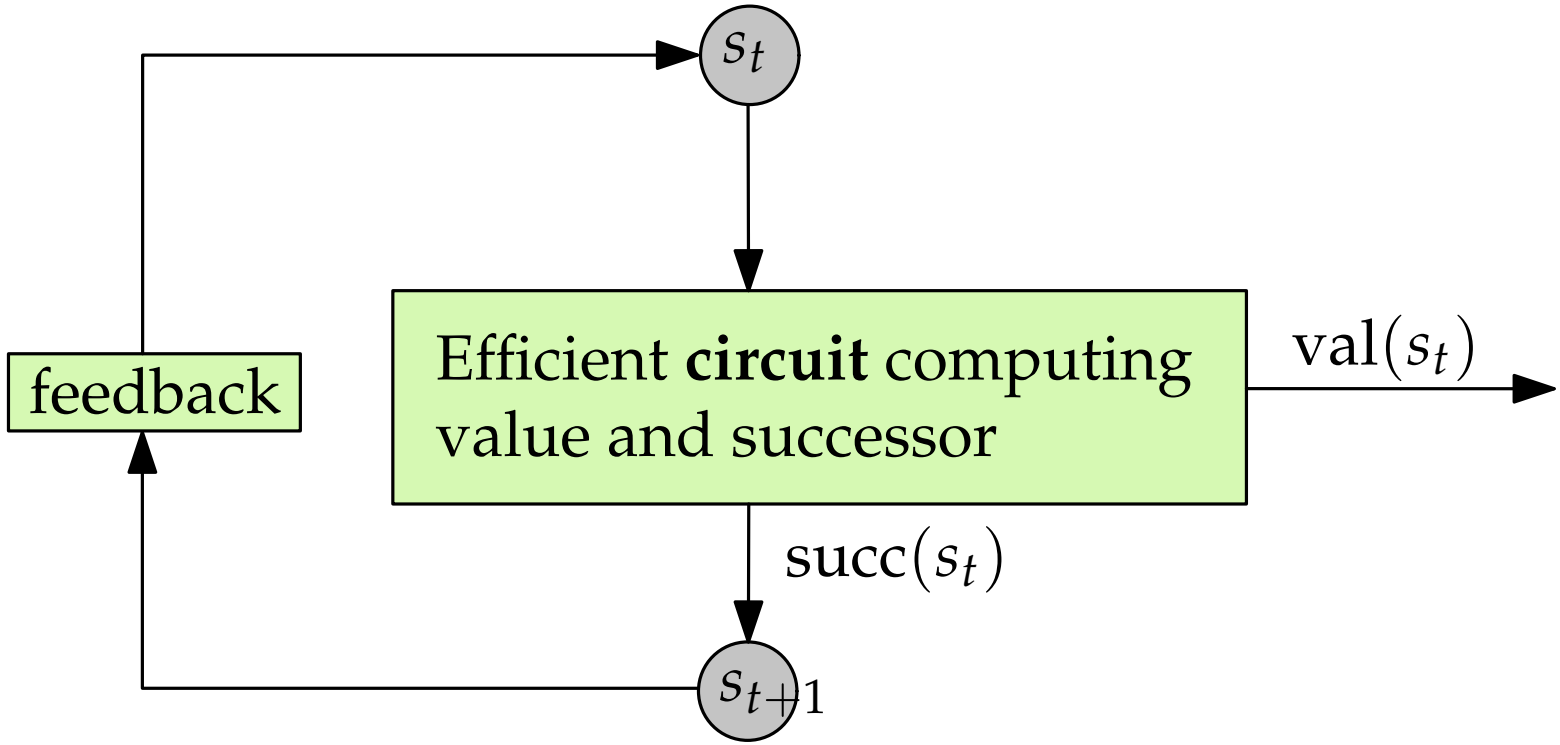

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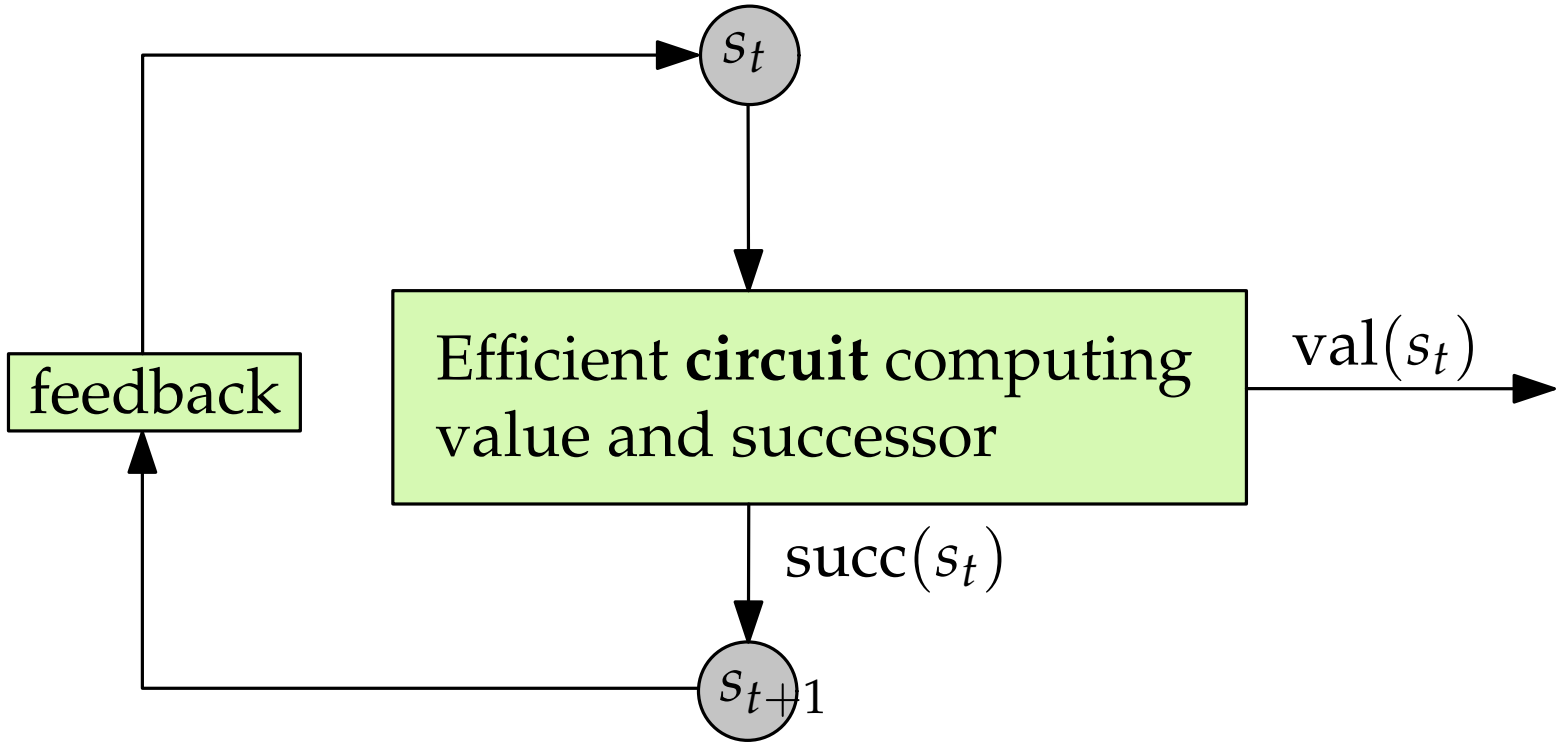

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PLS $\subseteq$ P not known, conjectured to be false.

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a propositional formula in conjunctive normal form
(a CNF formula)

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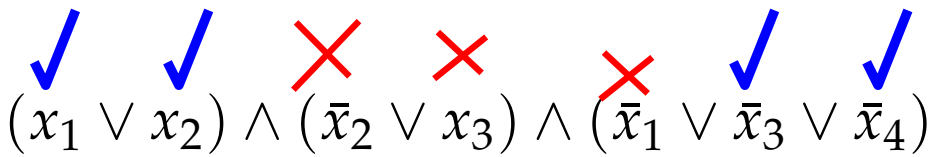

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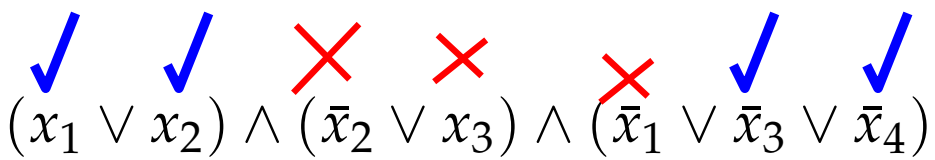

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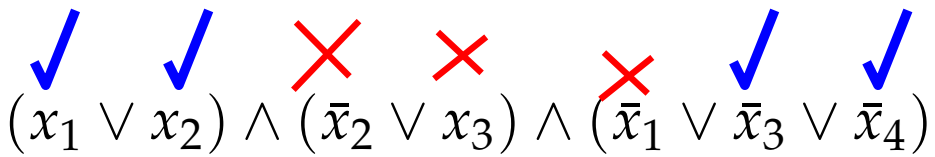

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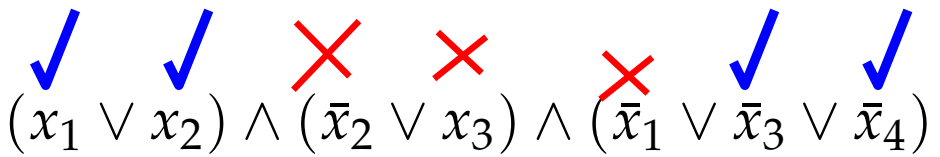

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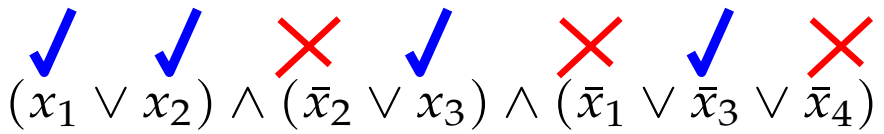

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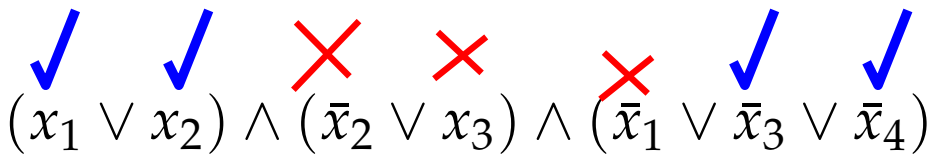

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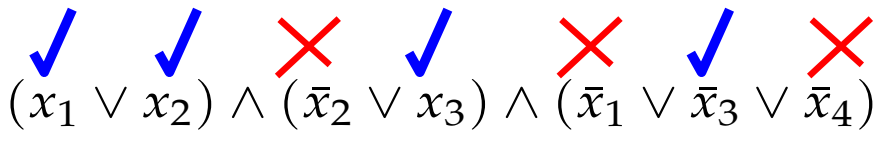

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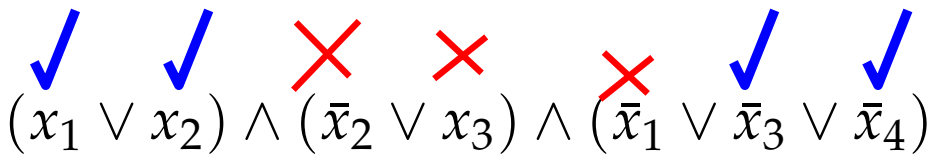
SATISFIABILITY (SAT)

a propositional formula in conjunctive normal form
(a CNF formula)

$$(x_1 \vee x_2) \wedge (\bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4)$$

$x_1 \mapsto 1$
$x_2 \mapsto 1$
$x_3 \mapsto 0$
$x_4 \mapsto 0$

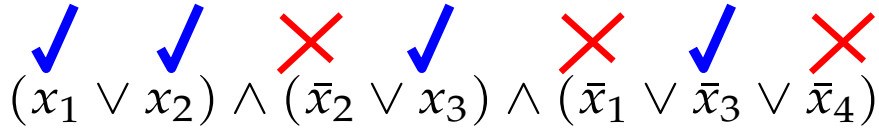
a truth assignment α



$F(\alpha) = 0$

$x_1 \mapsto 1$
$x_2 \mapsto 1$
$x_3 \mapsto 1$
$x_4 \mapsto 0$

a truth assignment β
a *satisfying assignment*



$F(\beta) = 1$

SATISFIABILITY (SAT)

a propositional formula in conjunctive normal form
(a CNF formula)

largest clause: 3

$$(x_1 \vee x_2) \wedge (\bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4)$$

$x_1 \mapsto 1$
$x_2 \mapsto 1$
$x_3 \mapsto 0$
$x_4 \mapsto 0$

a truth assignment α

✓ ✓ ✗ ✗ ✗ ✓ ✓

$$(x_1 \vee x_2) \wedge (\bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4)$$

$F(\alpha) = 0$

$x_1 \mapsto 1$
$x_2 \mapsto 1$
$x_3 \mapsto 1$
$x_4 \mapsto 0$

a truth assignment β

a *satisfying assignment*

✓ ✓ ✗ ✓ ✗ ✓ ✗

$$(x_1 \vee x_2) \wedge (\bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4)$$

$F(\beta) = 1$

3-SATISFIABILITY (3-SAT)

a propositional formula in conjunctive normal form
(a 3-CNF formula)

largest clause: 3

$$(x_1 \vee x_2) \wedge (\bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4)$$

$x_1 \mapsto 1$
$x_2 \mapsto 1$
$x_3 \mapsto 0$
$x_4 \mapsto 0$

a truth assignment α

✓ ✓ ✗ ✗ ✗ ✓ ✓
$(x_1 \vee x_2) \wedge (\bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4)$

$F(\alpha) = 0$

$x_1 \mapsto 1$
$x_2 \mapsto 1$
$x_3 \mapsto 1$
$x_4 \mapsto 0$

a truth assignment β

a *satisfying assignment*

✓ ✓ ✗ ✓ ✗ ✓ ✗
$(x_1 \vee x_2) \wedge (\bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4)$

$F(\beta) = 1$

Problem (k-SAT). Given a k-CNF formula F, find a truth assignment α such that $F(\alpha) = 1$.

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Problem (CIRCUIT-SAT). Given a Boolean circuit C, find a truth assignment α such that $C(\alpha) = 1$.

Problem (k-SAT). Given a k-CNF formula F, find a truth assignment α such that $F(\alpha) = 1$.

Problem (CIRCUIT-SAT). Given a Boolean circuit C, find a truth assignment α such that $C(\alpha) = 1$.

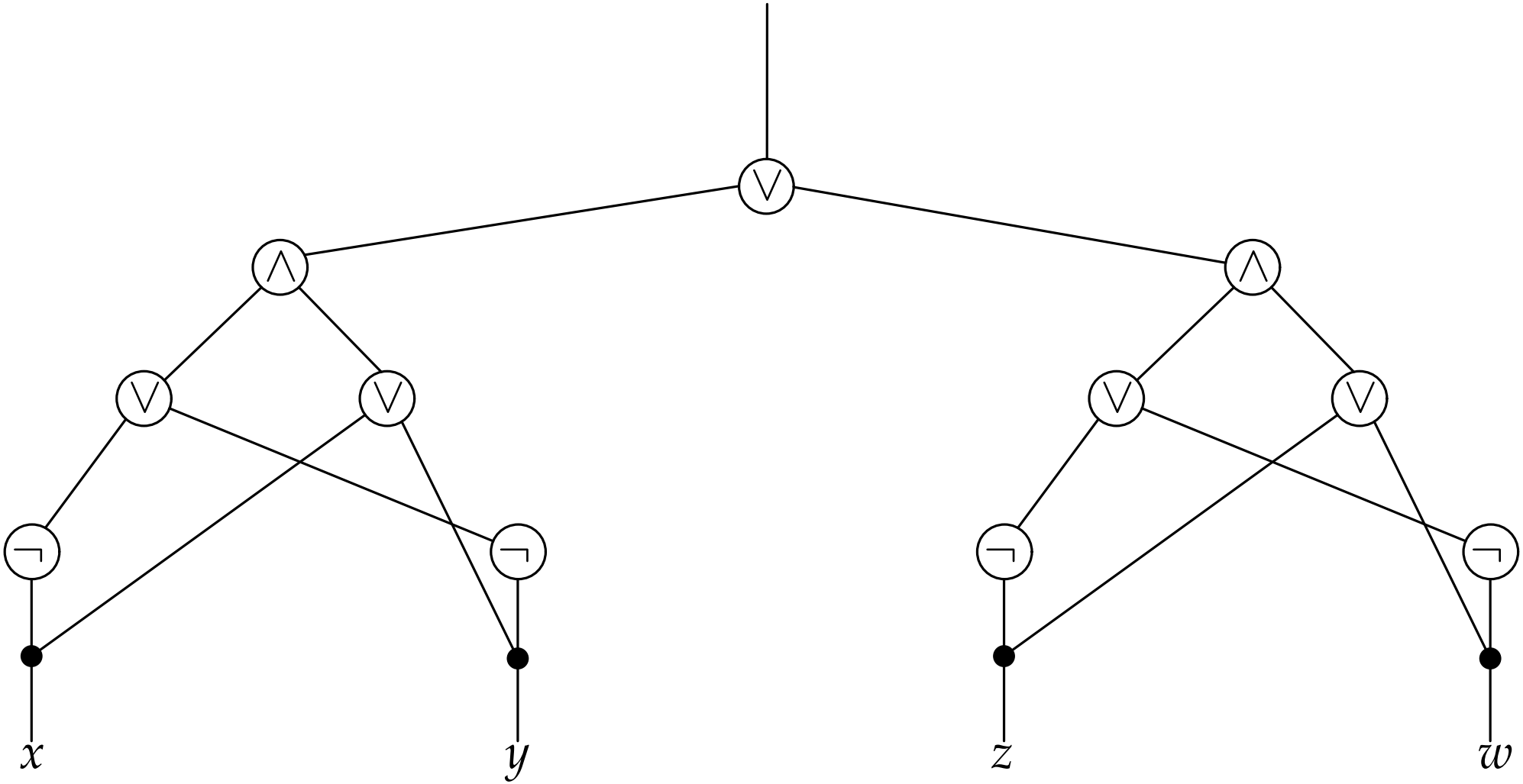
Theorem (Cook-Levin). CIRCUIT-SAT can be reduced to 3-SAT.

Problem (k-SAT). Given a k-CNF formula F, find a truth assignment α such that $F(\alpha) = 1$.

Problem (CIRCUIT-SAT). Given a Boolean circuit C, find a truth assignment α such that $C(\alpha) = 1$.

Theorem (Cook-Levin). CIRCUIT-SAT can be reduced to 3-SAT. In particular: we can transform a circuit $C(x)$ into a 3-CNF formula $F(x, y)$ such that

$$\forall x \in \{0,1\}^n : C(x) = 1 \iff \exists y \in \{0,1\}^m : F(x,y) = 1 \,.$$


∨
∧
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∨
∨
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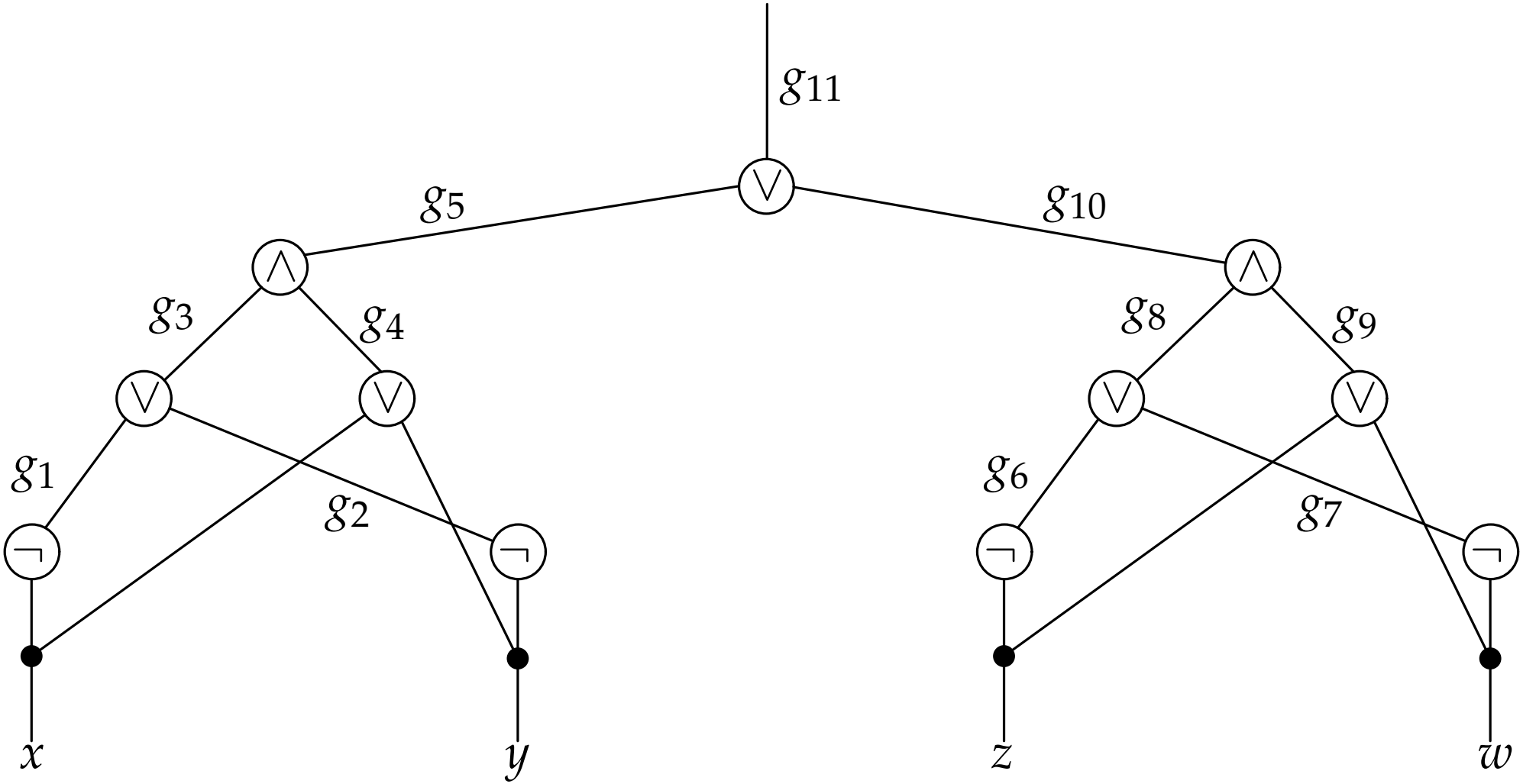
x
y
z
w


g_{11}
g_5
g_{10}
g_3
g_4
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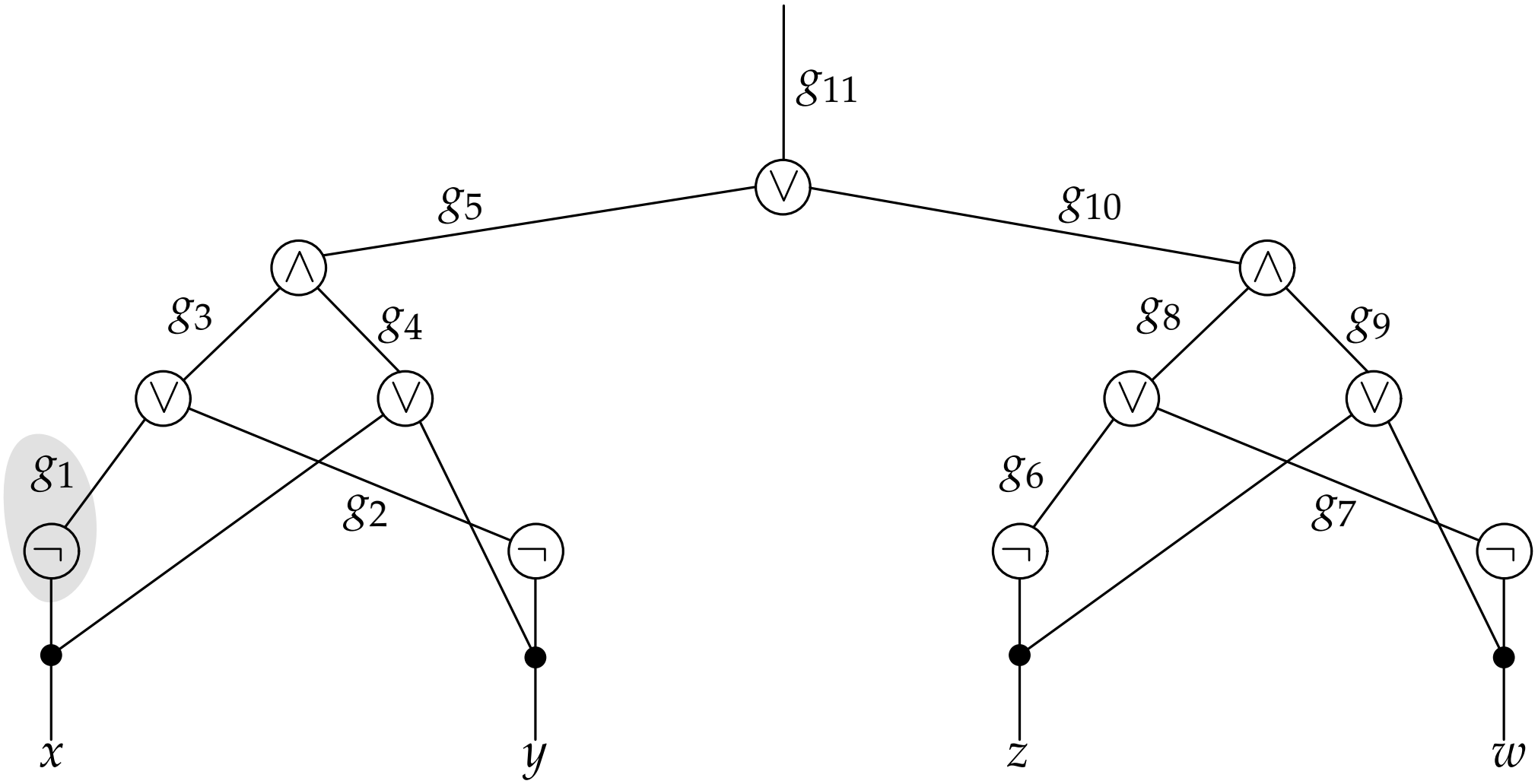
g_7
x
y
z
w


g_{11}
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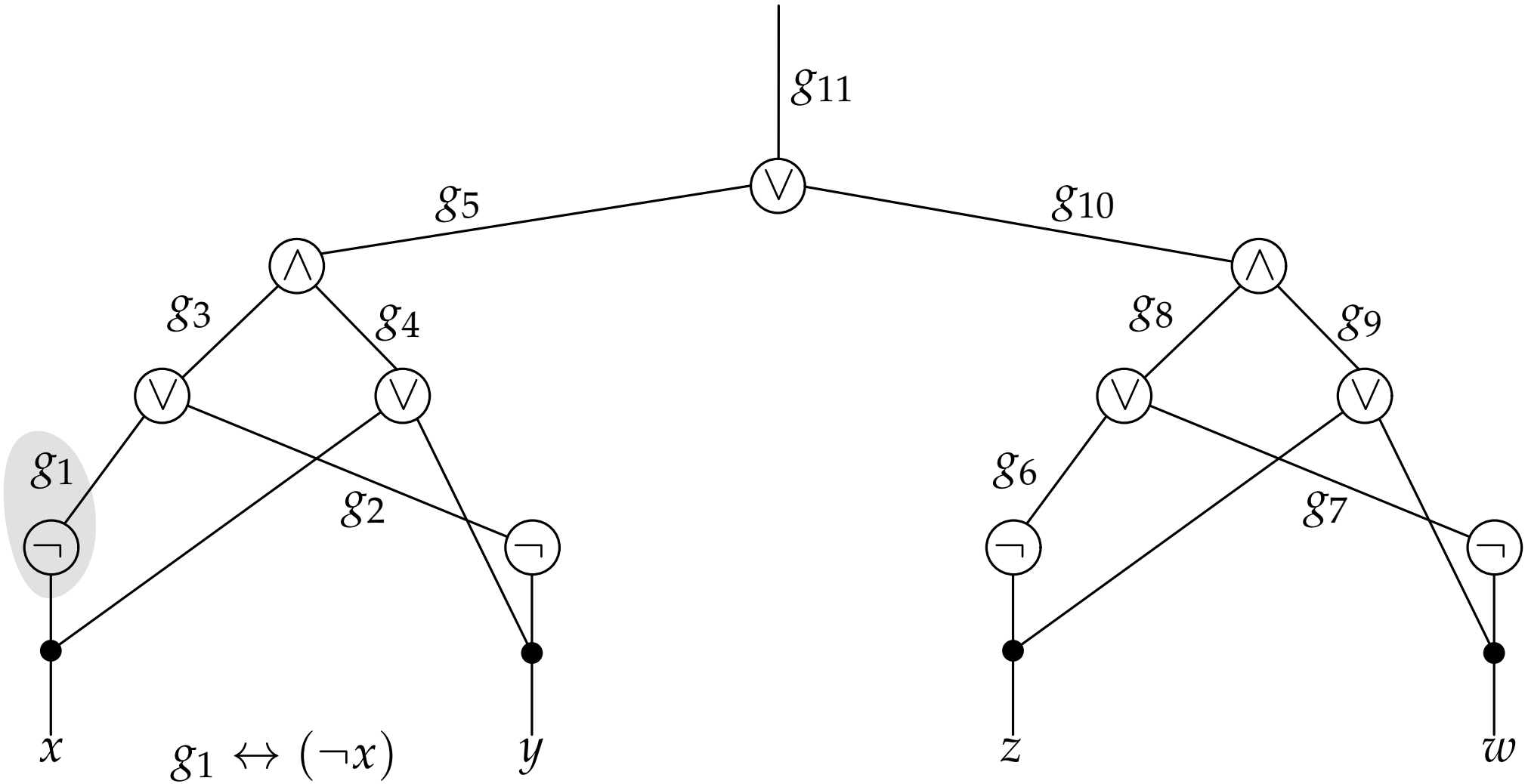
g_{11}
g_5
g_{10}
g_3
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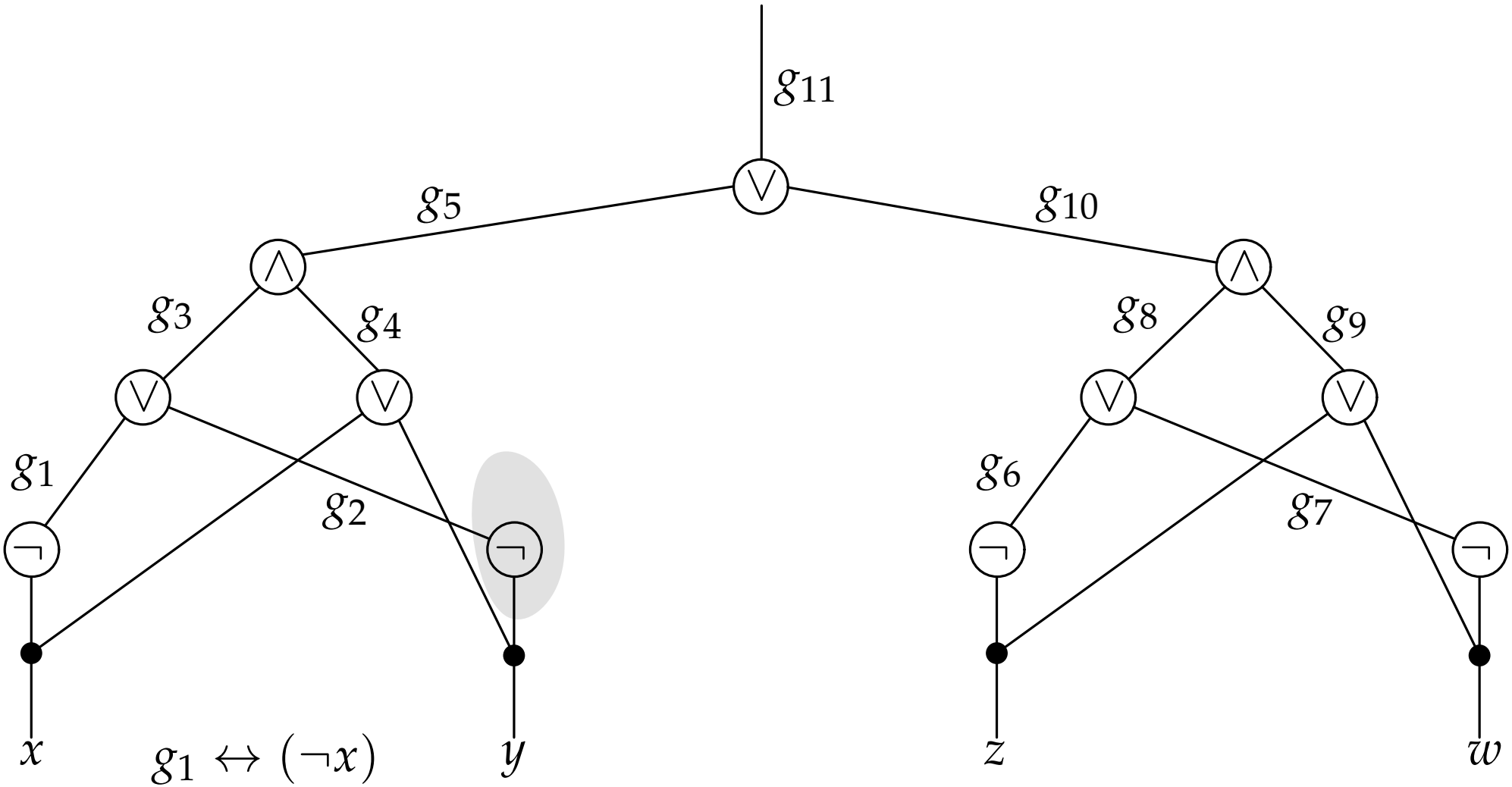
g_2
g_6
g_7
x
y
z
w
$g_1 \leftrightarrow (\neg x)$


g_{11}
g_5
g_{10}
g_3
g_4
g_8
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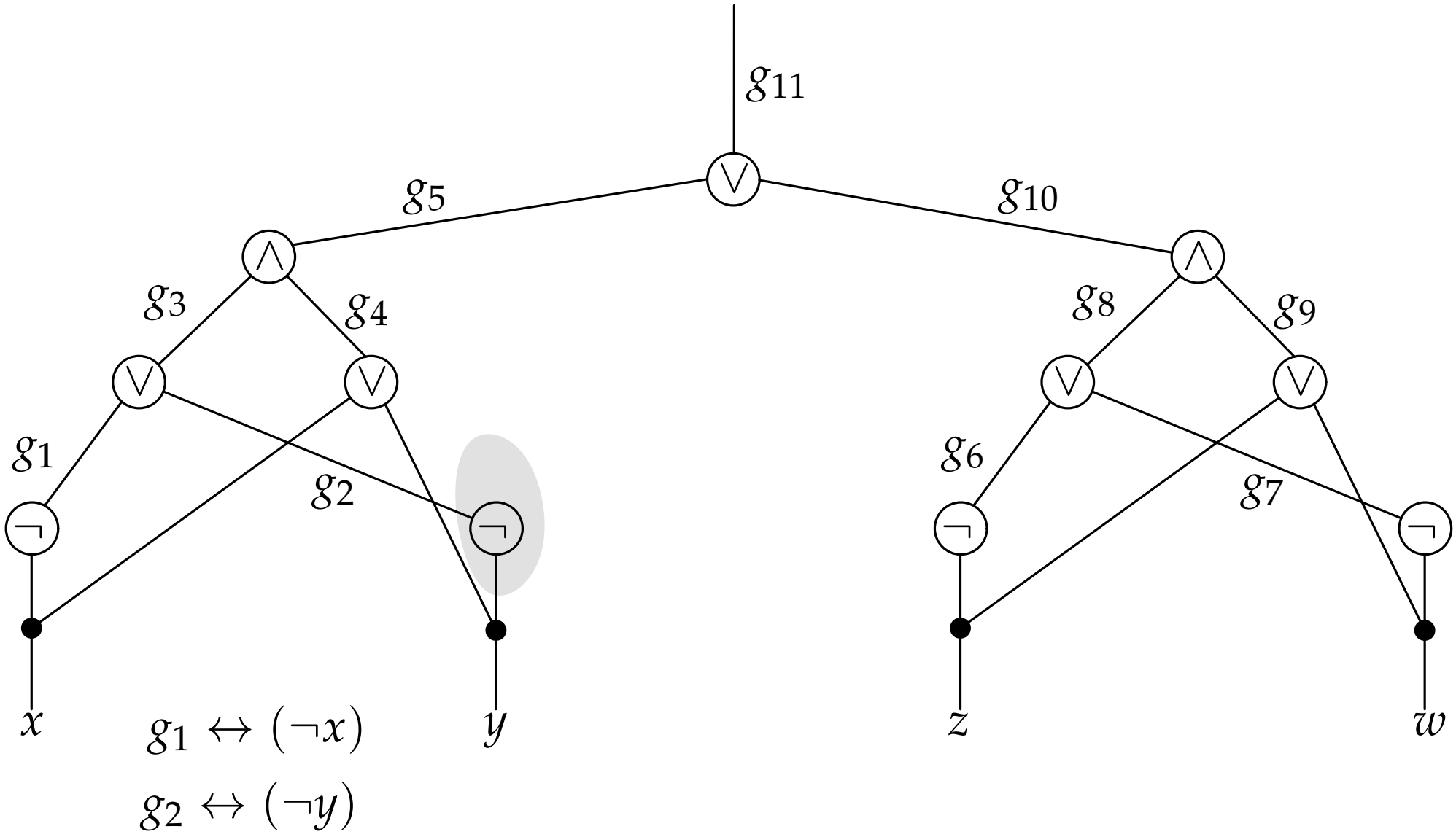
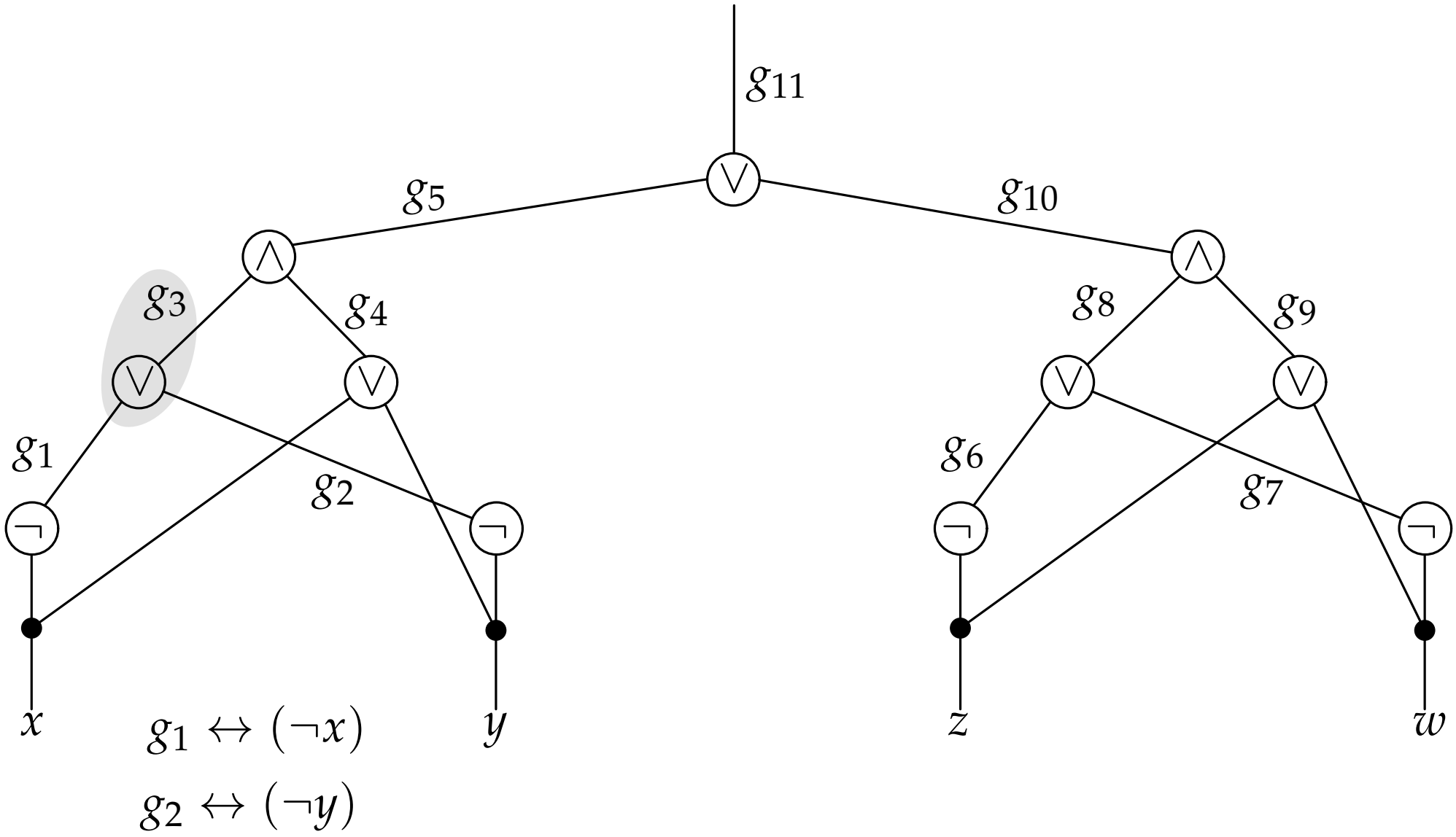
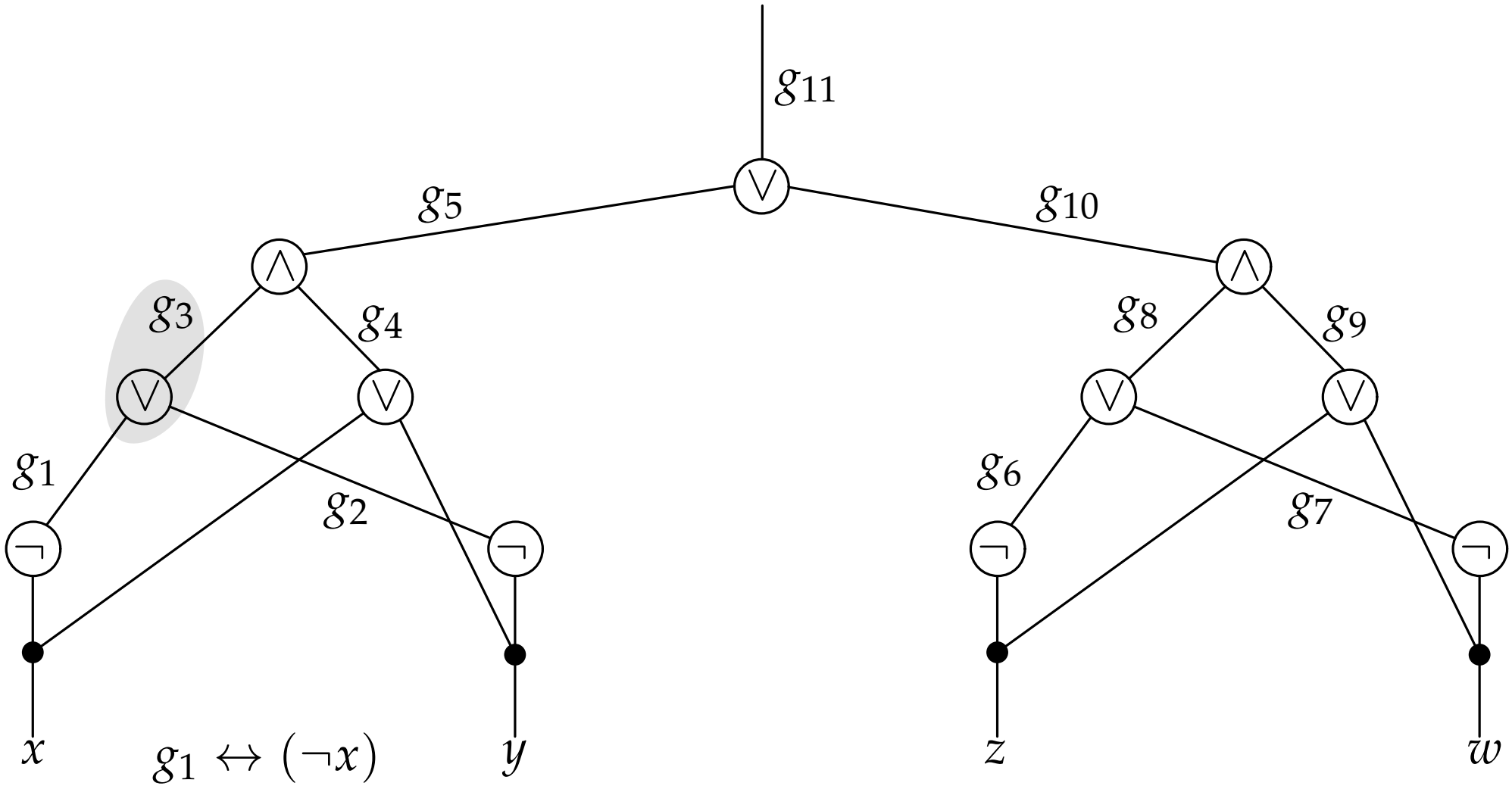
g_6
g_7
x
y
z
w
$g_1 \leftrightarrow (\neg x)$


g_{11}
g_5
g_{10}
g_3
g_4
g_8
g_9
g_1
g_2
g_6
g_7
x
y
z
w
$g_1 \leftrightarrow (\neg x)$
$g_2 \leftrightarrow (\neg y)$

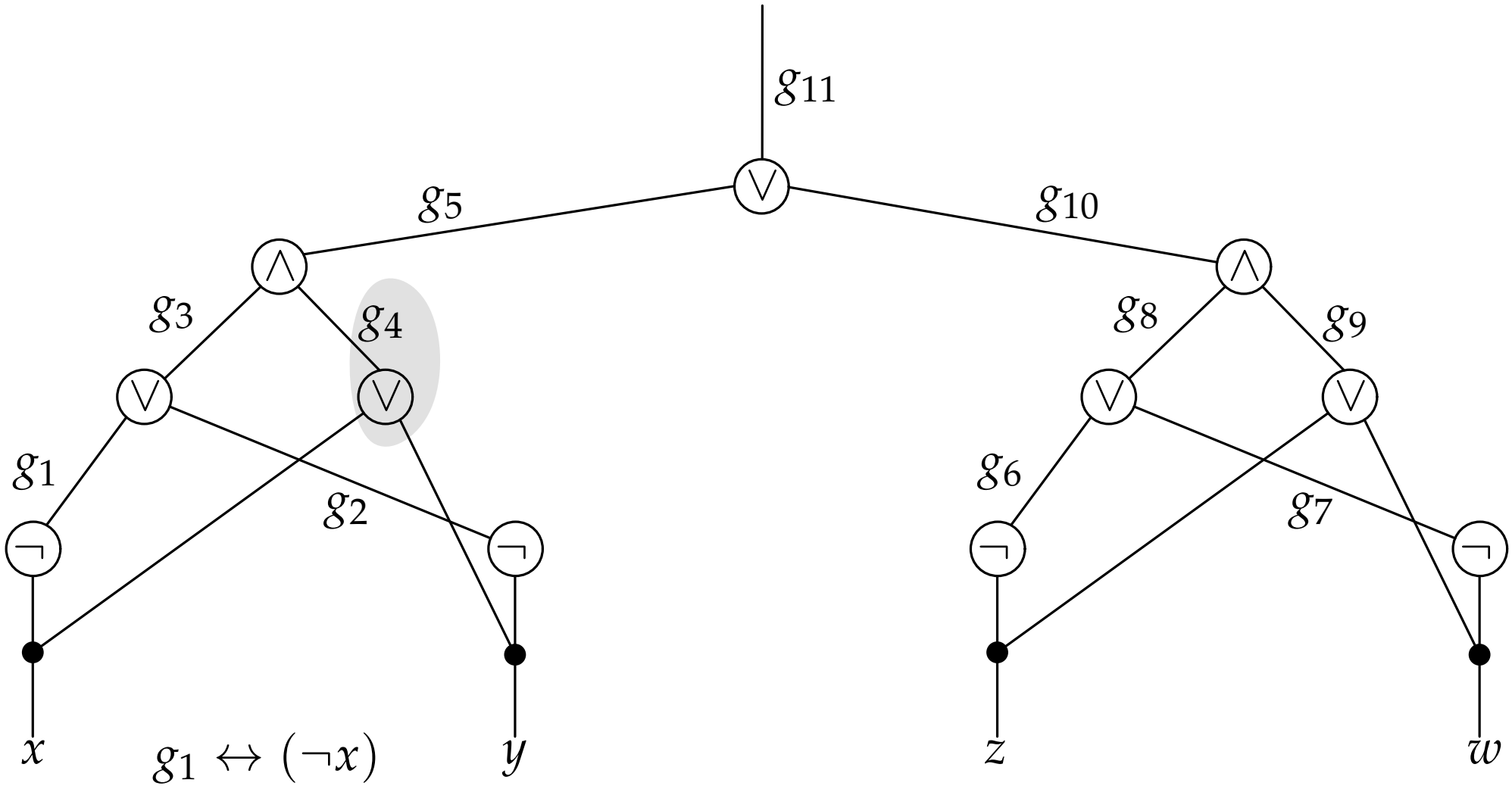

g_{11}
g_5
g_{10}
g_3
g_4
g_8
g_9
g_1
g_2
g_6
g_7
x
y
z
w
$g_1 \leftrightarrow (\neg x)$
$g_2 \leftrightarrow (\neg y)$



$$g_1 \leftrightarrow (\neg x)$$

$$g_2 \leftrightarrow (\neg y)$$

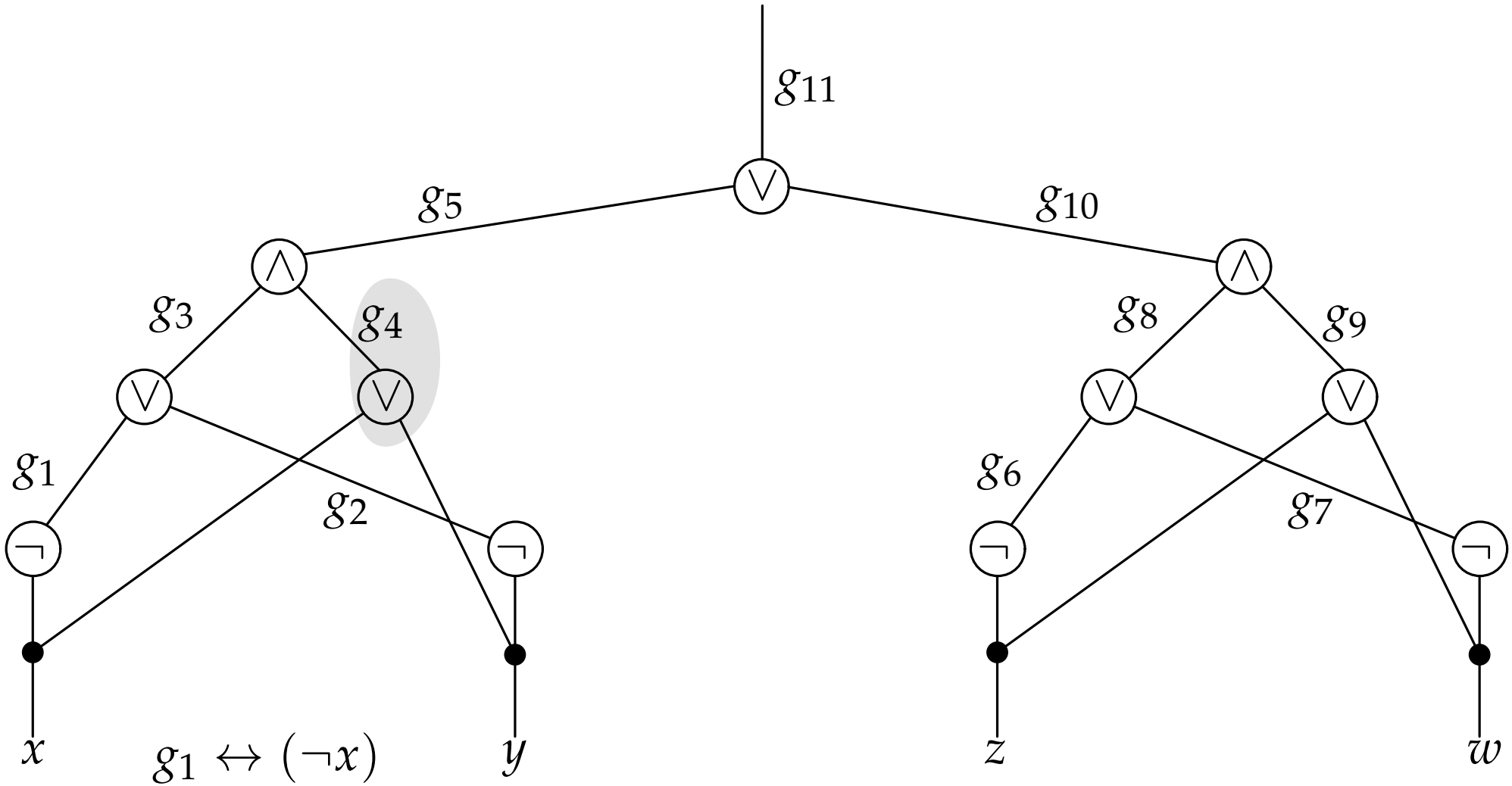
$$g_3 \leftrightarrow (g_1 \vee g_2)$$



$g_1 \leftrightarrow (\neg x)$

$g_2 \leftrightarrow (\neg y)$

$g_3 \leftrightarrow (g_1 \vee g_2)$

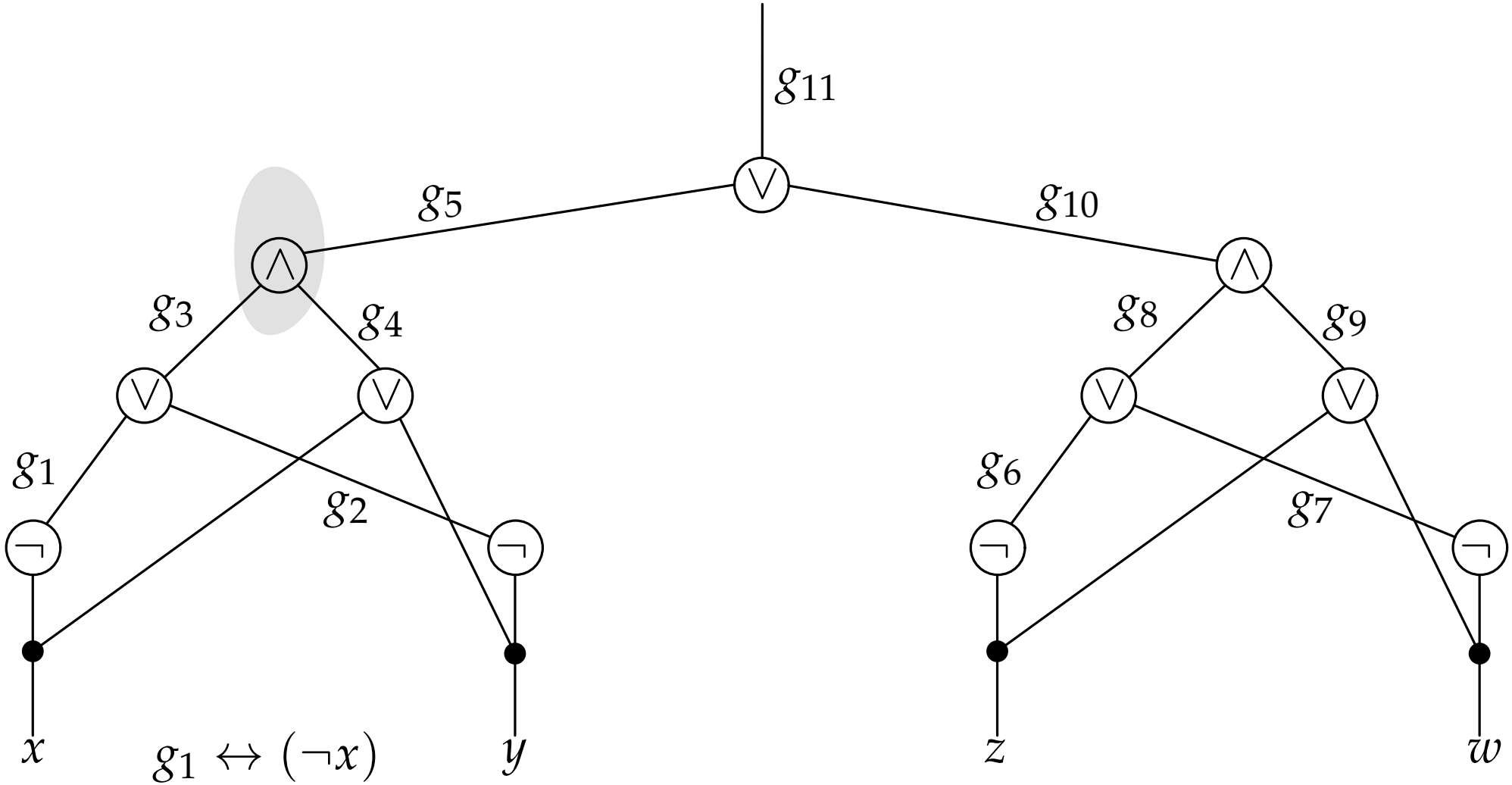


$$g_1 \leftrightarrow (\neg x)$$

$$g_2 \leftrightarrow (\neg y)$$

$$g_3 \leftrightarrow (g_1 \vee g_2)$$

$$g_4 \leftrightarrow (x \vee y)$$

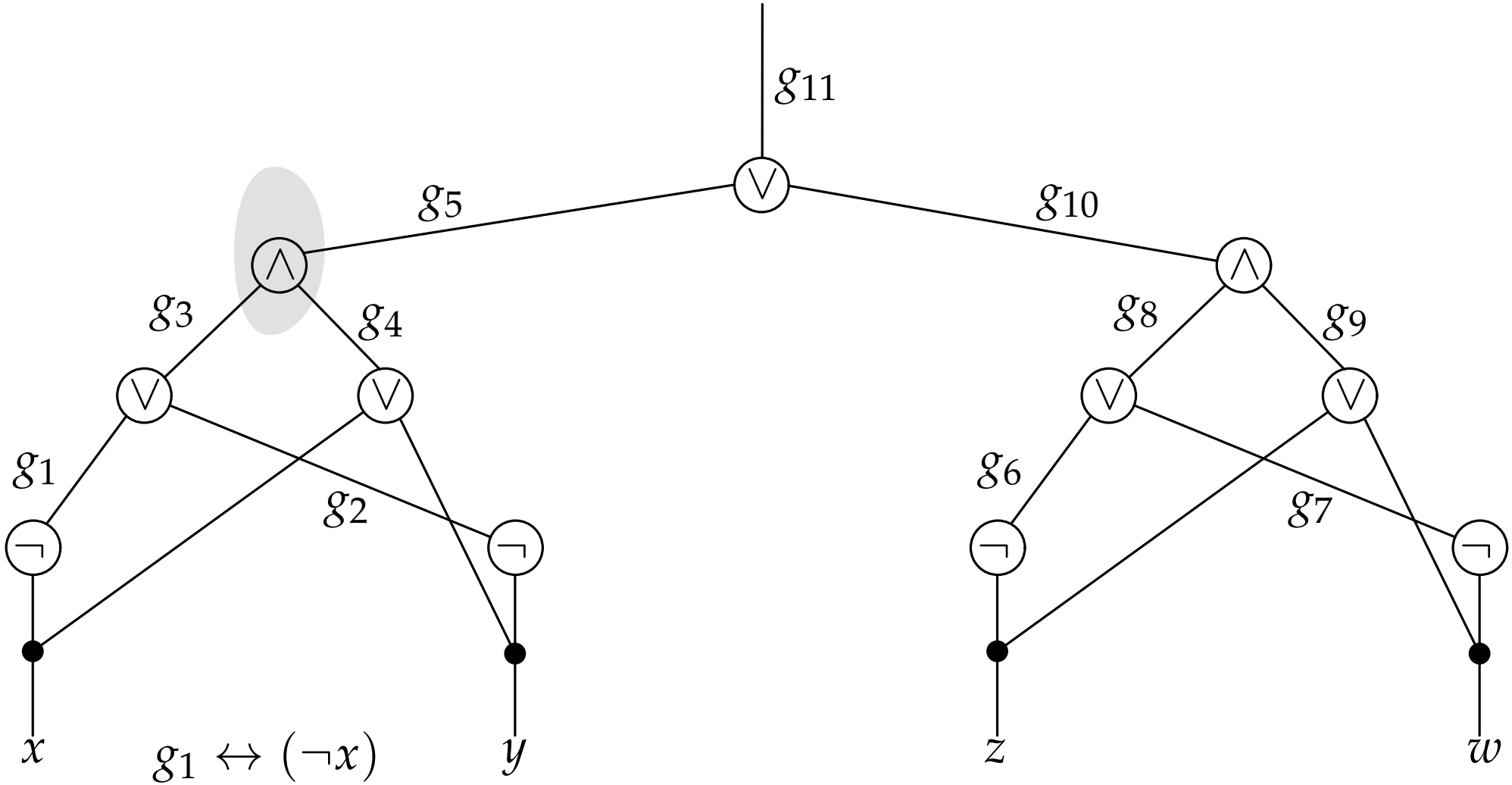


$$g_1 \leftrightarrow (\neg x)$$

$$g_2 \leftrightarrow (\neg y)$$

$$g_3 \leftrightarrow (g_1 \vee g_2)$$

$$g_4 \leftrightarrow (x \vee y)$$



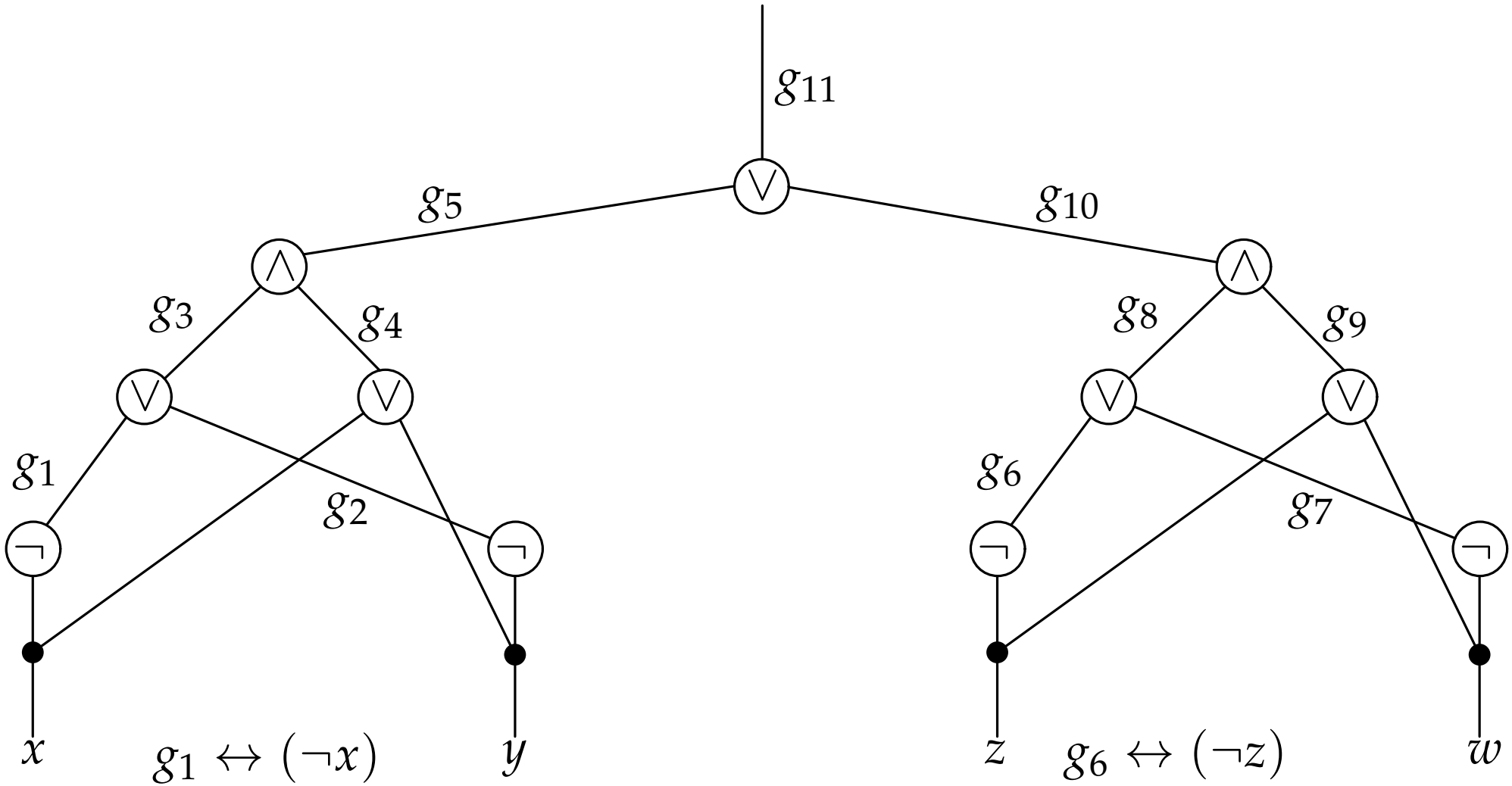
$$g_1 \leftrightarrow (\neg x)$$

$$g_2 \leftrightarrow (\neg y)$$

$$g_3 \leftrightarrow (g_1 \vee g_2)$$

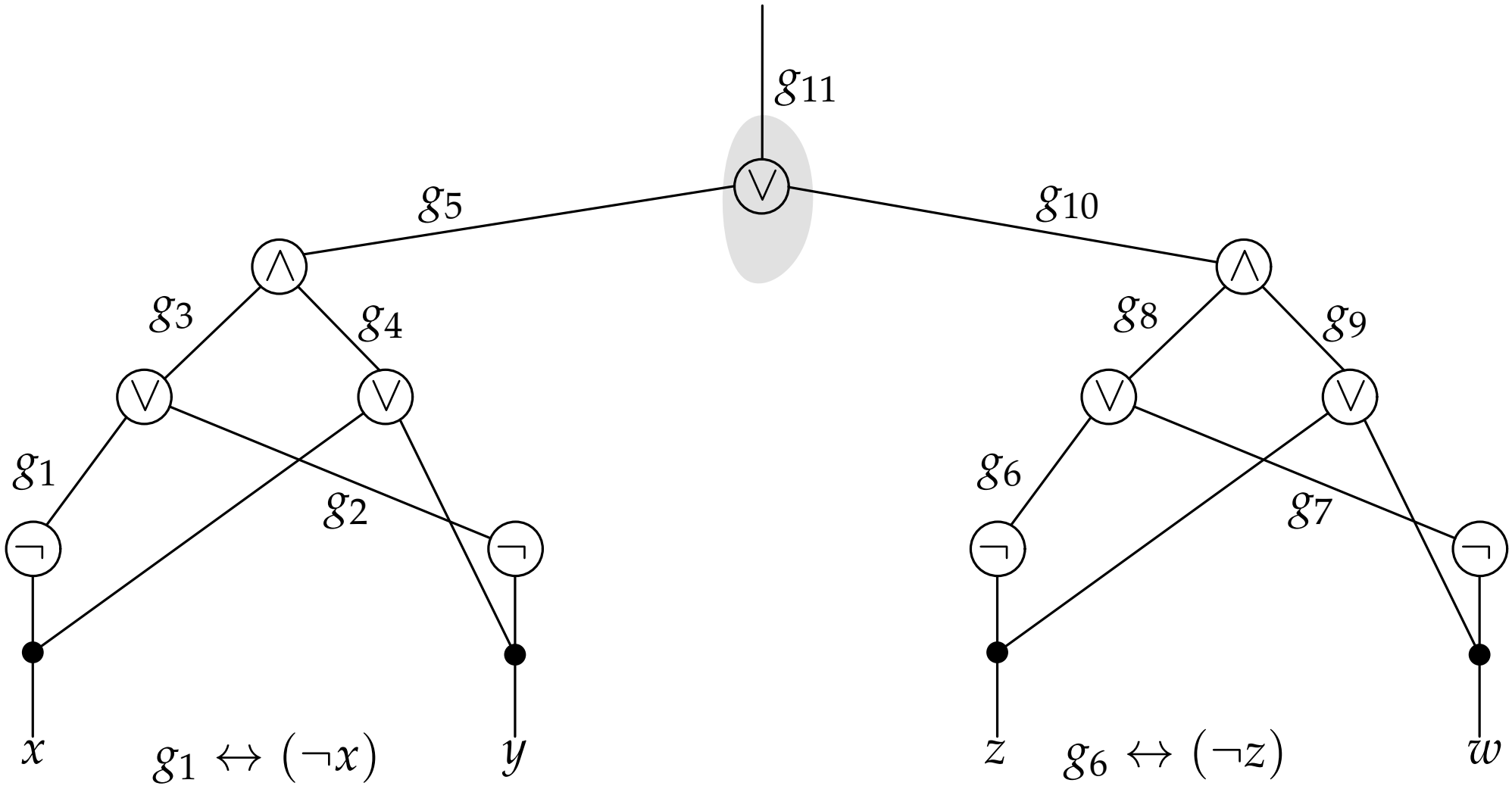
$$g_4 \leftrightarrow (x \vee y)$$

$$g_5 \leftrightarrow (g_3 \wedge g_4)$$



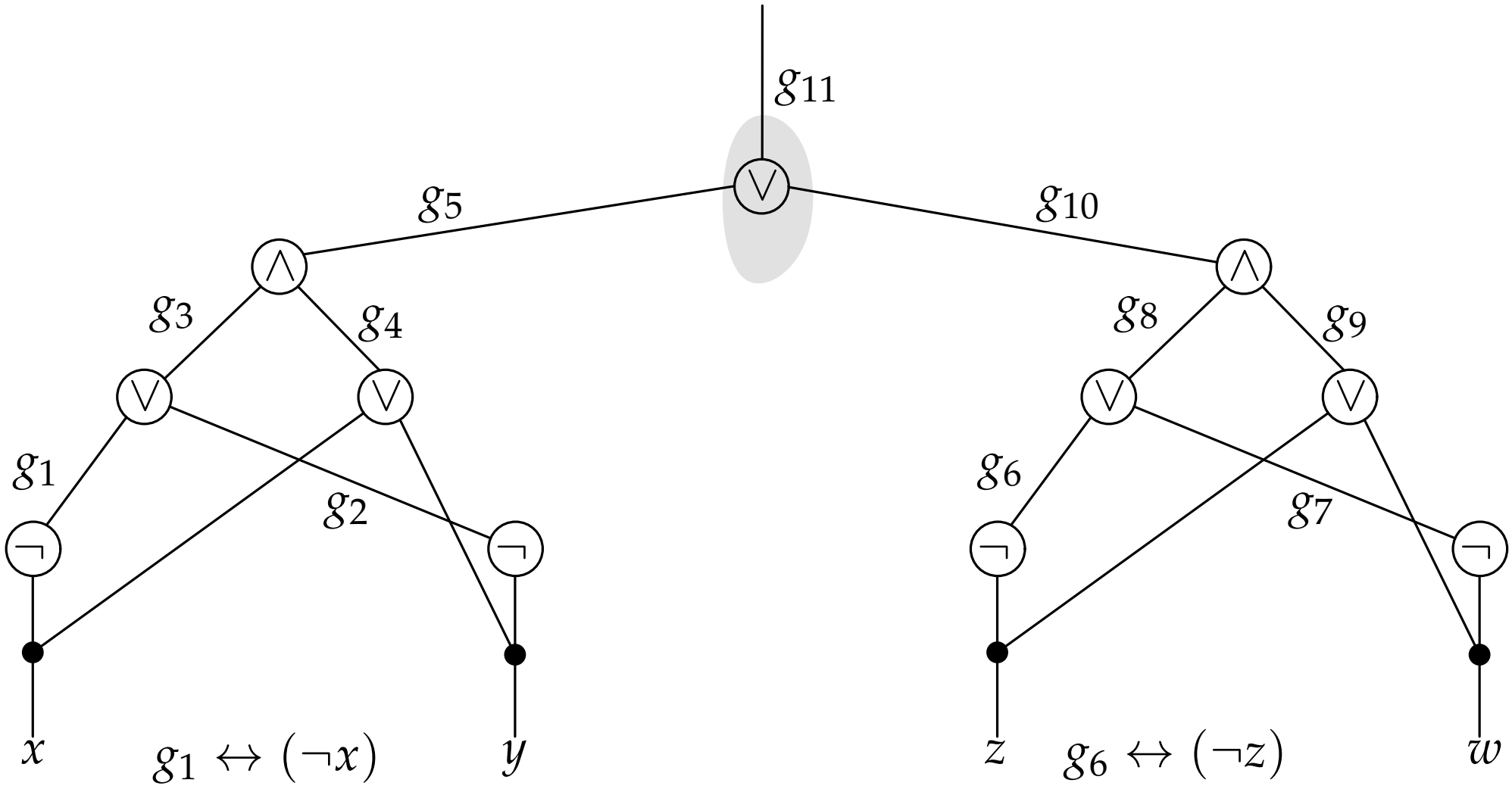
$$g_1 \leftrightarrow (\neg x)$$
$$g_2 \leftrightarrow (\neg y)$$
$$g_3 \leftrightarrow (g_1 \vee g_2)$$
$$g_4 \leftrightarrow (x \vee y)$$
$$g_5 \leftrightarrow (g_3 \wedge g_4)$$

$$g_6 \leftrightarrow (\neg z)$$
$$g_7 \leftrightarrow (\neg w)$$
$$g_8 \leftrightarrow (g_6 \vee g_7)$$
$$g_9 \leftrightarrow (z \vee w)$$
$$g_{10} \leftrightarrow (g_8 \wedge g_9)$$



$$g_1 \leftrightarrow (\neg x)$$
$$g_2 \leftrightarrow (\neg y)$$
$$g_3 \leftrightarrow (g_1 \vee g_2)$$
$$g_4 \leftrightarrow (x \vee y)$$
$$g_5 \leftrightarrow (g_3 \wedge g_4)$$

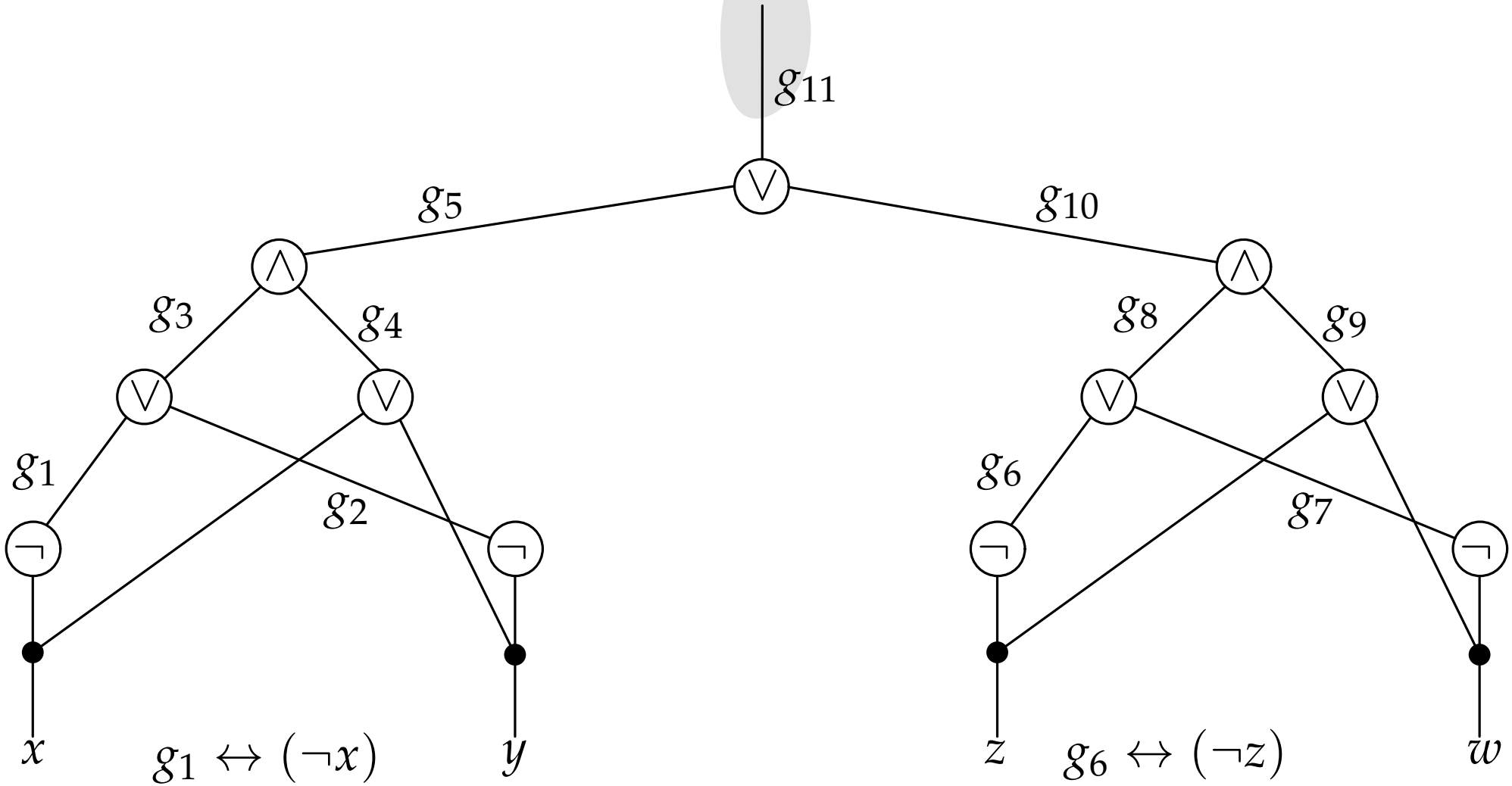
$$g_6 \leftrightarrow (\neg z)$$
$$g_7 \leftrightarrow (\neg w)$$
$$g_8 \leftrightarrow (g_6 \vee g_7)$$
$$g_9 \leftrightarrow (z \vee w)$$
$$g_{10} \leftrightarrow (g_8 \wedge g_9)$$



$$g_1 \leftrightarrow (\neg x)$$
$$g_2 \leftrightarrow (\neg y)$$
$$g_3 \leftrightarrow (g_1 \vee g_2)$$
$$g_4 \leftrightarrow (x \vee y)$$
$$g_5 \leftrightarrow (g_3 \wedge g_4)$$

$$g_{11} \leftrightarrow (g_5 \vee g_{10})$$

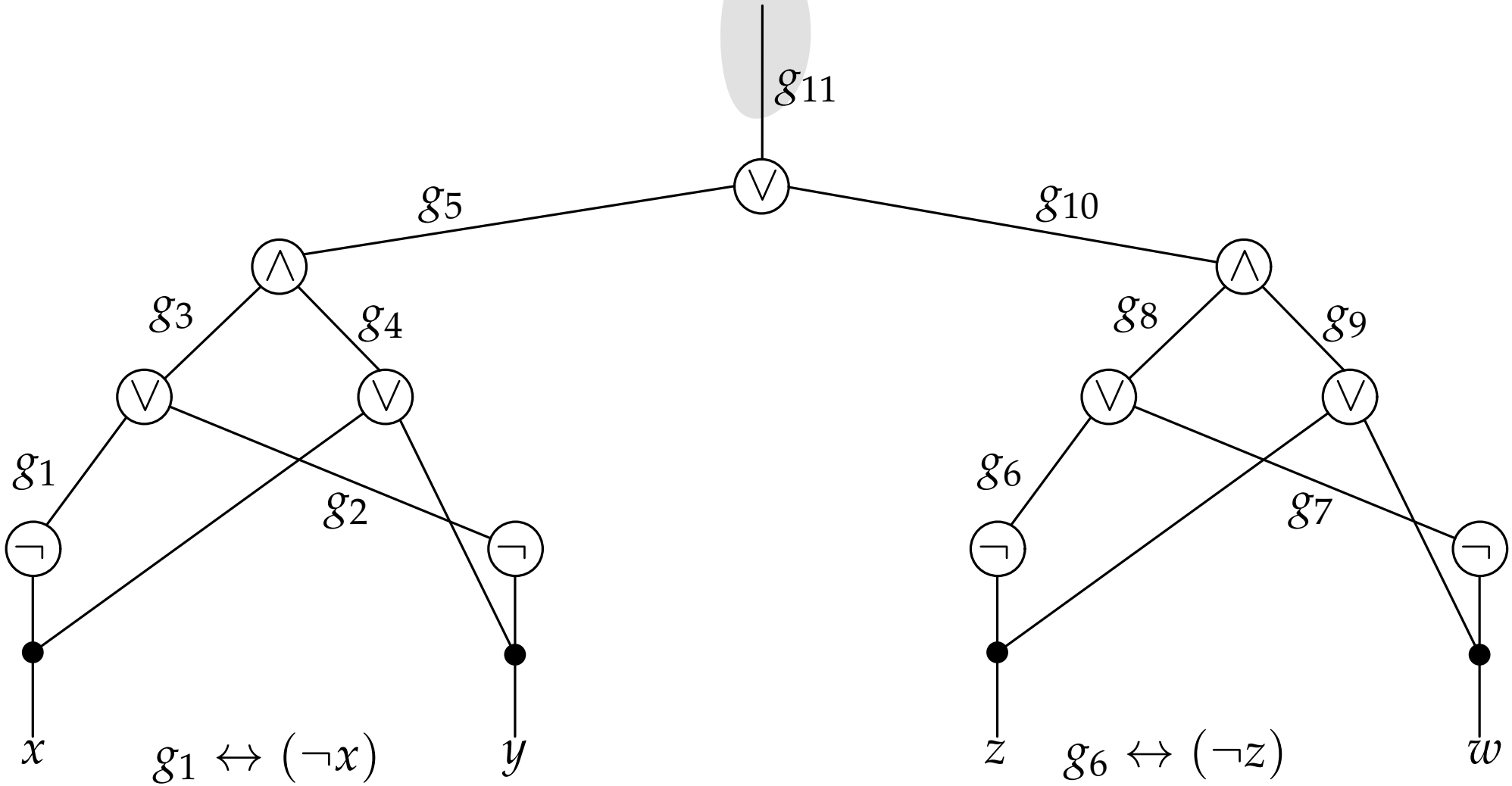
$$g_6 \leftrightarrow (\neg z)$$
$$g_7 \leftrightarrow (\neg w)$$
$$g_8 \leftrightarrow (g_6 \vee g_7)$$
$$g_9 \leftrightarrow (z \vee w)$$
$$g_{10} \leftrightarrow (g_8 \wedge g_9)$$



$$g_1 \leftrightarrow (\neg x)$$
$$g_2 \leftrightarrow (\neg y)$$
$$g_3 \leftrightarrow (g_1 \vee g_2)$$
$$g_4 \leftrightarrow (x \vee y)$$
$$g_5 \leftrightarrow (g_3 \wedge g_4)$$

$$g_{11} \leftrightarrow (g_5 \vee g_{10})$$

$$g_6 \leftrightarrow (\neg z)$$
$$g_7 \leftrightarrow (\neg w)$$
$$g_8 \leftrightarrow (g_6 \vee g_7)$$
$$g_9 \leftrightarrow (z \vee w)$$
$$g_{10} \leftrightarrow (g_8 \wedge g_9)$$

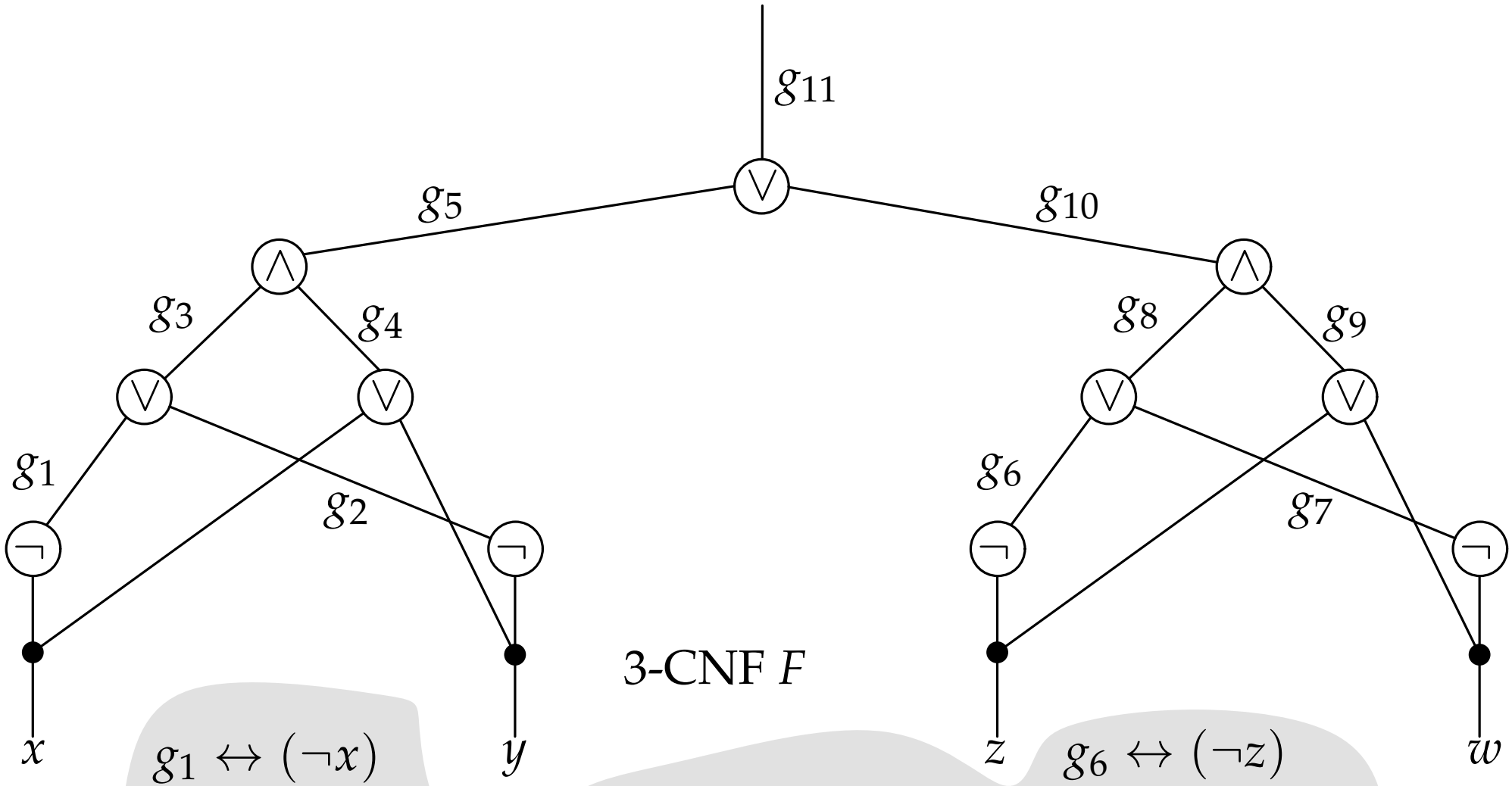


$$g_1 \leftrightarrow (\neg x)$$
$$g_2 \leftrightarrow (\neg y)$$
$$g_3 \leftrightarrow (g_1 \vee g_2)$$
$$g_4 \leftrightarrow (x \vee y)$$
$$g_5 \leftrightarrow (g_3 \wedge g_4)$$

$$g_{11} \leftrightarrow (g_5 \vee g_{10})$$

$$(g_{11})$$

$$g_6 \leftrightarrow (\neg z)$$
$$g_7 \leftrightarrow (\neg w)$$
$$g_8 \leftrightarrow (g_6 \vee g_7)$$
$$g_9 \leftrightarrow (z \vee w)$$
$$g_{10} \leftrightarrow (g_8 \wedge g_9)$$

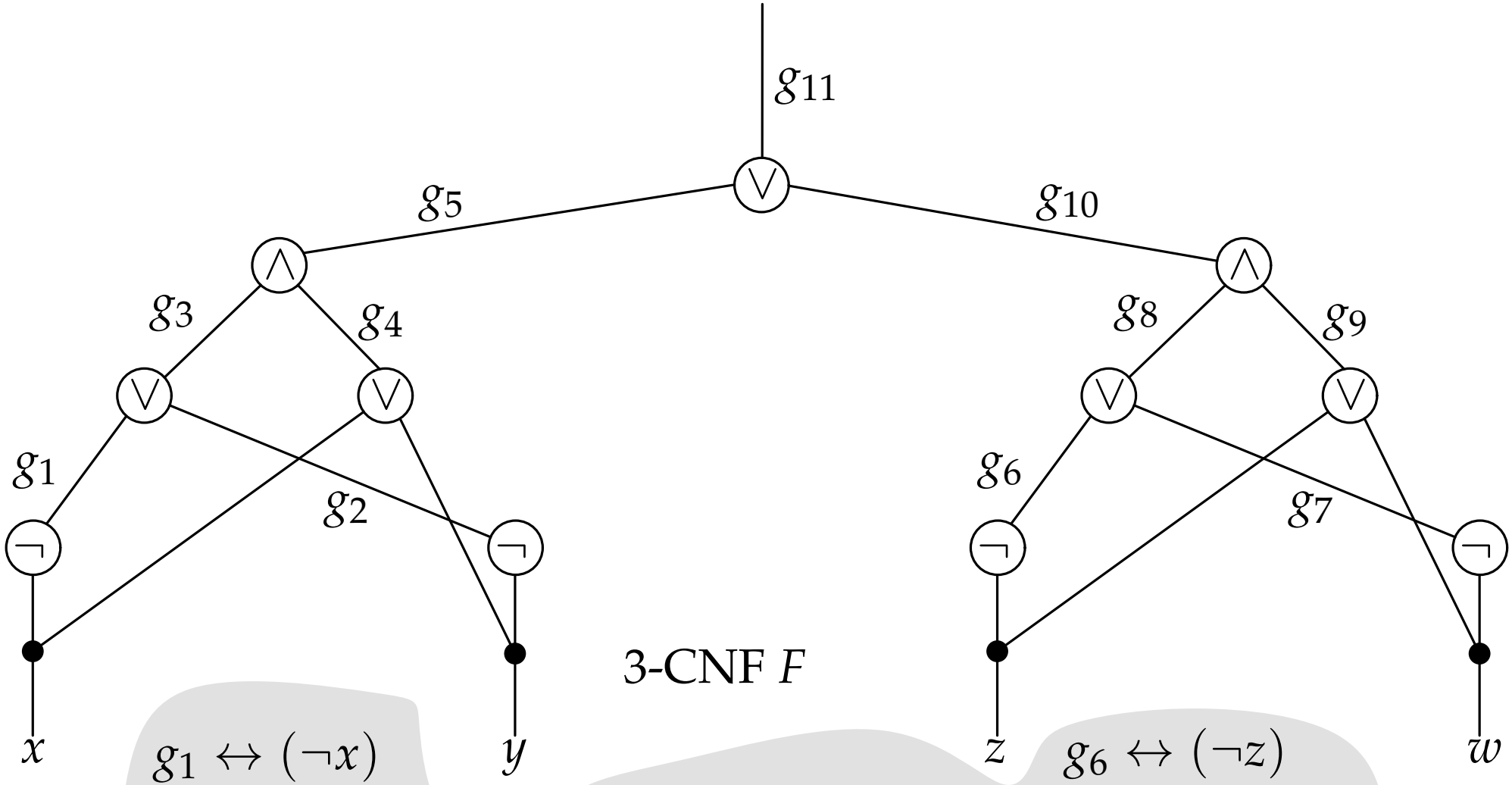


3-CNF F

$$g_1 \leftrightarrow (\neg x)$$
$$g_2 \leftrightarrow (\neg y)$$
$$g_3 \leftrightarrow (g_1 \vee g_2)$$
$$g_4 \leftrightarrow (x \vee y)$$
$$g_5 \leftrightarrow (g_3 \wedge g_4)$$

$$g_{11} \leftrightarrow (g_5 \vee g_{10})$$
$$(g_{11})$$

$$g_6 \leftrightarrow (\neg z)$$
$$g_7 \leftrightarrow (\neg w)$$
$$g_8 \leftrightarrow (g_6 \vee g_7)$$
$$g_9 \leftrightarrow (z \vee w)$$
$$g_{10} \leftrightarrow (g_8 \wedge g_9)$$



3-CNF F

$$g_1 \leftrightarrow (\neg x)$$
$$g_2 \leftrightarrow (\neg y)$$
$$g_3 \leftrightarrow (g_1 \vee g_2)$$
$$g_4 \leftrightarrow (x \vee y)$$
$$g_5 \leftrightarrow (g_3 \wedge g_4)$$

$$g_{11} \leftrightarrow (g_5 \vee g_{10})$$
$$(g_{11})$$
$$C(x,y,z,w) = 1$$
$$\Longleftrightarrow$$
$$\exists \vec{g} : F(x,y,z,w,\vec{g}) = 1$$

$$g_6 \leftrightarrow (\neg z)$$
$$g_7 \leftrightarrow (\neg w)$$
$$g_8 \leftrightarrow (g_6 \vee g_7)$$
$$g_9 \leftrightarrow (z \vee w)$$
$$g_{10} \leftrightarrow (g_8 \wedge g_9)$$

CIRCUIT-SAT to 3-SAT

$$C(x, y, z, w) = 1$$
$$\Longleftrightarrow$$
$$\exists \vec{g} : F(x, y, z, w, \vec{g}) = 1$$

CIRCUIT-SAT to 3-SAT

can solve 3-SAT $\Longrightarrow$ can solve CIRCUIT-SAT

$$C(x, y, z, w) = 1$$
$$\Longleftrightarrow$$
$$\exists \vec{g} : F(x, y, z, w, \vec{g}) = 1$$

CIRCUIT-SAT to 3-SAT

can solve 3-SAT $\Longrightarrow$ can solve CIRCUIT-SAT

Theorem (Cook-Levin). 3-SAT is NP-complete.

$$C(x, y, z, w) = 1$$
$$\Longleftrightarrow$$
$$\exists \vec{g} : F(x, y, z, w, \vec{g}) = 1$$

Max-SAT

$$(x \vee y) \quad (x \vee z) \quad (\bar{x} \vee \bar{y}) \quad (\bar{x} \vee \bar{z}) \quad (y \vee z) \quad (x \vee \bar{y} \vee \bar{z})$$

Max-SAT

32	16	8	4	2	1
$(x \vee y)$	$(x \vee z)$	$(\bar{x} \vee \bar{y})$	$(\bar{x} \vee \bar{z})$	$(y \vee z)$	$(x \vee \bar{y} \vee \bar{z})$

Max-SAT

32	16	8	4	2	1
$(x \vee y)$	$(x \vee z)$	$(\bar{x} \vee \bar{y})$	$(\bar{x} \vee \bar{z})$	$(y \vee z)$	$(x \vee \bar{y} \vee \bar{z})$

x	y	z	weight of satisfied clauses
1	1	1	

Max-SAT

32	16	8	4	2	1
$(x \vee y)$	$(x \vee z)$	$(\bar{x} \vee \bar{y})$	$(\bar{x} \vee \bar{z})$	$(y \vee z)$	$(x \vee \bar{y} \vee \bar{z})$

x	y	z	weight of satisfied clauses
1	1	1	

Max-SAT

32	16	8	4	2	1
$(x \vee y)$	$(x \vee z)$	$(\bar{x} \vee \bar{y})$	$(\bar{x} \vee \bar{z})$	$(y \vee z)$	$(x \vee \bar{y} \vee \bar{z})$

x	y	z	weight of satisfied clauses
1	1	1	

Max-SAT

32	16	8	4	2	1
$(x \vee y)$	$(x \vee z)$	$(\bar{x} \vee \bar{y})$	$(\bar{x} \vee \bar{z})$	$(y \vee z)$	$(x \vee \bar{y} \vee \bar{z})$

x	y	z	weight of satisfied clauses
1	1	1	51

Max-SAT

32	16	8	4	2	1
$(x \vee y)$	$(x \vee z)$	$(\bar{x} \vee \bar{y})$	$(\bar{x} \vee \bar{z})$	$(y \vee z)$	$(x \vee \bar{y} \vee \bar{z})$

x	y	z	weight of satisfied clauses
1	1	1	51
1	1	0	

Max-SAT

32	16	8	4	2	1
$(x \vee y)$	$(x \vee z)$	$(\bar{x} \vee \bar{y})$	$(\bar{x} \vee \bar{z})$	$(y \vee z)$	$(x \vee \bar{y} \vee \bar{z})$

x	y	z	weight of satisfied clauses
1	1	1	51
1	1	0	

Max-SAT

32	16	8	4	2	1
$(x \vee y)$	$(x \vee z)$	$(\bar{x} \vee \bar{y})$	$(\bar{x} \vee \bar{z})$	$(y \vee z)$	$(x \vee \bar{y} \vee \bar{z})$

x	y	z	weight of satisfied clauses
1	1	1	51
1	1	0	55

Max-SAT

32	16	8	4	2	1
$(x \vee y)$	$(x \vee z)$	$(\bar{x} \vee \bar{y})$	$(\bar{x} \vee \bar{z})$	$(y \vee z)$	$(x \vee \bar{y} \vee \bar{z})$

x	y	z	weight of satisfied clauses
1	1	1	51
1	1	0	55
1	0	0	

Max-SAT

32	16	8	4	2	1
$(x \vee y)$	$(x \vee z)$	$(\bar{x} \vee \bar{y})$	$(\bar{x} \vee \bar{z})$	$(y \vee z)$	$(x \vee \bar{y} \vee \bar{z})$

x	y	z	weight of satisfied clauses
1	1	1	51
1	1	0	55
1	0	0	

Max-SAT

32	16	8	4	2	1
$(x \vee y)$	$(x \vee z)$	$(\bar{x} \vee \bar{y})$	$(\bar{x} \vee \bar{z})$	$(y \vee z)$	$(x \vee \bar{y} \vee \bar{z})$

x	y	z	weight of satisfied clauses
1	1	1	51
1	1	0	55
1	0	0	61

Max-SAT

32	16	8	4	2	1
$(x \vee y)$	$(x \vee z)$	$(\bar{x} \vee \bar{y})$	$(\bar{x} \vee \bar{z})$	$(y \vee z)$	$(x \vee \bar{y} \vee \bar{z})$

x	y	z	weight of satisfied clauses
1	1	1	51
1	1	0	55
1	0	0	61

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0	0	0	13

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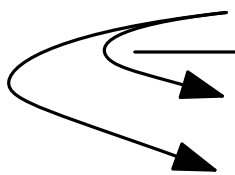
1	0	0	61
0	0	0	13
1	1	0	



Max-SAT

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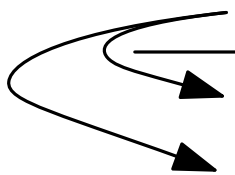

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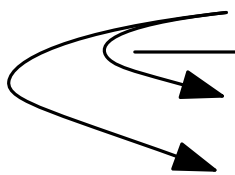

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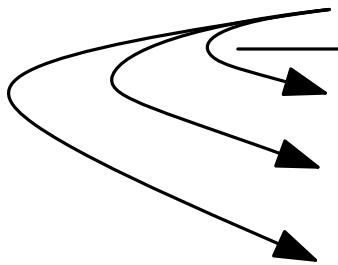
1	0	0	61
0	0	0	13
1	1	0	55
1	0	1	



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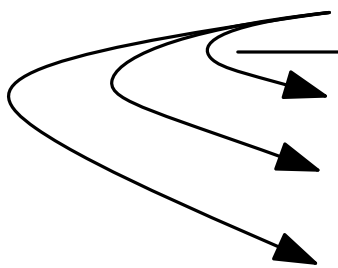

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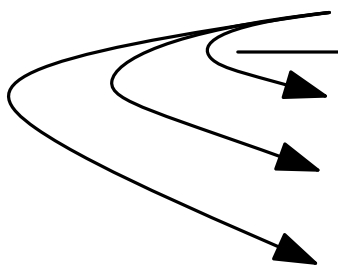
1	1	1	51
1	1	0	55
1	0	0	61
0	0	0	13
1	1	0	55
1	0	1	59



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0	0	0	13
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1	0	1	59

local optimum

Max-SAT

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1	0	1	59
0	1	1	

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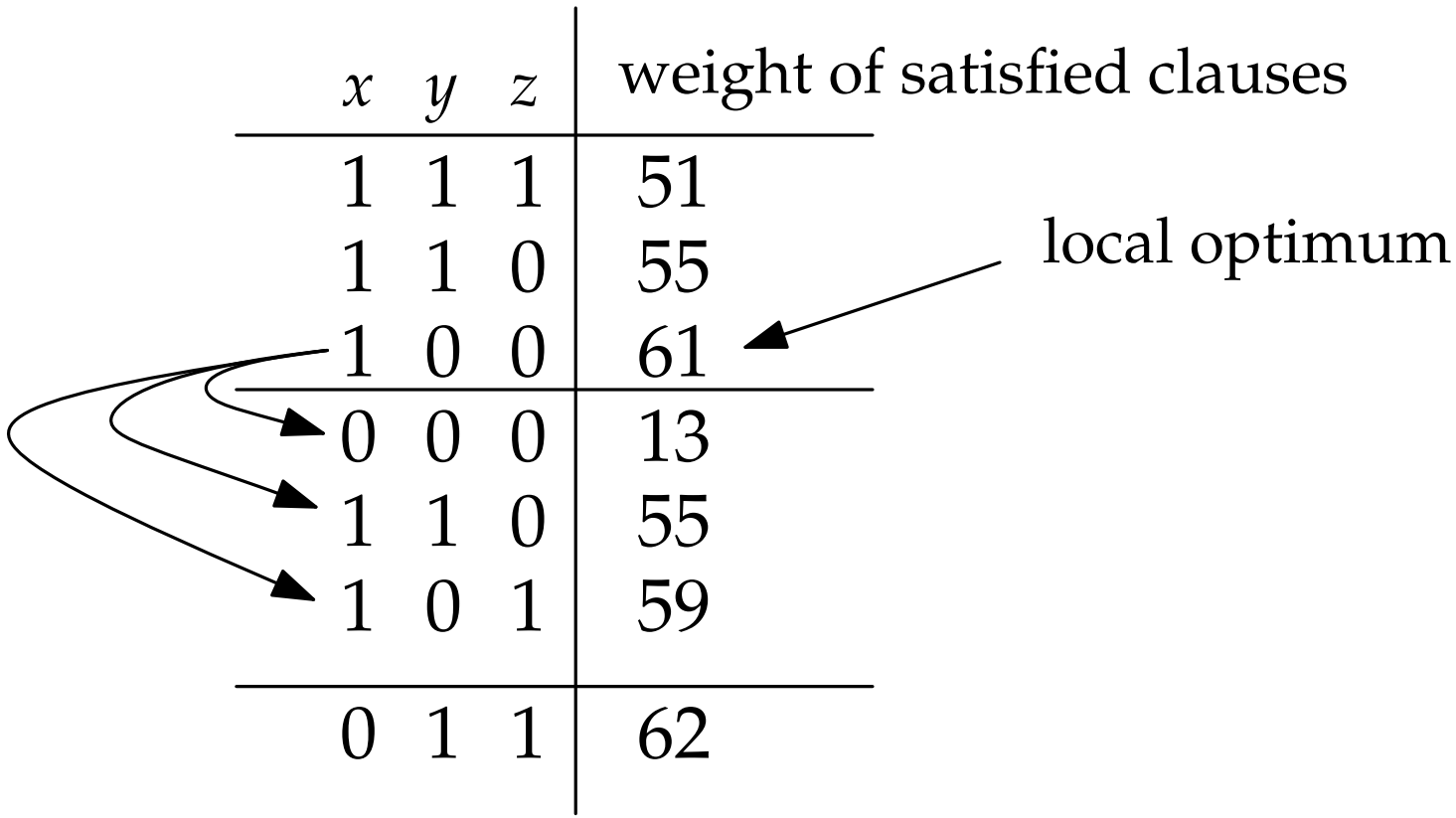
1	1	0	55
1	0	1	59
0	1	1	62



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1	1	0	55	
1	0	0	61	local optimum
0	0	0	13	
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1	0	1	59	
0	1	1	62	global optimum

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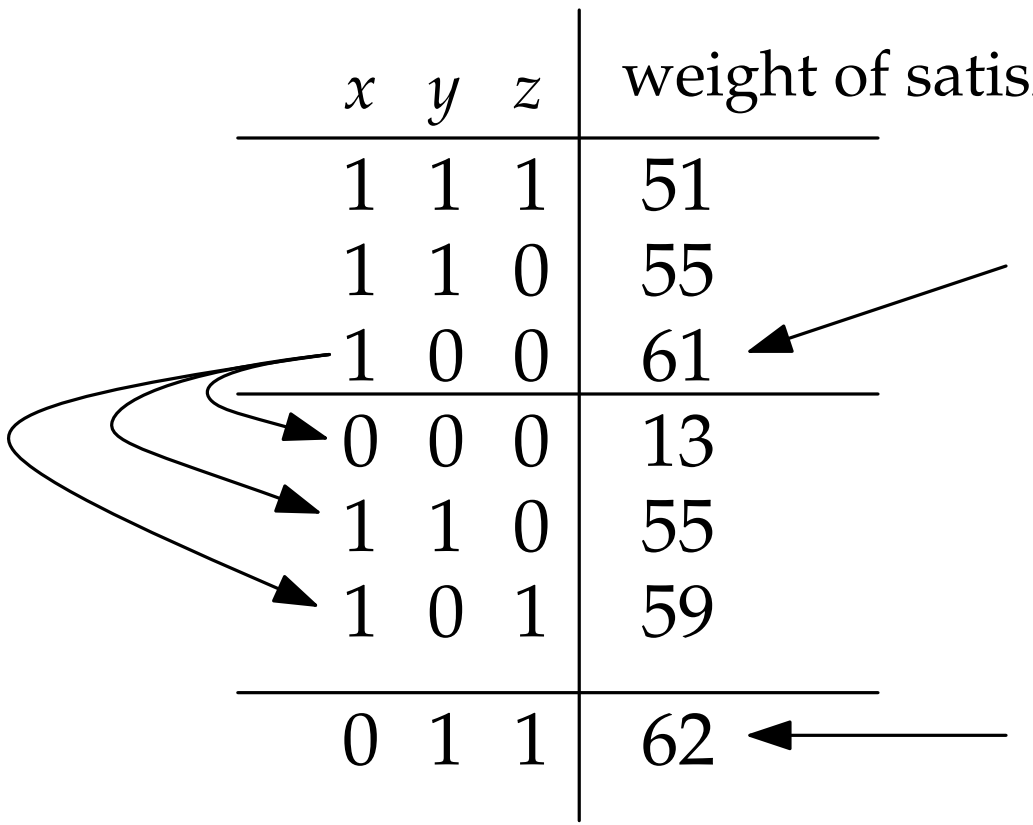

local optimum

global optimum

NP-hard to find (Max-SAT)

Max-SAT

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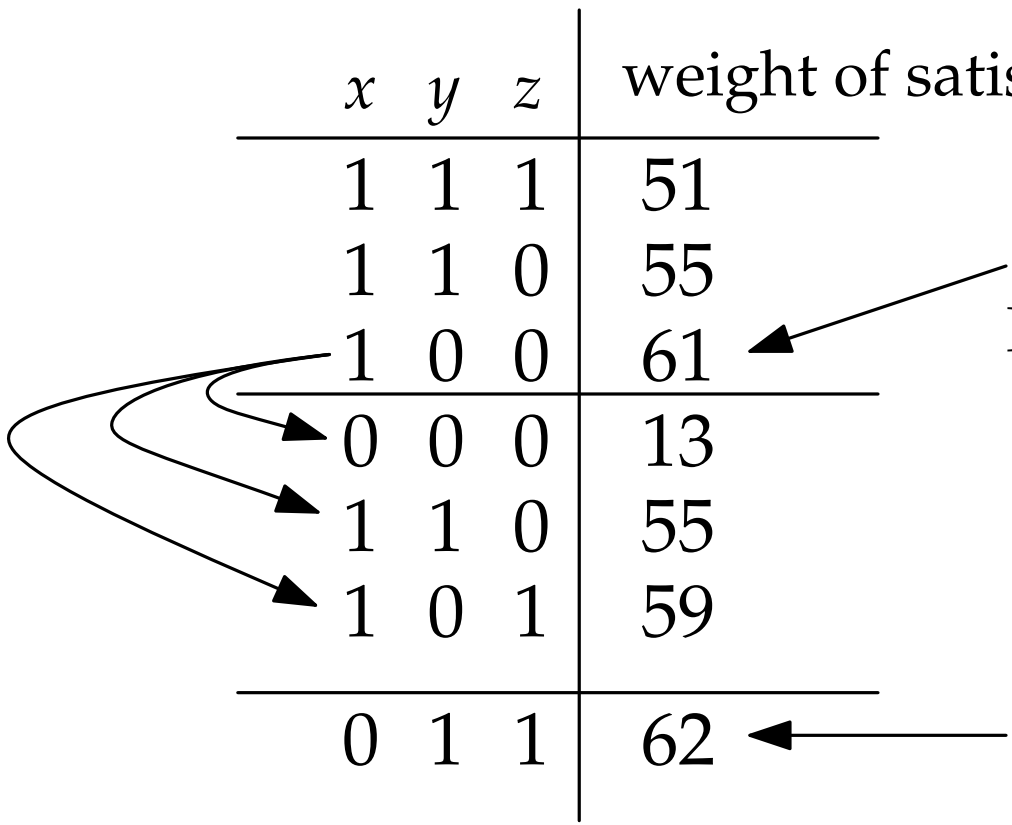

local optimum

How hard can it be???

global optimum

NP-hard to find (Max-SAT)

Local Max-k-SAT

Theorem (Johnson, Papadimitriou, Yannakakis). Local Max-k-SAT is PLS-complete.

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Proof. Take an instance C of ITERATECIRCUIT and build a k-CNF-formula F so that "locally optimizing" F mimics iterating the circuit.

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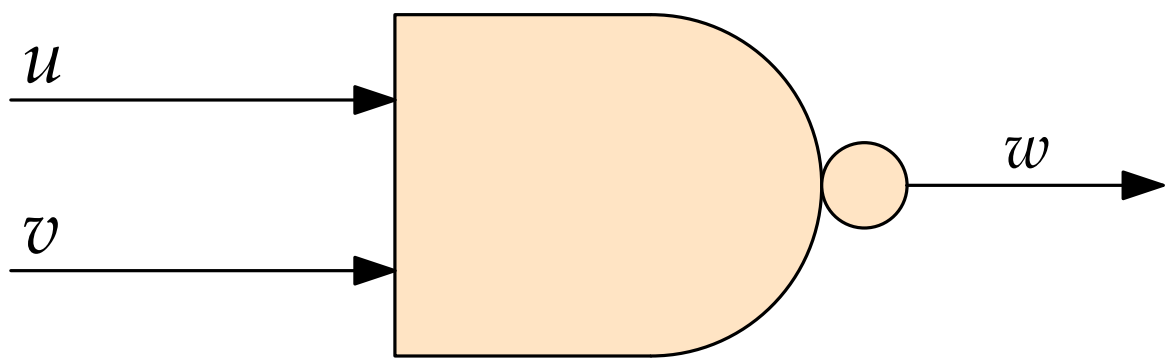
In words: finding a local optimum of F implicitly evaluates the circuit C.

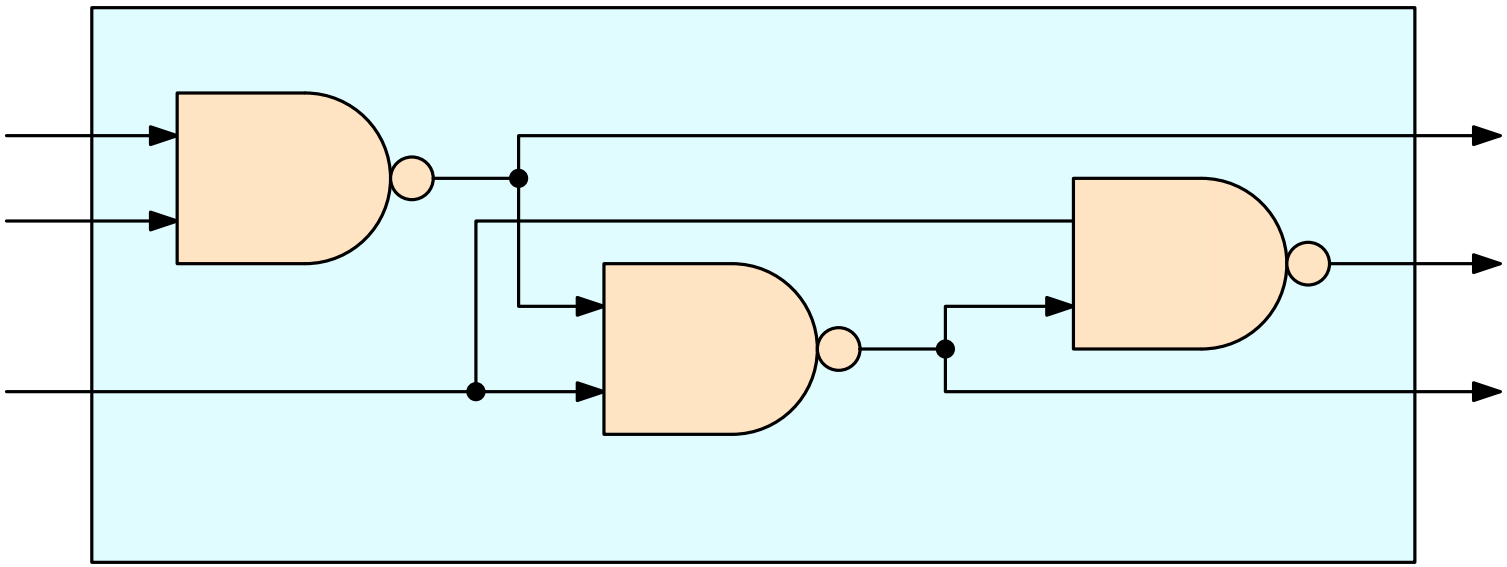
First step: make sure C uses only NAND gates.

This is a NAND gate



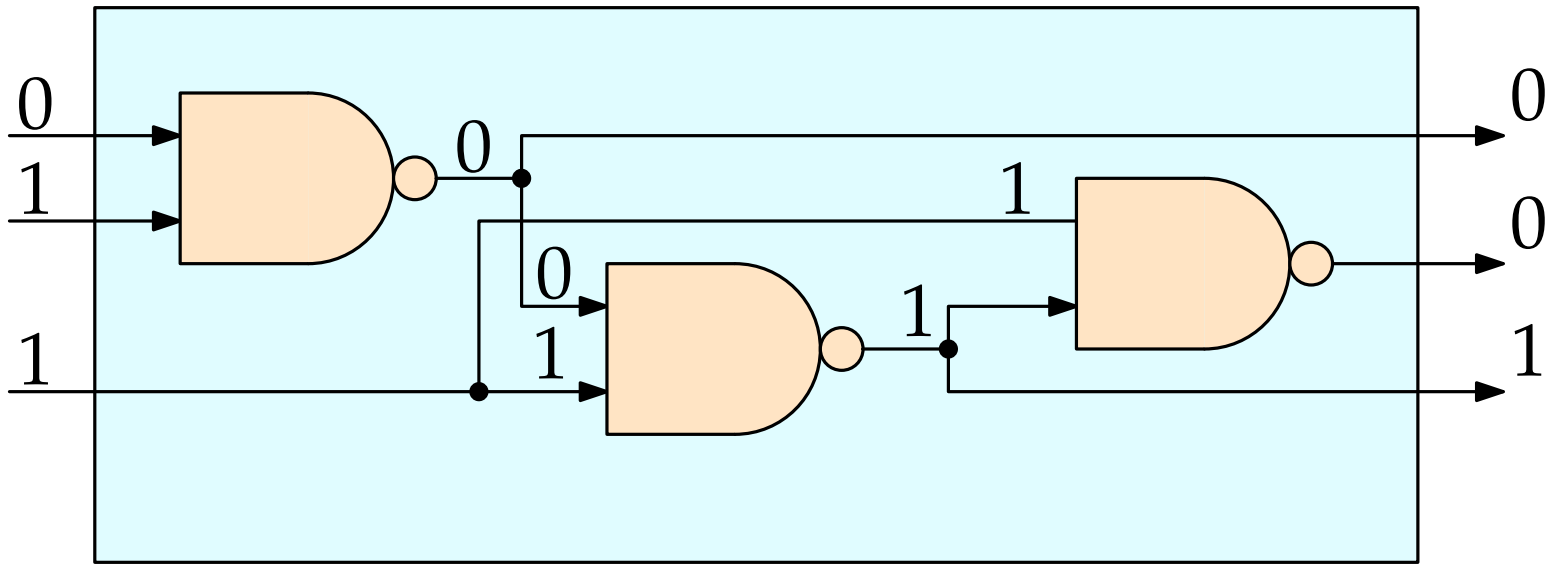
u	v	w
0	0	1
0	1	1
1	0	1
1	1	0

This is a circuit of NAND gates



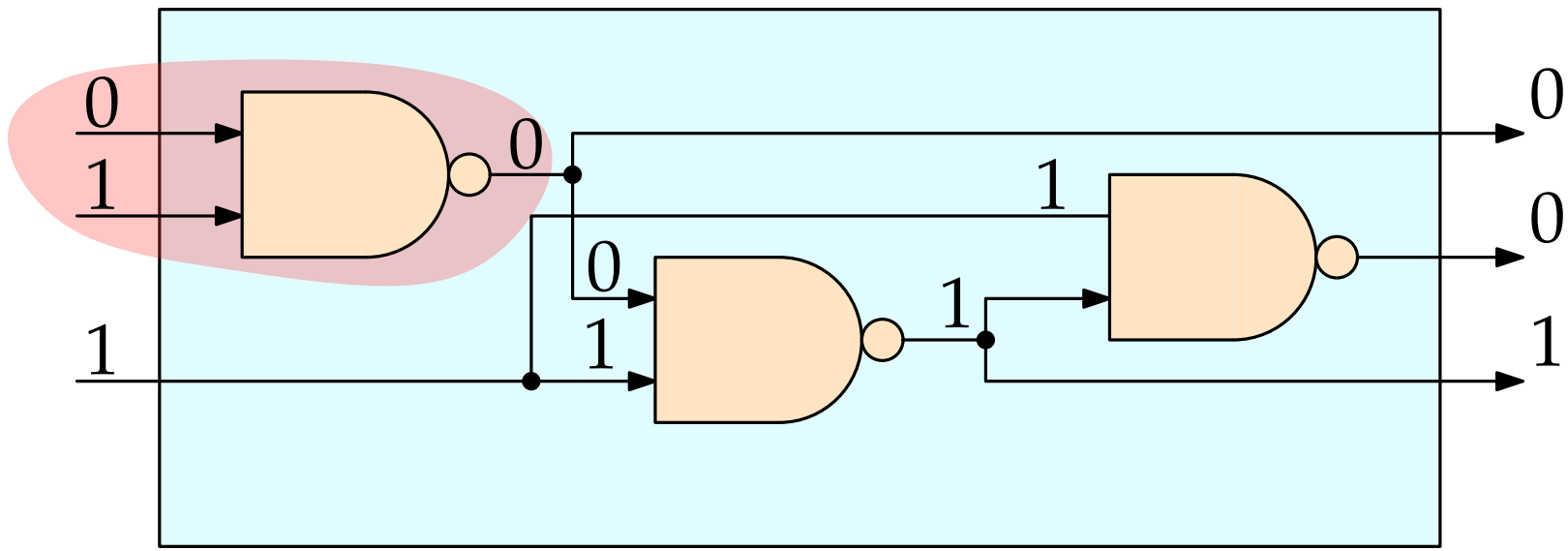
u	v	w
0	0	1
0	1	1
1	0	1
1	1	0

This is a configuration of the circuit



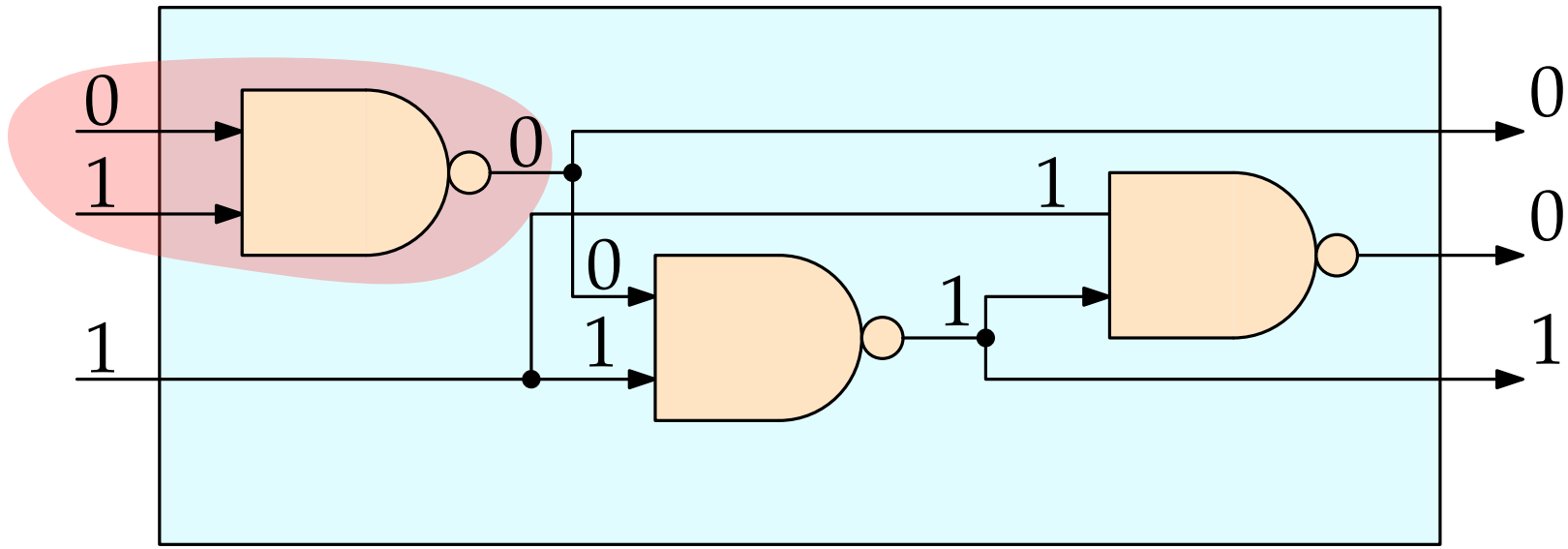
u	v	w
0	0	1
0	1	1
1	0	1
1	1	0

This is a configuration of the circuit
Wait, what?



u	v	w
0	0	1
0	1	1
1	0	1
1	1	0

This is a configuration of the circuit
Wait, what?
Yes! We allow gate outputs to be incorrect.



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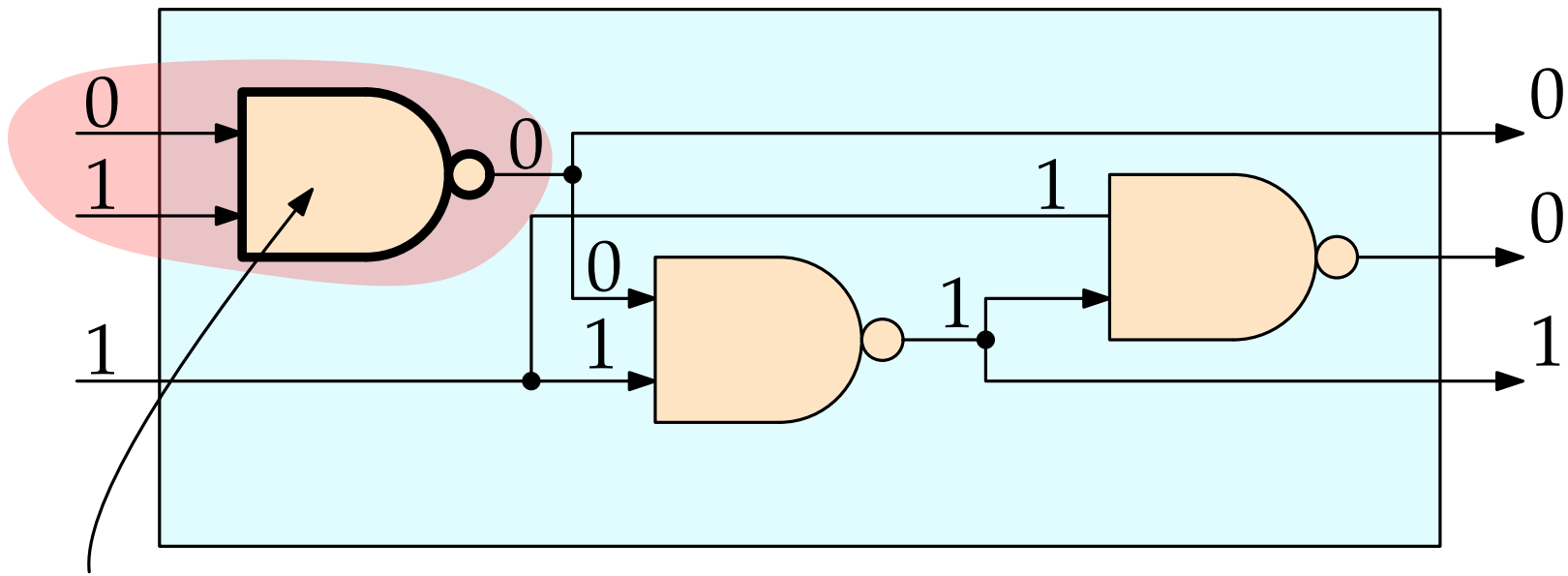
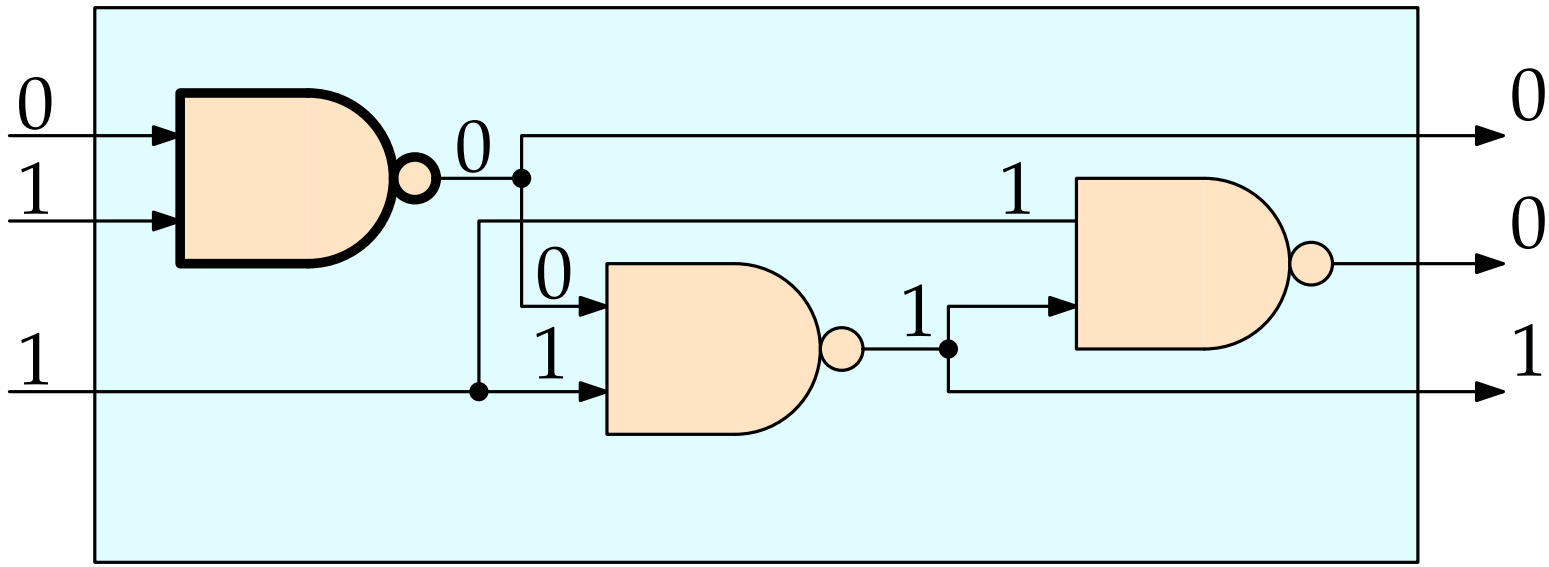
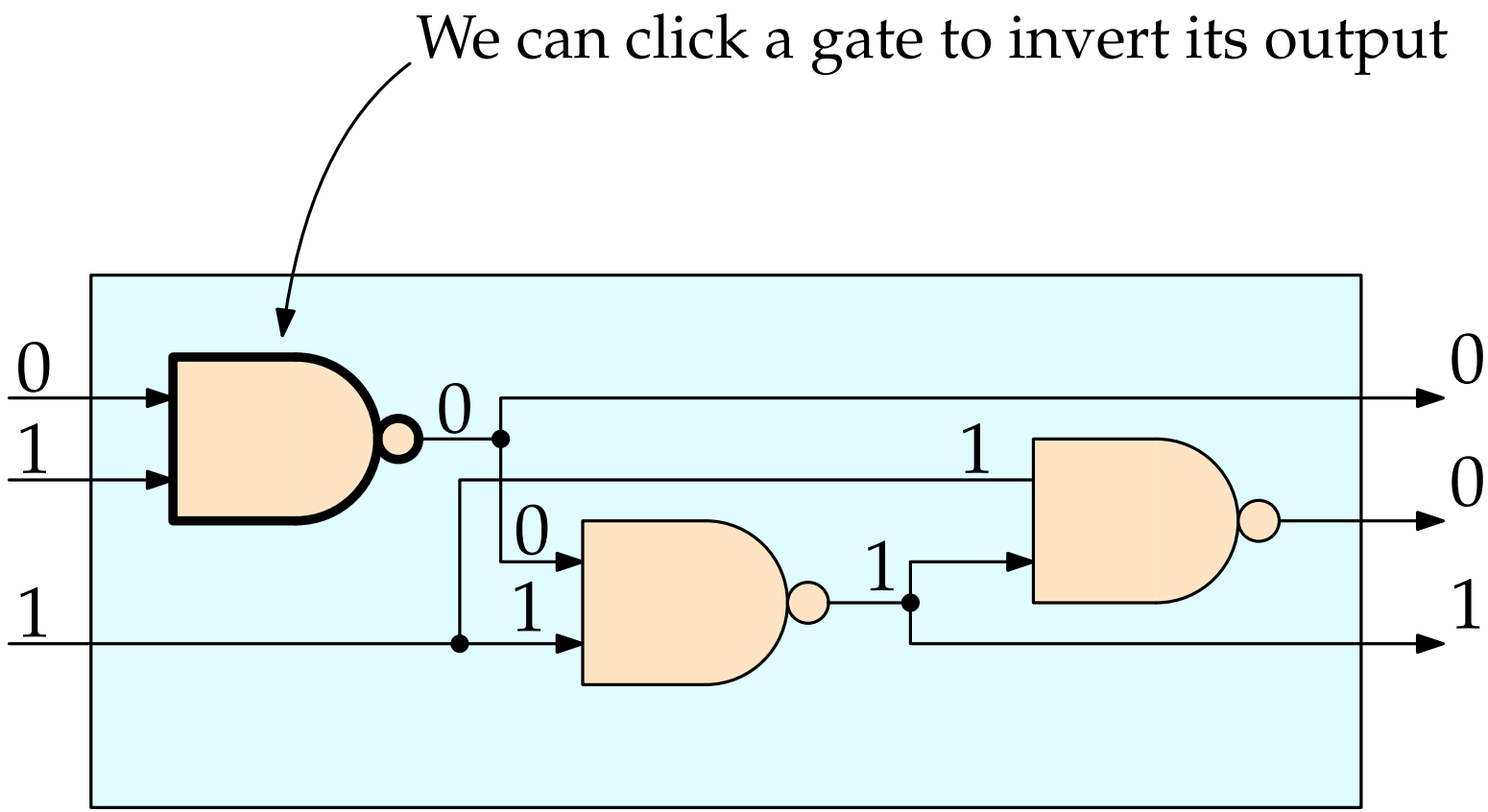
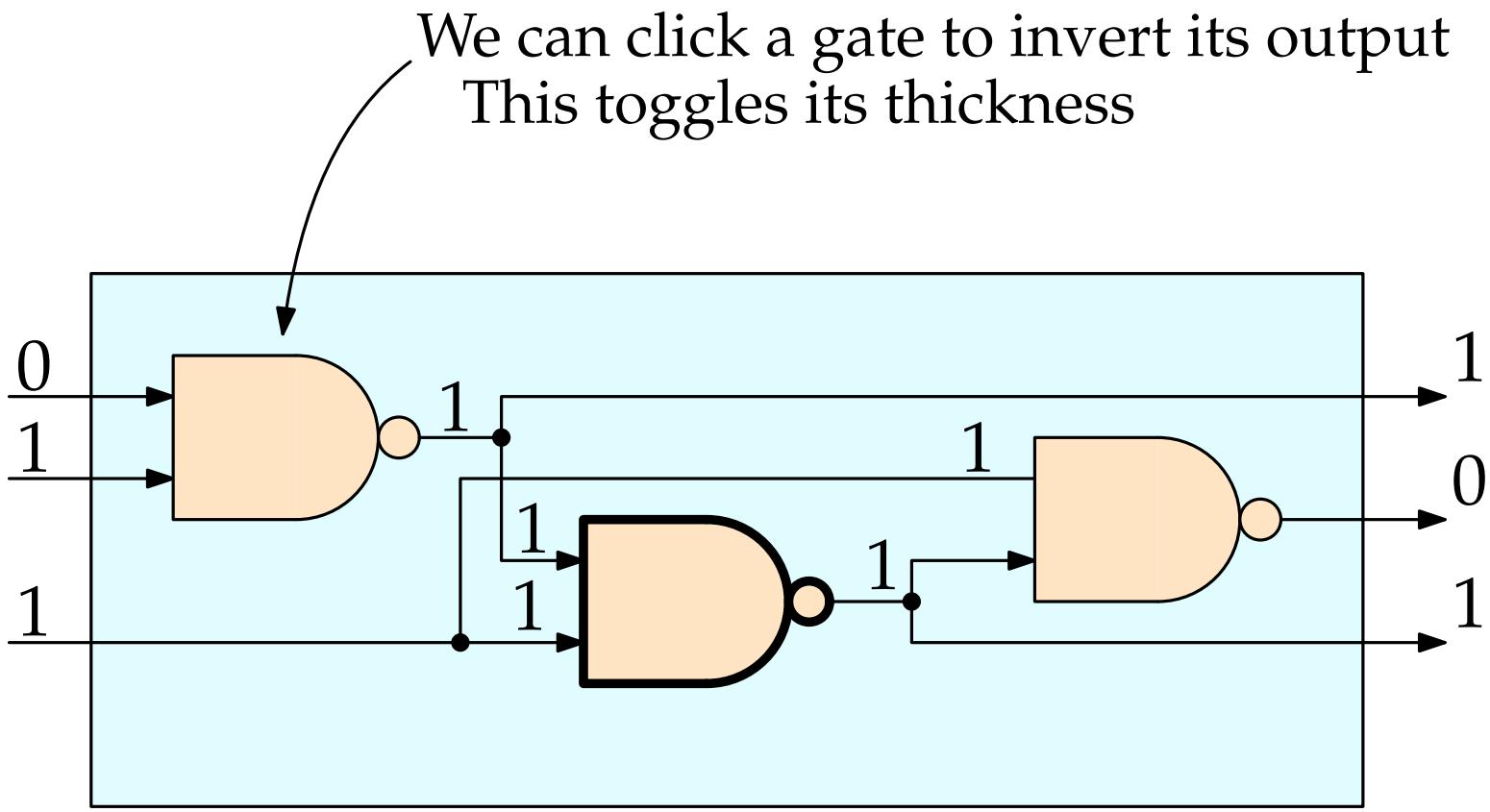


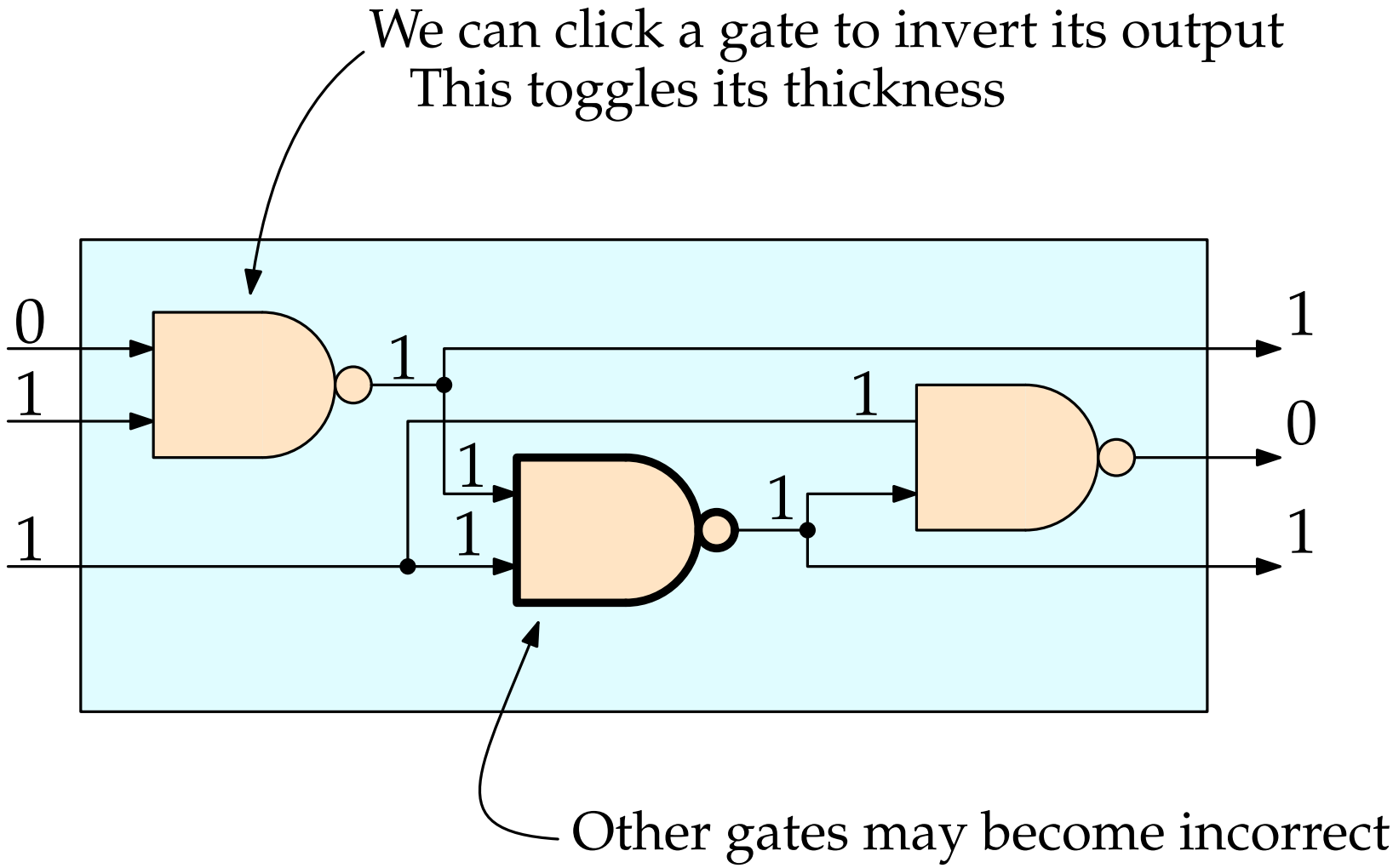
Incorrect gate = thick

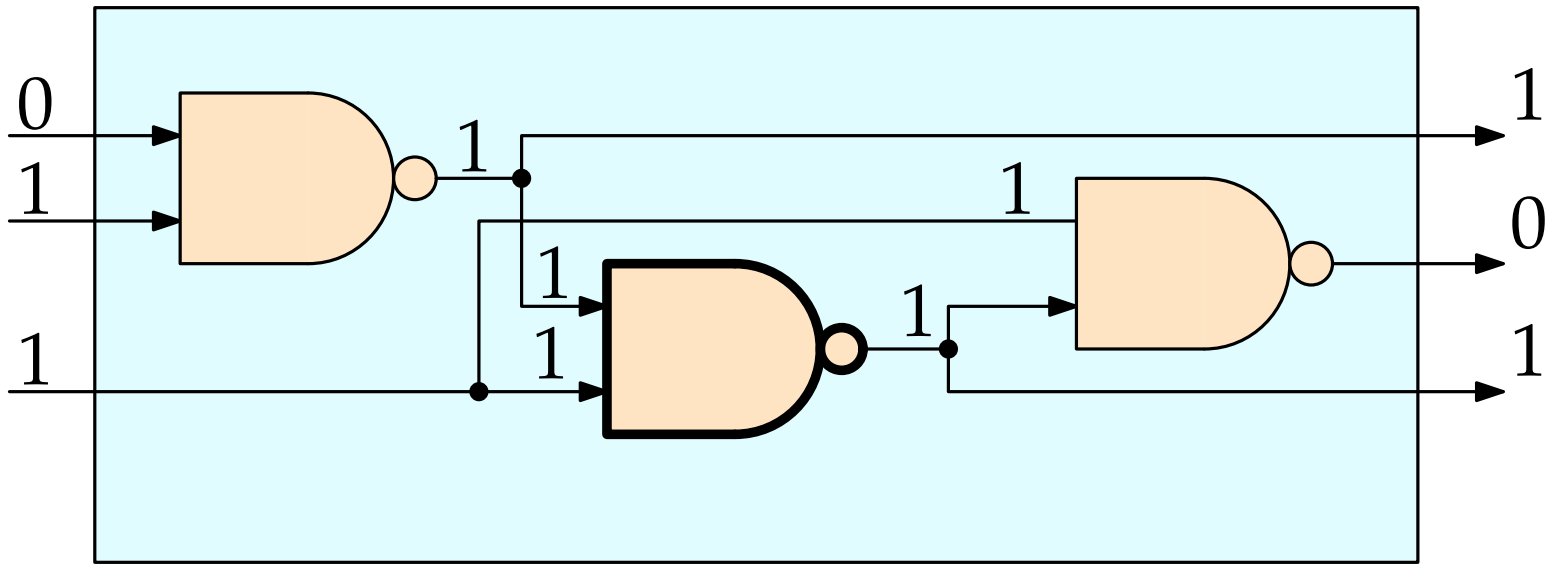
u	v	w
0	0	1
0	1	1
1	0	1
1	1	0

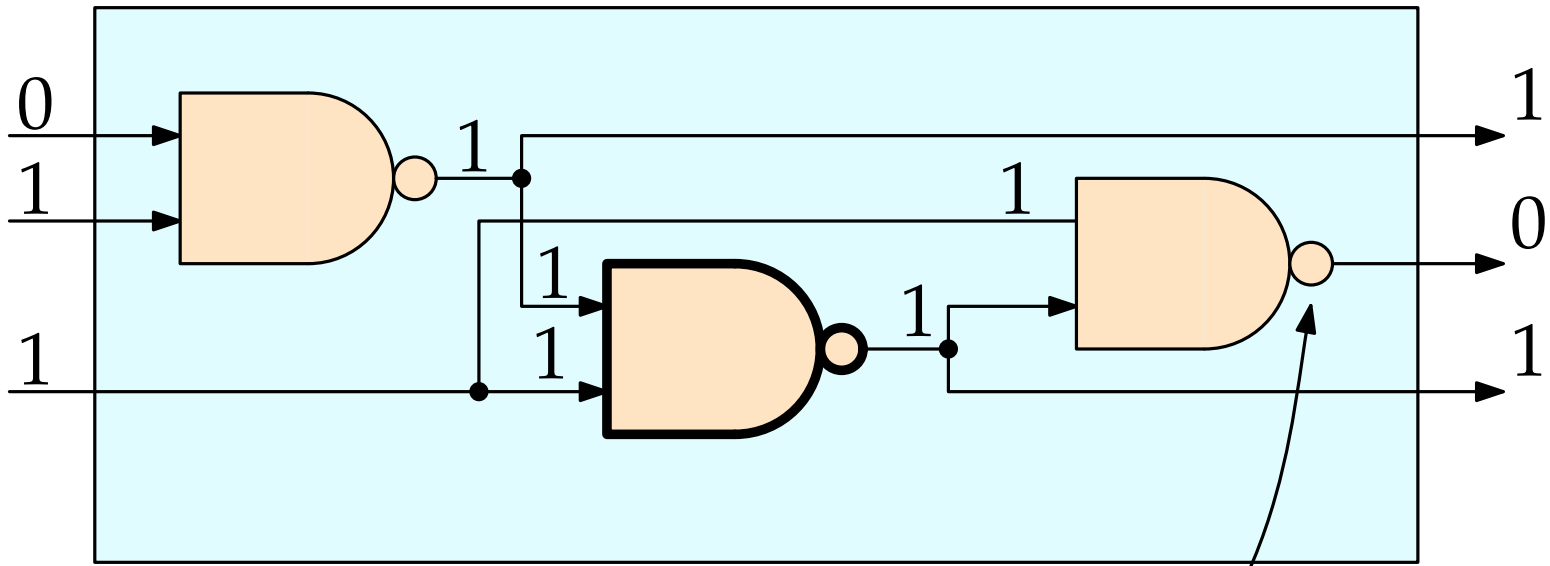

0
1
1
0
0
1
0
1
1
0
0
1


We can click a gate to invert its output
0
1
1
0
0
1
1
0
1
0
1

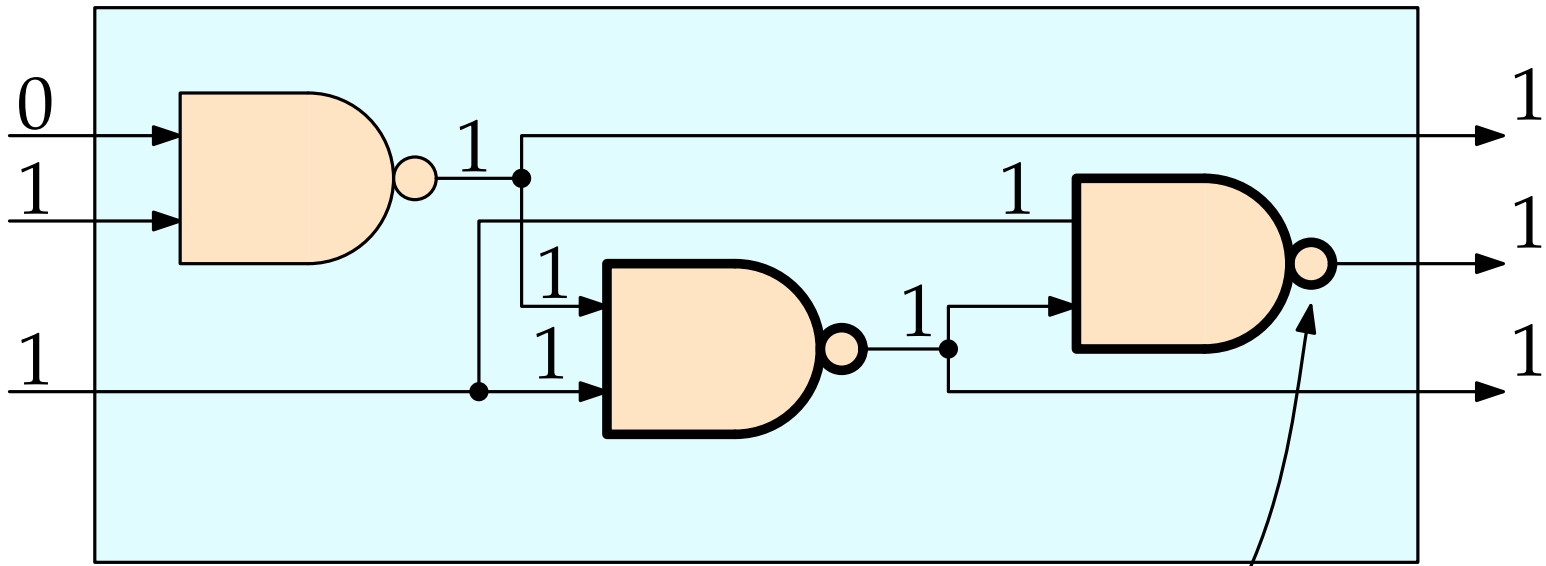

We can click a gate to invert its output
This toggles its thickness
0
1
1
1
1
1
1
1
1
0
1


We can click a gate to invert its output
This toggles its thickness
0
1
1
1
1
1
1
1
1
0
1
Other gates may become incorrect


0
1
1
1
1
1
1
1
1
1
0
1



We can also click a gate
that's currently correct



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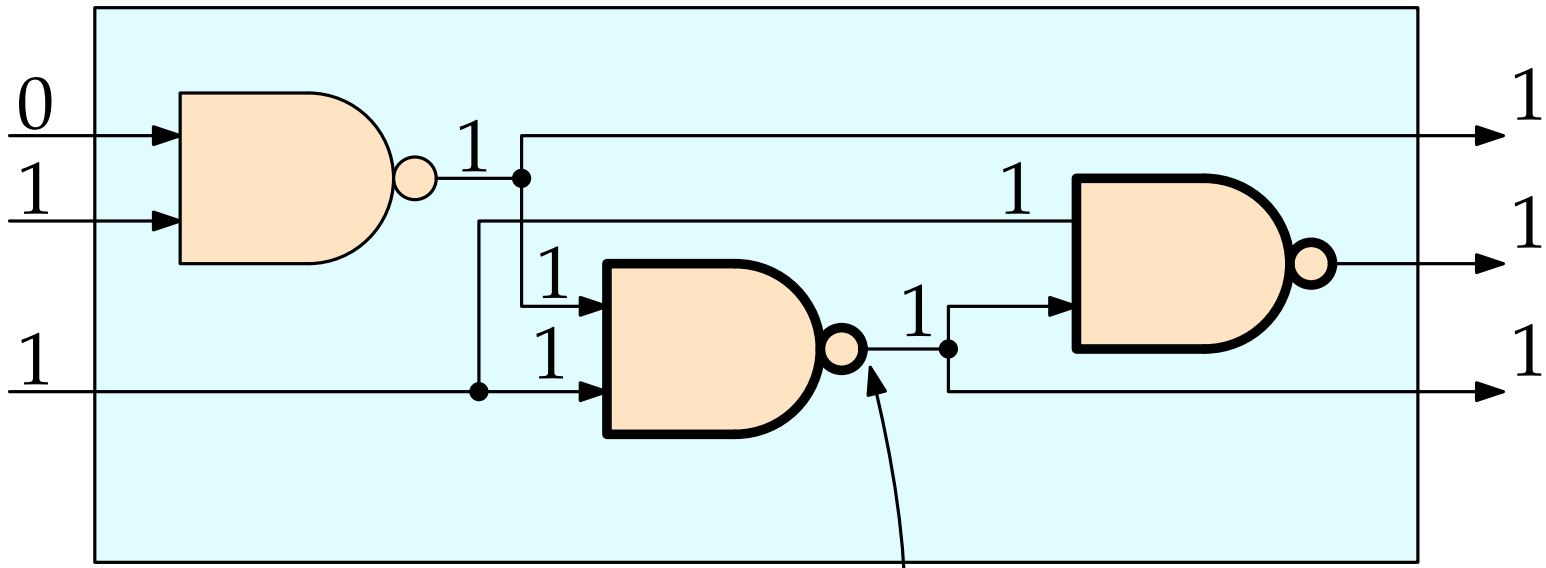
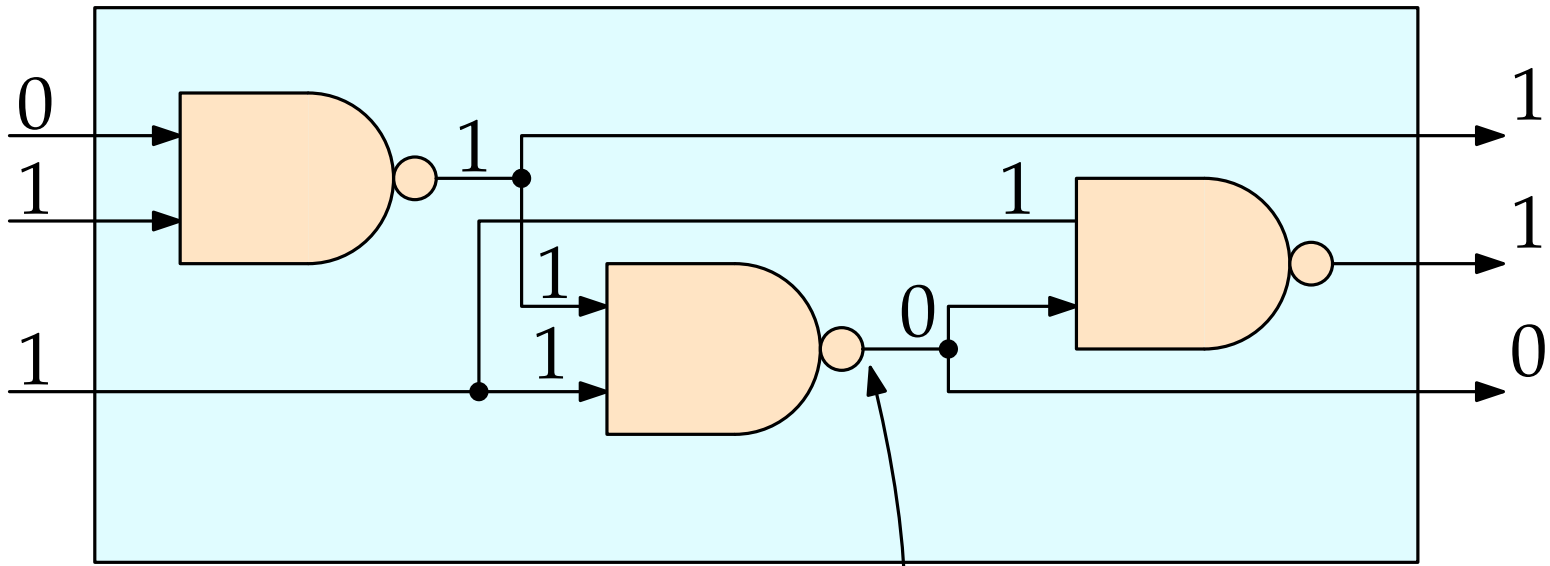


But ultimately we want all gates to be correct, so we click here



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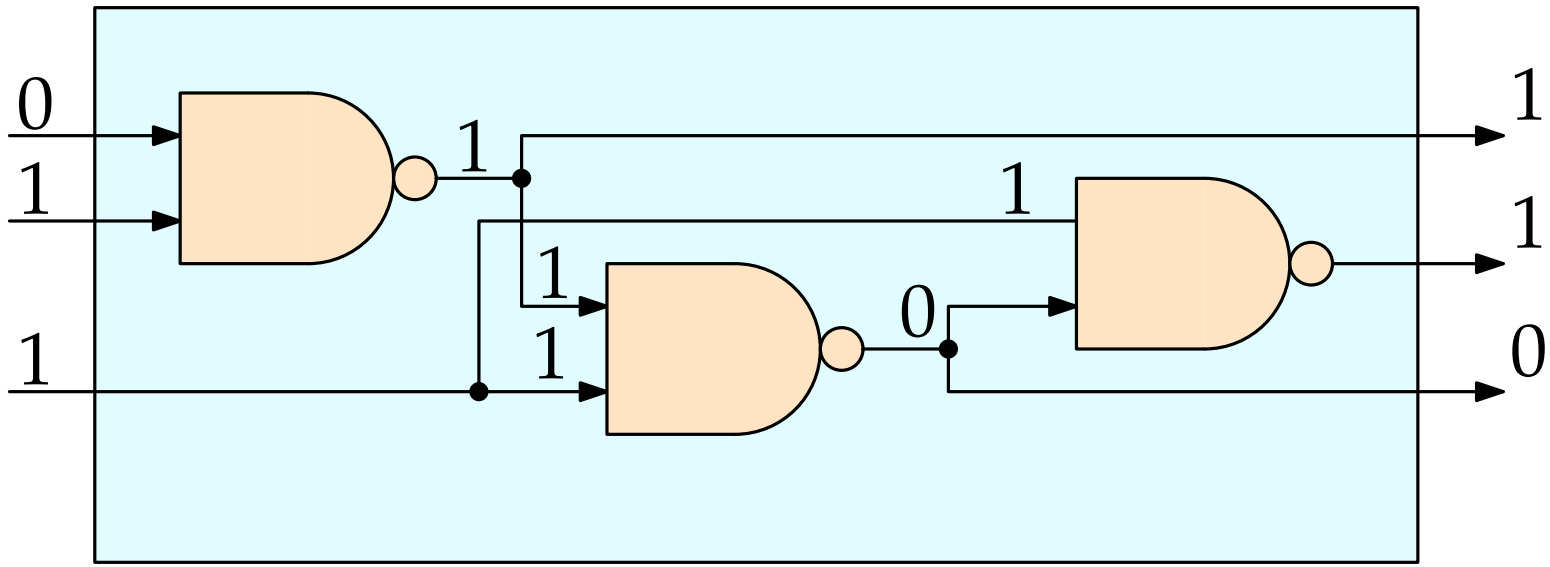
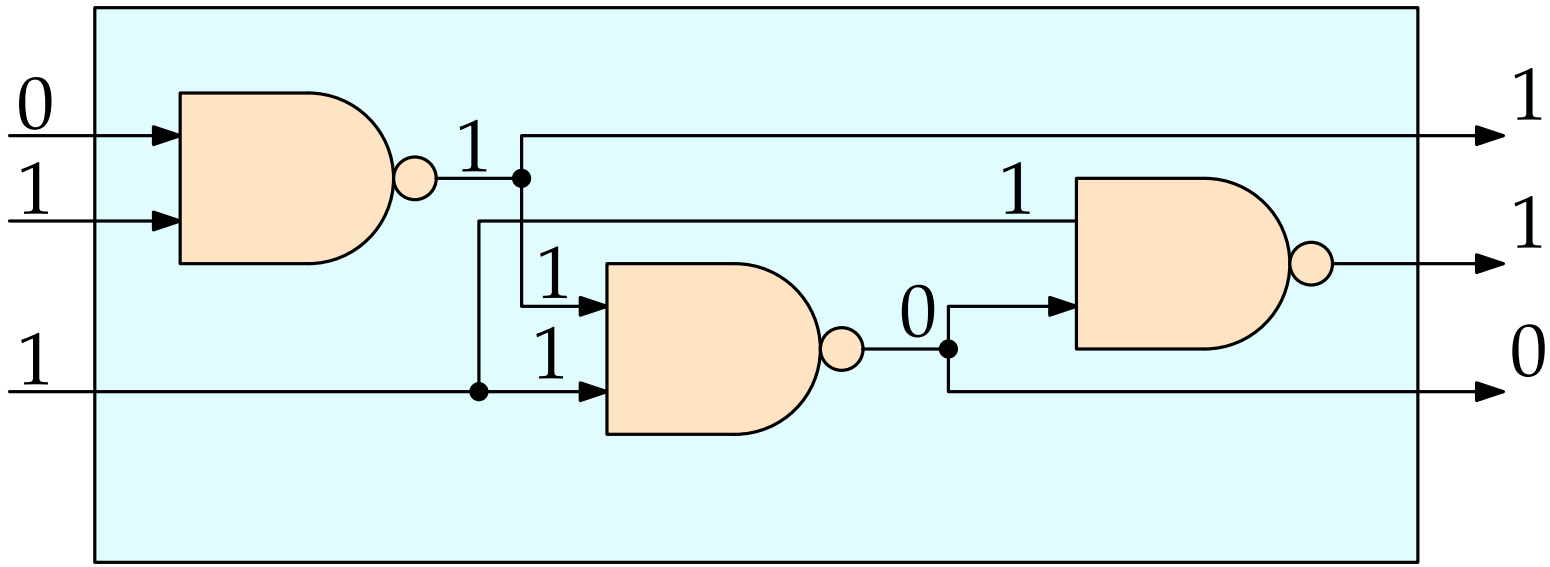
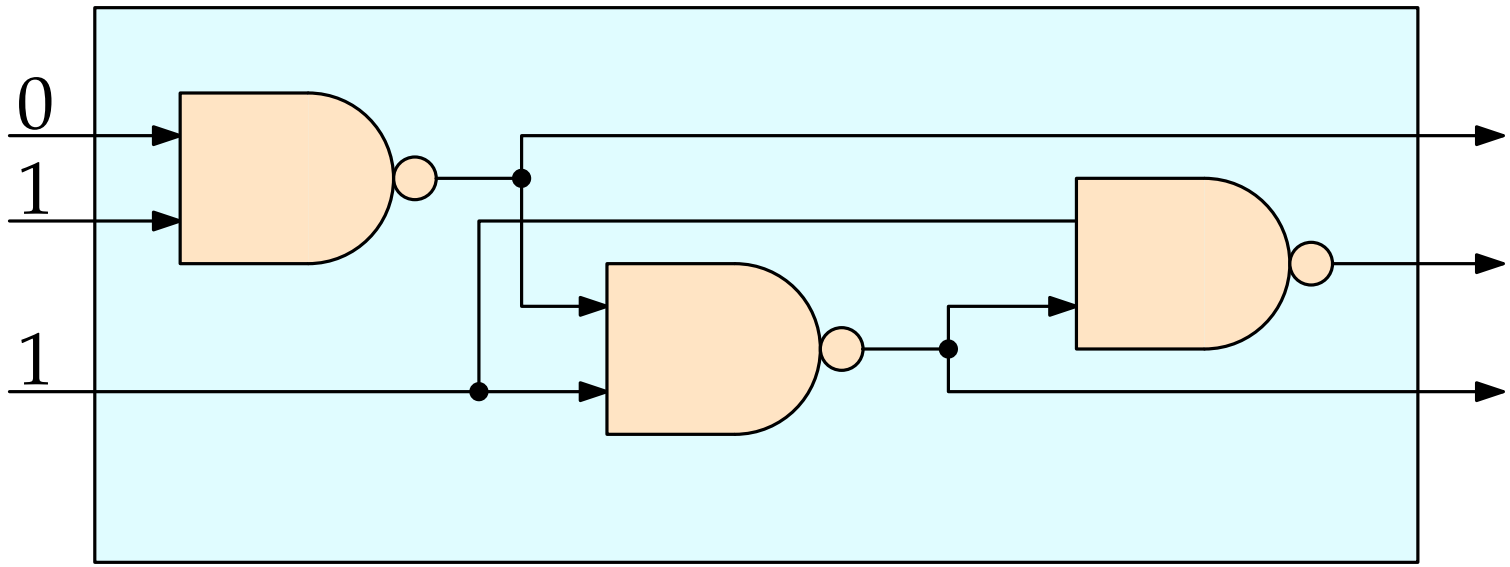
When all gates are correct, then the output is correct.

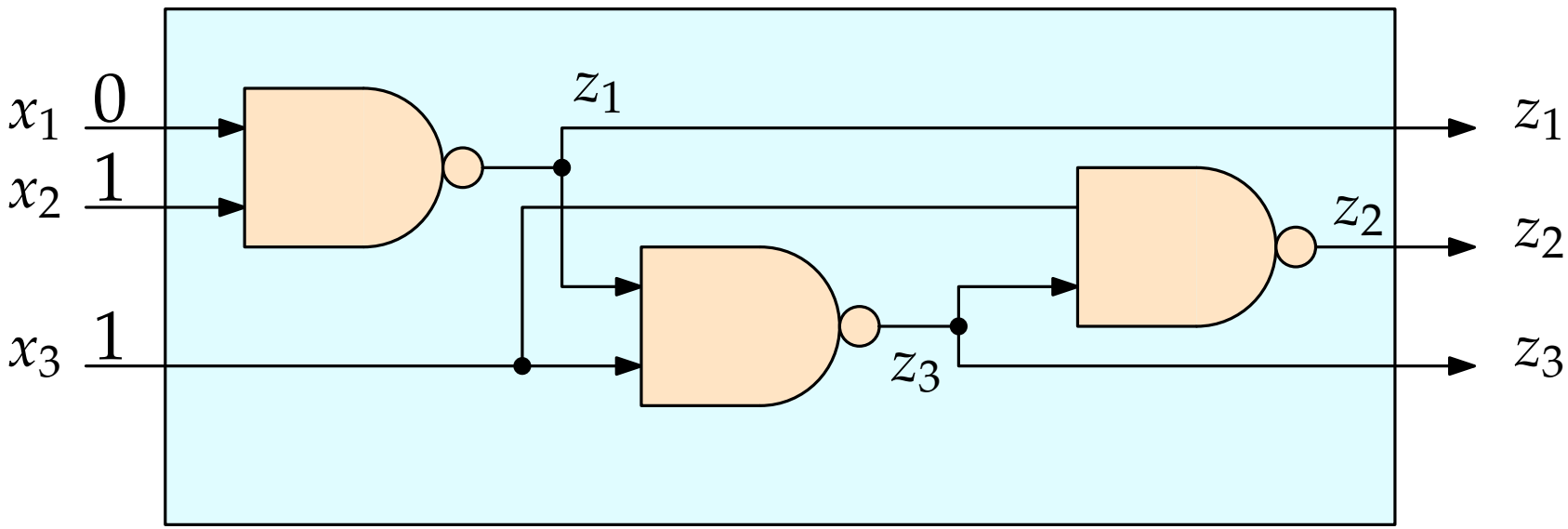


When all gates are correct, then the output is correct.

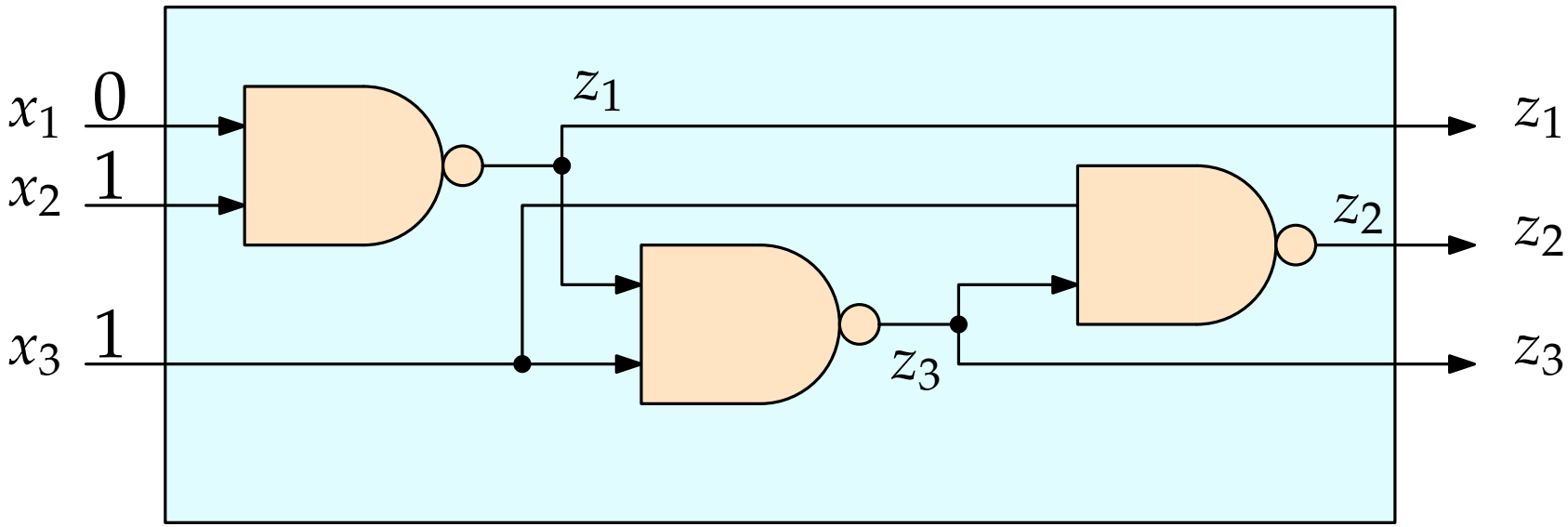


Now build a 3-CNF formula to simulate this!


0
1
1


x_1 0
x_2 1
x_3 1
z_1
z_2
z_3
z_1
z_2
z_3

Here, every gate output is also a circuit output. But that need not the case in general.



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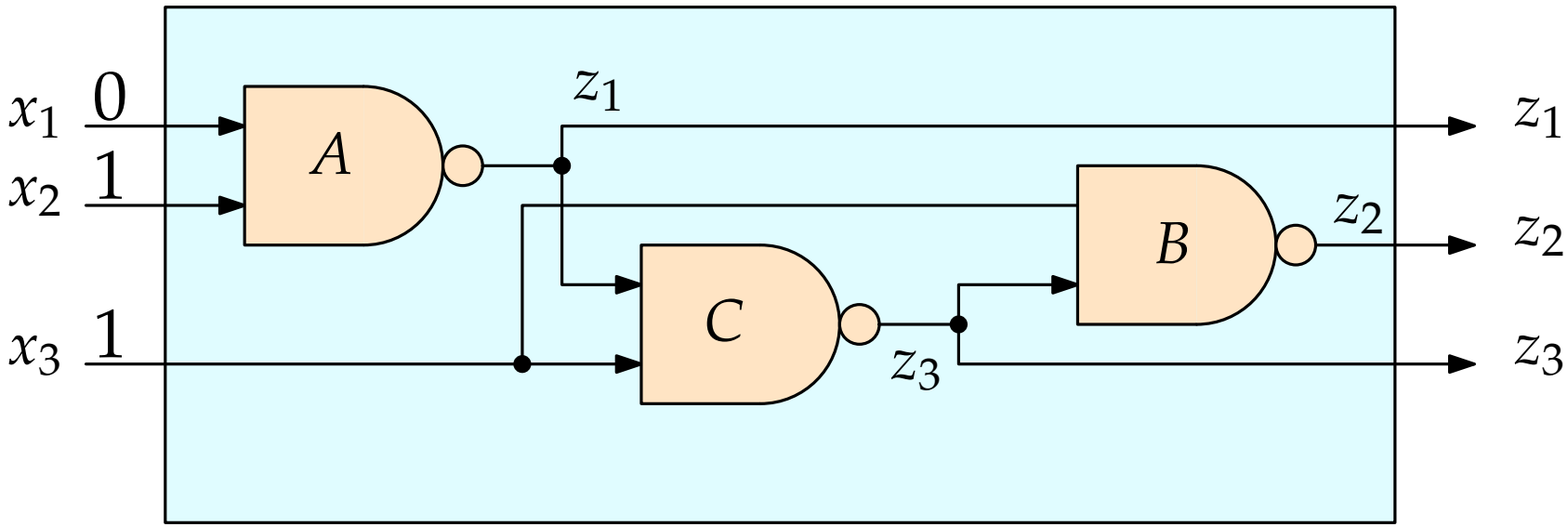

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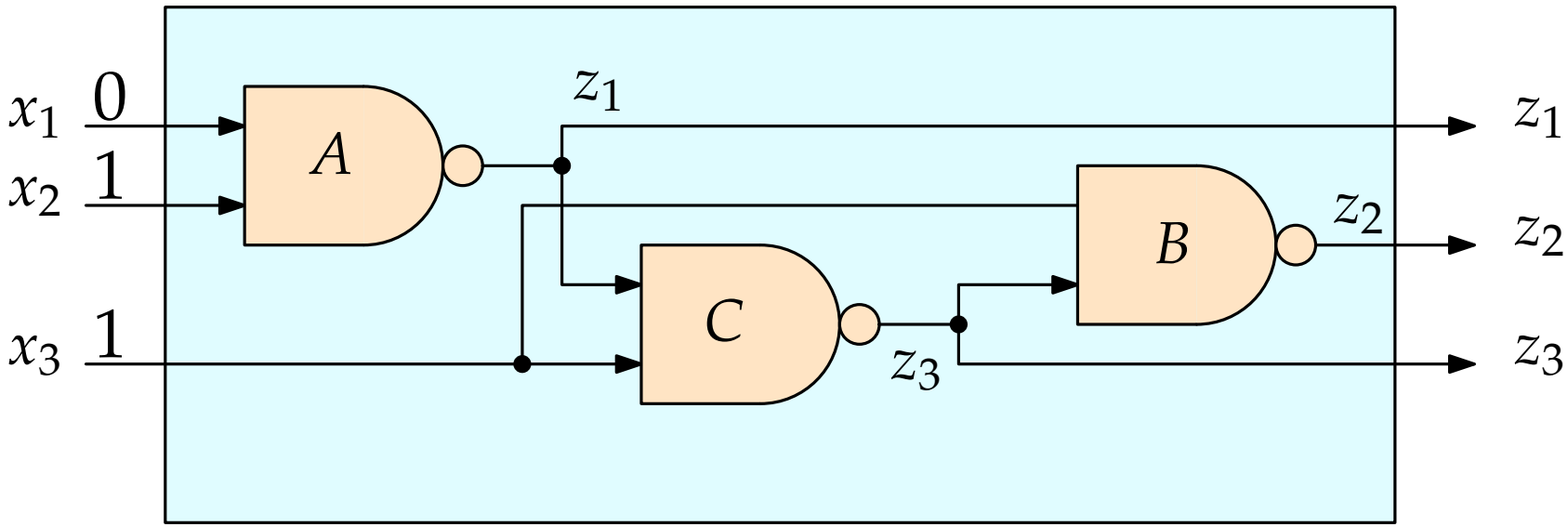
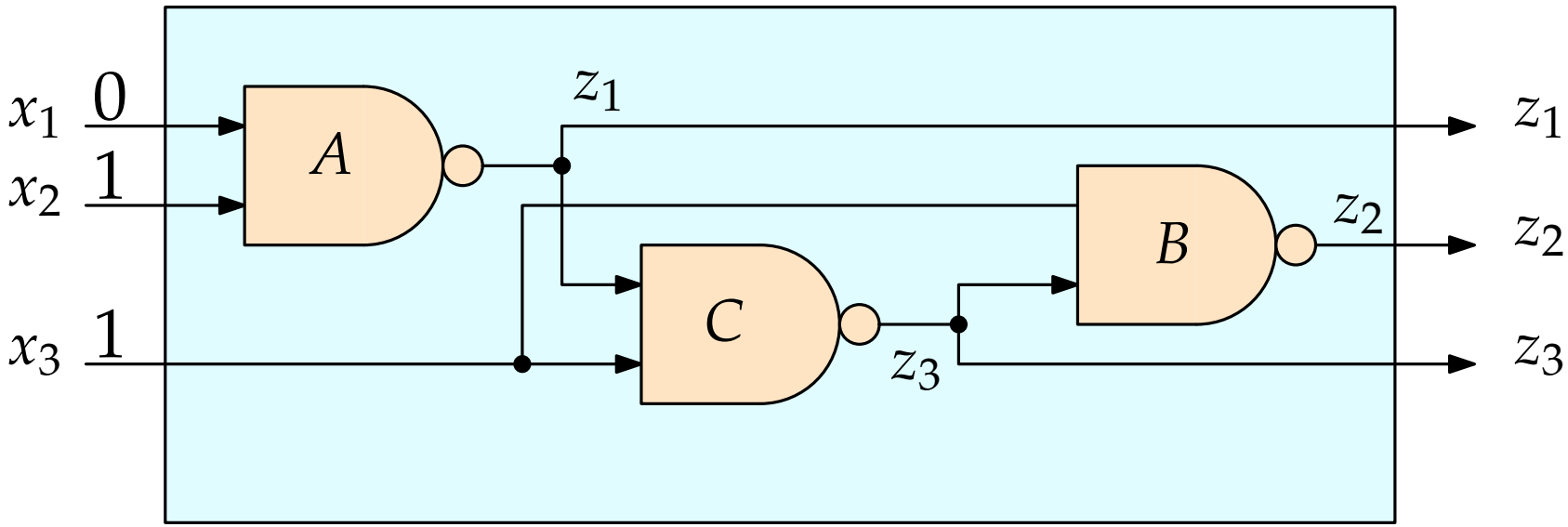


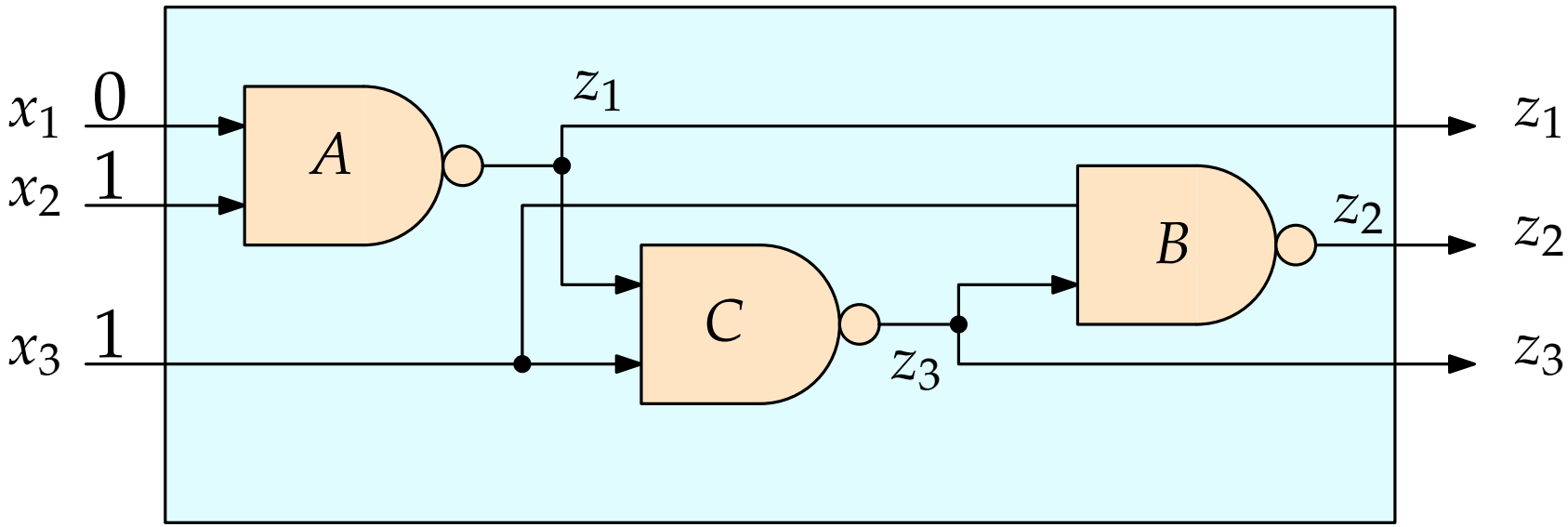
$(\bar{x}_1)$ (x_2) (x_3)

Here, every gate output is also a circuit output. But that need not the case in general.



$$(\bar{x}_1) \quad (x_2) \quad (x_3) \quad (z_1 \leftrightarrow \neg(x_1 \wedge x_2))$$
$$(z_3 \leftrightarrow \neg(z_1 \wedge x_3))$$
$$(z_2 \leftrightarrow \neg(x_3 \wedge z_3))$$

Here, every gate output is also a circuit output. But that need not the case in general.

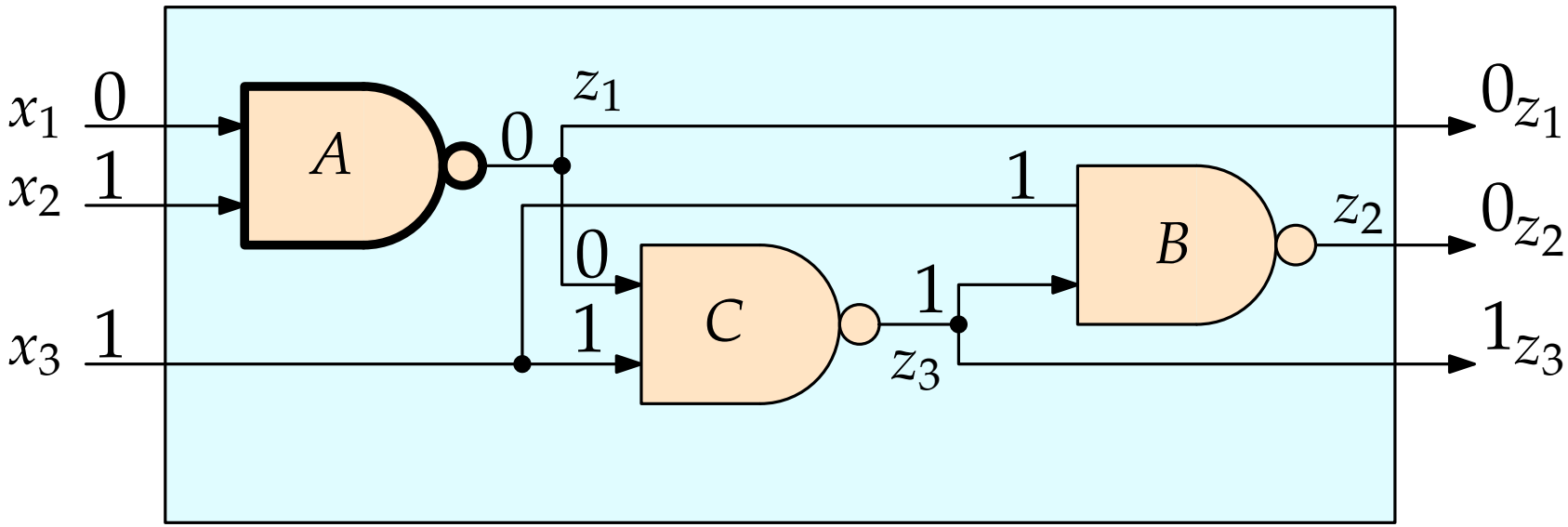


$(\bar{x}_1)$ (x_2) (x_3) $(z_1 \leftrightarrow \neg(x_1 \wedge x_2))$ 4

32 16 8 $(z_3 \leftrightarrow \neg(z_1 \wedge x_3))$ 2

$(z_2 \leftrightarrow \neg(x_3 \wedge z_3))$ 1

Here, every gate output is also a circuit output. But that need not the case in general.



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32 16 8 $(z_3 \leftrightarrow \neg(z_1 \wedge x_3))$ 2

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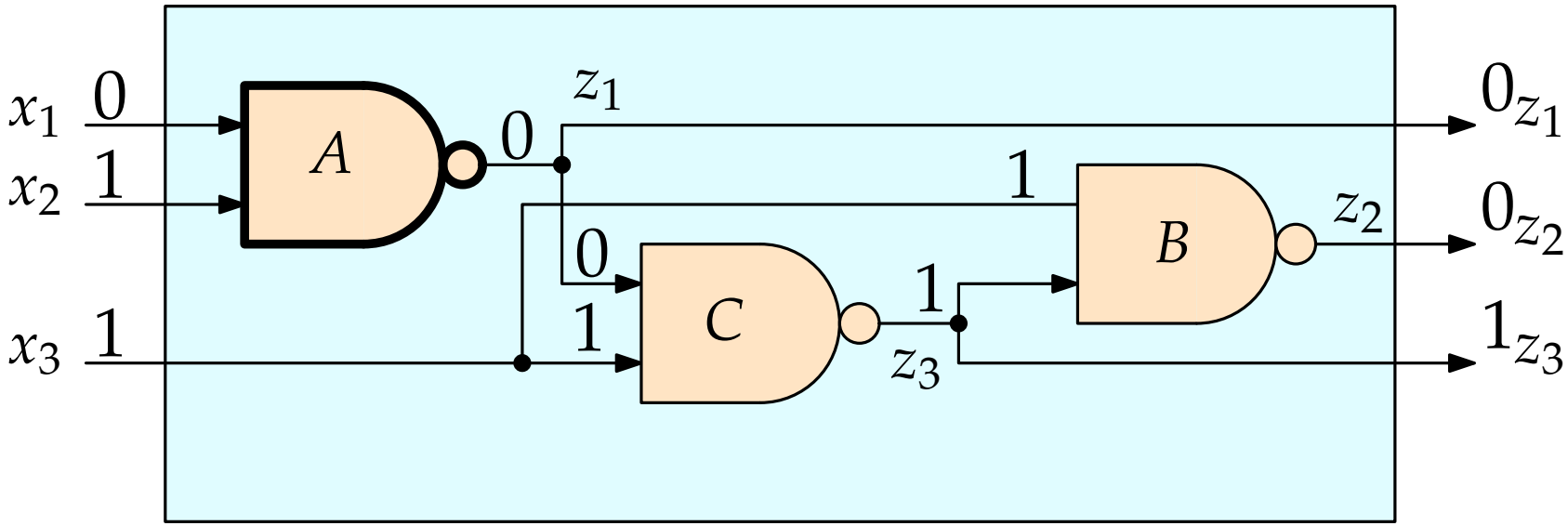

$(\bar{x}_1)$ (x_2) (x_3) $(z_1 \leftrightarrow \neg(x_1 \wedge x_2))$ 4

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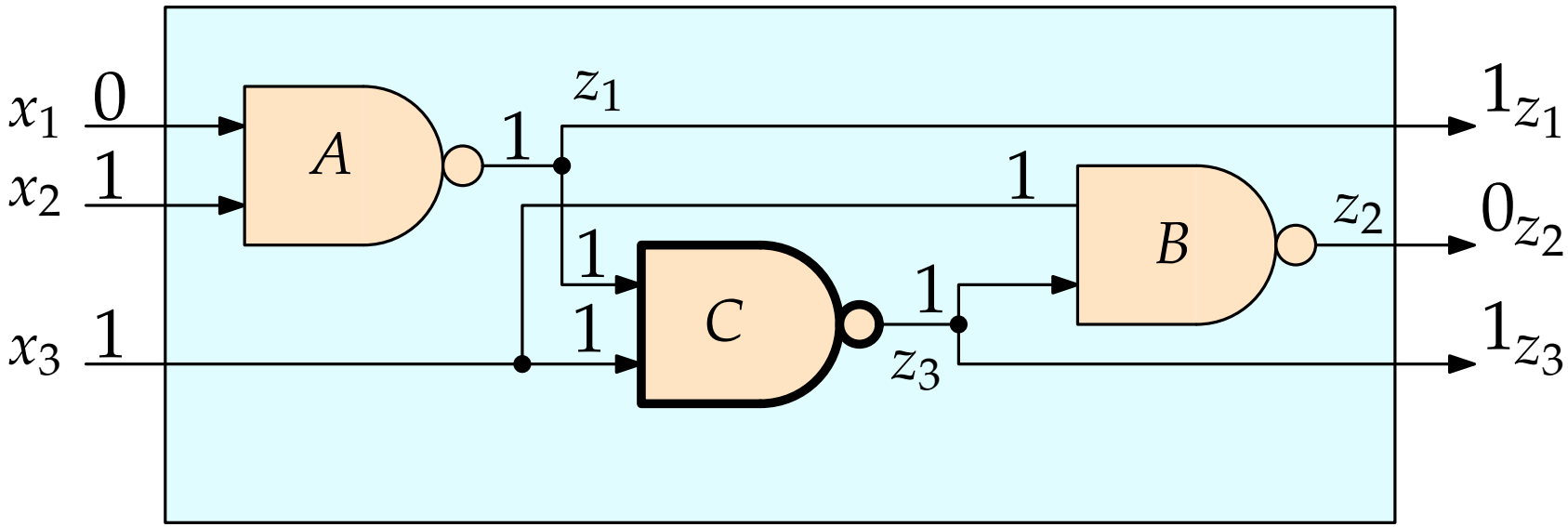

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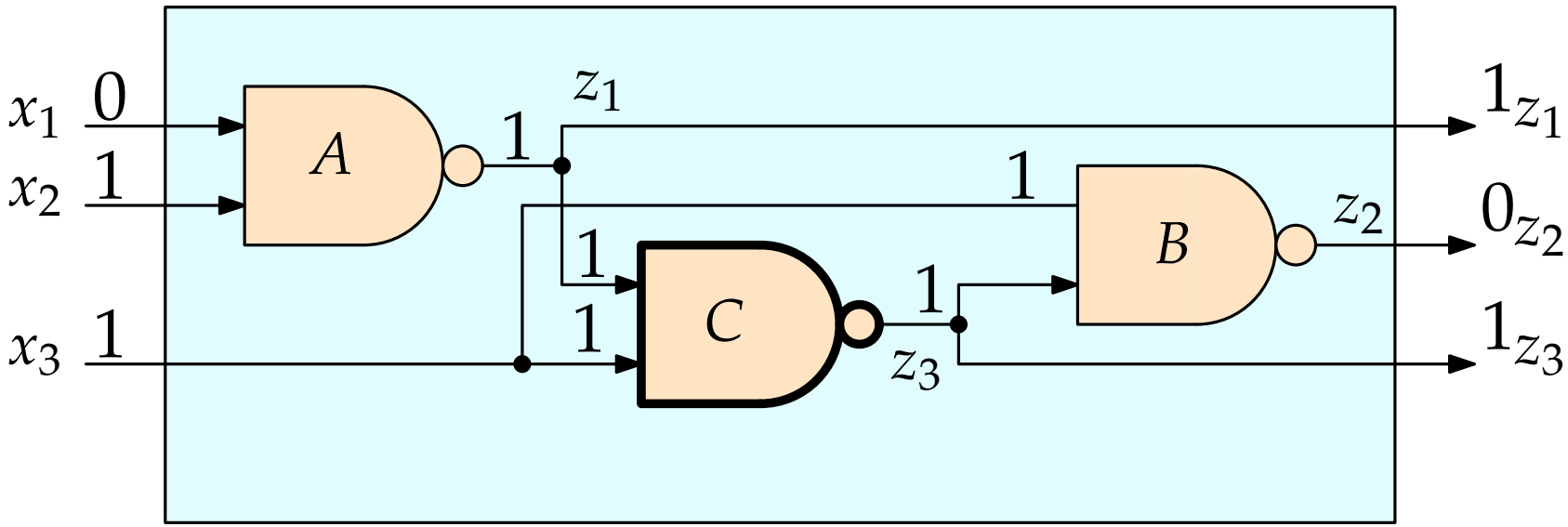

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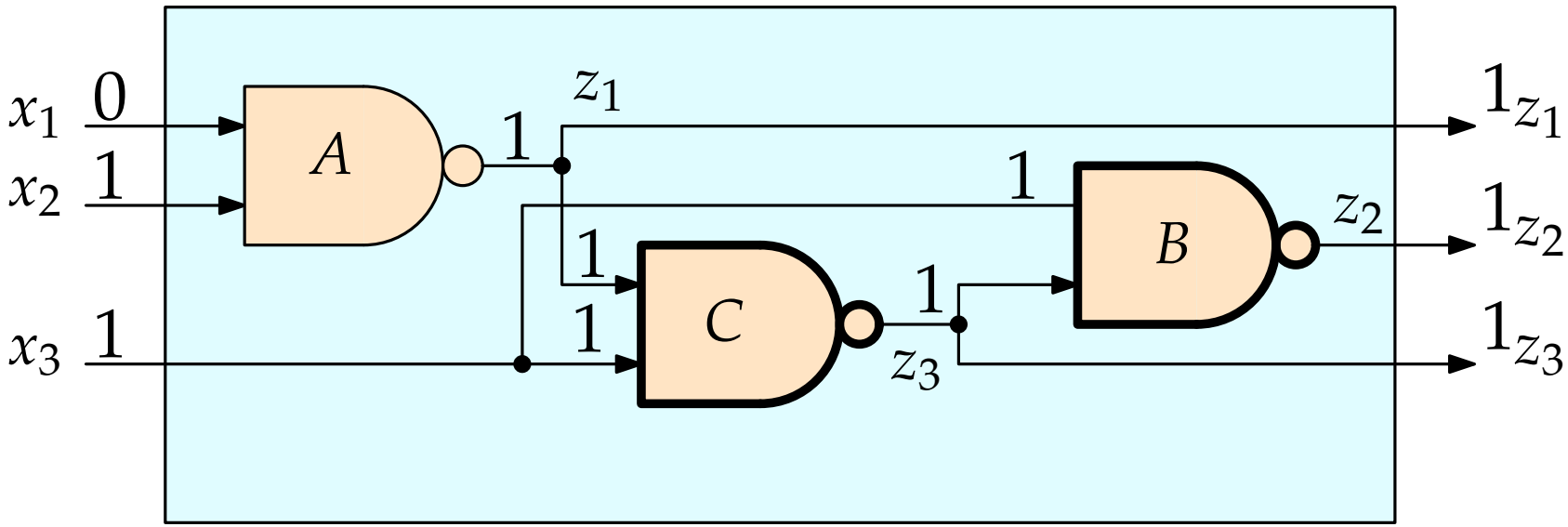

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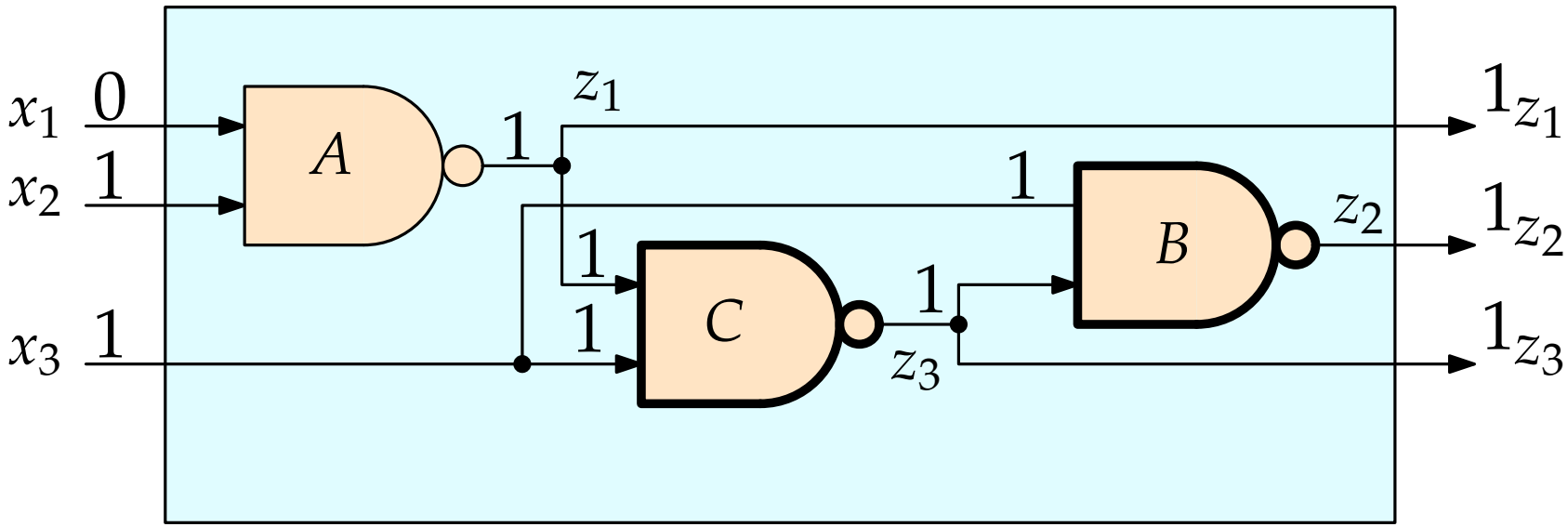

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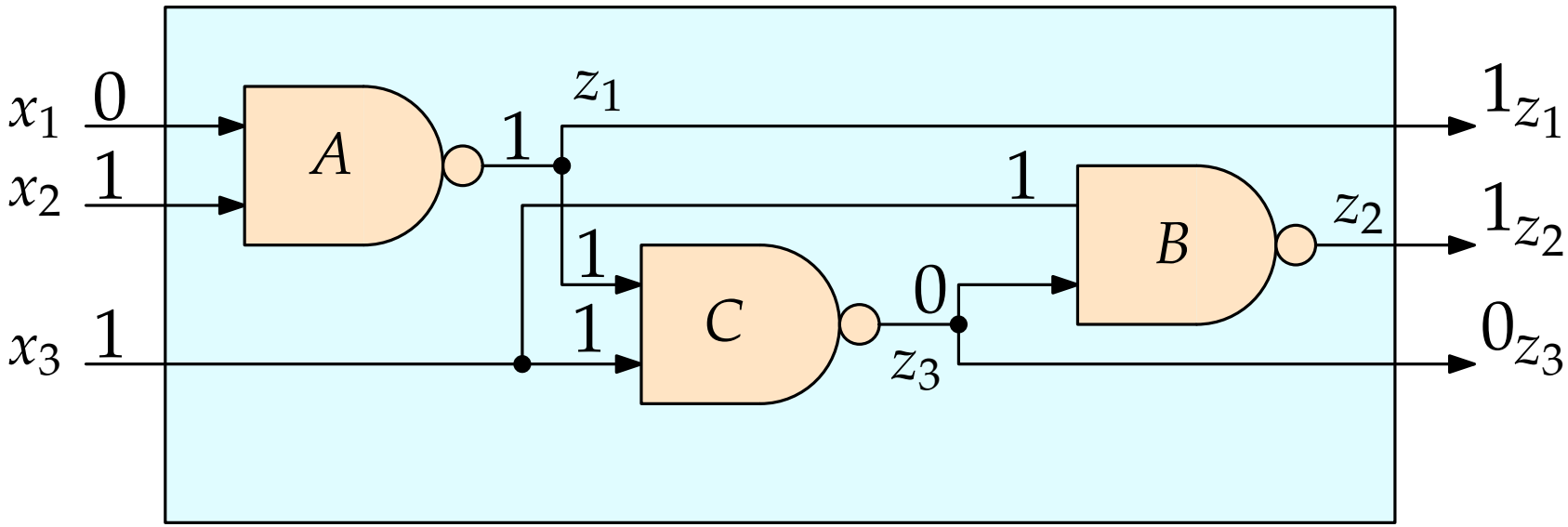

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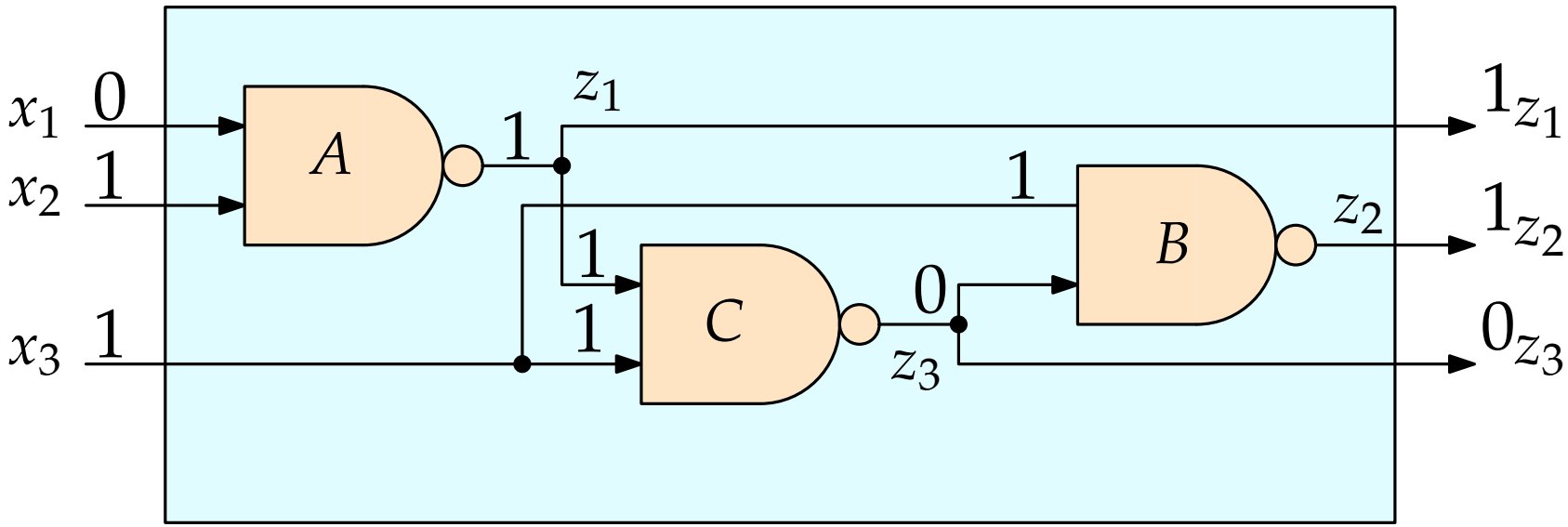

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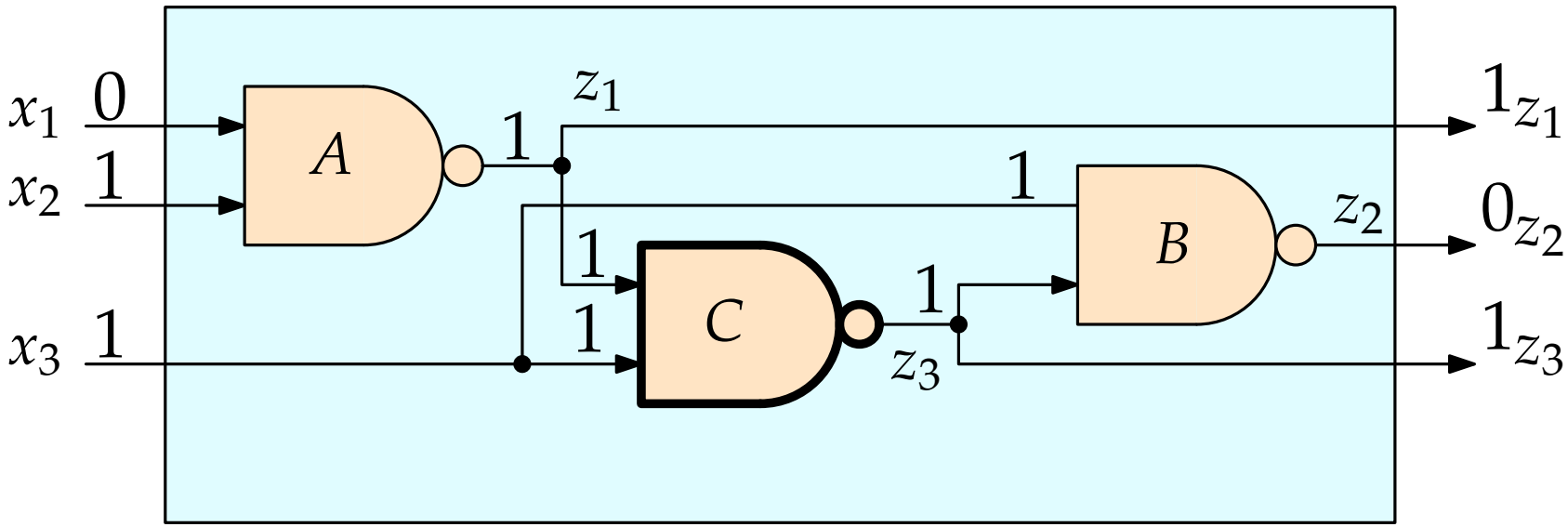
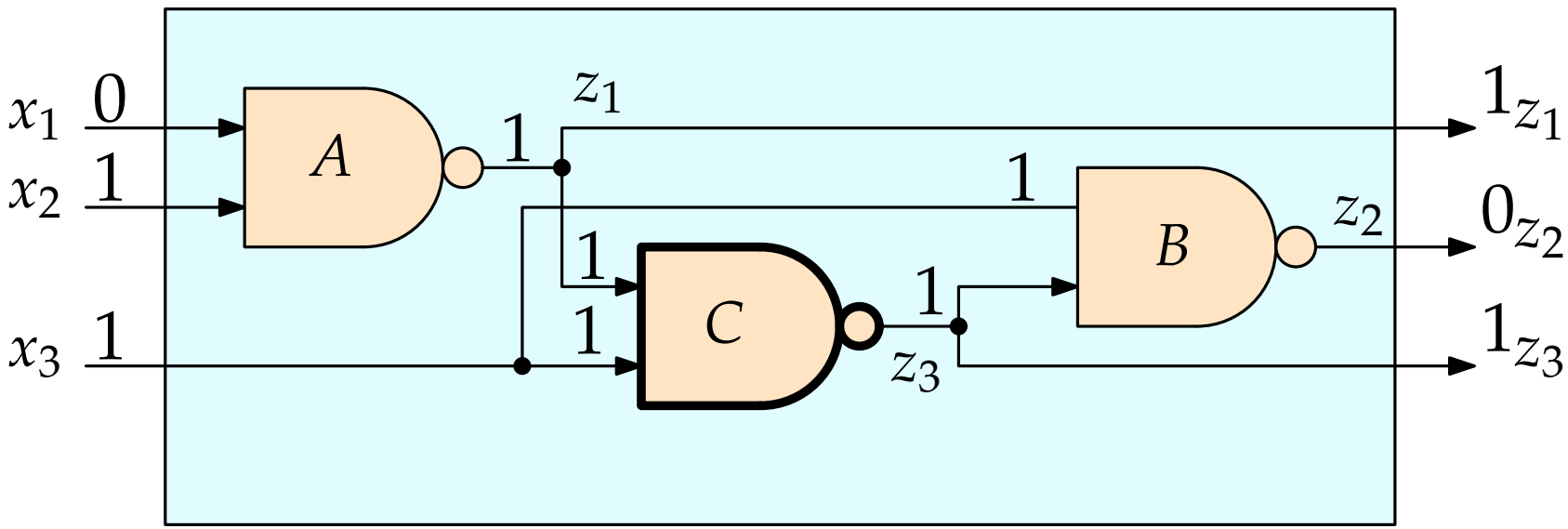
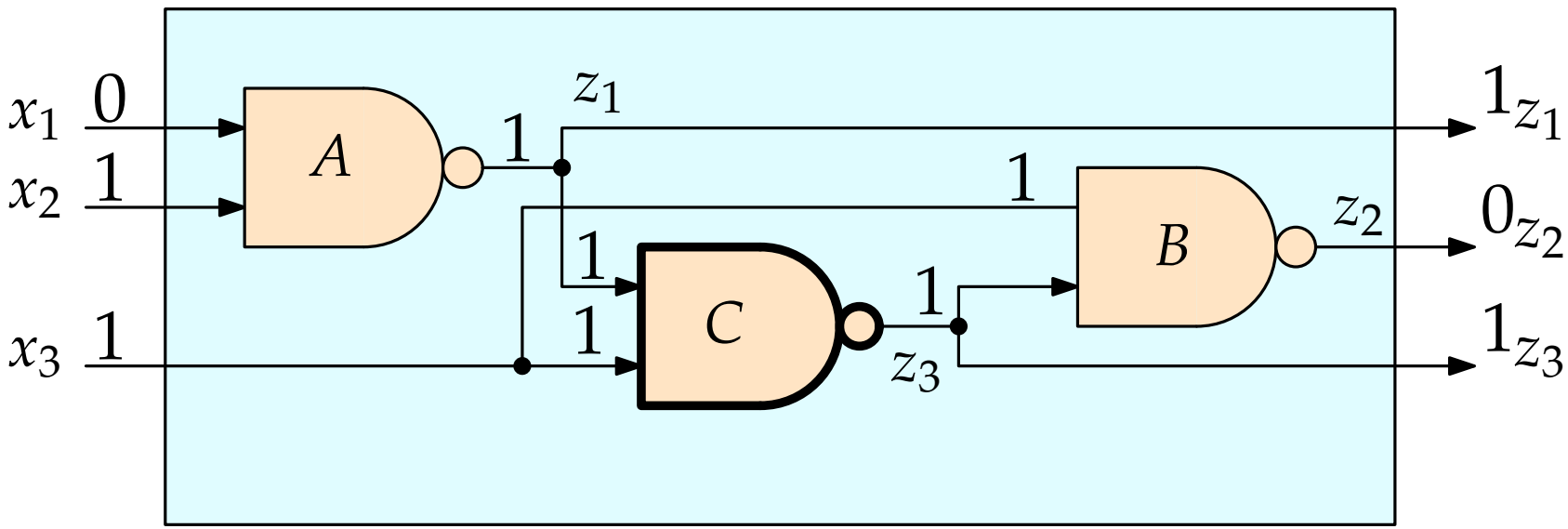
$(z_2 \leftrightarrow \neg(x_3 \wedge z_3))$ 1

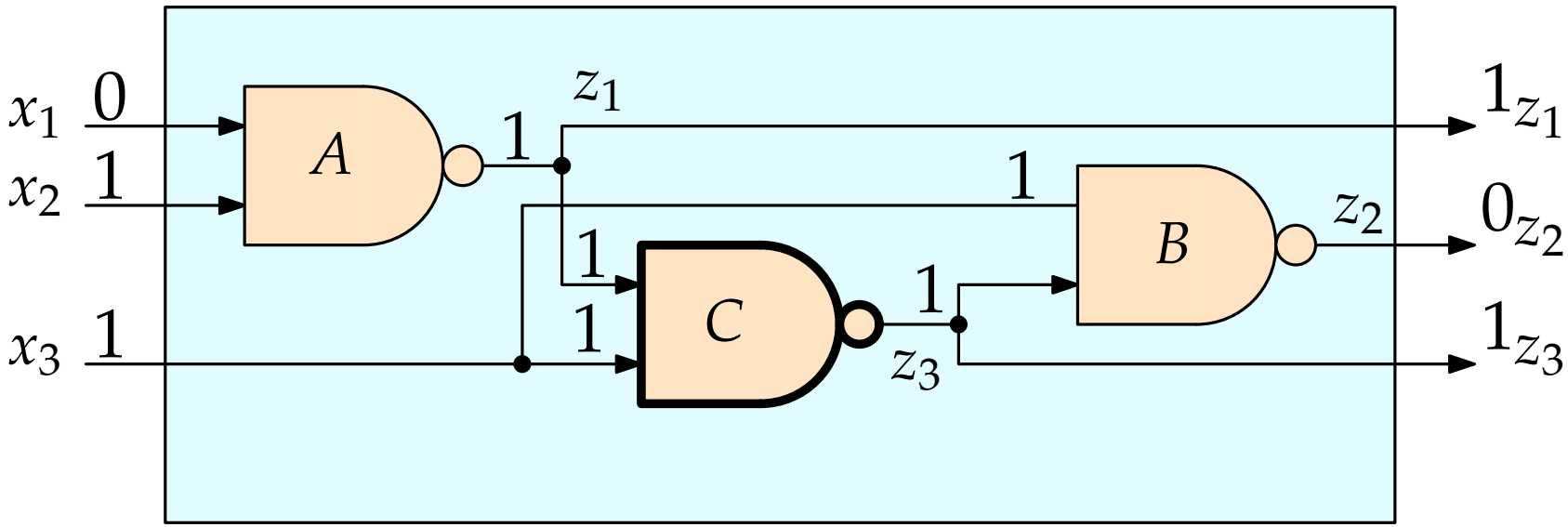

x_1 0
x_2 1
x_3 1
A
B
C
1
1
1
1
1
z_1
z_2
z_3
1_{z_1}
0_{z_2}
1_{z_3}



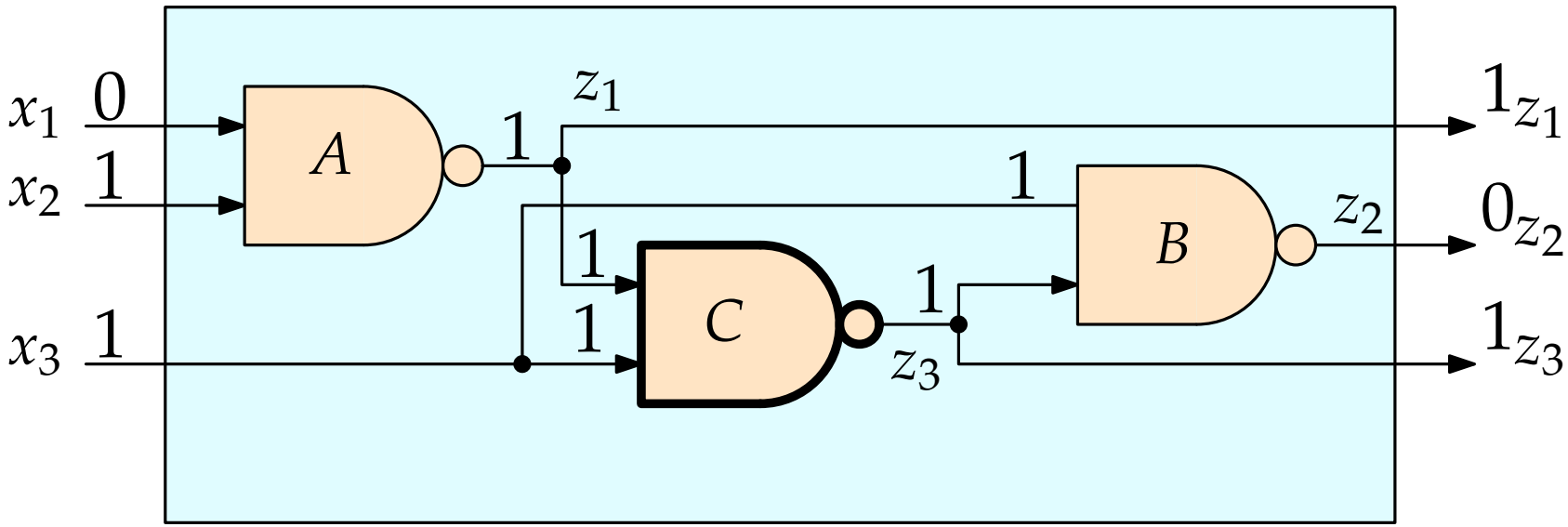
As long as a gate as violated, click the leftmost violated gate.



As long as a gate as violated, click the leftmost violated gate. This will satisfy its clause C.



As long as a gate as violated, click the leftmost violated gate.
This will satisfy its clause C.
Downstream clauses might become violated.



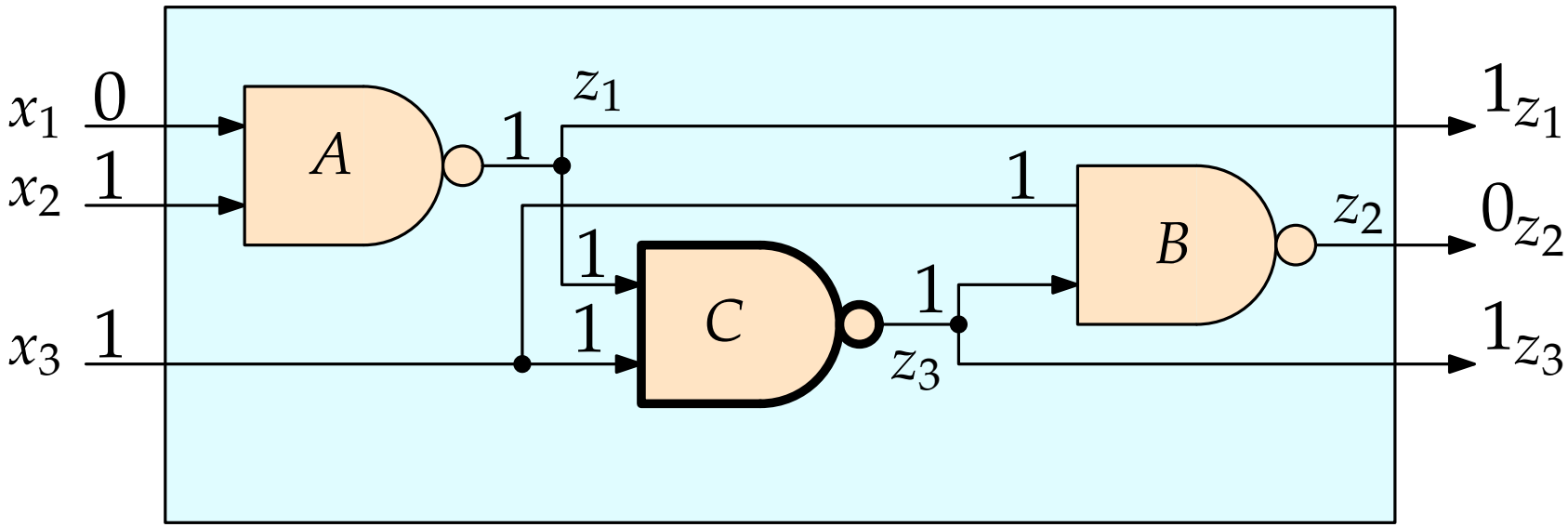
As long as a gate as violated, click the leftmost violated gate.

This will satisfy its clause C.

Downstream clauses might become violated.

It's fine because C is more valuable than all downstream clauses combined.

Values of $\vec{z}$ in the local optimum are the correct evaluation of C.



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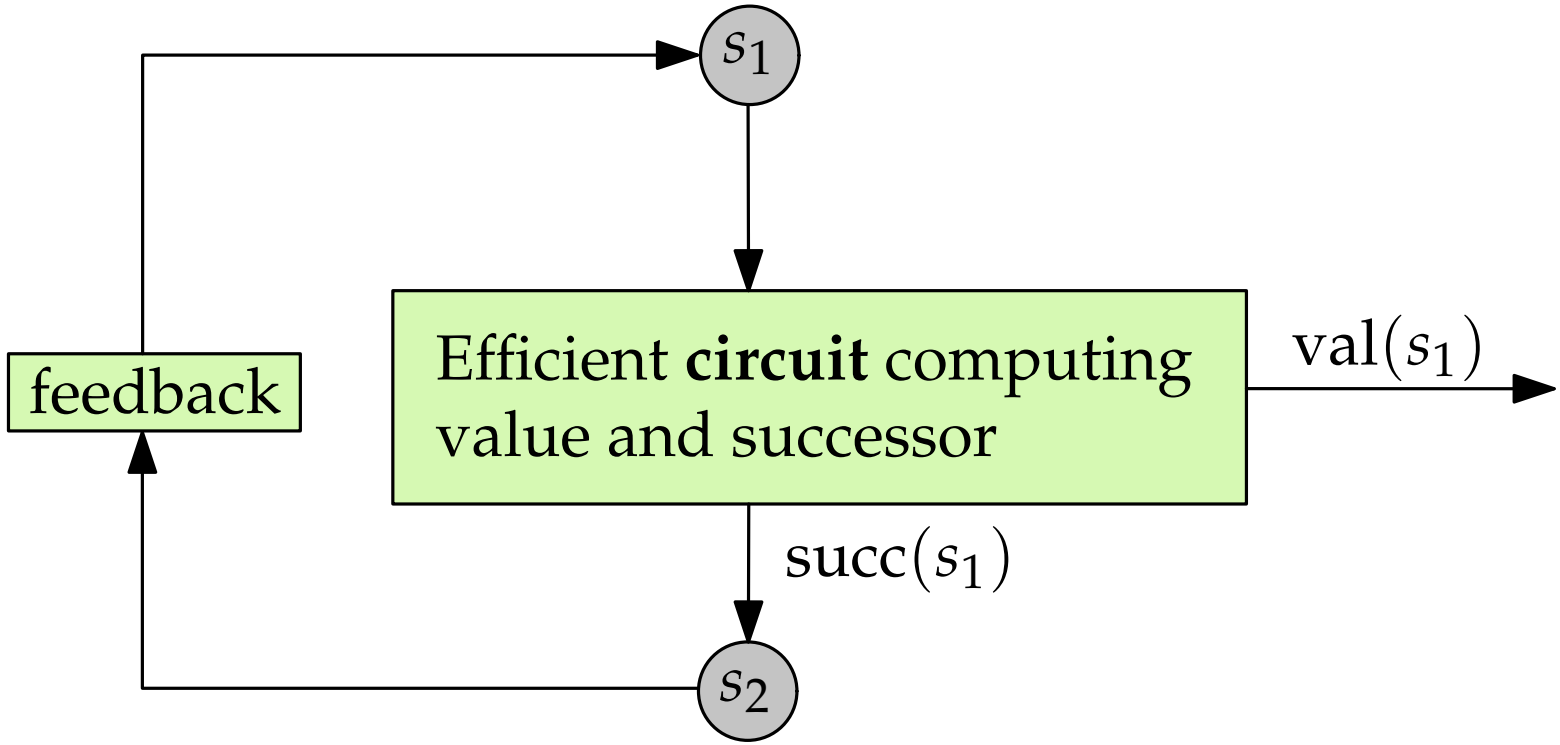
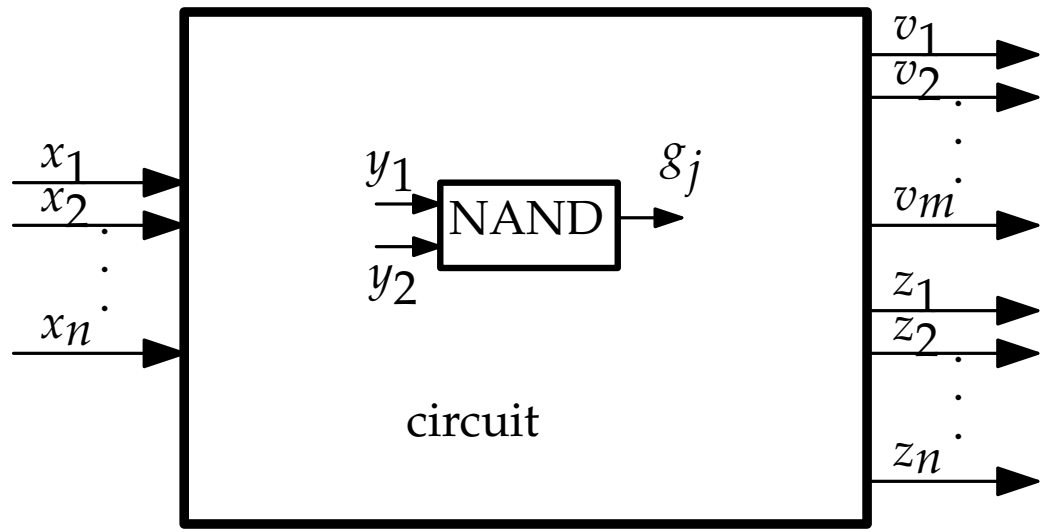
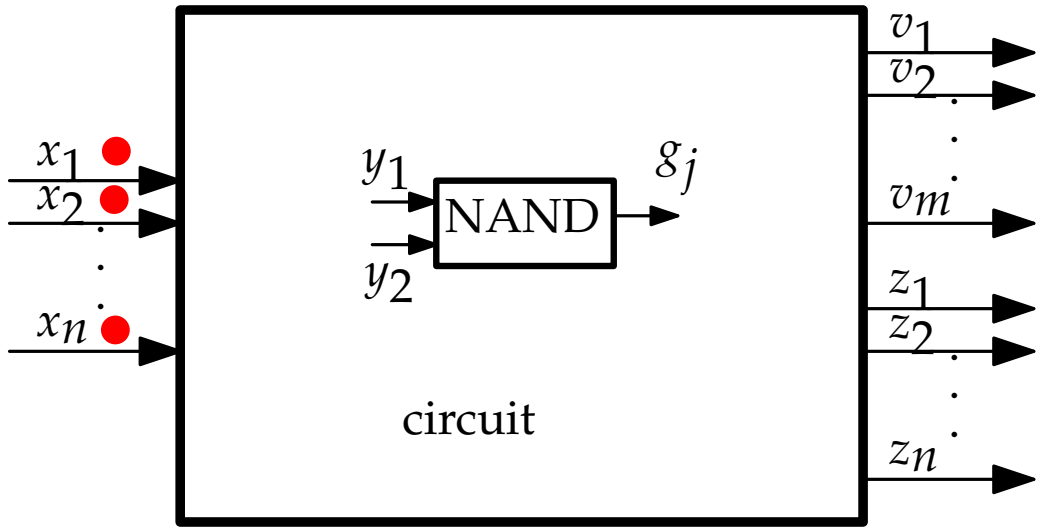
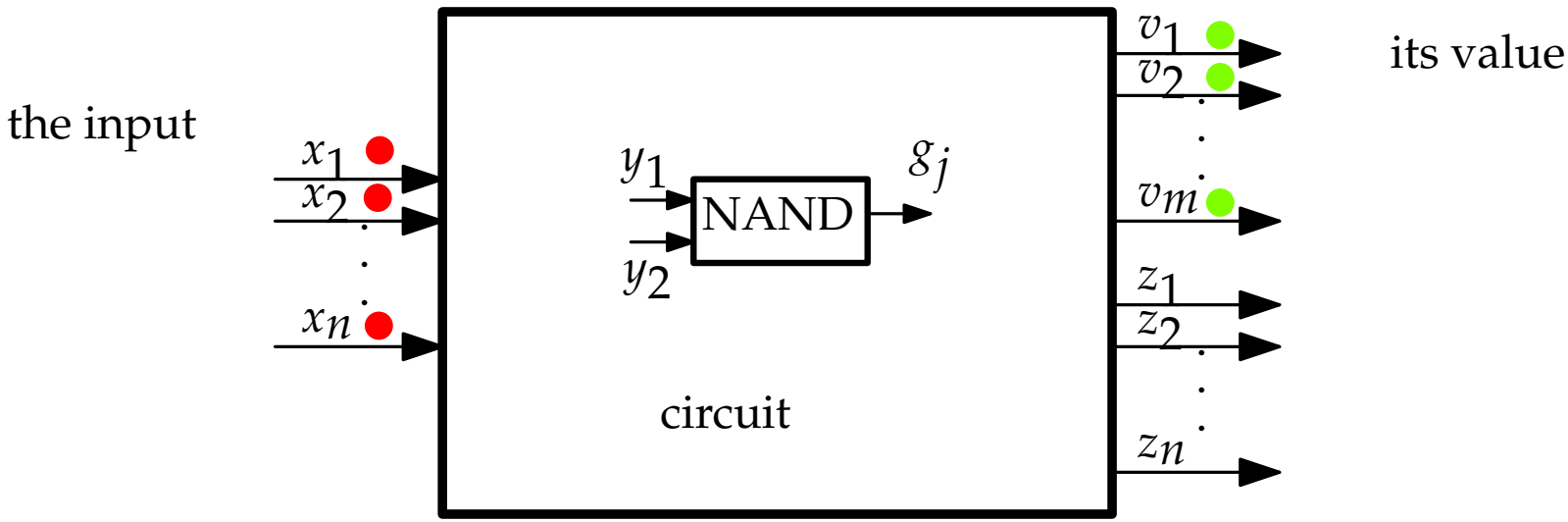
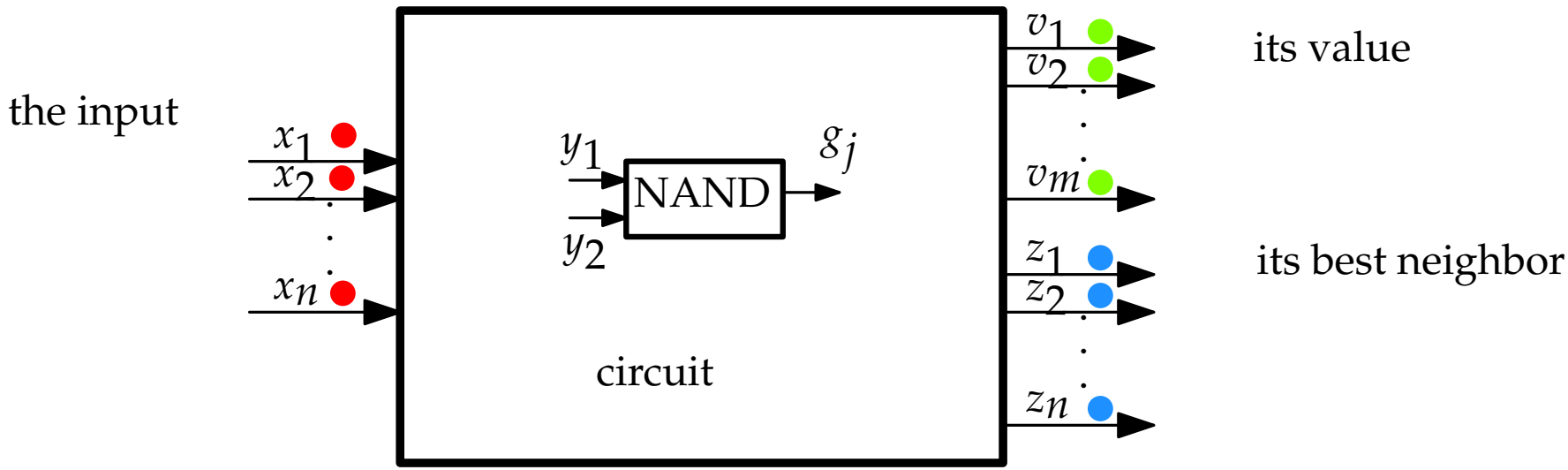

s_1
Efficient **circuit** computing value and successor
val(s_1)
feedback
succ(s_1)
s_2

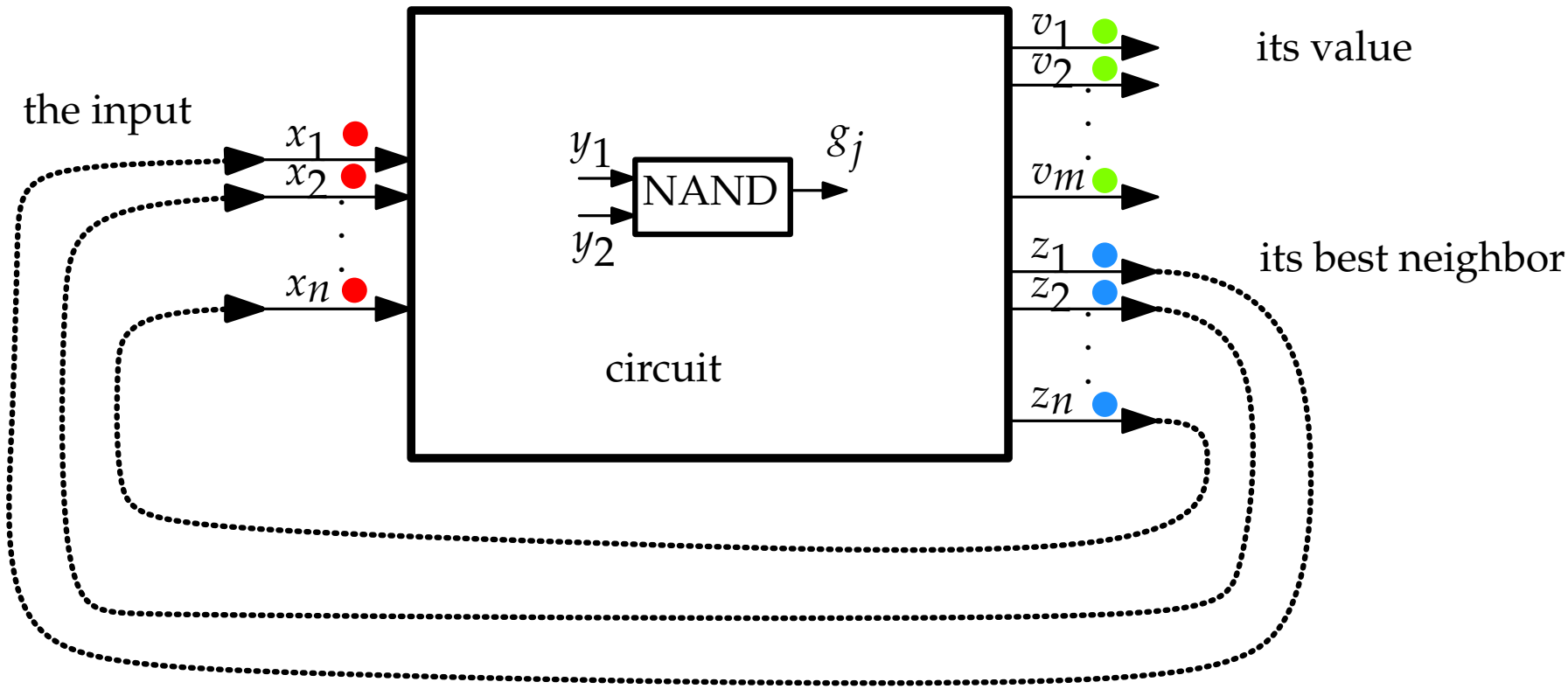

x_1
x_2
x_n
y_1
y_2
NAND
g_j
circuit
v_1
v_2
v_m
z_1
z_2
z_n

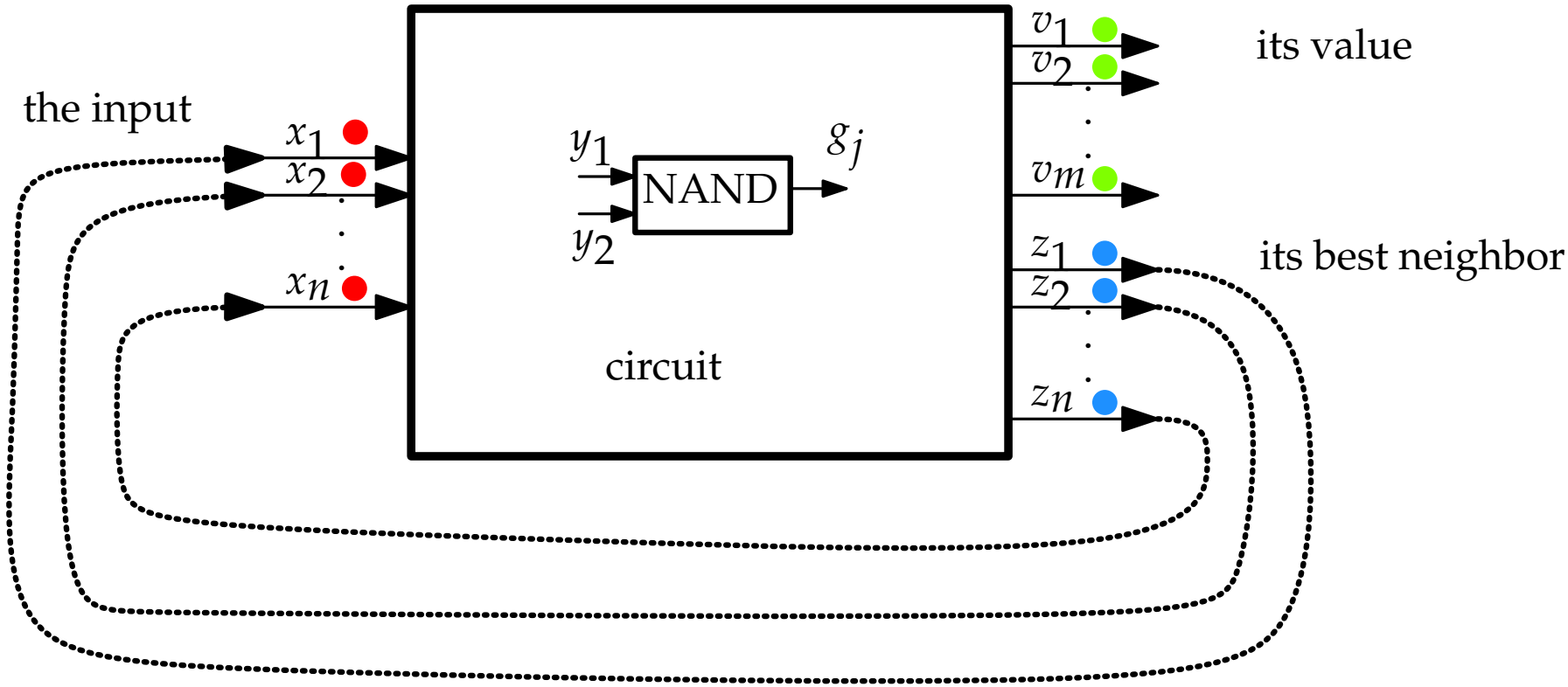
the input



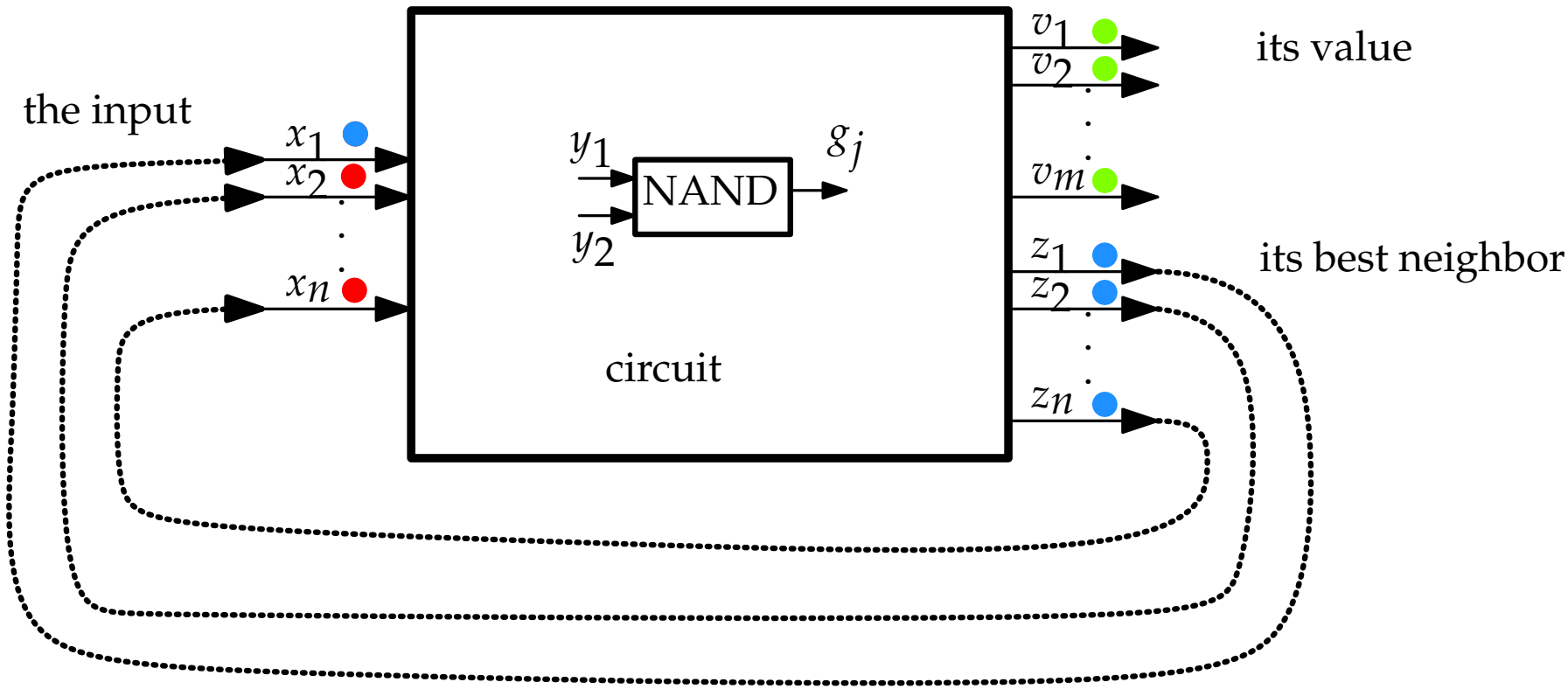

the input
x_1
x_2
x_n
y_1
y_2
NAND
g_j
circuit
v_1
v_2
v_m
z_1
z_2
z_n
its value


the input
x_1
x_2
x_n
y_1
y_2
NAND
g_j
circuit
v_1
v_2
v_m
z_1
z_2
z_n
its value
its best neighbor


the input
x_1
x_2
x_n
y_1
y_2
NAND
g_j
circuit
v_1
v_2
v_m
its value
z_1
z_2
z_n
its best neighbor



Want: Once $\vec{v}$ and $\vec{z}$ are correct, the "user" should have an incentive to flip the x_j to match the z_j to iterate the evaluation.



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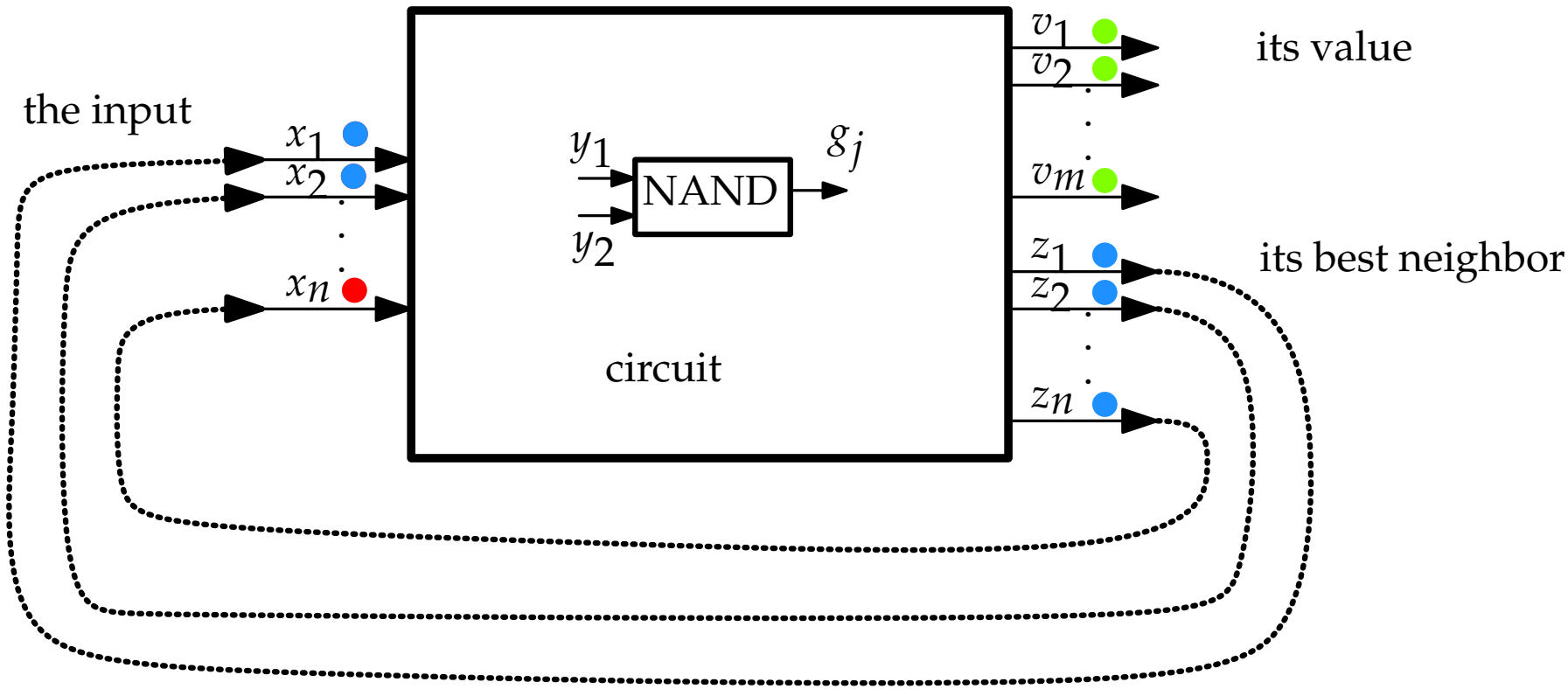

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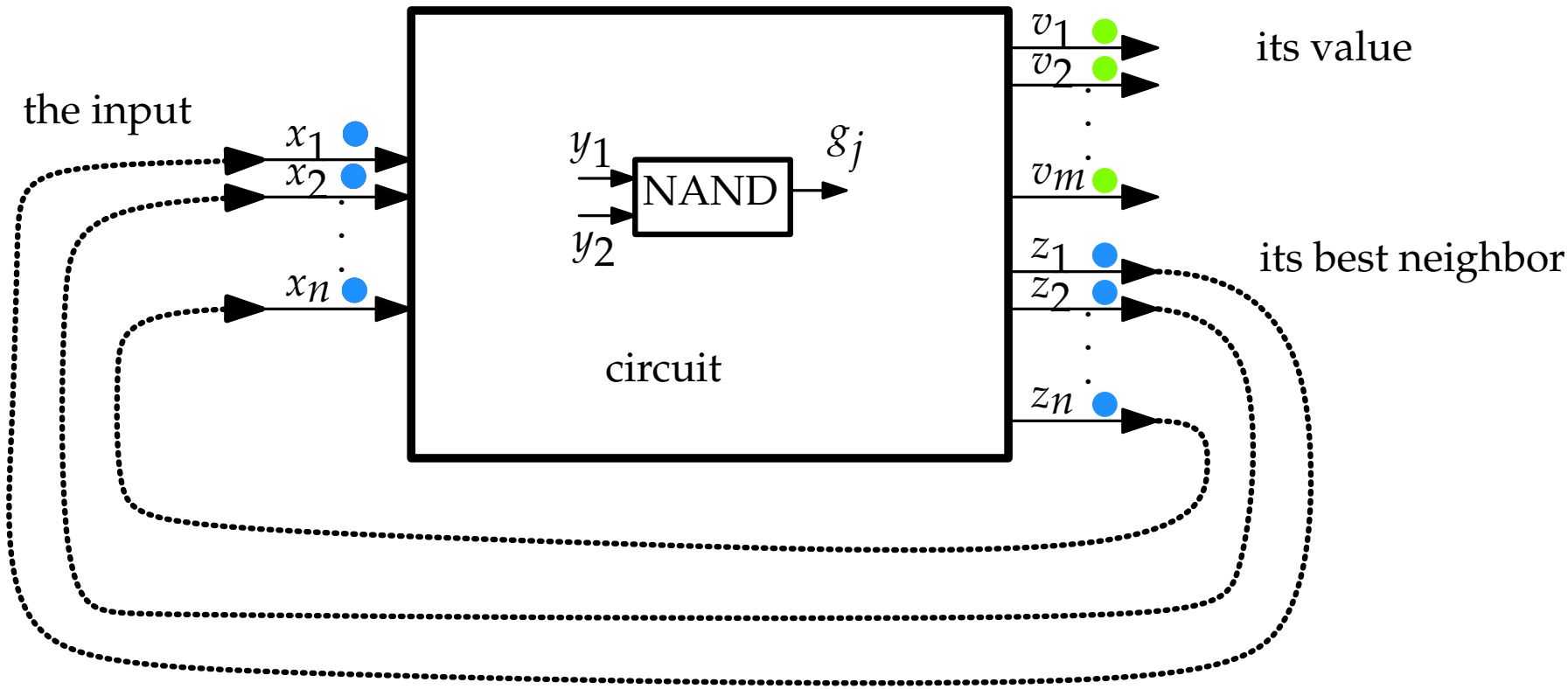
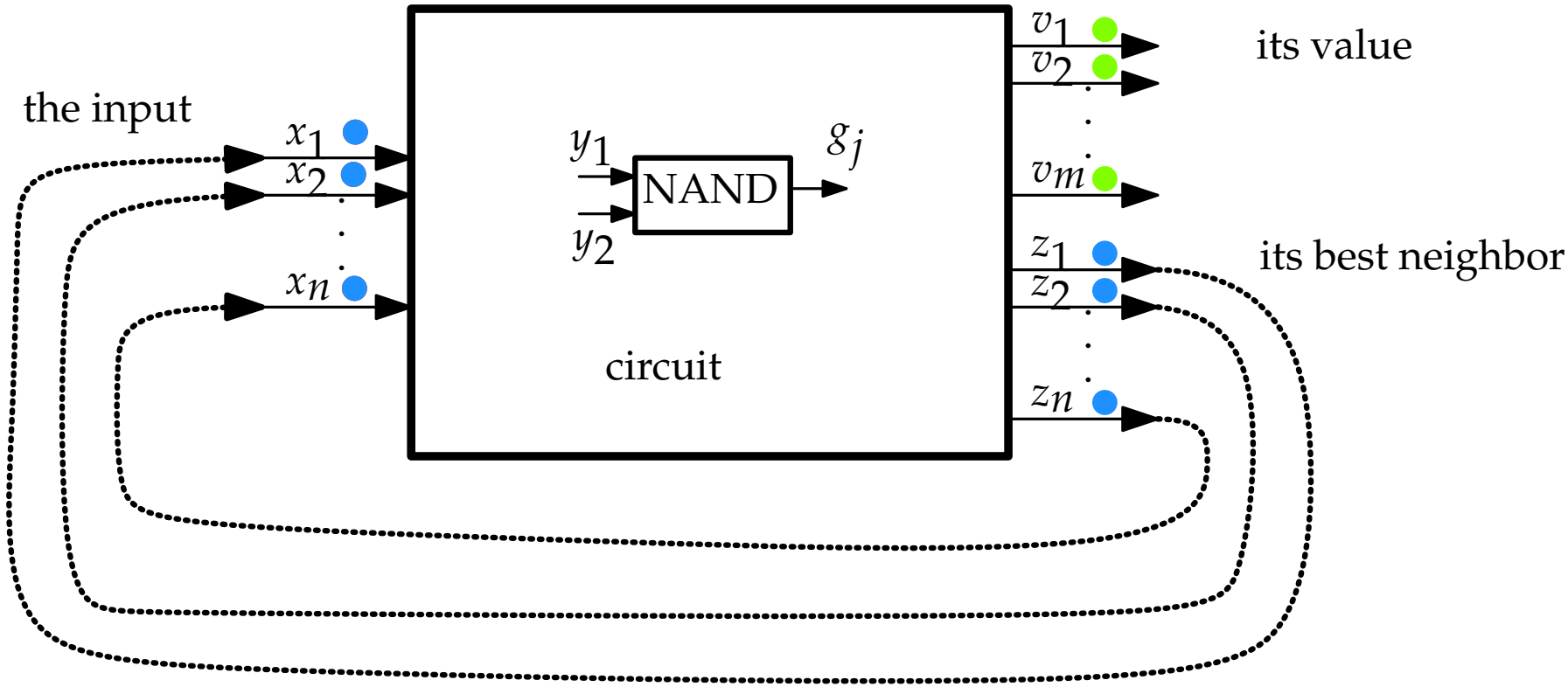
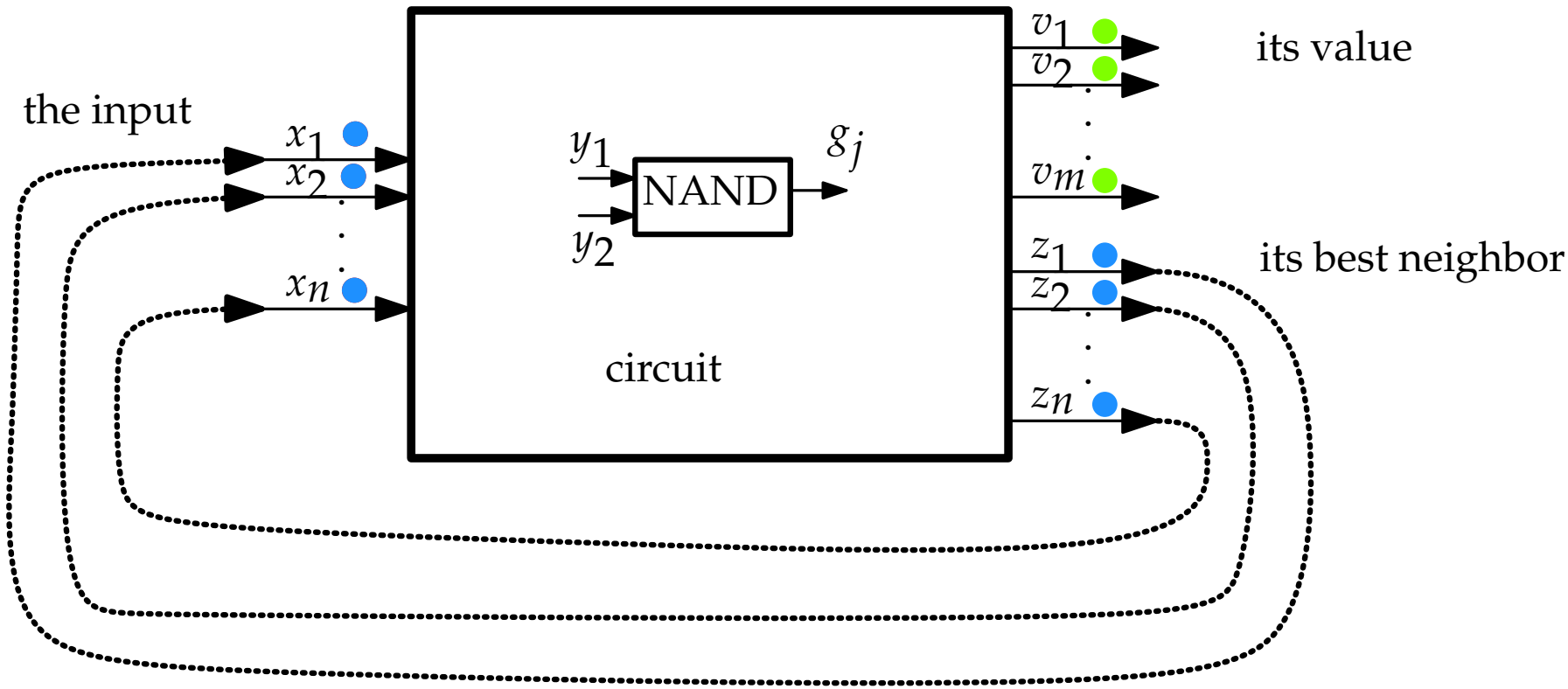


Want: Once $\vec{v}$ and $\vec{z}$ are correct, the "user" should have an incentive to flip the x_j to match the z_j to iterate the evaluation. But this will violate gates!



Want: Once $\vec{v}$ and $\vec{z}$ are correct, the "user" should have an incentive to flip the x_j to match the z_j to iterate the evaluation. But this will violate gates!

And should only work if $\text{val}(\vec{z})$ is better than $\text{val}(\vec{x})$!



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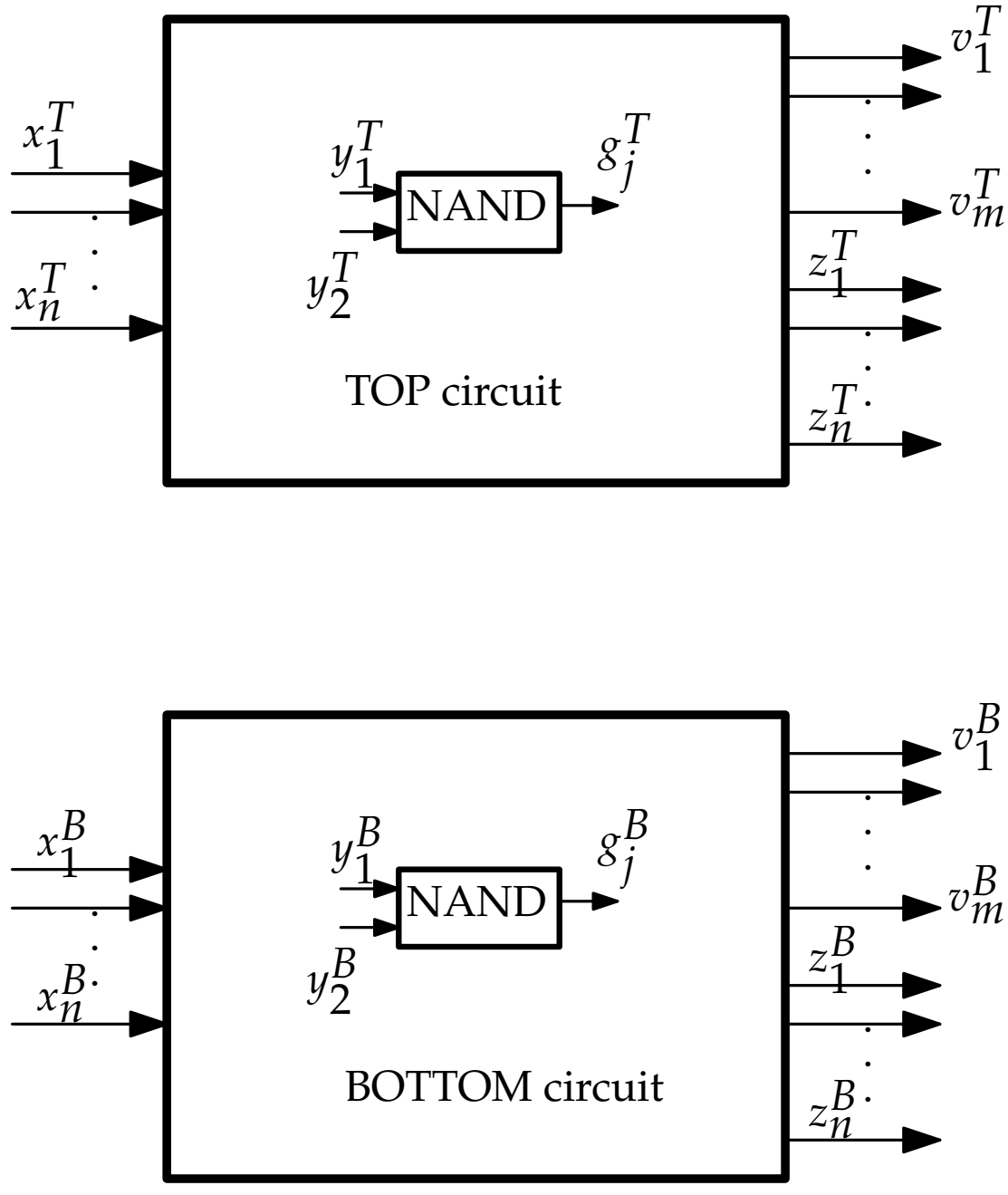
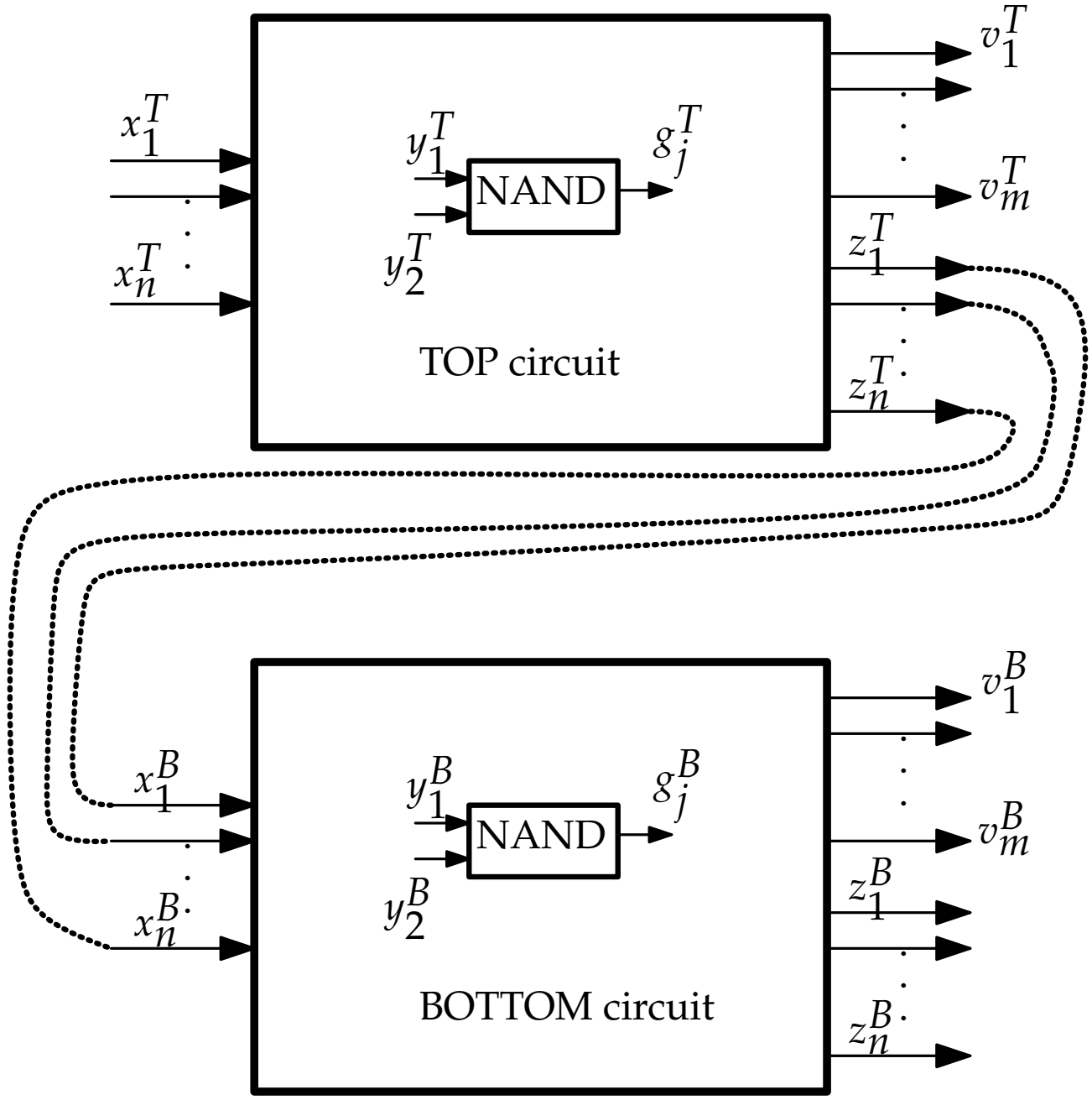
And should only work if $\text{val}(\vec{z})$ is better than $\text{val}(\vec{x})$!

But we haven't evaluated $\text{val}(\vec{z})$ yet. How could we?

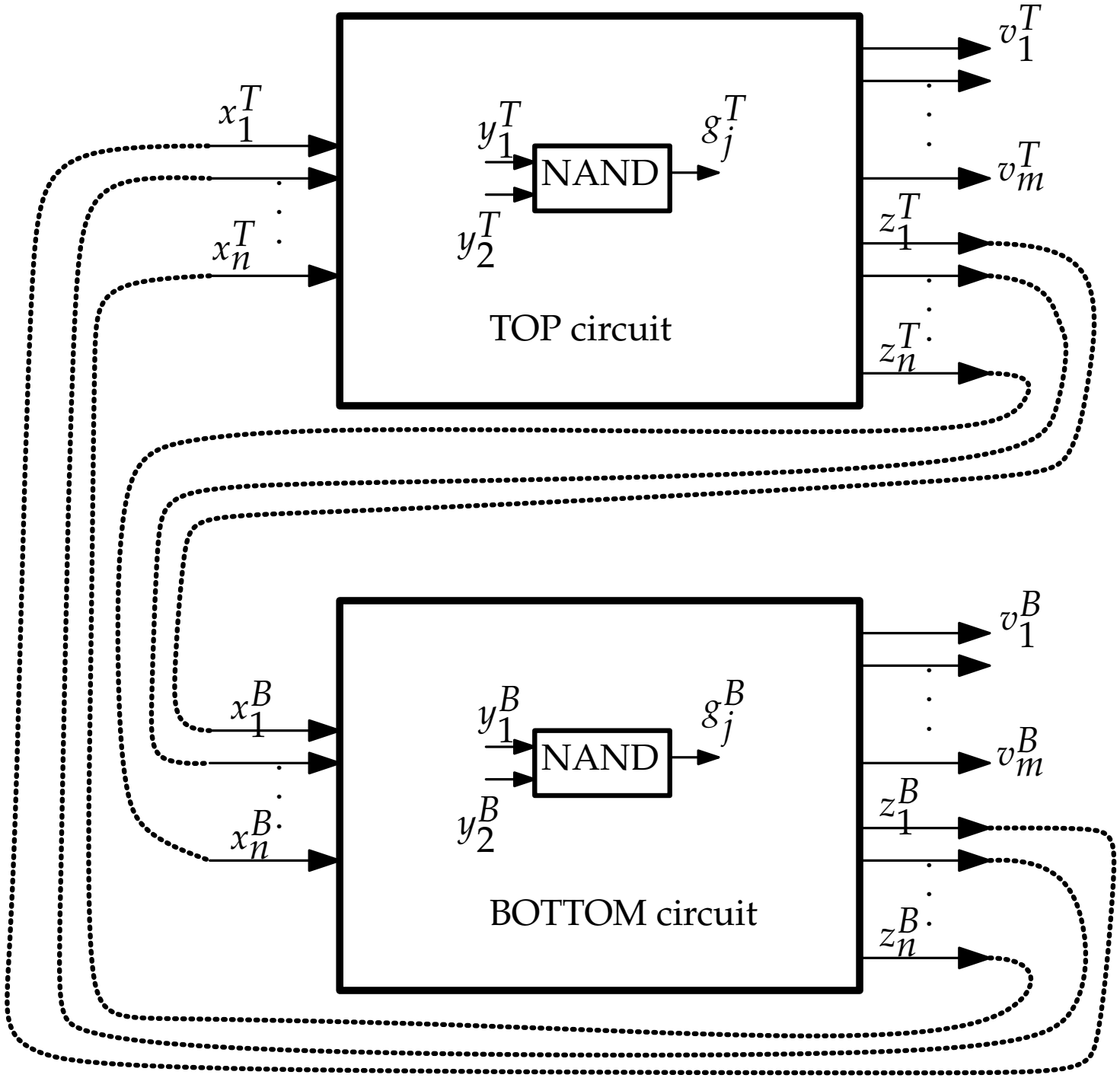
Krentels' Circuit Layout



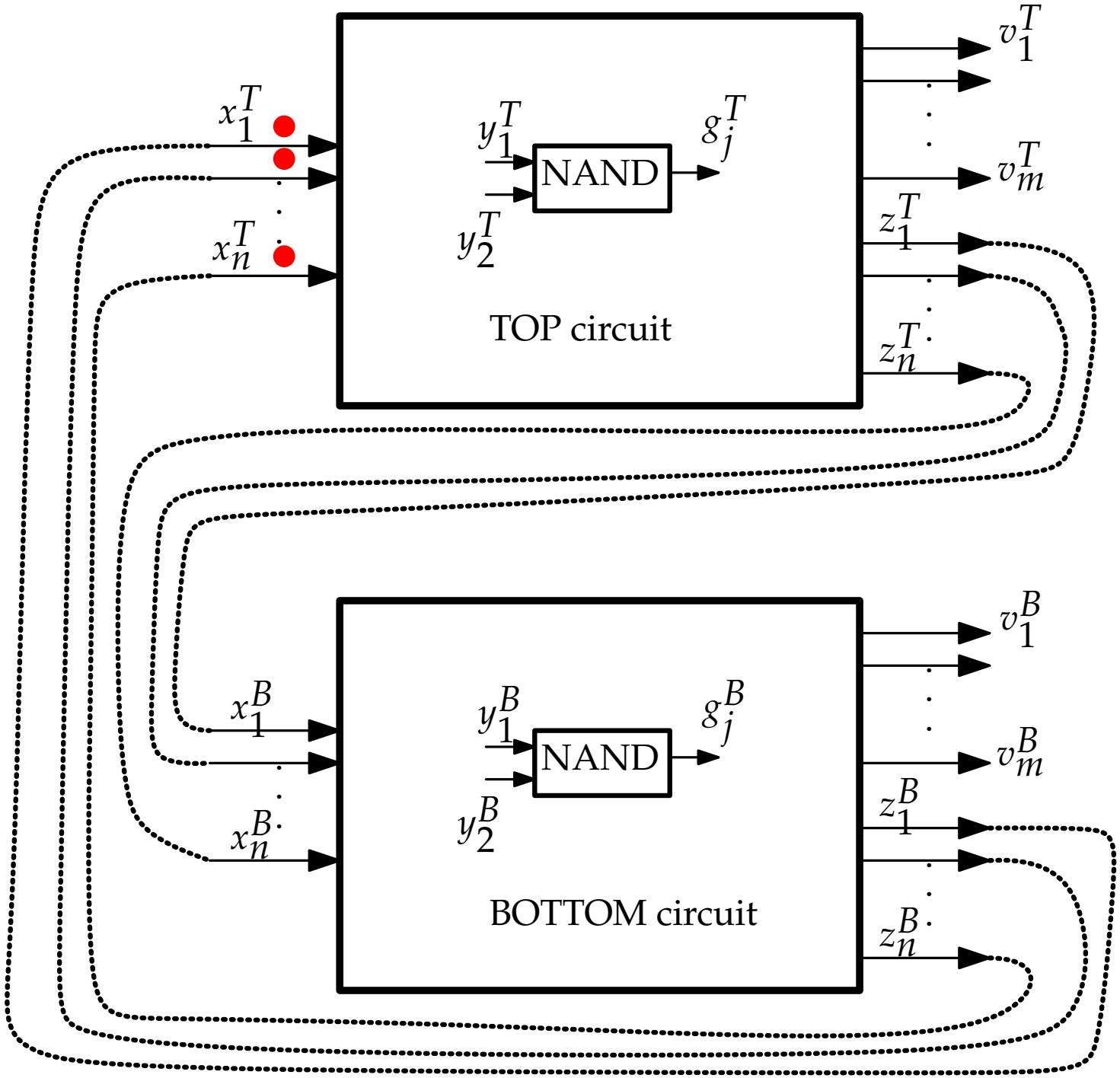
Krentels' Circuit Layout



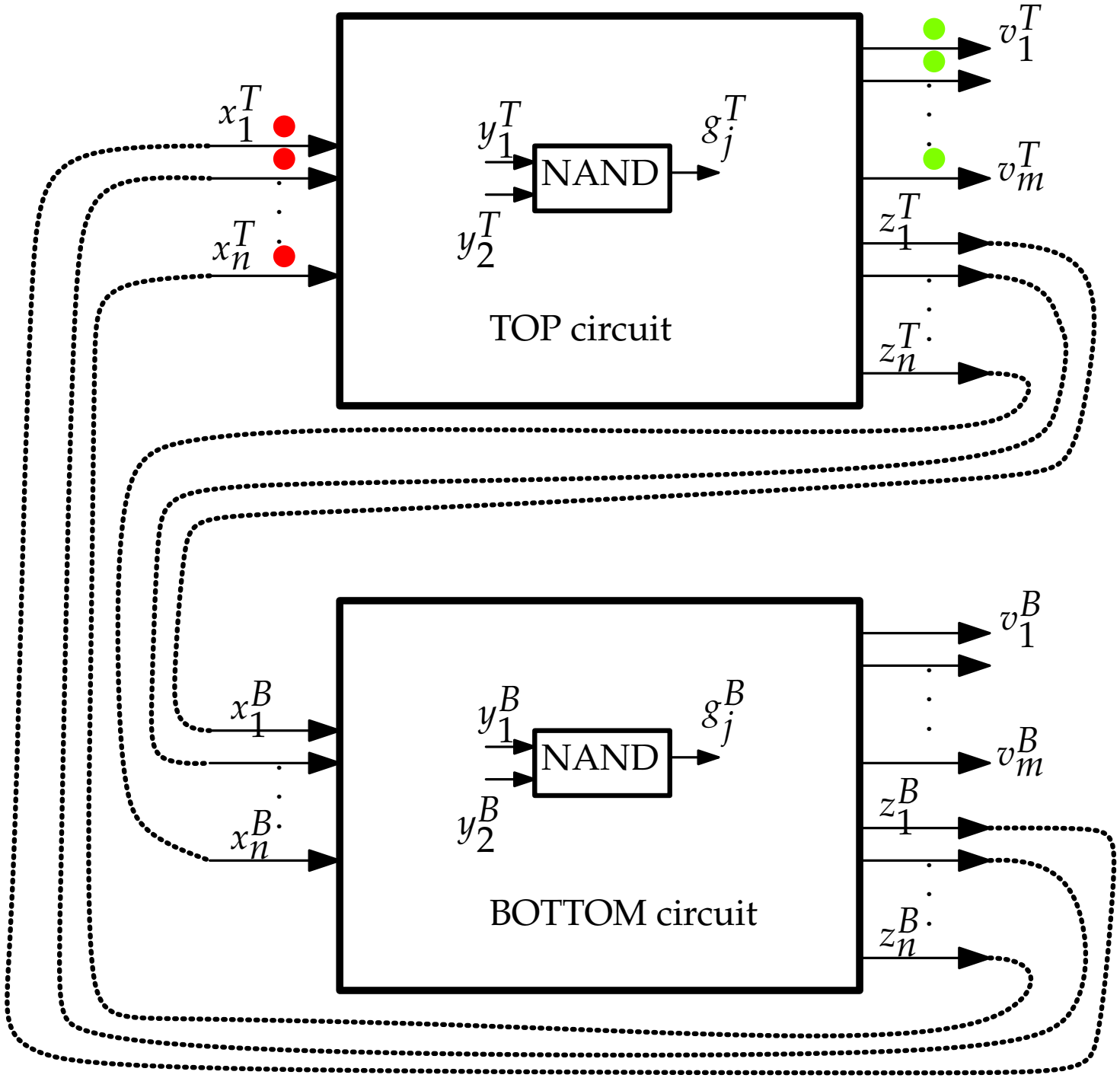
Krentels' Circuit Layout



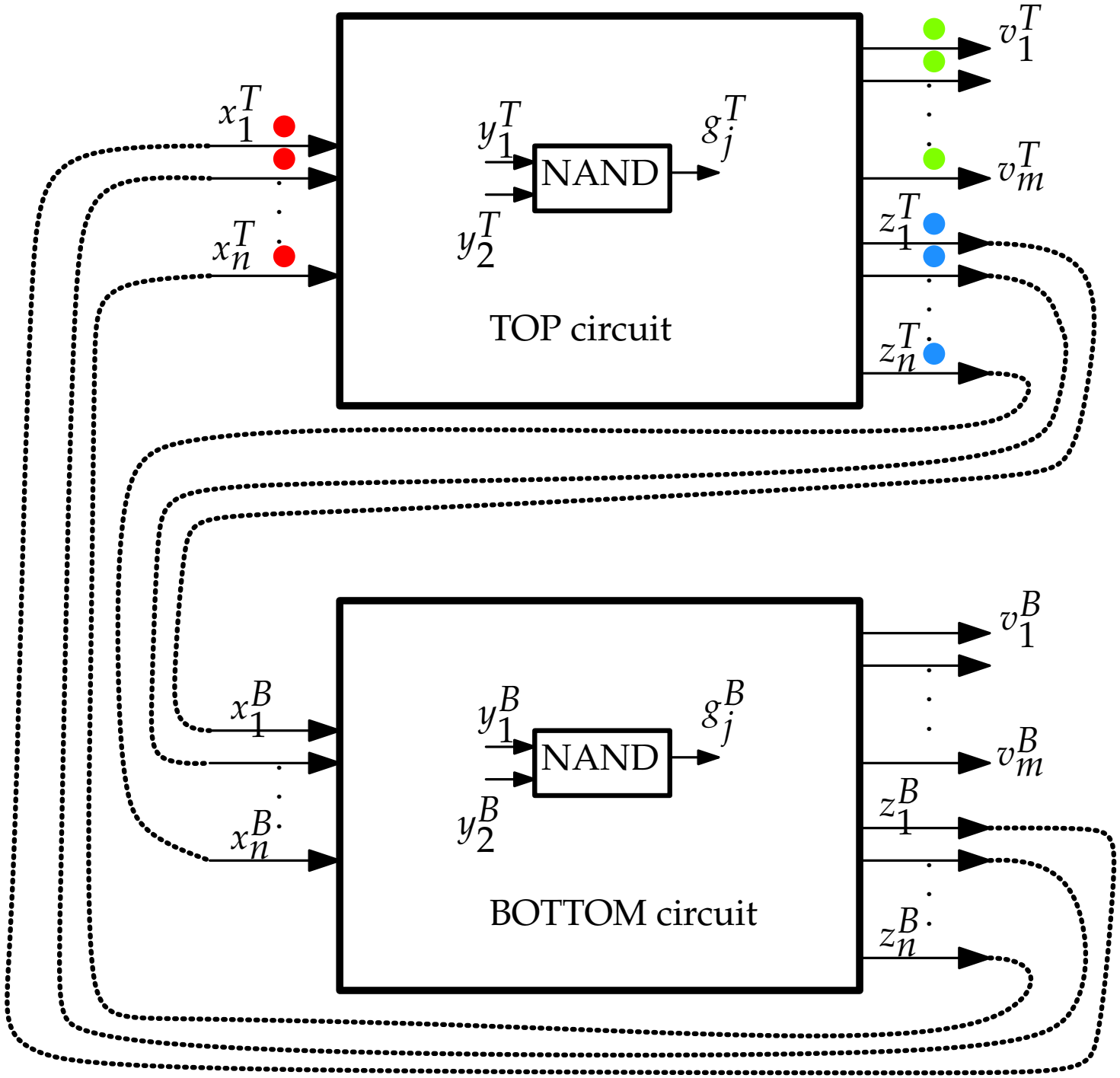
Krentels' Circuit Layout



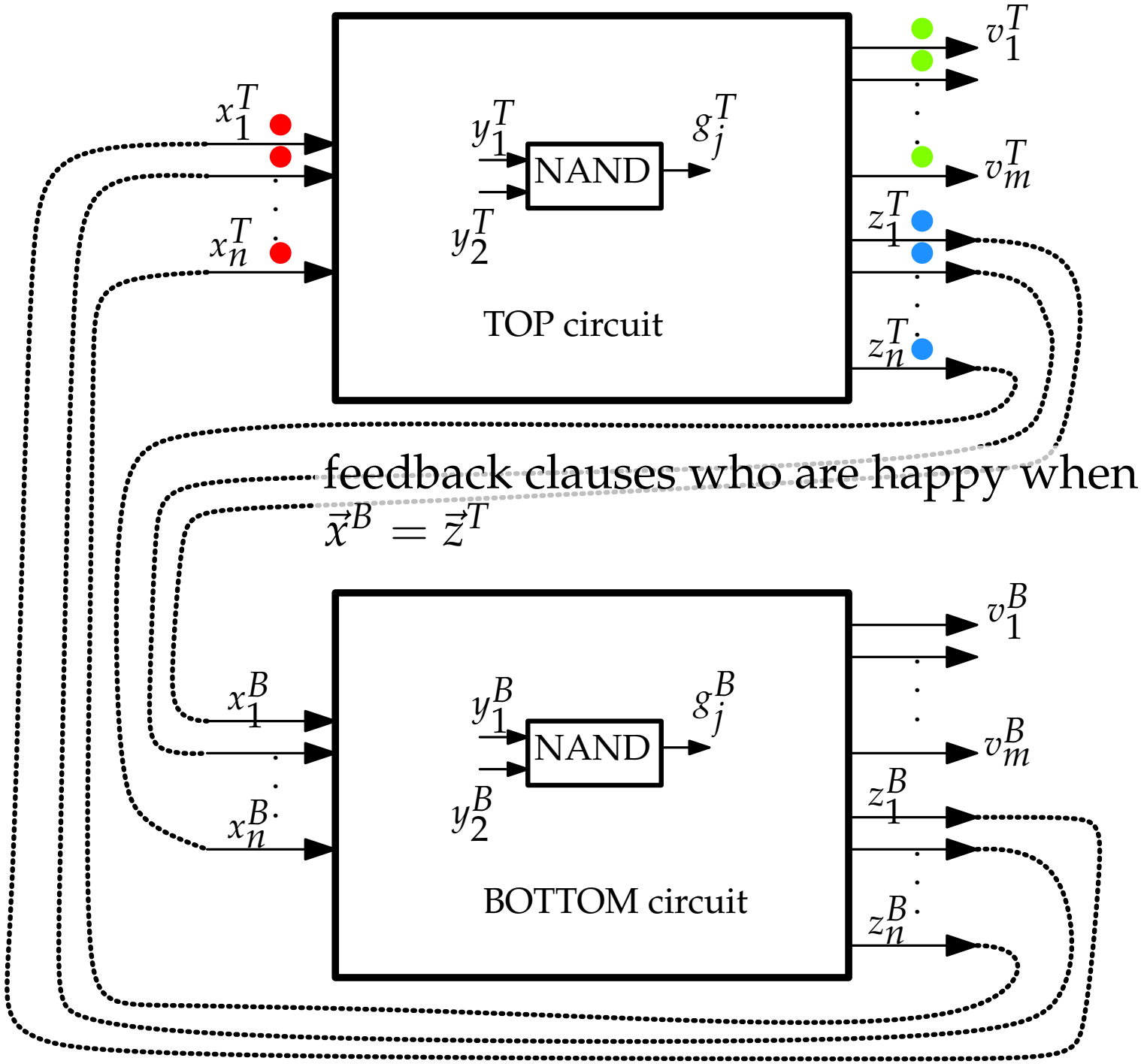
Krentels' Circuit Layout



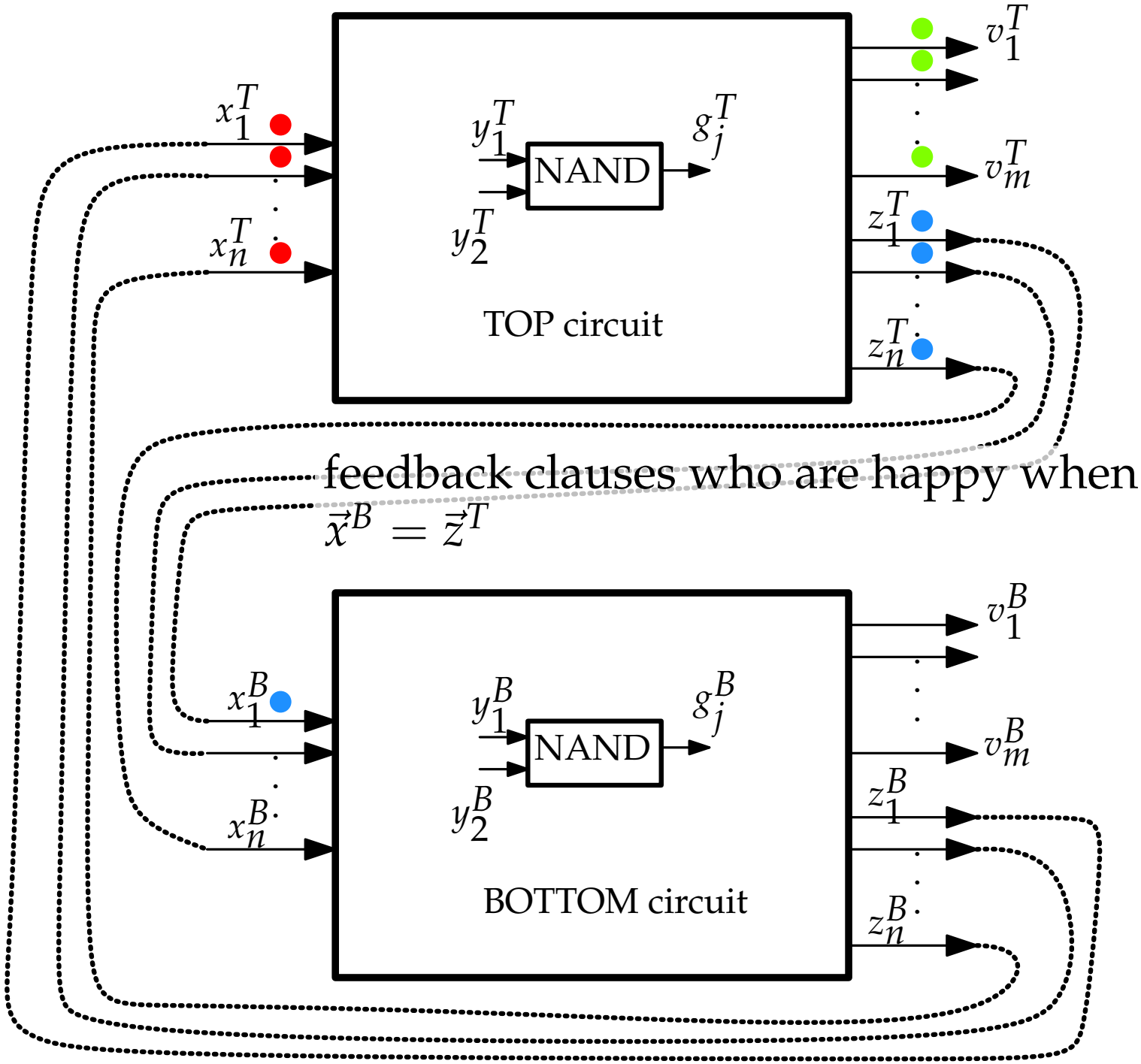
Krentels' Circuit Layout



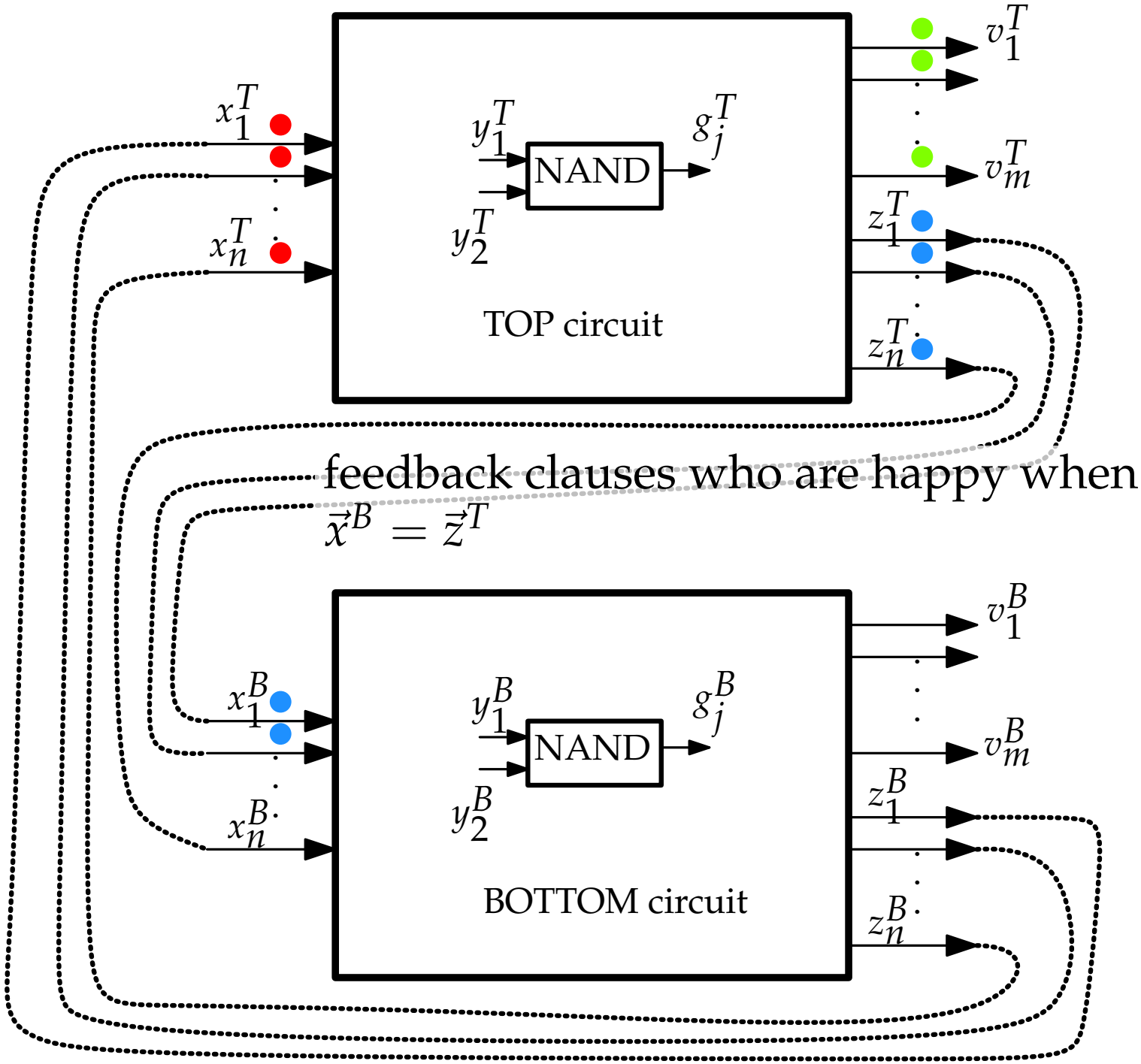
Krentels' Circuit Layout



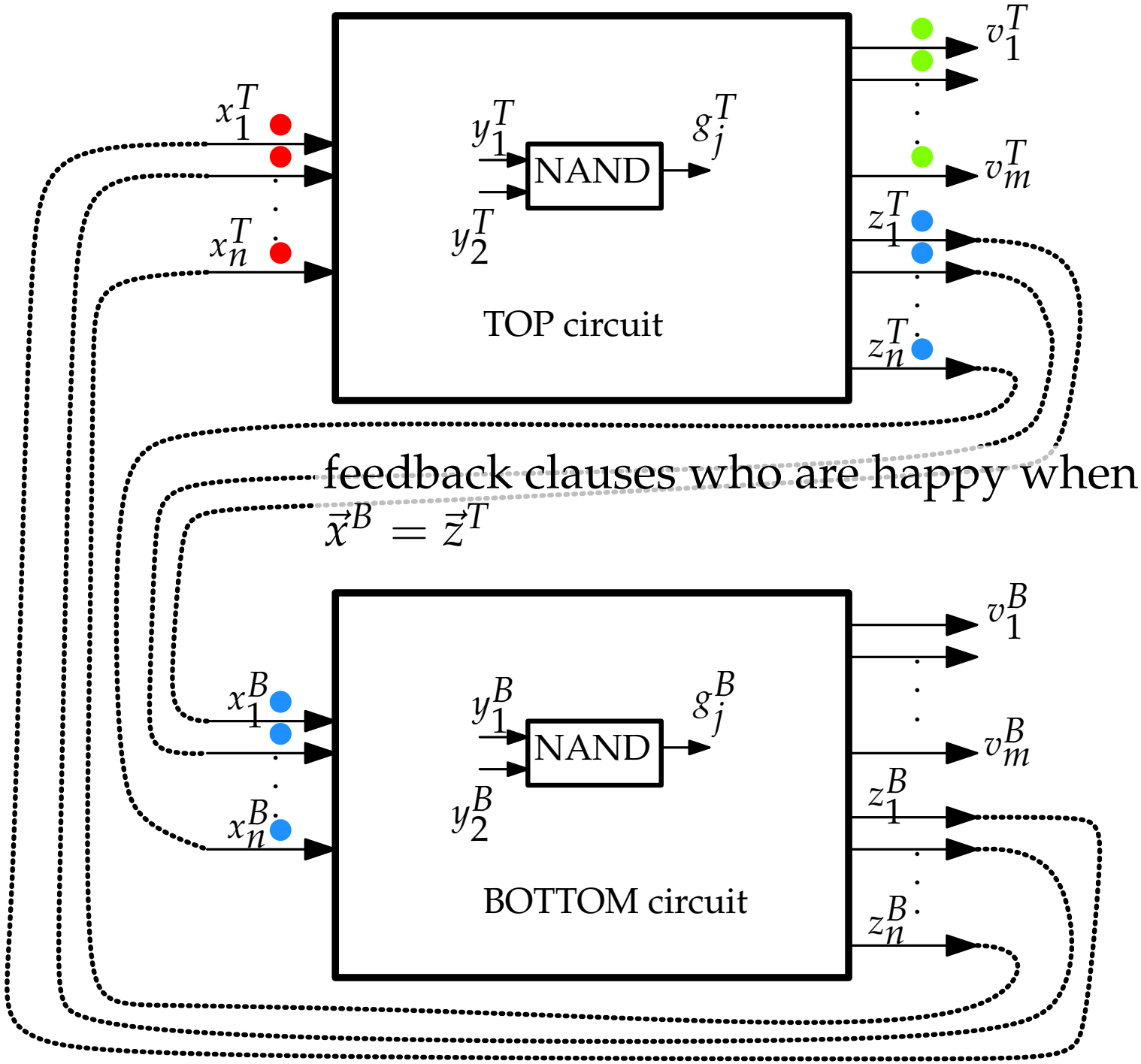
Krentels' Circuit Layout



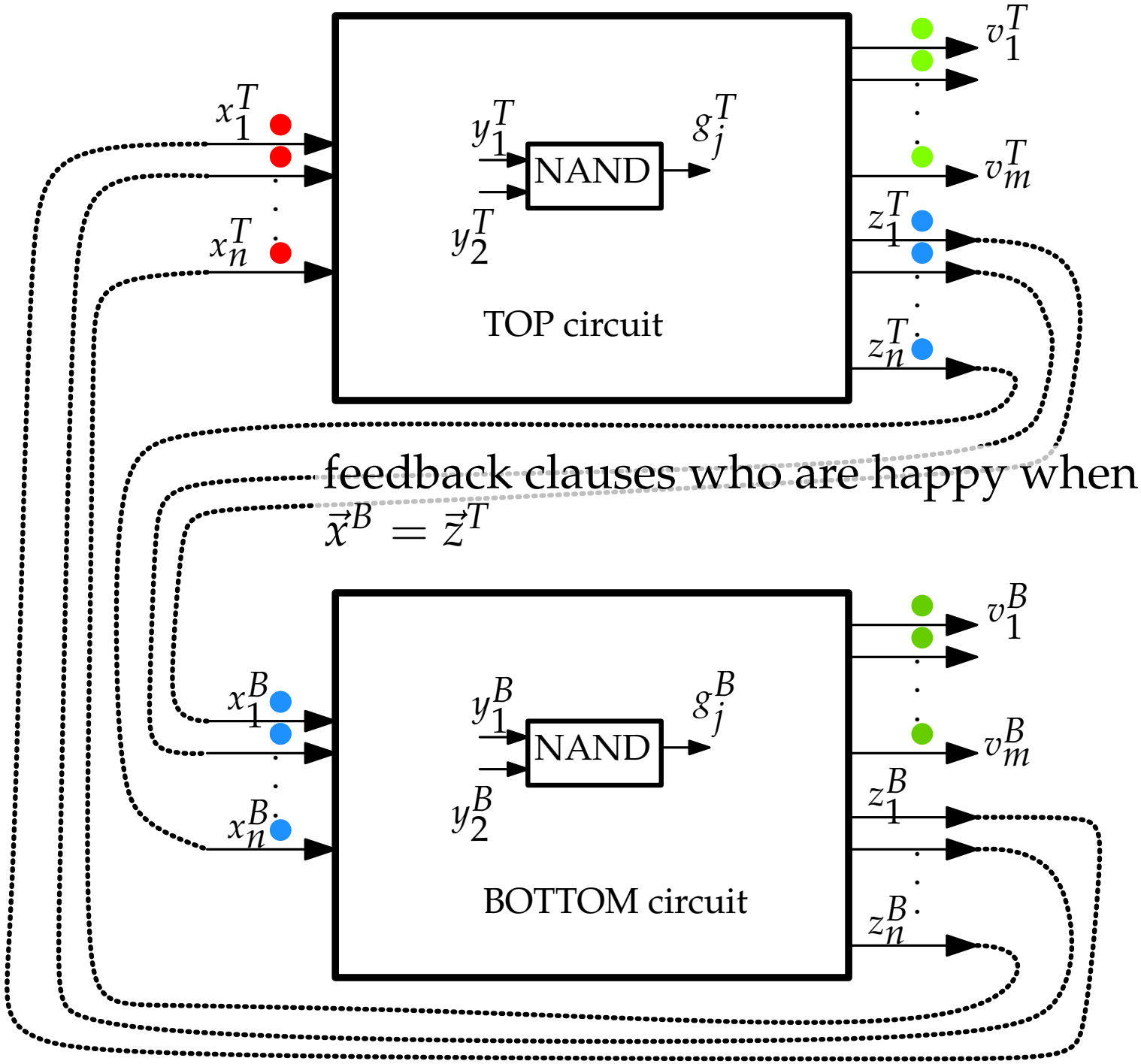
Krentels' Circuit Layout



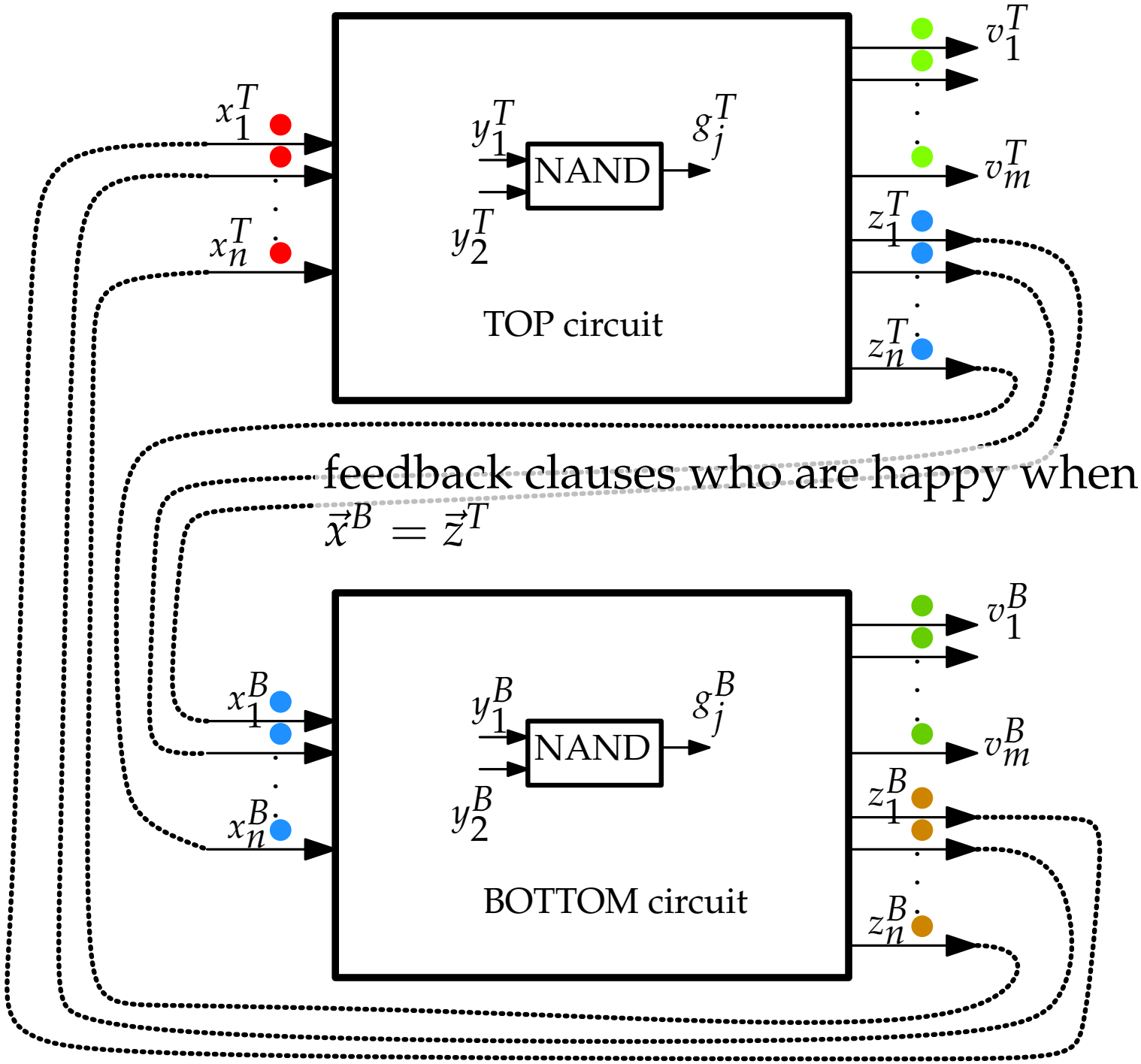
Krentels' Circuit Layout



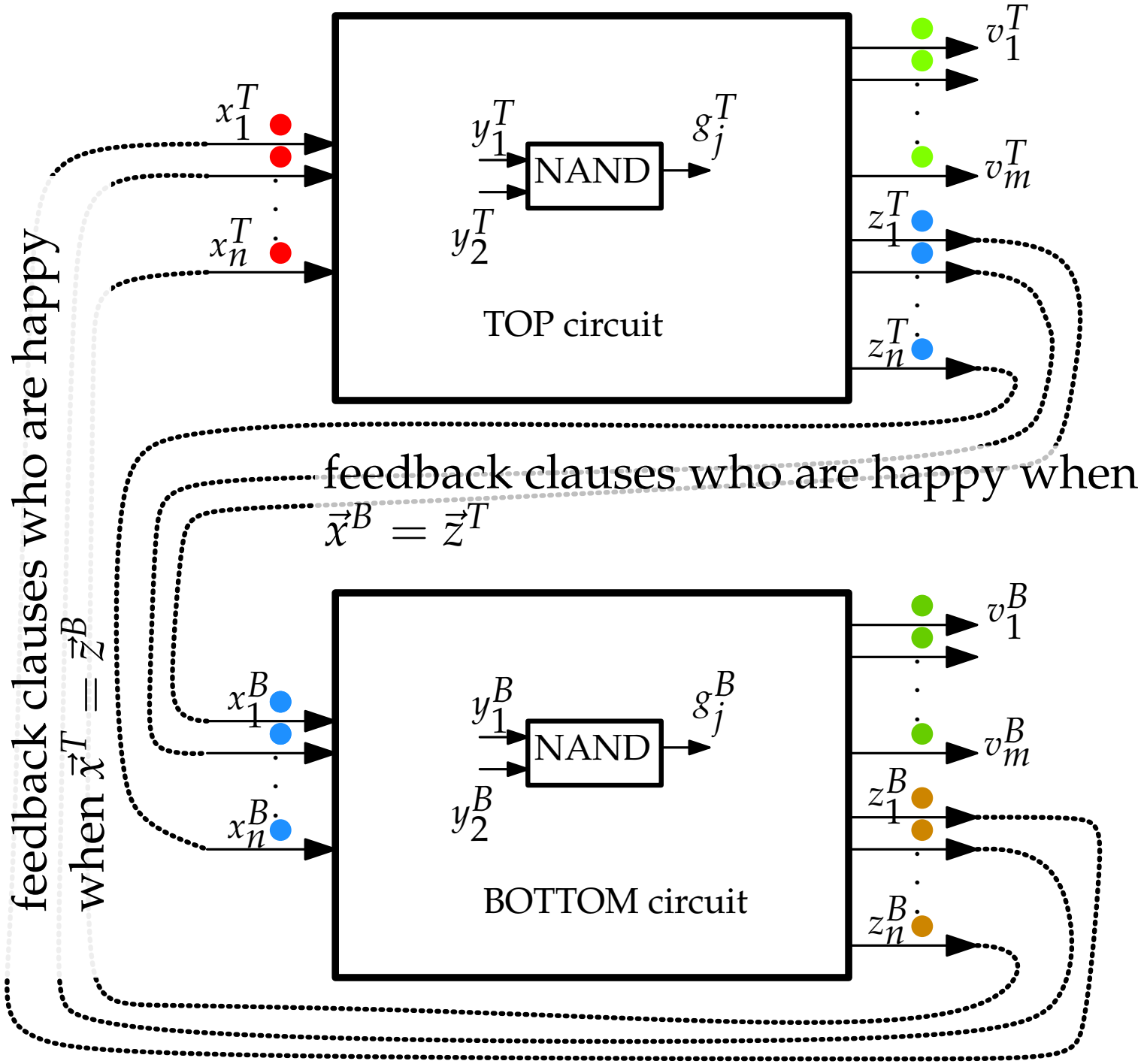
Krentels' Circuit Layout

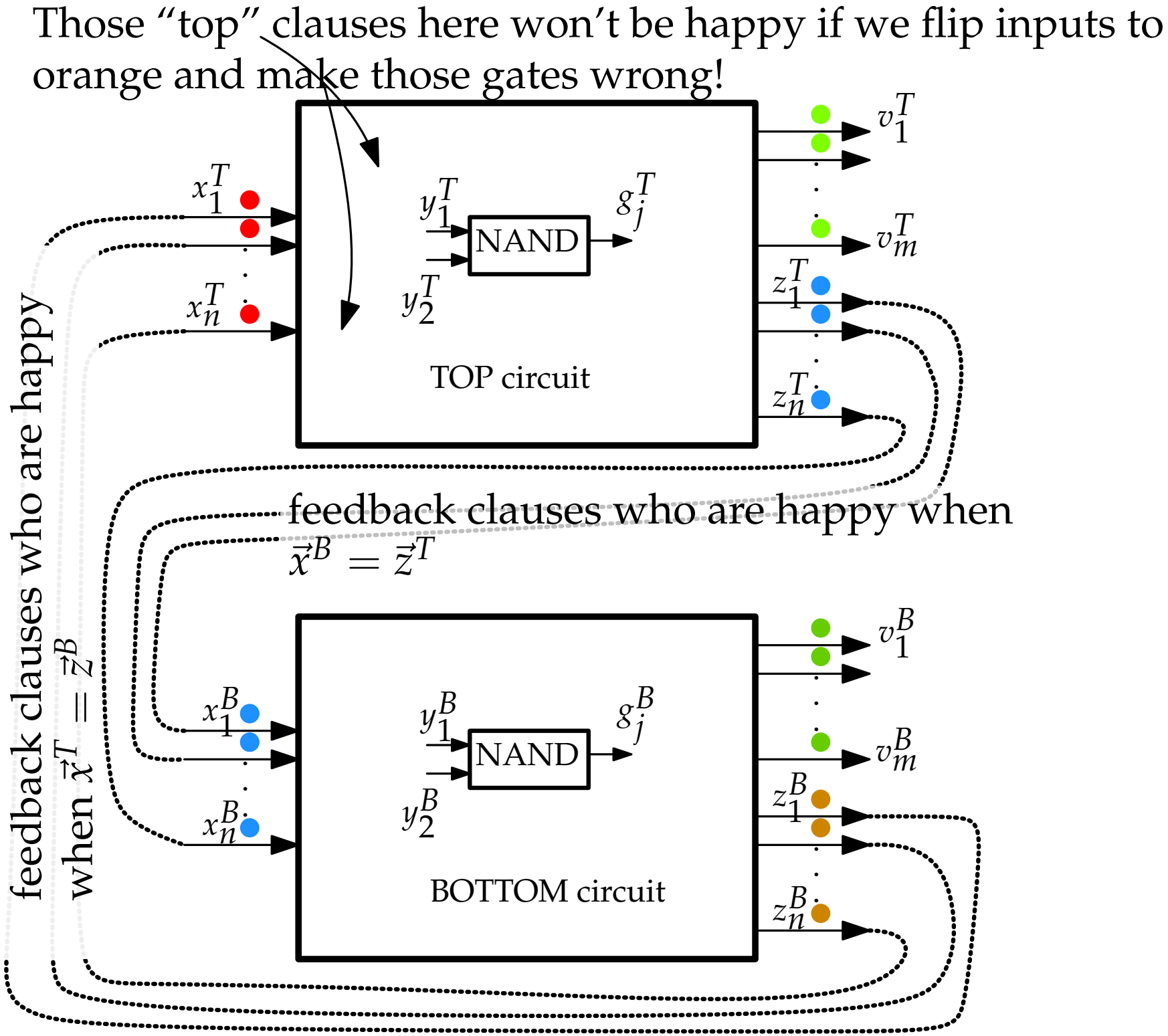


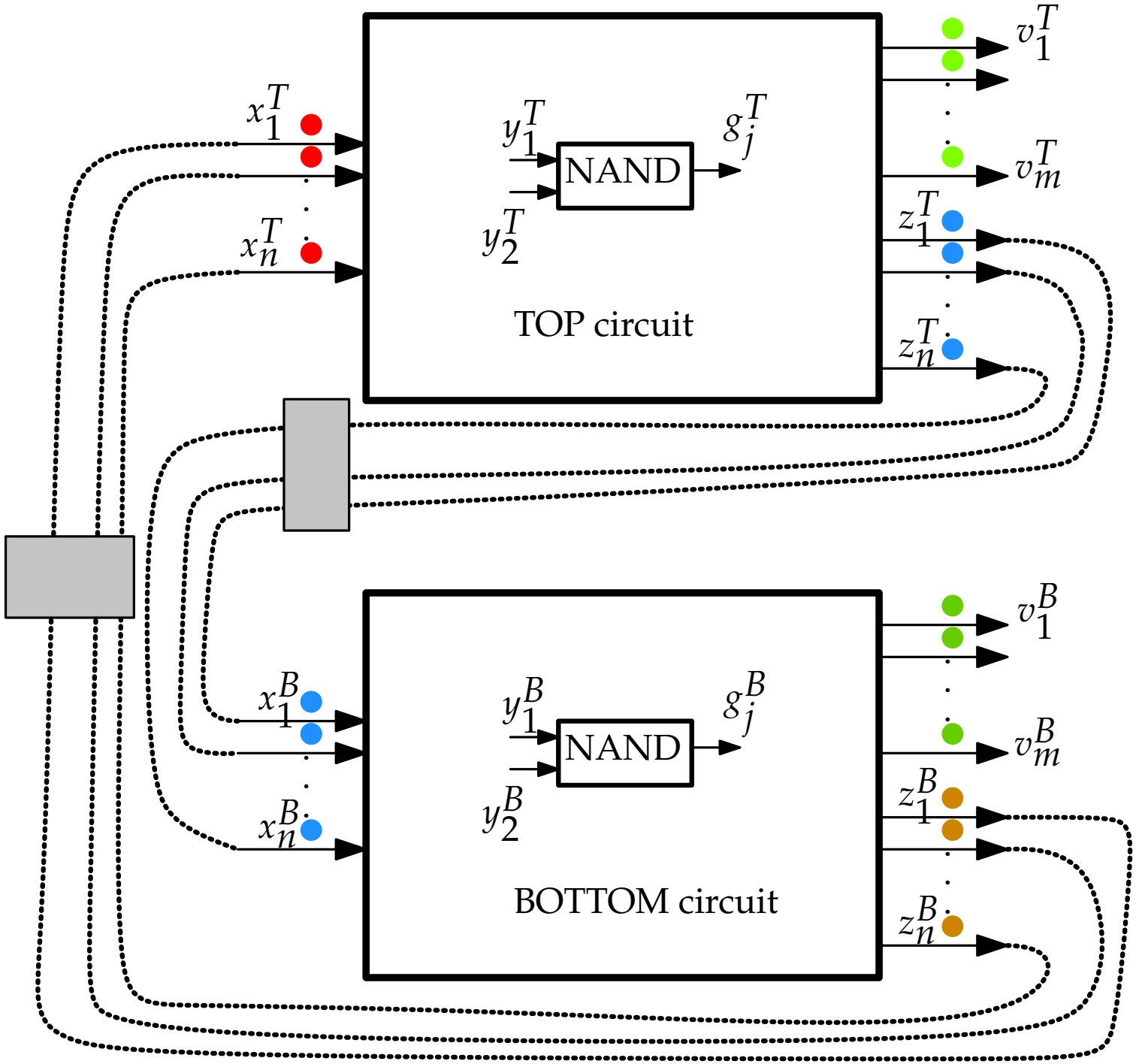
Krentels' Circuit Layout

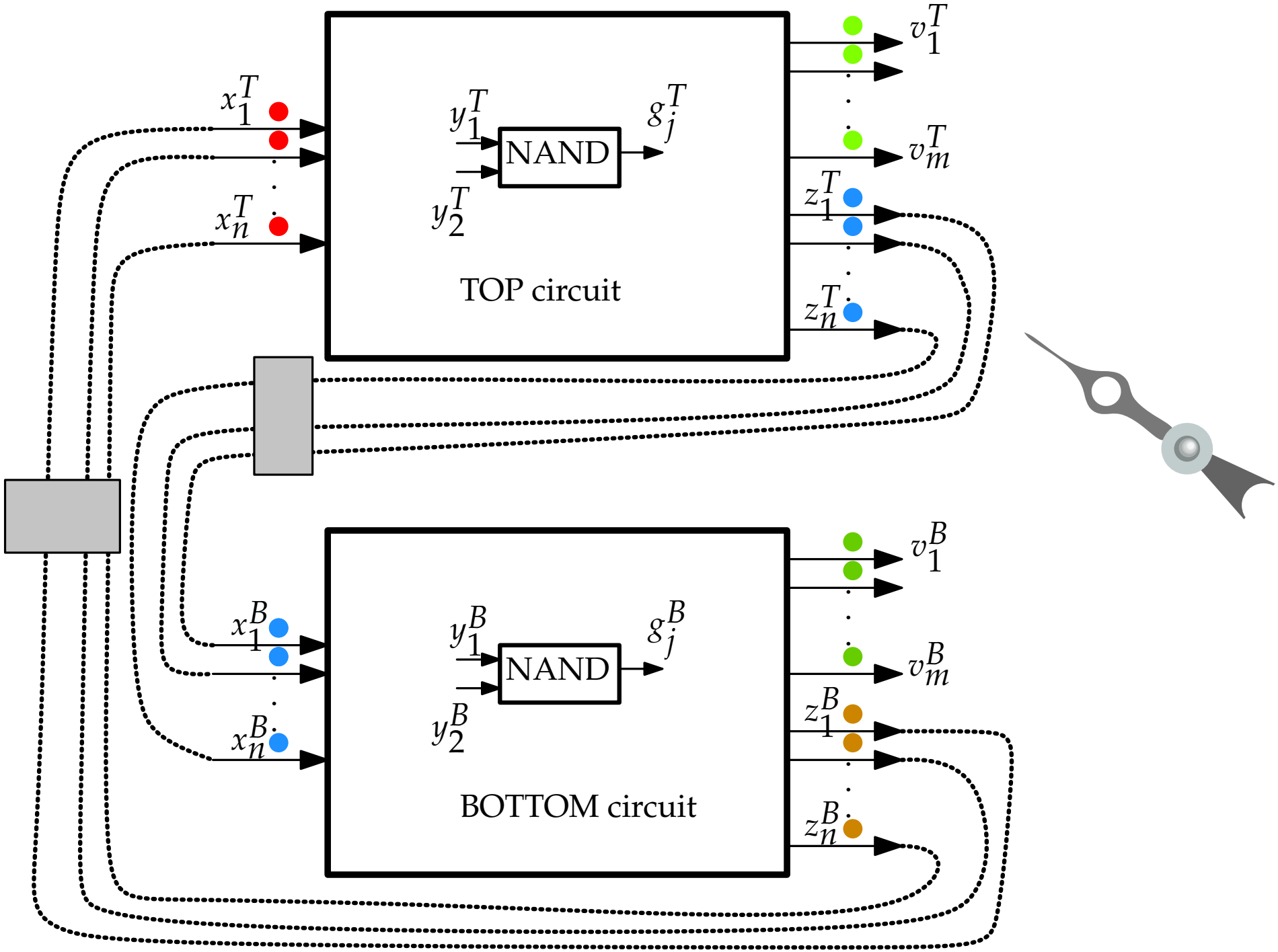


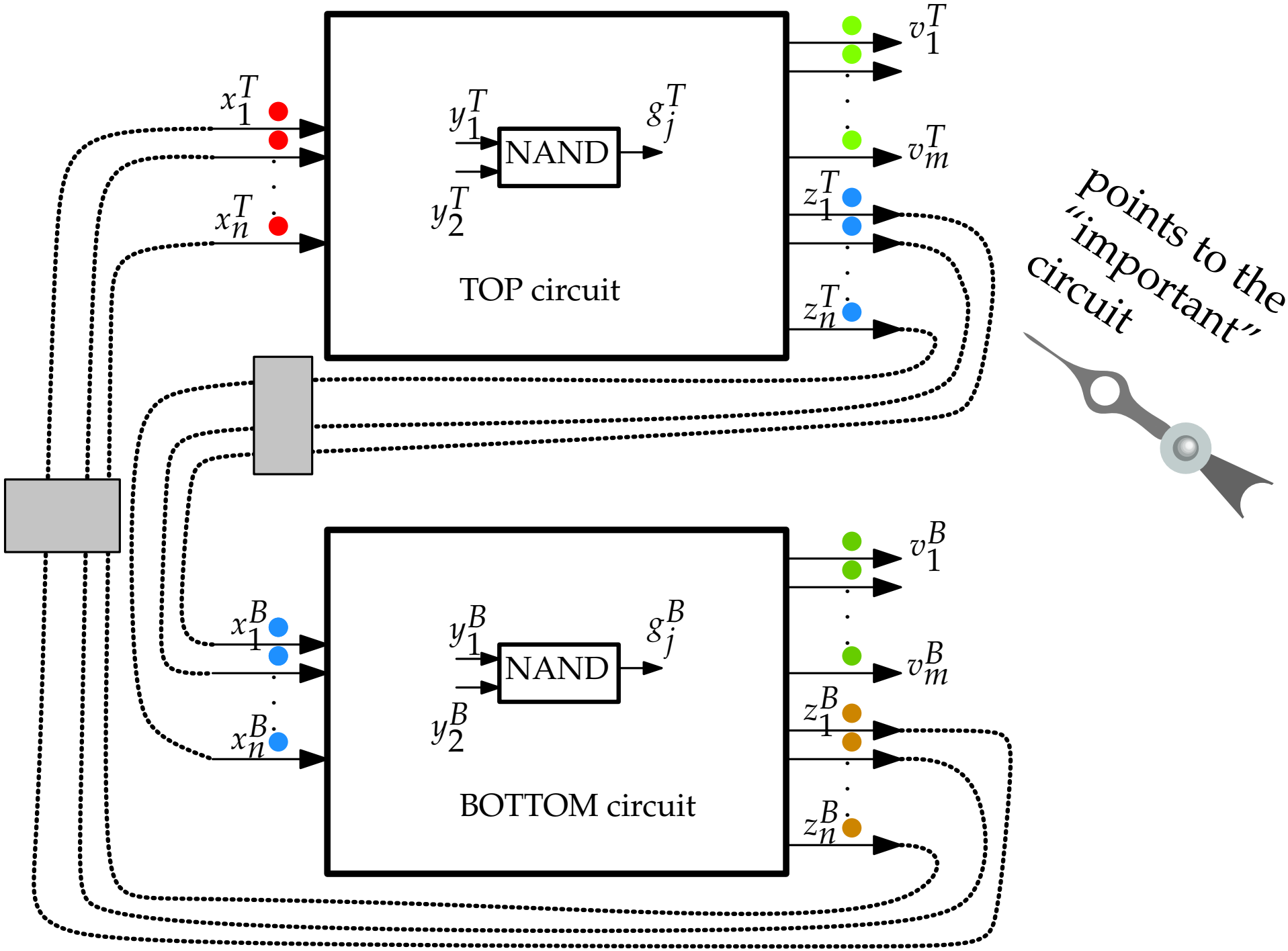
Krentels' Circuit Layout

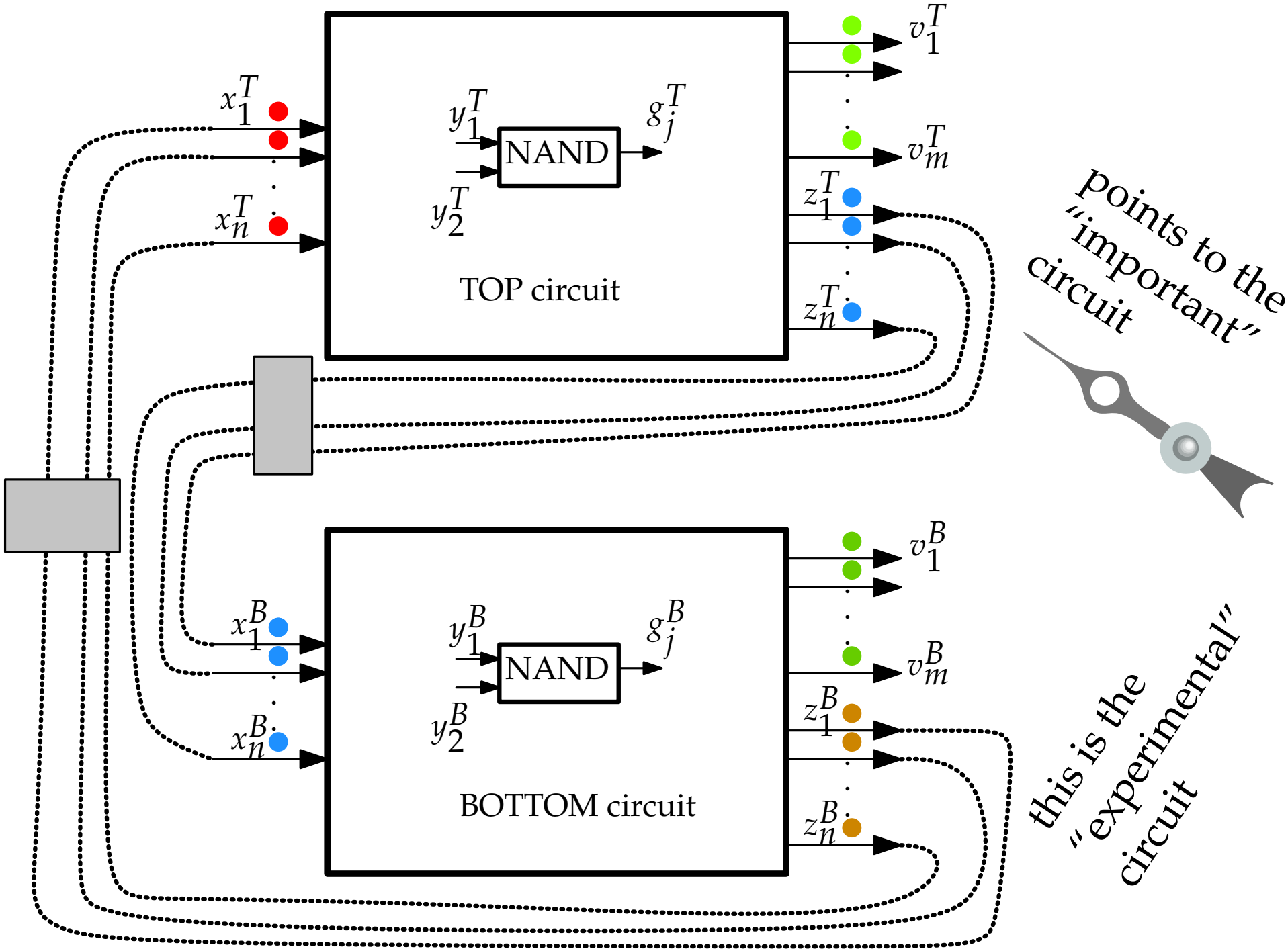


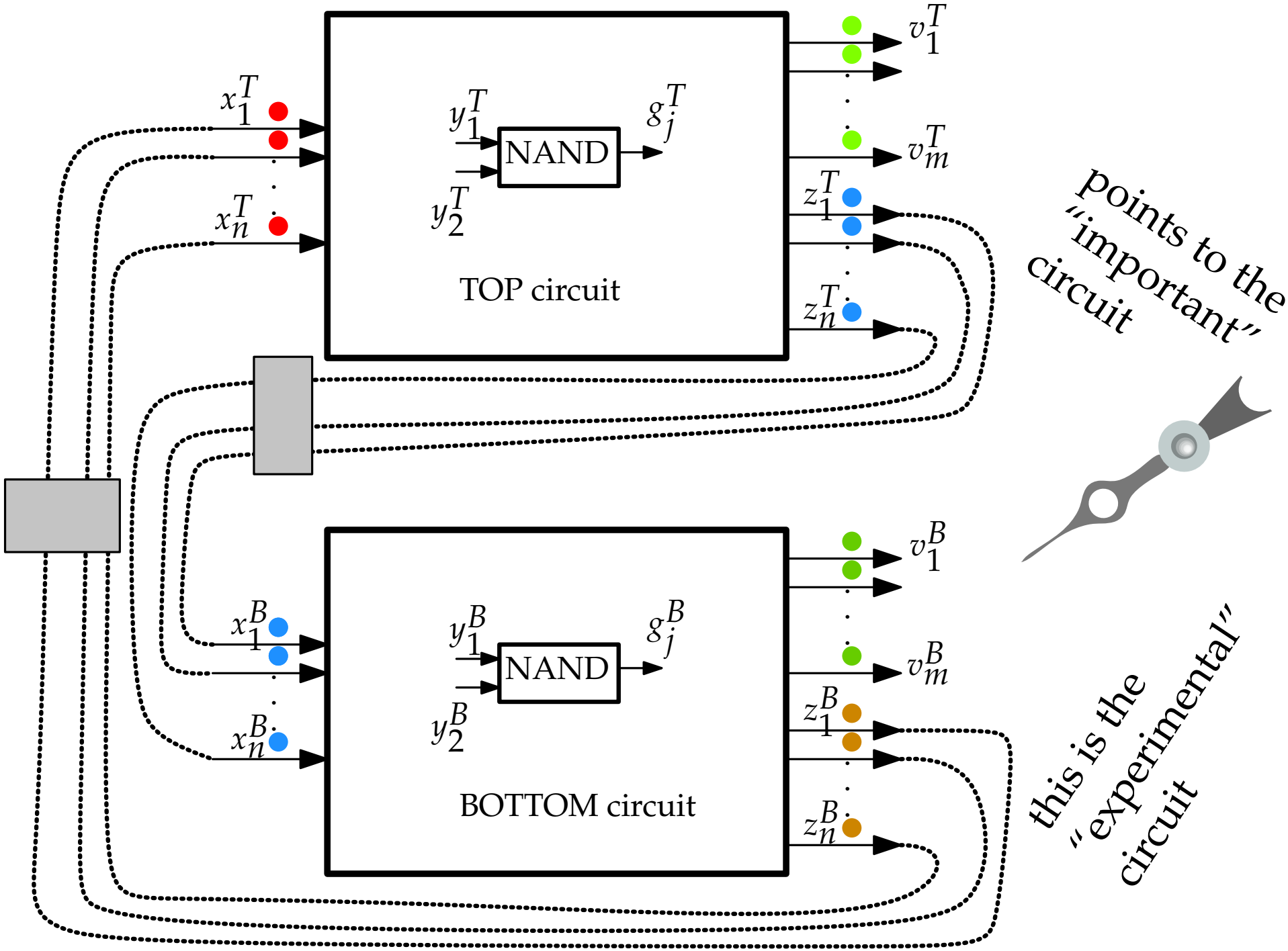

Those "top" clauses here won't be happy if we flip inputs to orange and make those gates wrong!
x_1^T
x_n^T
y_1^T
y_2^T
NAND
g_j^T
TOP circuit
v_1^T
v_m^T
z_1^T
z_n^T
feedback clauses who are happy when $\vec{x}^B = \vec{z}^T$
feedback clauses who are happy when $\vec{x}^T = \vec{z}^B$
x_1^B
x_n^B
y_1^B
y_2^B
NAND
g_j^B
BOTTOM circuit
v_1^B
v_m^B
z_1^B
z_n^B

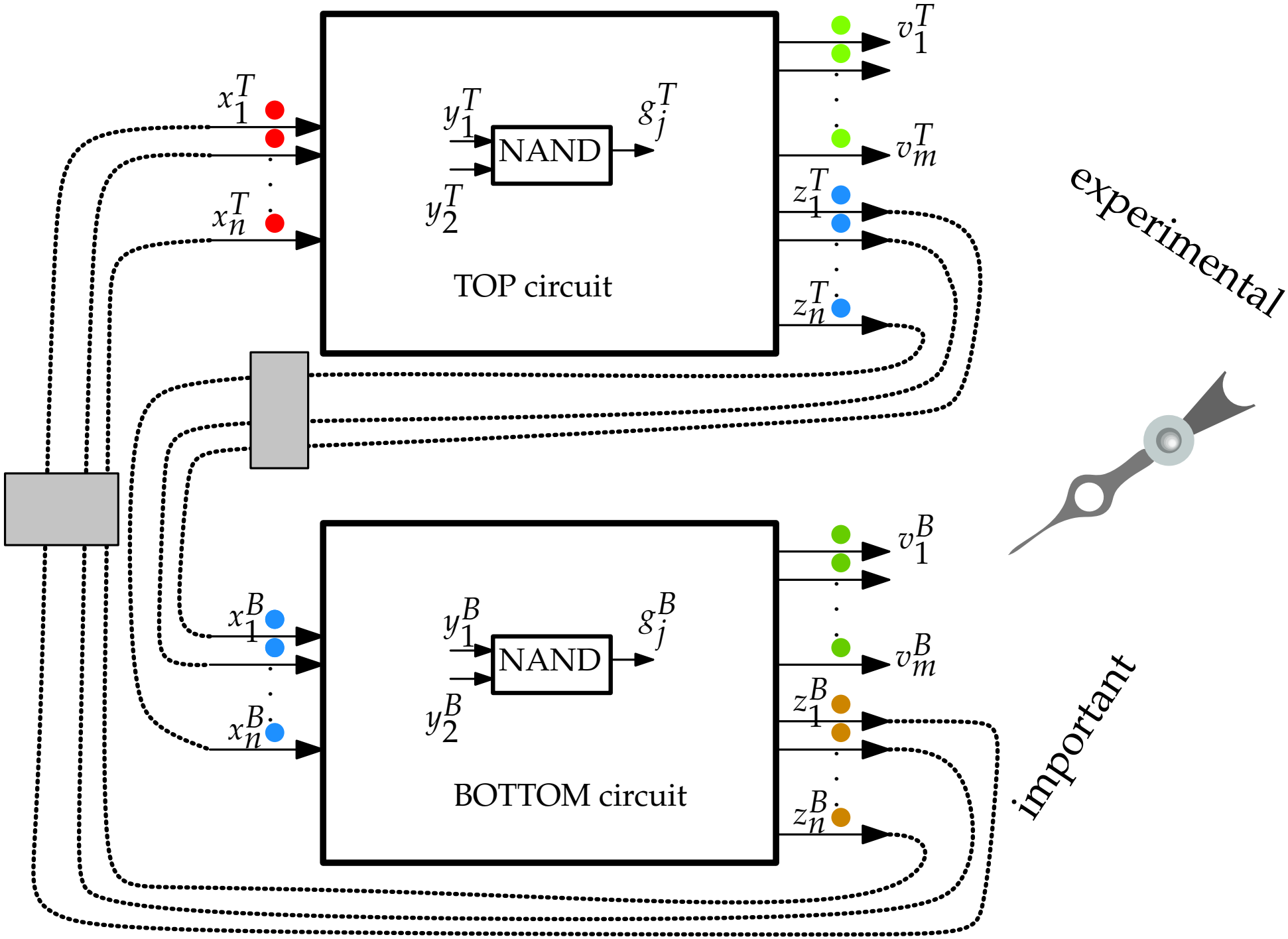

x_1^T
x_n^T
y_1^T
y_2^T
NAND
g_j^T
TOP circuit
v_1^T
v_m^T
z_1^T
z_n^T
x_1^B
x_n^B
y_1^B
y_2^B
NAND
g_j^B
BOTTOM circuit
v_1^B
v_m^B
z_1^B
z_n^B

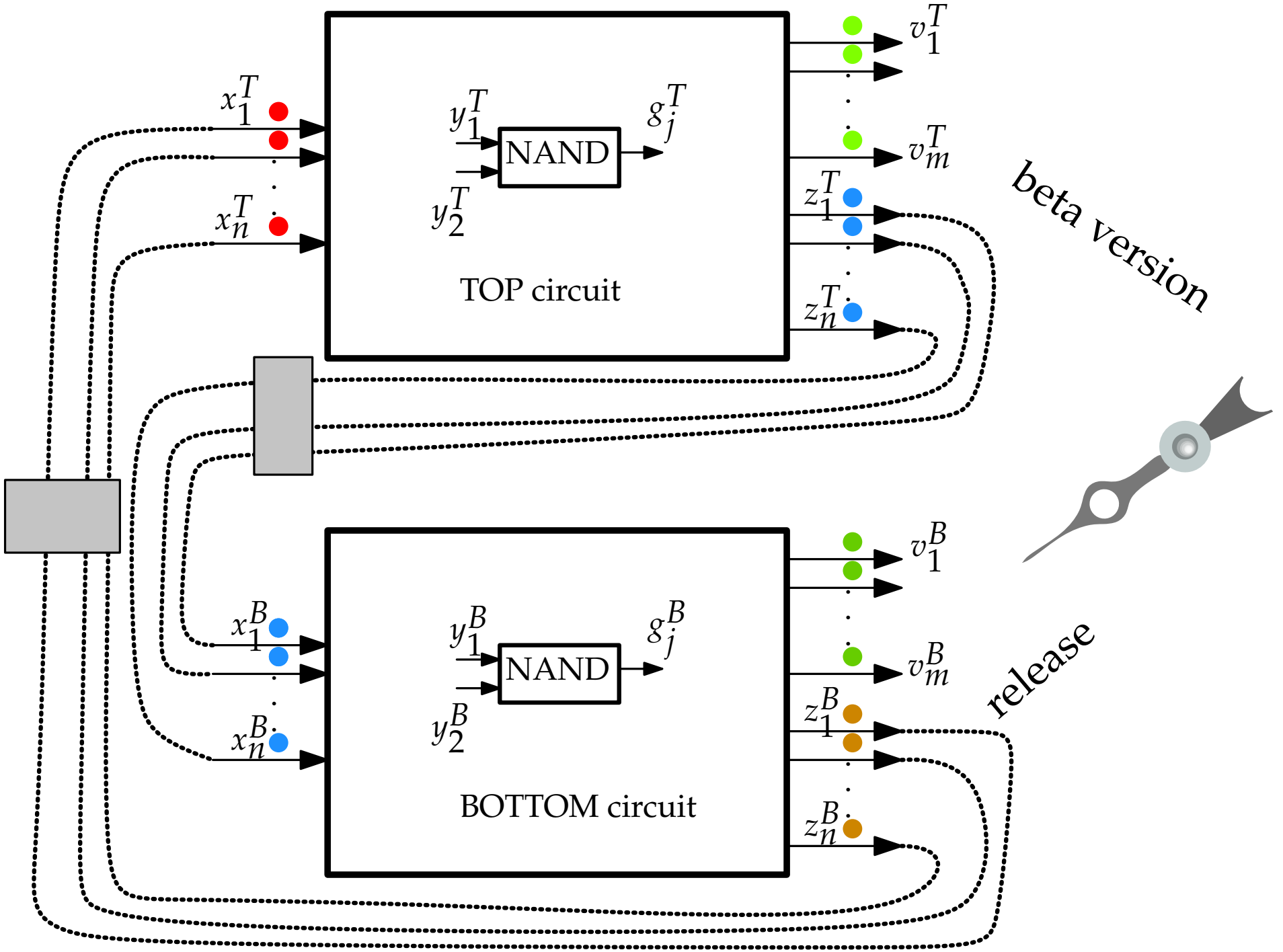

x_1^T
x_n^T
y_1^T
y_2^T
NAND
g_j^T
TOP circuit
v_1^T
v_m^T
z_1^T
z_n^T
x_1^B
x_n^B
y_1^B
y_2^B
NAND
g_j^B
BOTTOM circuit
v_1^B
v_m^B
z_1^B
z_n^B

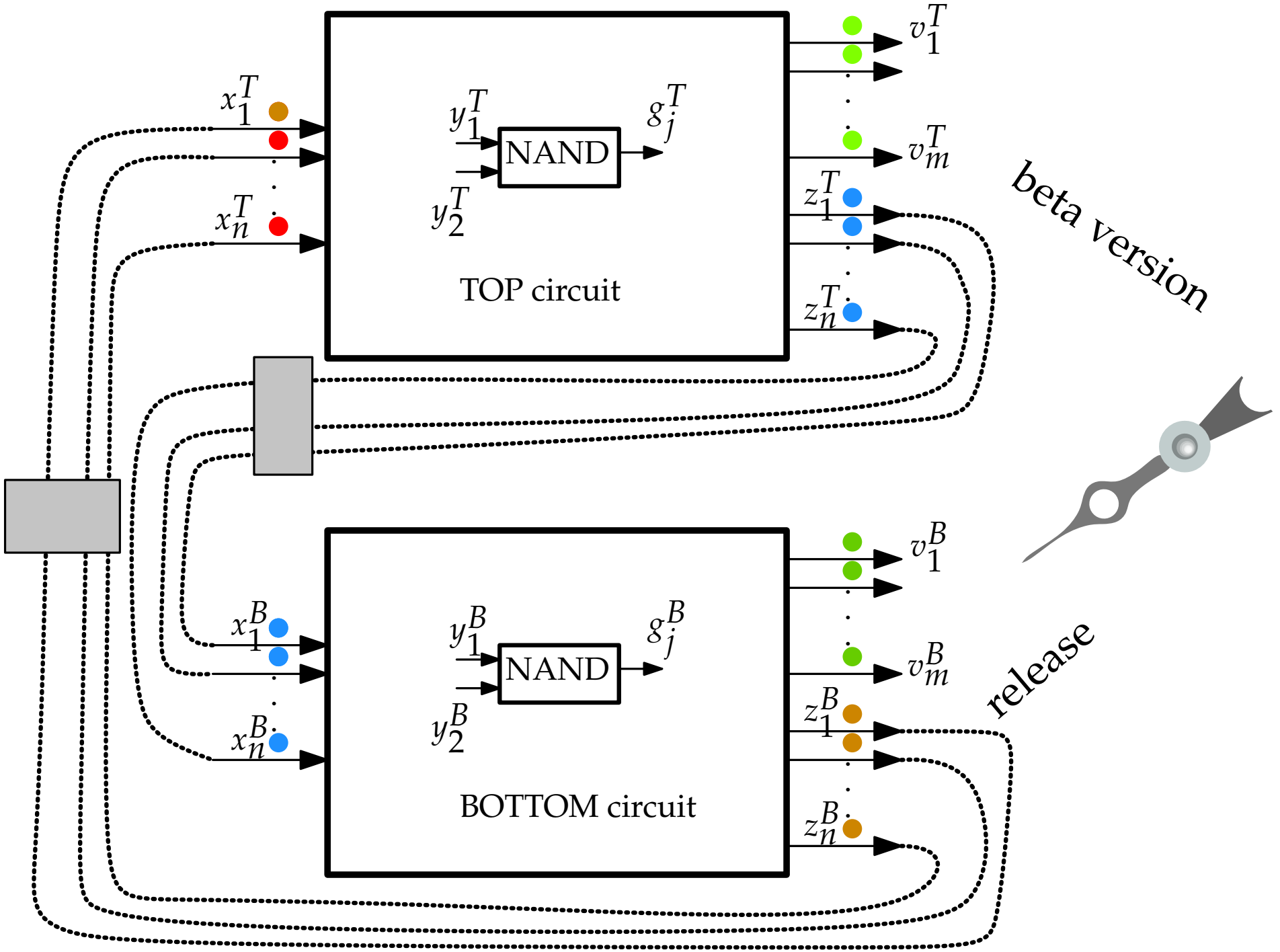

x_1^T
x_n^T
y_1^T
y_2^T
NAND
g_j^T
TOP circuit
v_1^T
v_m^T
z_1^T
z_n^T
points to the "important" circuit
x_1^B
x_n^B
y_1^B
y_2^B
NAND
g_j^B
BOTTOM circuit
v_1^B
v_m^B
z_1^B
z_n^B

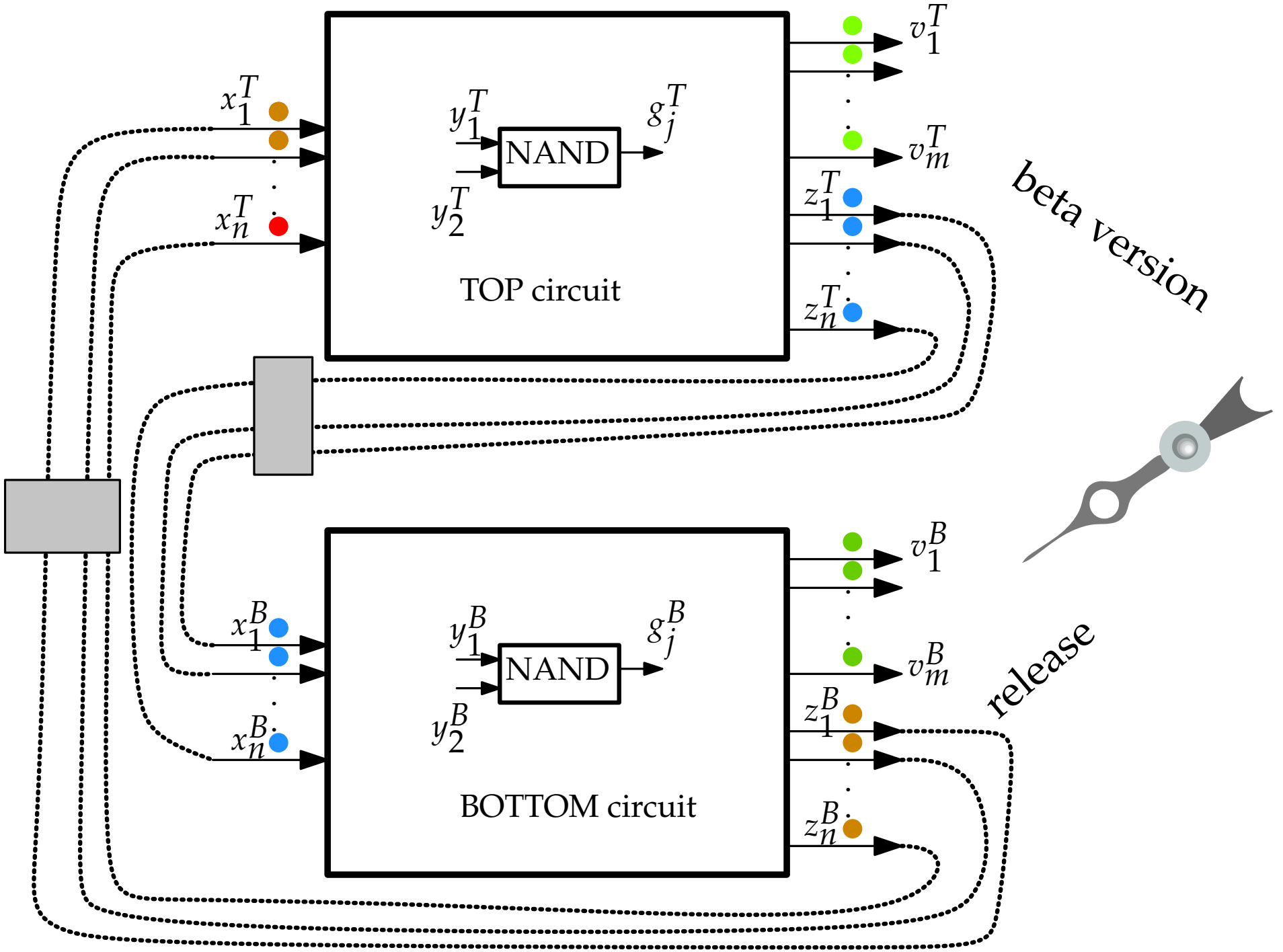

x_1^T
x_n^T
y_1^T
y_2^T
NAND
g_j^T
TOP circuit
v_1^T
v_m^T
z_1^T
z_n^T
points to the "important" circuit
x_1^B
x_n^B
y_1^B
y_2^B
NAND
g_j^B
BOTTOM circuit
v_1^B
v_m^B
z_1^B
z_n^B
this is the "experimental" circuit

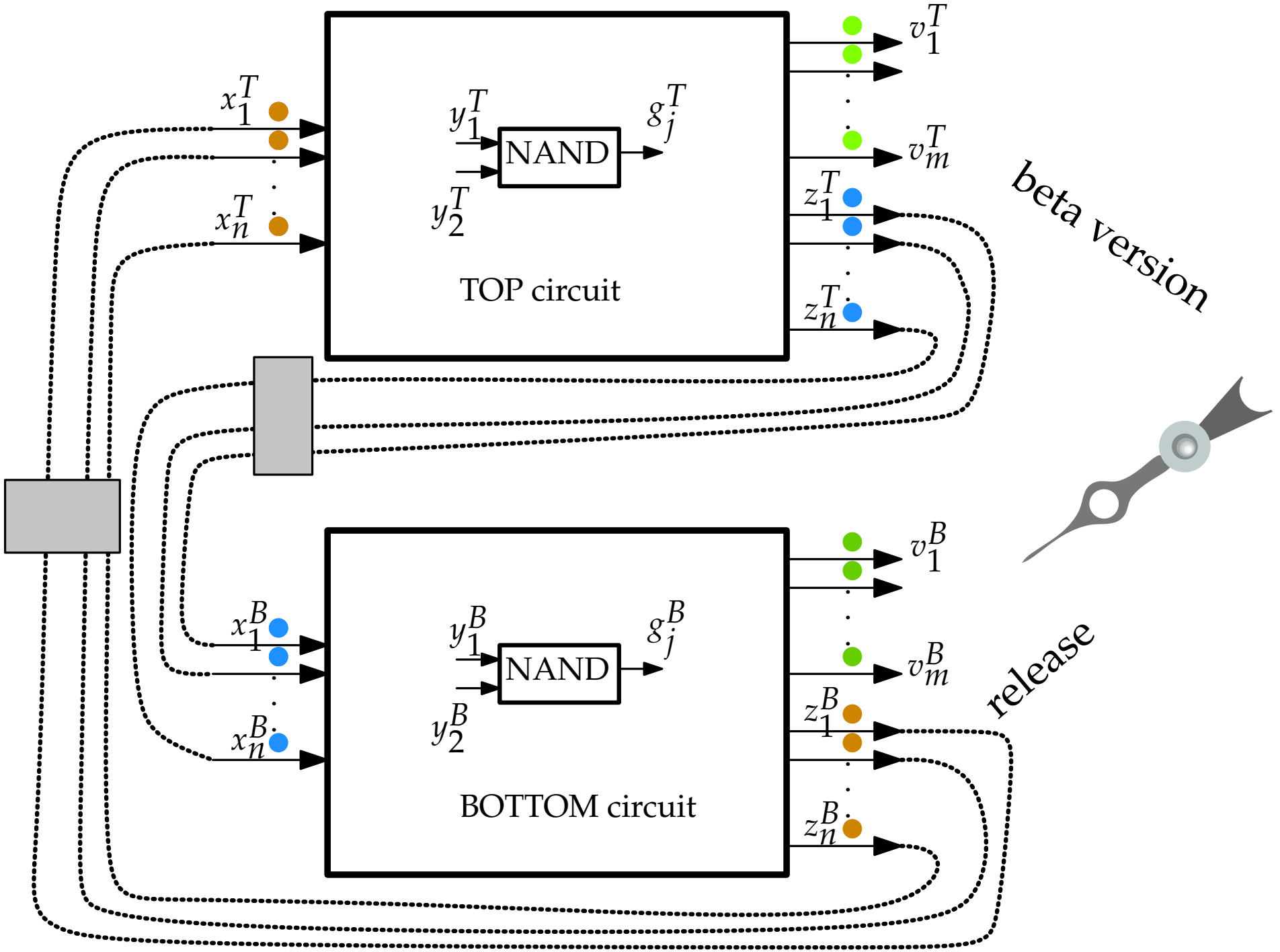

x_1^T
x_n^T
y_1^T
y_2^T
NAND
g_j^T
TOP circuit
v_1^T
v_m^T
z_1^T
z_n^T
points to the "important" circuit
x_1^B
x_n^B
y_1^B
y_2^B
NAND
g_j^B
BOTTOM circuit
v_1^B
v_m^B
z_1^B
z_n^B
this is the "experimental" circuit

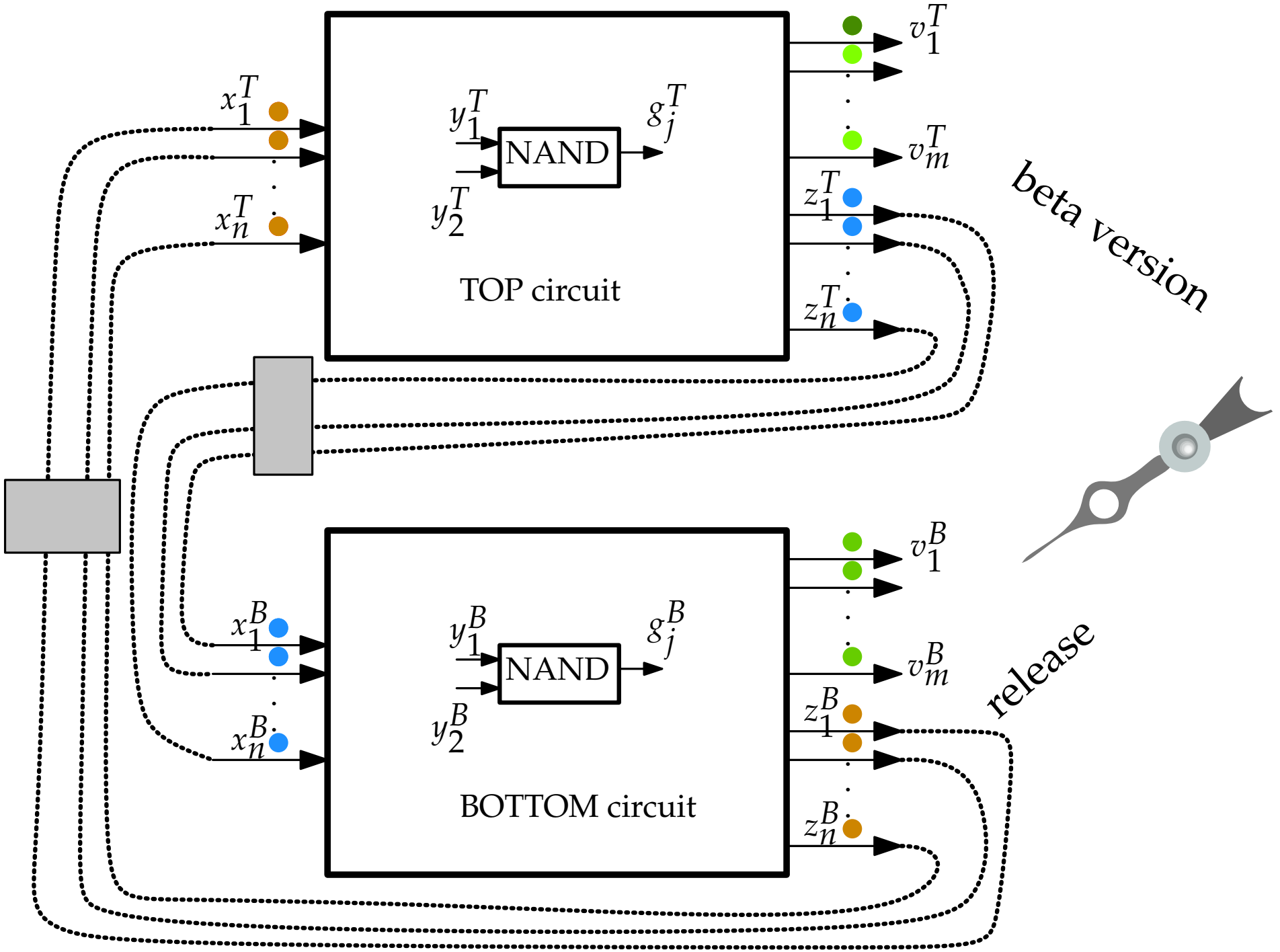

x_1^T
x_n^T
y_1^T
y_2^T
NAND
g_j^T
TOP circuit
v_1^T
v_m^T
z_1^T
z_n^T
x_1^B
x_n^B
y_1^B
y_2^B
NAND
g_j^B
BOTTOM circuit
v_1^B
v_m^B
z_1^B
z_n^B
experimental
important

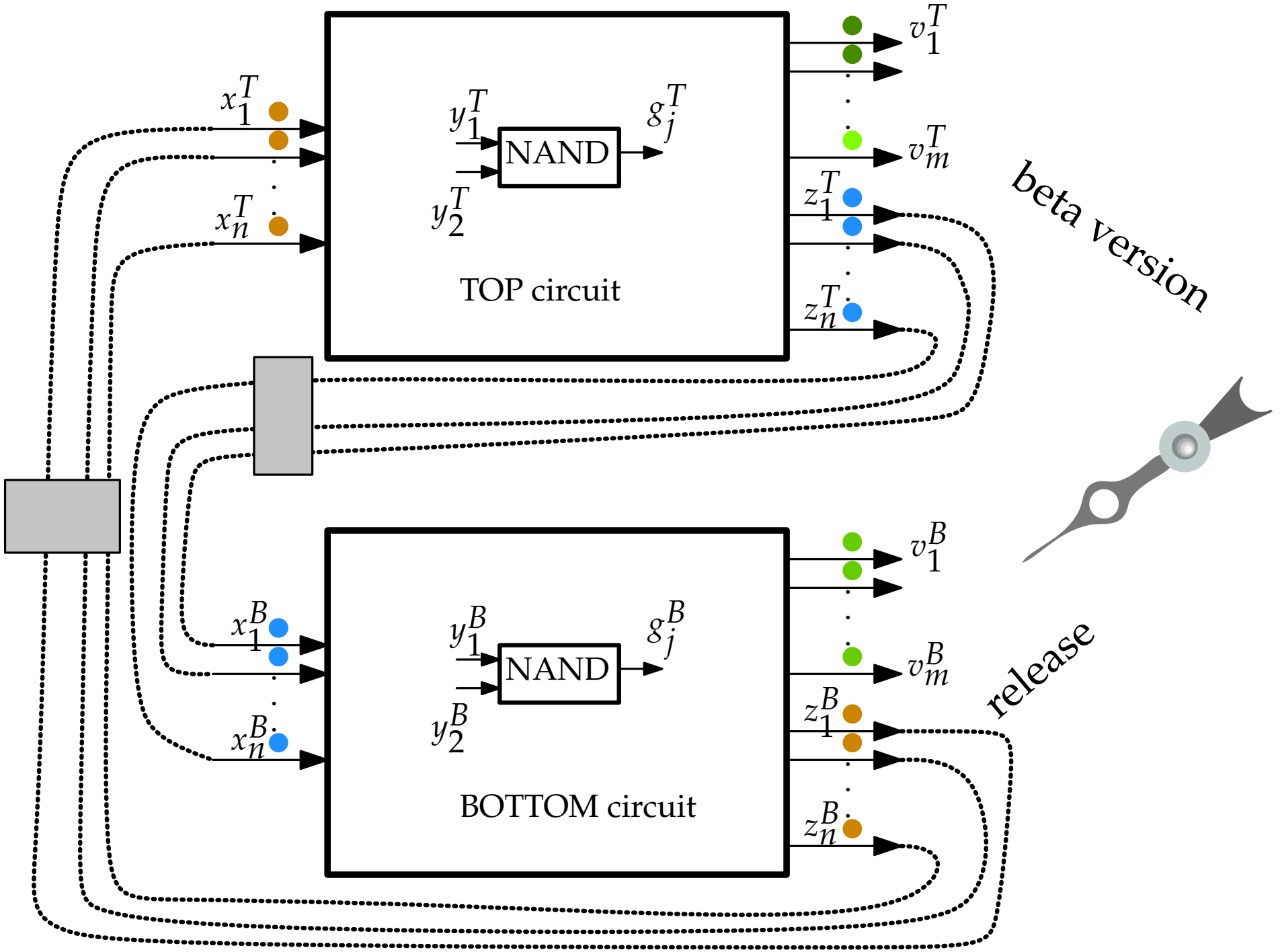

x_1^T
x_n^T
y_1^T
y_2^T
NAND
g_j^T
TOP circuit
v_1^T
v_m^T
z_1^T
z_n^T
beta version
x_1^B
x_n^B
y_1^B
y_2^B
NAND
g_j^B
BOTTOM circuit
v_1^B
v_m^B
z_1^B
z_n^B
release

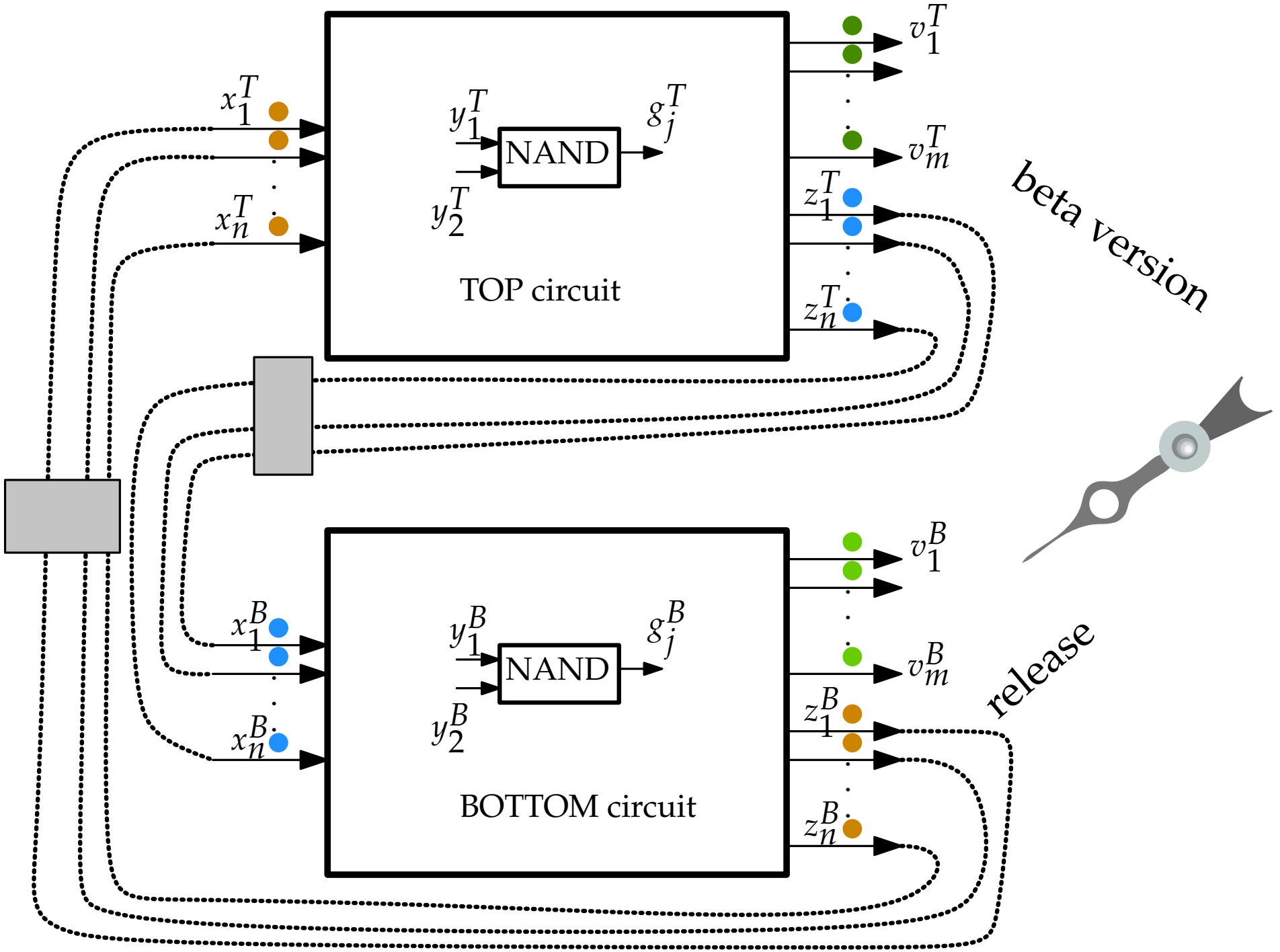

x_1^T
x_n^T
y_1^T
y_2^T
NAND
g_j^T
TOP circuit
v_1^T
v_m^T
z_1^T
z_n^T
beta version
x_1^B
x_n^B
y_1^B
y_2^B
NAND
g_j^B
BOTTOM circuit
v_1^B
v_m^B
z_1^B
z_n^B
release

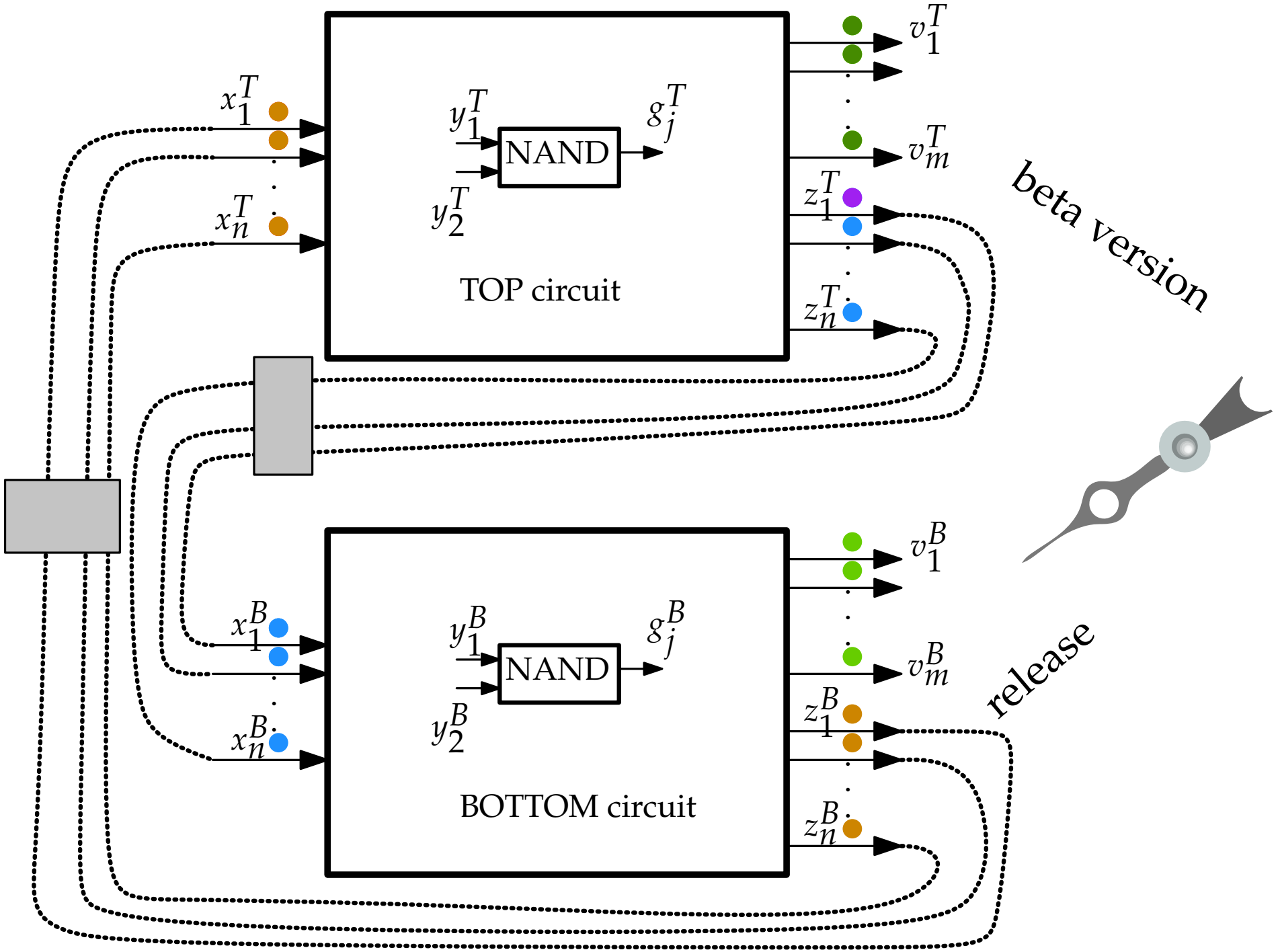

x_1^T
x_n^T
y_1^T
y_2^T
NAND
g_j^T
TOP circuit
v_1^T
v_m^T
z_1^T
z_n^T
beta version
x_1^B
x_n^B
y_1^B
y_2^B
NAND
g_j^B
BOTTOM circuit
v_1^B
v_m^B
z_1^B
z_n^B
release

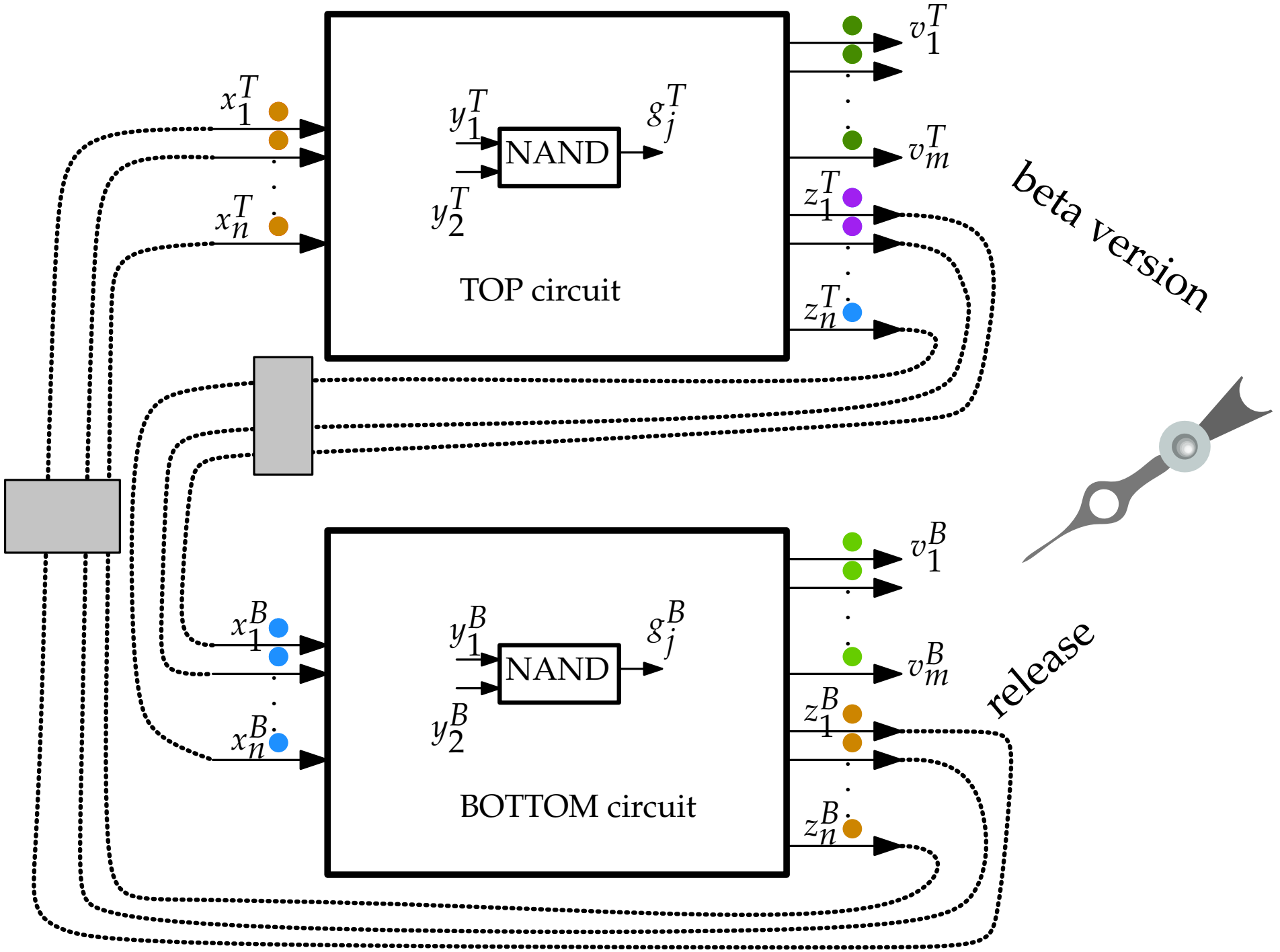

x_1^T
x_n^T
y_1^T
y_2^T
NAND
g_j^T
TOP circuit
v_1^T
v_m^T
z_1^T
z_n^T
beta version
x_1^B
x_n^B
y_1^B
y_2^B
NAND
g_j^B
BOTTOM circuit
v_1^B
v_m^B
z_1^B
z_n^B
release

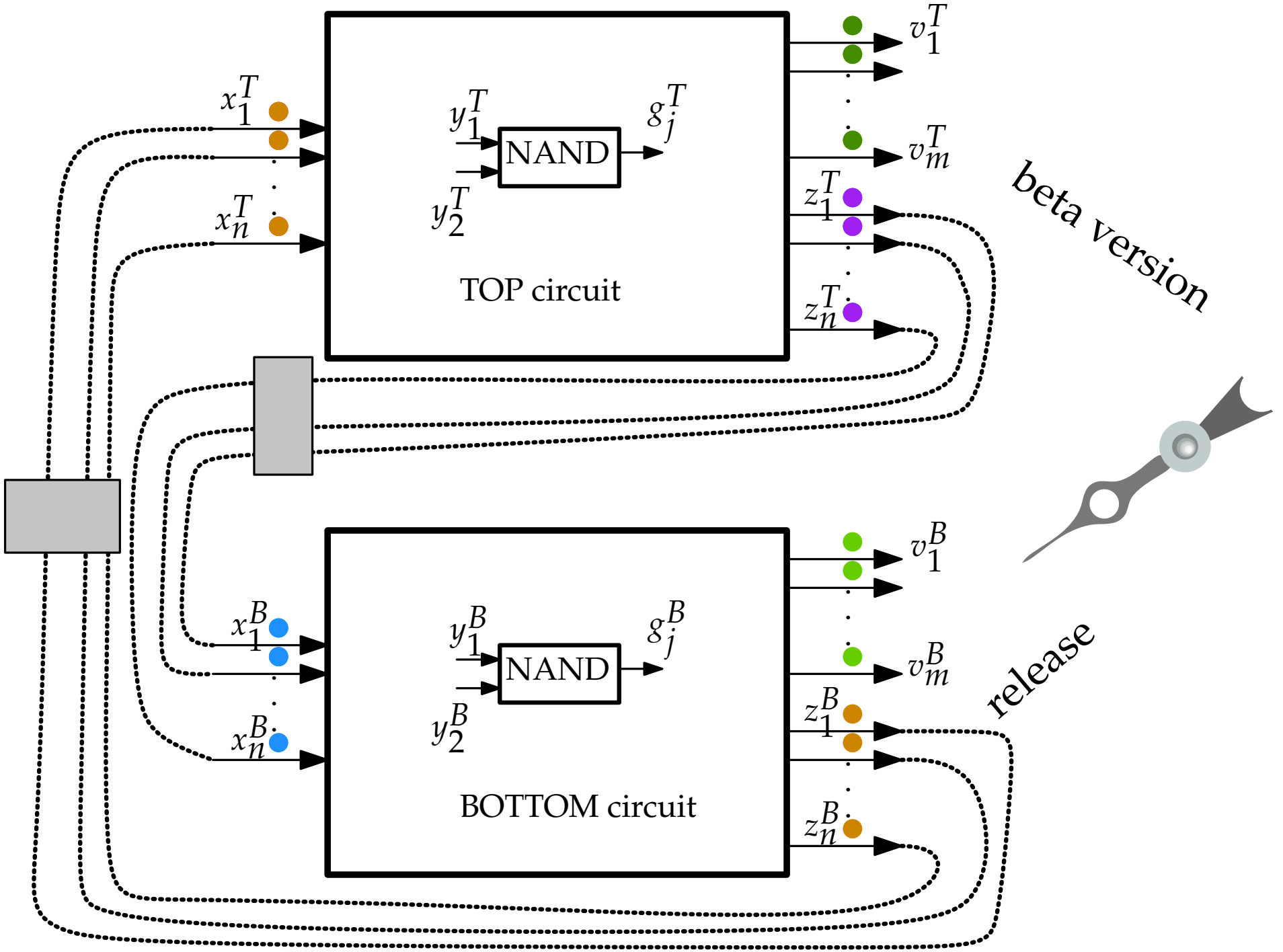

x_1^T
x_n^T
y_1^T
y_2^T
NAND
g_j^T
TOP circuit
v_1^T
v_m^T
z_1^T
z_n^T
beta version
x_1^B
x_n^B
y_1^B
y_2^B
NAND
g_j^B
BOTTOM circuit
v_1^B
v_m^B
z_1^B
z_n^B
release

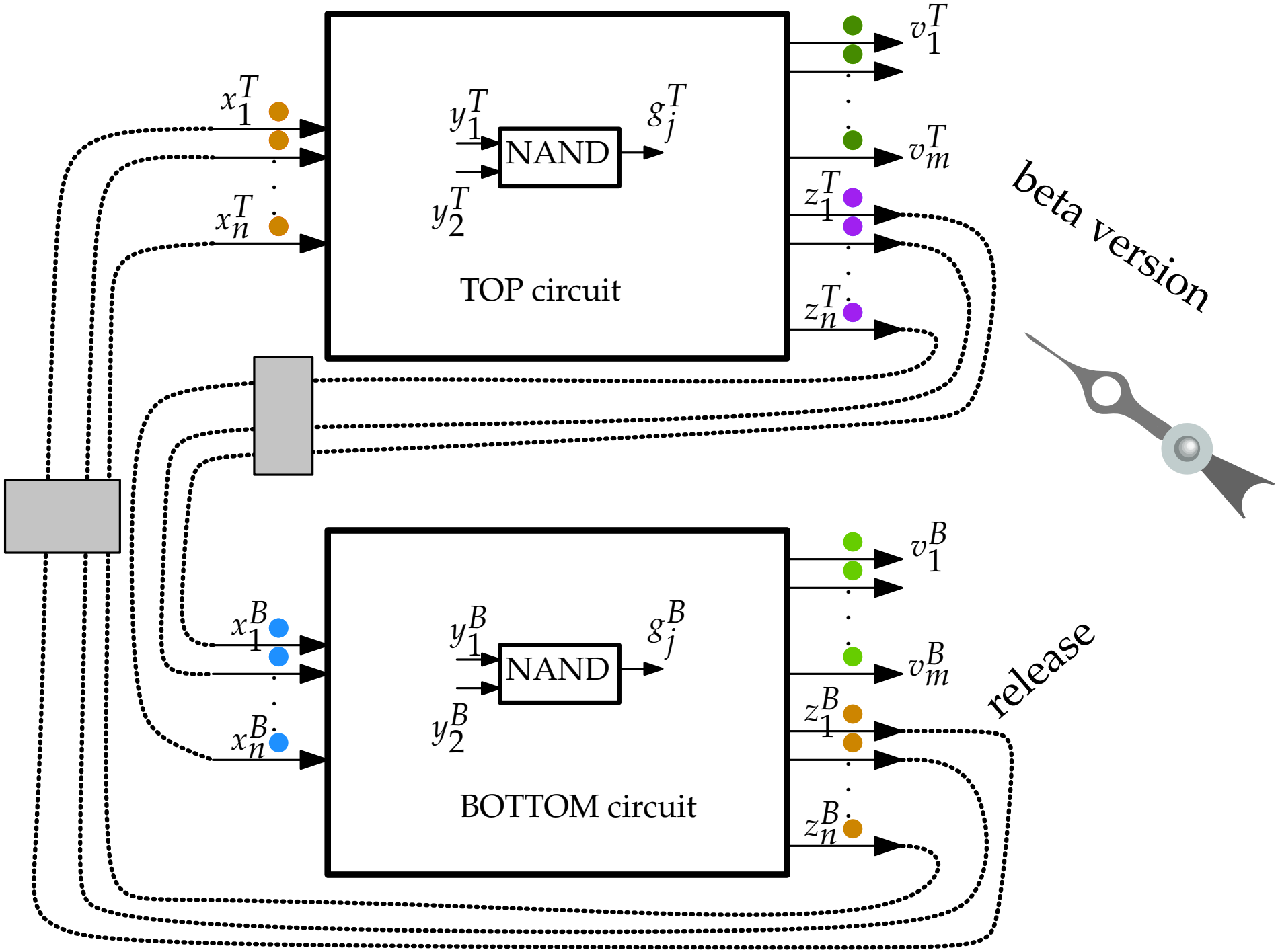

x_1^T
x_n^T
y_1^T
y_2^T
NAND
g_j^T
TOP circuit
v_1^T
v_m^T
z_1^T
z_n^T
beta version
x_1^B
x_n^B
y_1^B
y_2^B
NAND
g_j^B
BOTTOM circuit
v_1^B
v_m^B
z_1^B
z_n^B
release

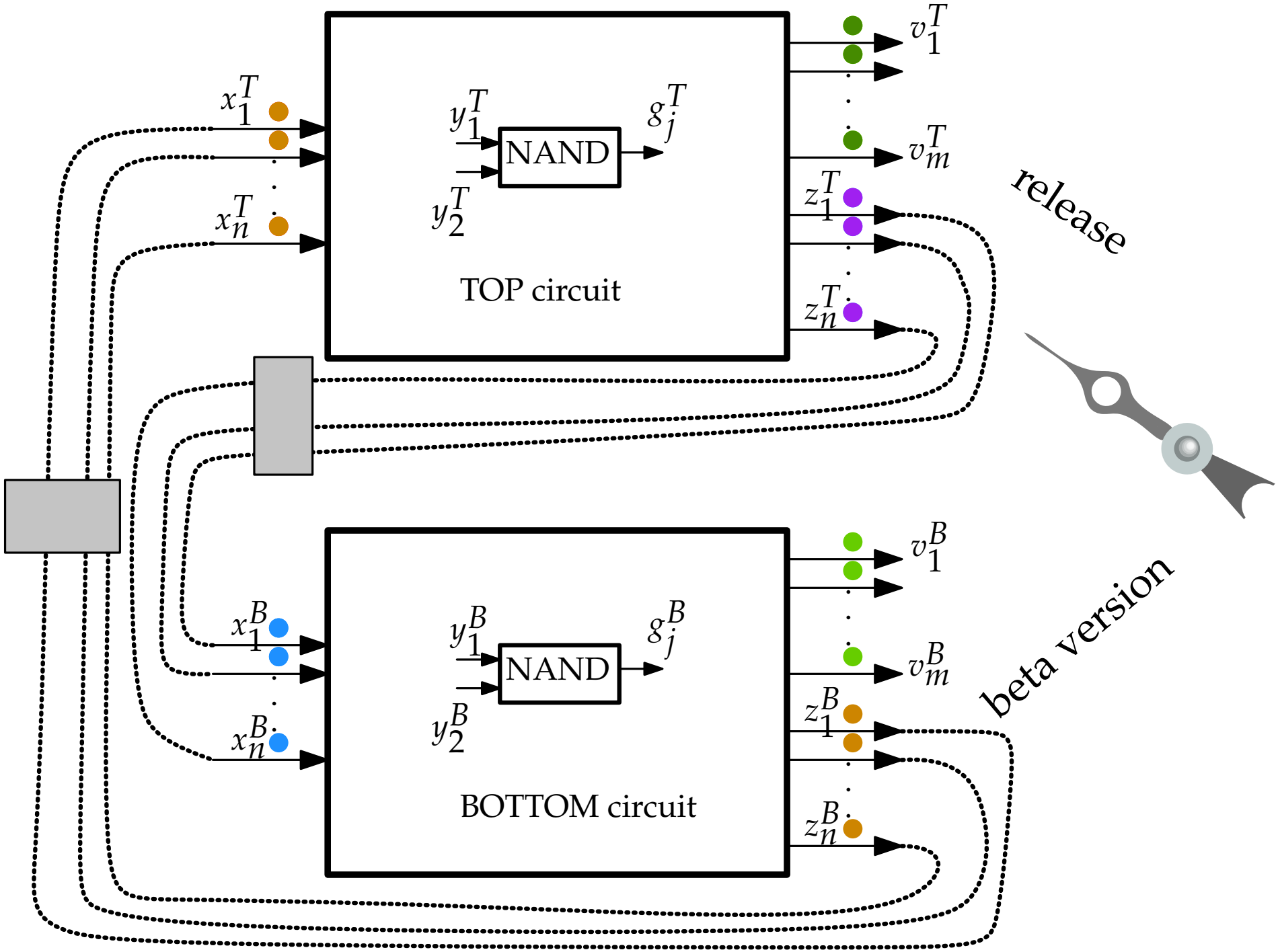

x_1^T
x_n^T
y_1^T
y_2^T
NAND
g_j^T
TOP circuit
v_1^T
v_m^T
z_1^T
z_n^T
beta version
x_1^B
x_n^B
y_1^B
y_2^B
NAND
g_j^B
BOTTOM circuit
v_1^B
v_m^B
z_1^B
z_n^B
release


x_1^T
x_n^T
y_1^T
y_2^T
NAND
g_j^T
TOP circuit
v_1^T
v_m^T
z_1^T
z_n^T
beta version
x_1^B
x_n^B
y_1^B
y_2^B
NAND
g_j^B
BOTTOM circuit
v_1^B
v_m^B
z_1^B
z_n^B
release

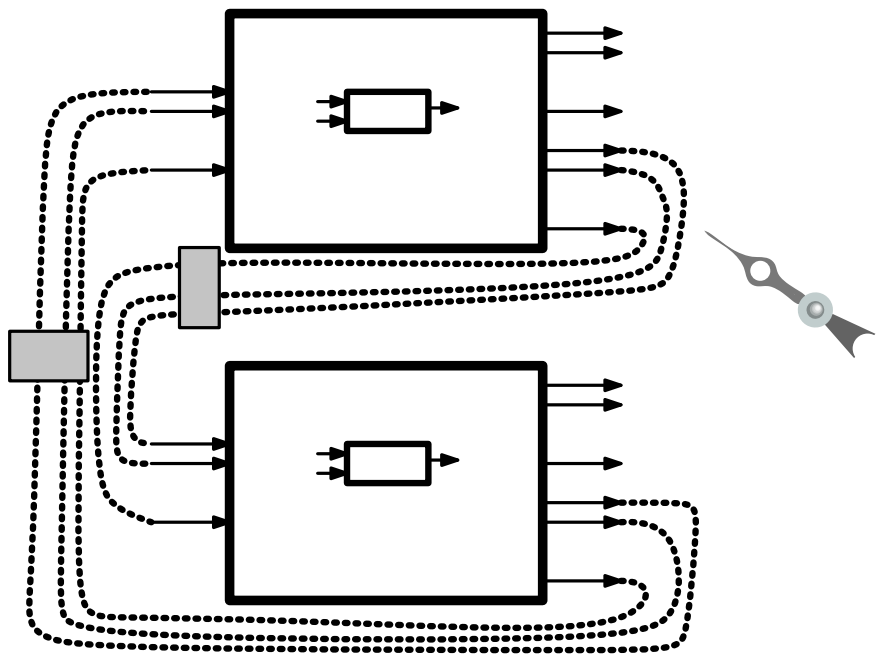

x_1^T
x_n^T
y_1^T
y_2^T
NAND
g_j^T
TOP circuit
v_1^T
v_m^T
z_1^T
z_n^T
beta version
x_1^B
x_n^B
y_1^B
y_2^B
NAND
g_j^B
BOTTOM circuit
v_1^B
v_m^B
z_1^B
z_n^B
release


x_1^T
x_n^T
y_1^T
y_2^T
NAND
g_j^T
TOP circuit
v_1^T
v_m^T
z_1^T
z_n^T
beta version
x_1^B
x_n^B
y_1^B
y_2^B
NAND
g_j^B
BOTTOM circuit
v_1^B
v_m^B
z_1^B
z_n^B
release


x_1^T
x_n^T
y_1^T
y_2^T
NAND
g_j^T
TOP circuit
v_1^T
v_m^T
z_1^T
z_n^T
beta version
x_1^B
x_n^B
y_1^B
y_2^B
NAND
g_j^B
BOTTOM circuit
v_1^B
v_m^B
z_1^B
z_n^B
release

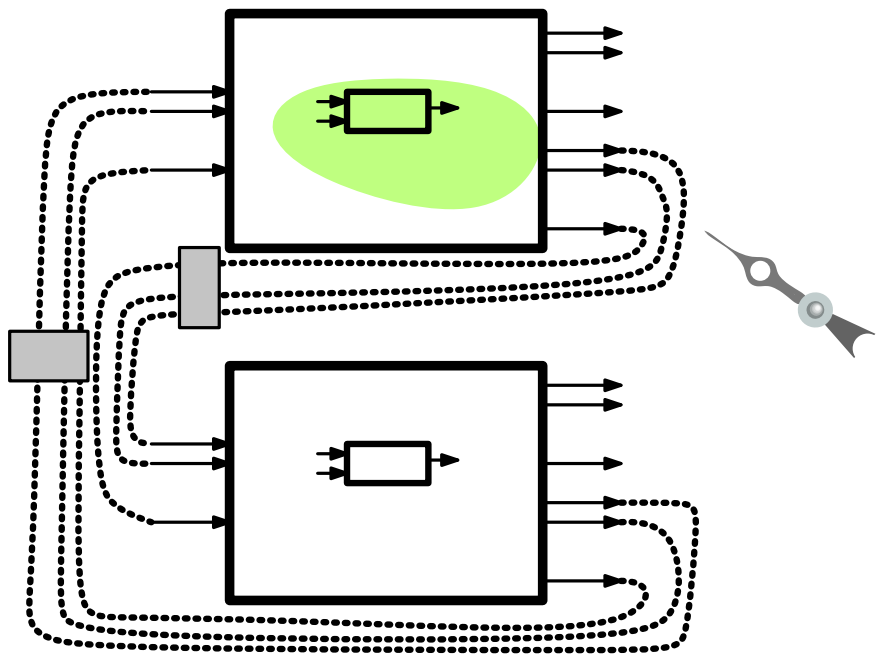

x_1^T
x_n^T
y_1^T
y_2^T
NAND
g_j^T
TOP circuit
v_1^T
v_m^T
z_1^T
z_n^T
release
x_1^B
x_n^B
y_1^B
y_2^B
NAND
g_j^B
BOTTOM circuit
v_1^B
v_m^B
z_1^B
z_n^B
beta version

Let's build a formula encoding all of this mechanism. Design four clause groups:



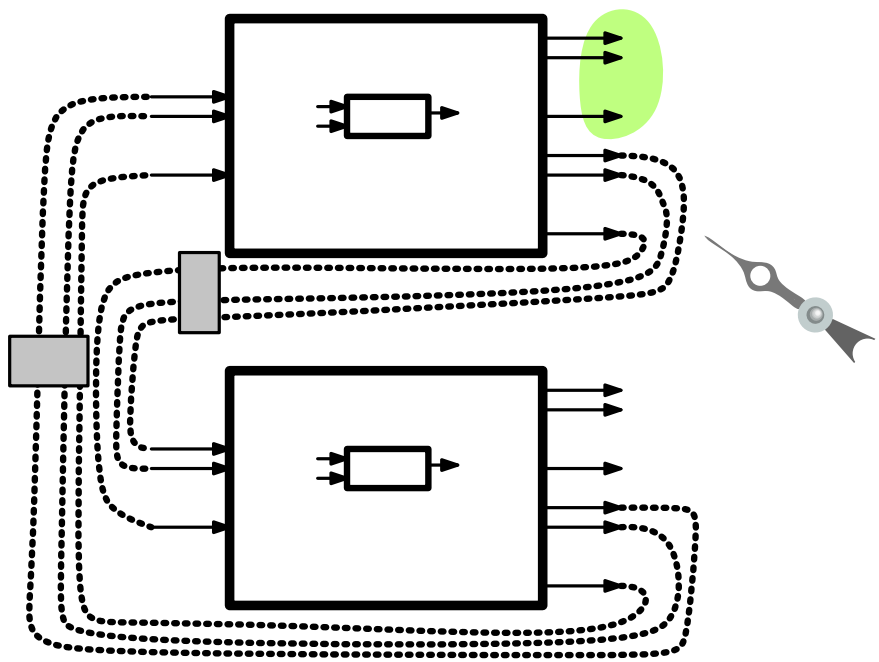
Let's build a formula encoding all of this mechanism. Design four clause groups:

- Group 1: All gates in the *release* circuit should be correct.



Let's build a formula encoding all of this mechanism. Design four clause groups:

- Group 1: All gates in the *release* circuit should be correct.
- Group 2: The output of the *release* circuit should be big.



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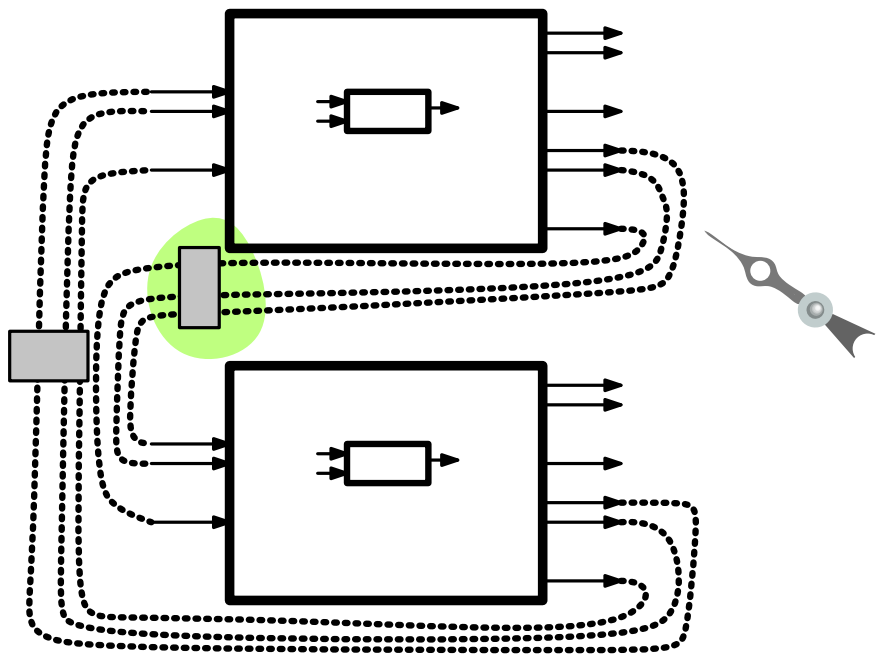

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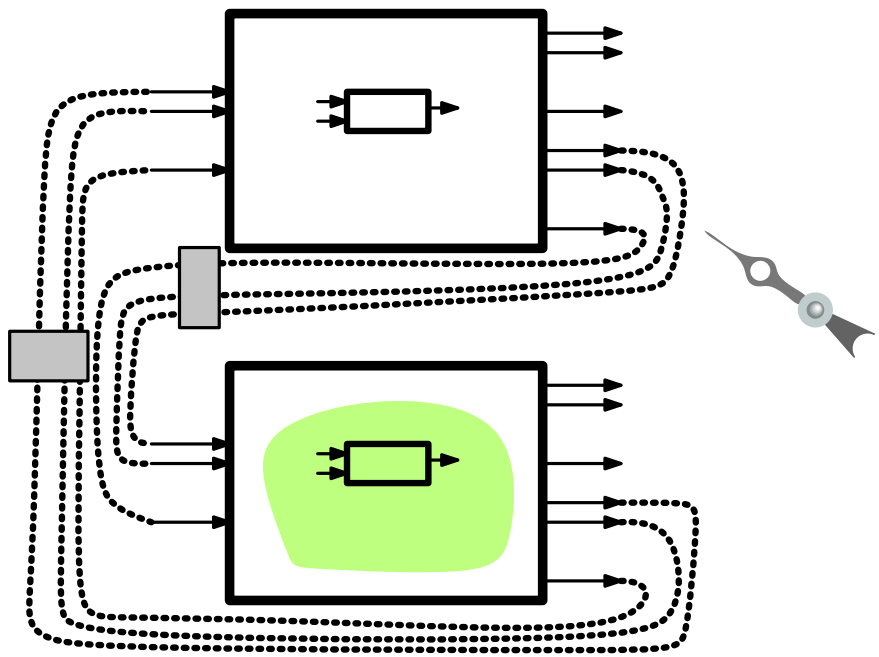

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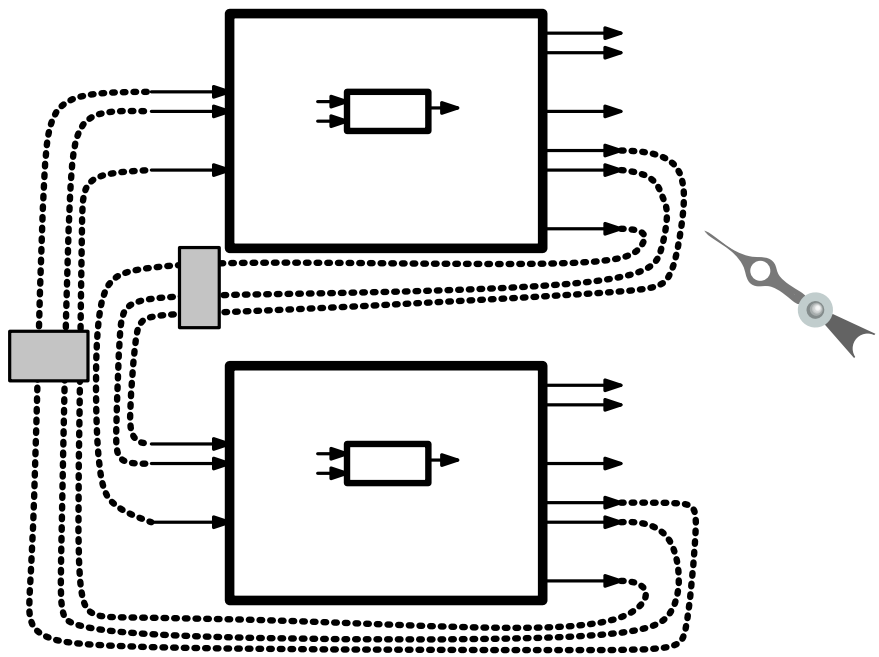

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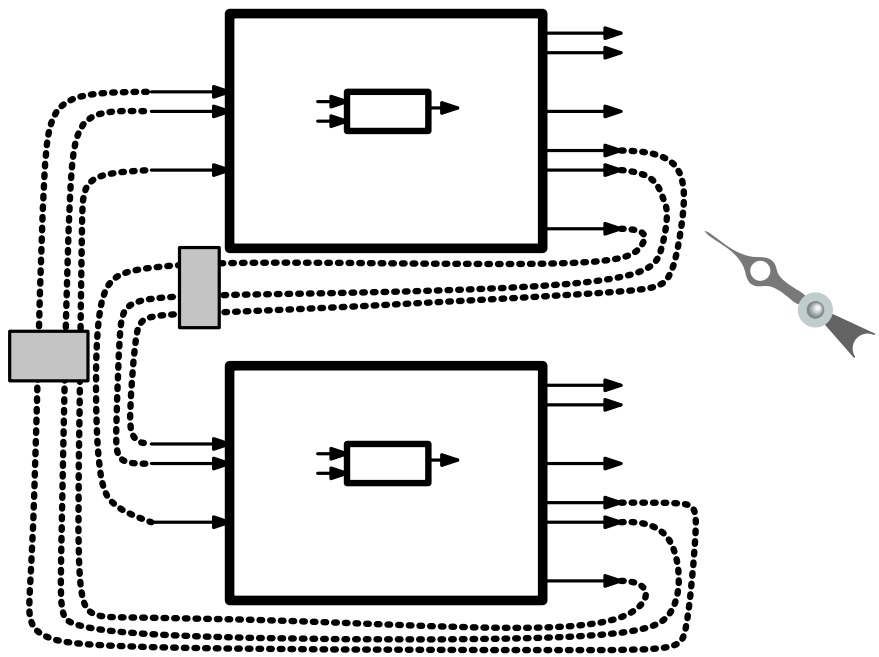

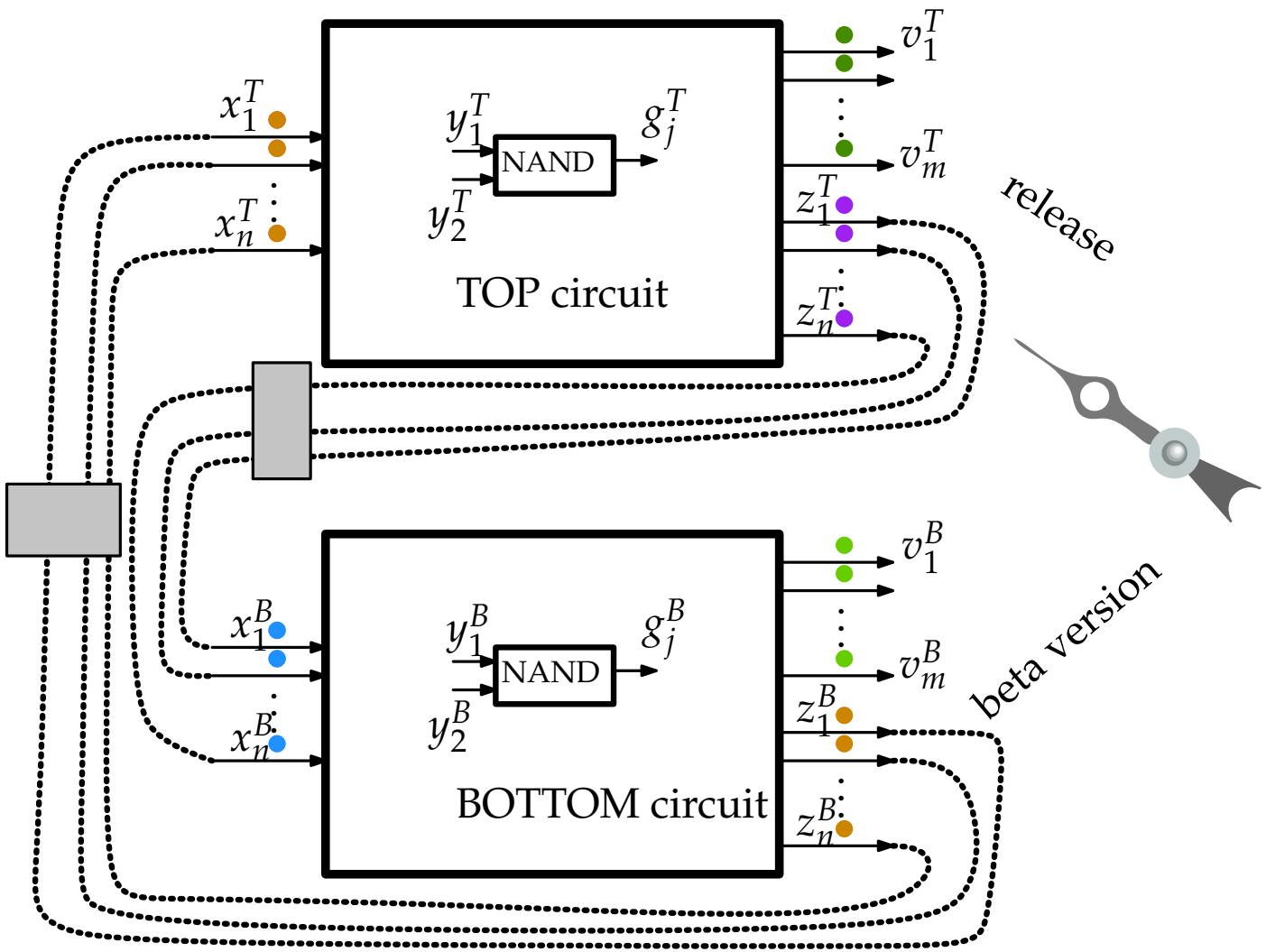
x_1^T
x_n^T
y_1^T
y_2^T
NAND
g_j^T
TOP circuit
v_1^T
v_m^T
z_1^T
z_n^T
release
x_1^B
x_n^B
y_1^B
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NAND
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BOTTOM circuit
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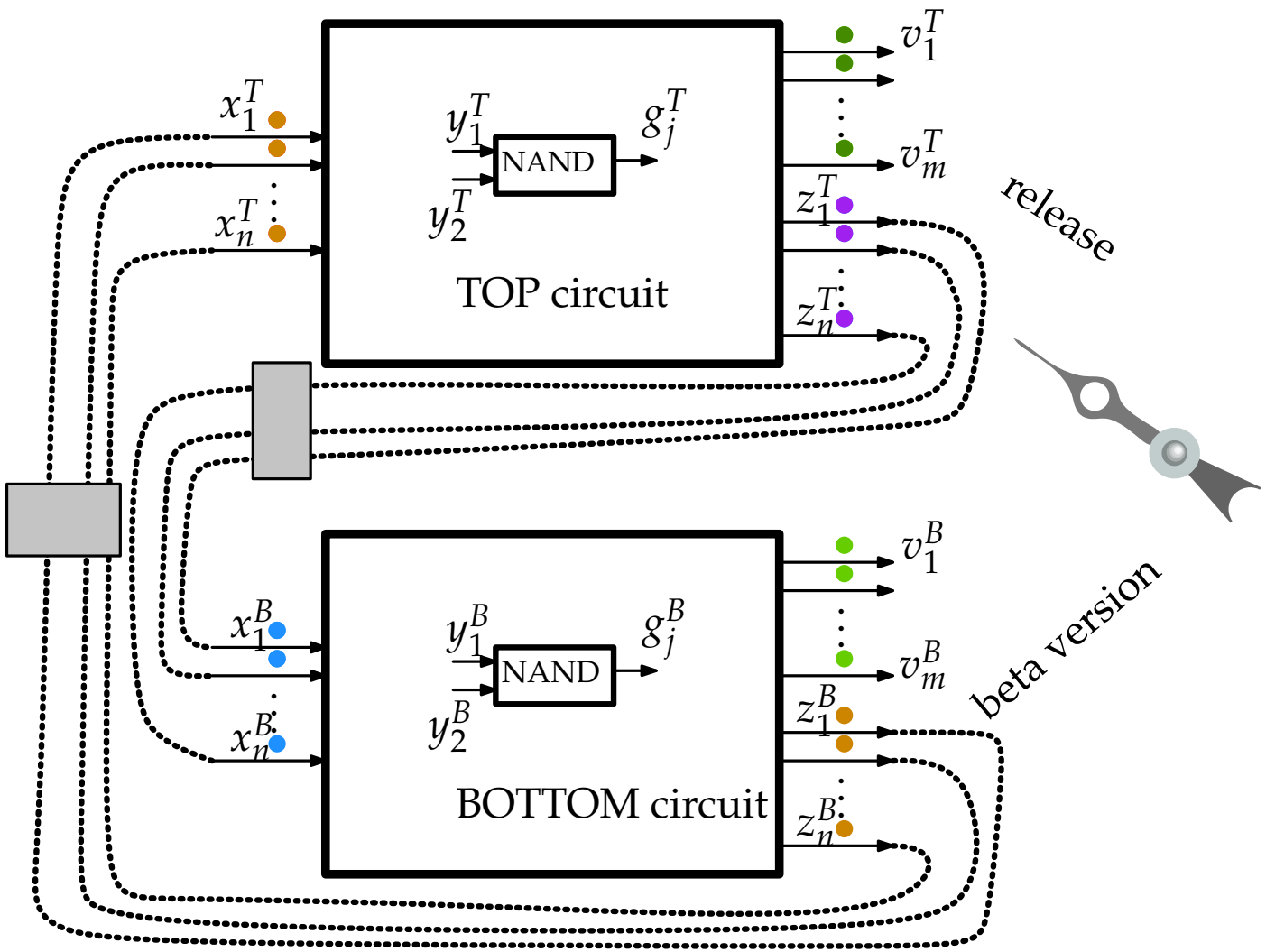
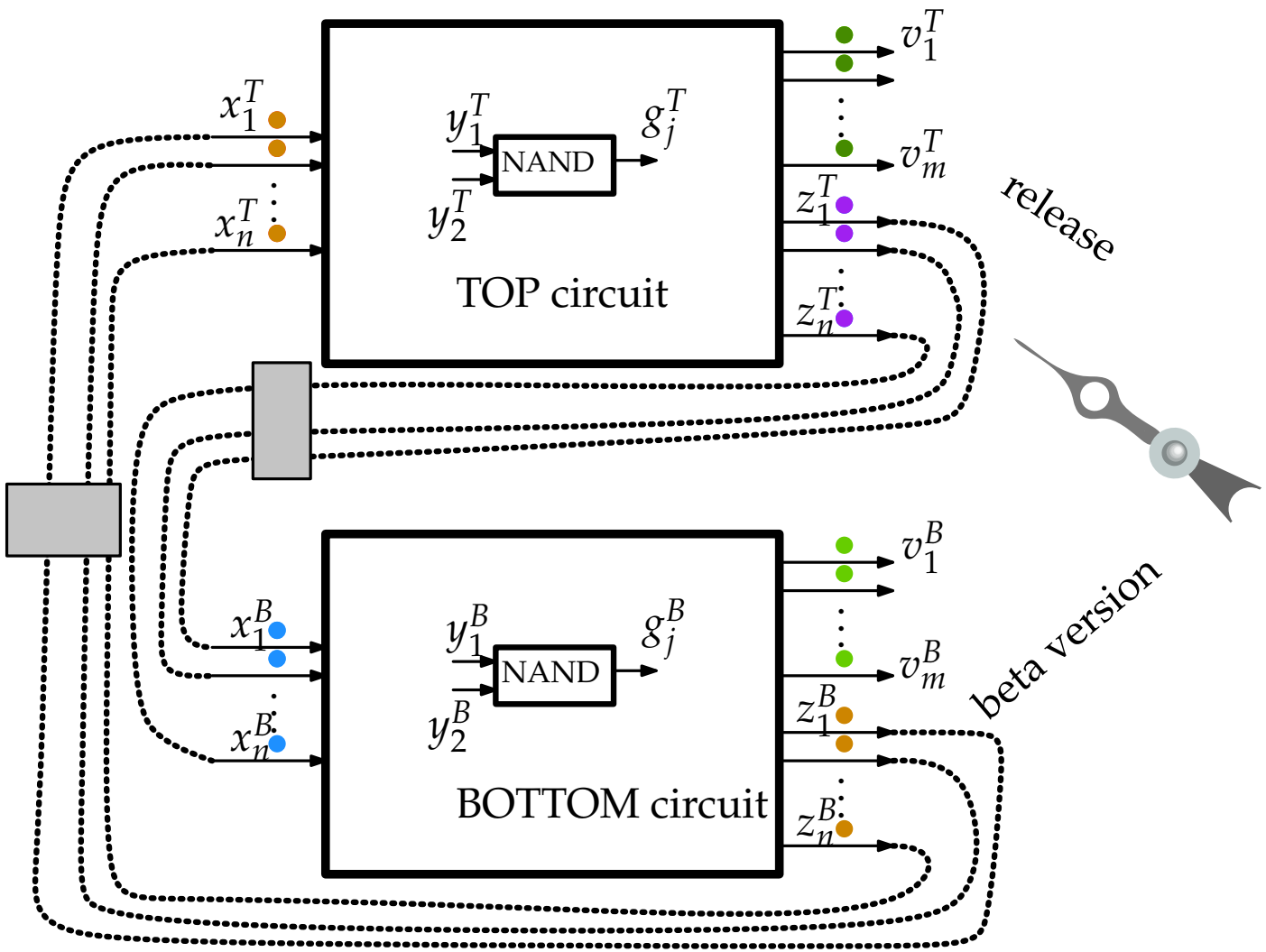
z_n^B
beta version

It would be good if v_i^T were 1 if TOP is the release.



It would be good if v_i^T were 1 if TOP is the release.
If TOP is beta version it wouldn't matter.



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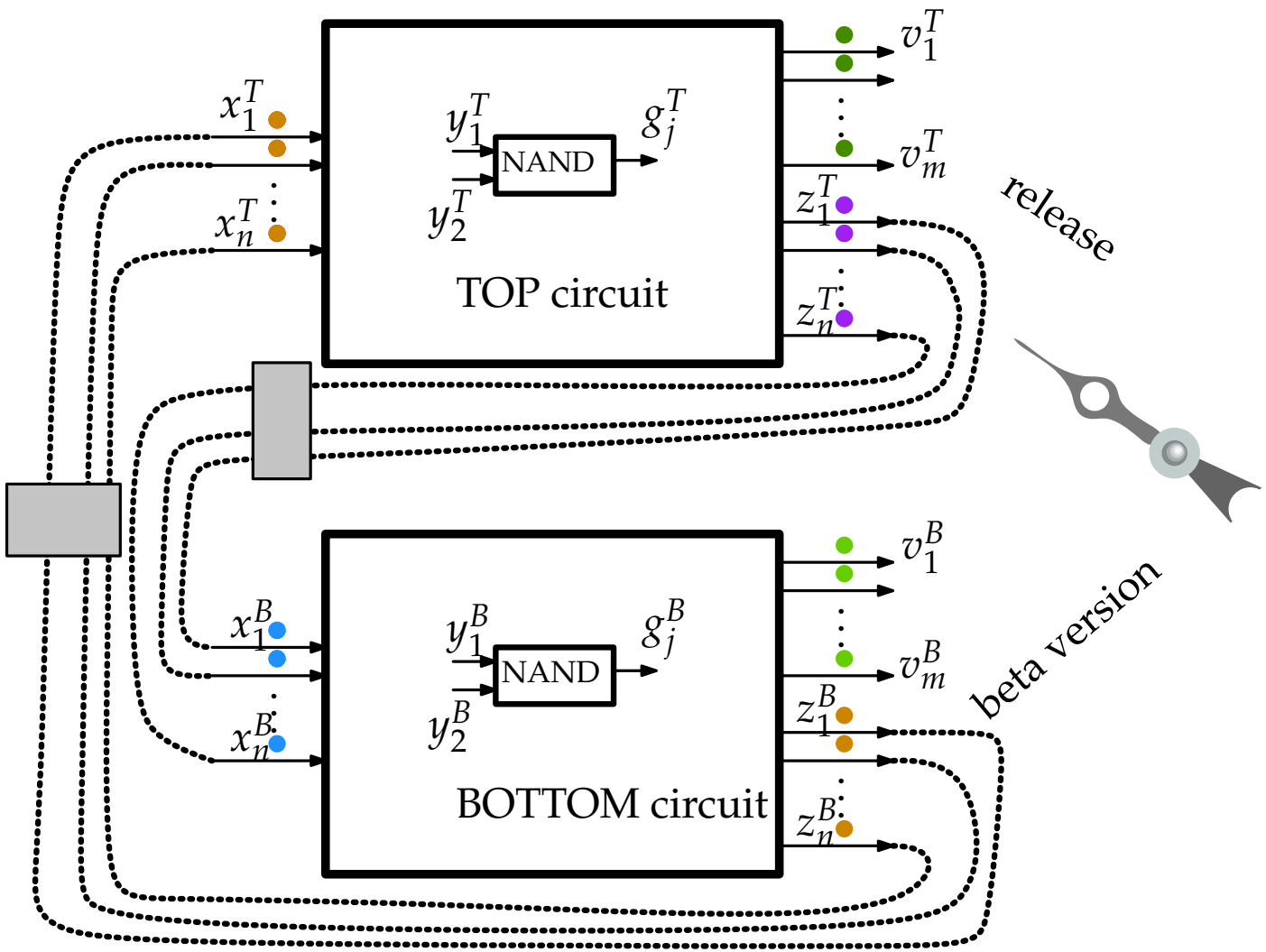

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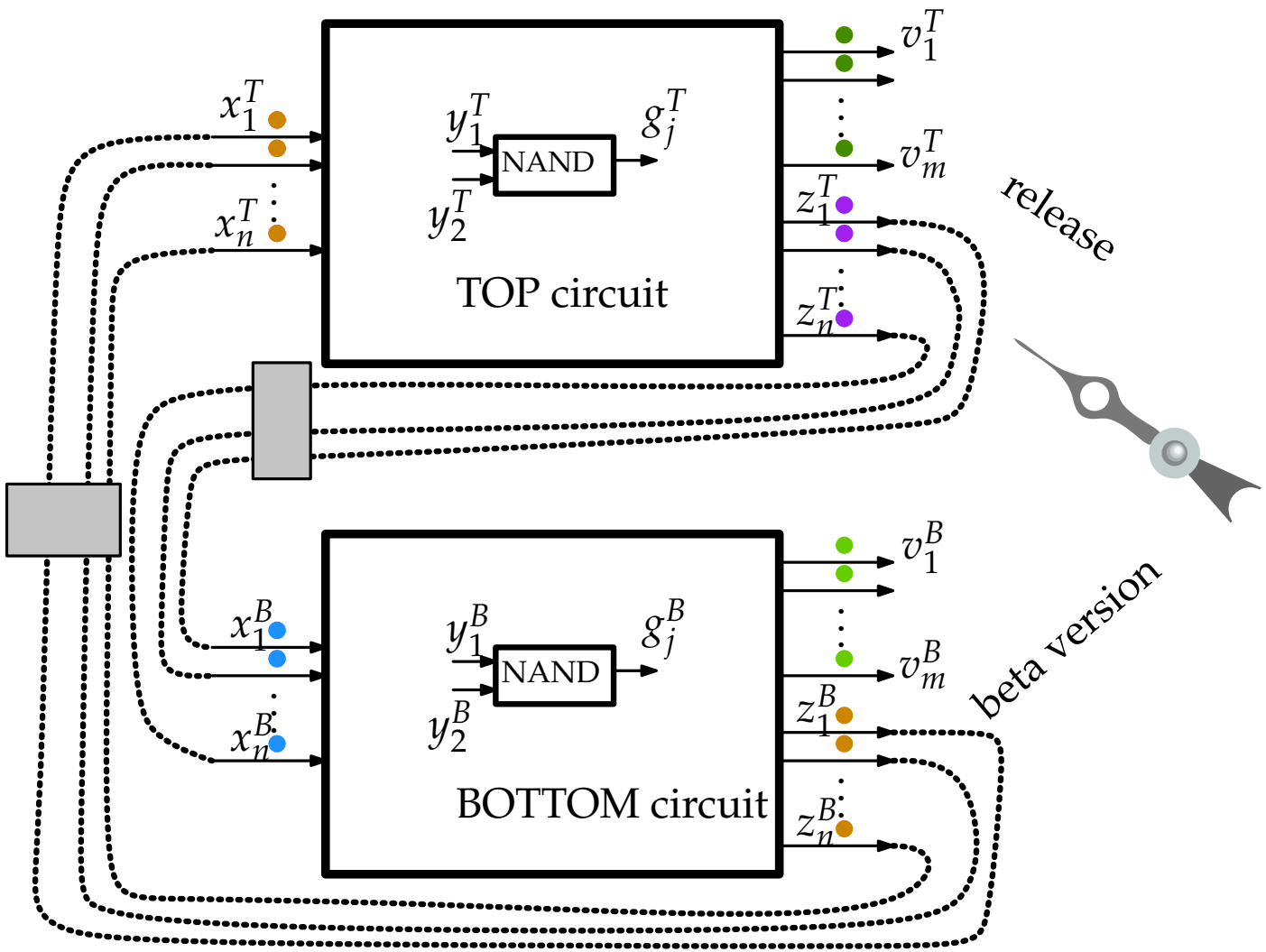

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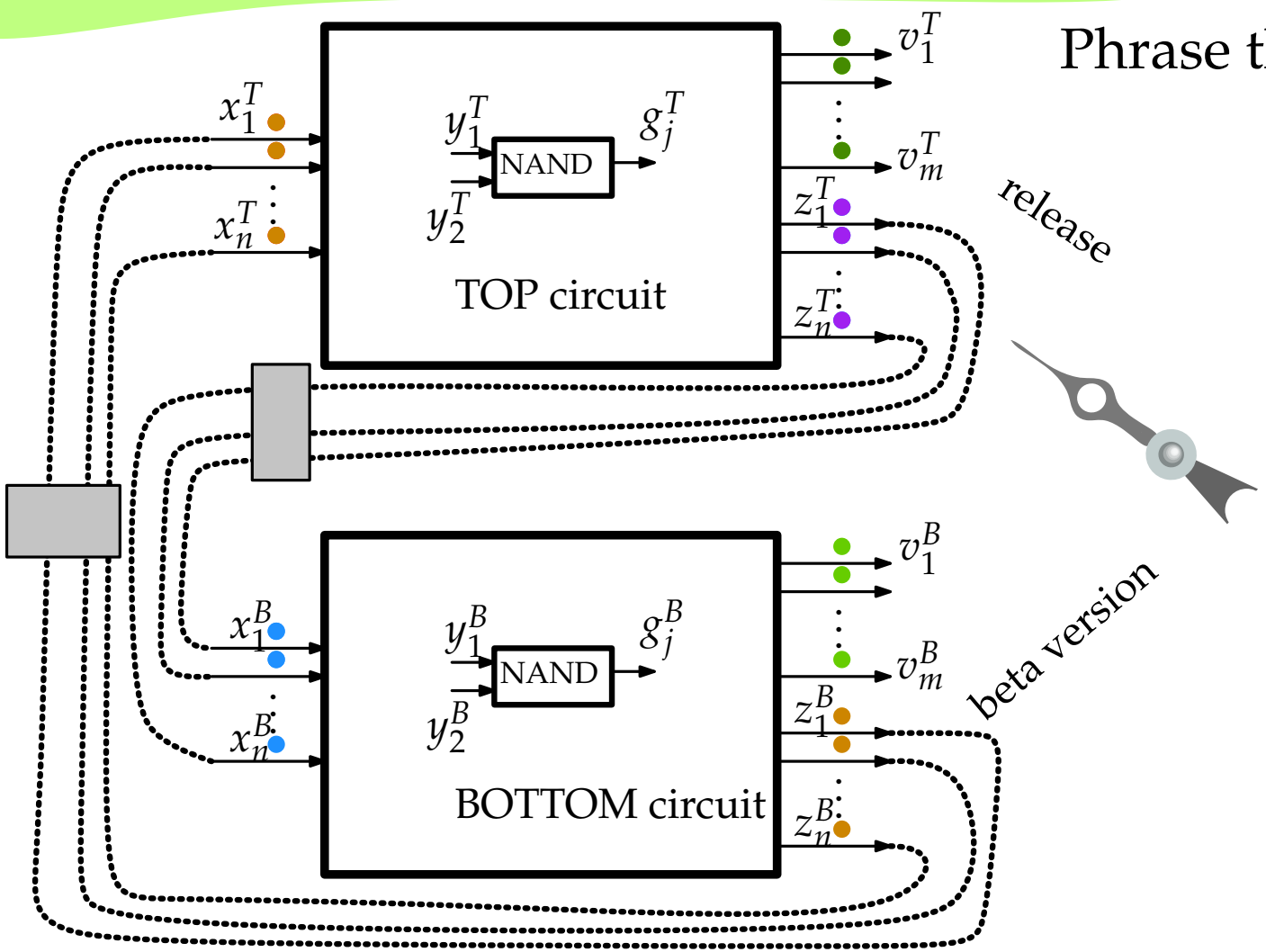
Phrase this as a 3-clasue.



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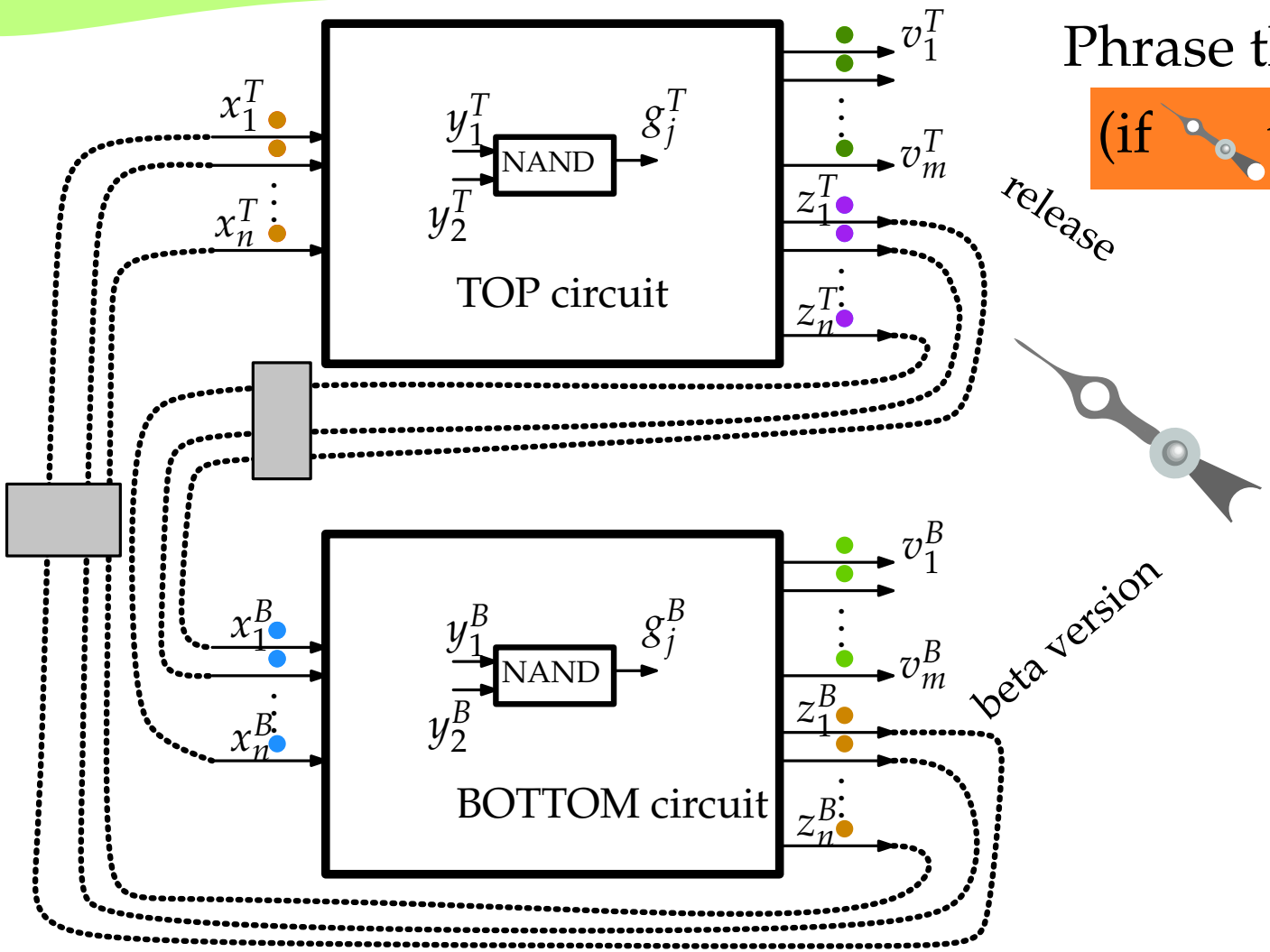
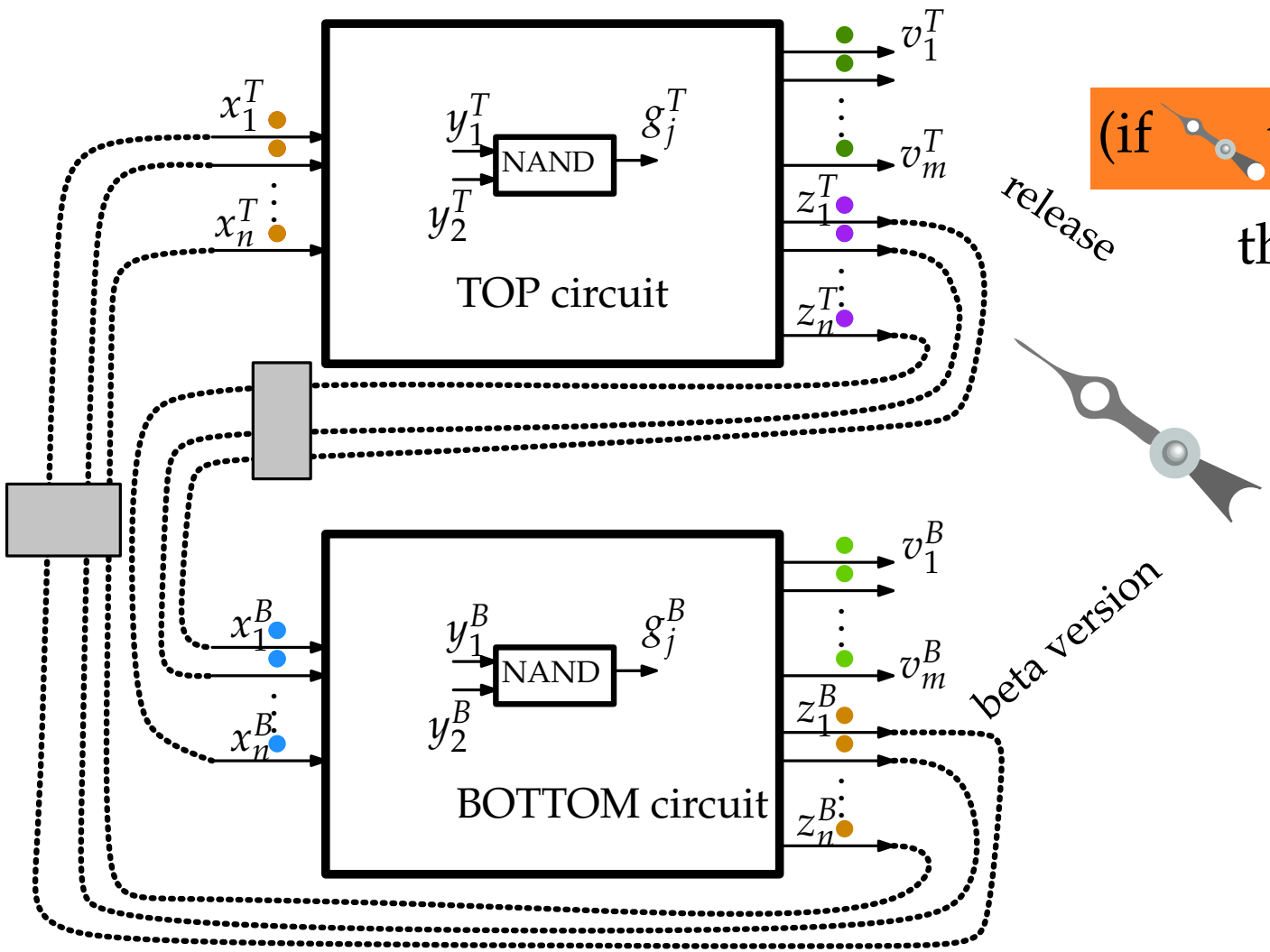


Phrase this as a 3-clasue.

(if then x_i^T else x_i^B)



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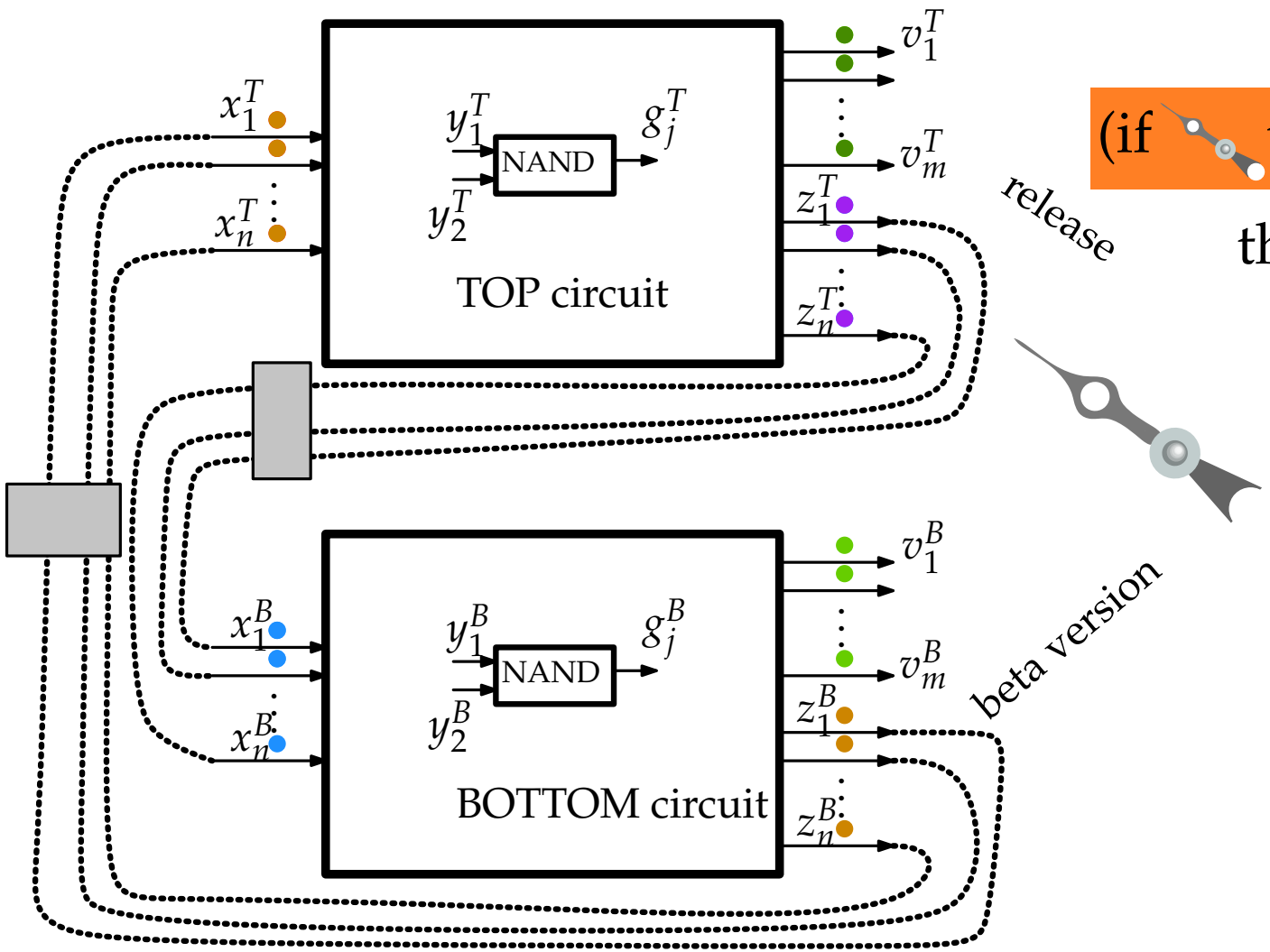
that's not a clause

$(\ \rightarrow x_i^T) \wedge (\ \rightarrow x_i^B)$



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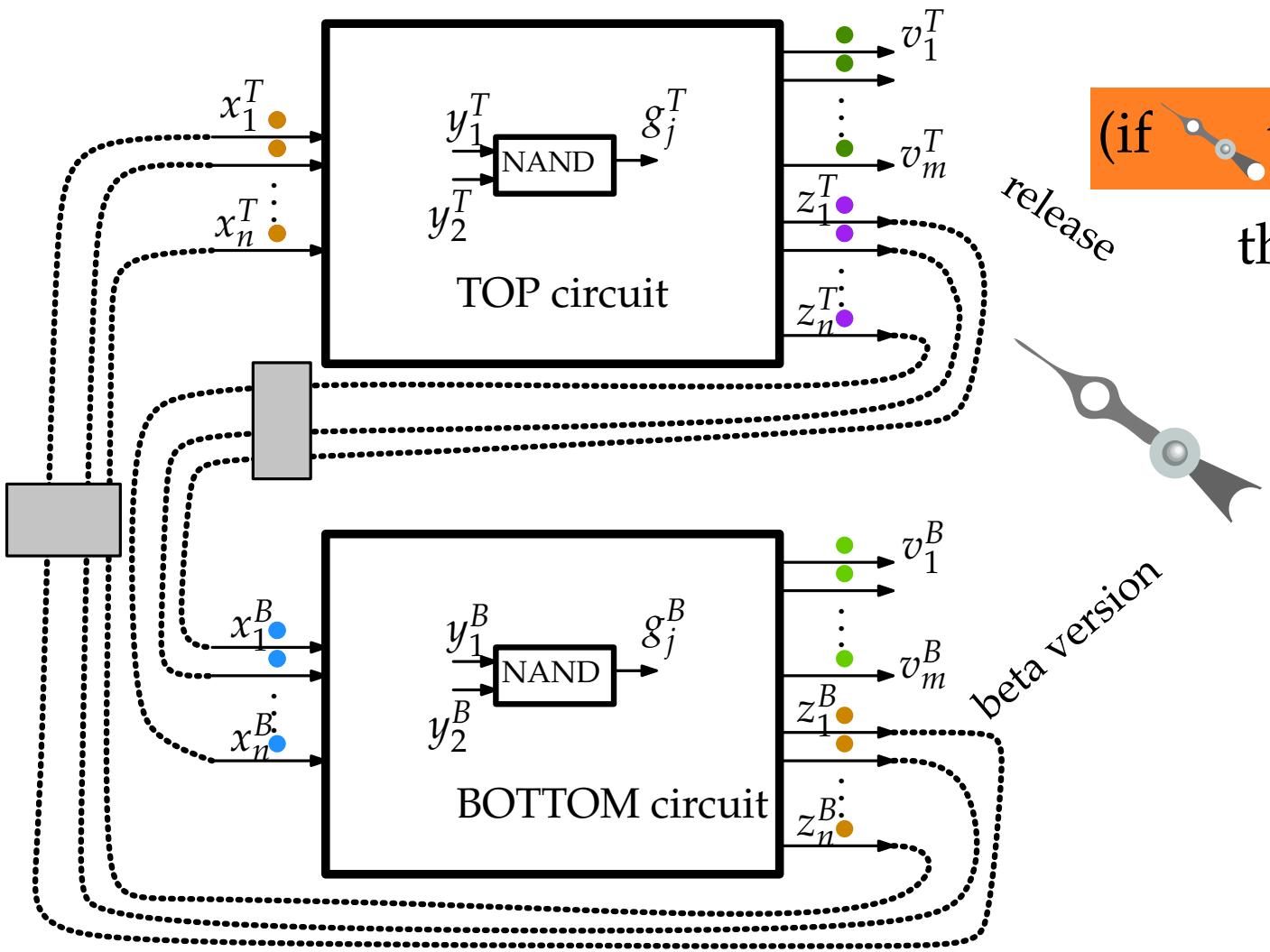

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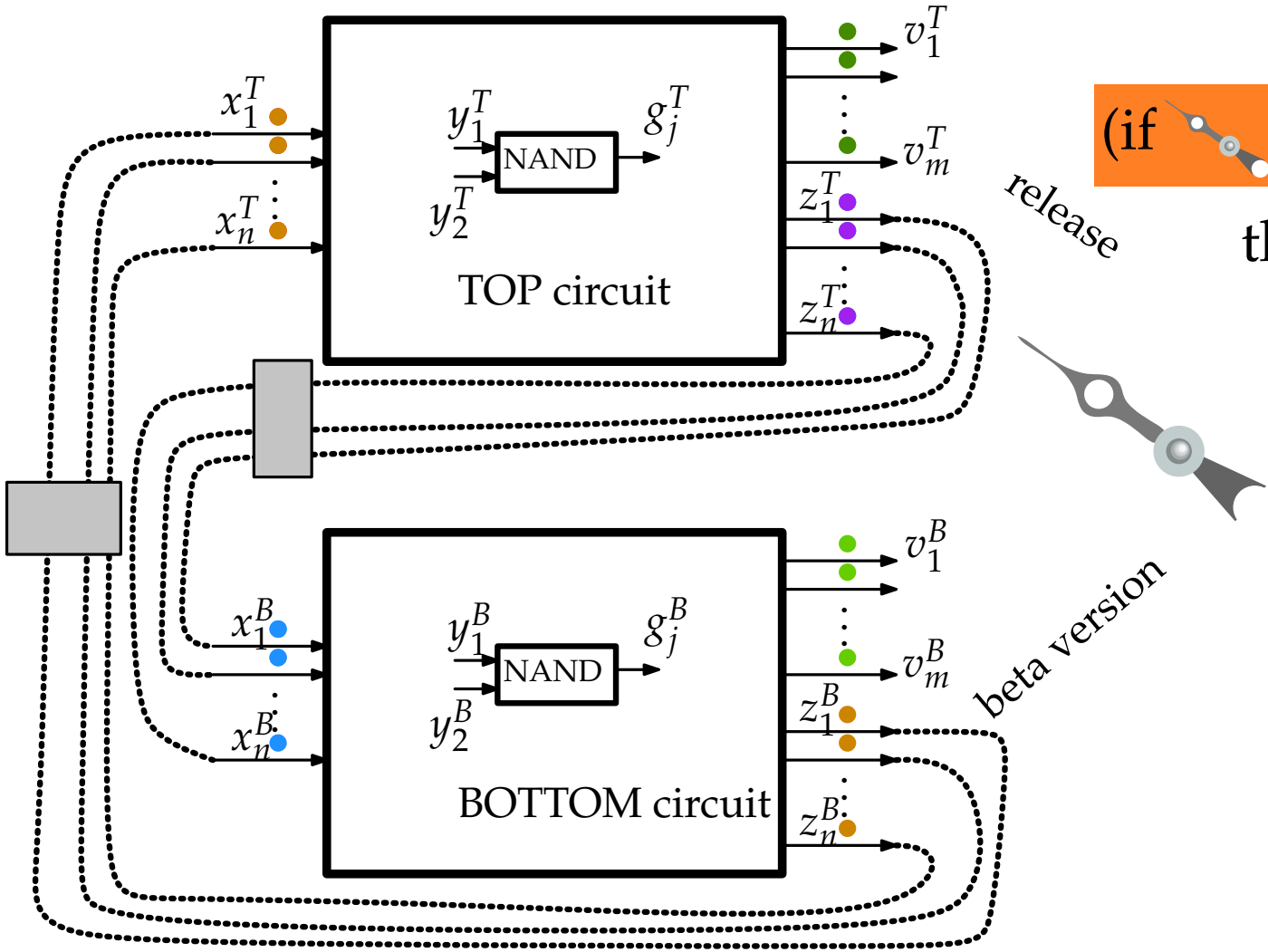

$(\text{if } \text{} \text{ then } x_i^T \text{ else } x_i^B)$

that's not a clause

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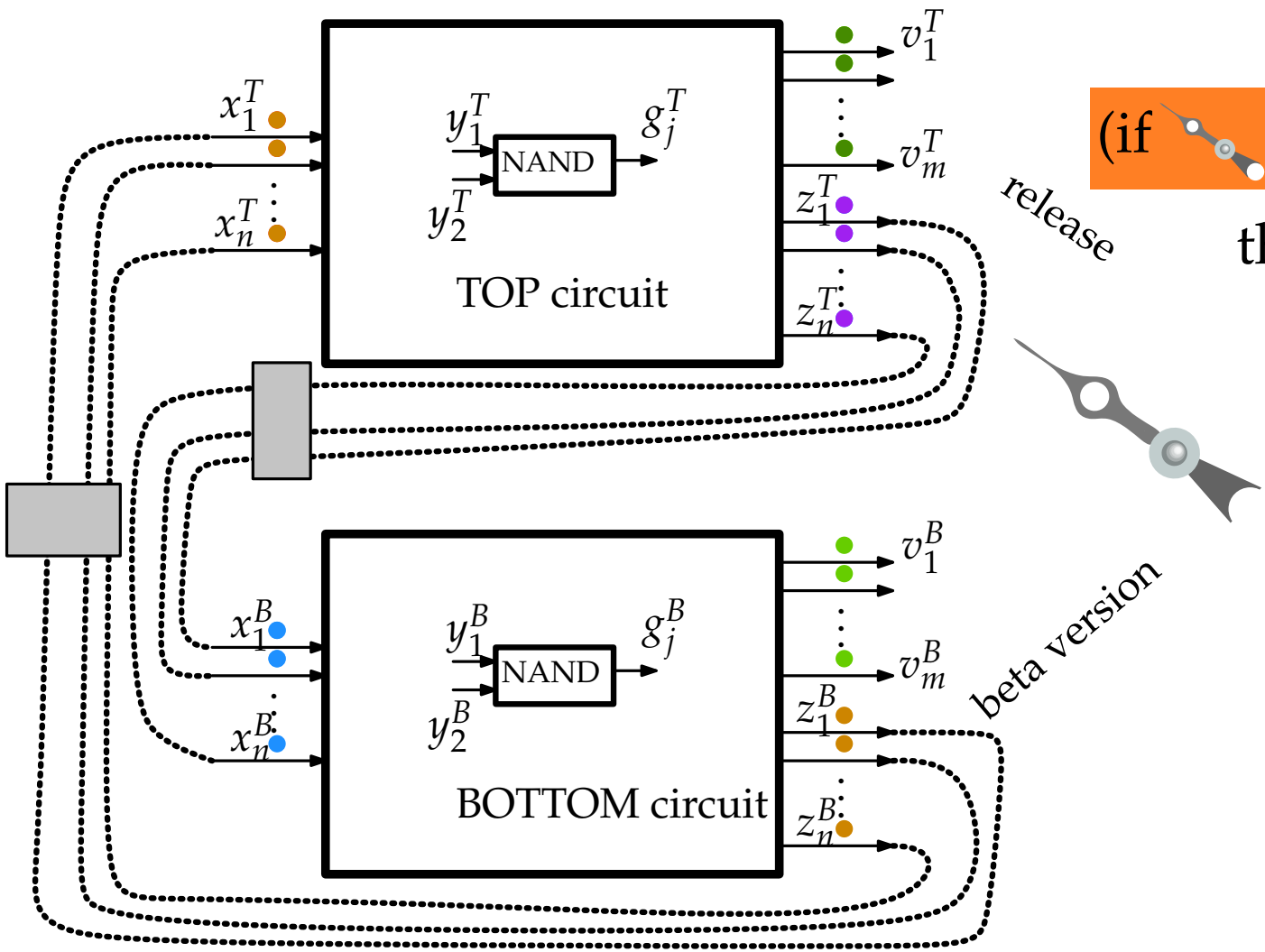
It's two 2-clauses!

Both have equal weight!
But we want: each clause weighs more than all downstream clauses combined!



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that's not a clause

Theorem (Johnson, Papadimitriou, Yannakakis). Local Max-k-SAT is PLS-complete.

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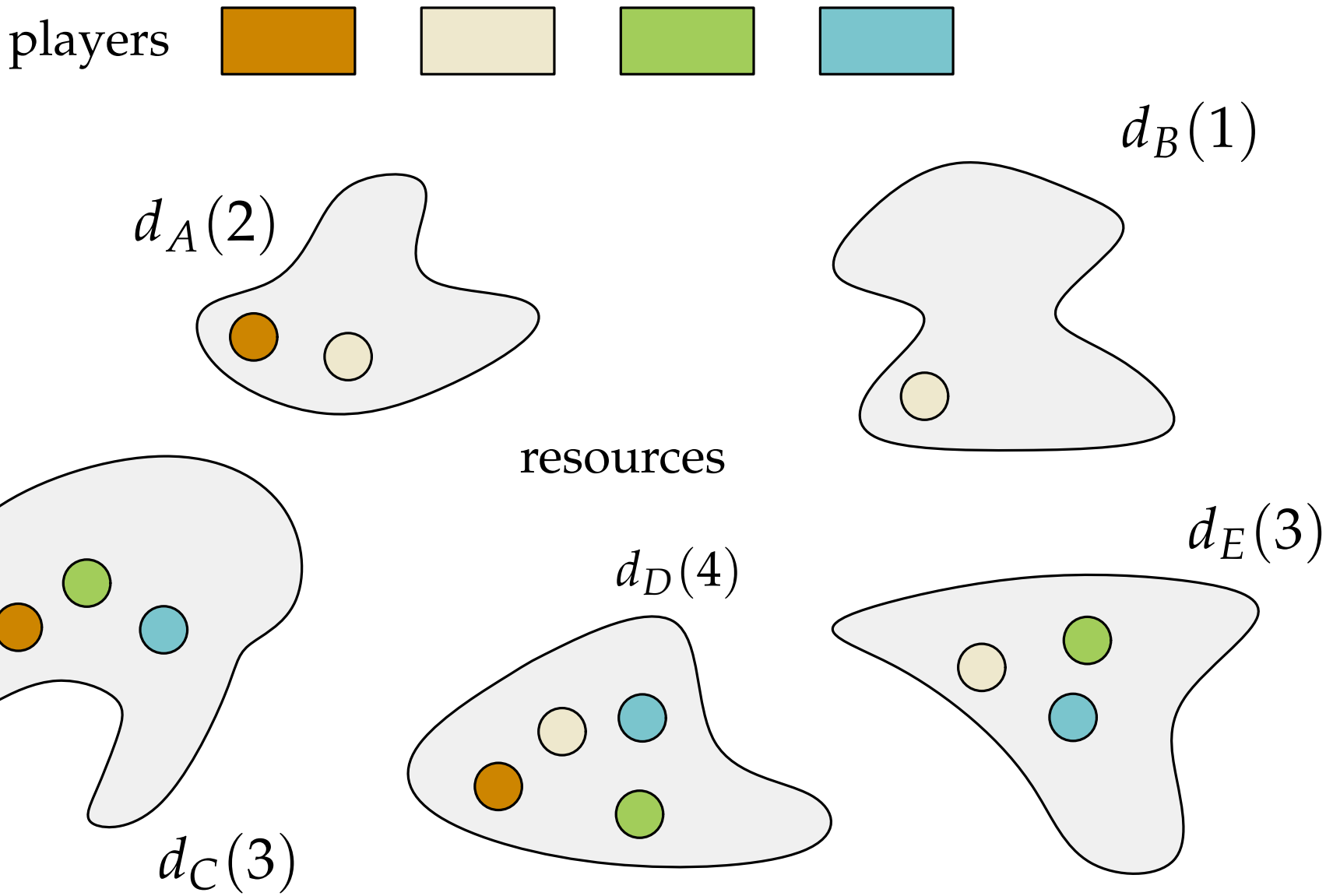
From k-SAT to Congestion Games

Theorem (Rosenthal). Every congestion game has a pure Nash equilibrium.



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$$(x \vee y) \quad (x \vee z) \quad (\bar{x} \vee \bar{y}) \quad (\bar{x} \vee \bar{z}) \quad (y \vee z) \quad (x \vee \bar{y} \vee \bar{z})$$

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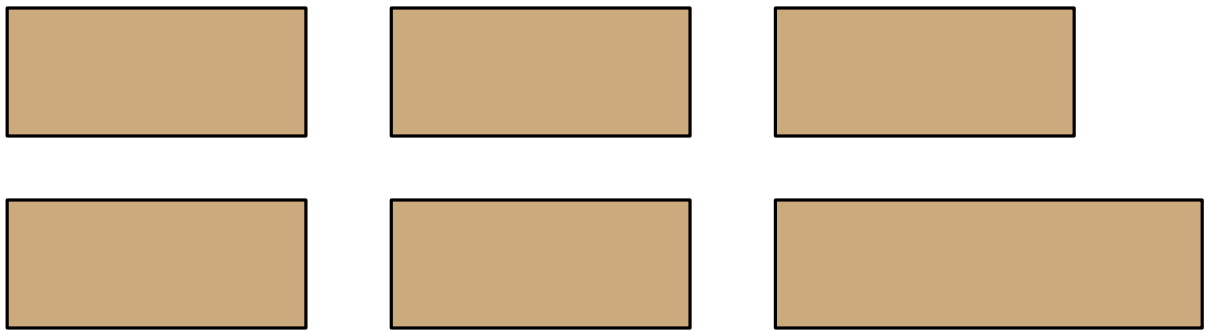

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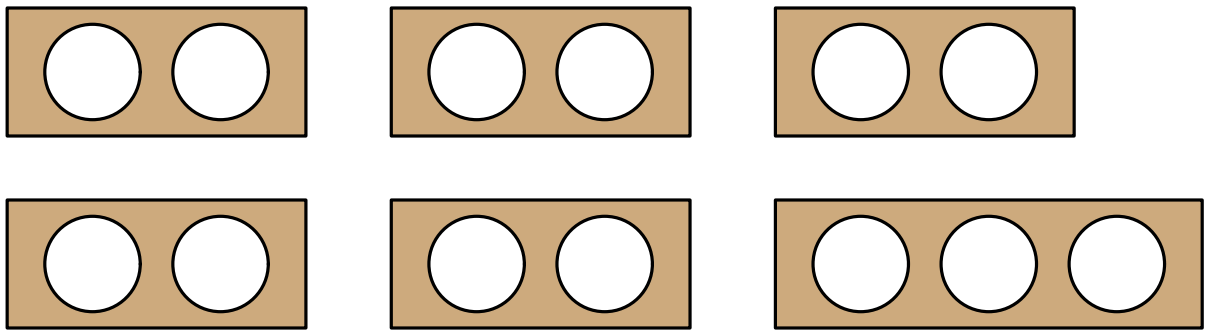

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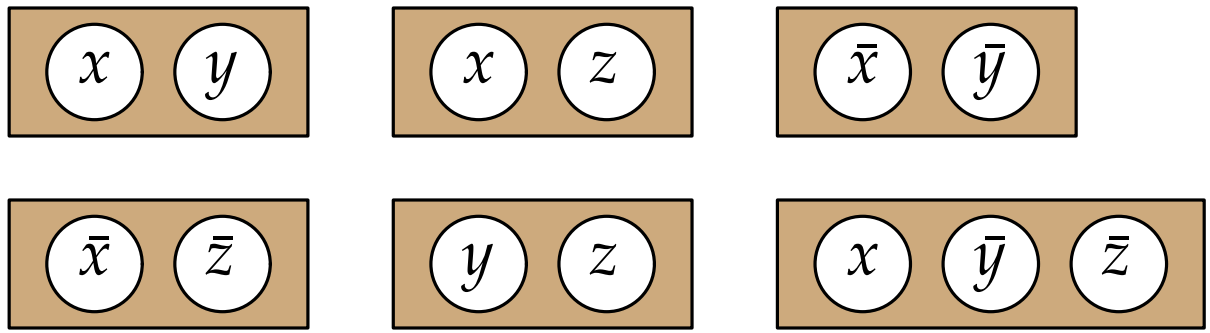

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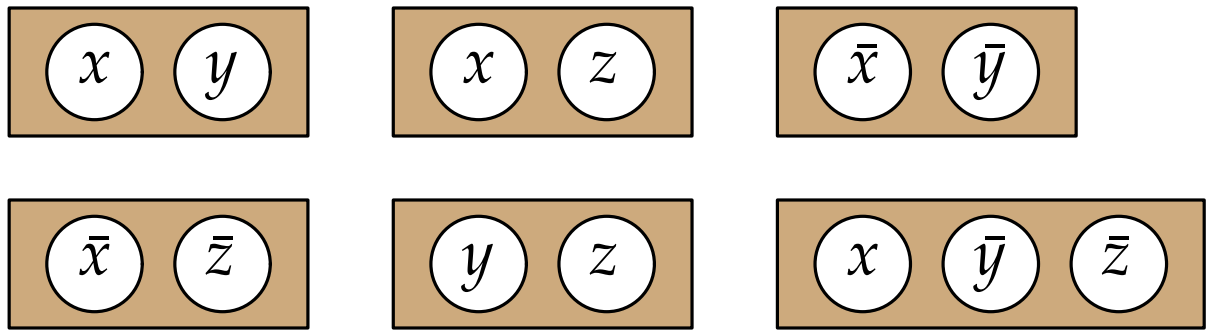

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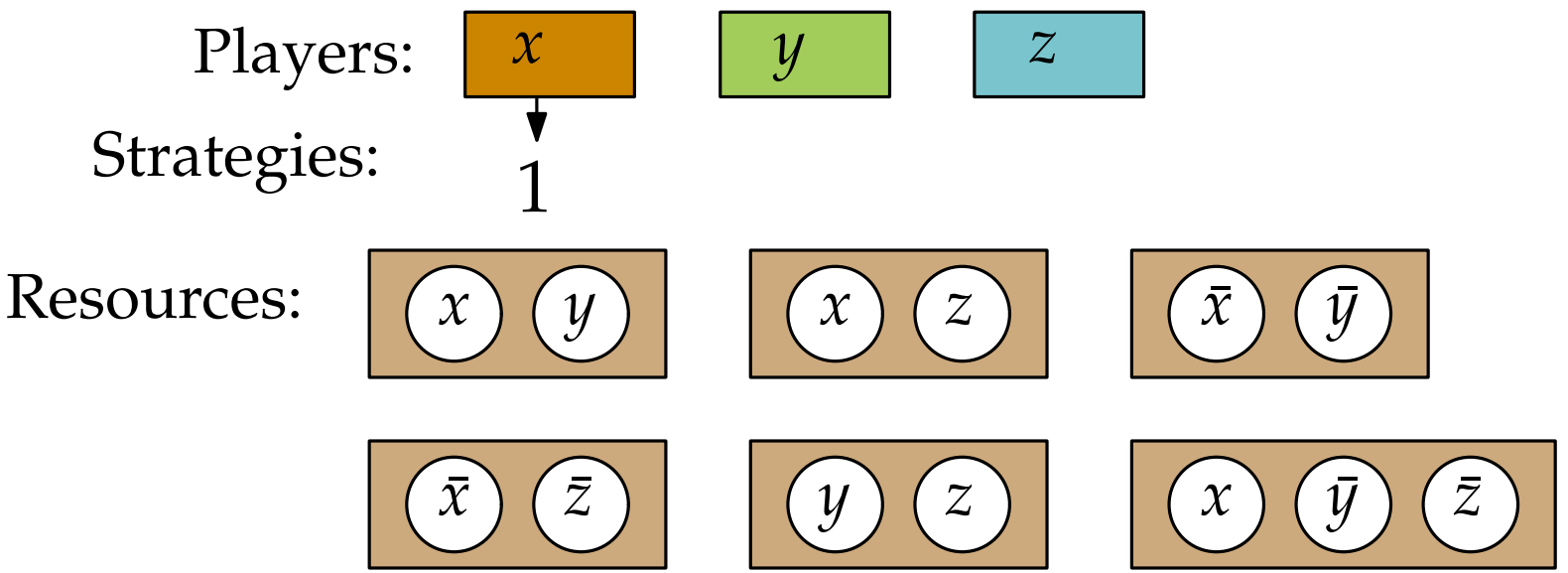

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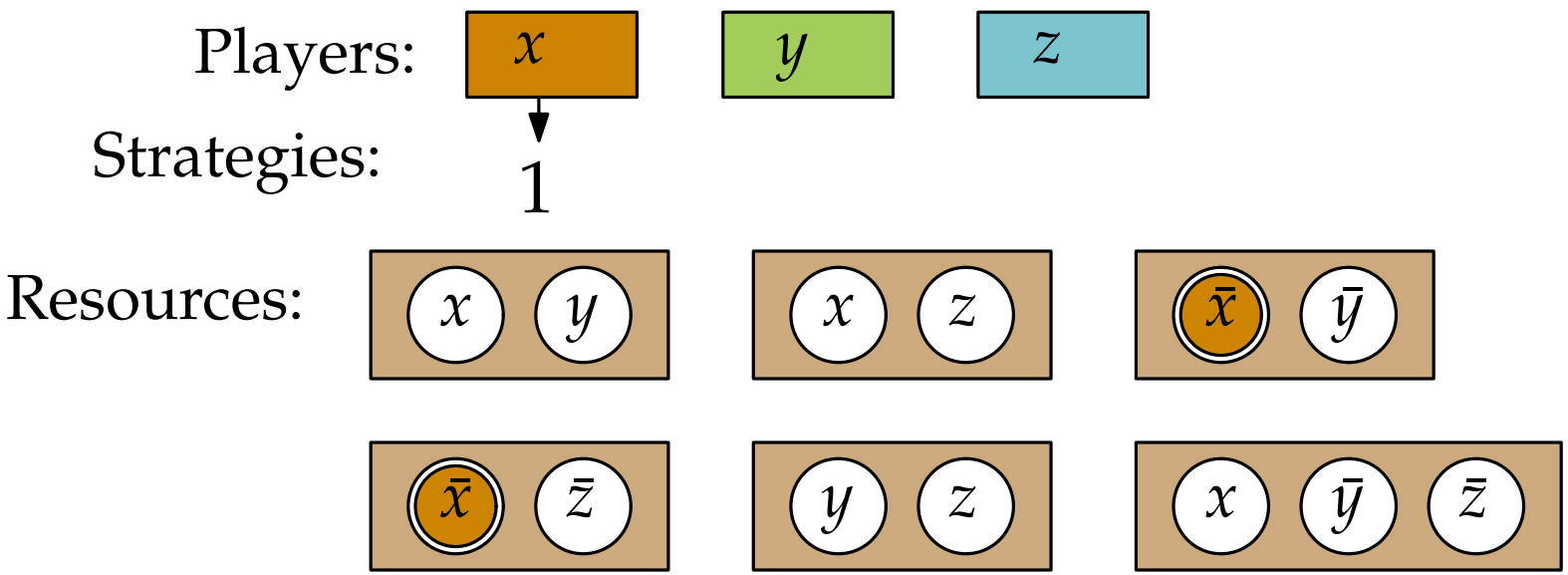

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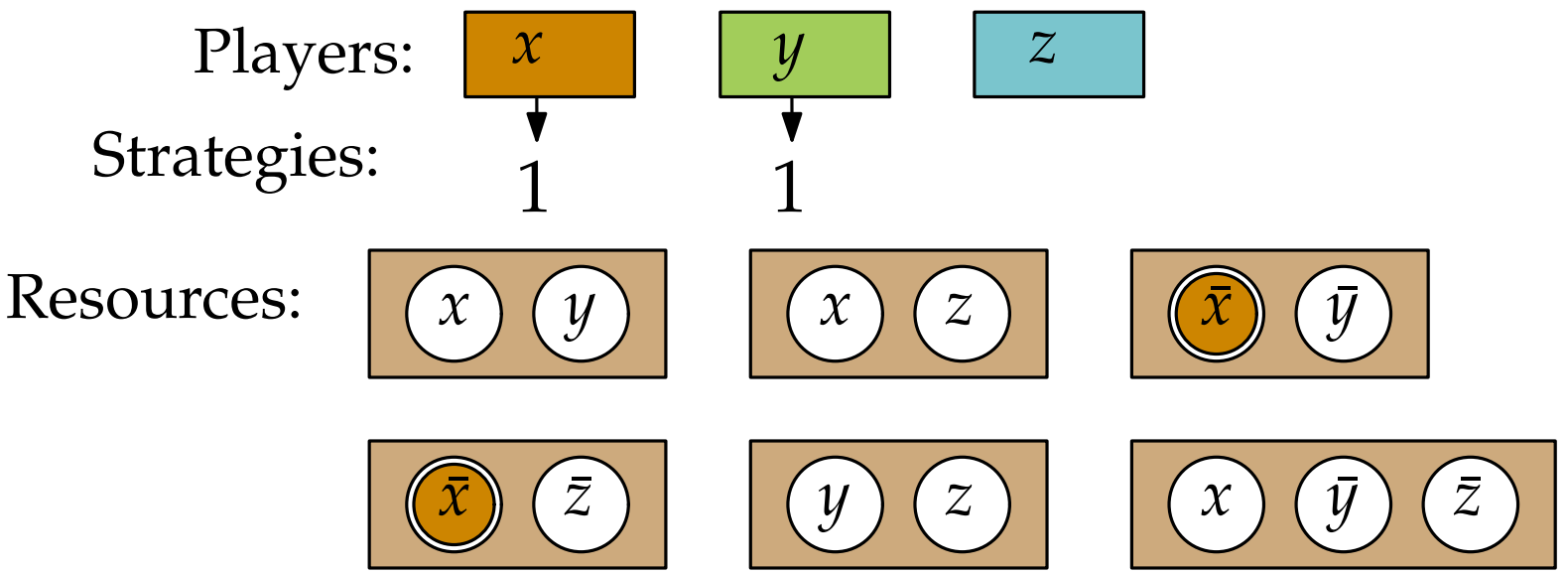

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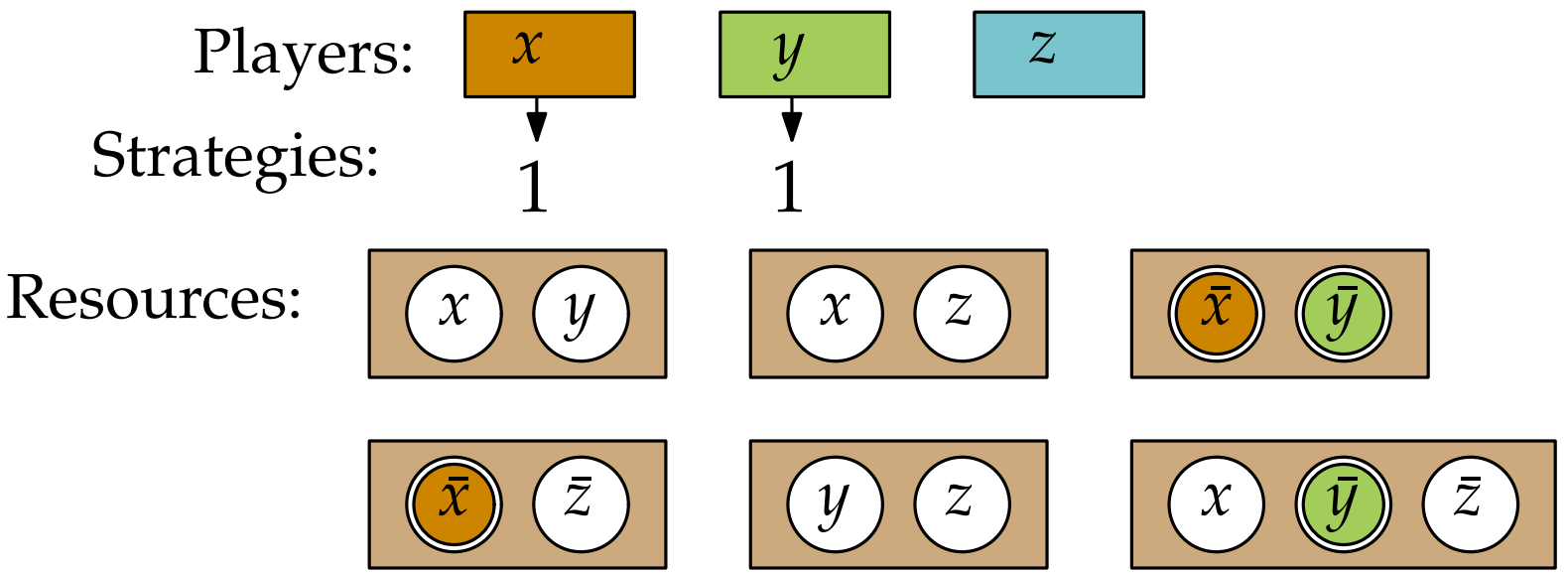

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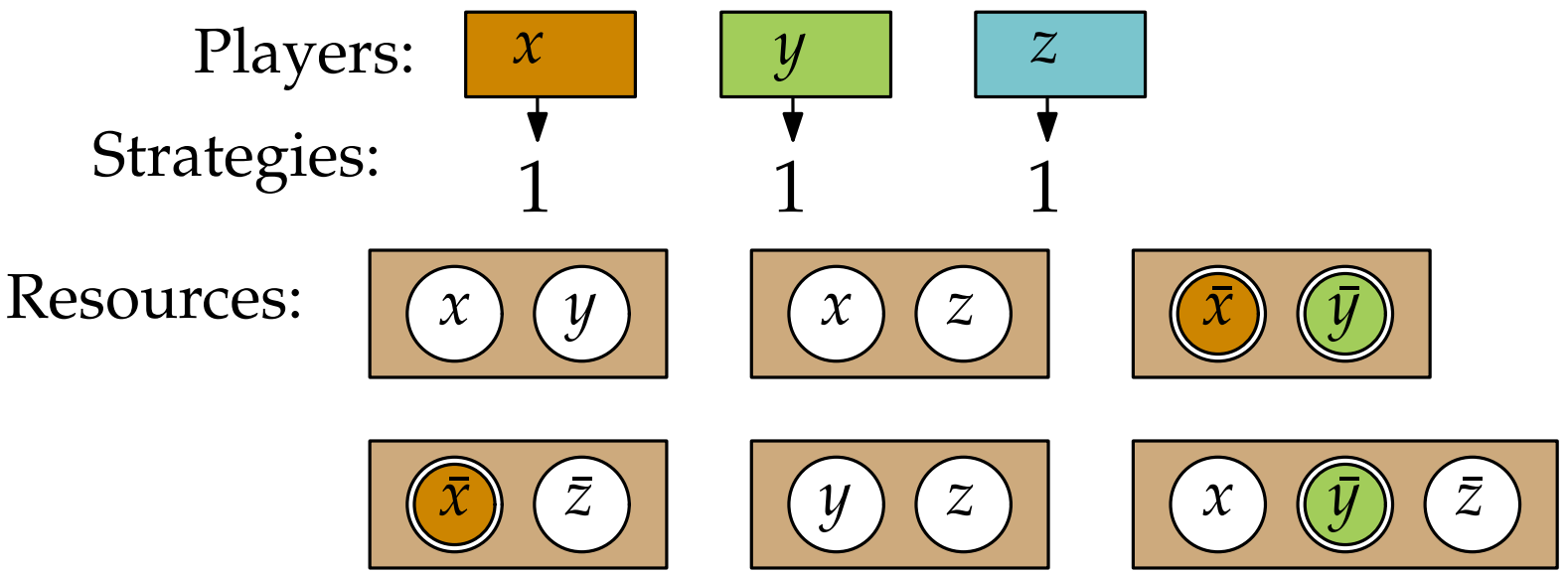

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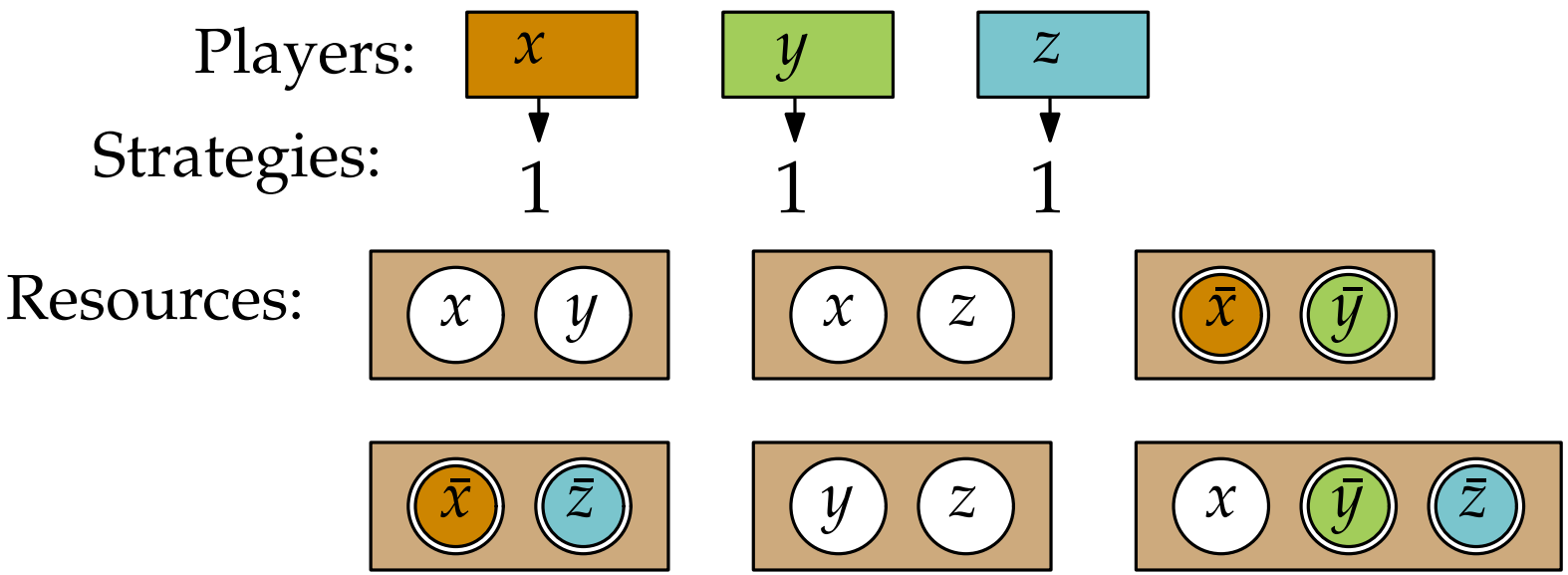

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Theorem (Ackermann and Skopalik, 2008). Finding a pure Nash equilibrium in congestion games is PLS-complete.

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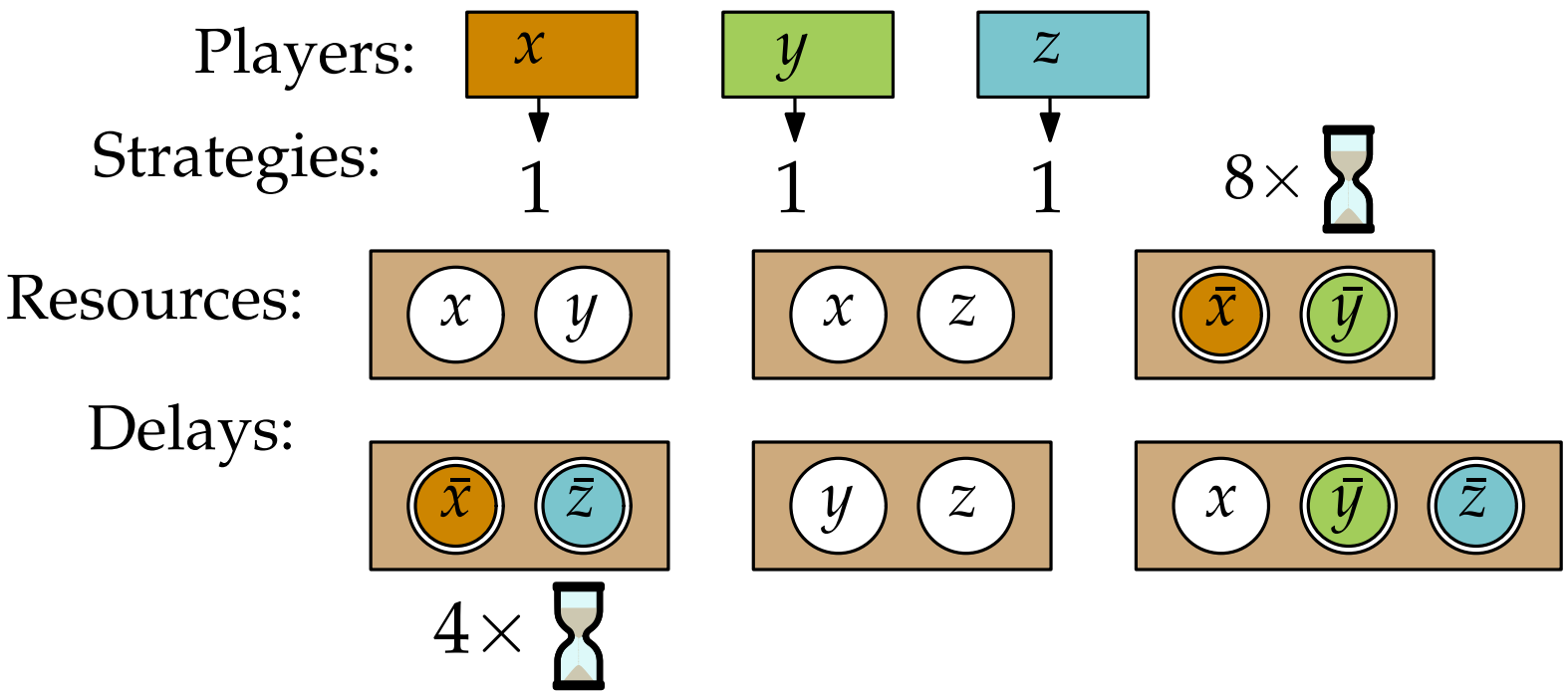

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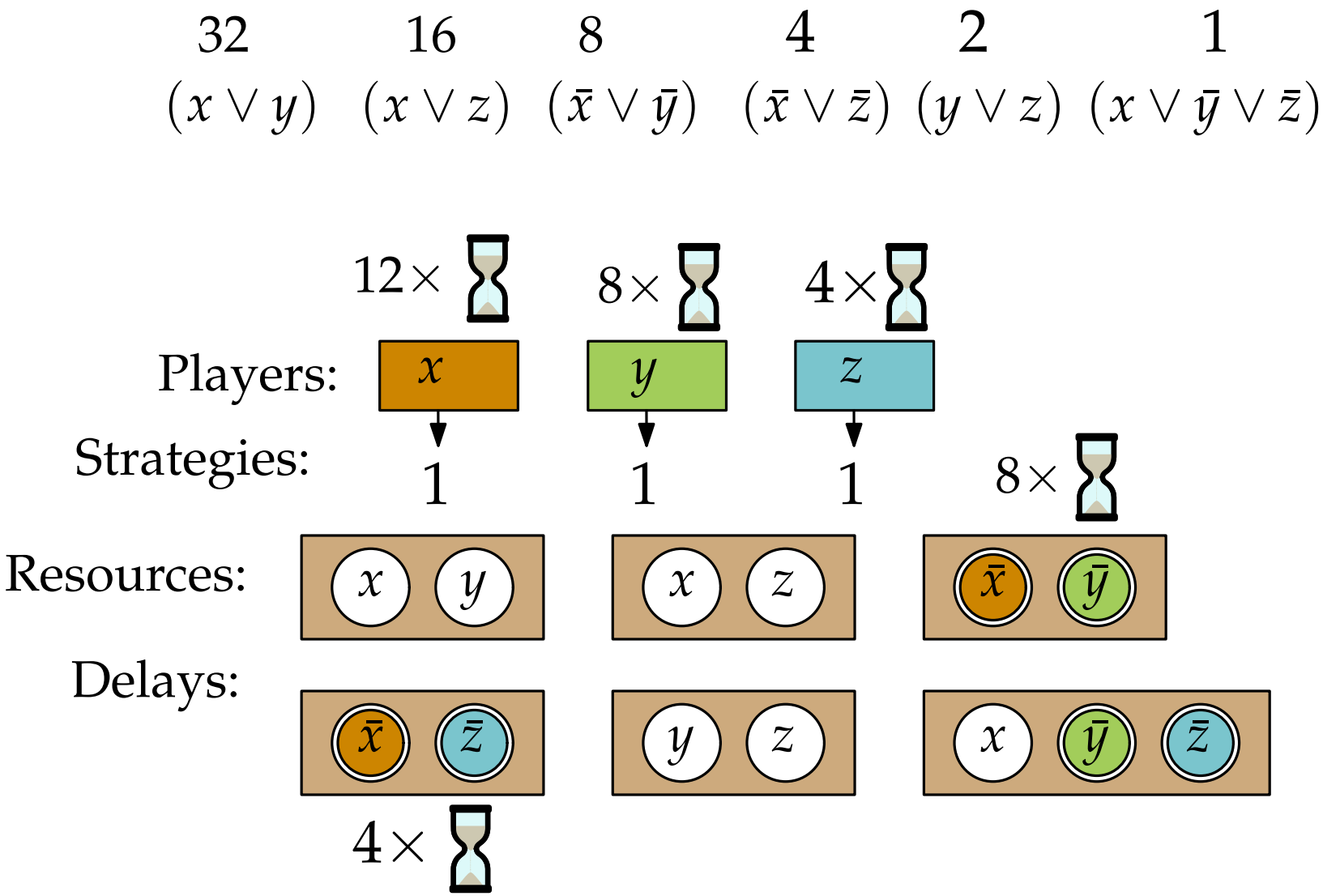

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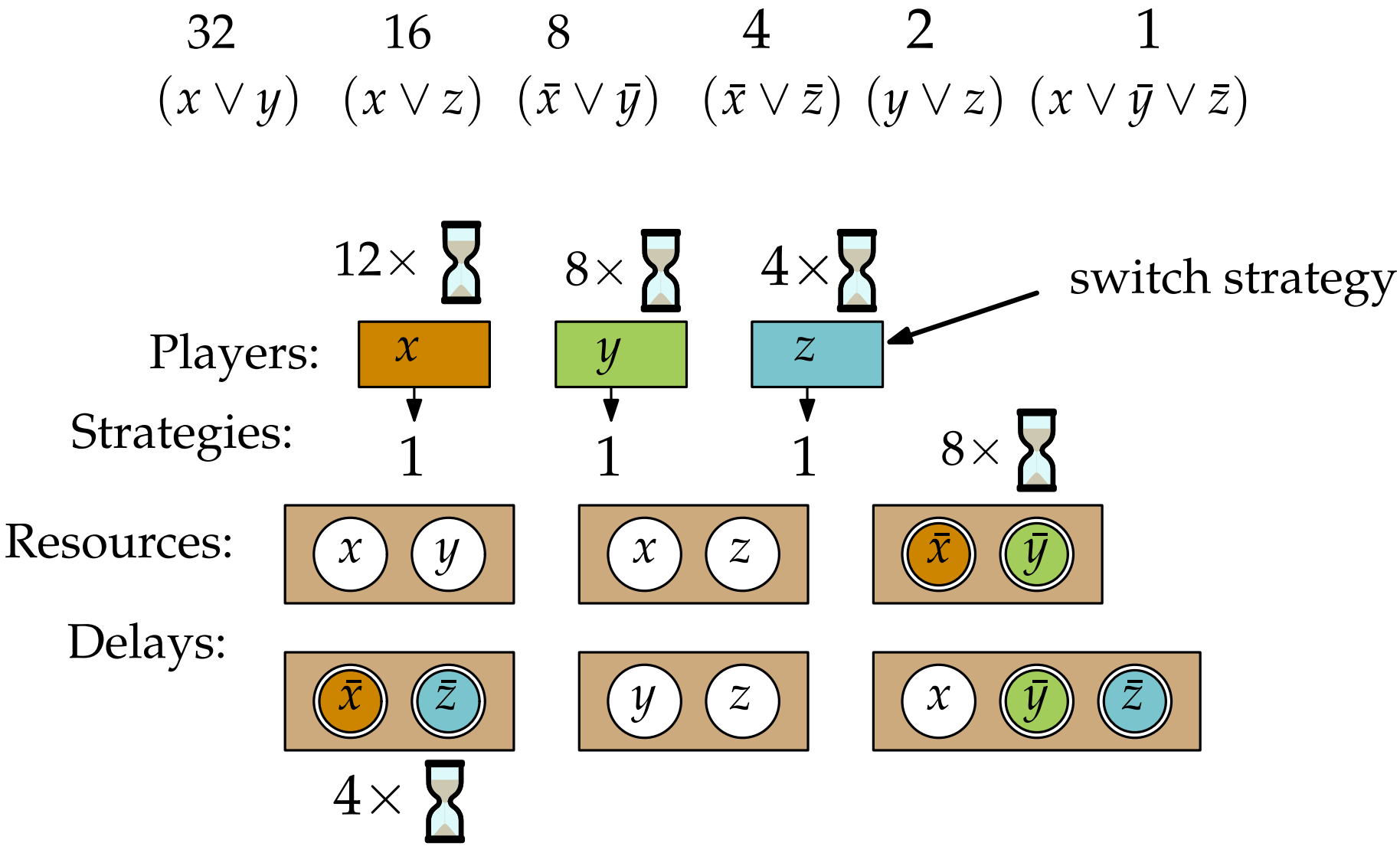

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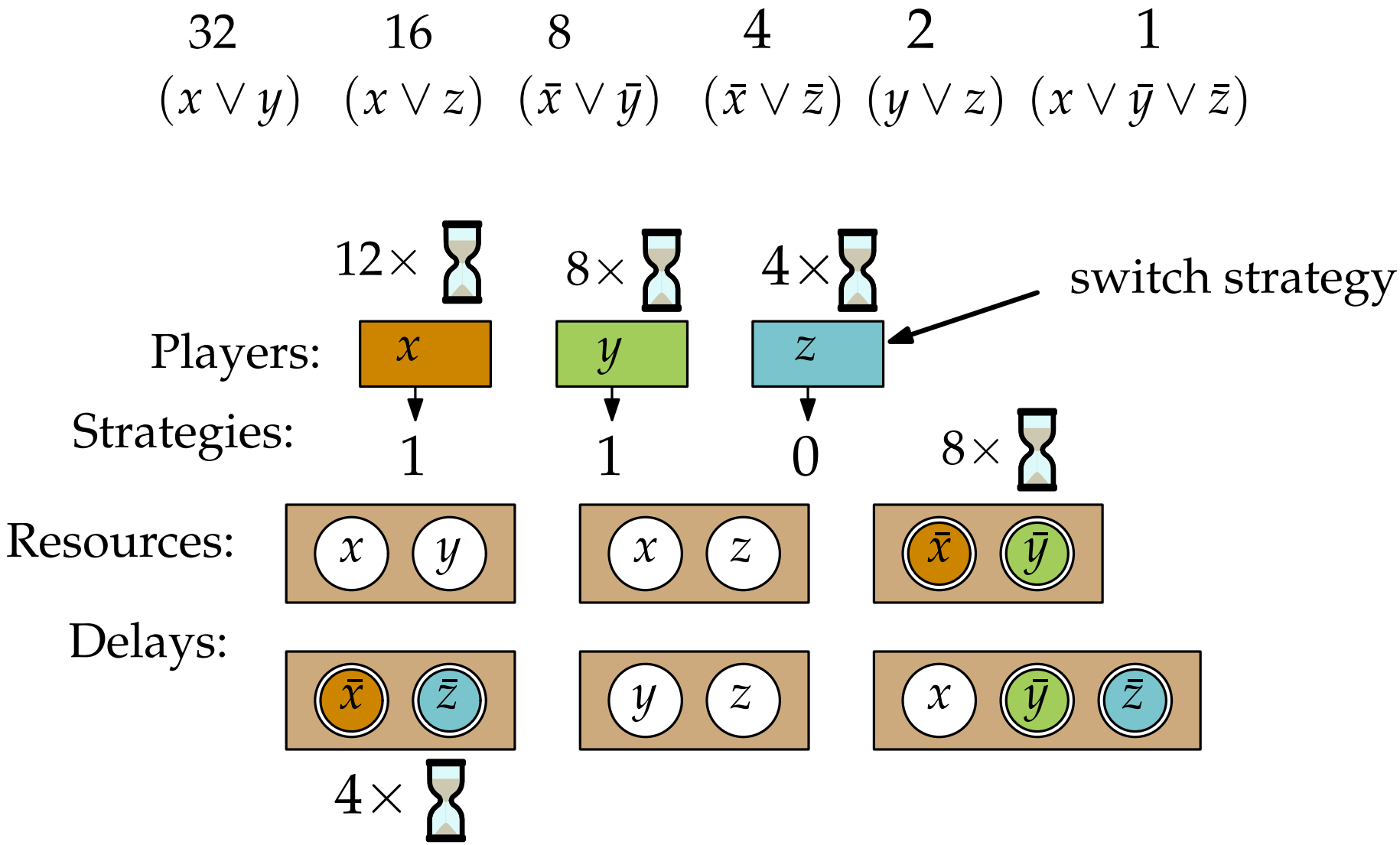

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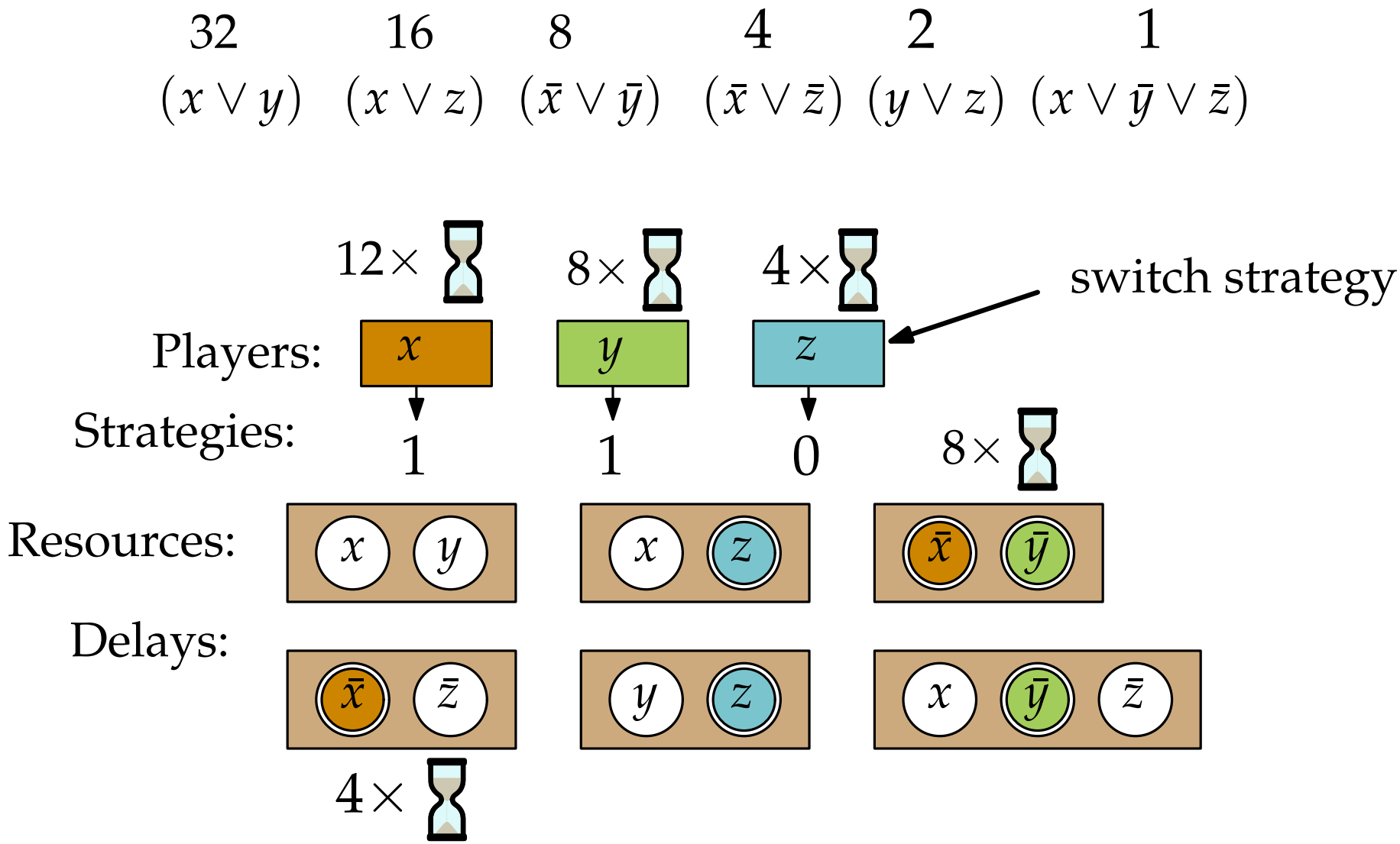

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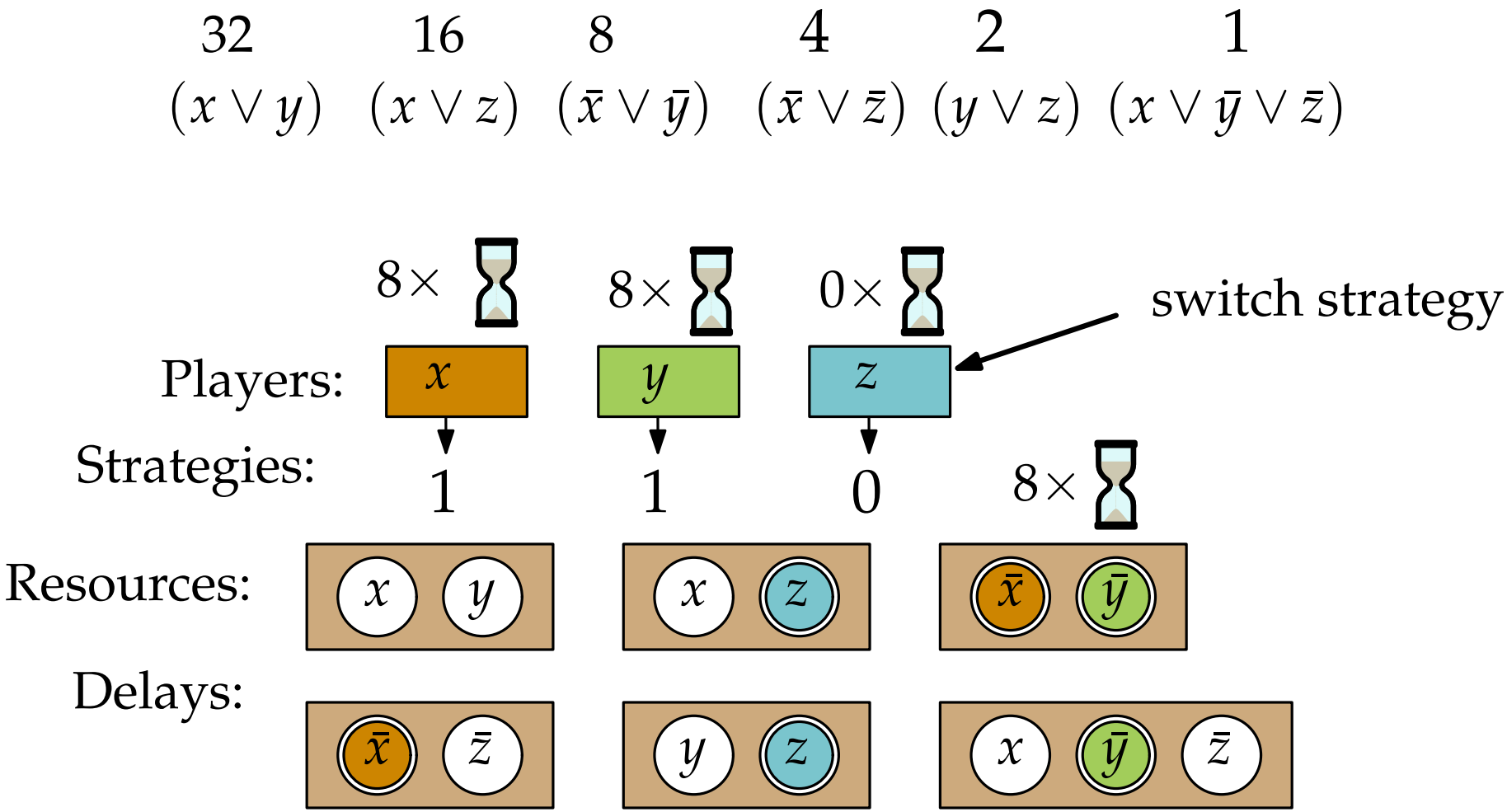

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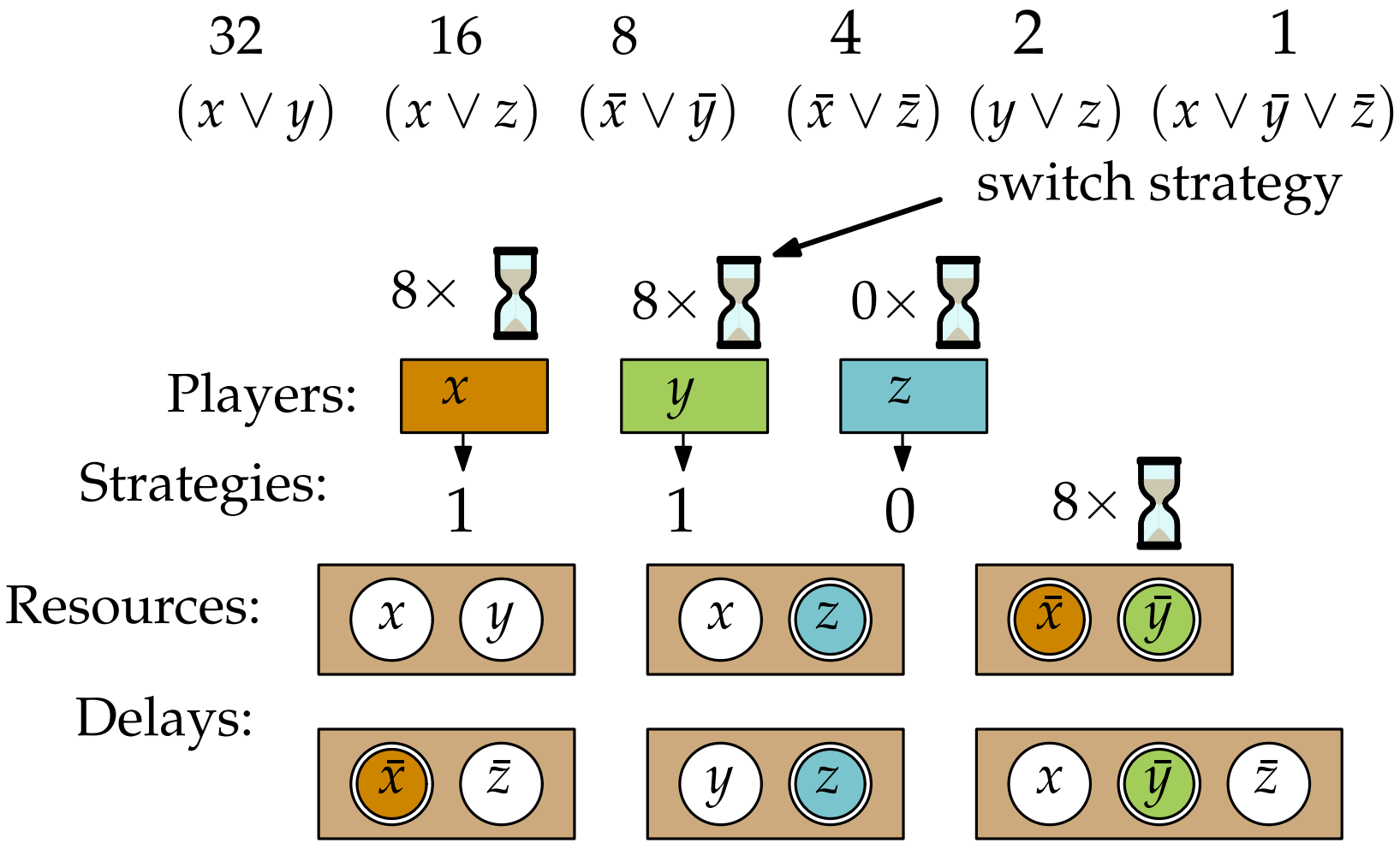

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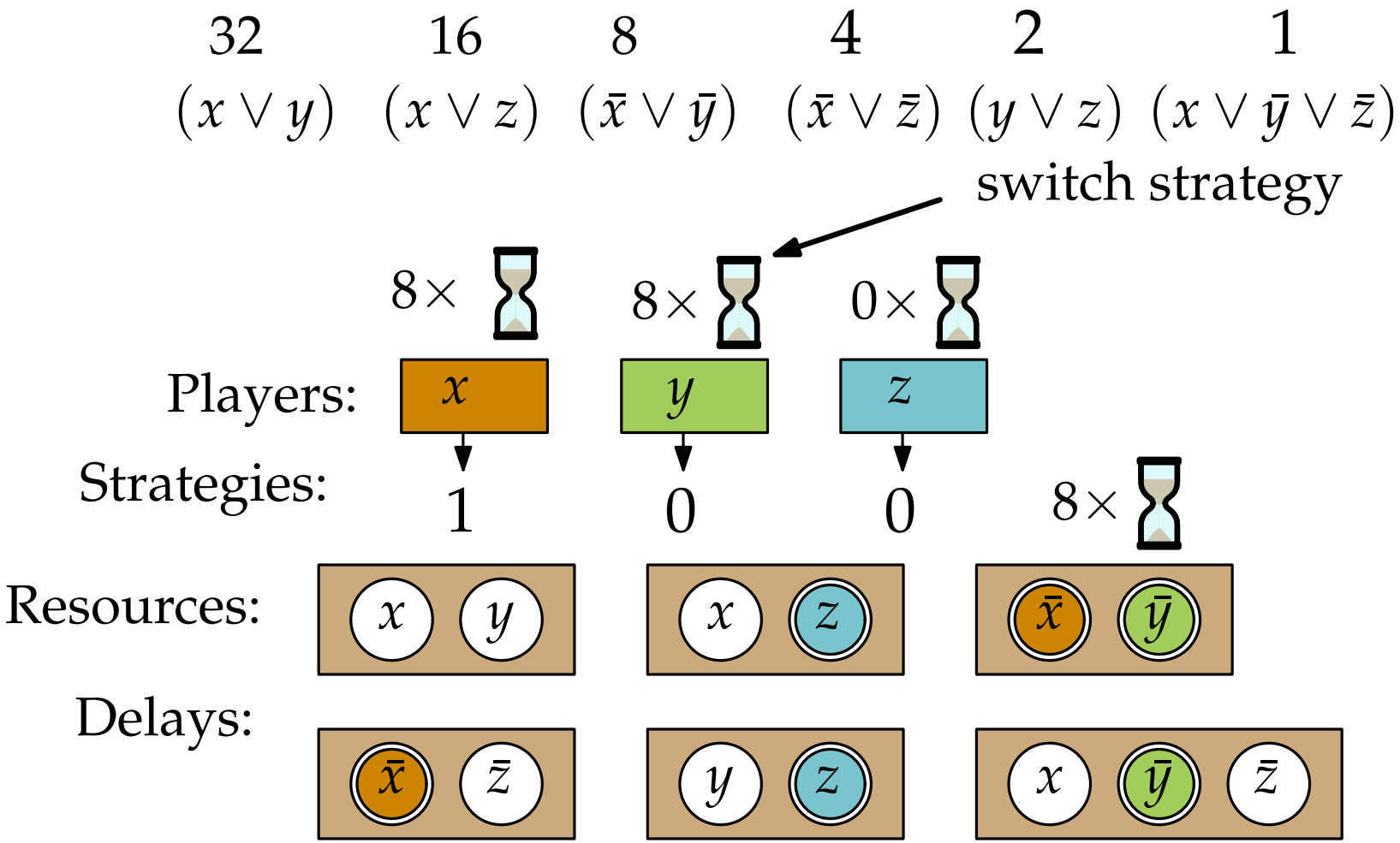

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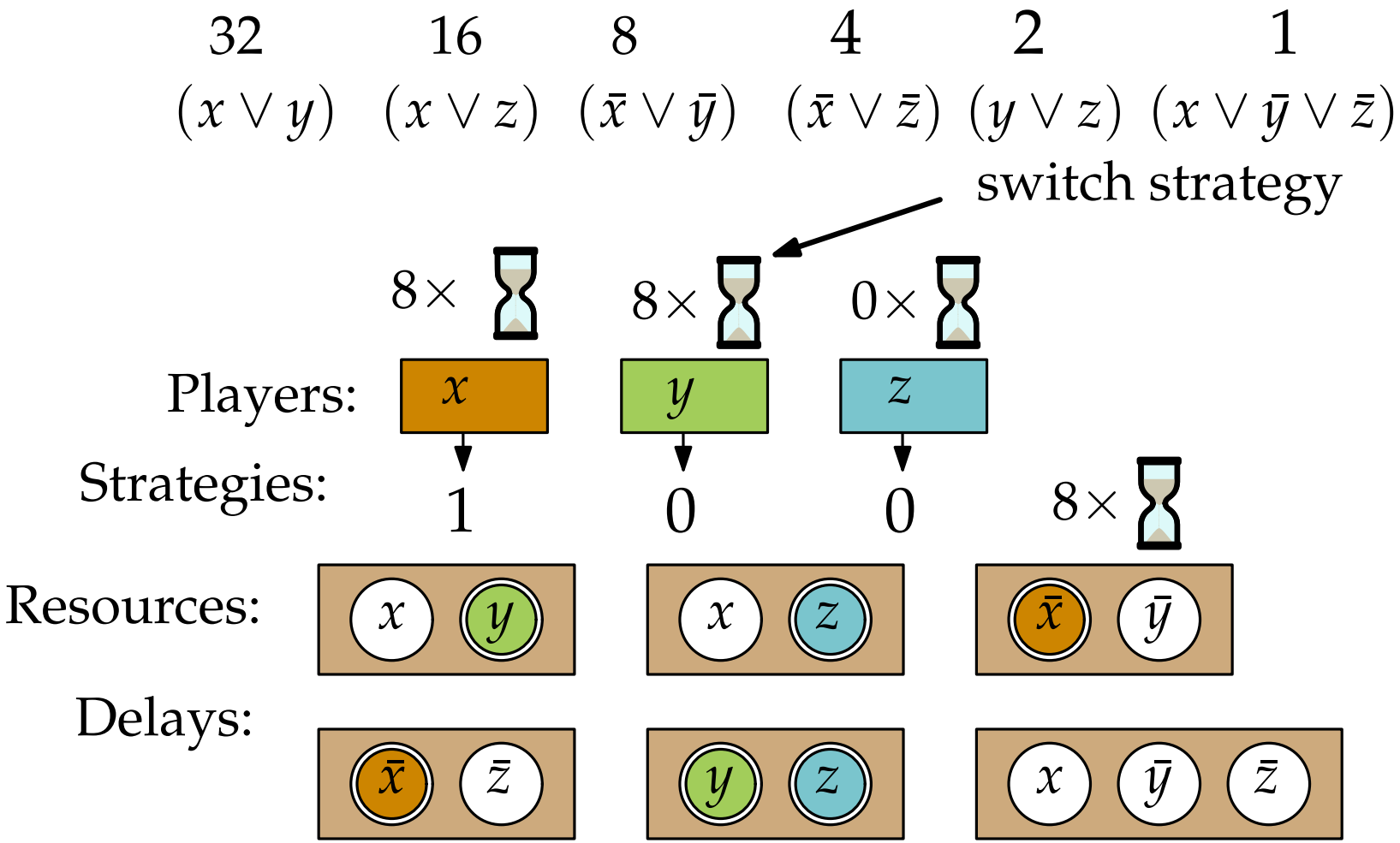

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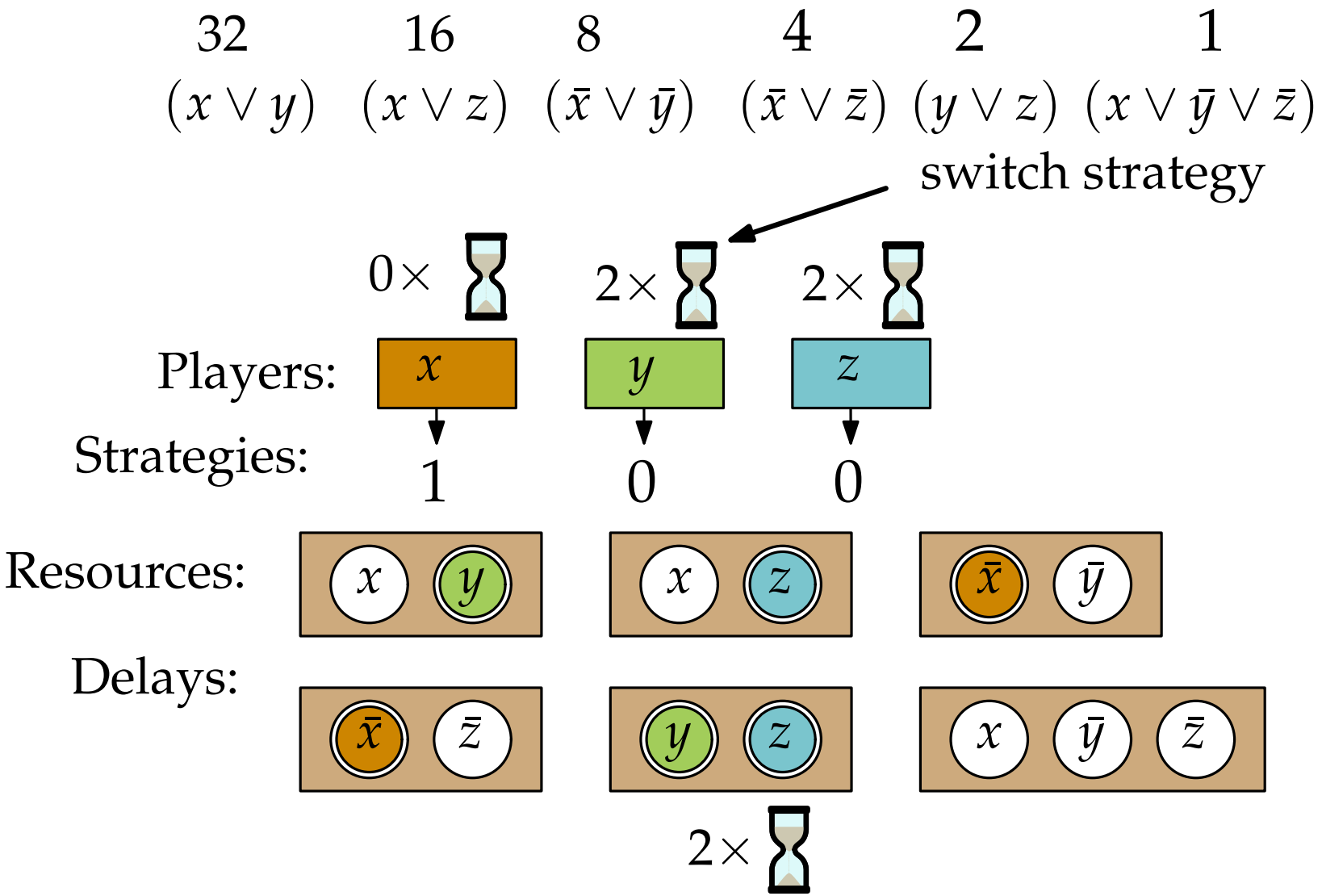

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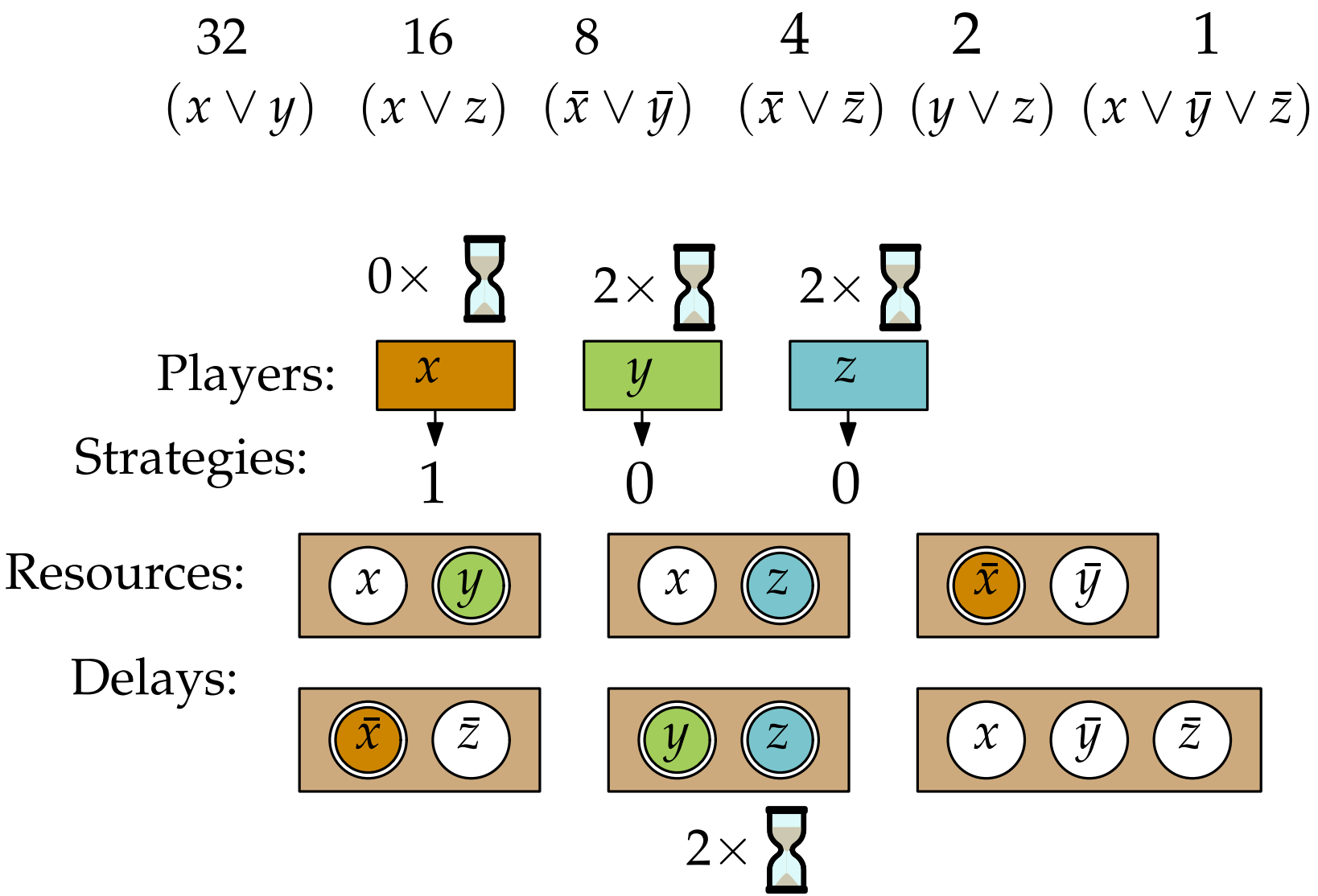

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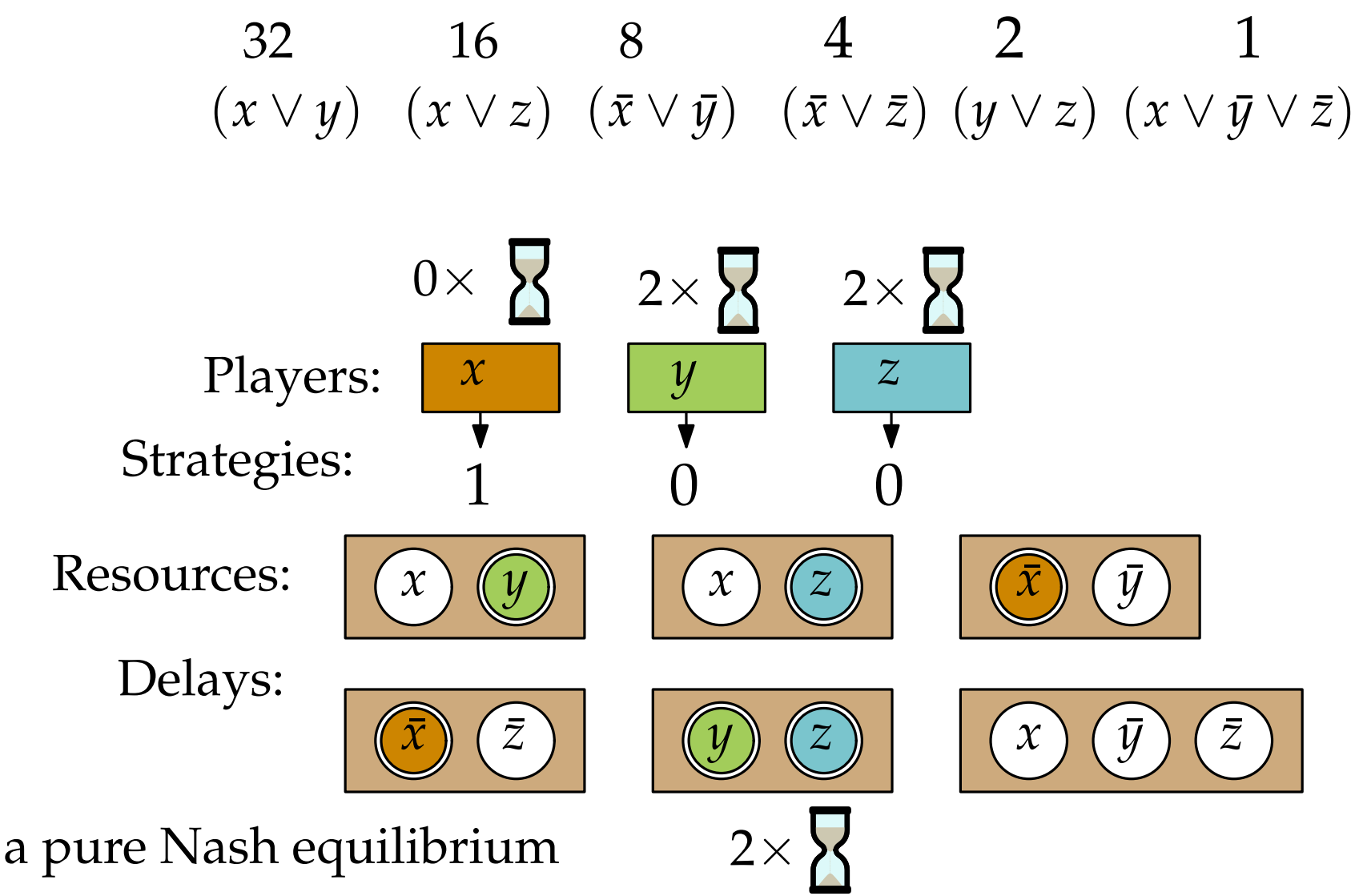

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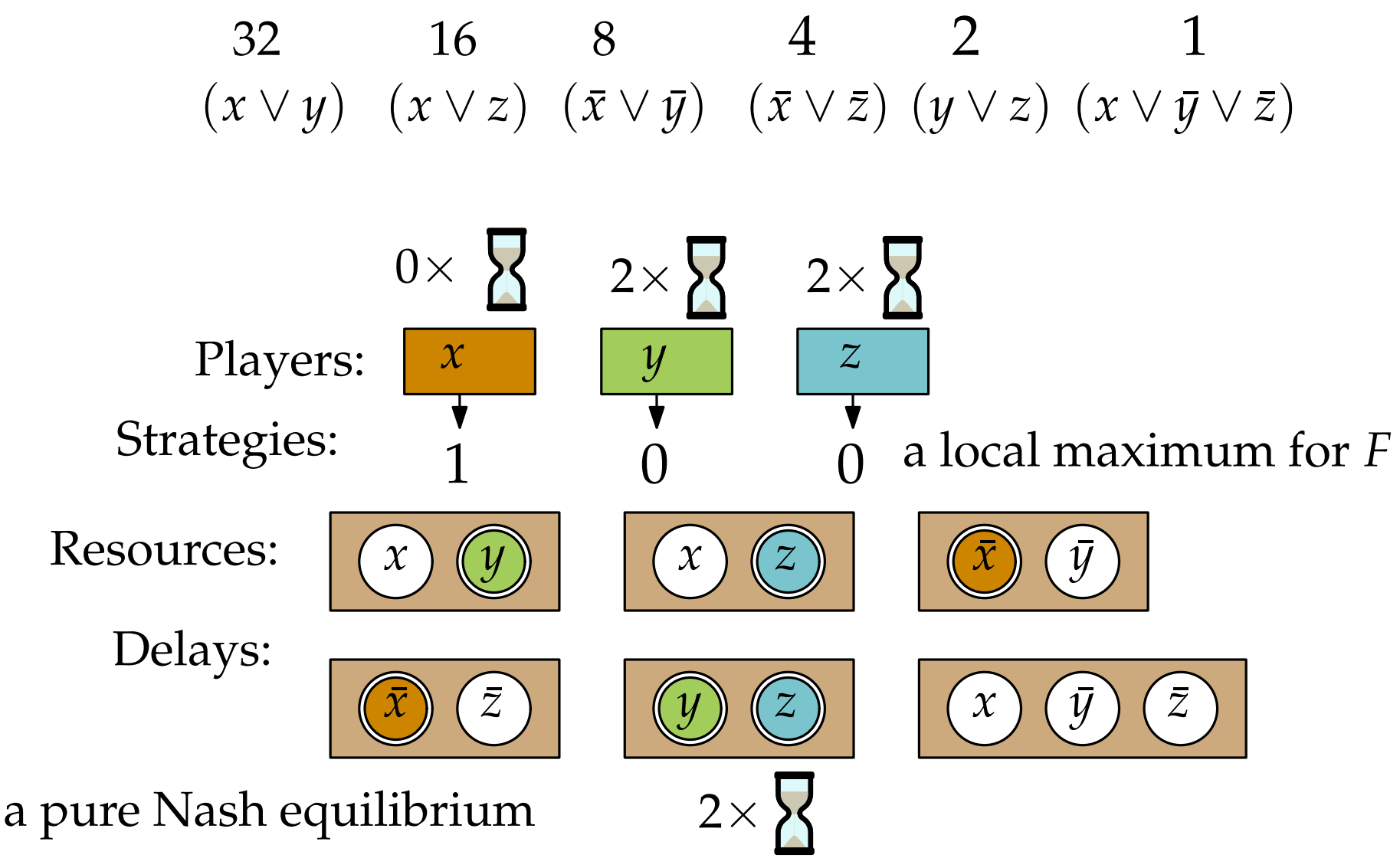

From k-SAT to Congestion Games

Nash equilibrium. $\forall$ player p: Switching strategy does not improve her delay.

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Very simple proof (S., Tantow).

From k-SAT to Congestion Games

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Proof.

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Proof. Consider an instance of Local Max-k-SAT:

C_1 C_2 C_3 C_4 C_5 C_6 C_7

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Which is better? No need for computation! Of course β is better!

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Proof. Consider an instance of Local Max-k-SAT:

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	729	243	81	27	9	3	1
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Which is better? No need for computation! Of course β is better!

More than twice as heavy than those guys!

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$F =$ 

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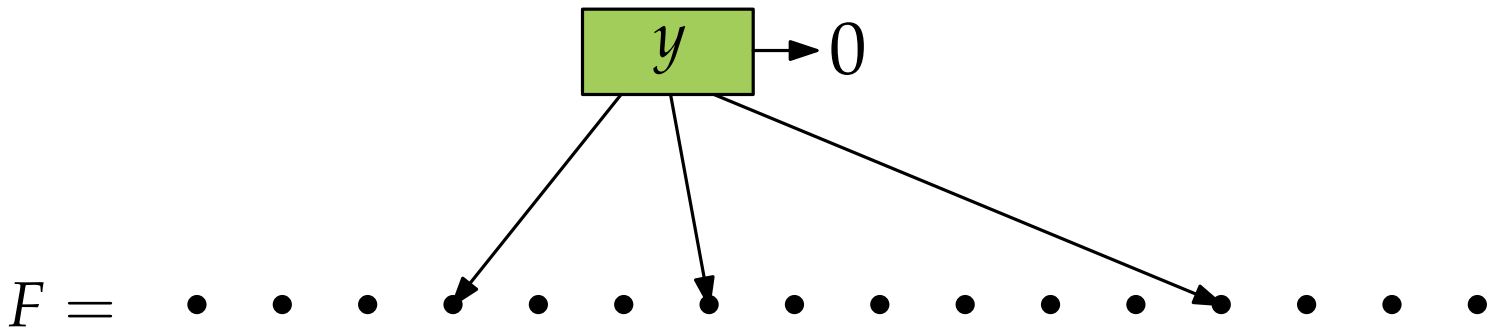

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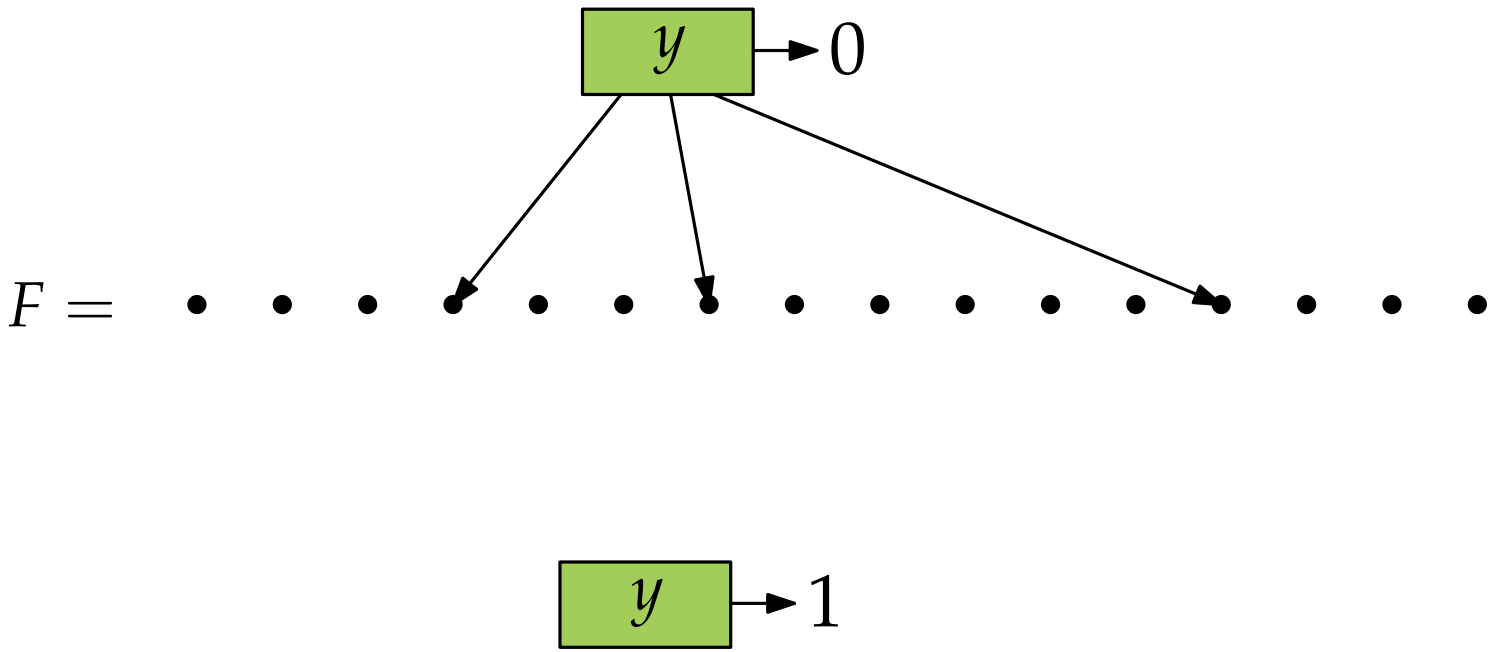

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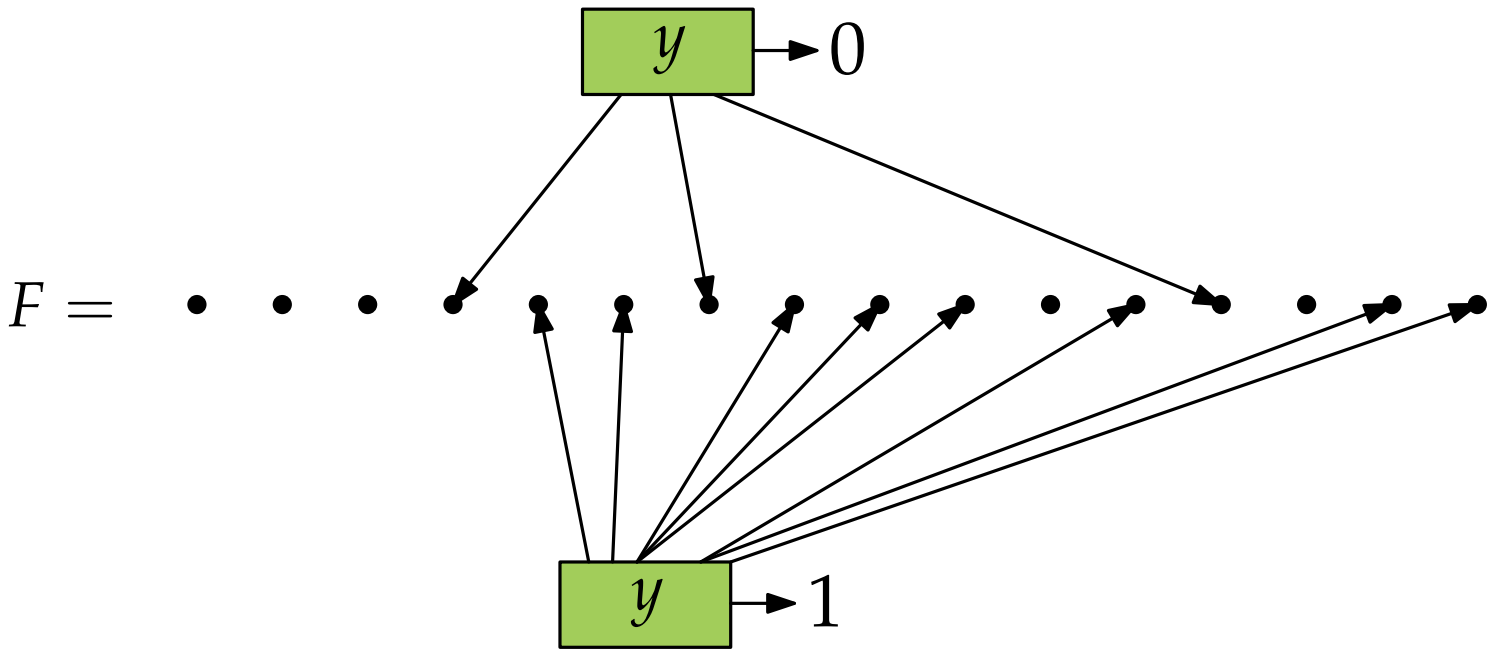

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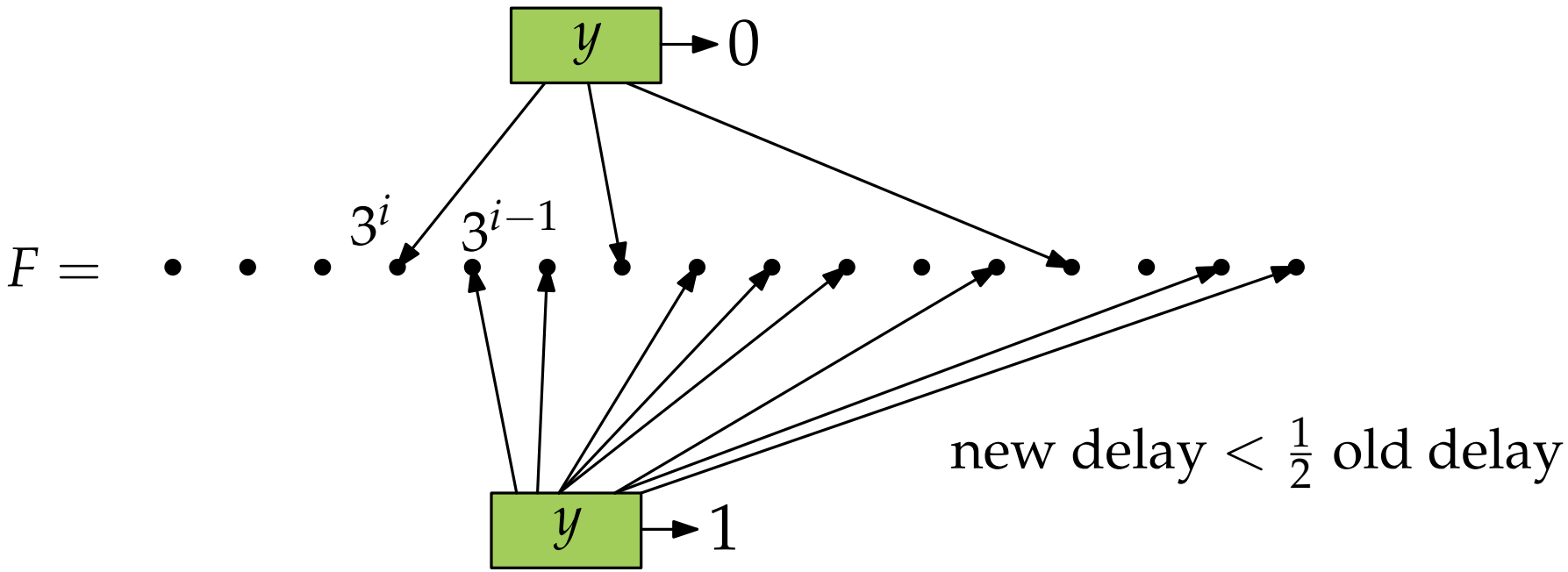

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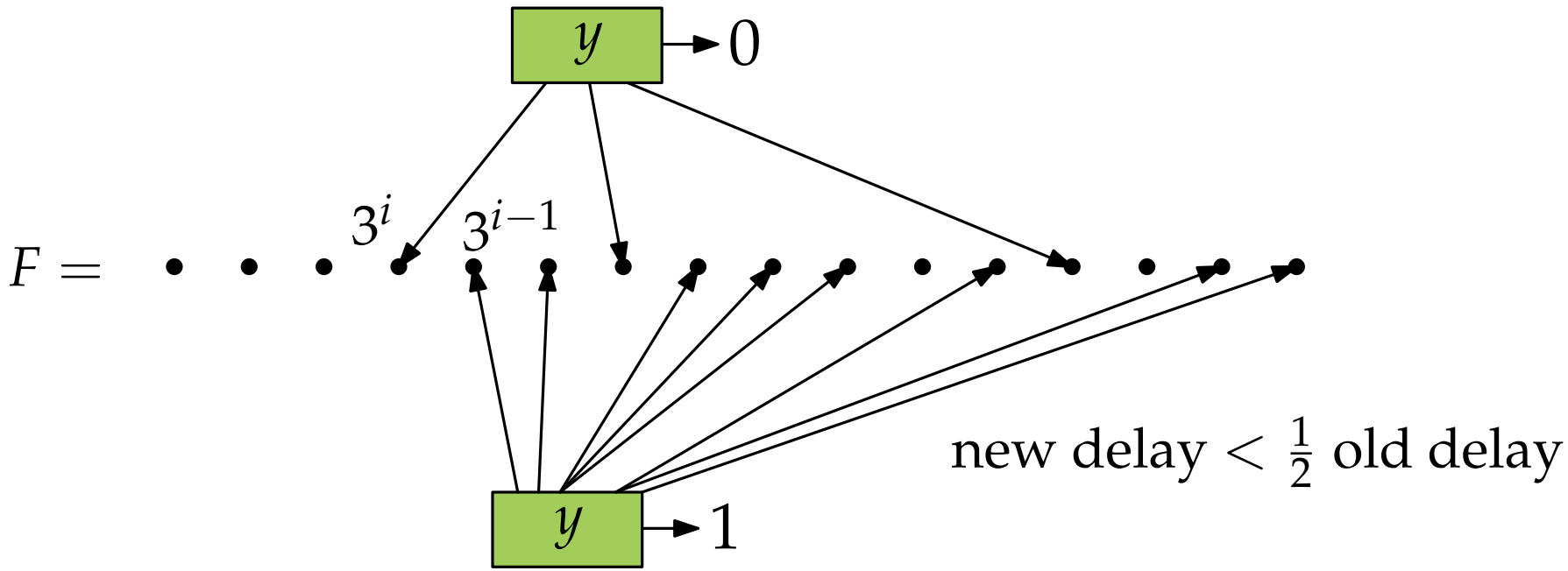

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A 2-Nash equilibrium will give us a locally optimal truth assignment!

Gràcies per la vostra atenció!

¡Gracias por su atención!

Danke für Eure Aufmerksamkeit!

Thank you for your attention!