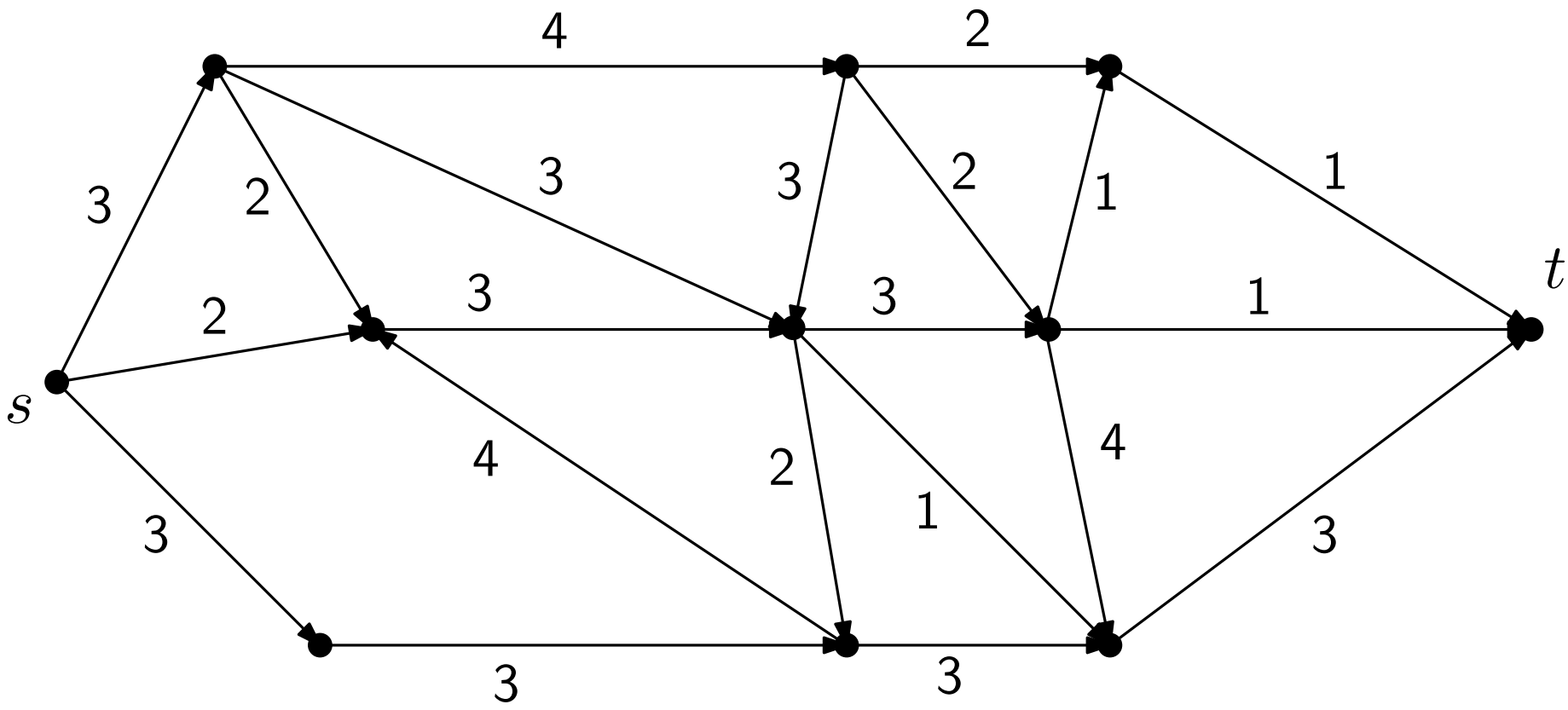
Dinic' Max Flow Algorithm

Slides by Dominik Scheder

Part I

Dinic' Algorithm in General Flow Networks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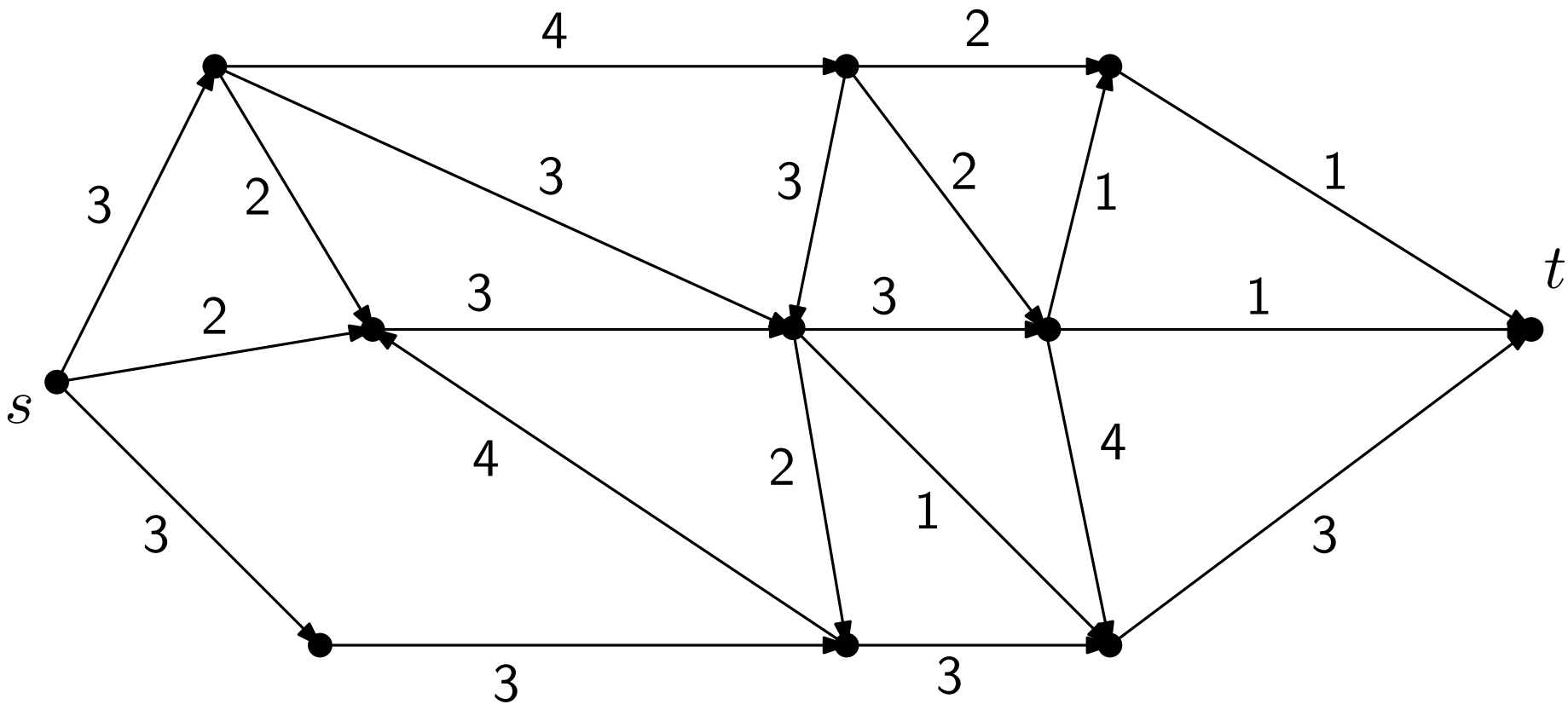
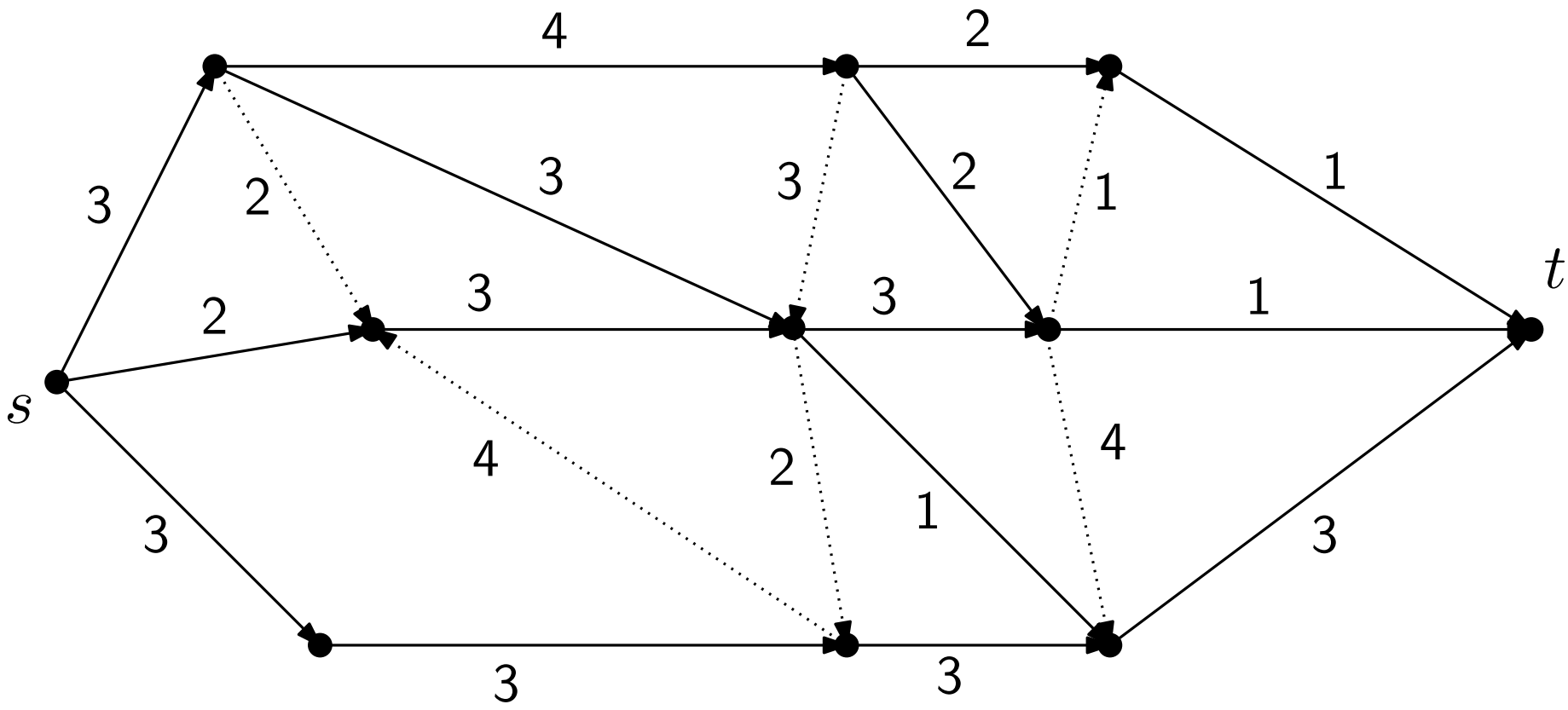


Find out which edges lie on a shortest s-t-path,



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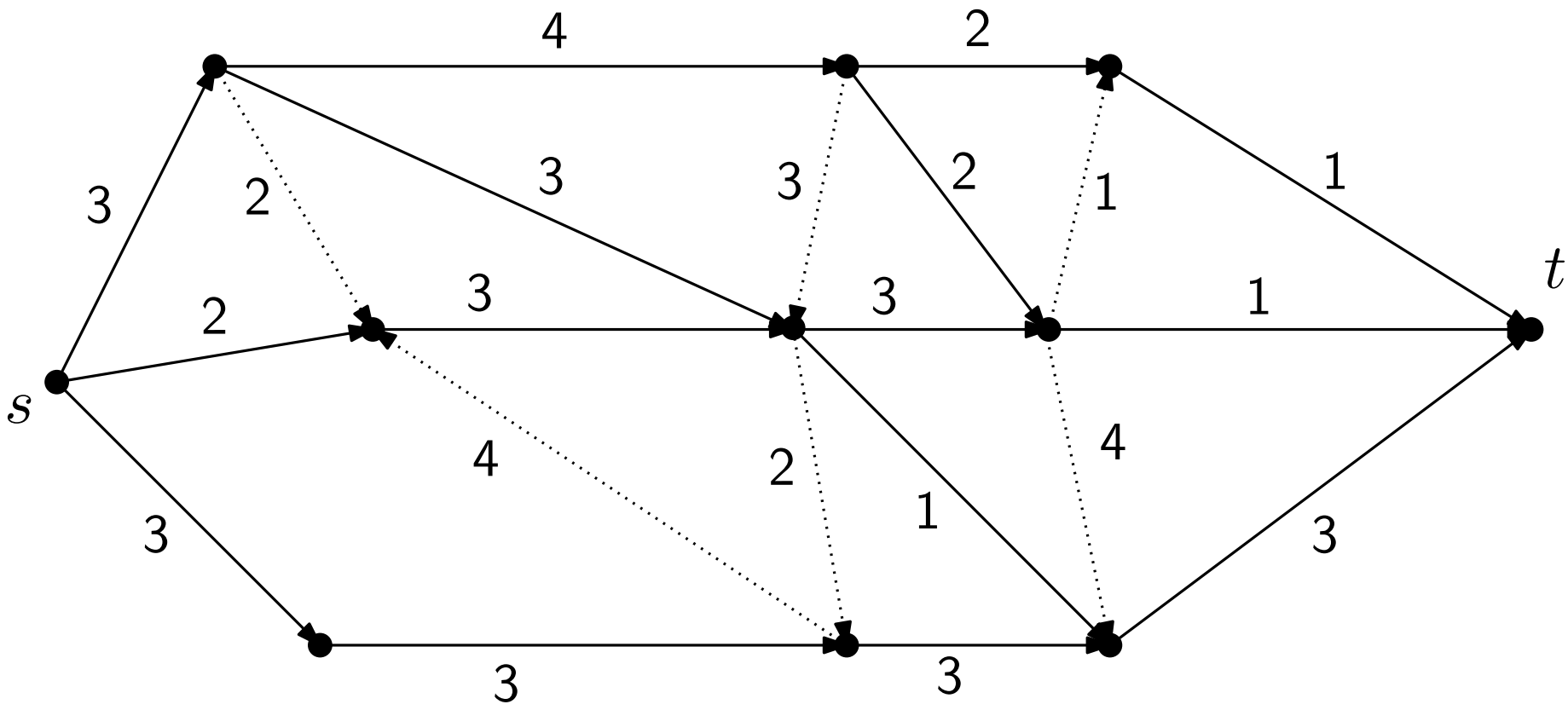
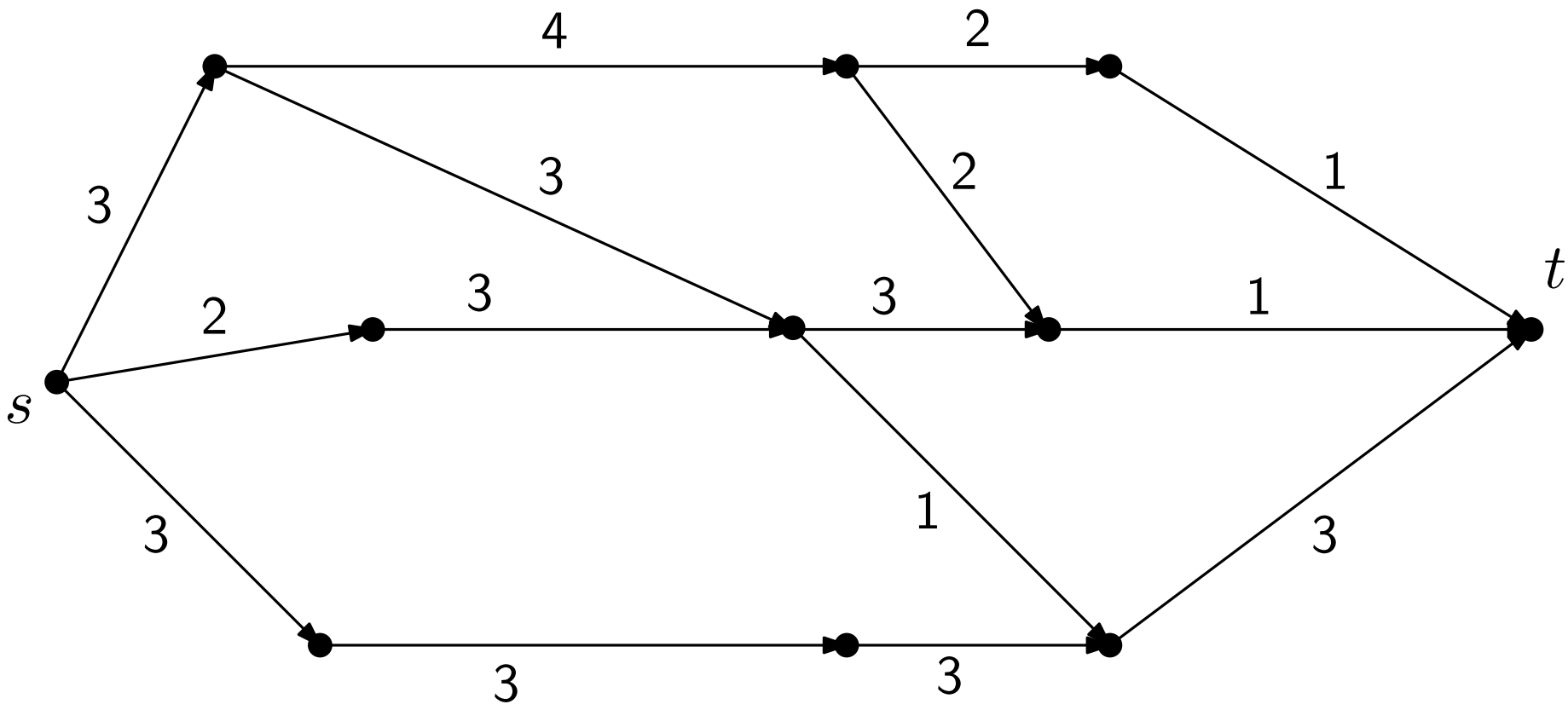


Find out which edges lie on a shortest s-t-path,

and forget the rest for the time being.



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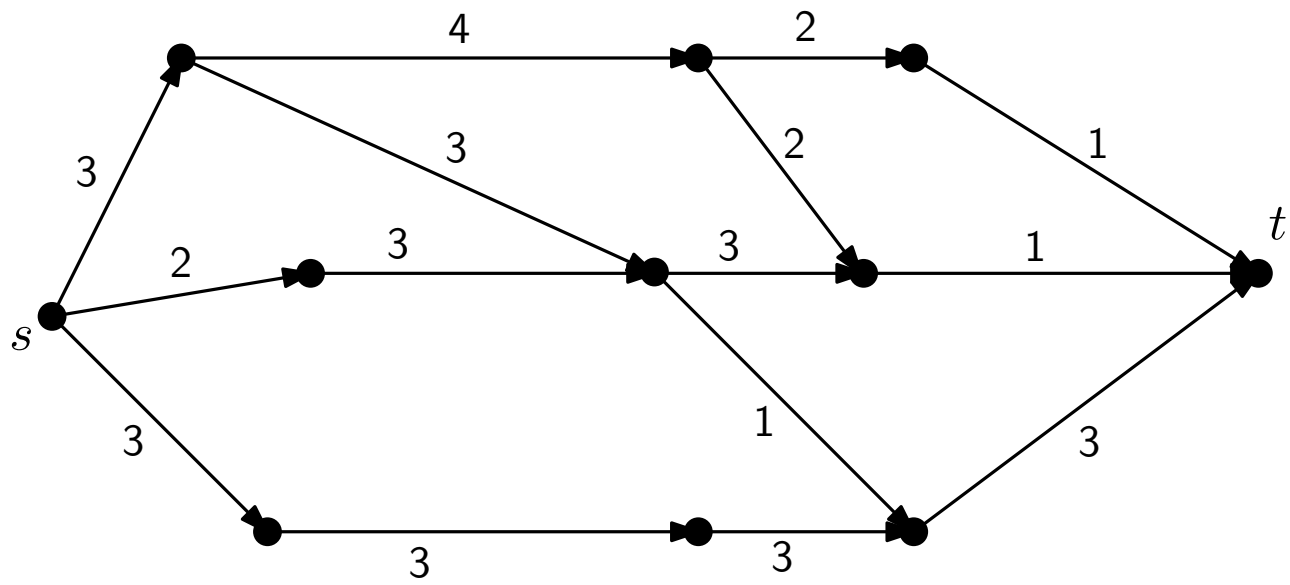
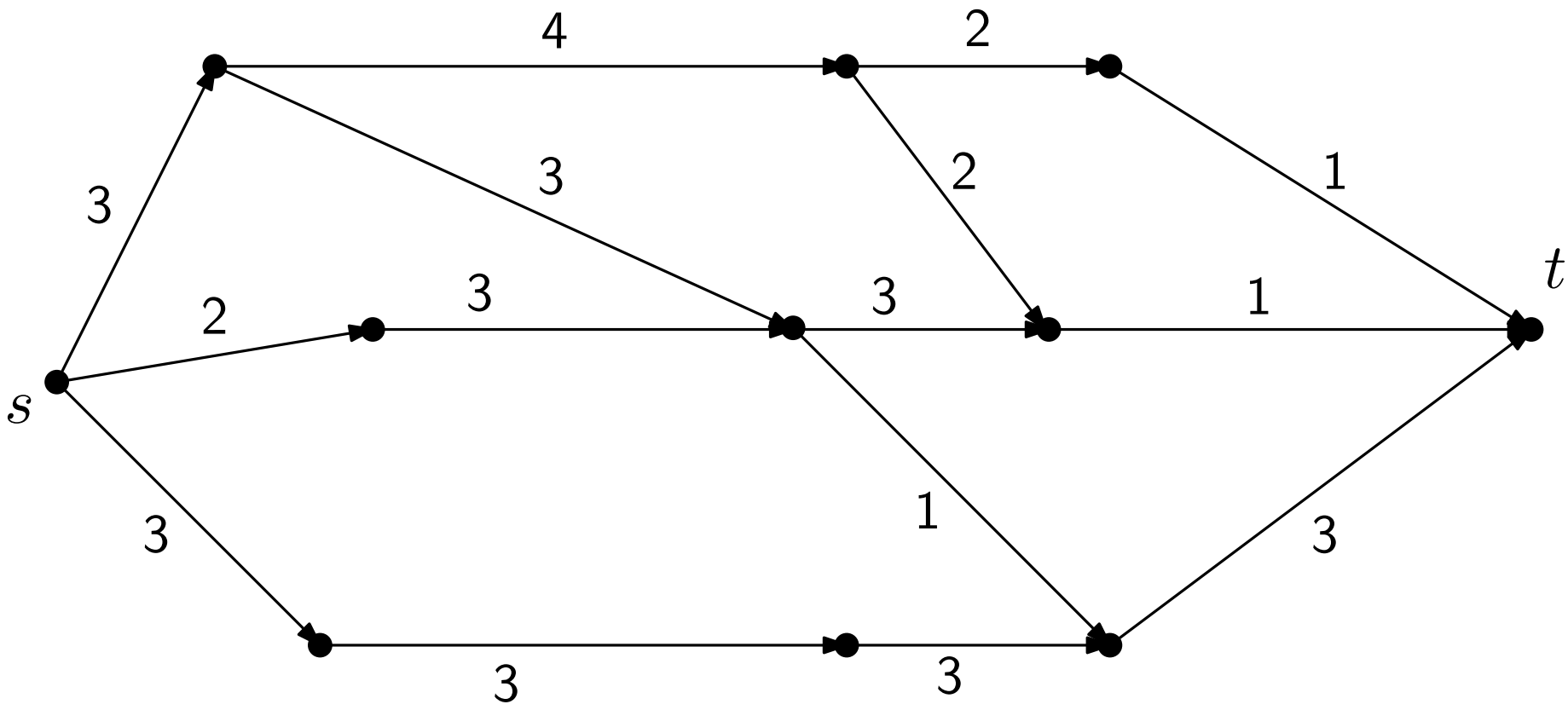


Now lets greedily route as much flow as possible in this network!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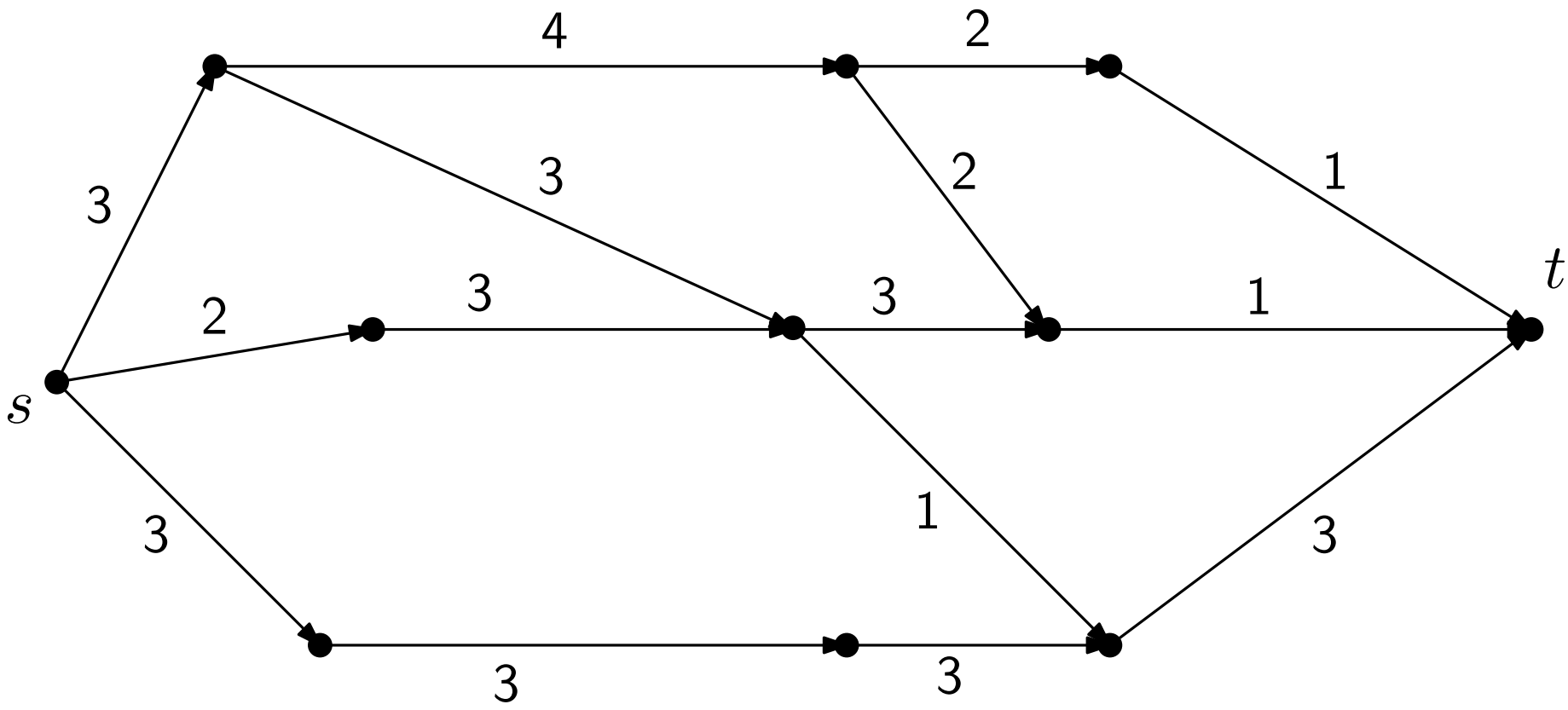
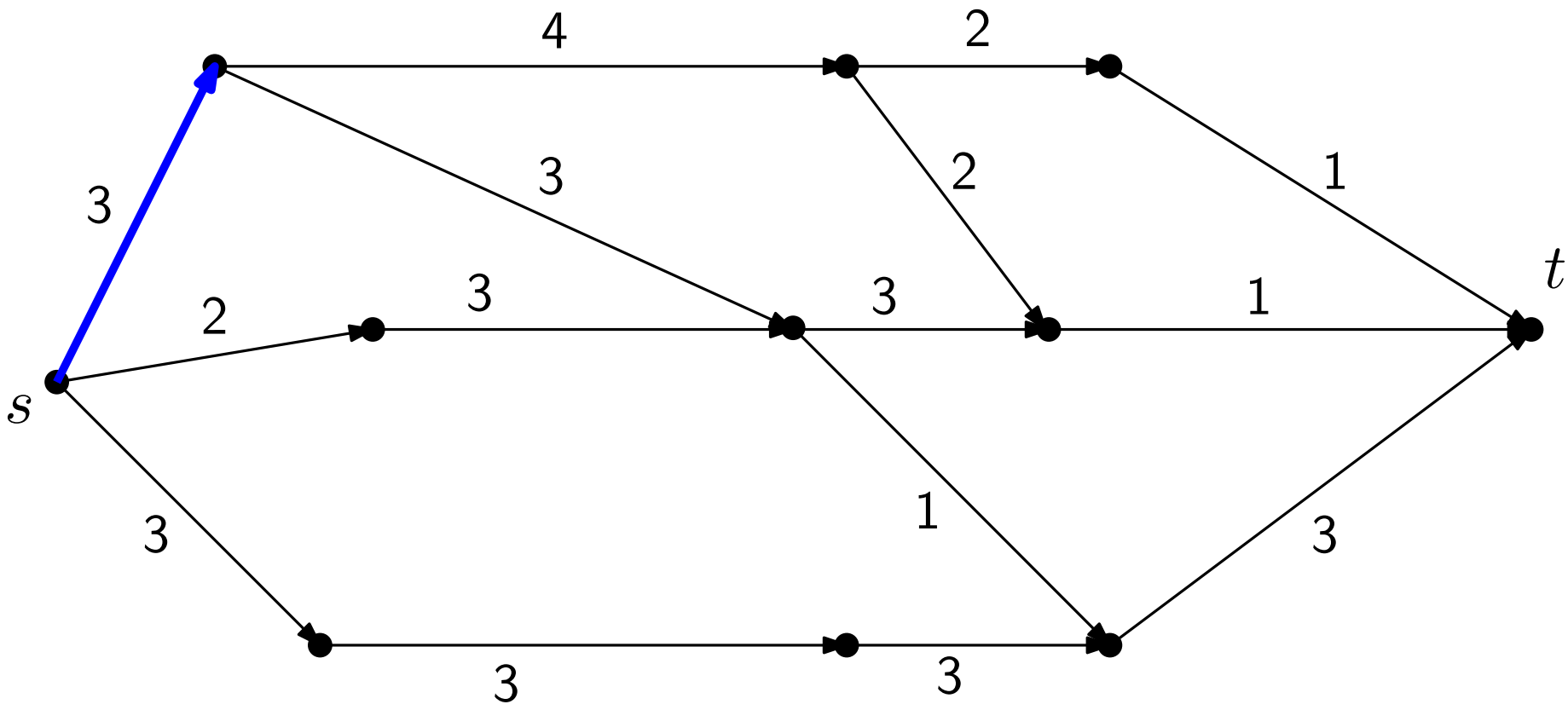


Find an s-t-path with depth-first search.



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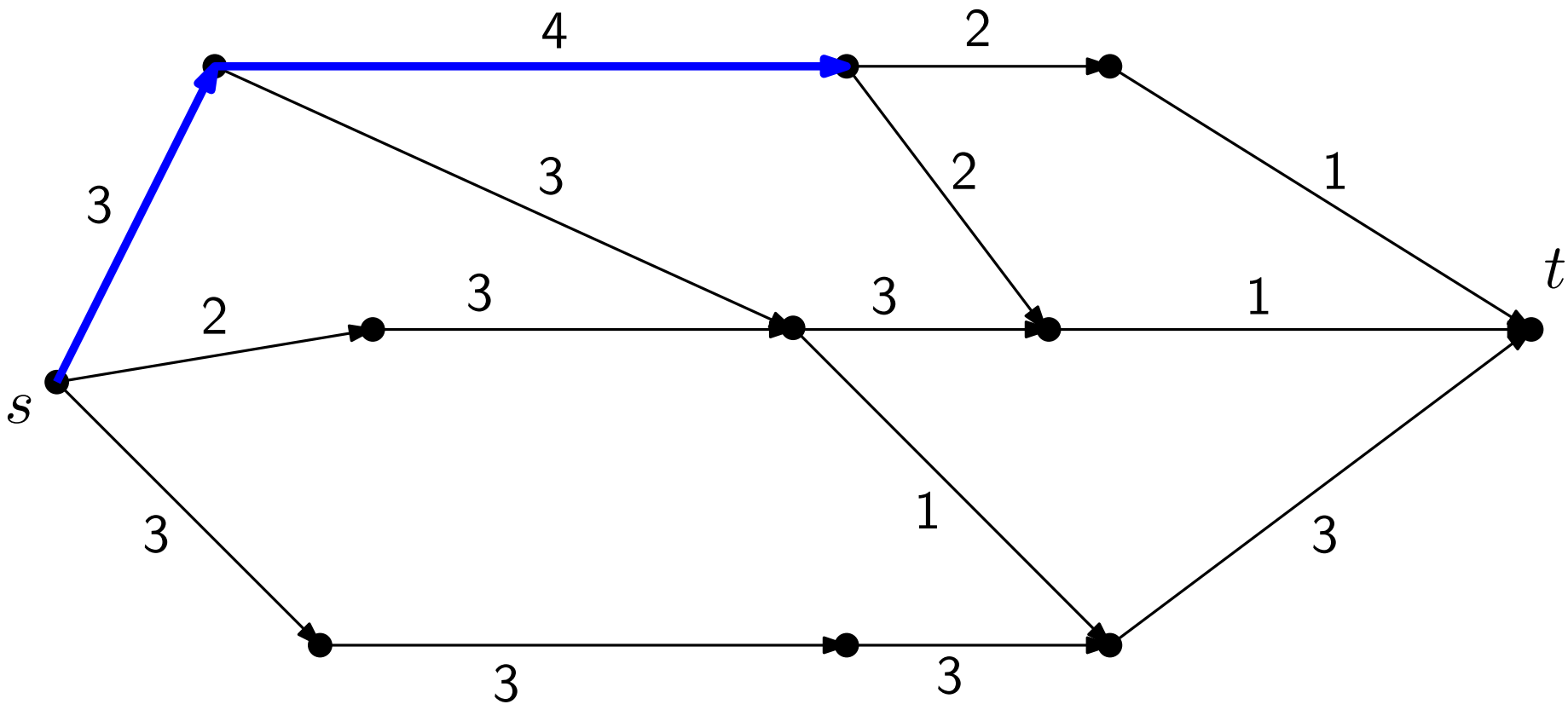

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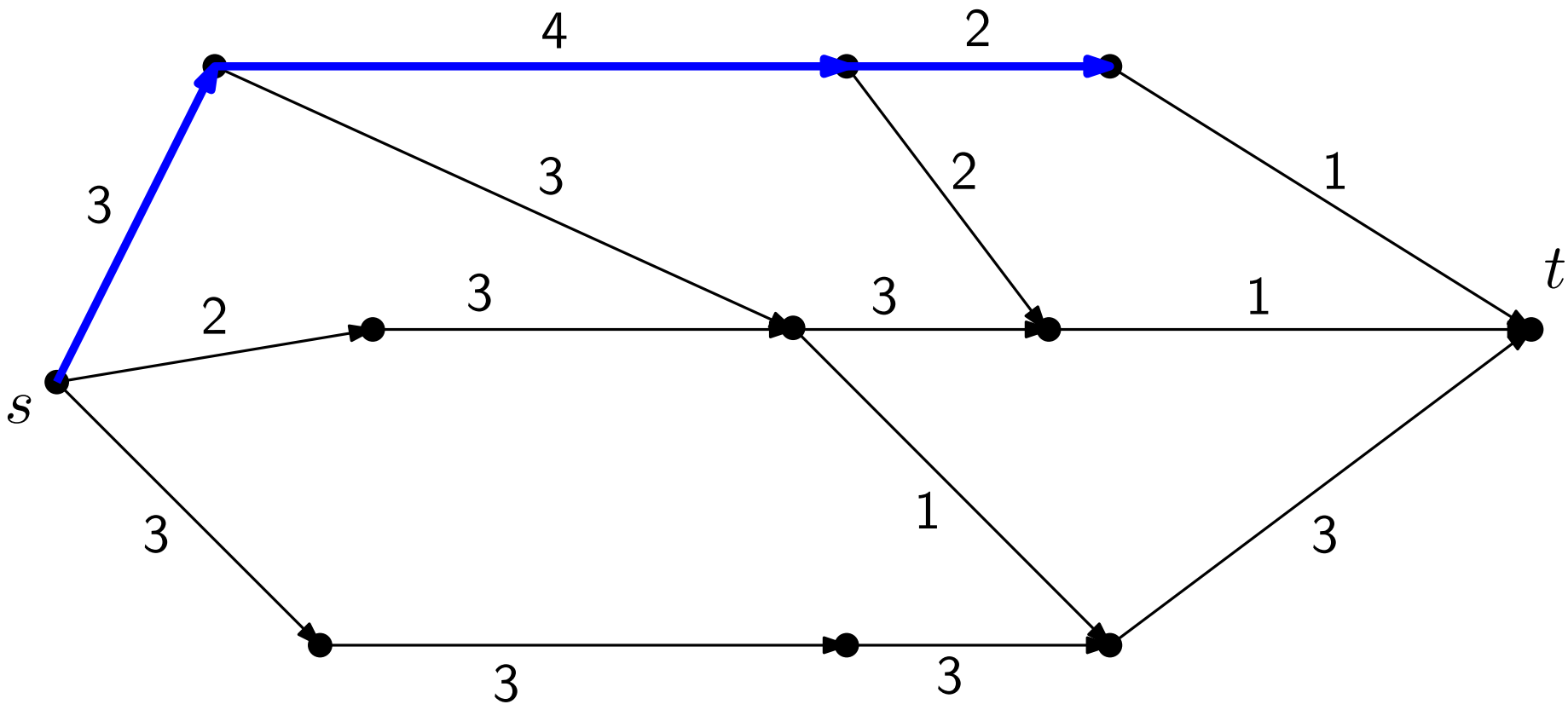

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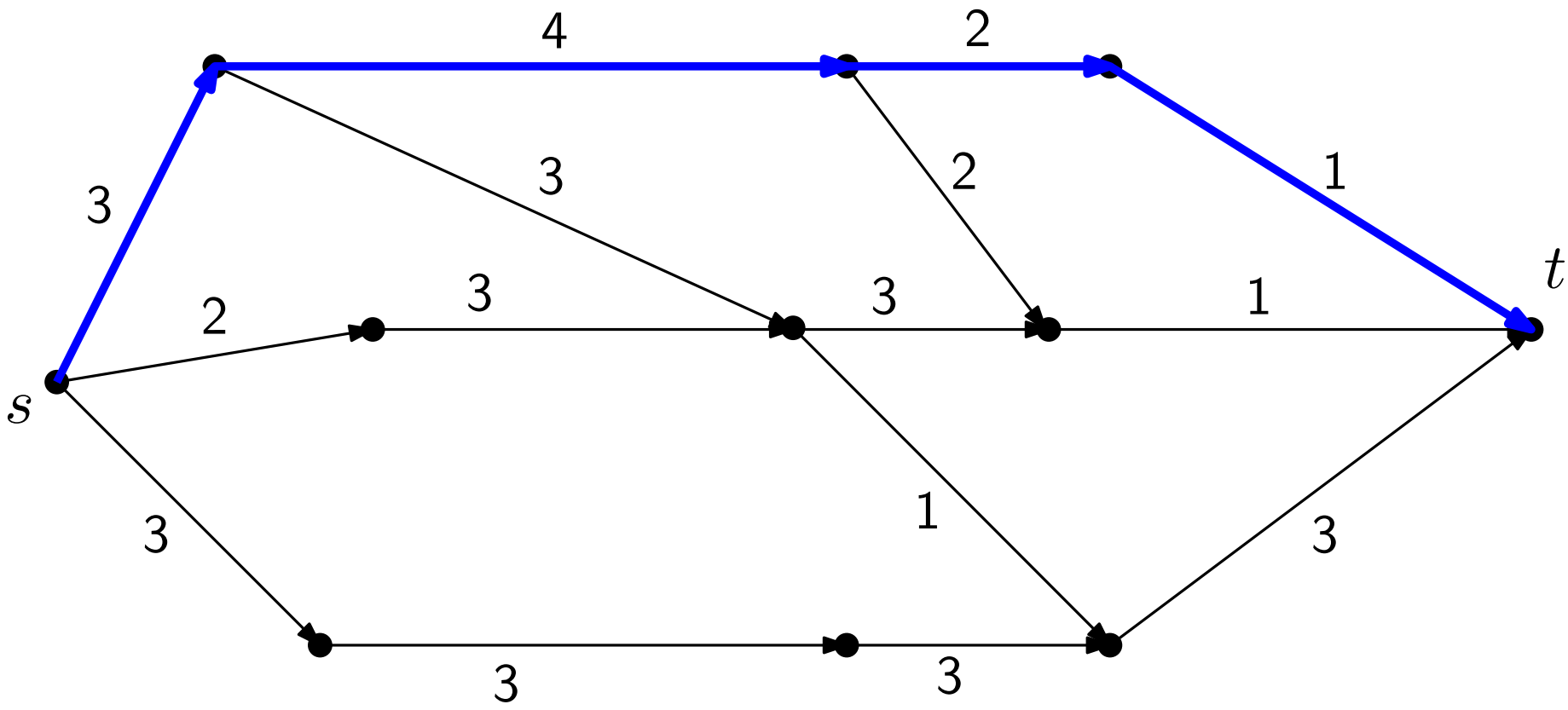

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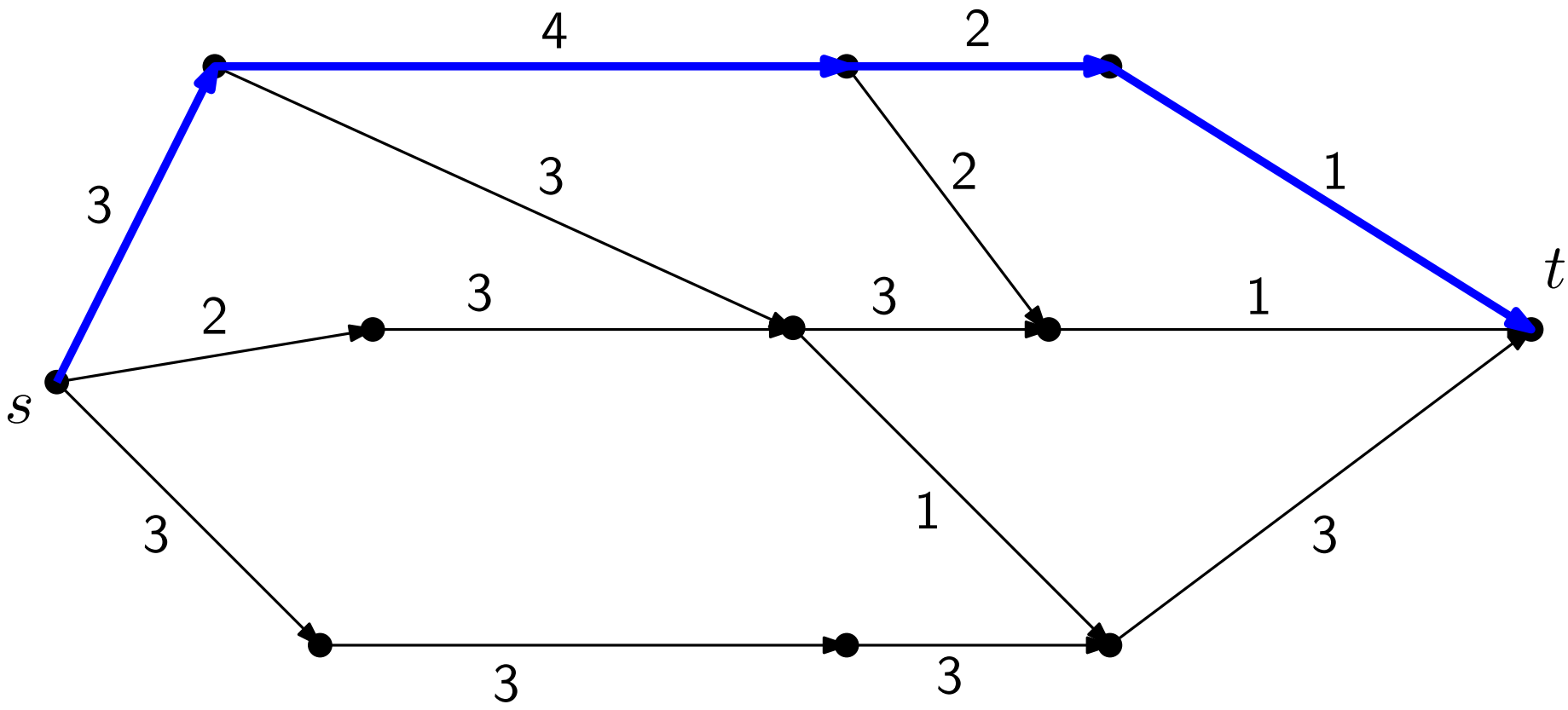
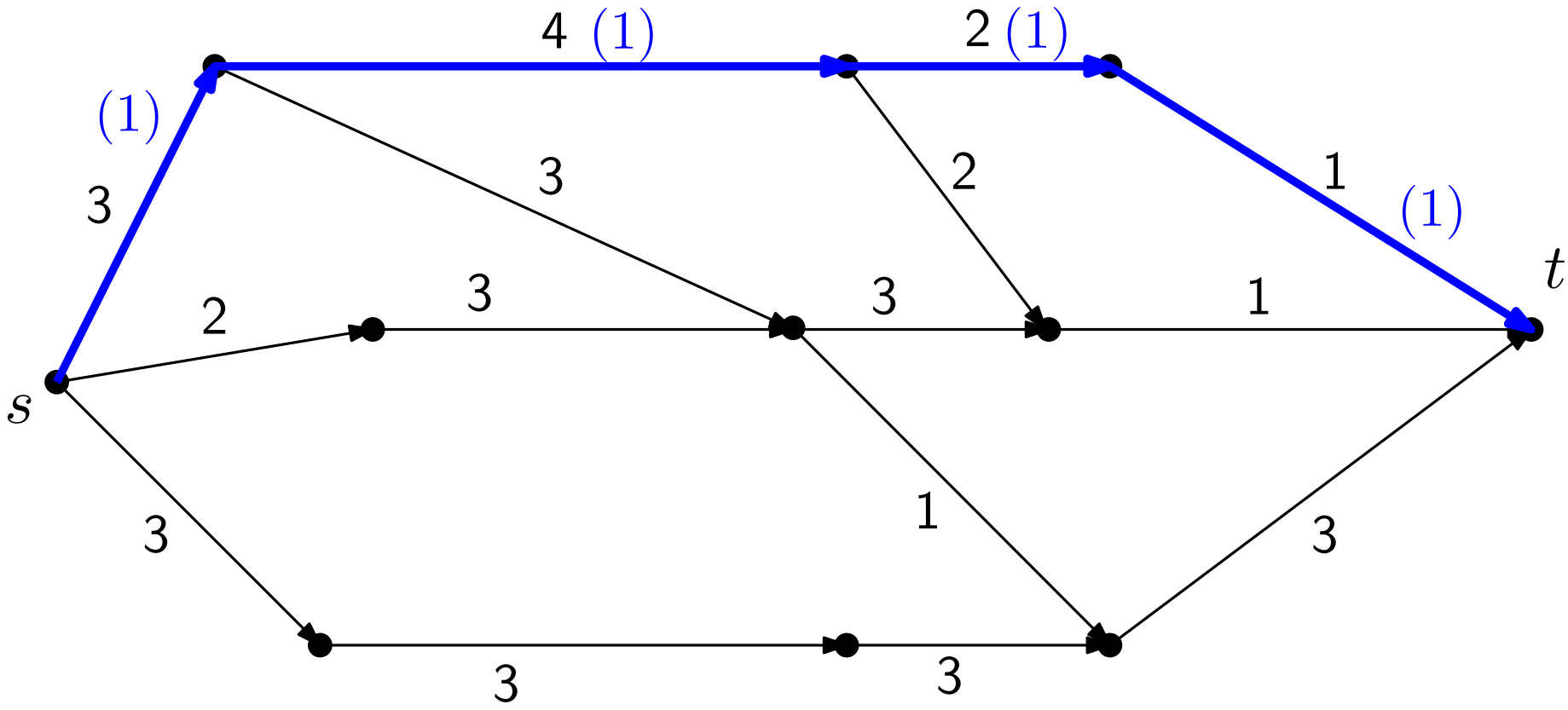


Find an s-t-path with depth-first search.

Route as much flow through it as possible: 1 unit.



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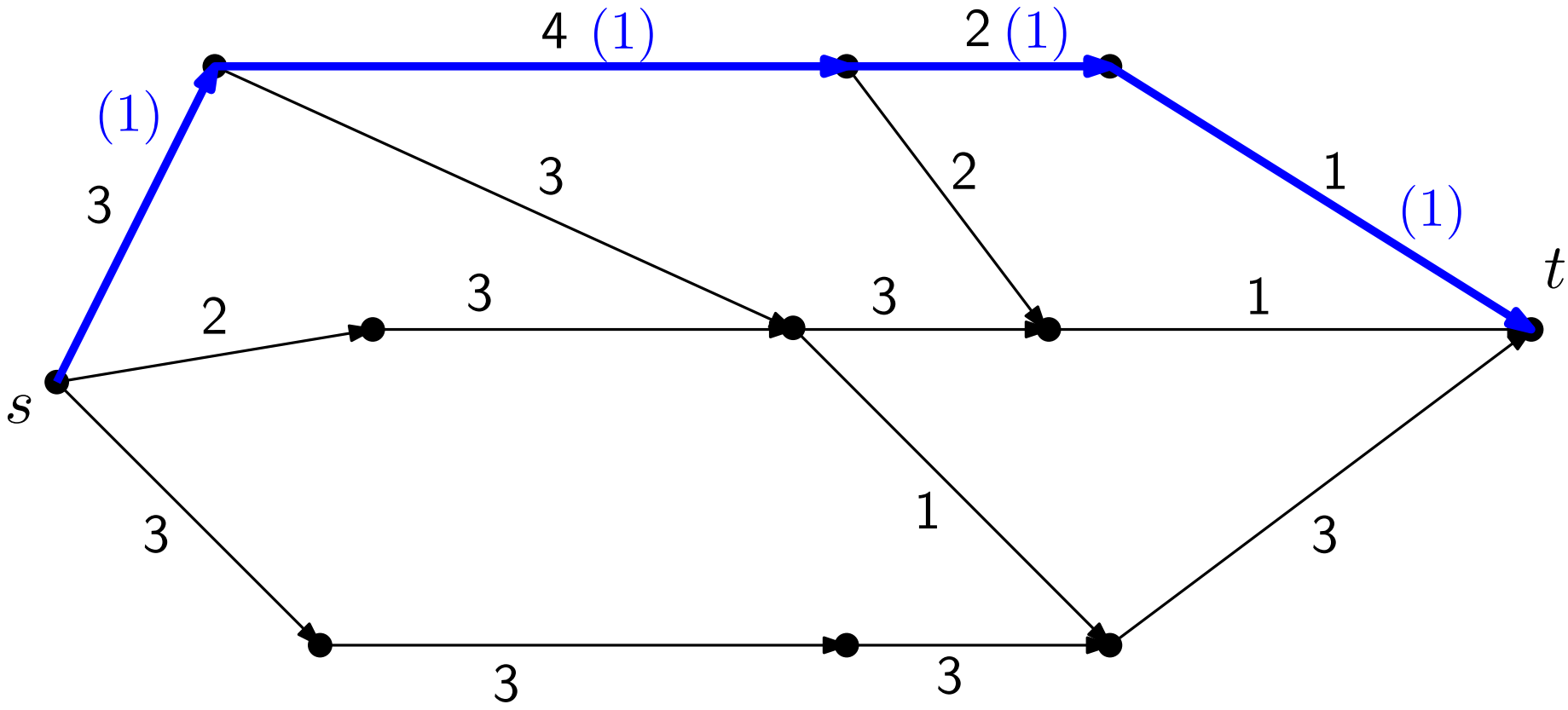

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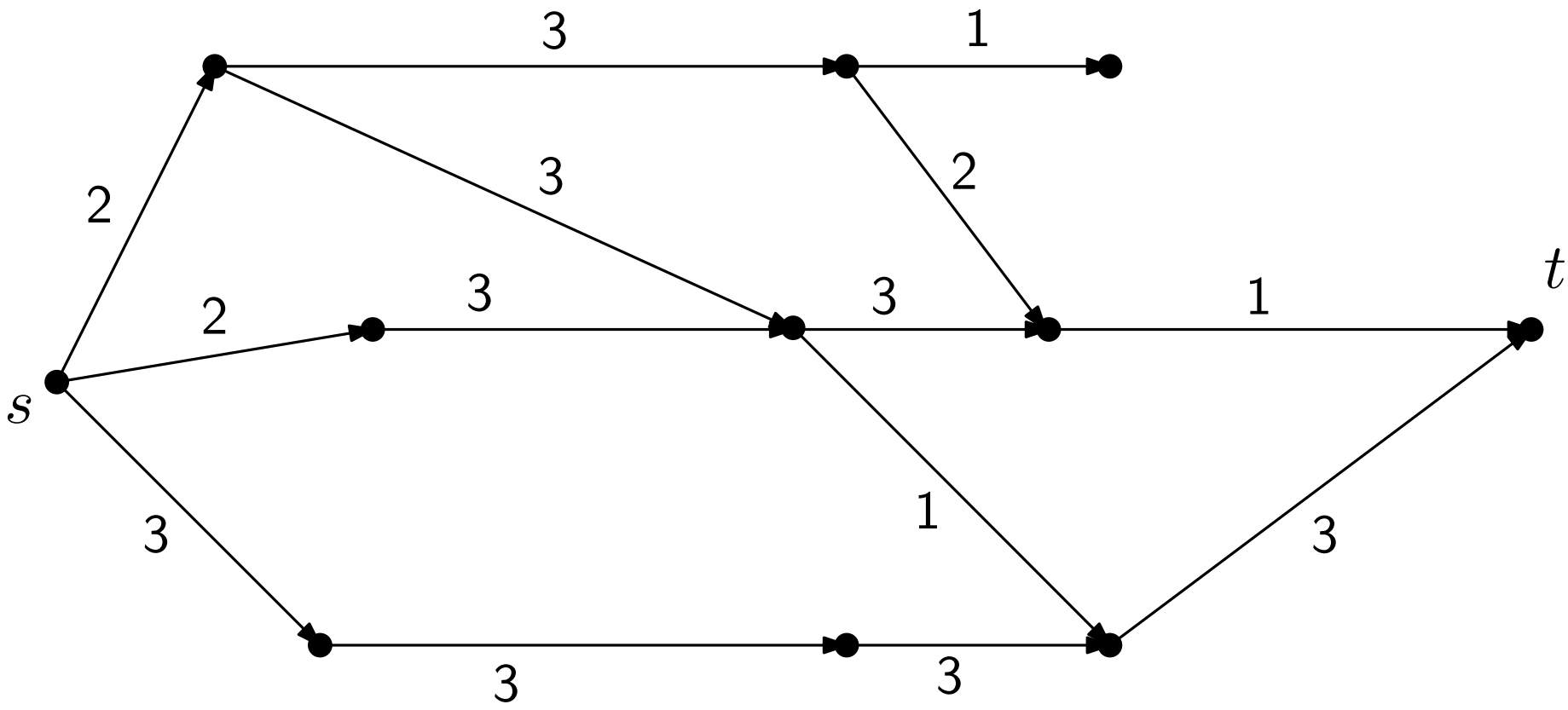

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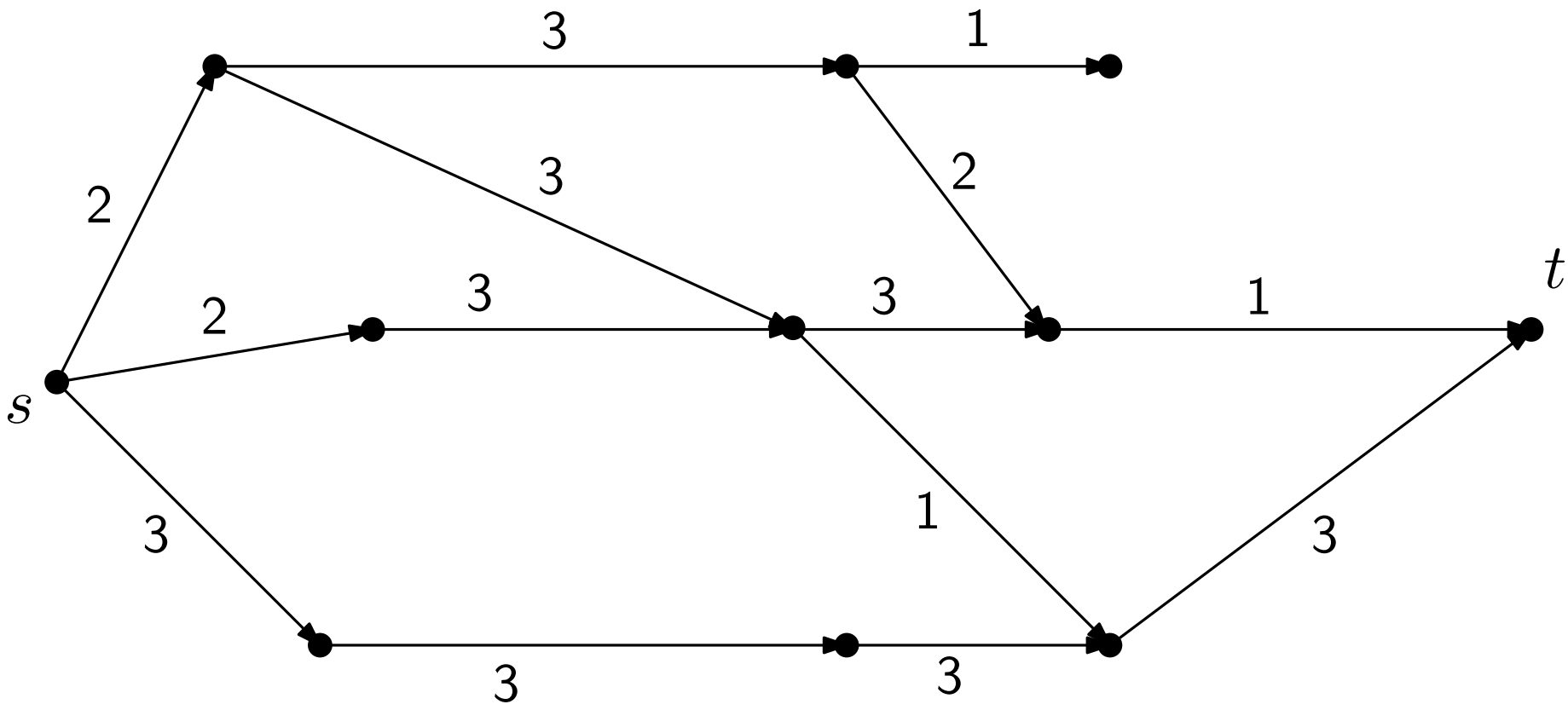

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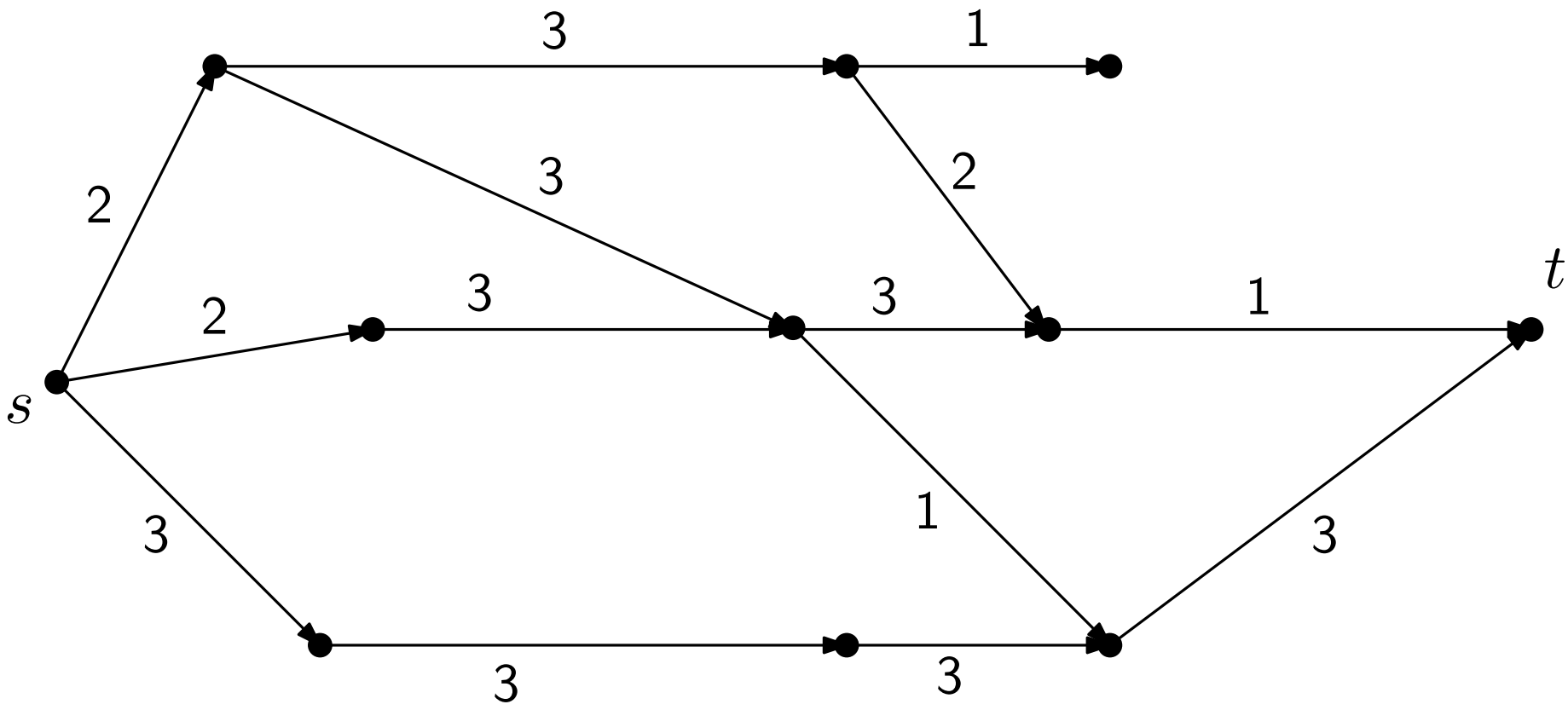
Update capacities. Repeat!



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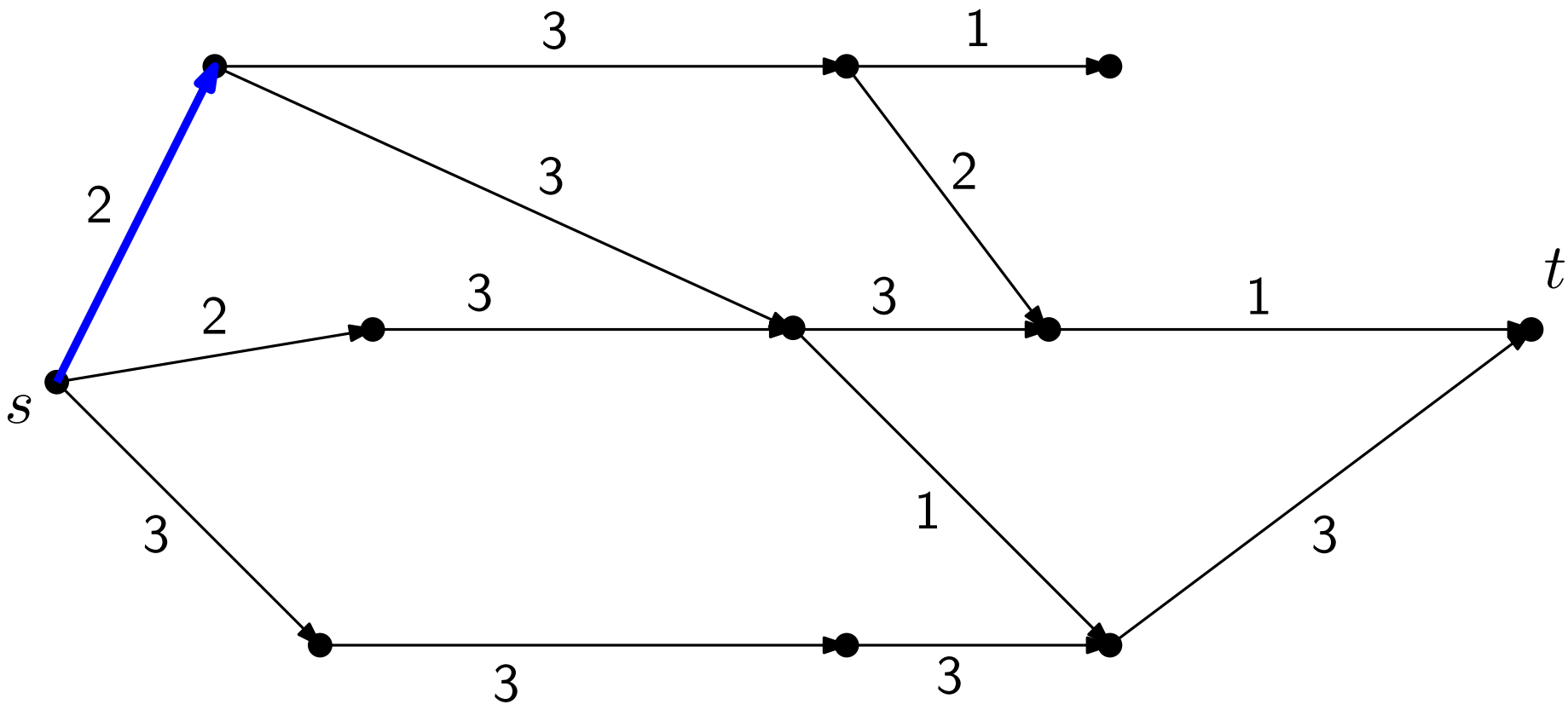

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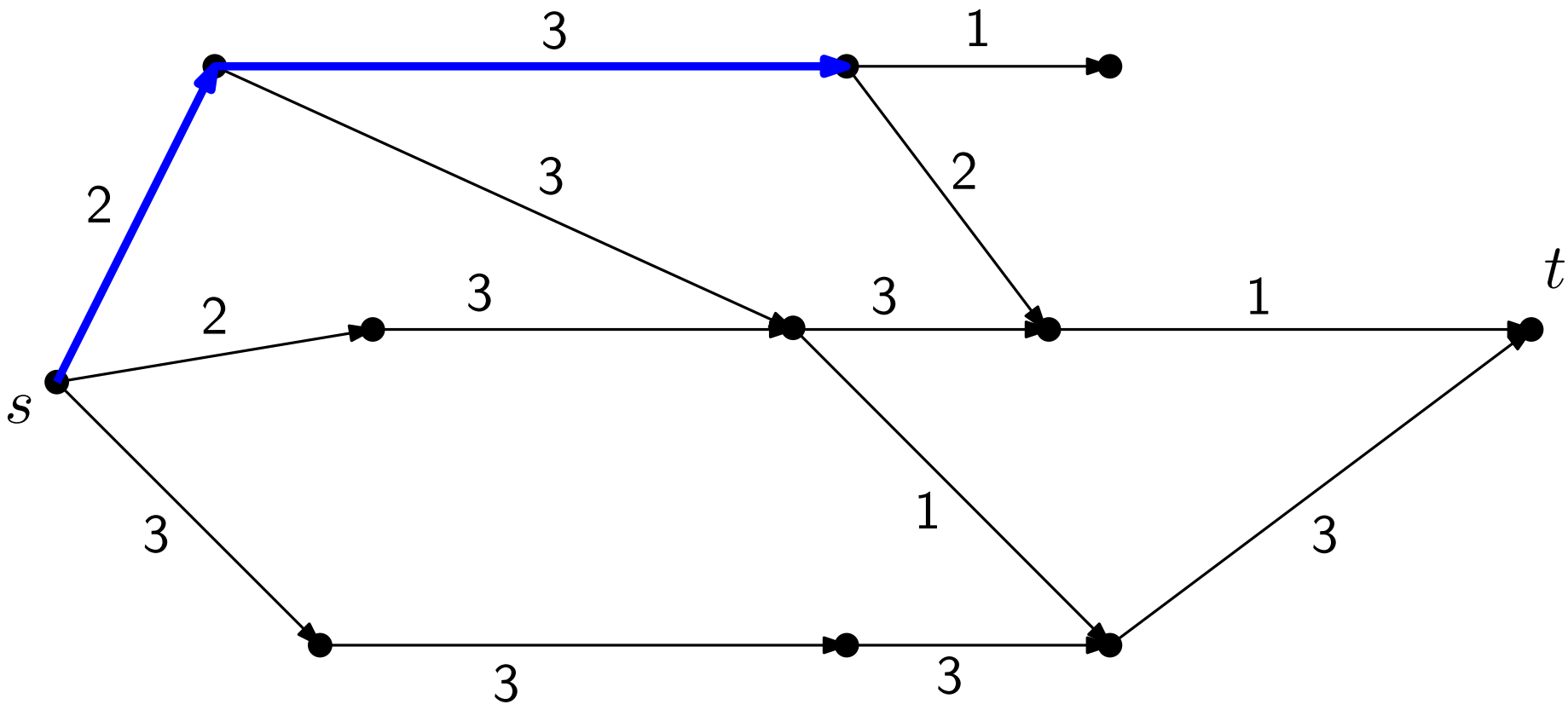

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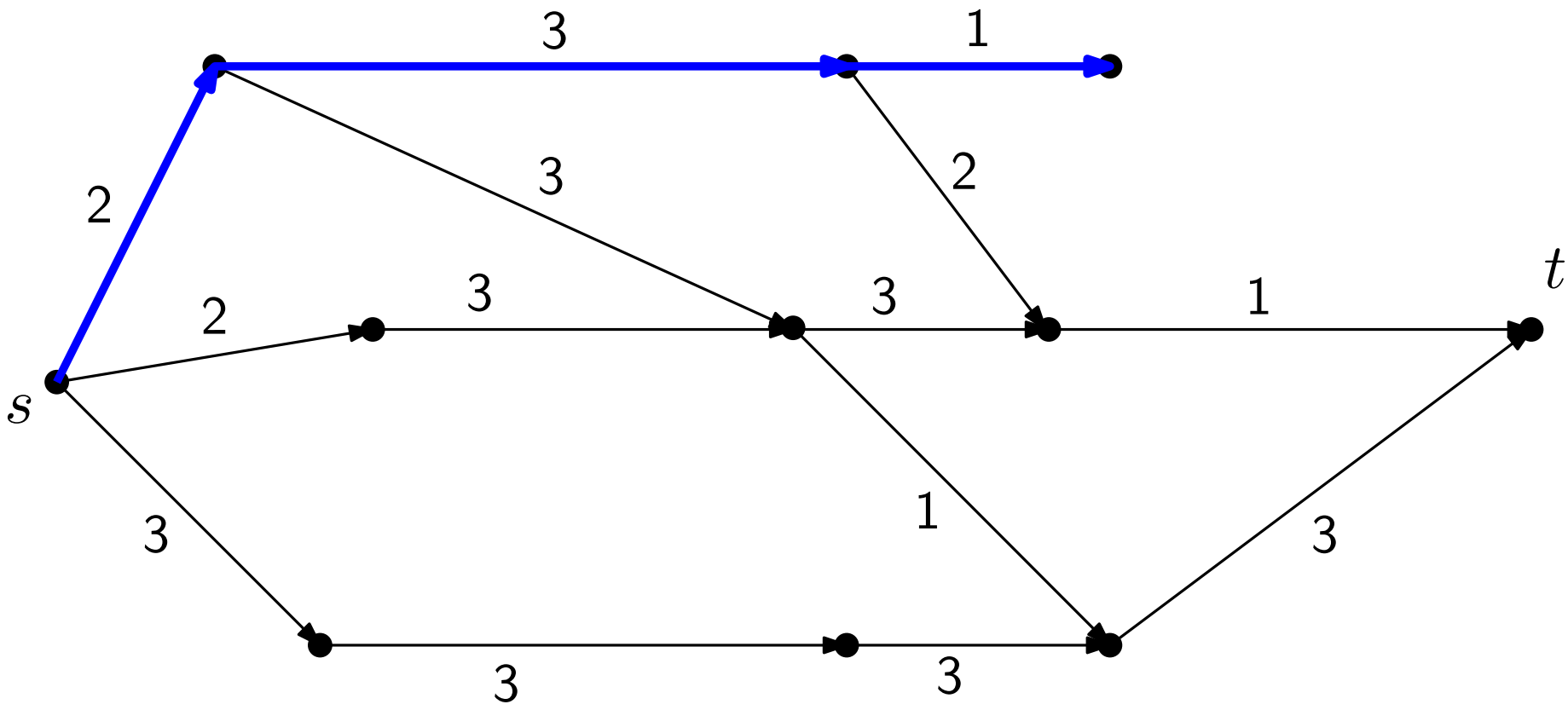

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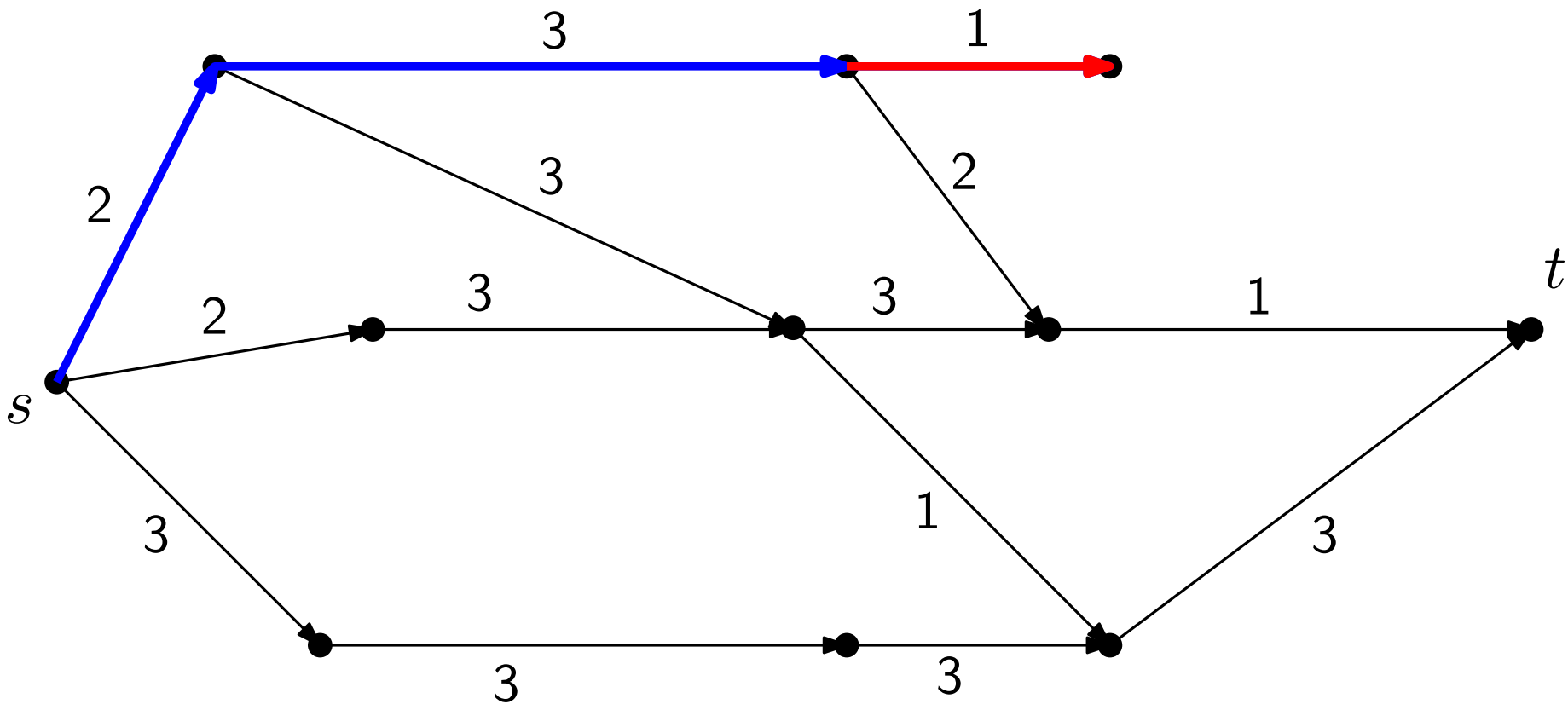

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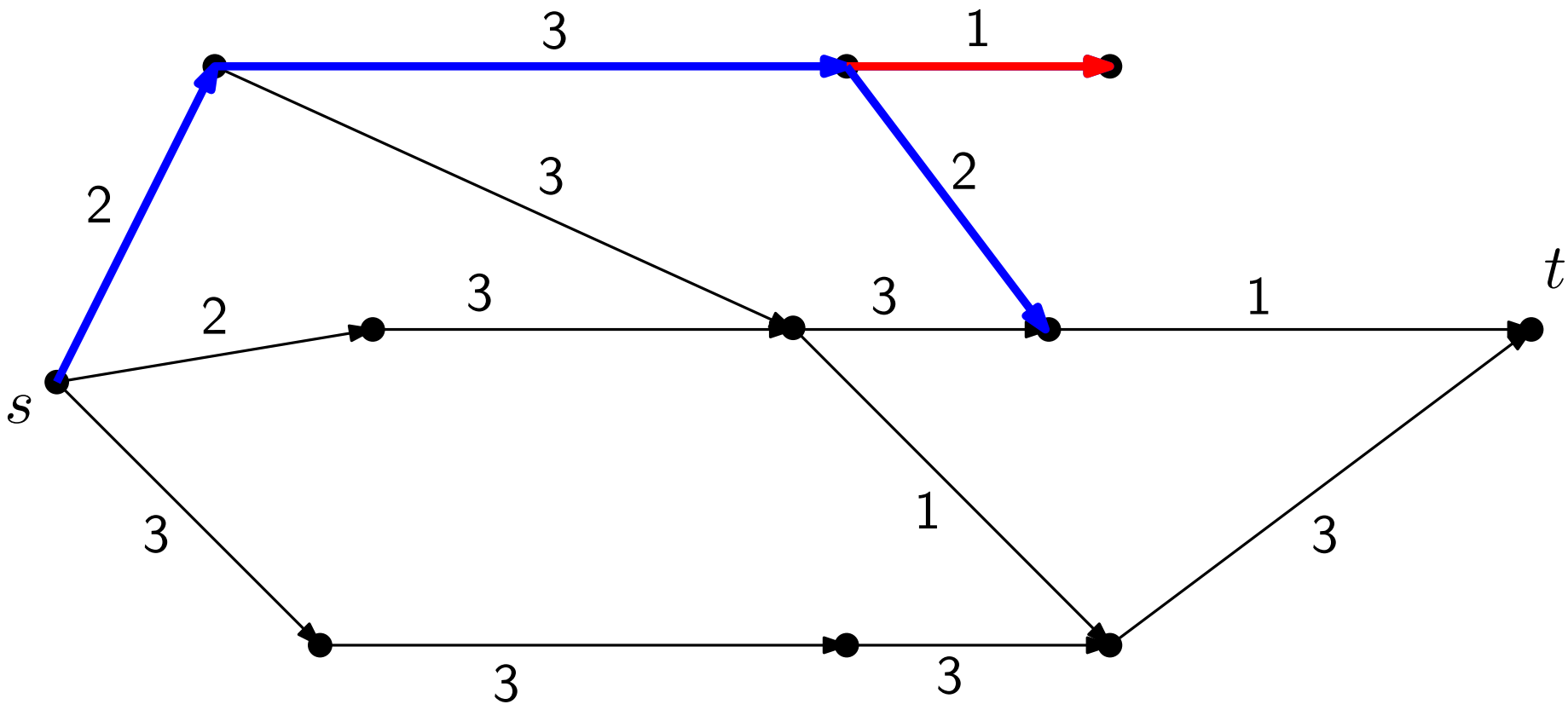

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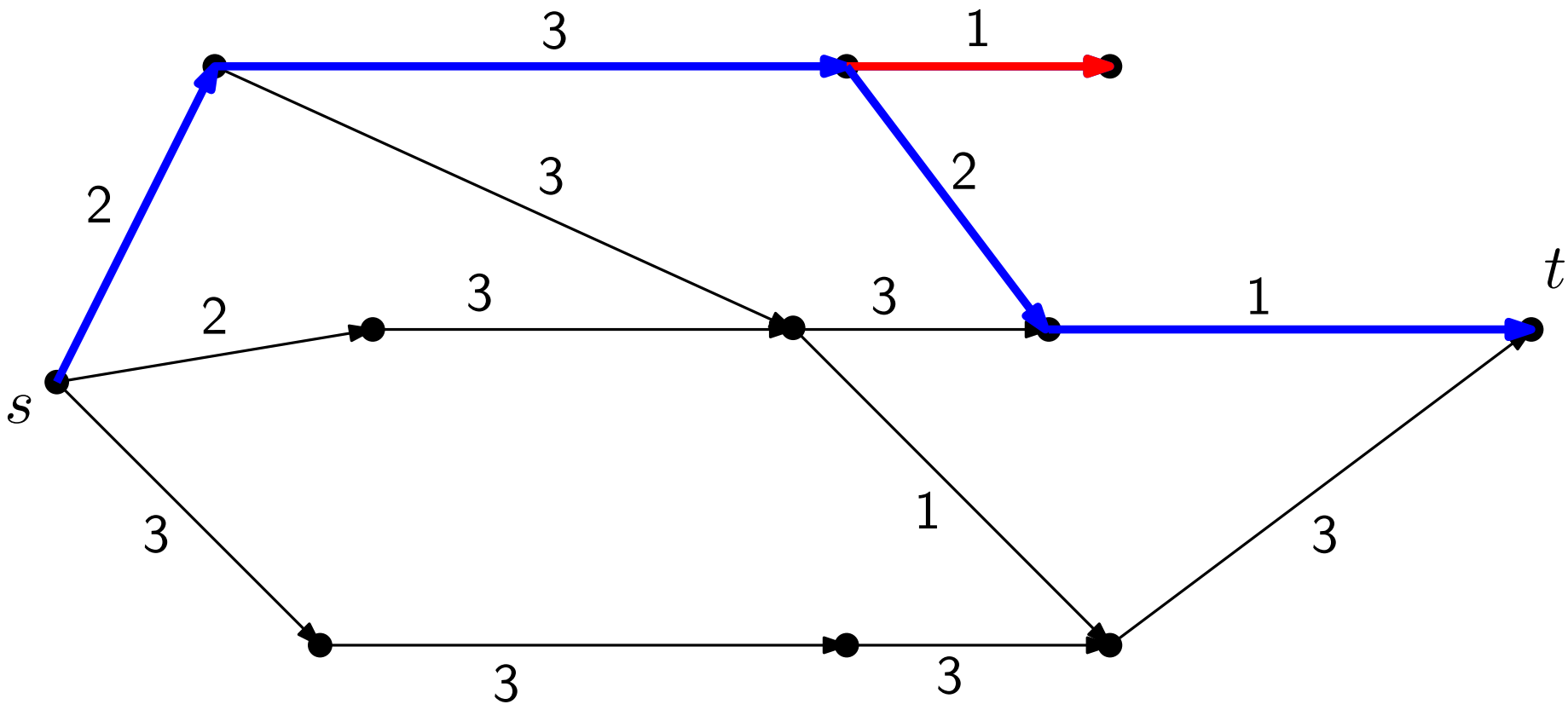

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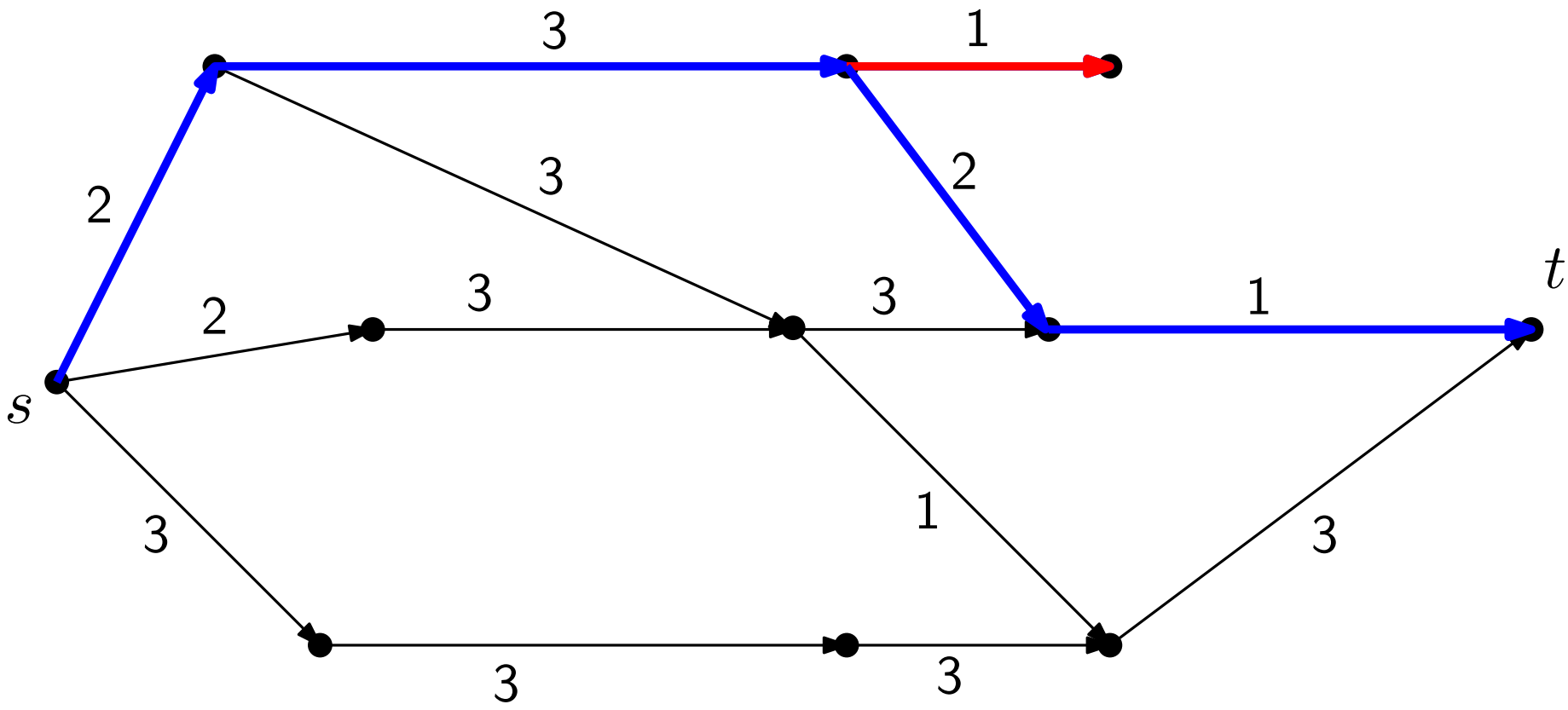

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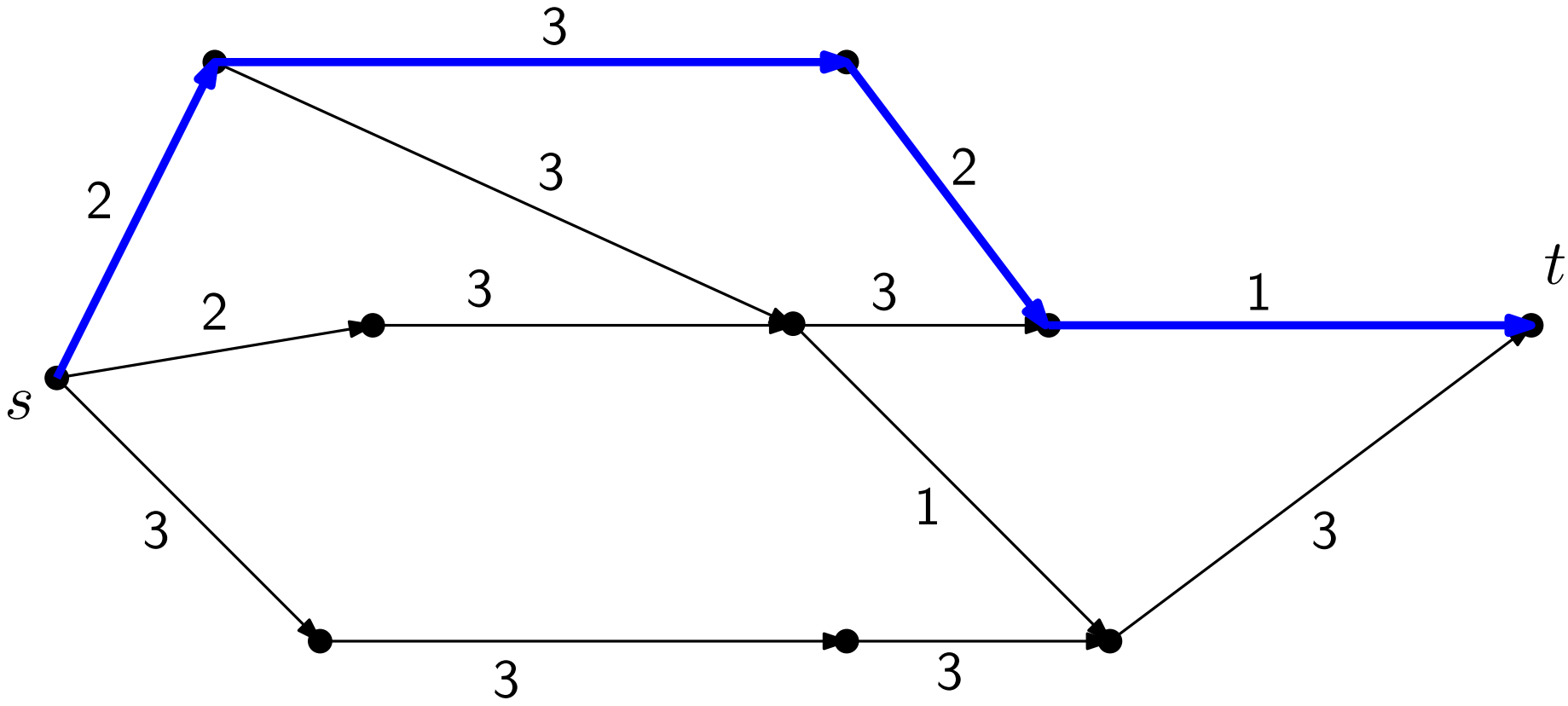
Delete dead ends.



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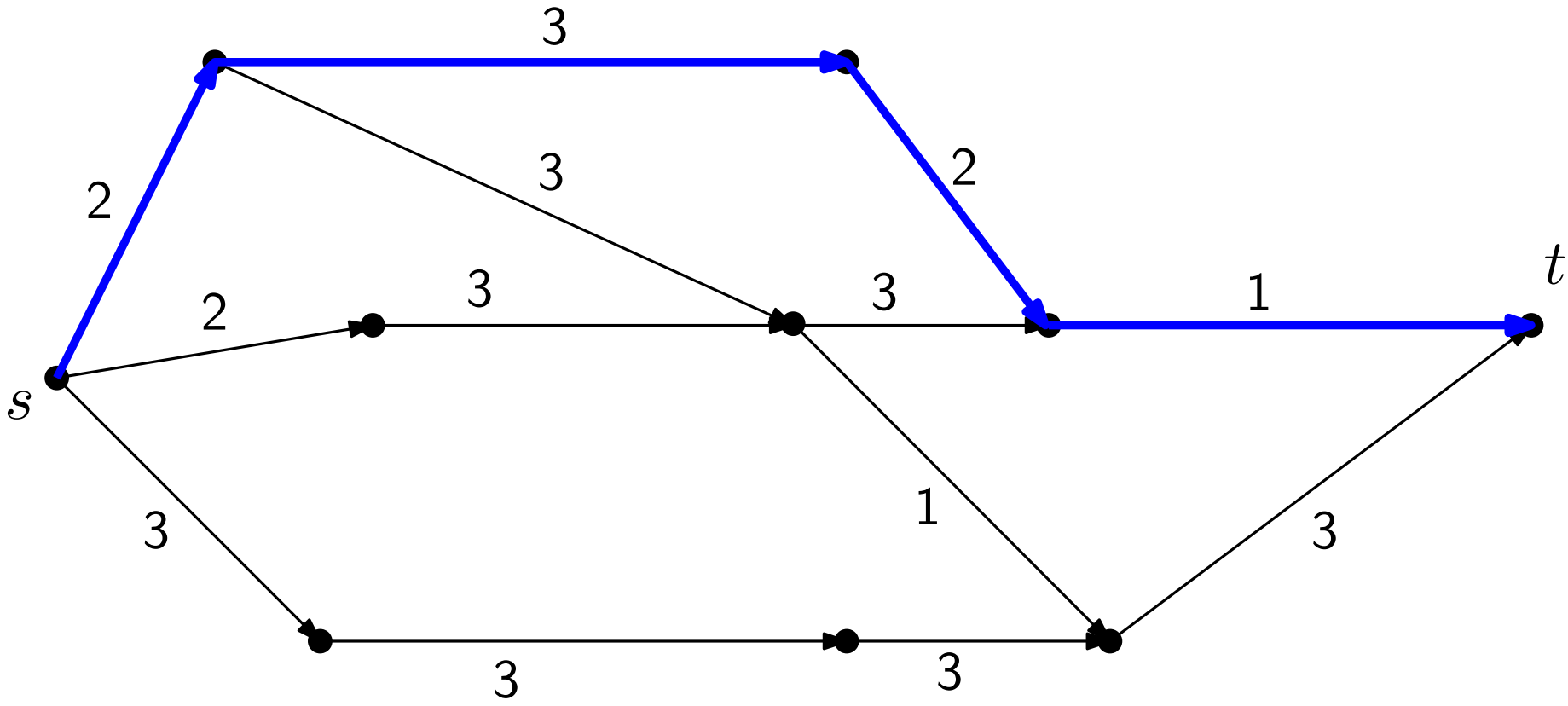

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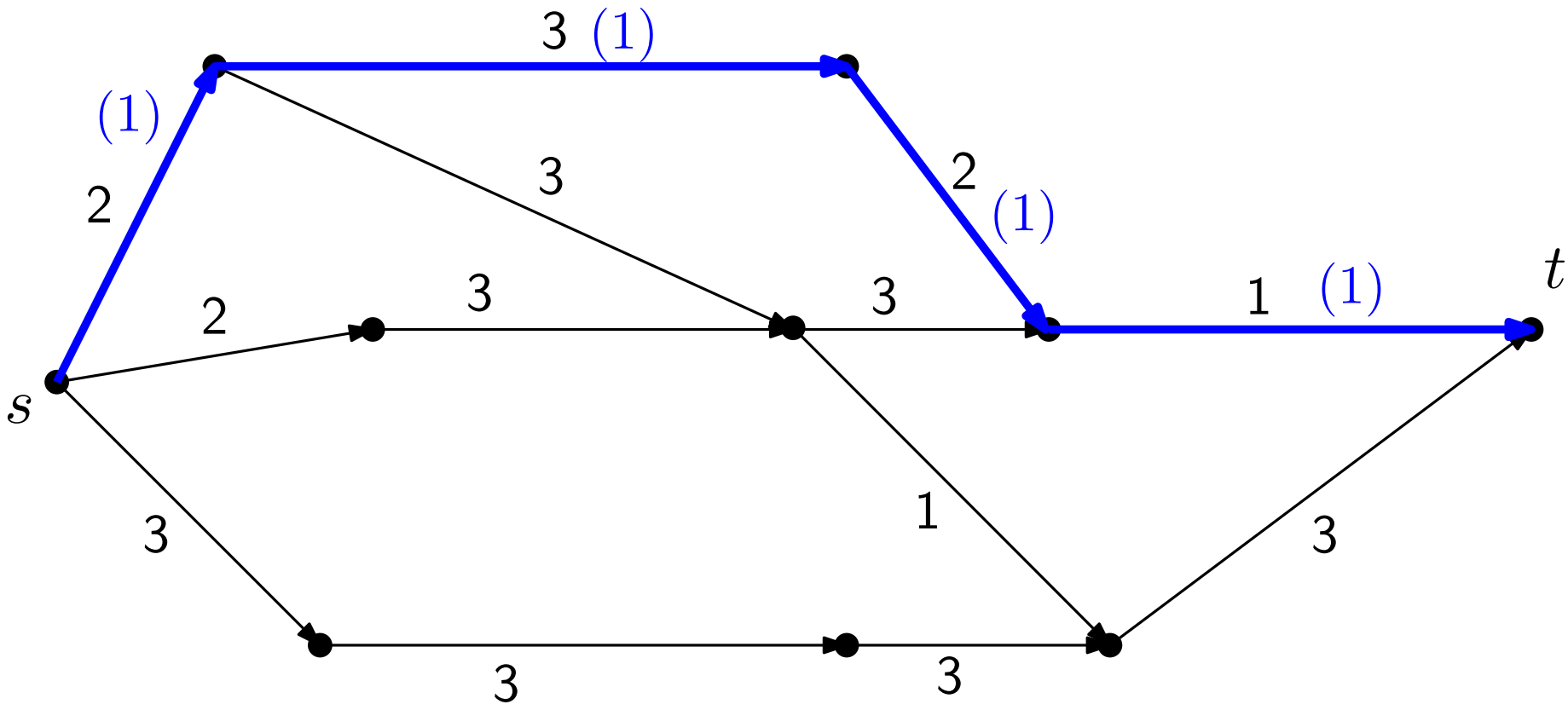
Route flow.



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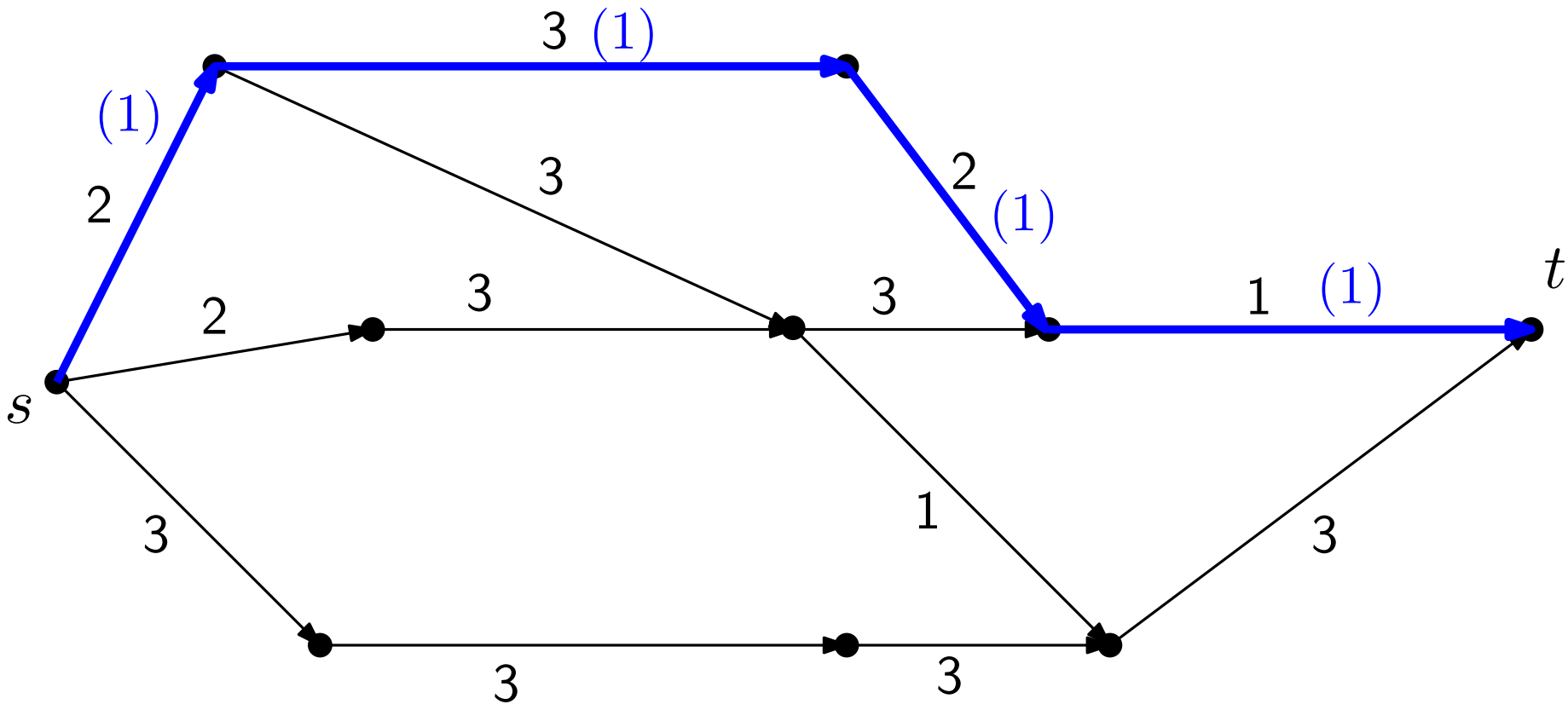

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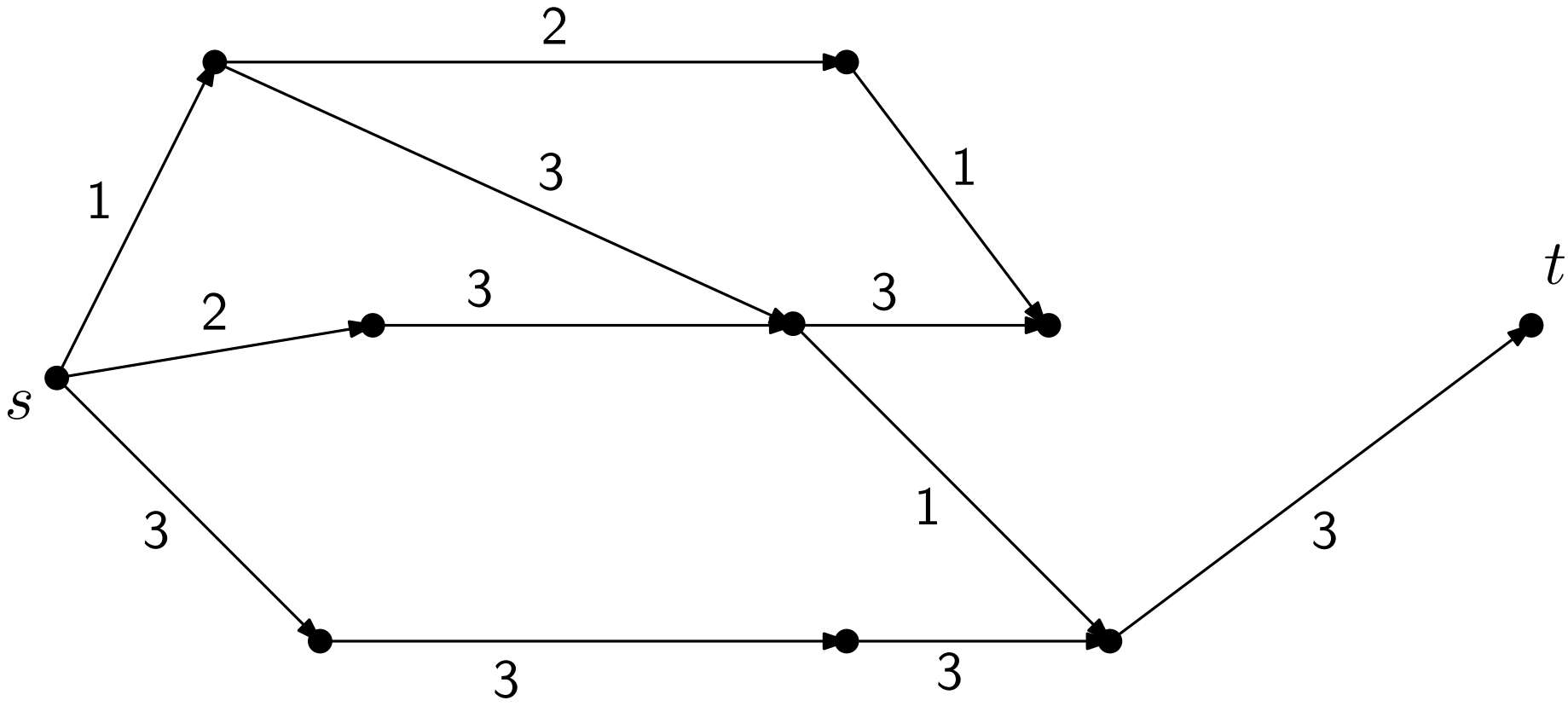

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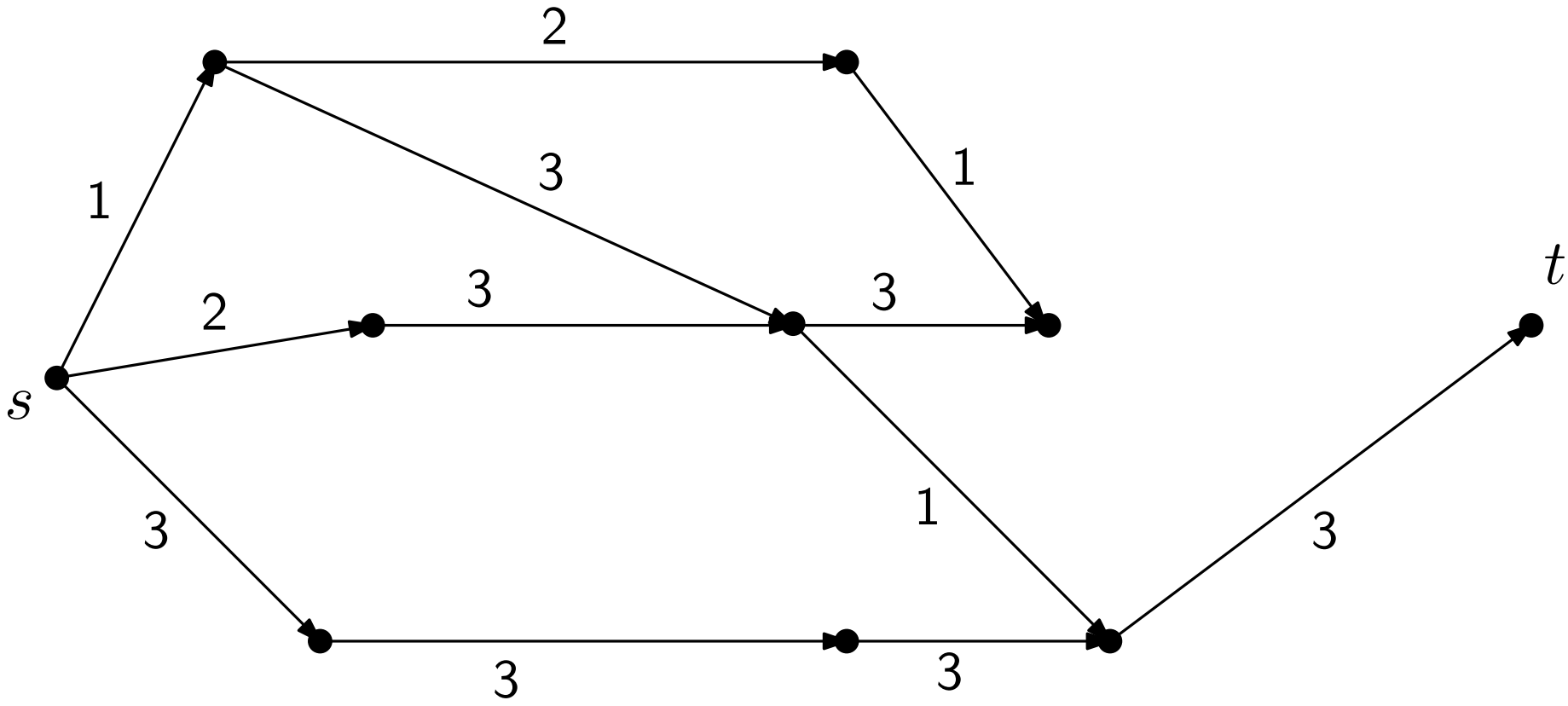

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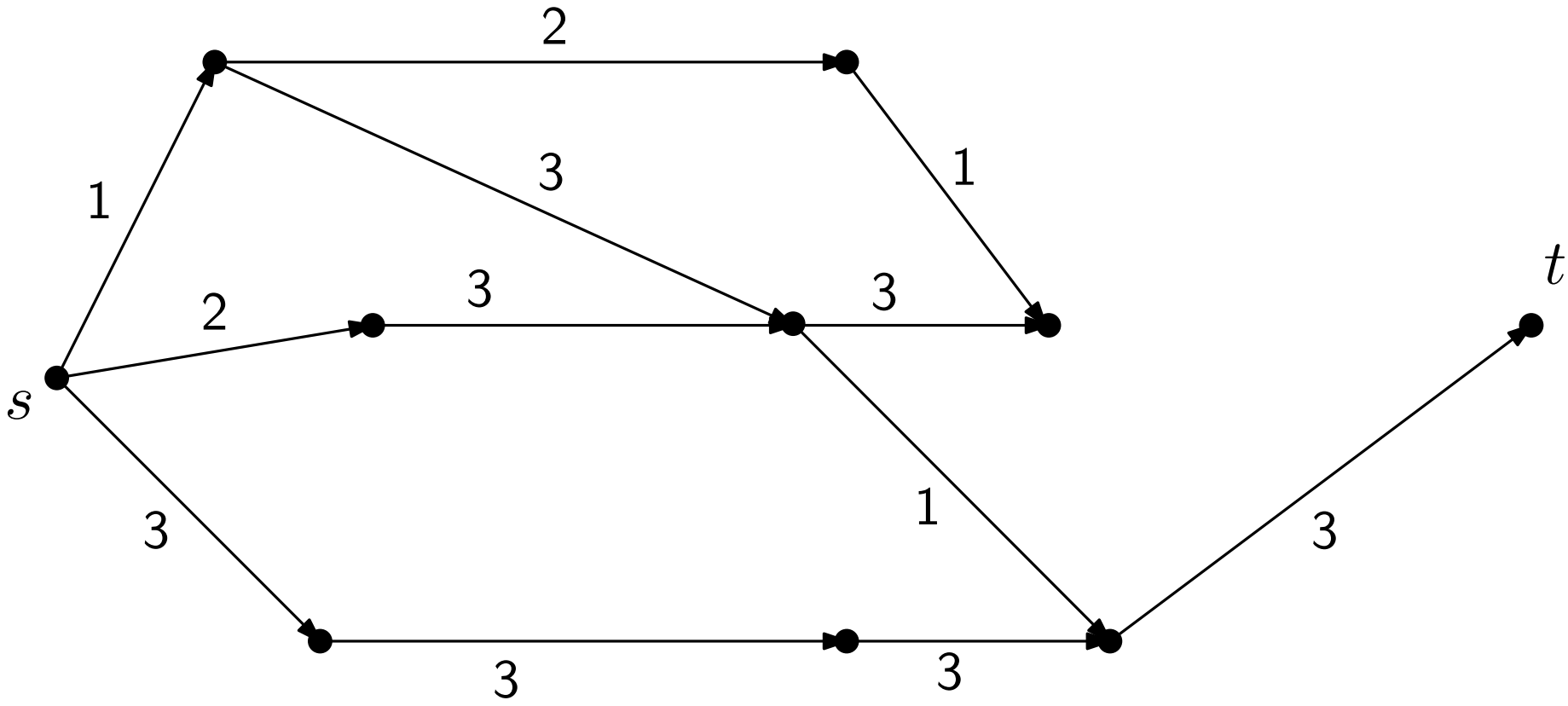

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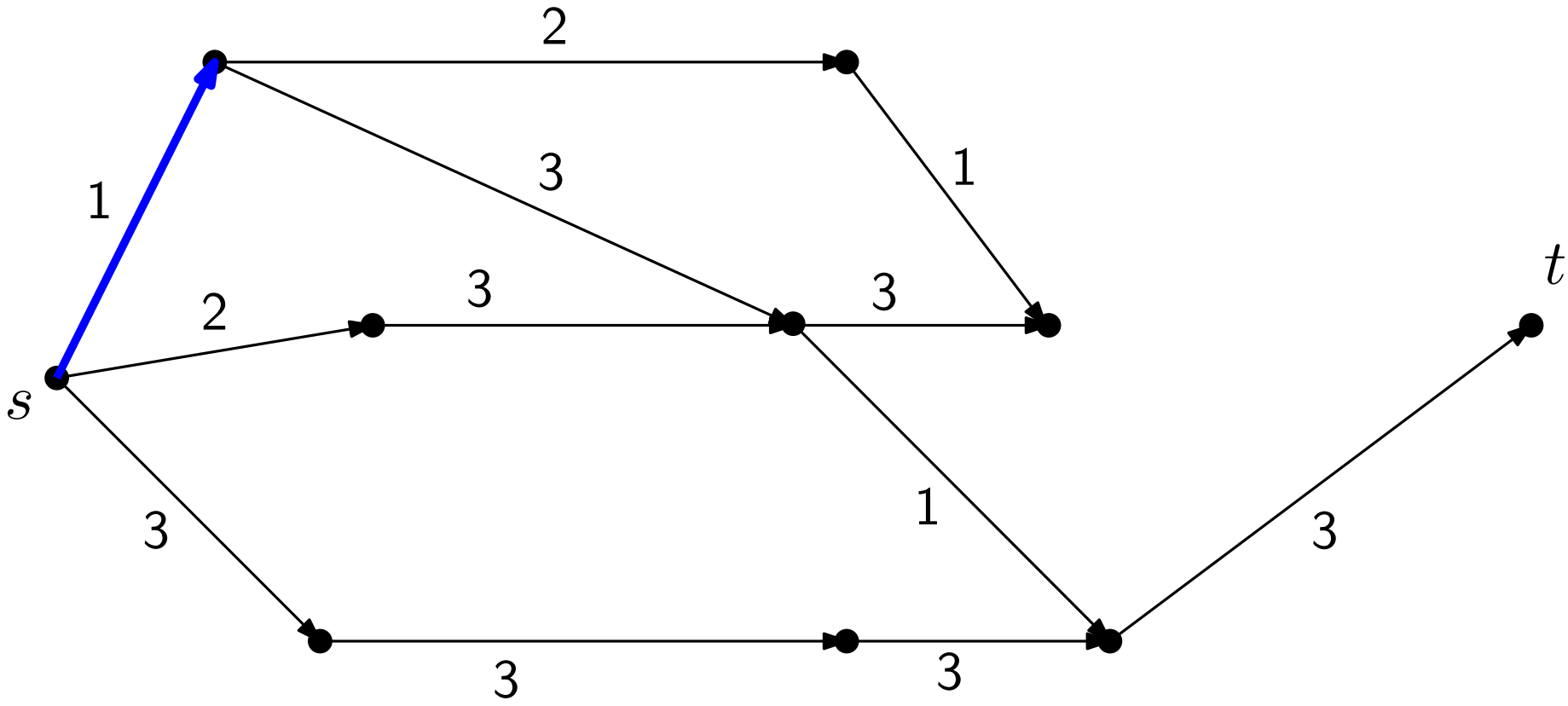

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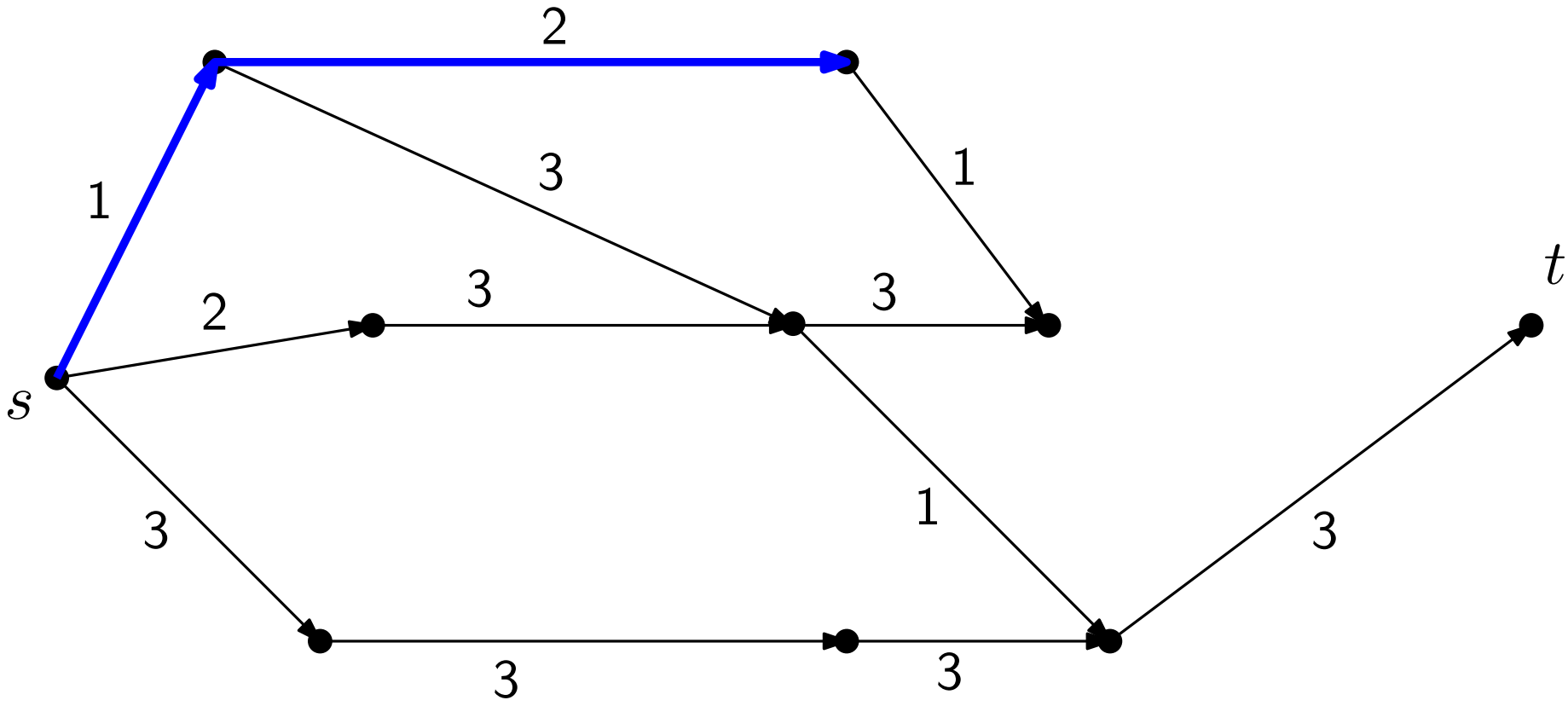

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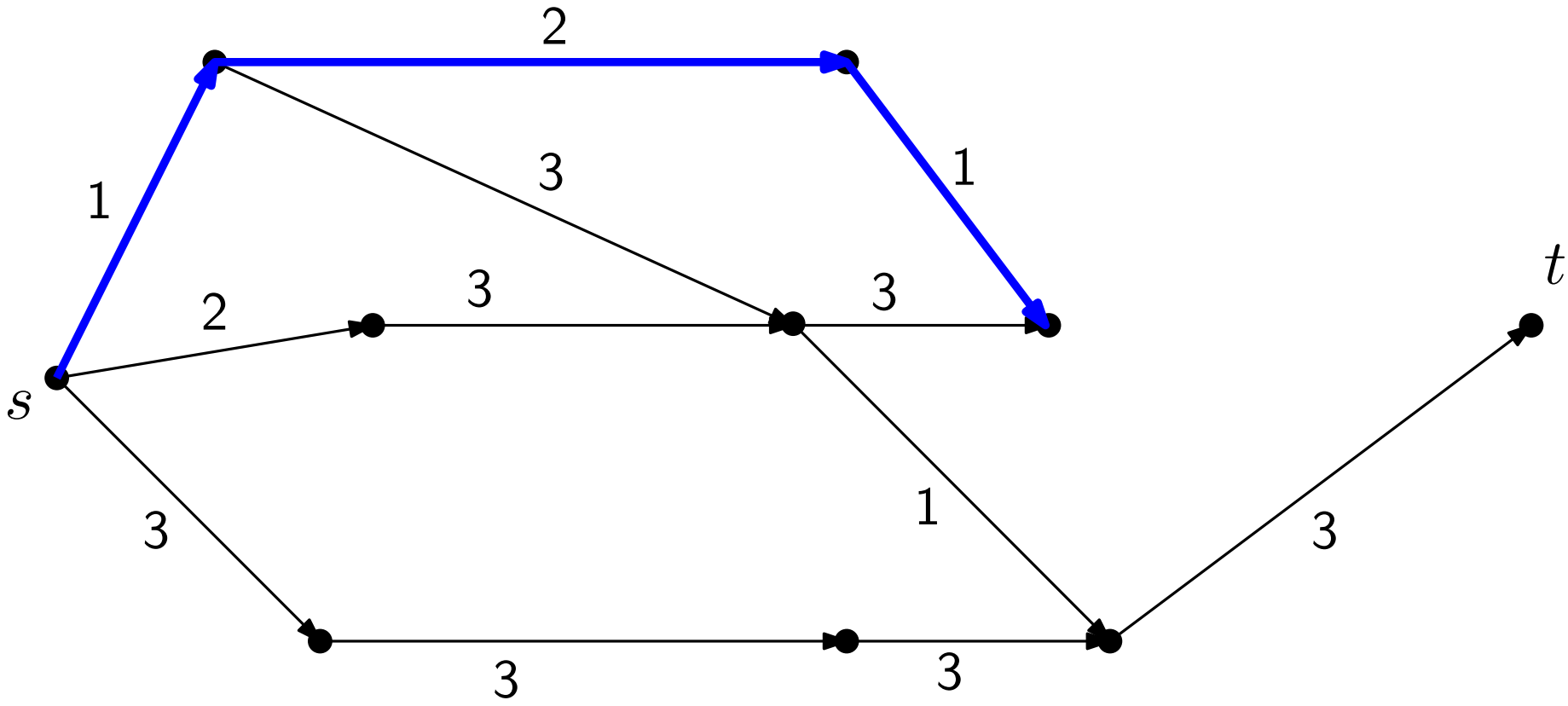

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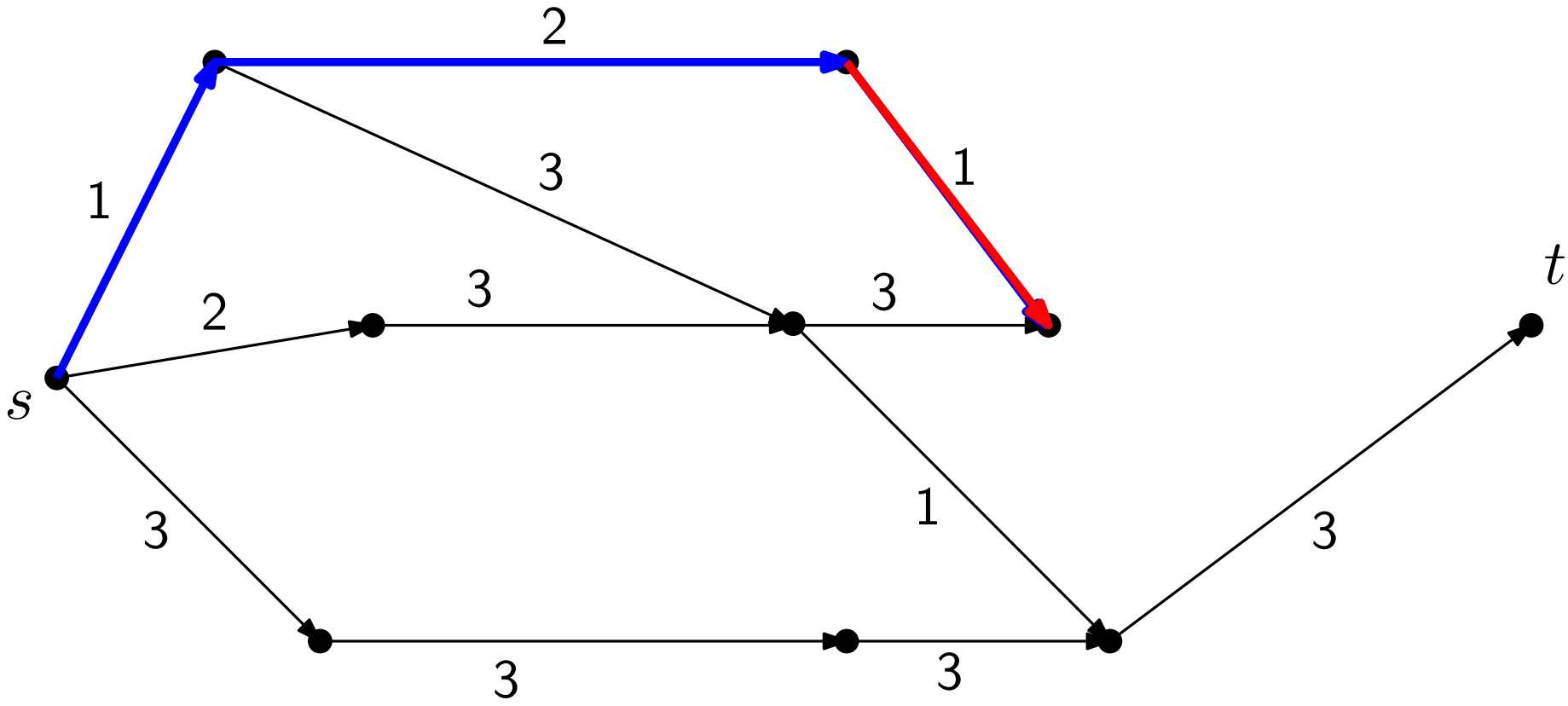

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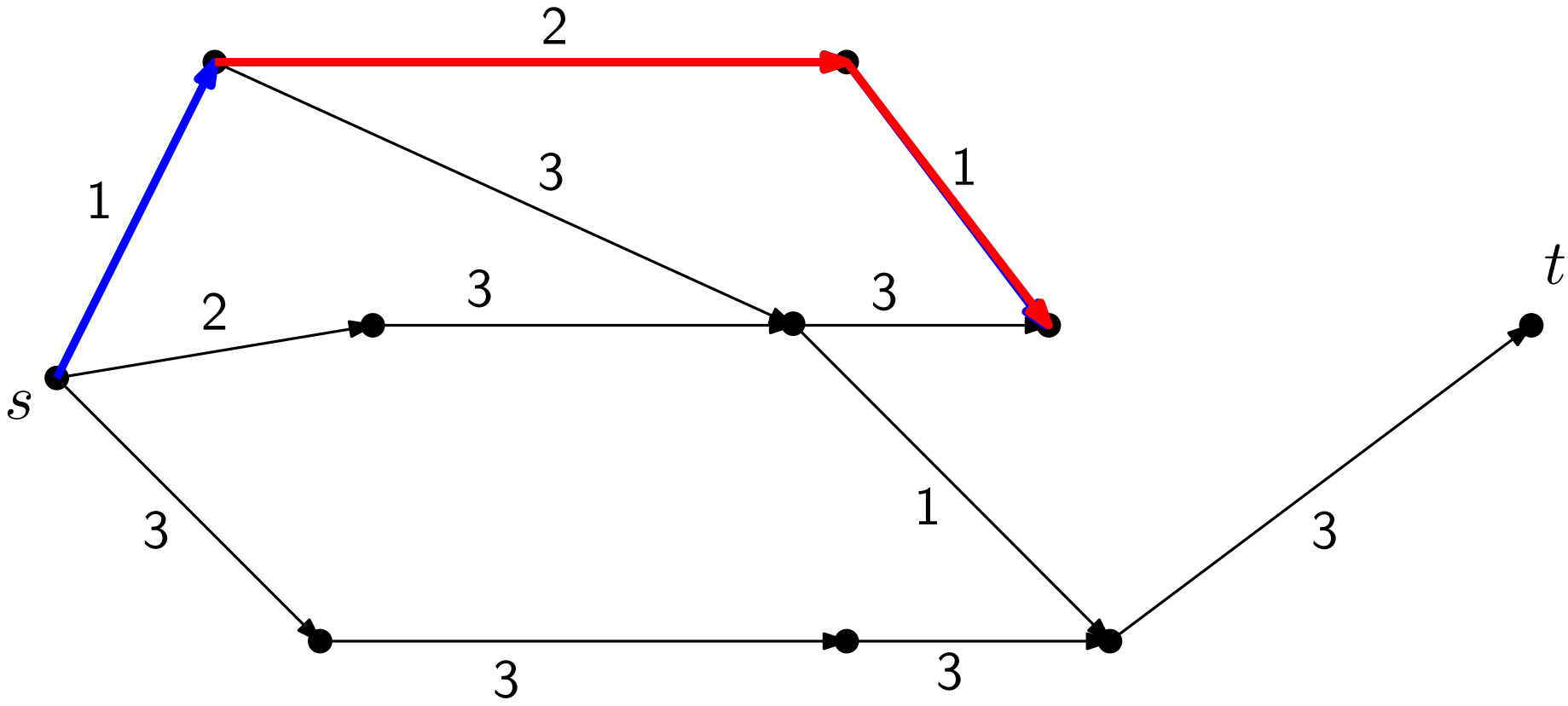

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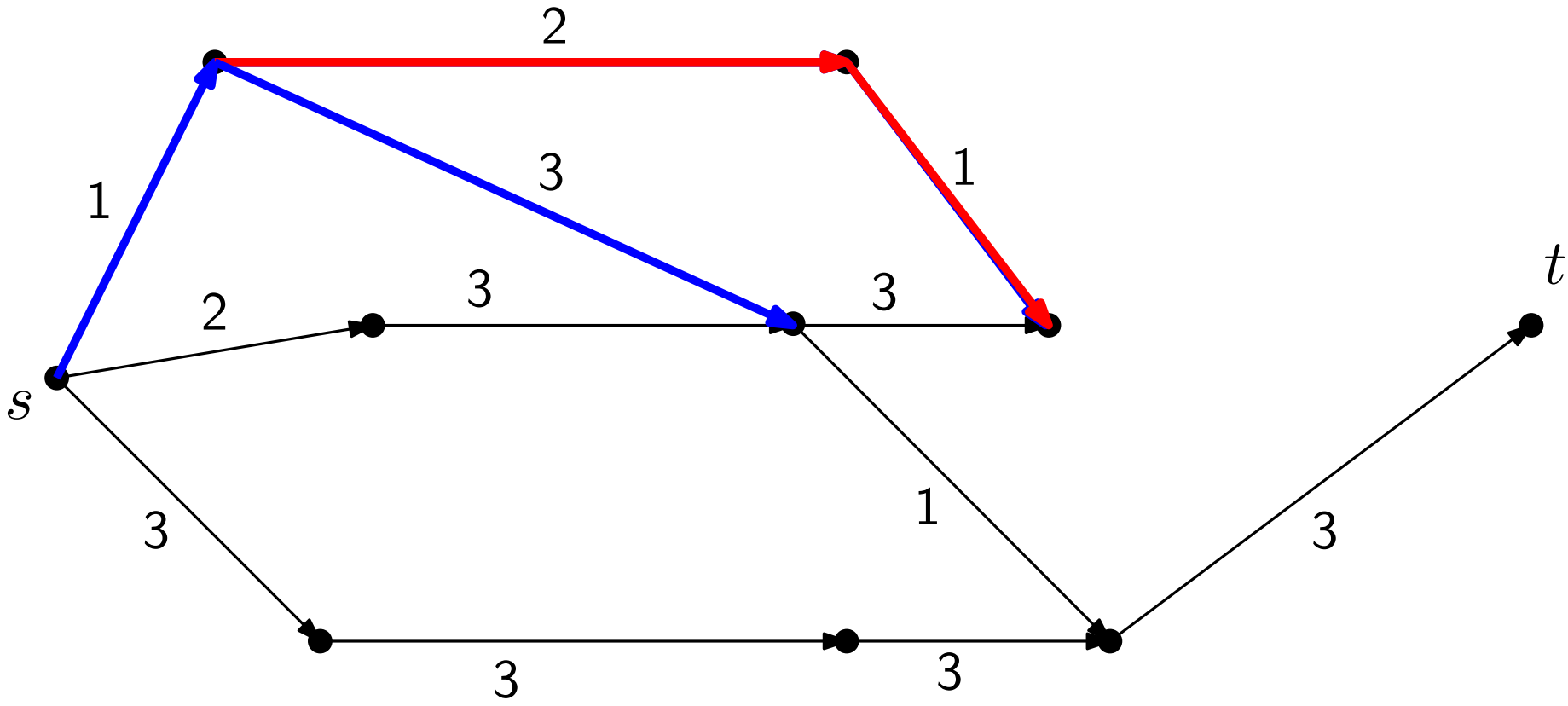

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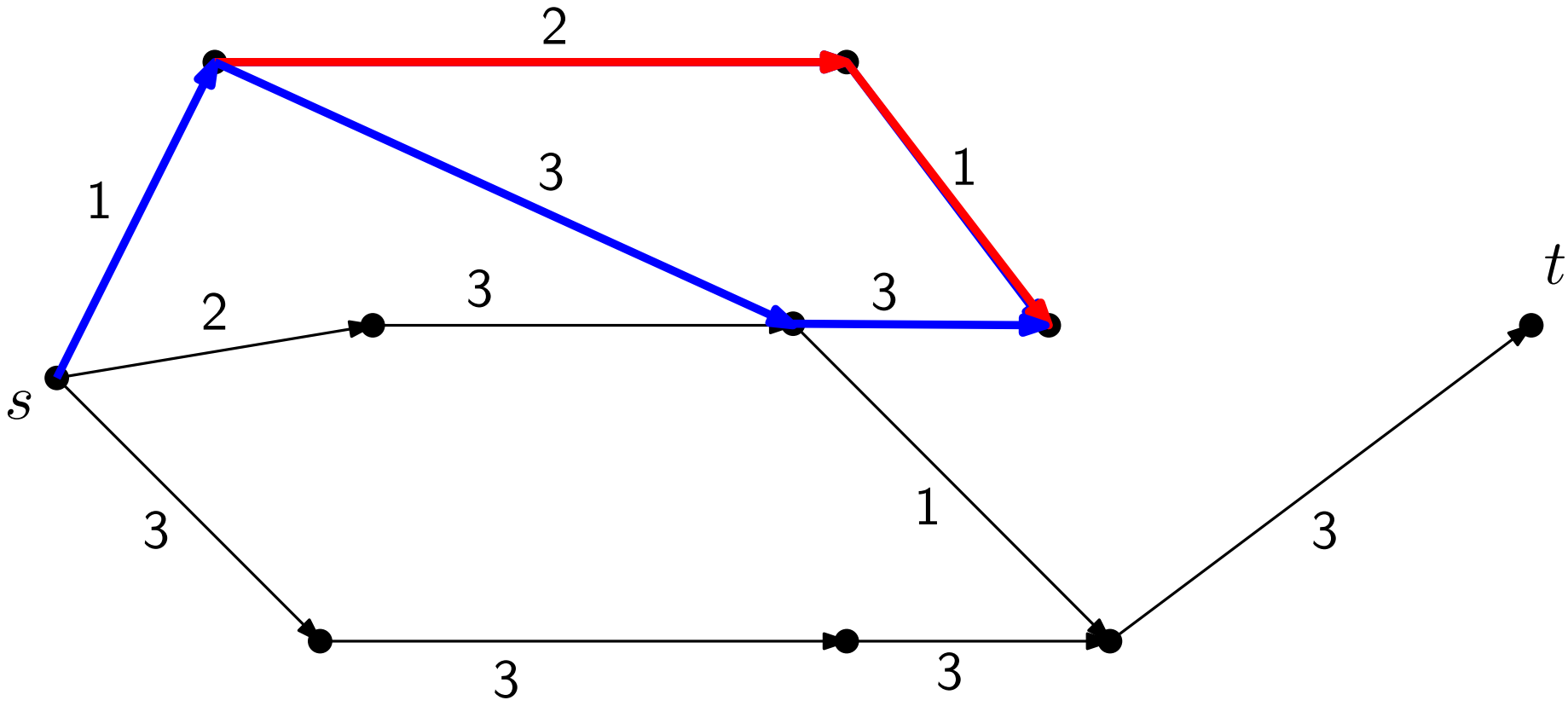

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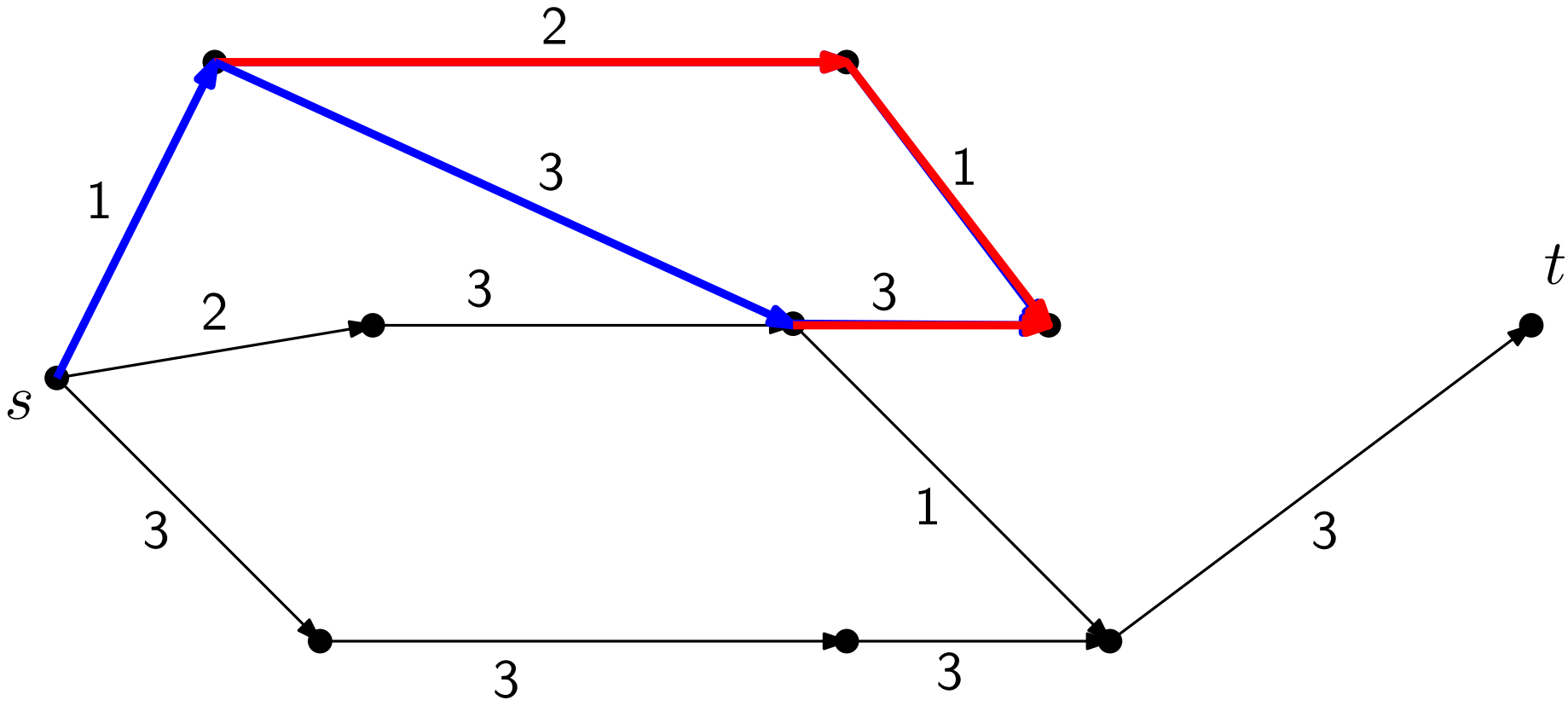

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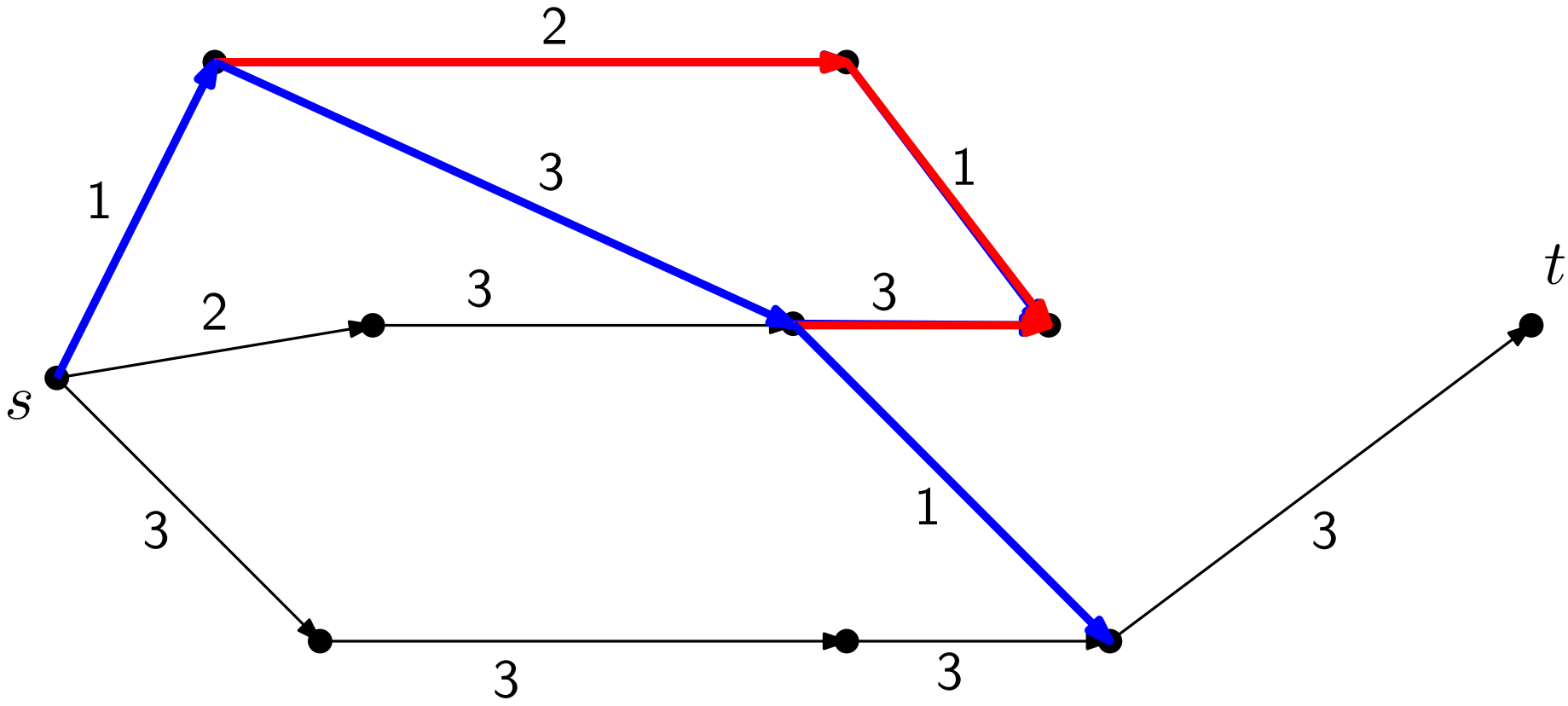

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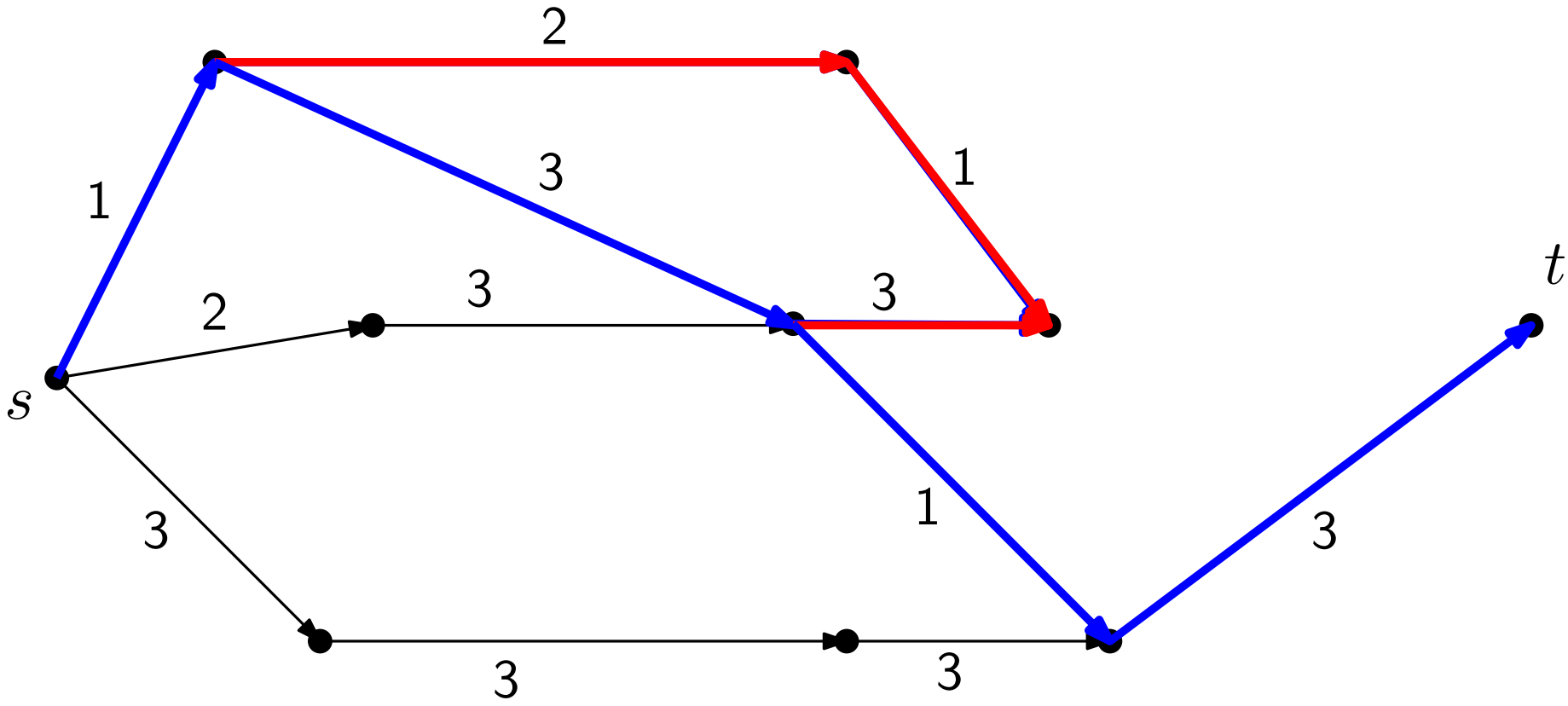

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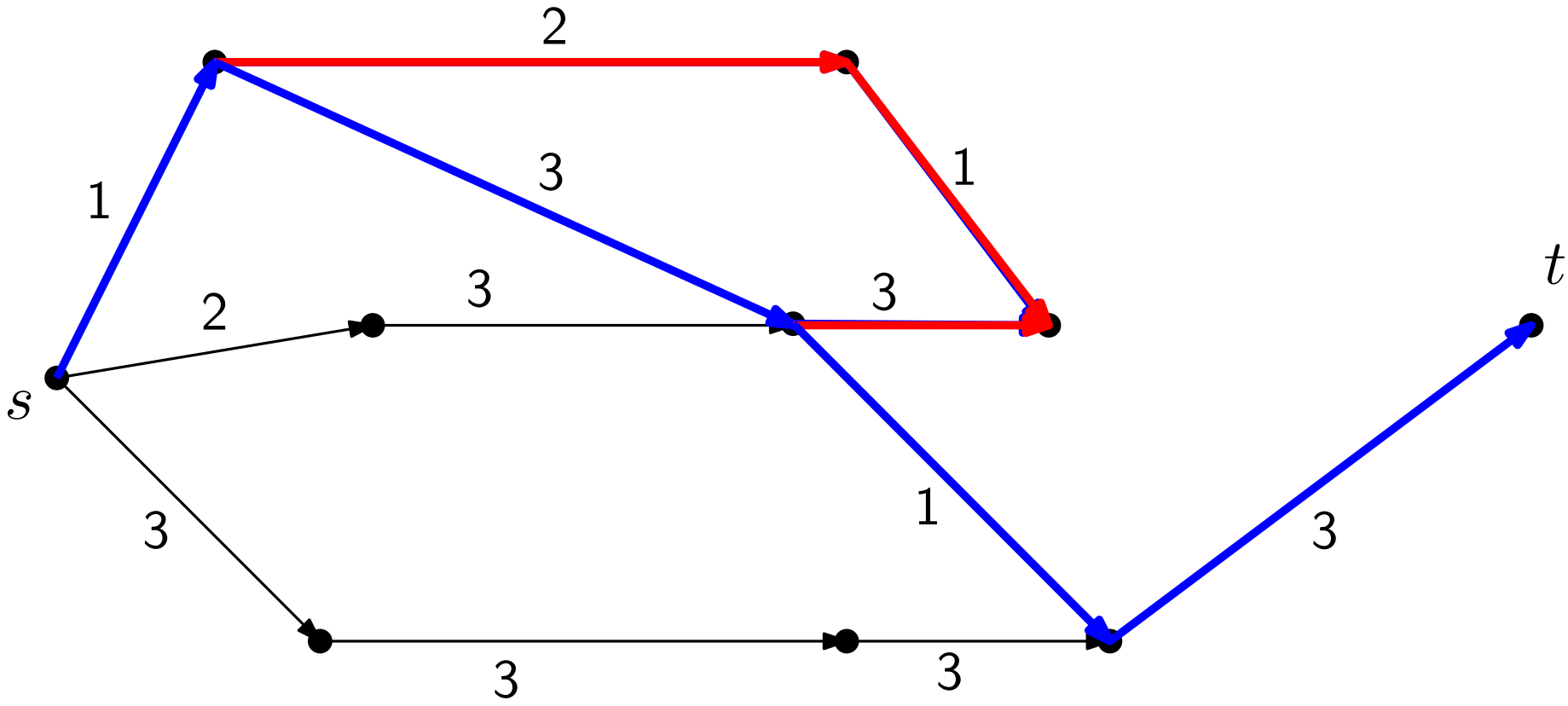

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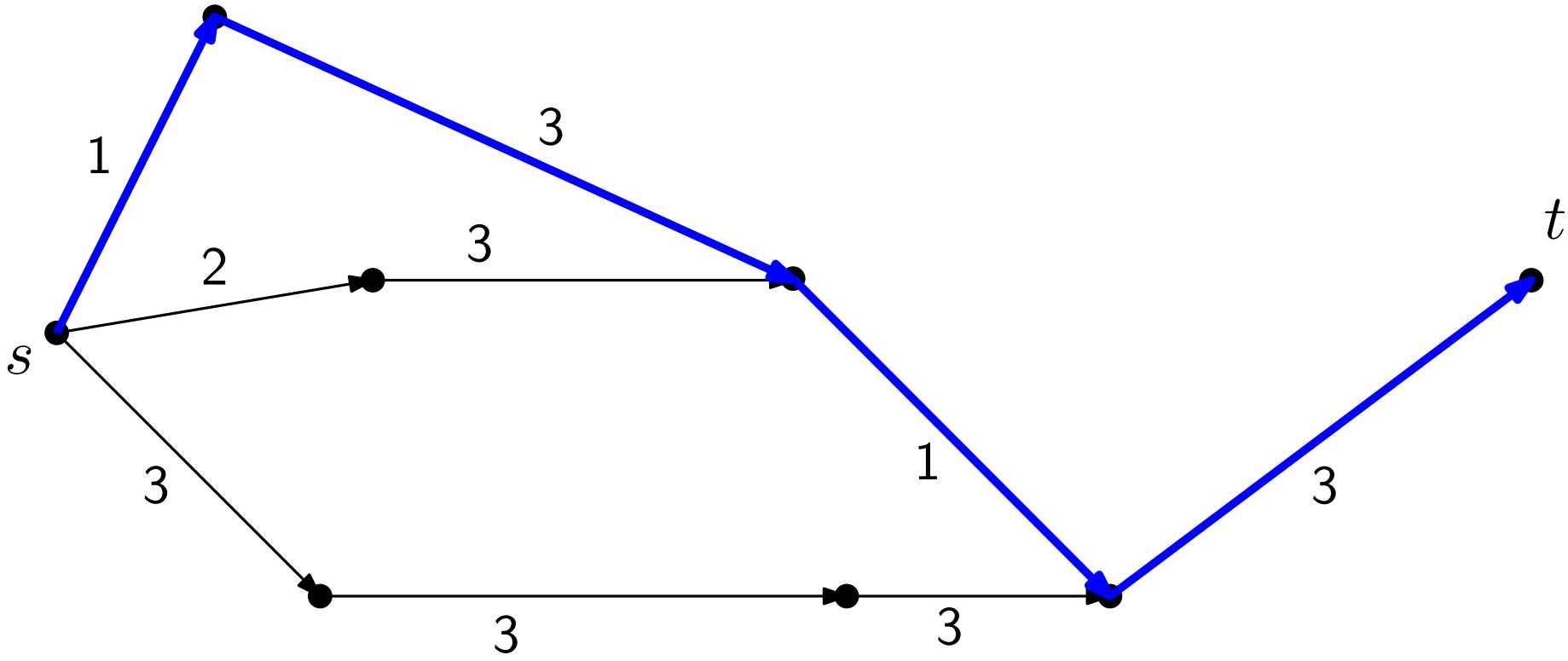

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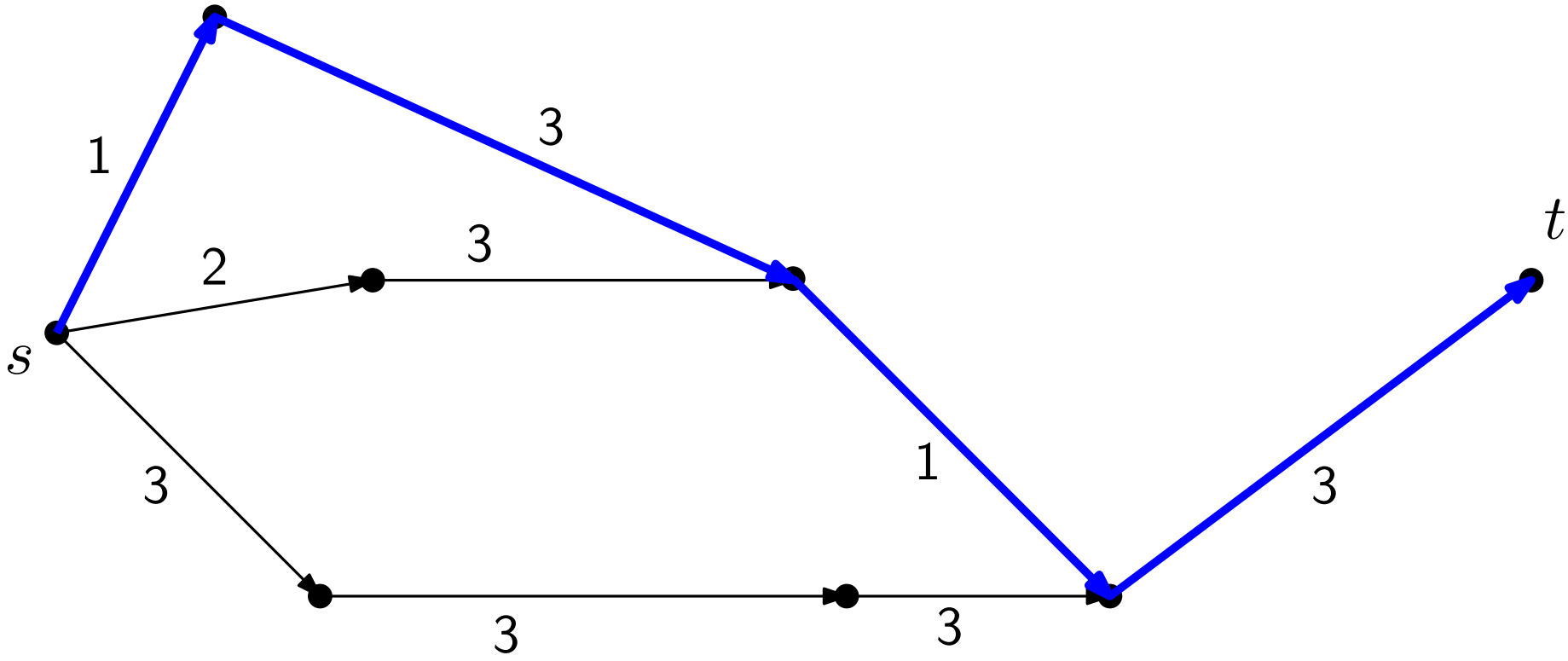

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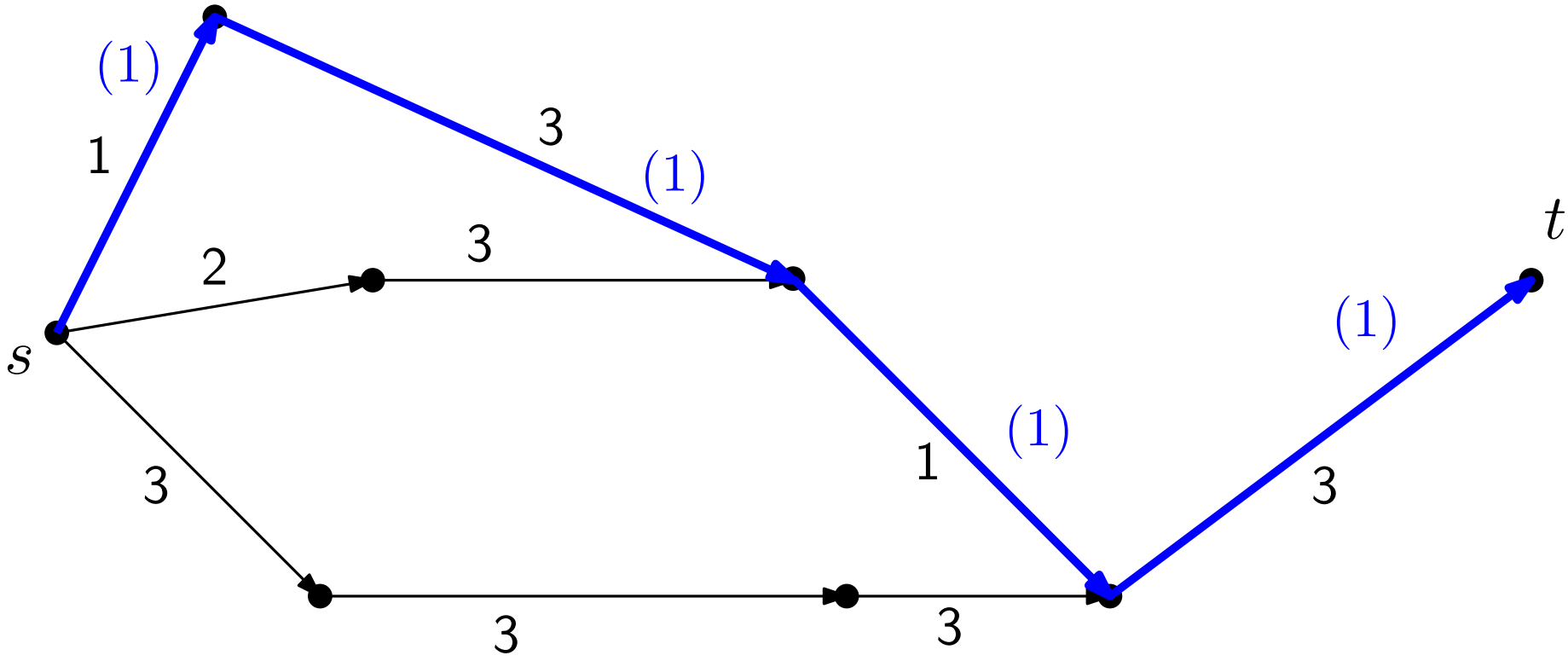
Route as much flow as possible.



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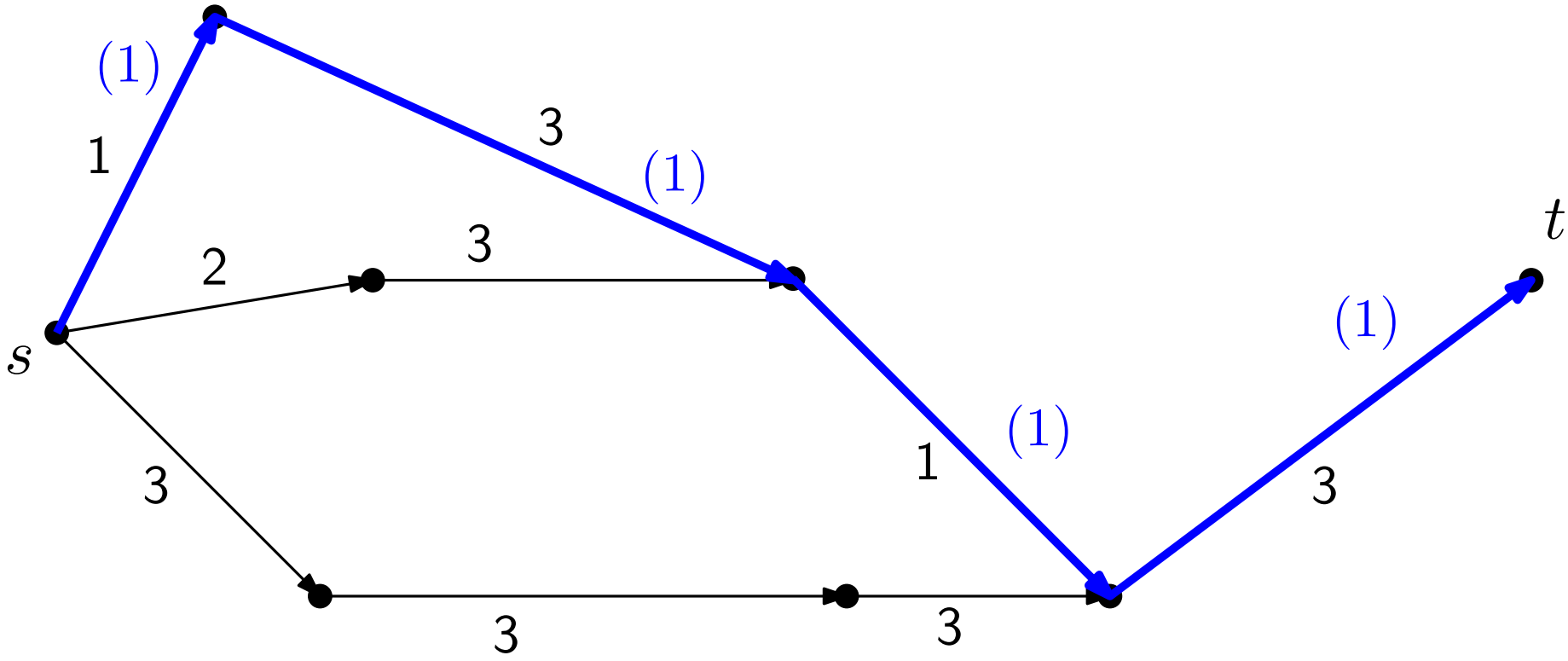

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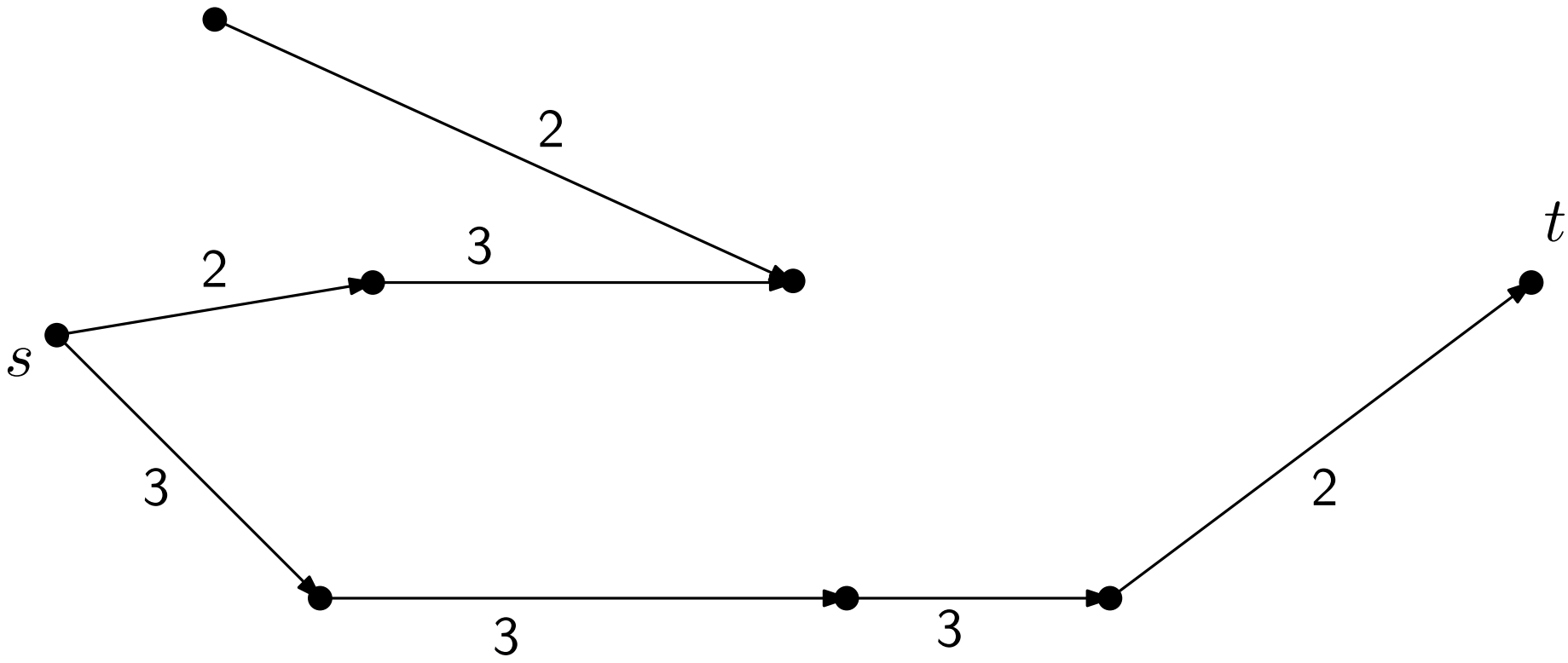
Route as much flow as possible.
Update capacities.



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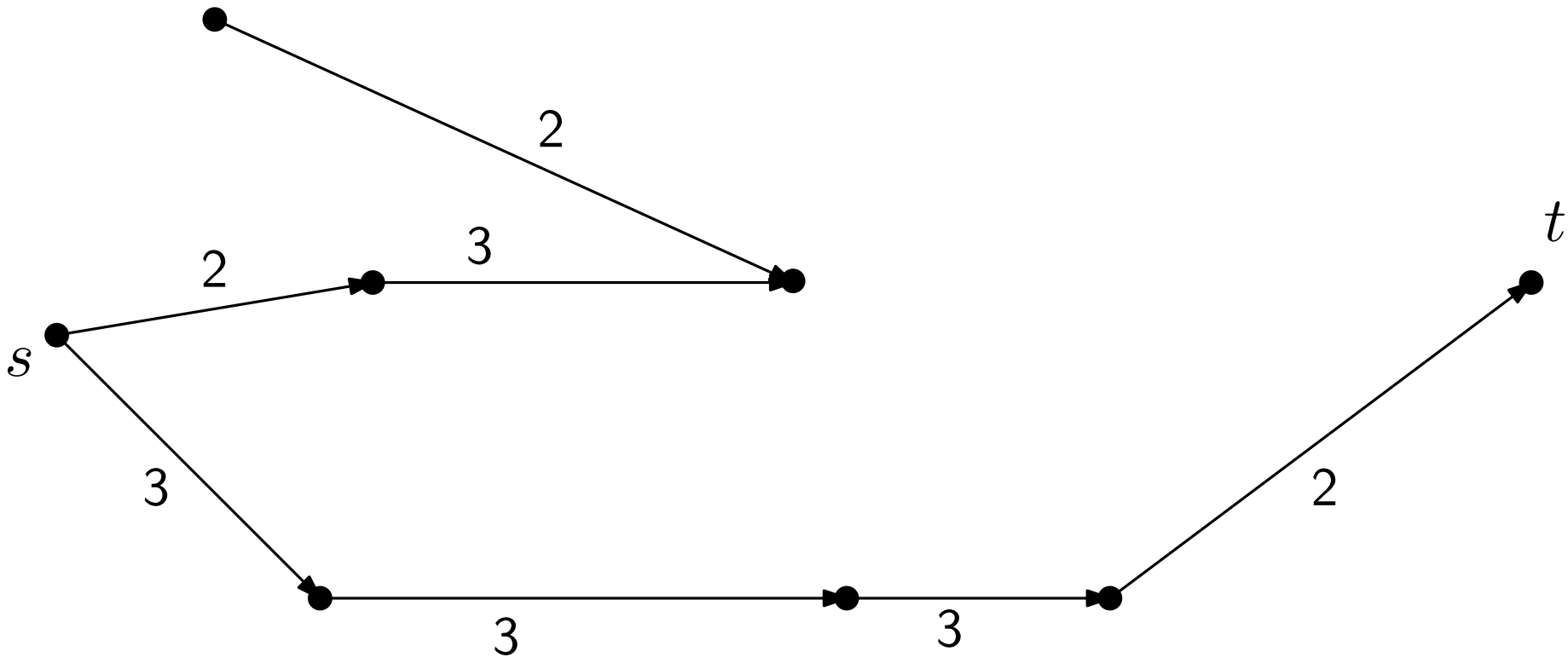

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Delete dead ends.
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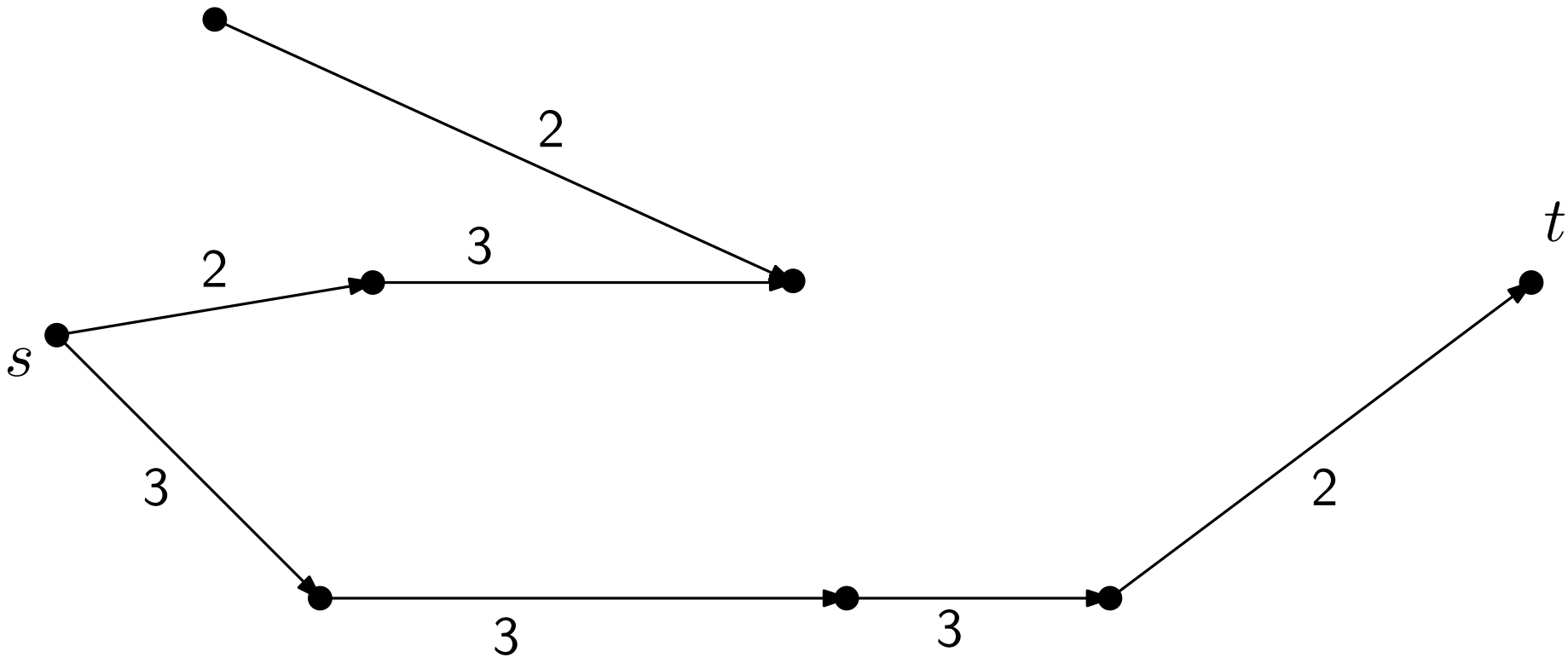
Update capacities. Repeat!



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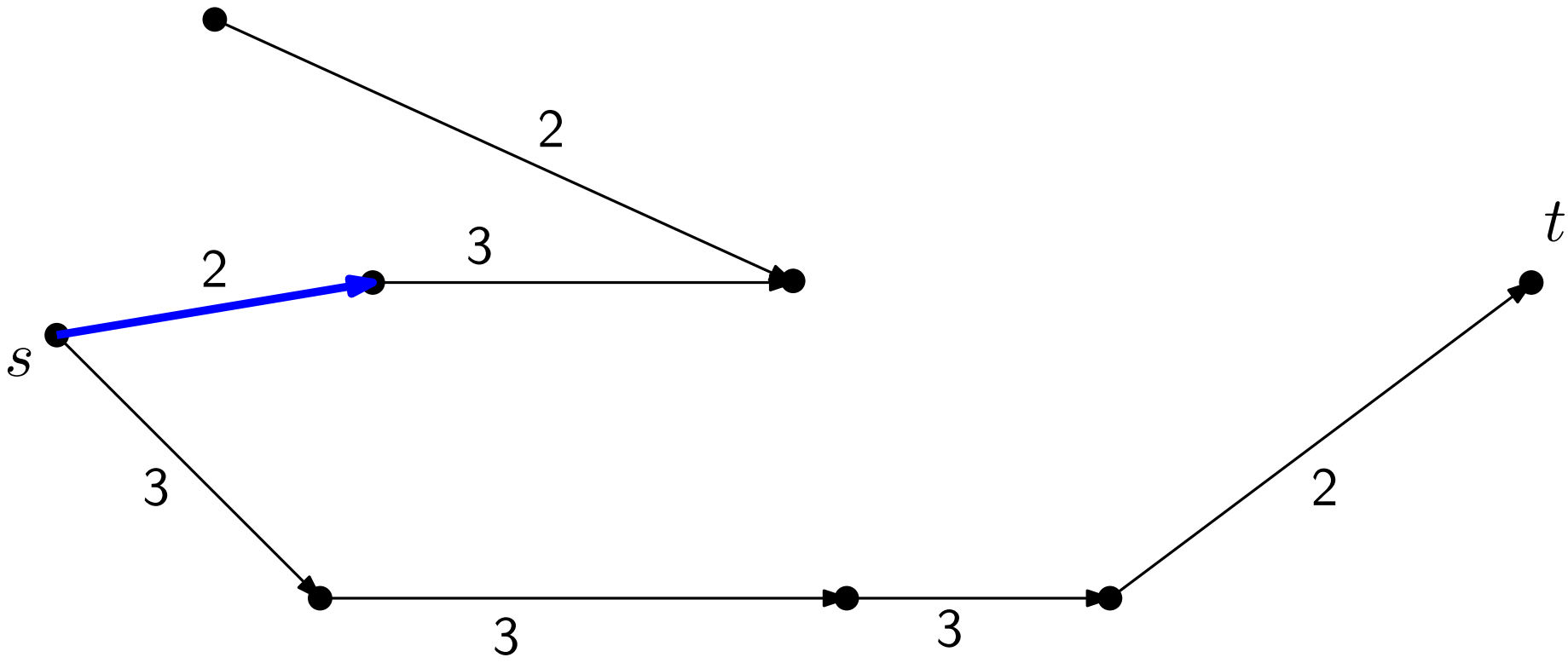

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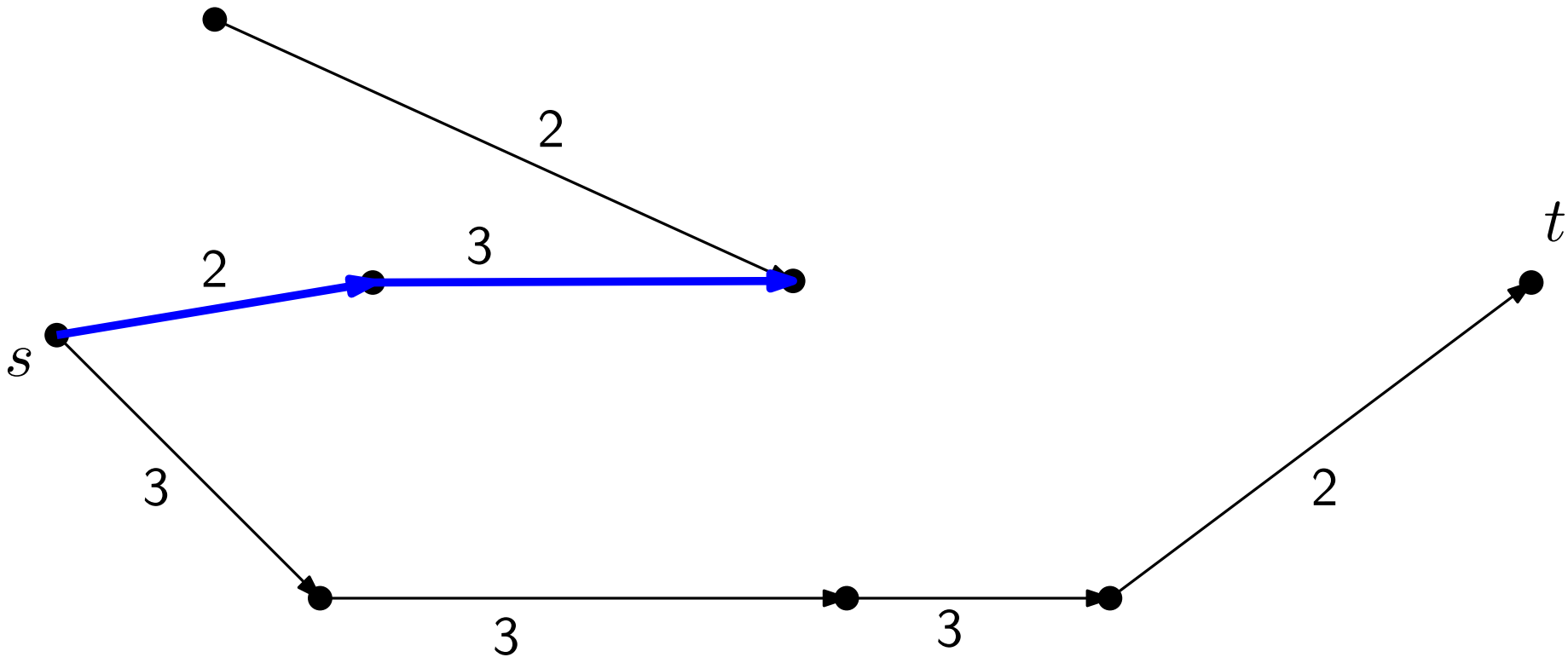

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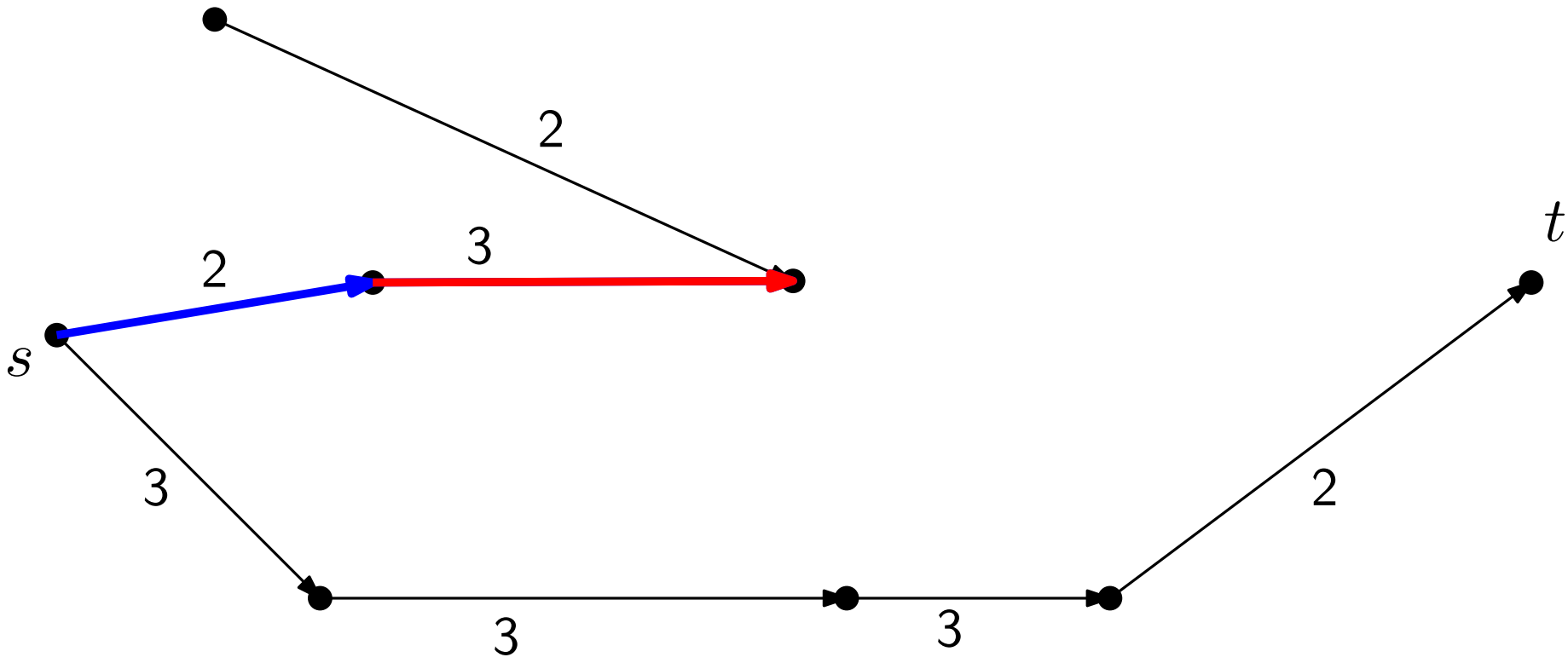

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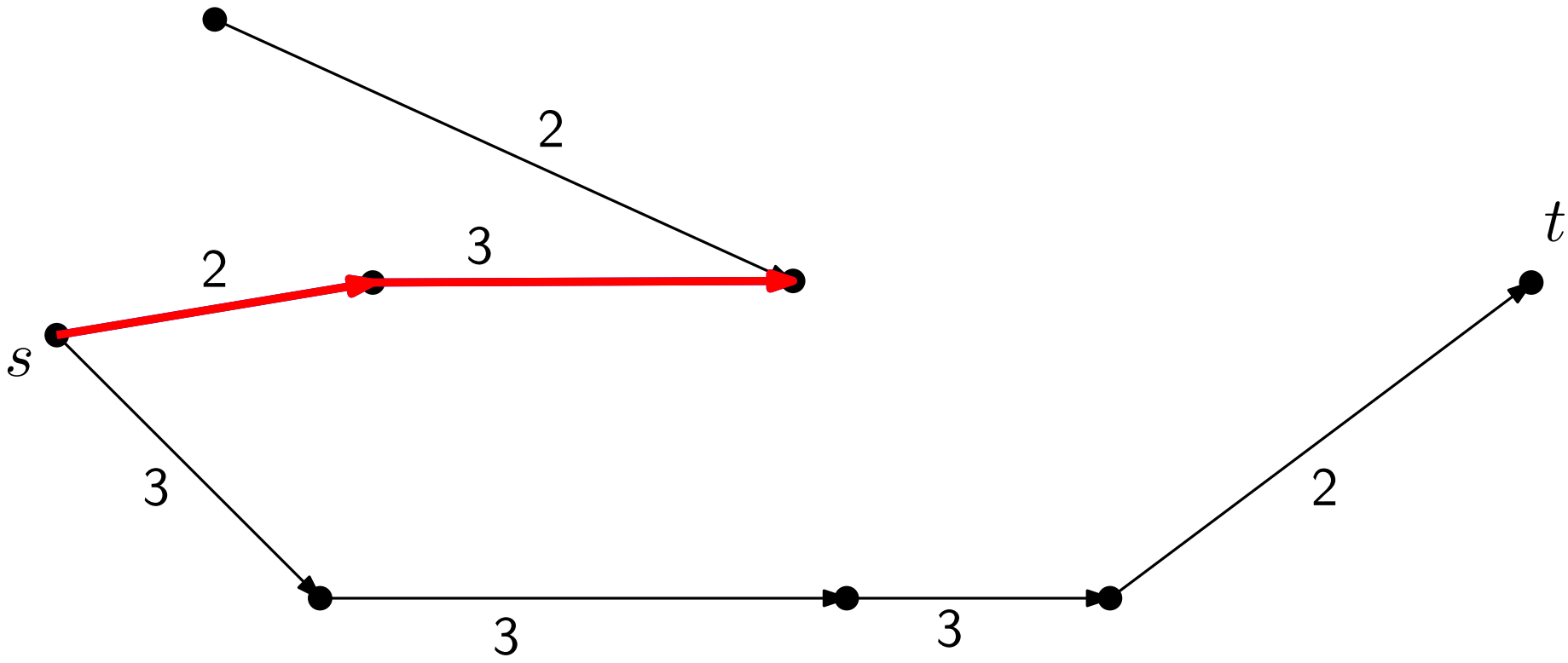

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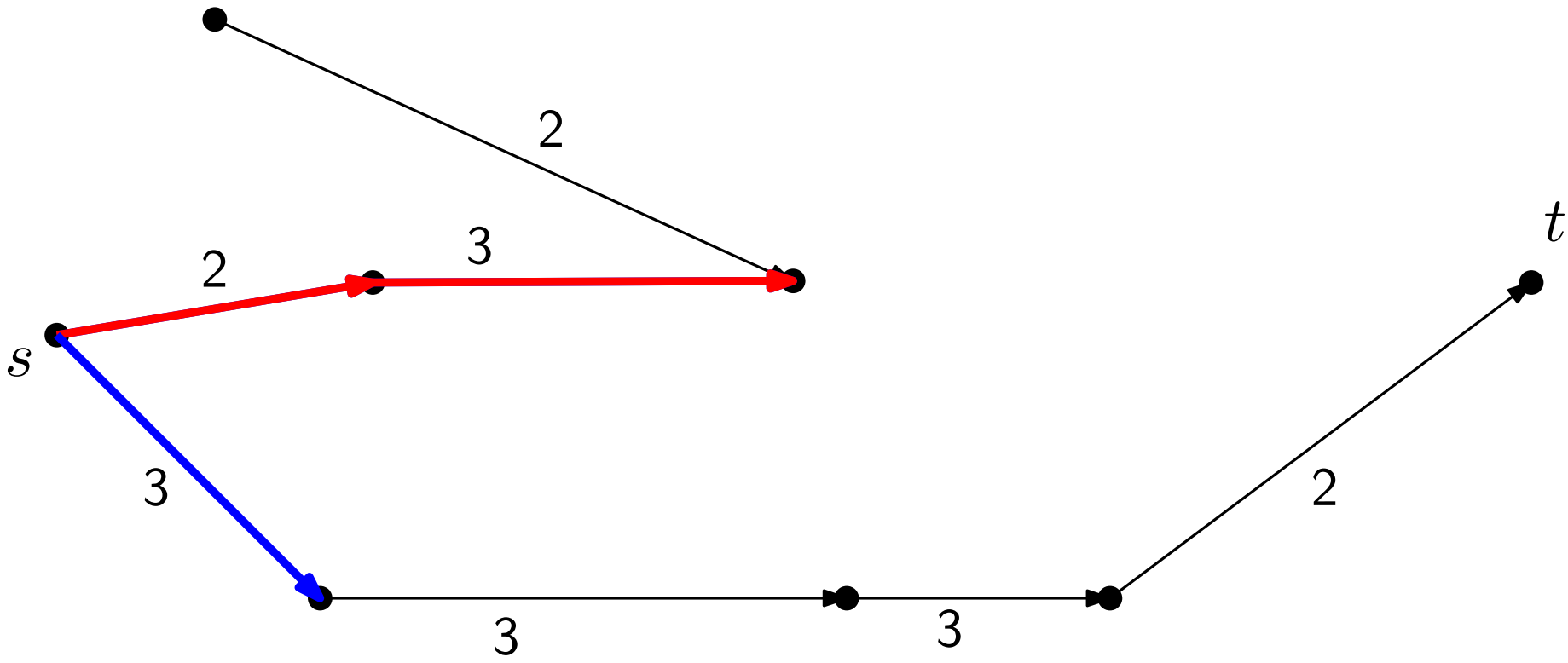

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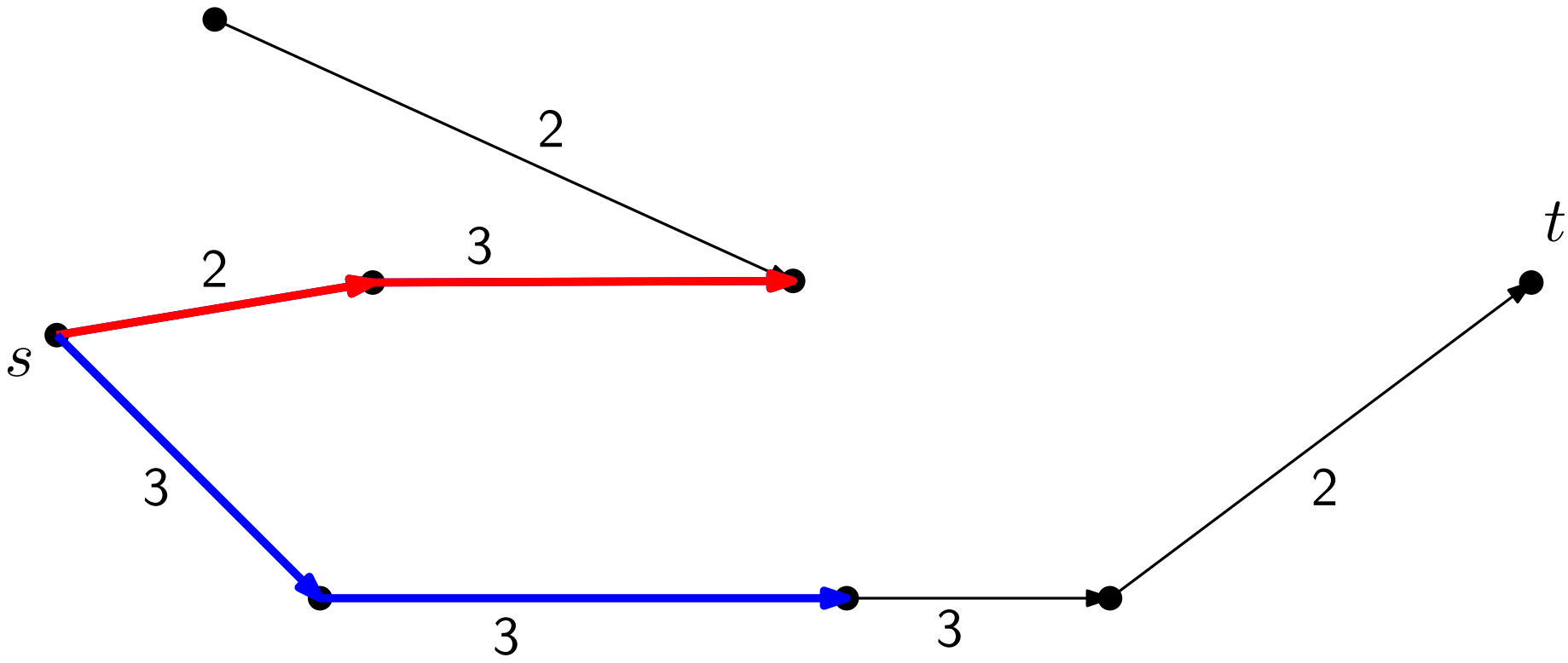

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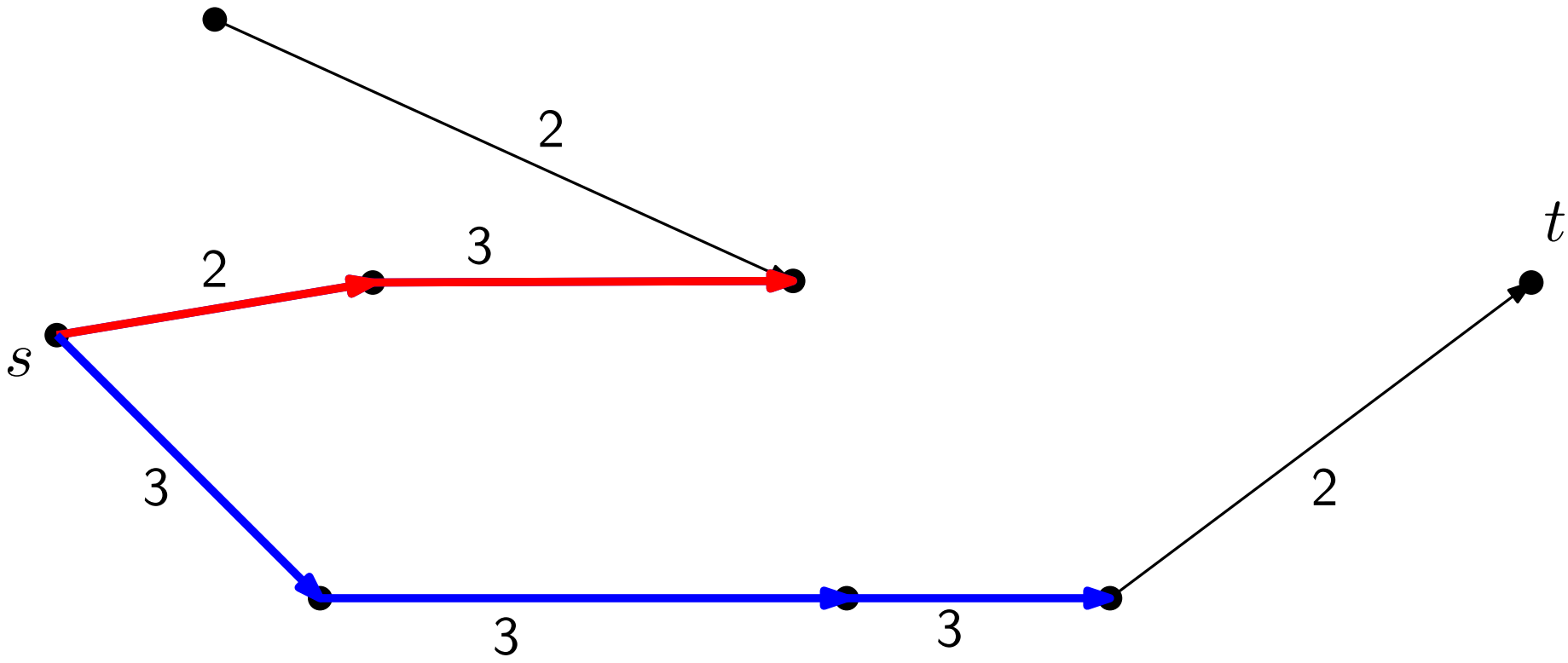

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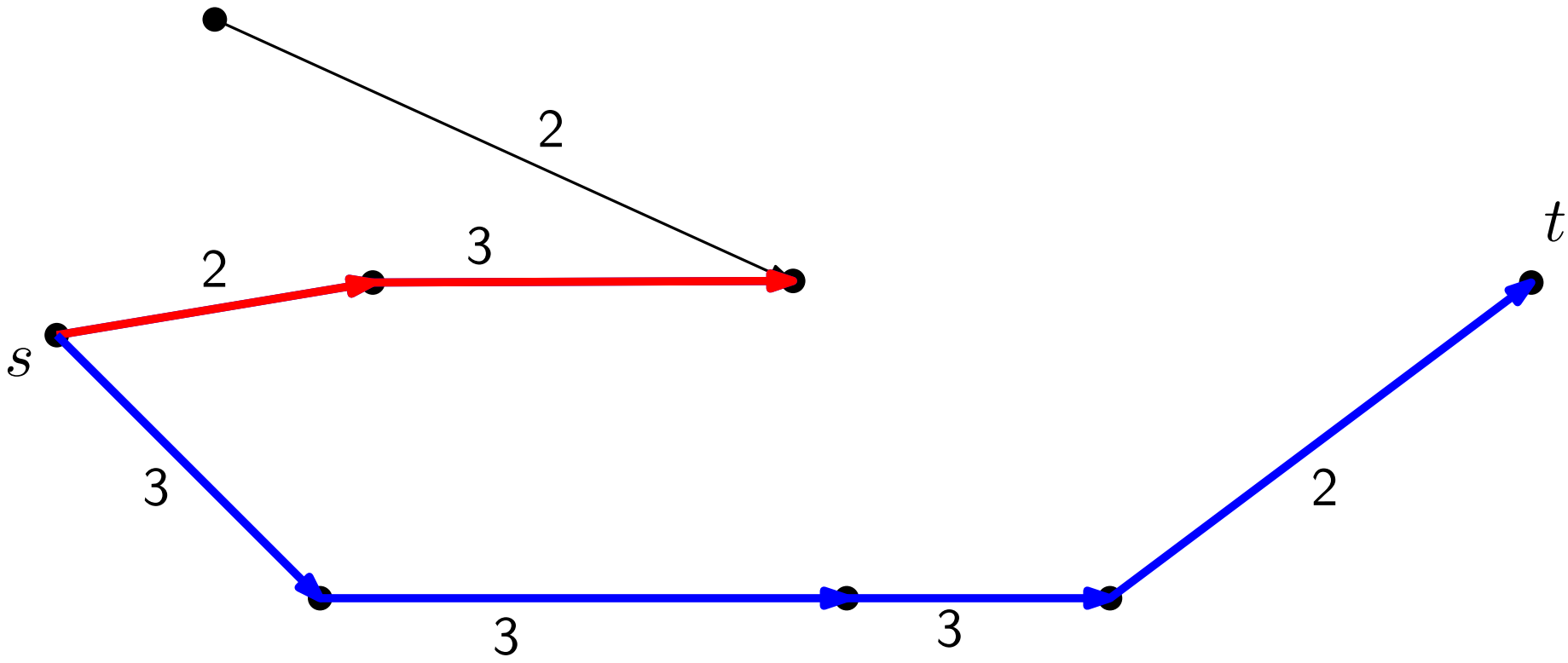
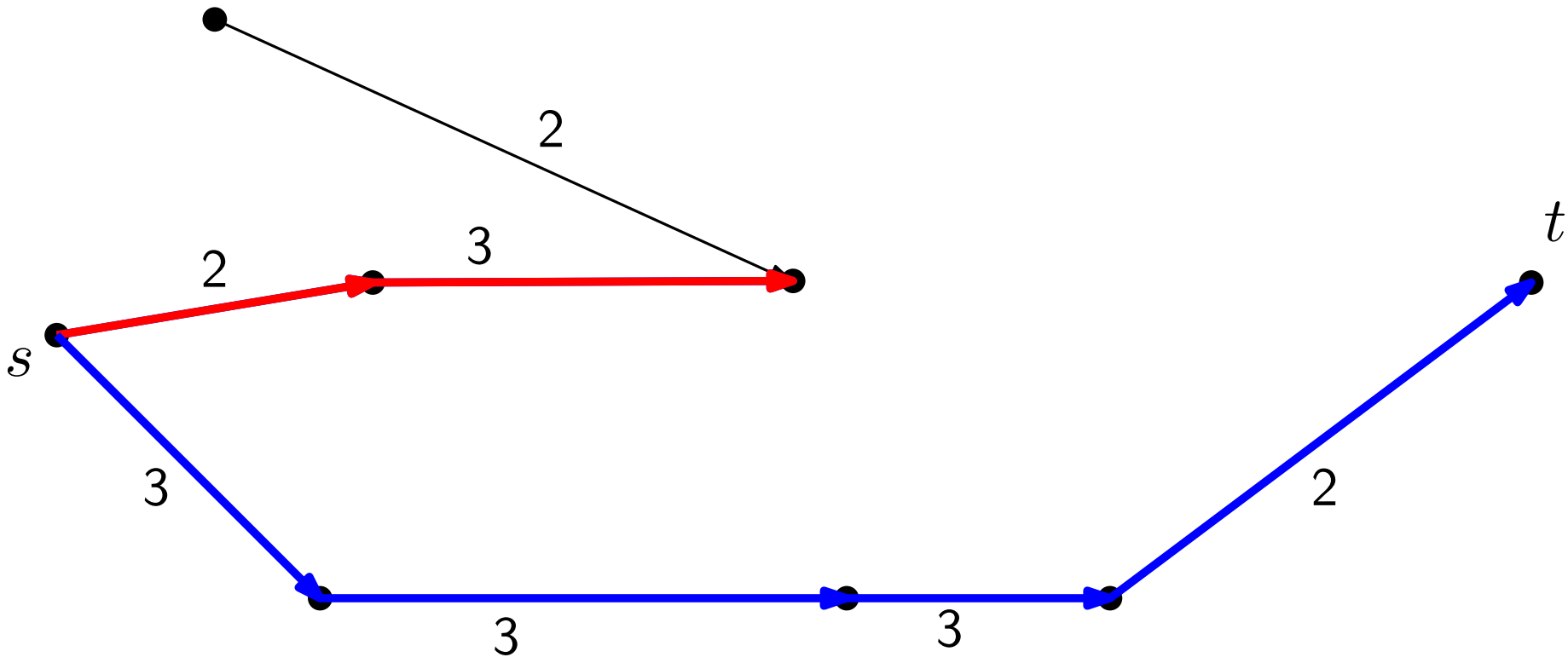
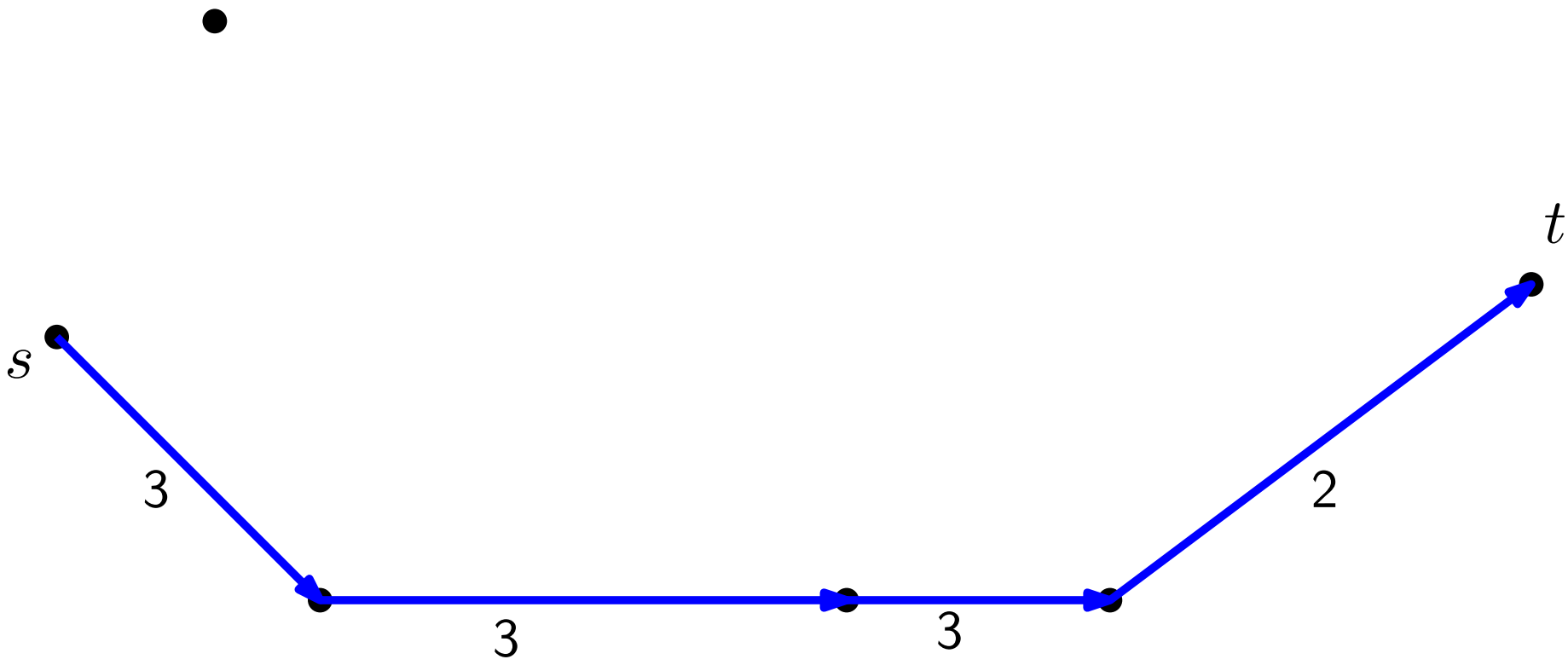


Find an s-t-path with depth-first search.



Find an s-t-path with depth-first search.
Delete dead ends.



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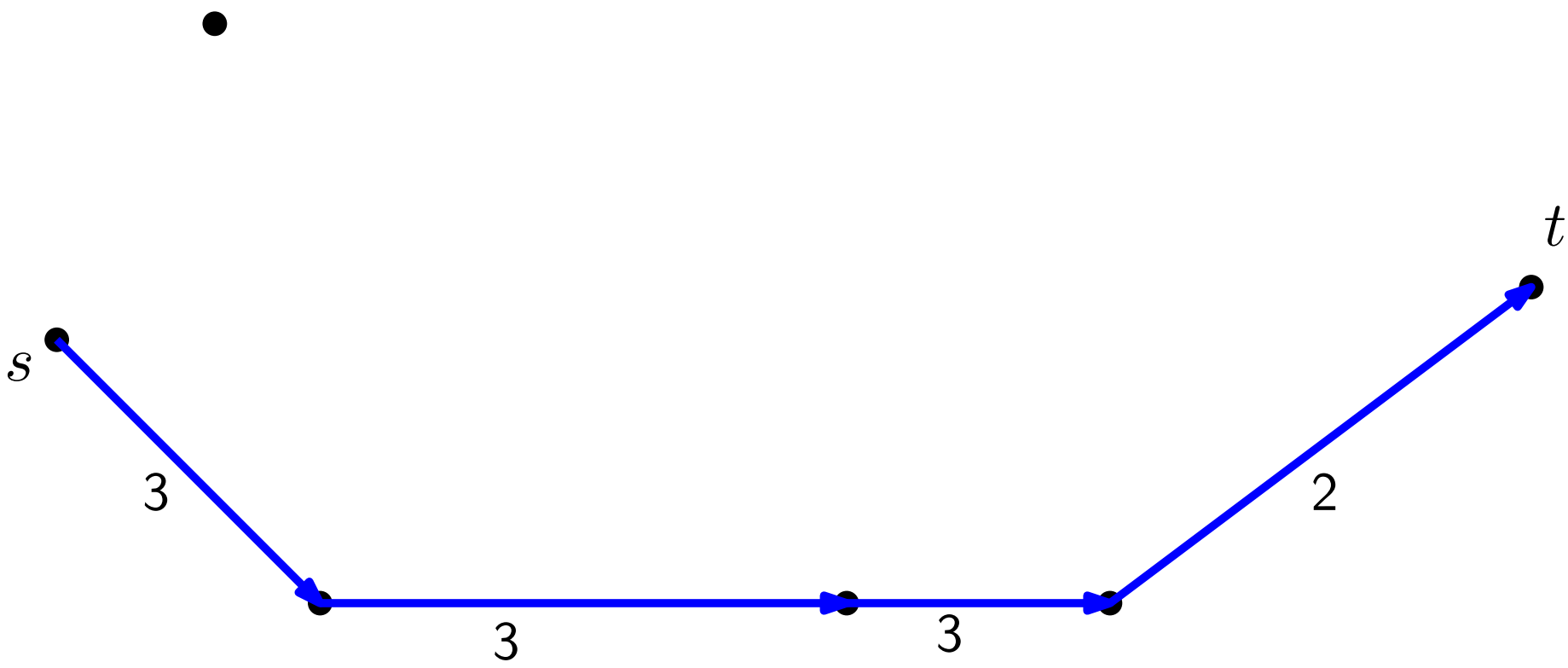
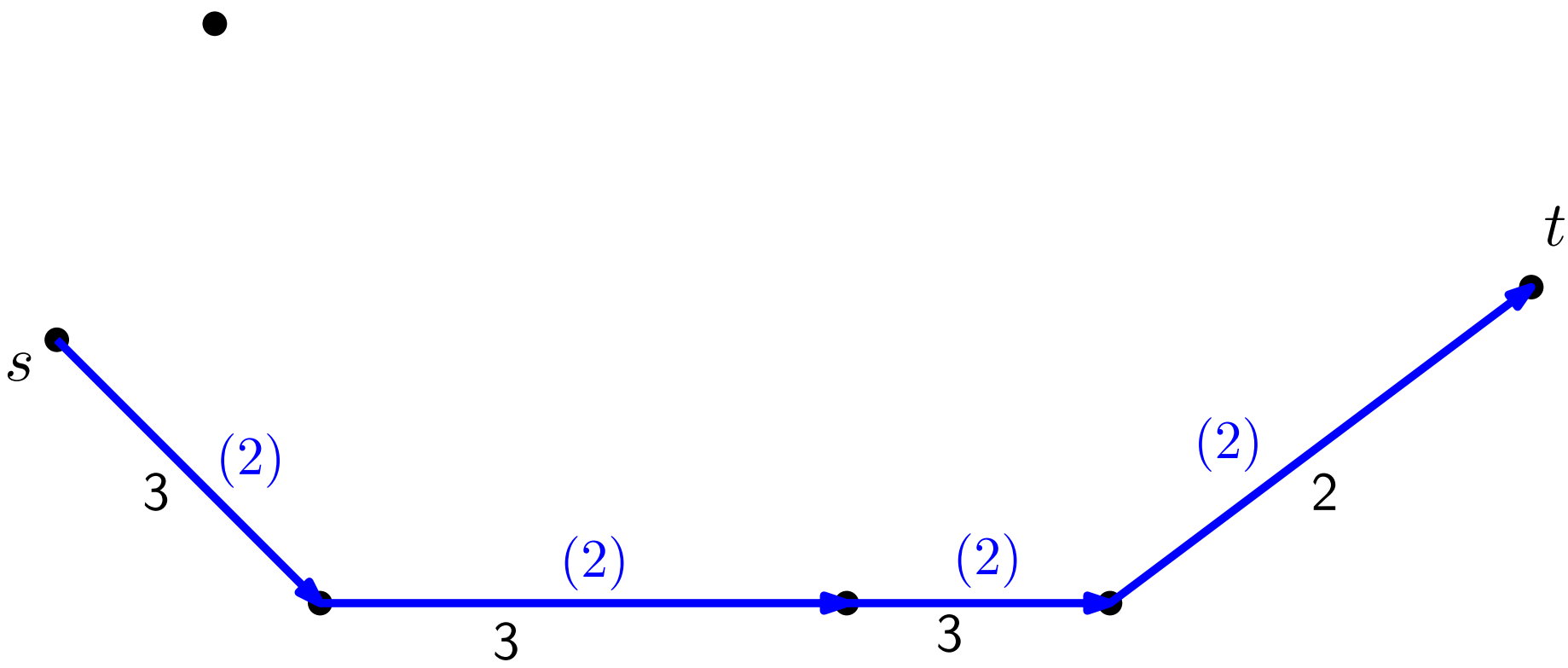


Find an s-t-path with depth-first search.
Delete dead ends.
Route as much flow as possible.



Find an s-t-path with depth-first search.
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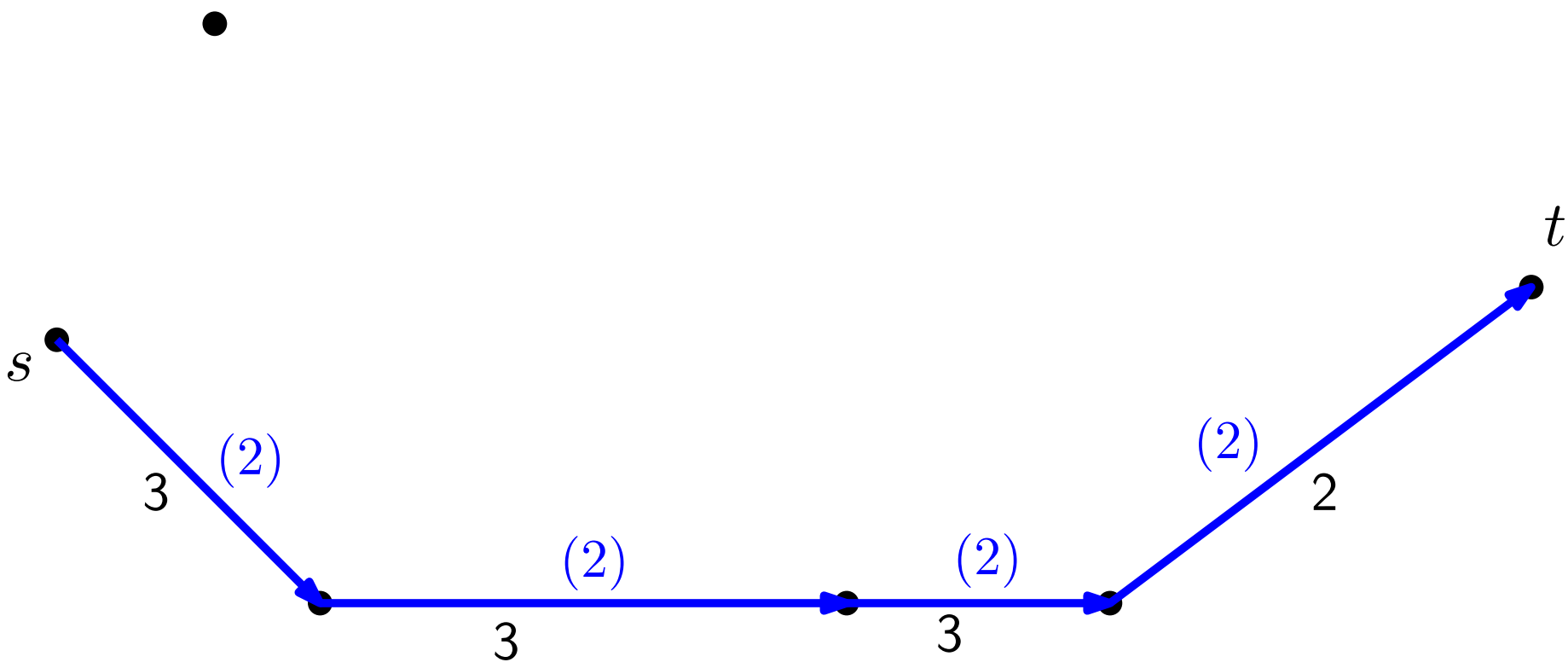
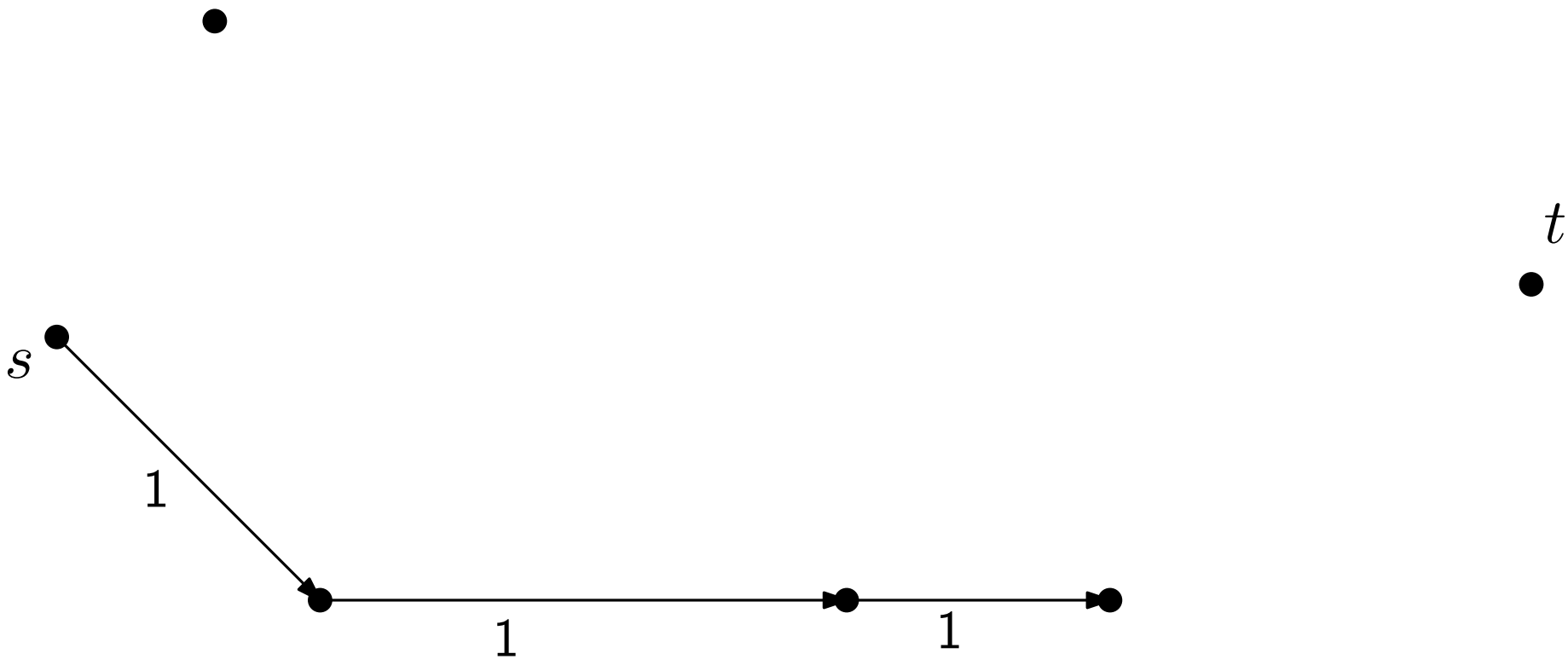
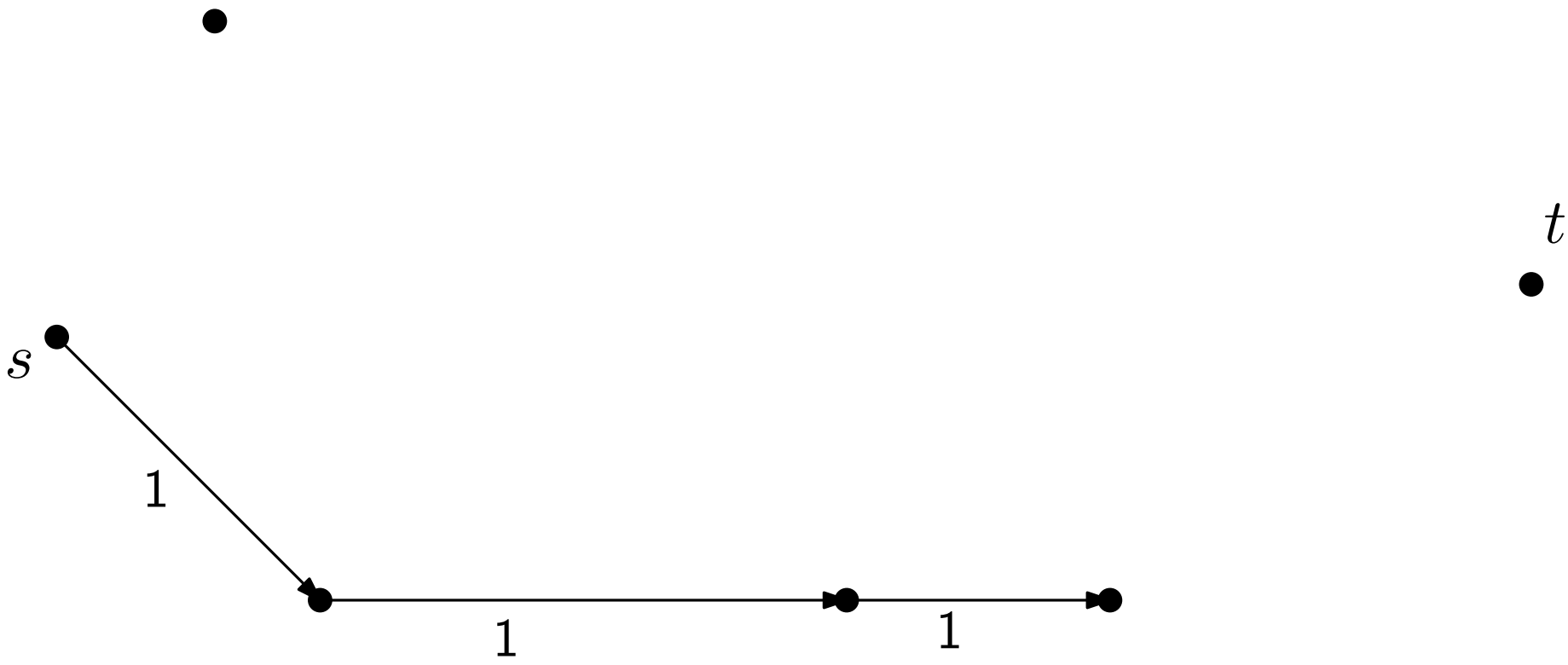


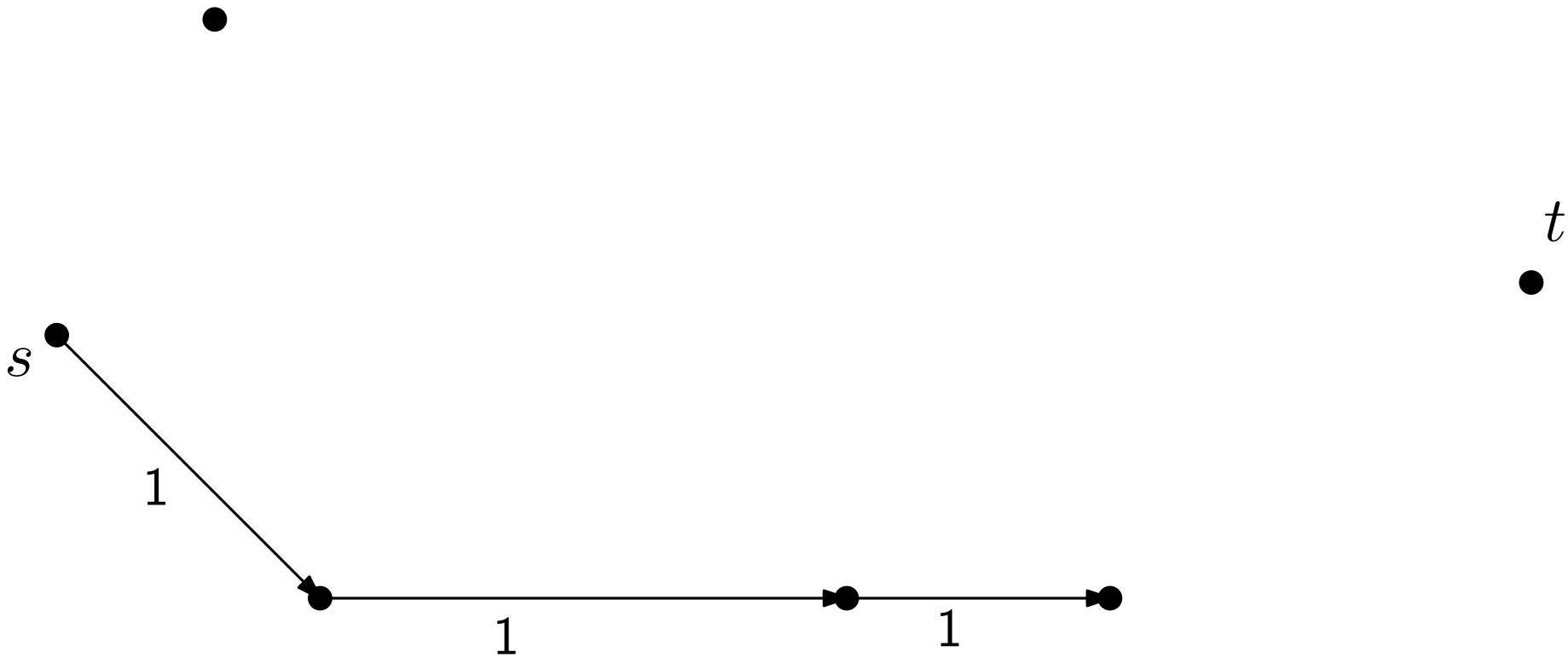
Find an s-t-path with depth-first search.
Delete dead ends.
Route as much flow as possible.
Update capacities.



Find an s-t-path with depth-first search.
Delete dead ends.
Route as much flow as possible.
Update capacities.



Find an s-t-path with depth-first search.
Delete dead ends.
Route as much flow as possible.
Update capacities. Repeat.



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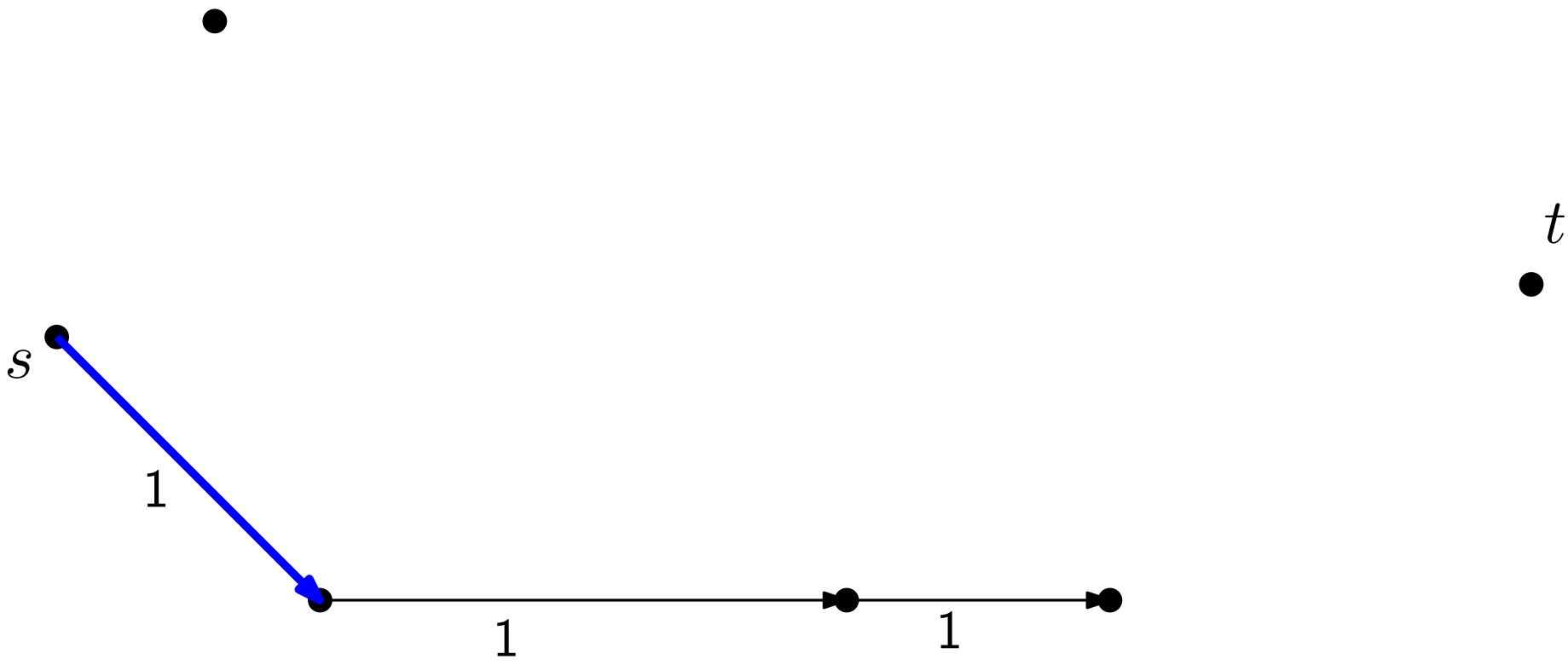

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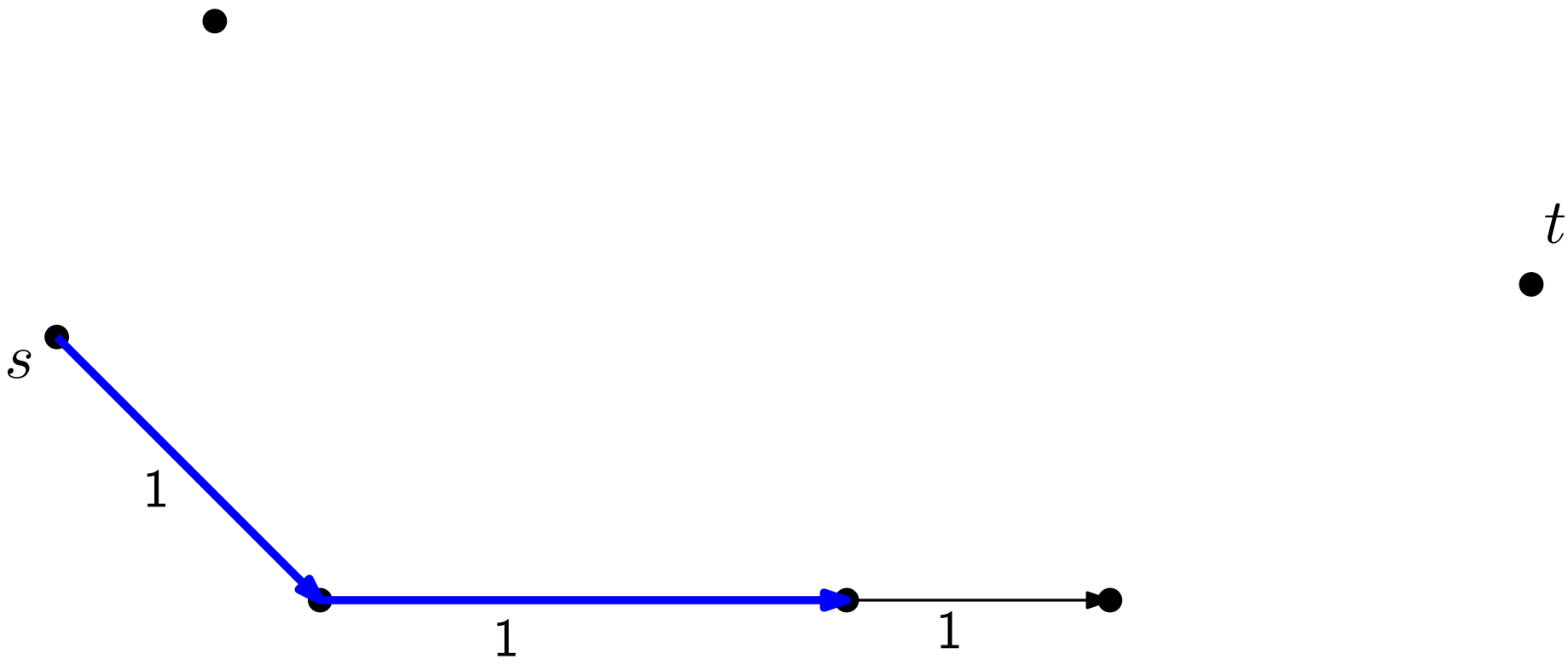

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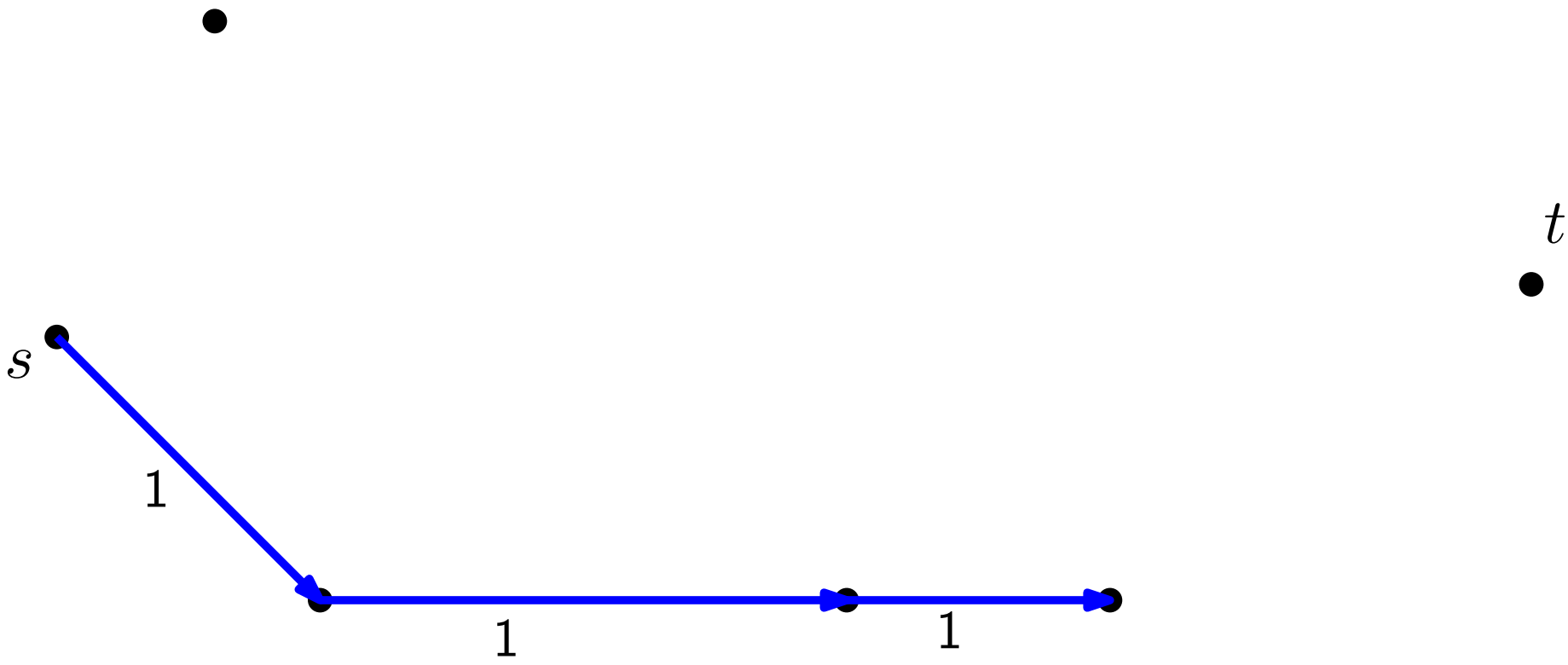

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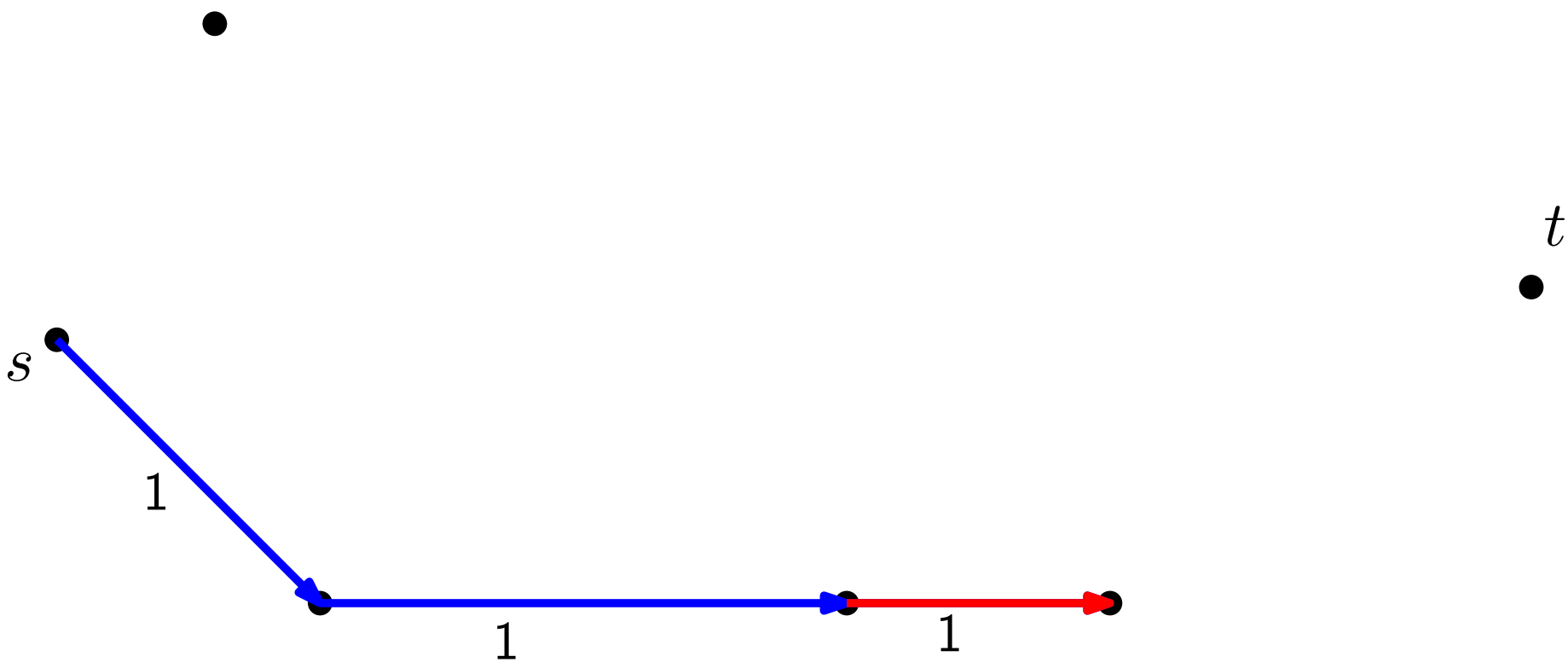
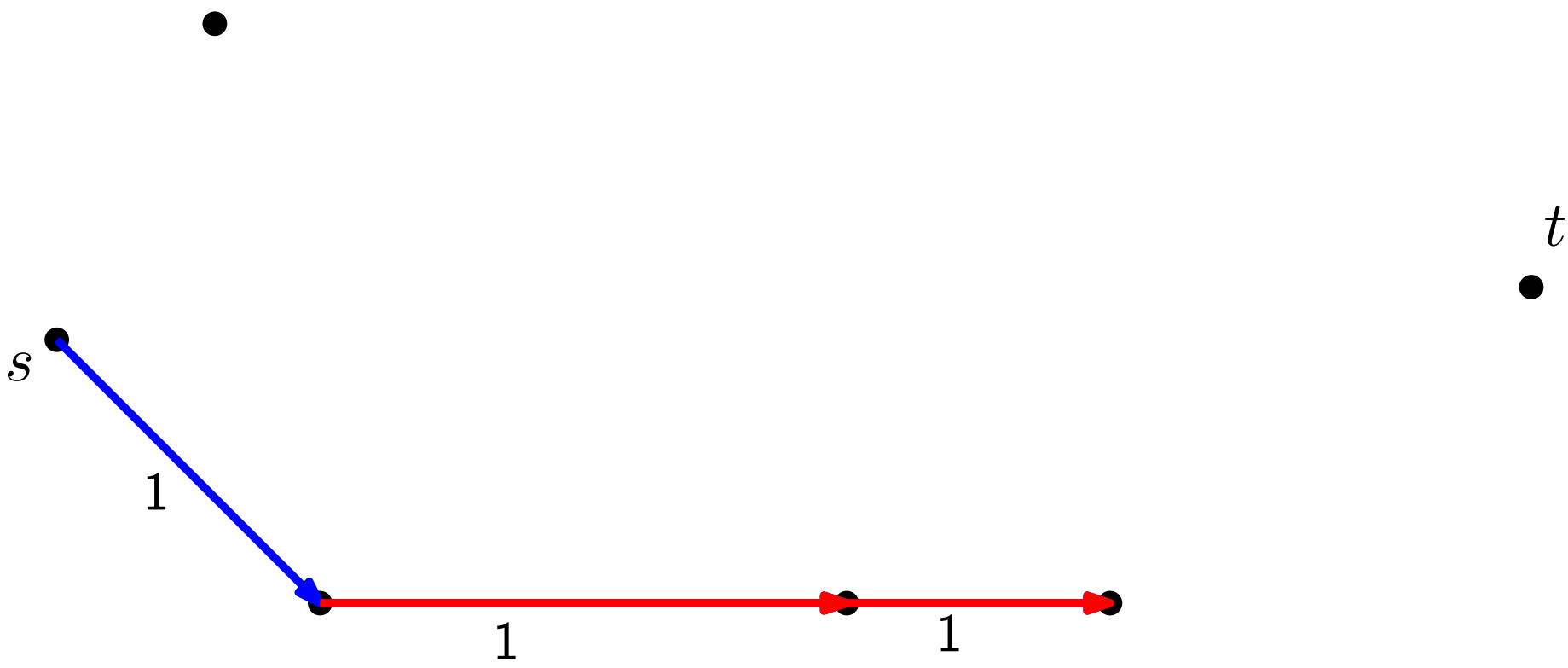
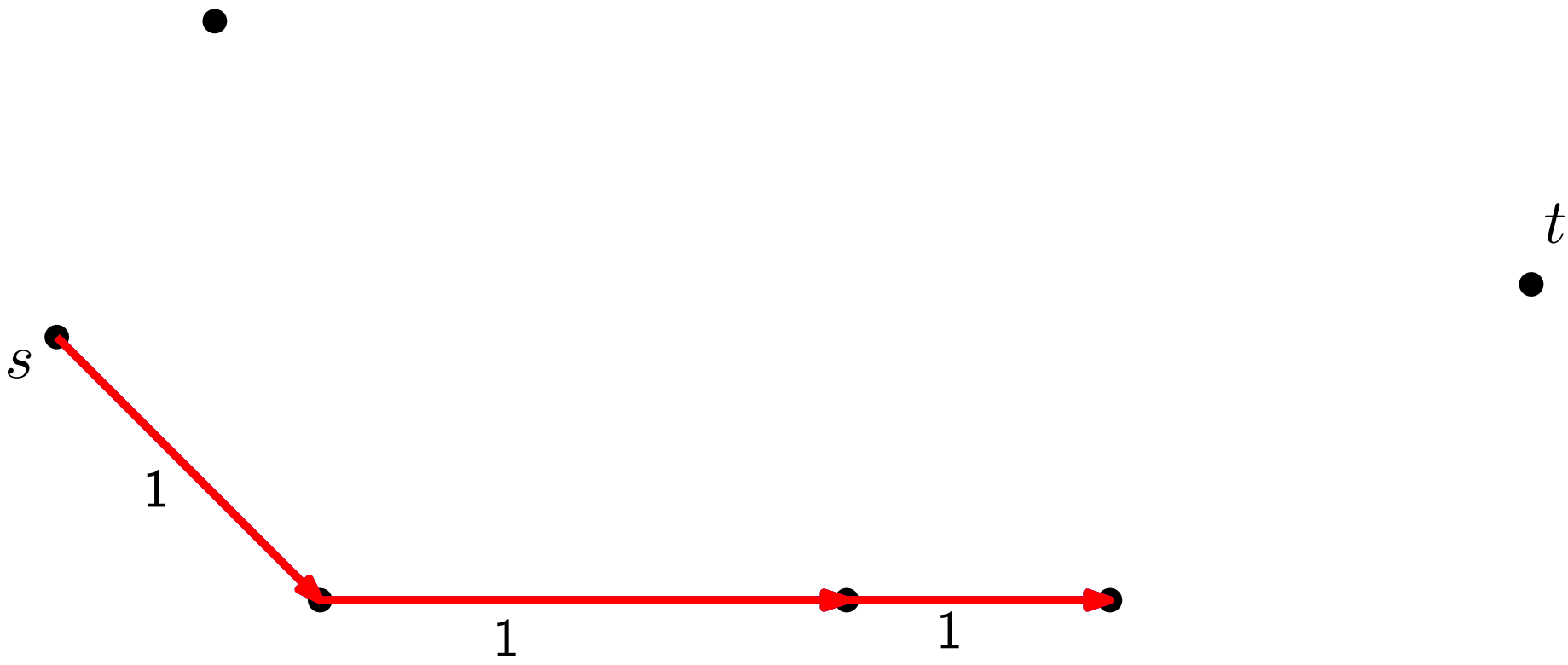


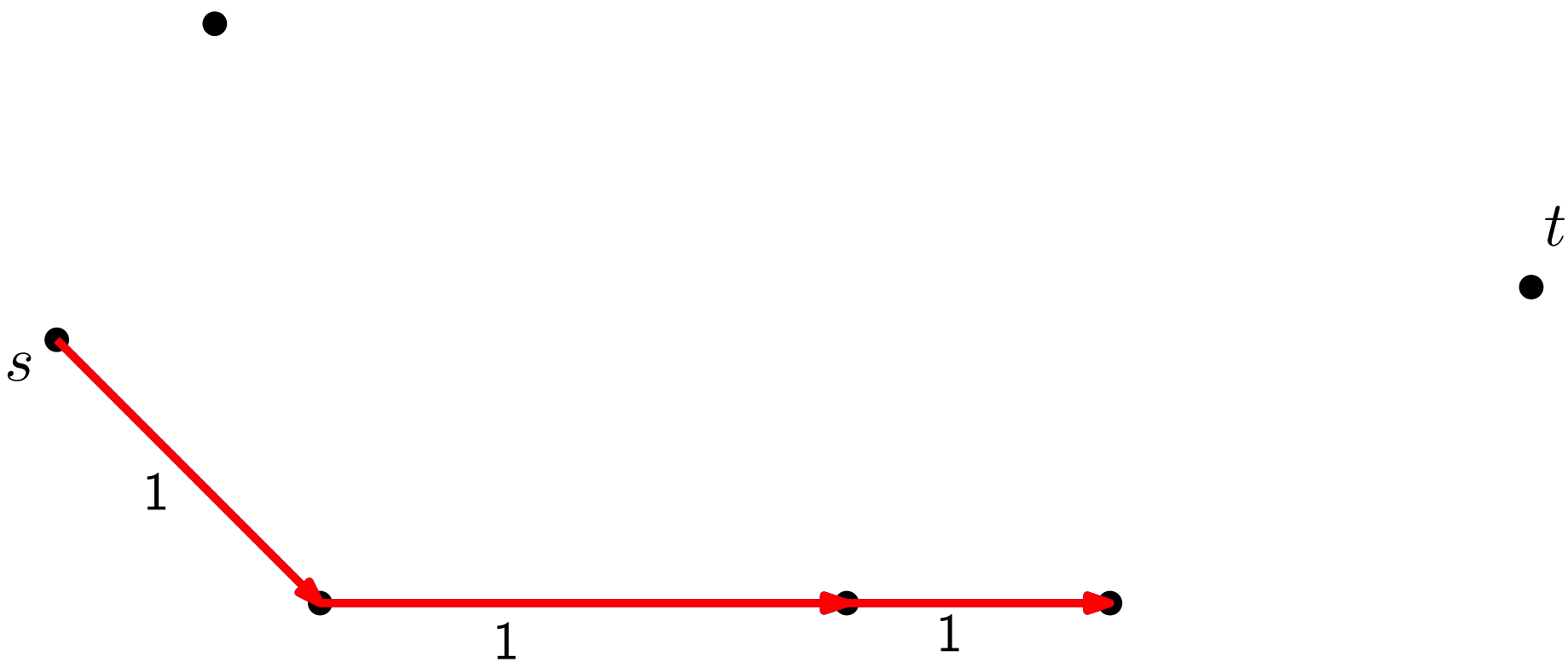
Find an s-t-path with depth-first search.



Find an s-t-path with depth-first search.

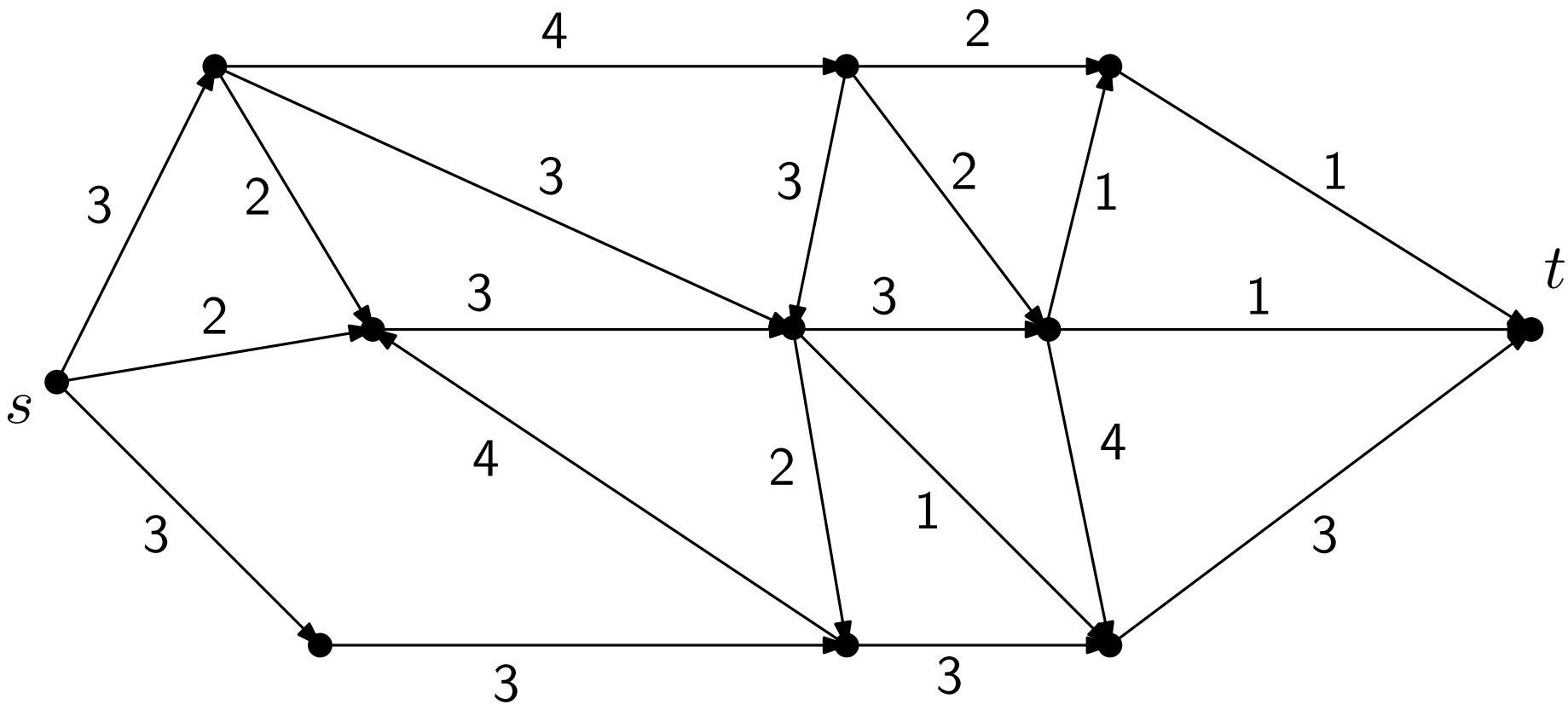


Find an s-t-path with depth-first search.



Find an s-t-path with depth-first search.
Delete dead ends.

Find an s-t-path with depth-first search.
Delete dead ends.
The source s has been deleted. The phase ends.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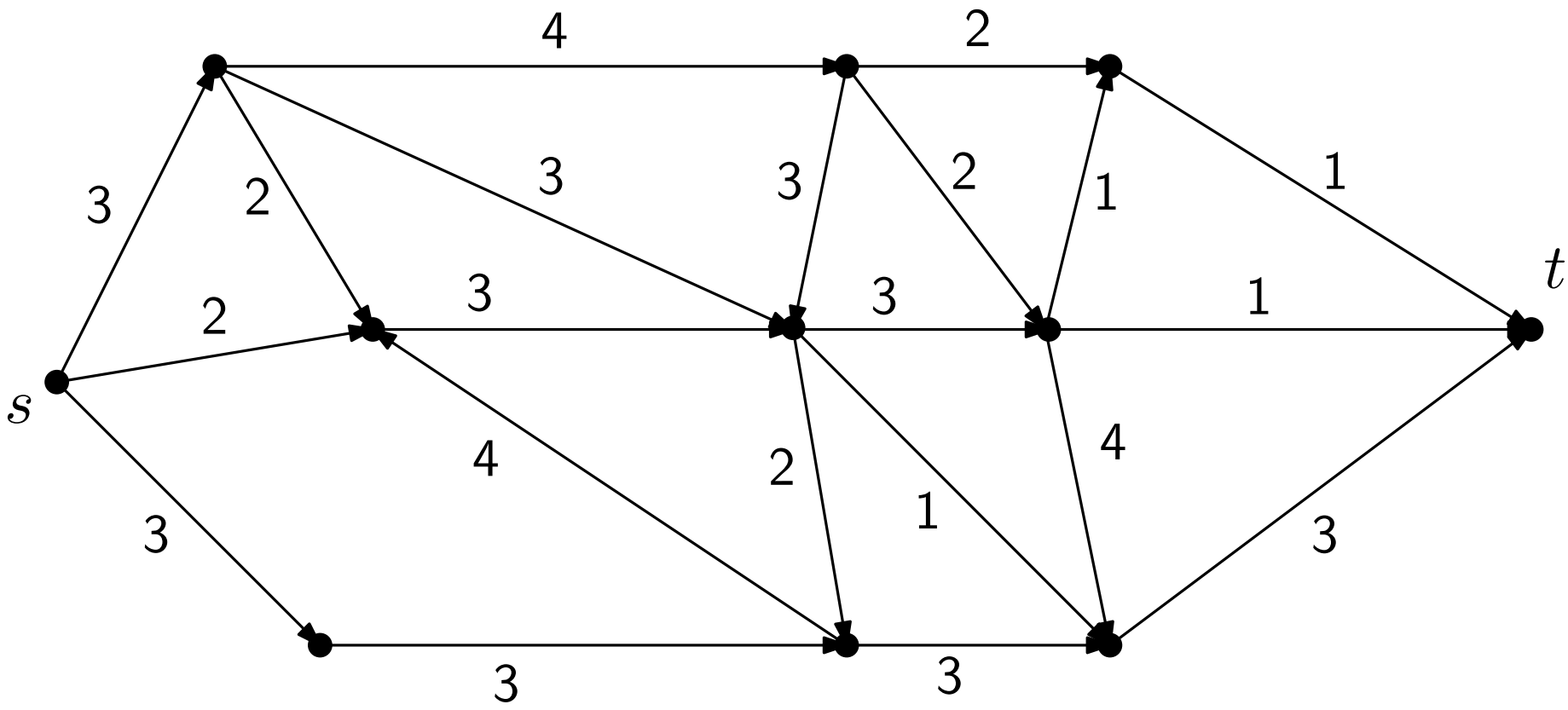
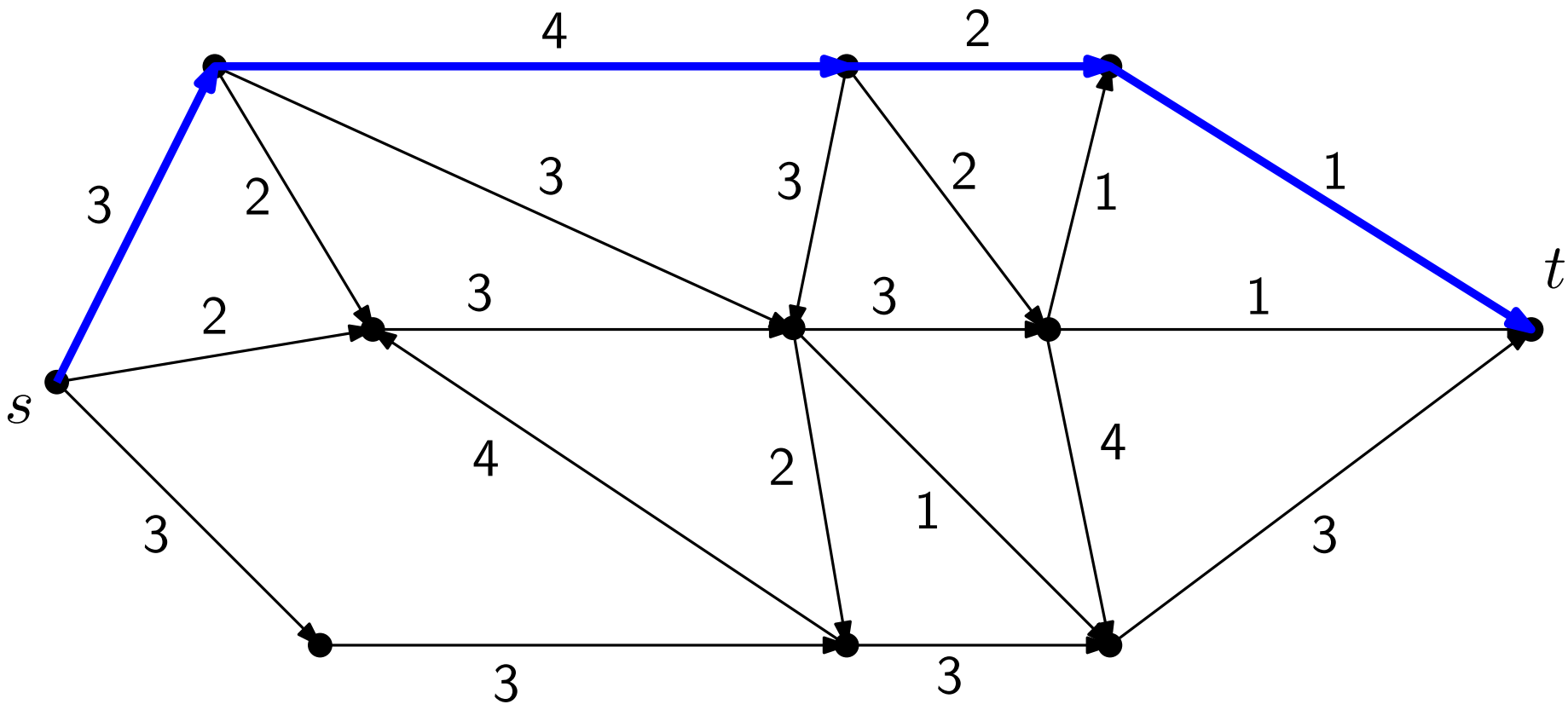


Collect the flow of this phase.



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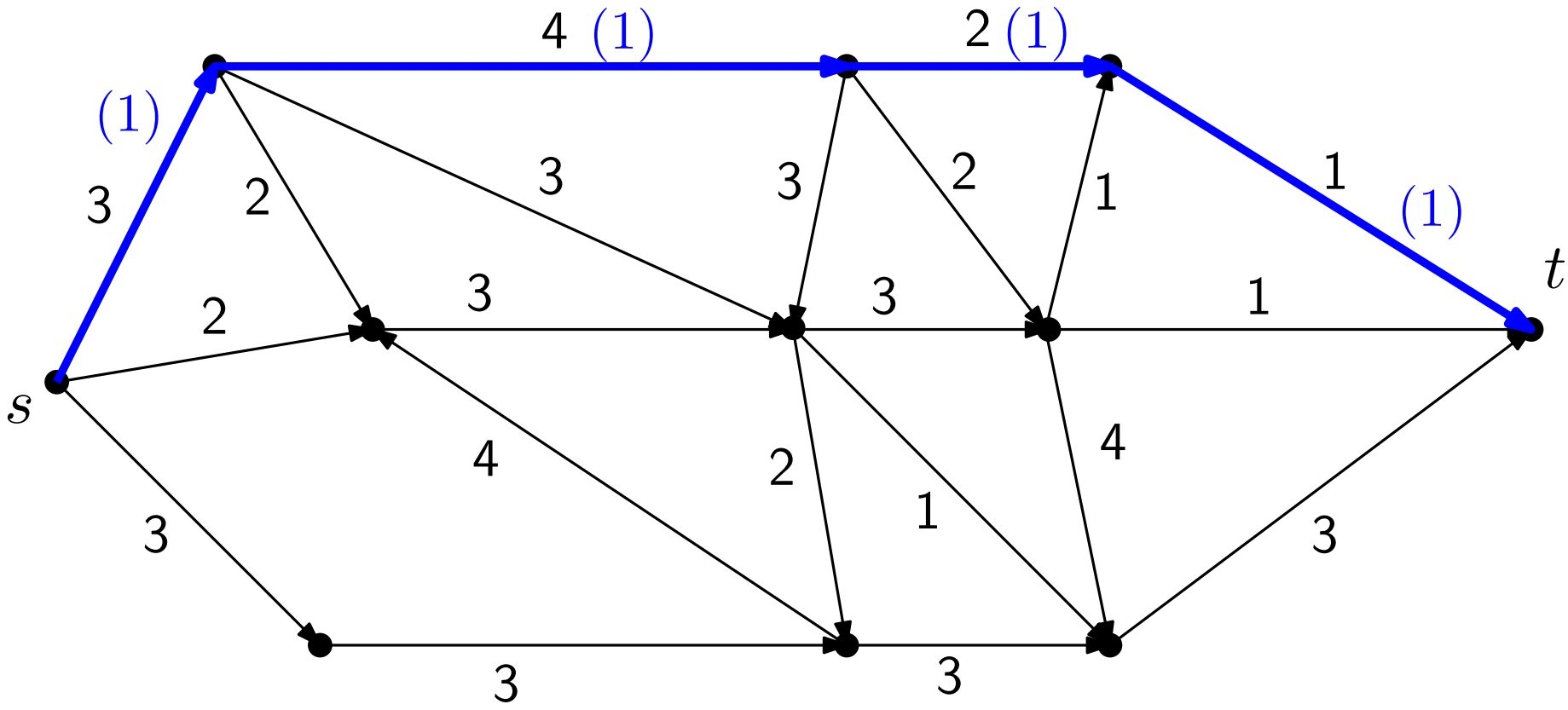

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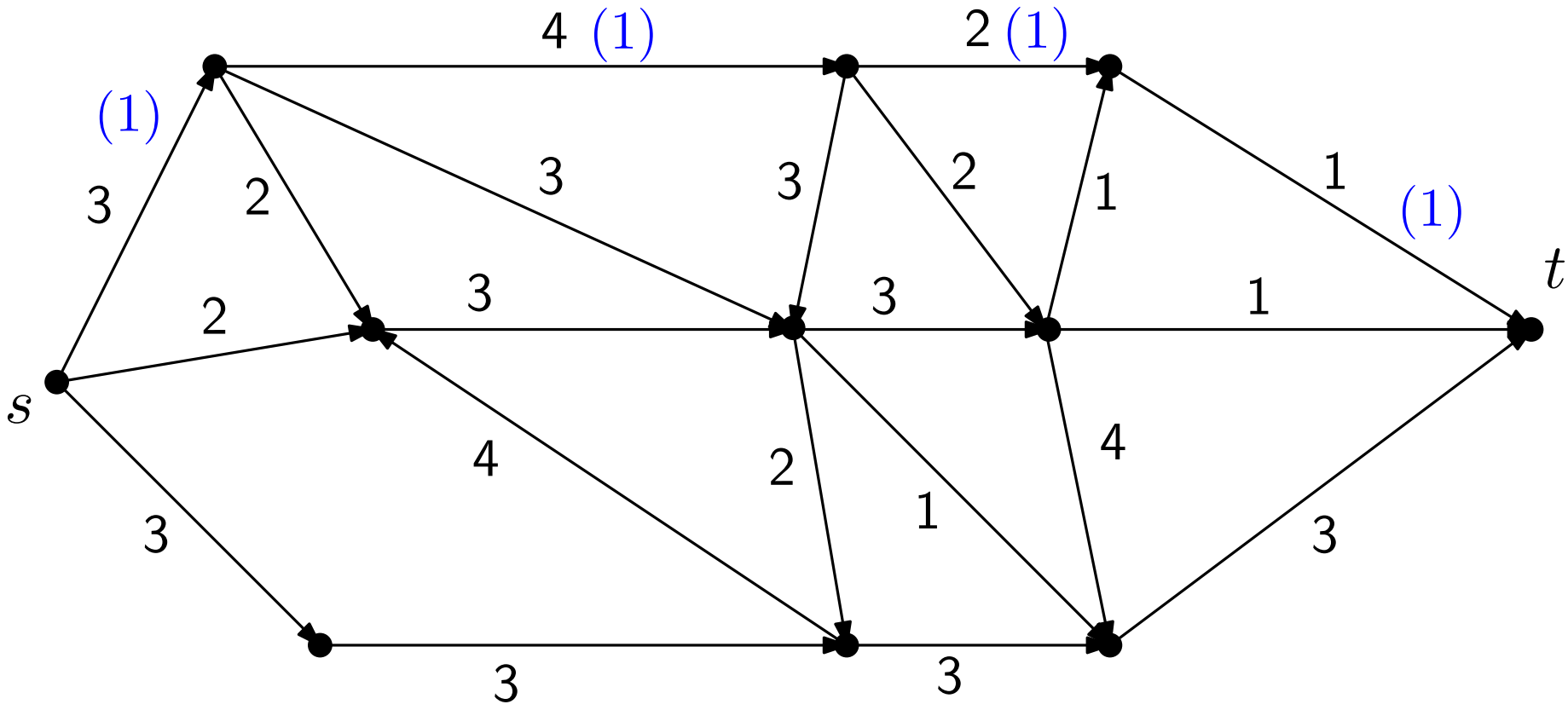

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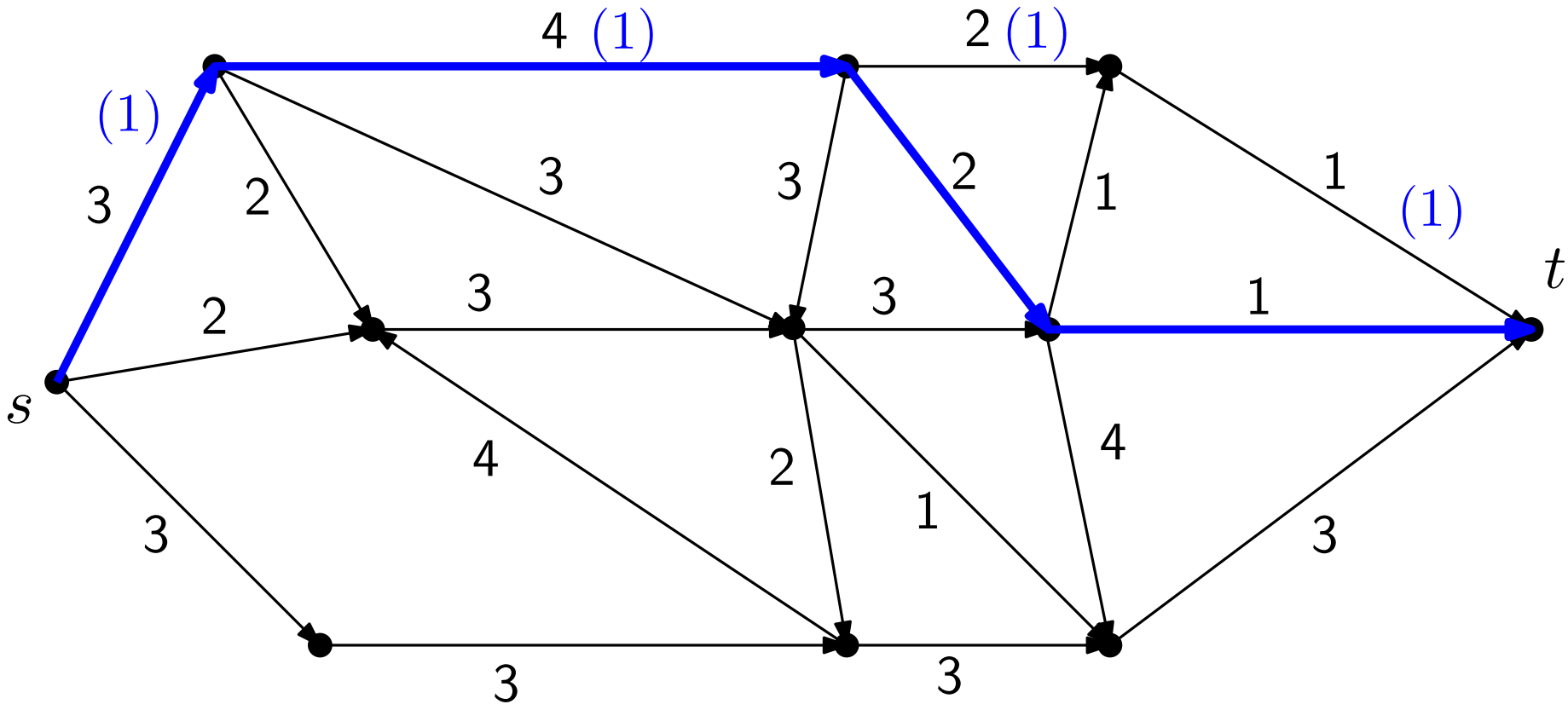

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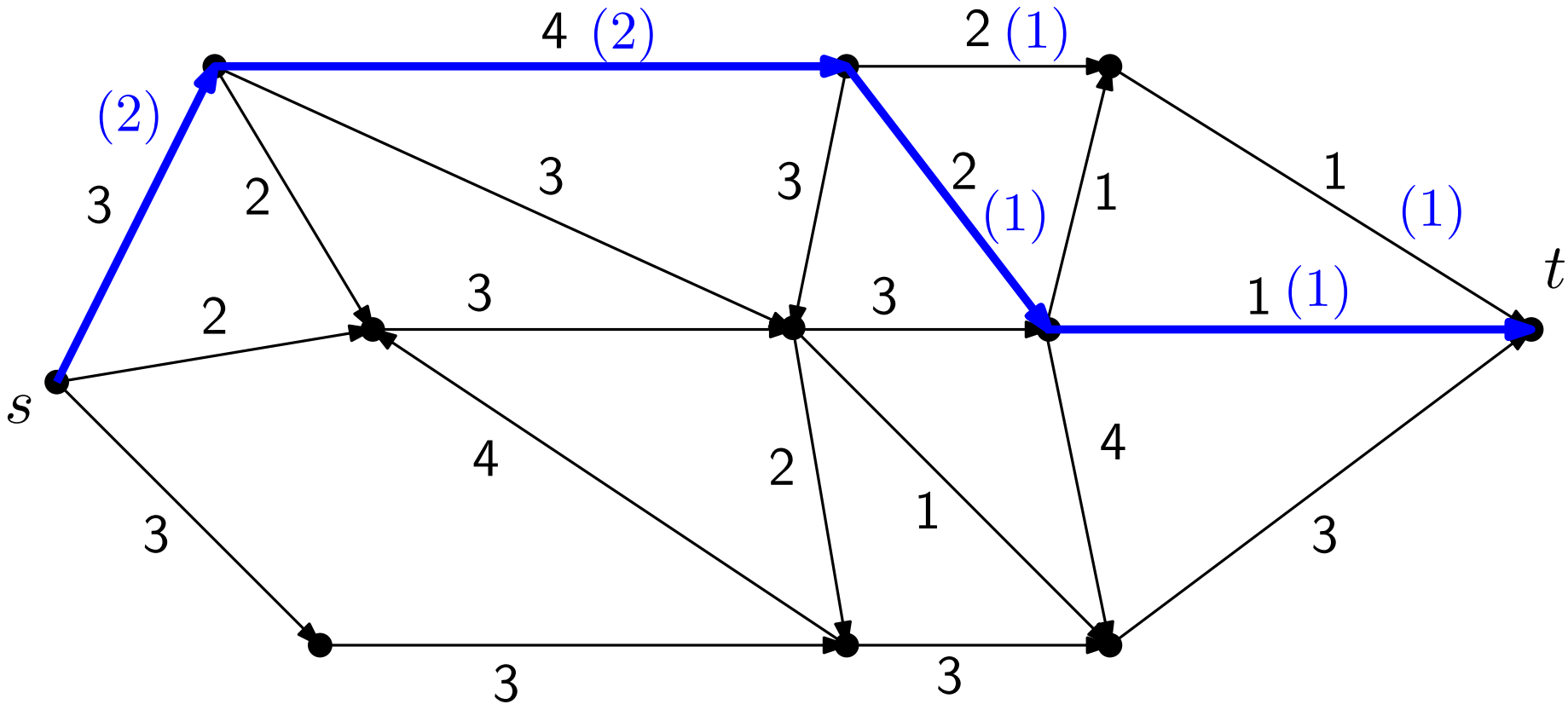

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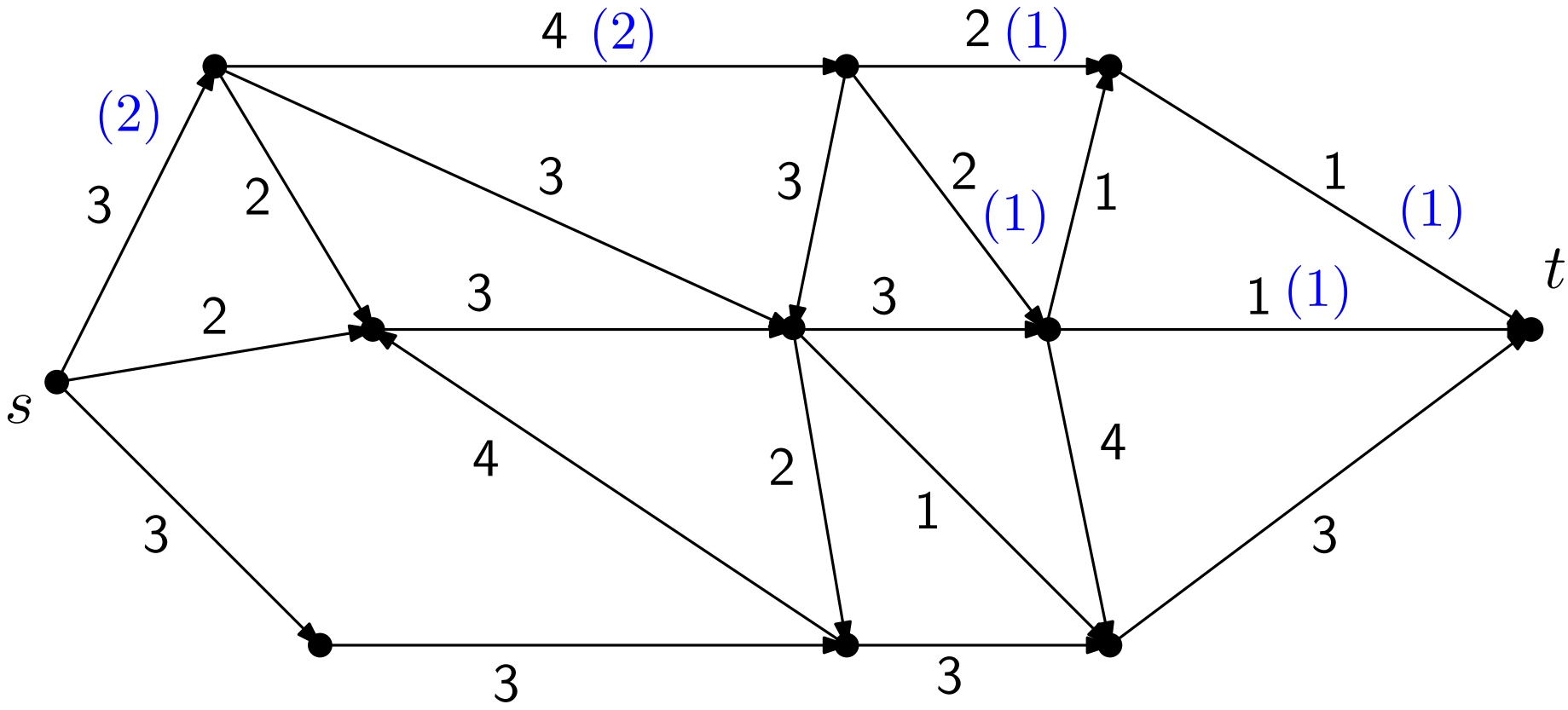

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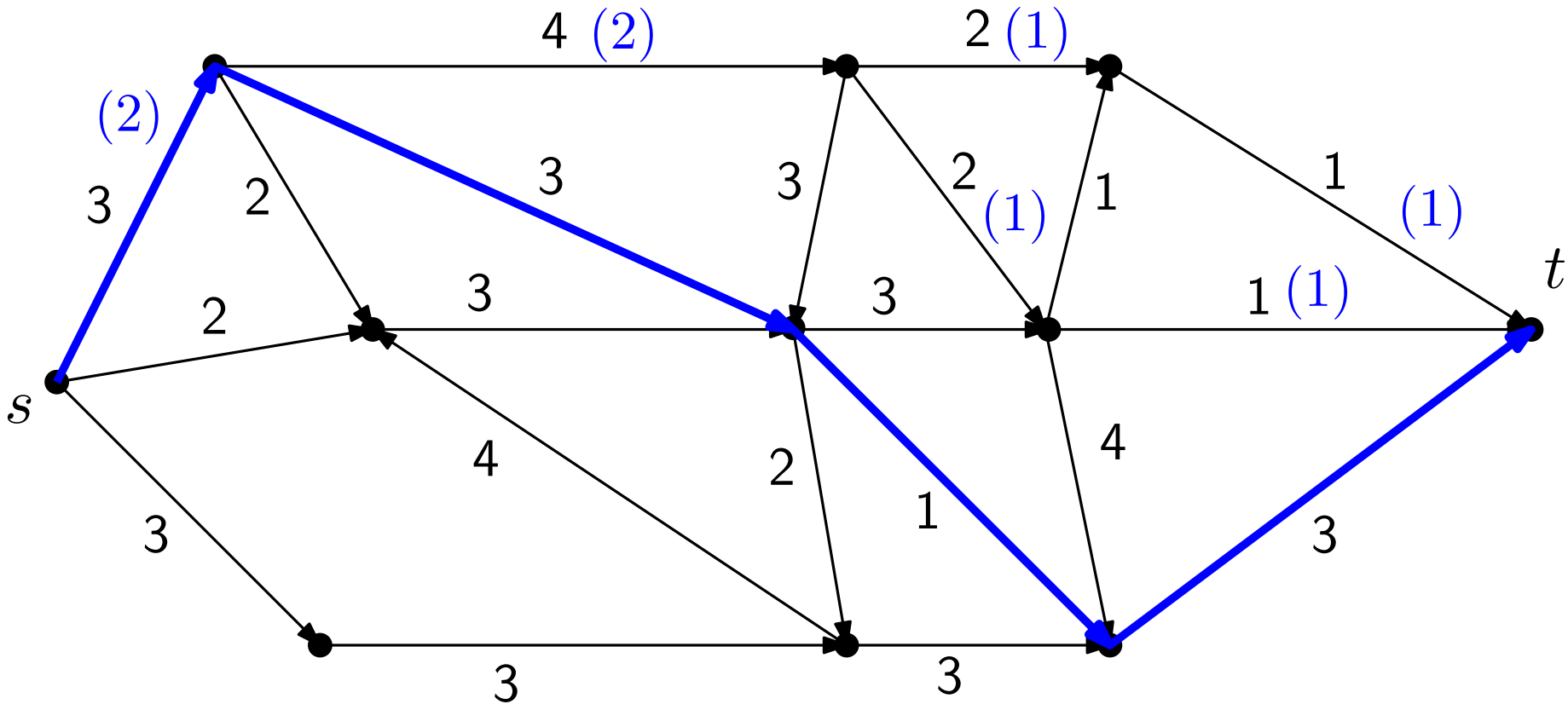

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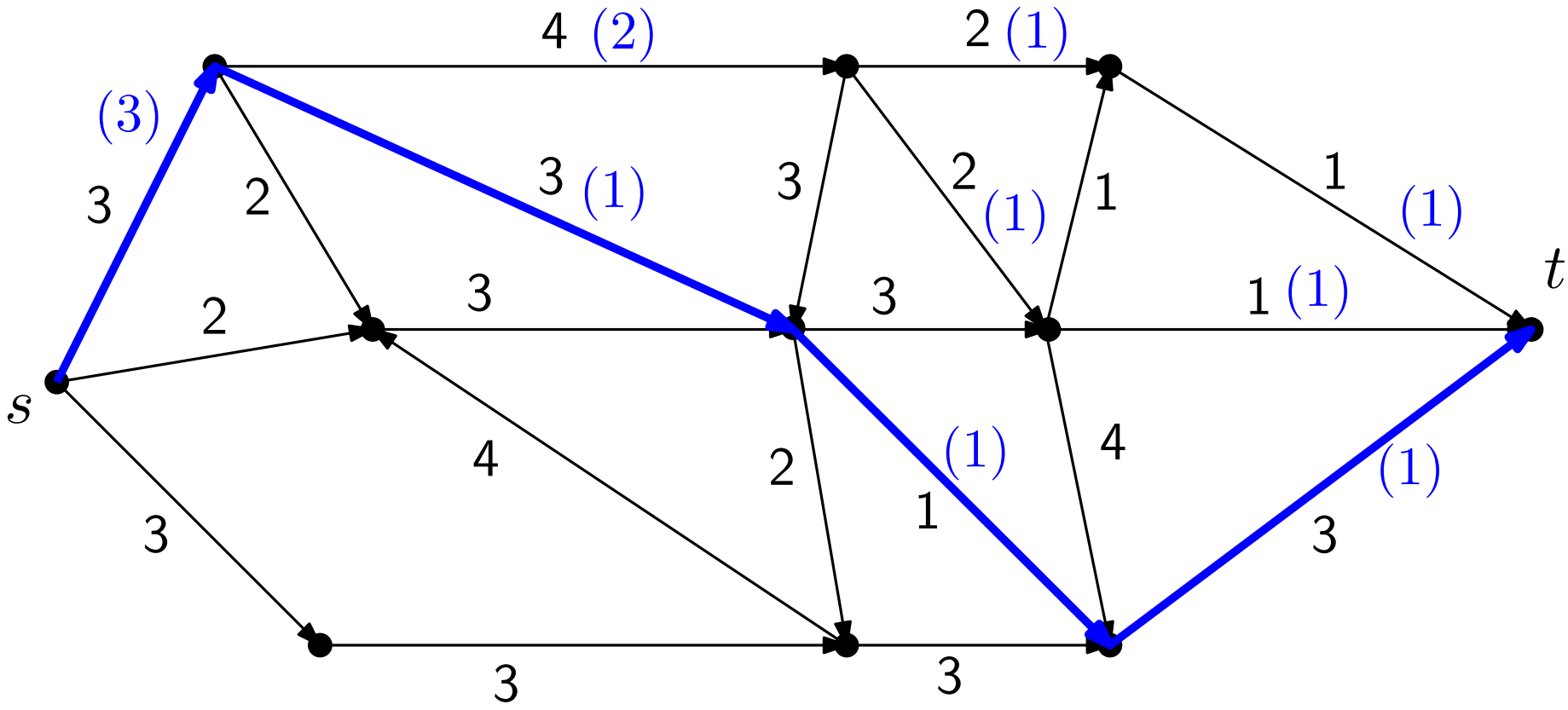

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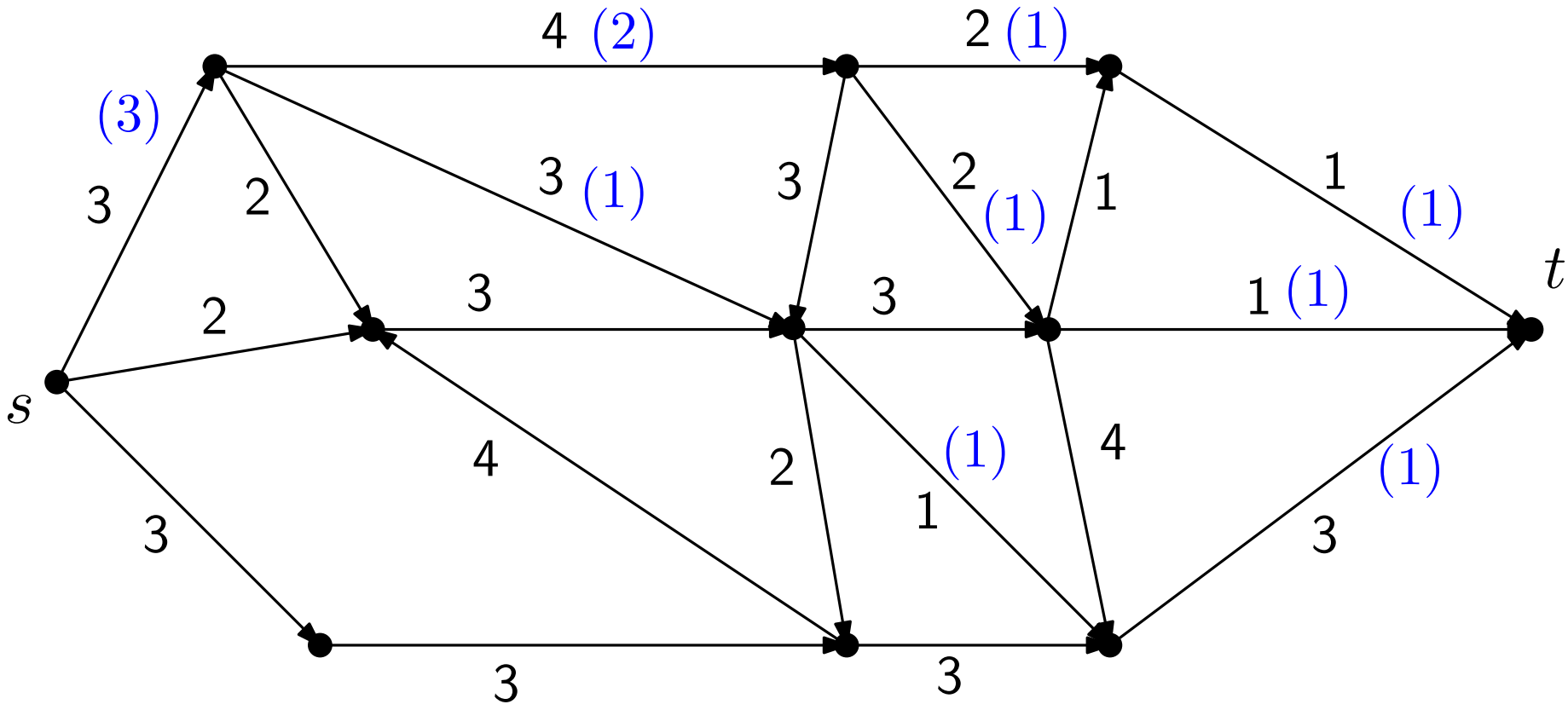

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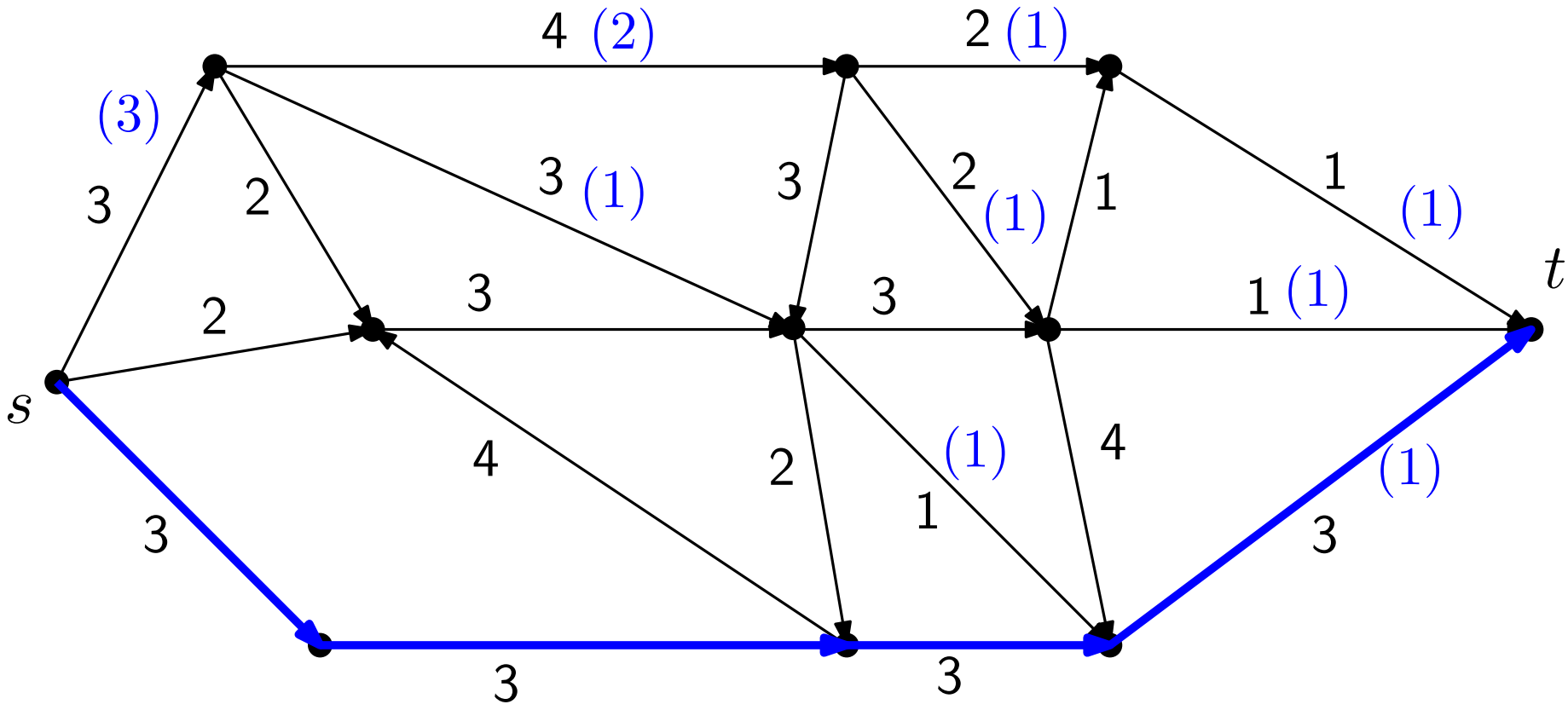

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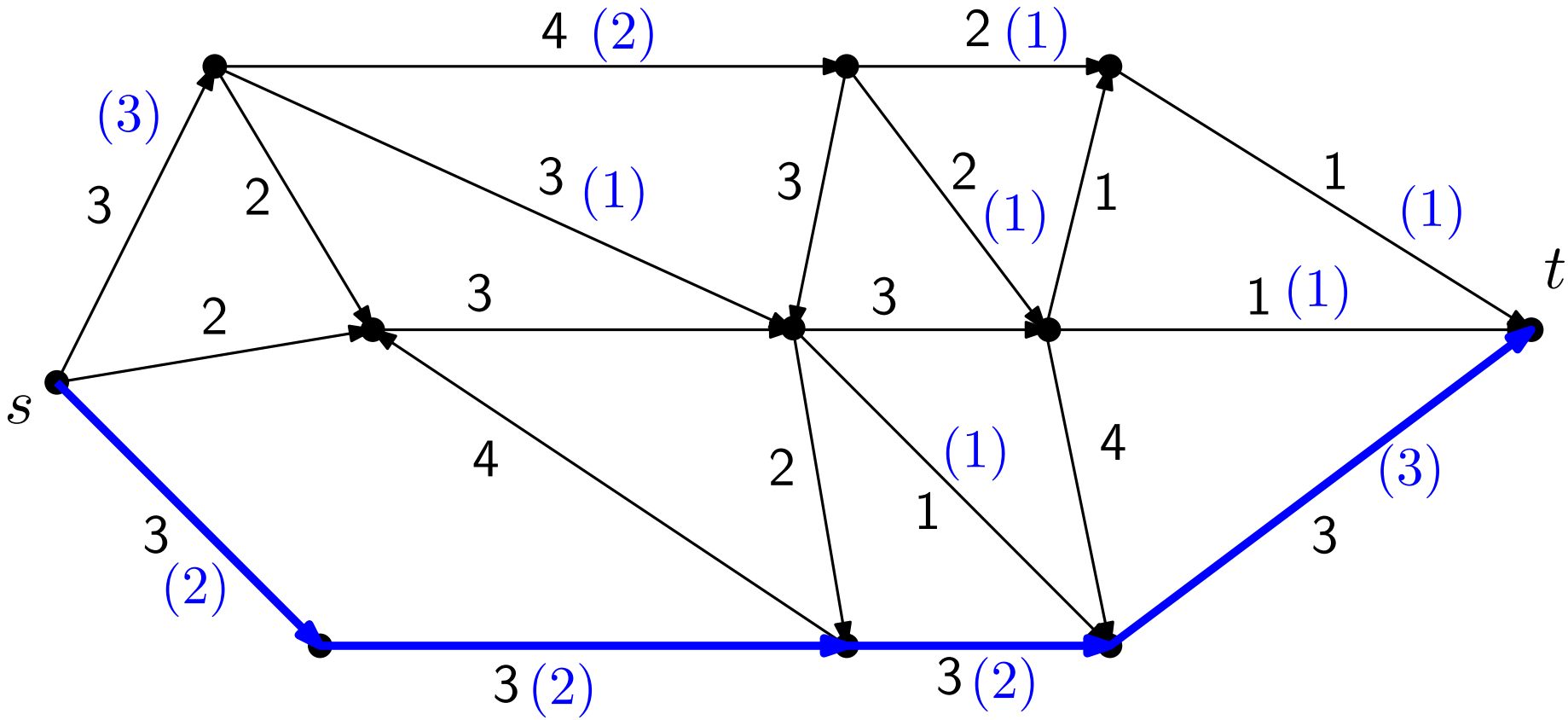

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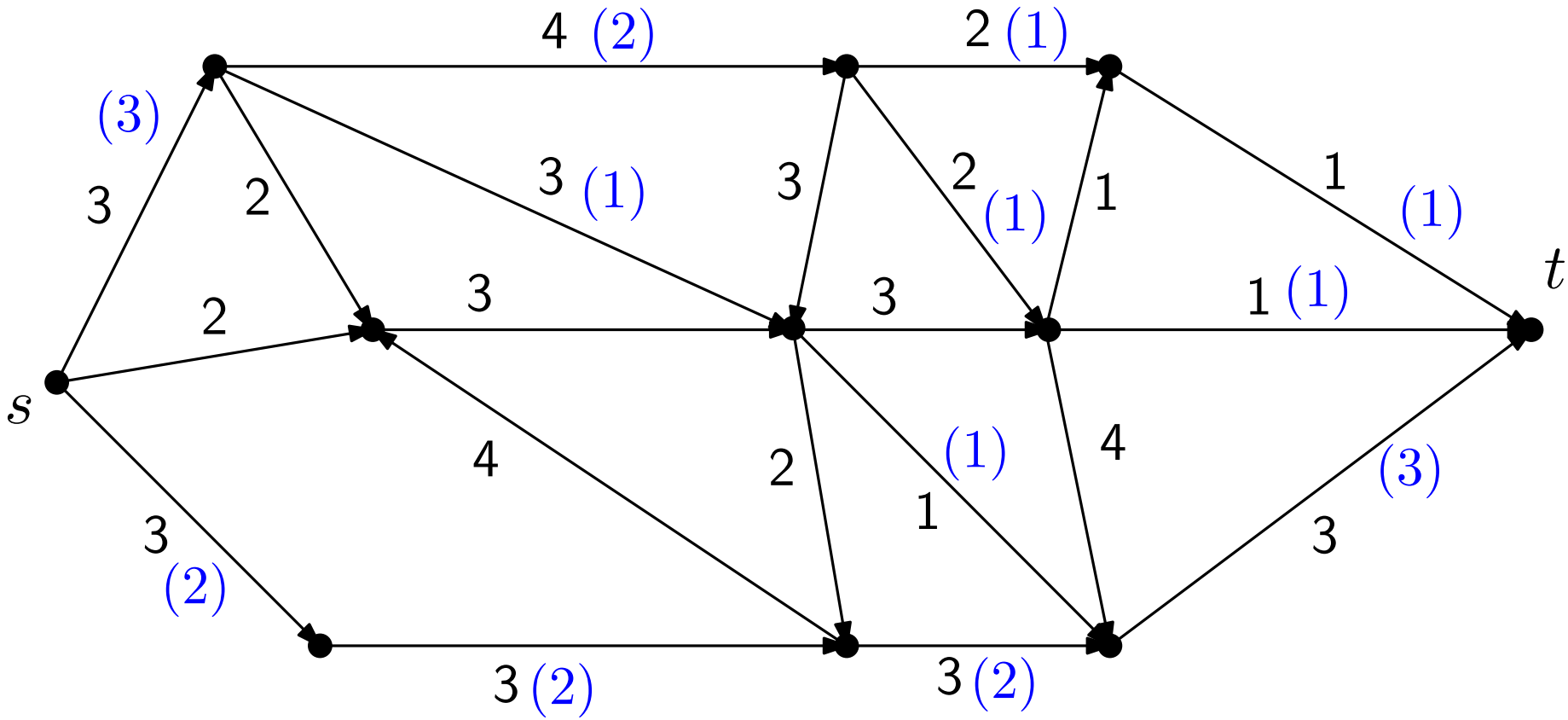

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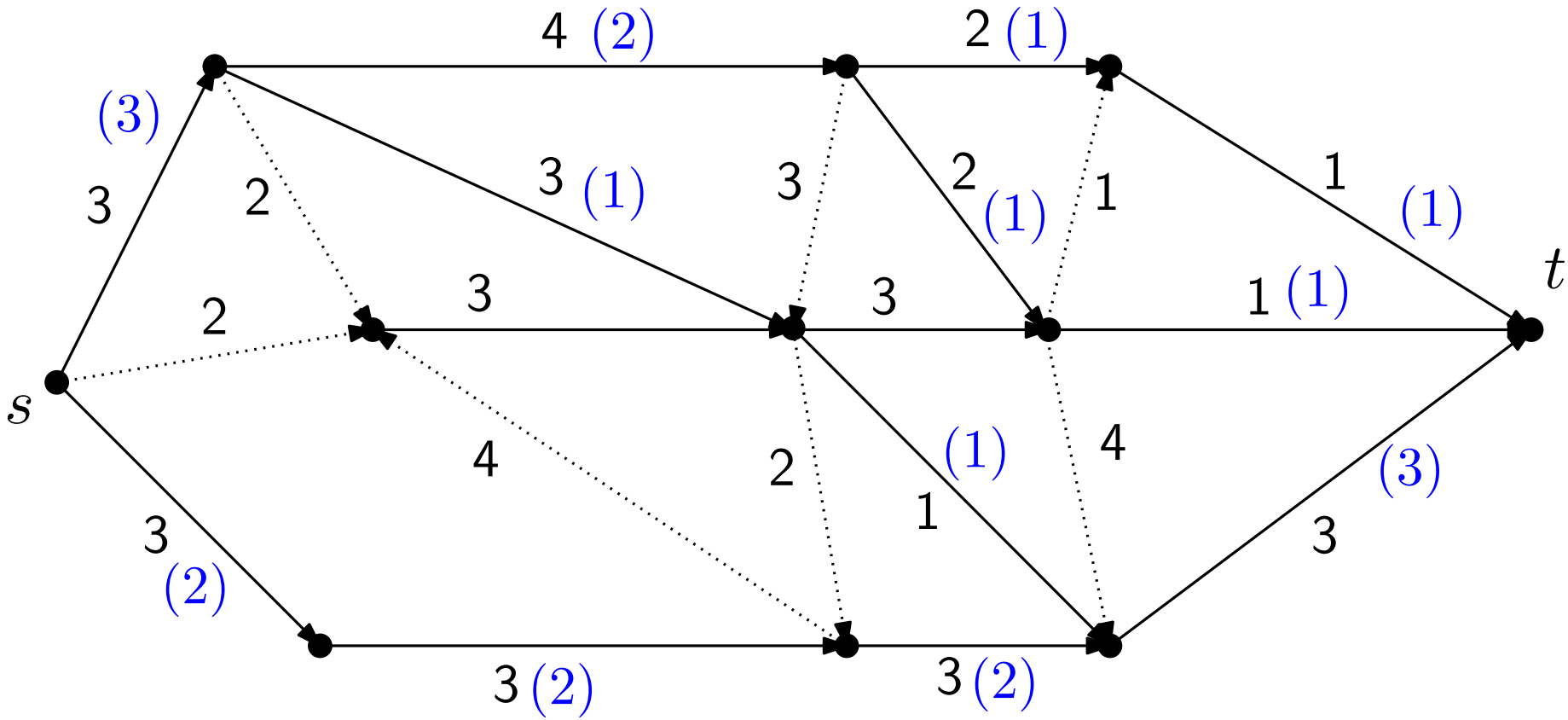
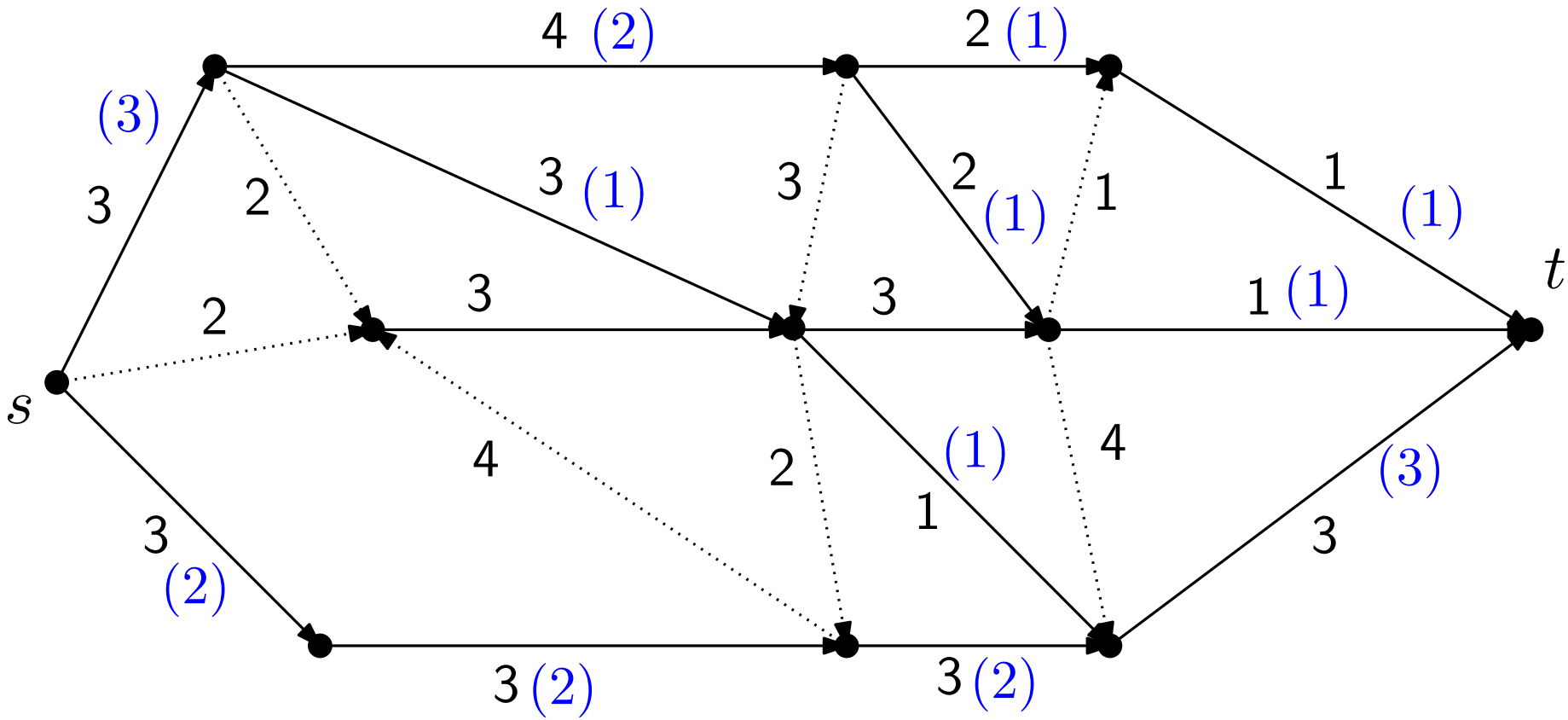


Collect the flow of this phase.

Observation: Only edges on a shortest s-t-path carry flow.



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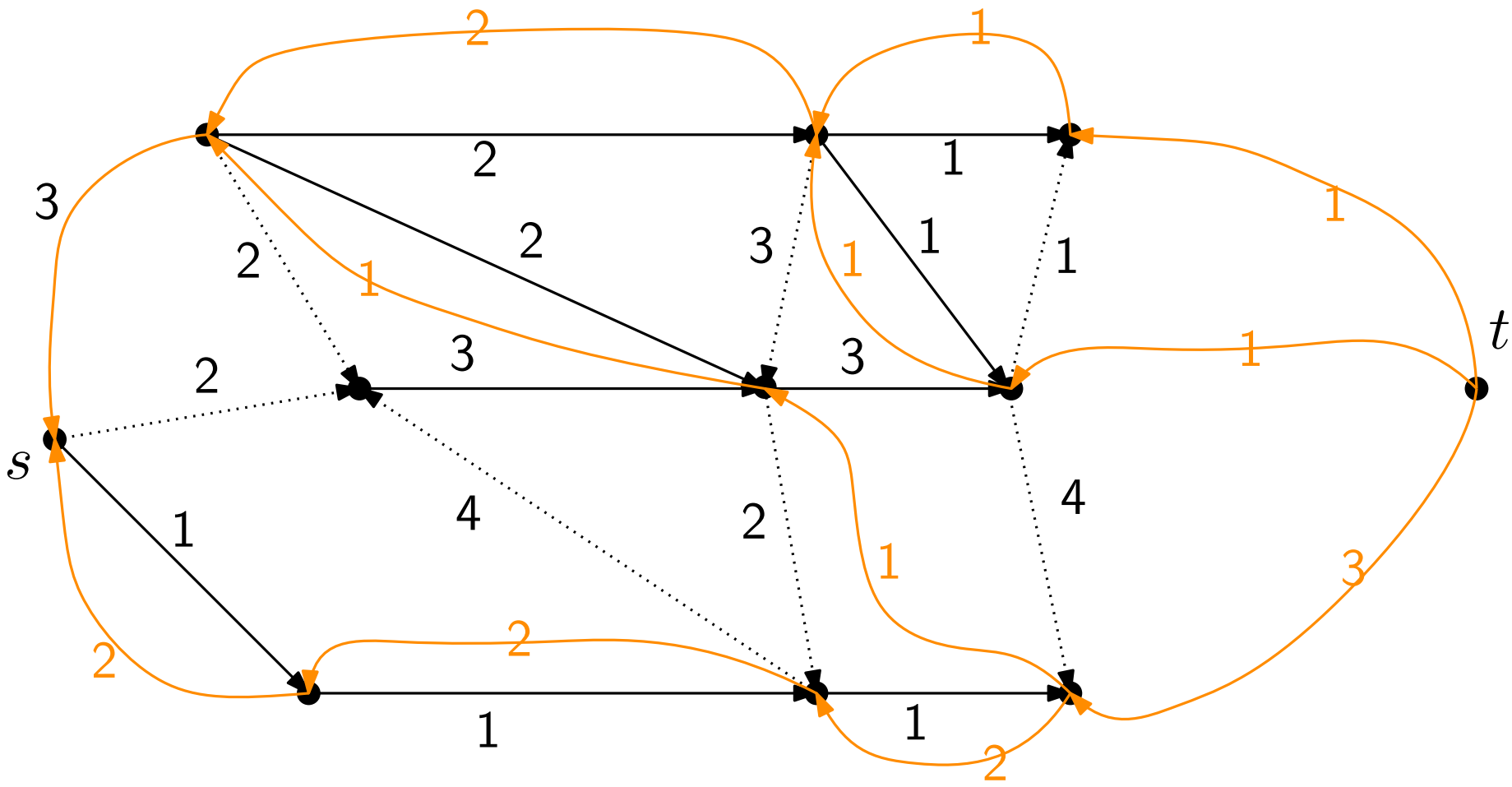
Build residual network (and introduce back edges)!



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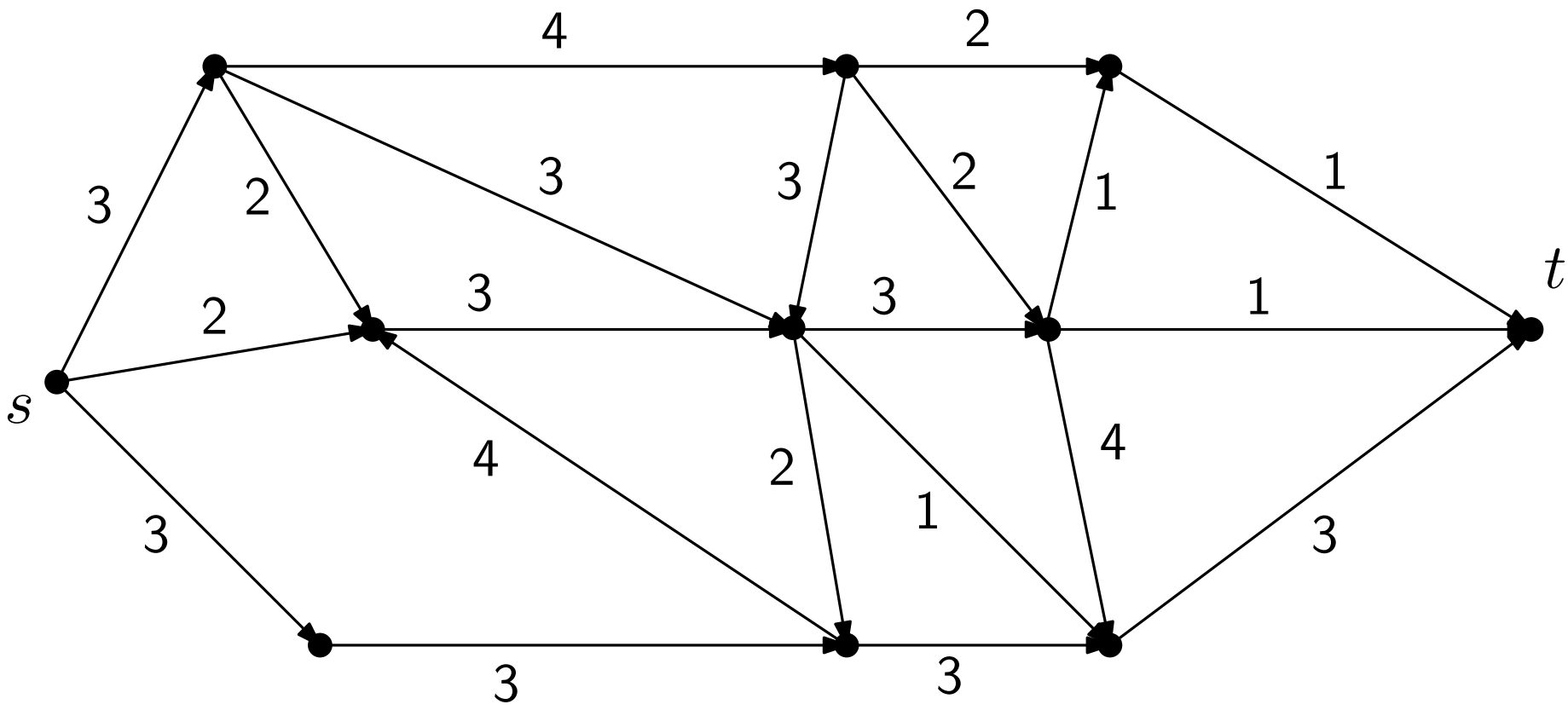
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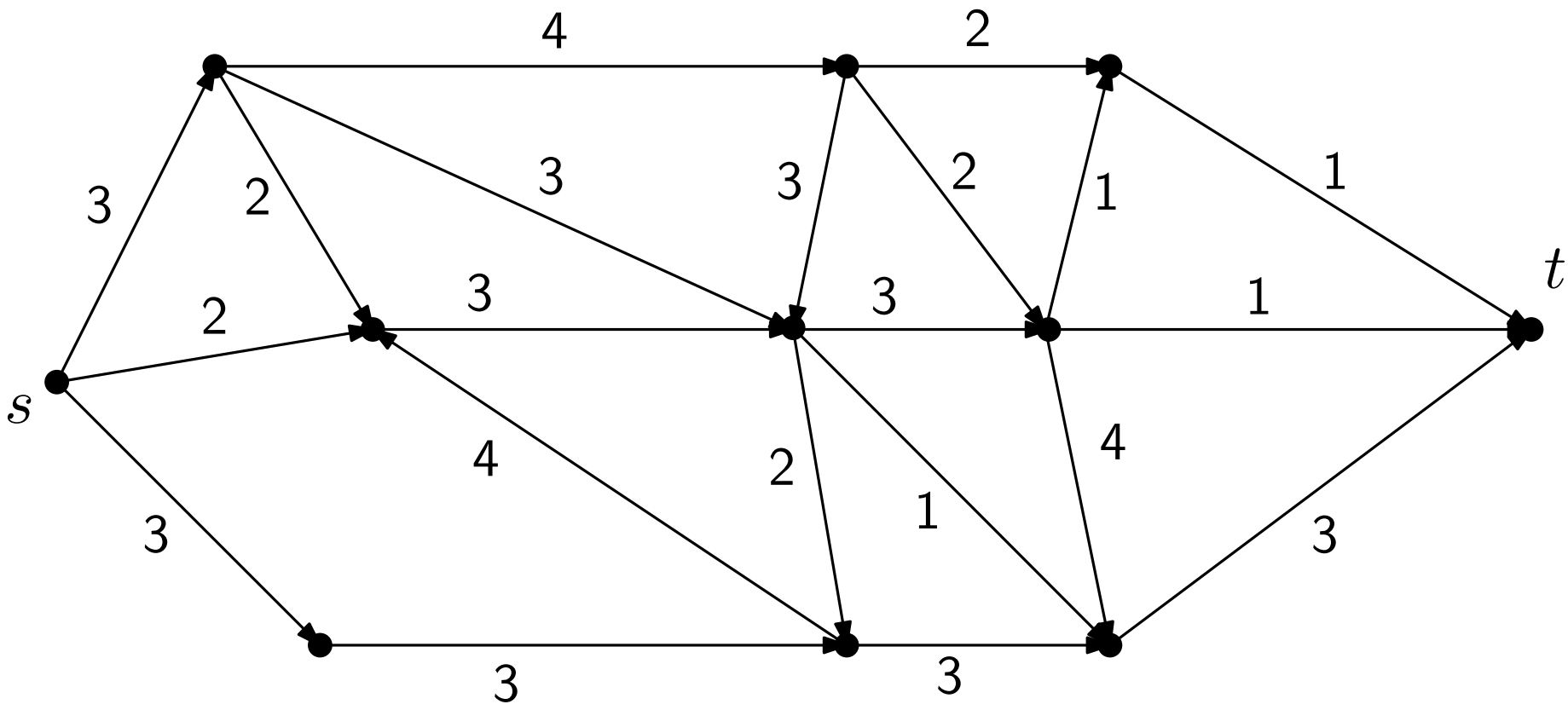
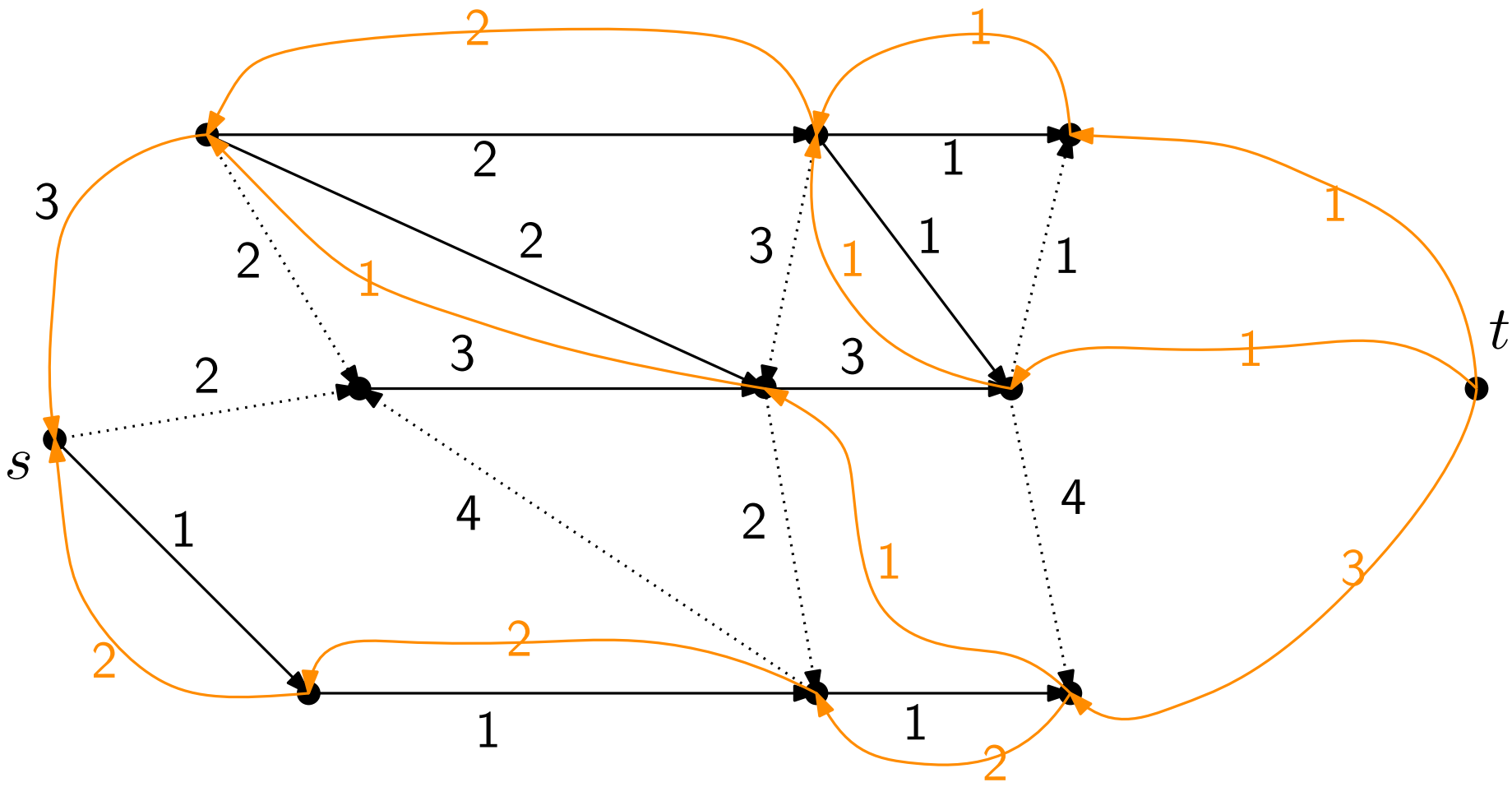


dist$(s, t) = 4$ in the original network.



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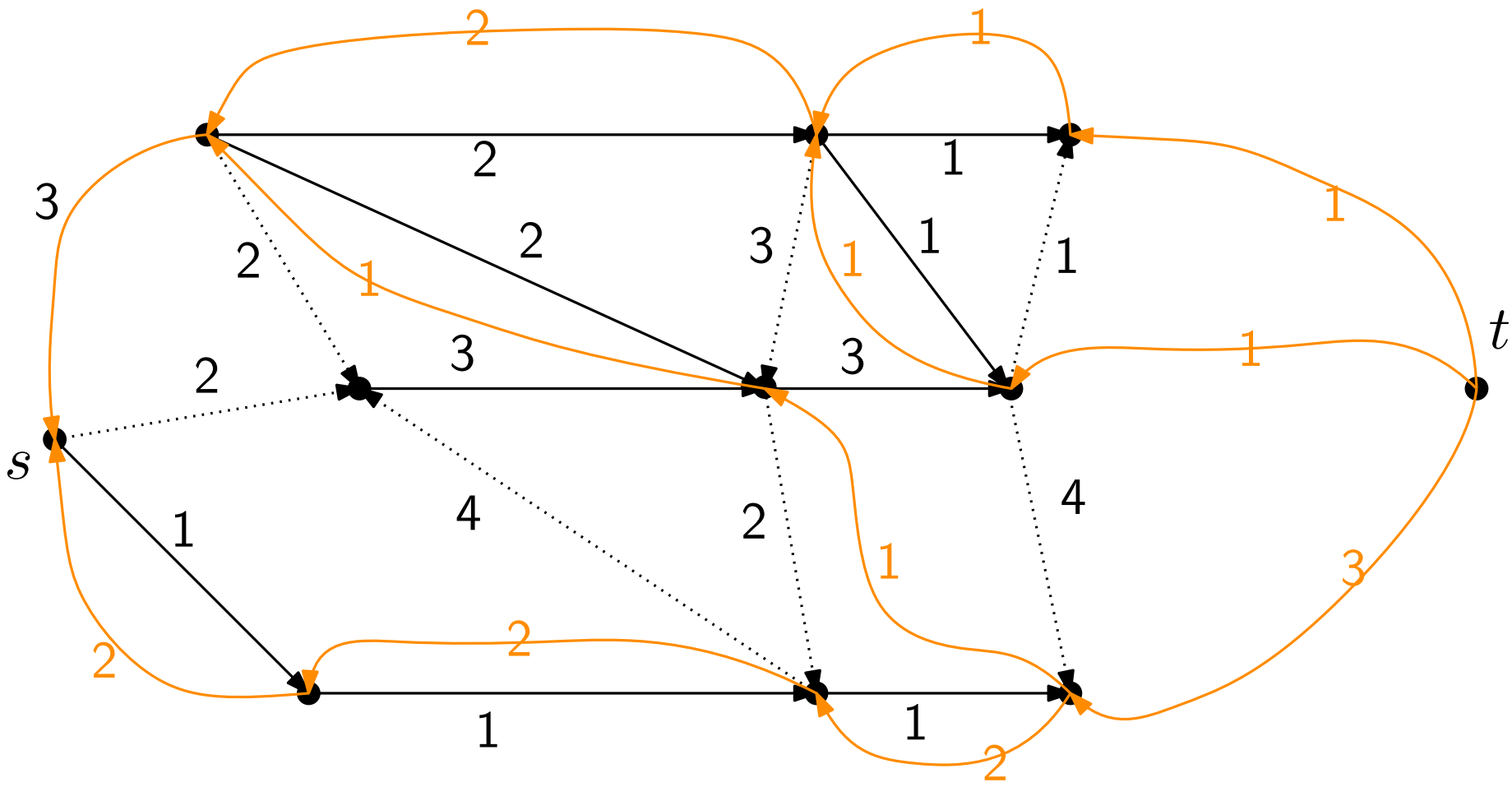
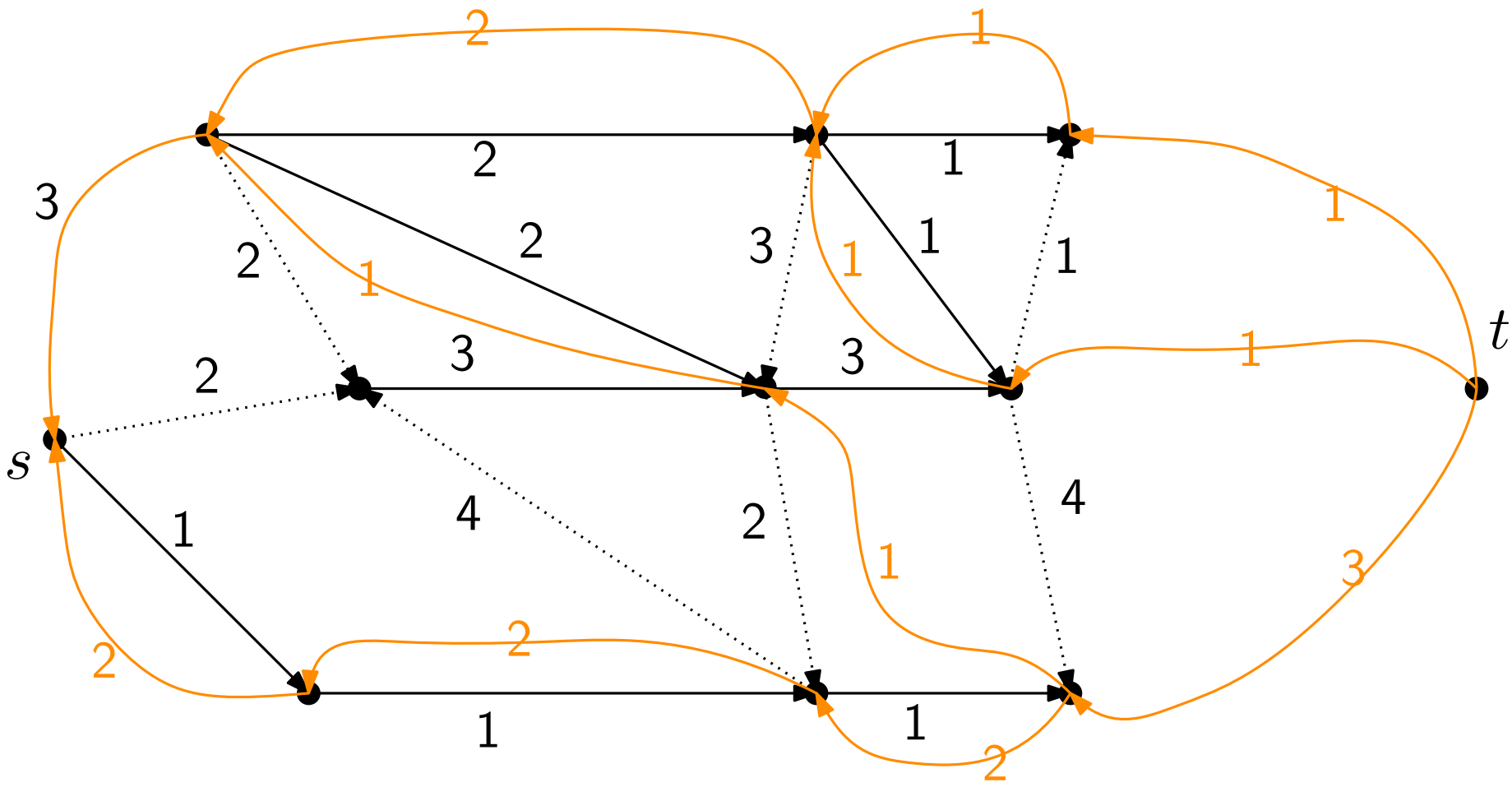


dist$(s, t) = 4$ in the original network.

Back edges go backwards. Not part of any length-4-path.



dist$(s, t) = 4$ in the original network.

Back edges go backwards. Not part of any length-4-path.

dist$(s, t) > 4$ in the residual network.

Runtime Analysis

- dist(s, t) increases in every phase.

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  At most $n - 2$ phases.

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  How many?

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  How many?

  How much time does each take?

Claim. Each phase performs at most m depth-first searches.

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Proof.

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- There are at most m edges in the network.

Claim. Each phase performs at most m depth-first searches.

Proof.

- There are at most m edges in the network.
- After every depth-first search we delete at least one edge.

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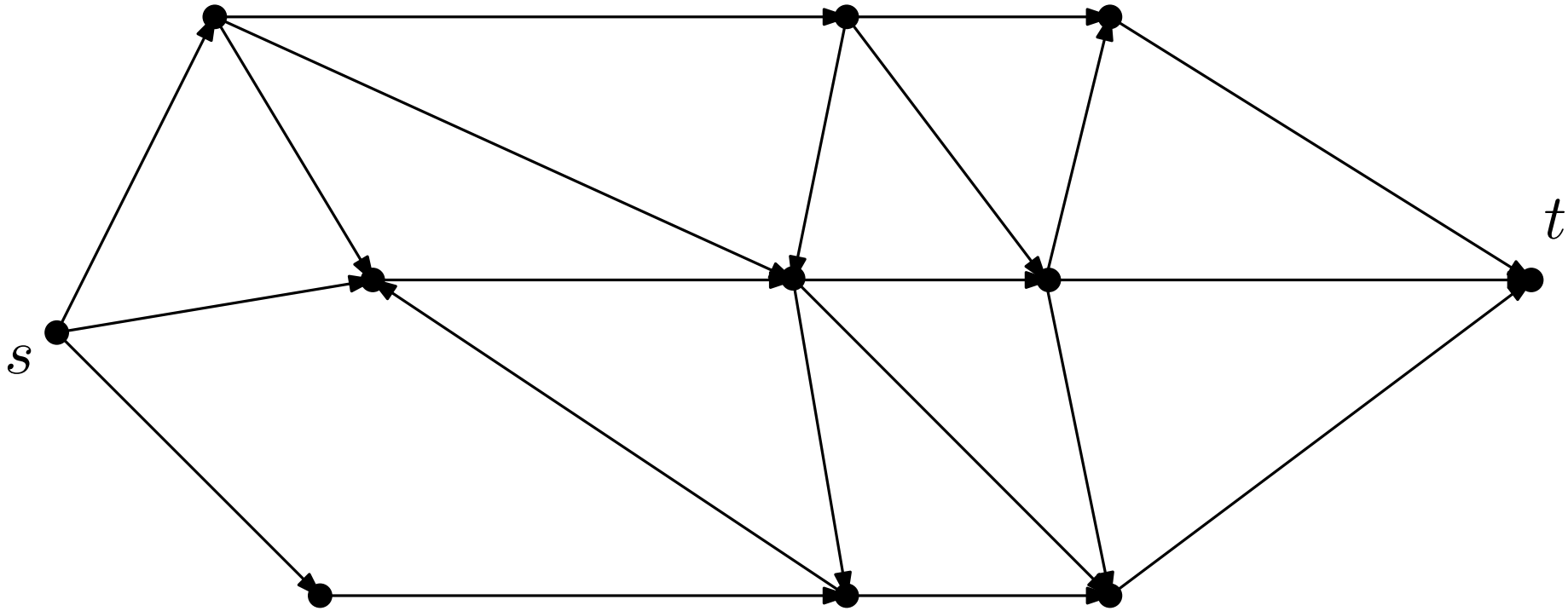
Dinic' algorithm performs at most $n - 2$ phases.

Dinic' algorithm performs $O(n^2 m)$ steps.

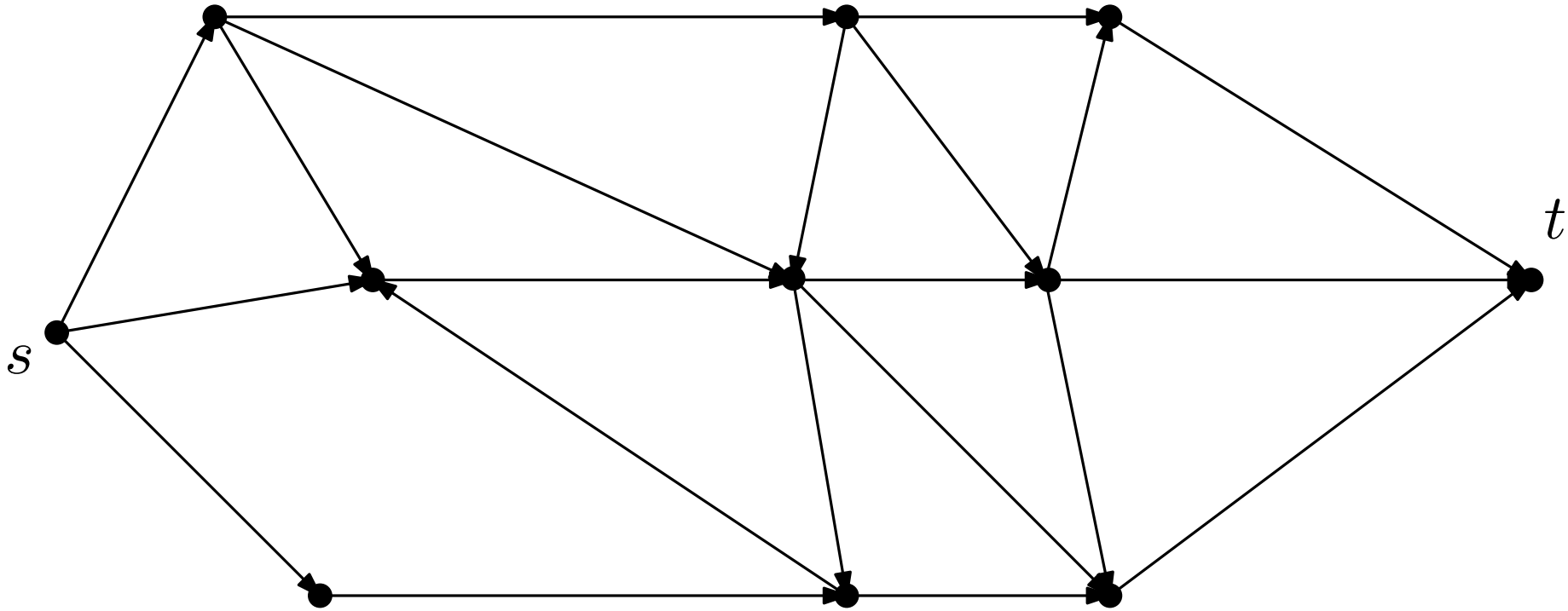
Theorem. Dinic' algorithm finds a maximum flow in $O(n^2m)$ steps.

Part II

Dinic' Algorithm in Unit Capacity Networks



All capacities are 1



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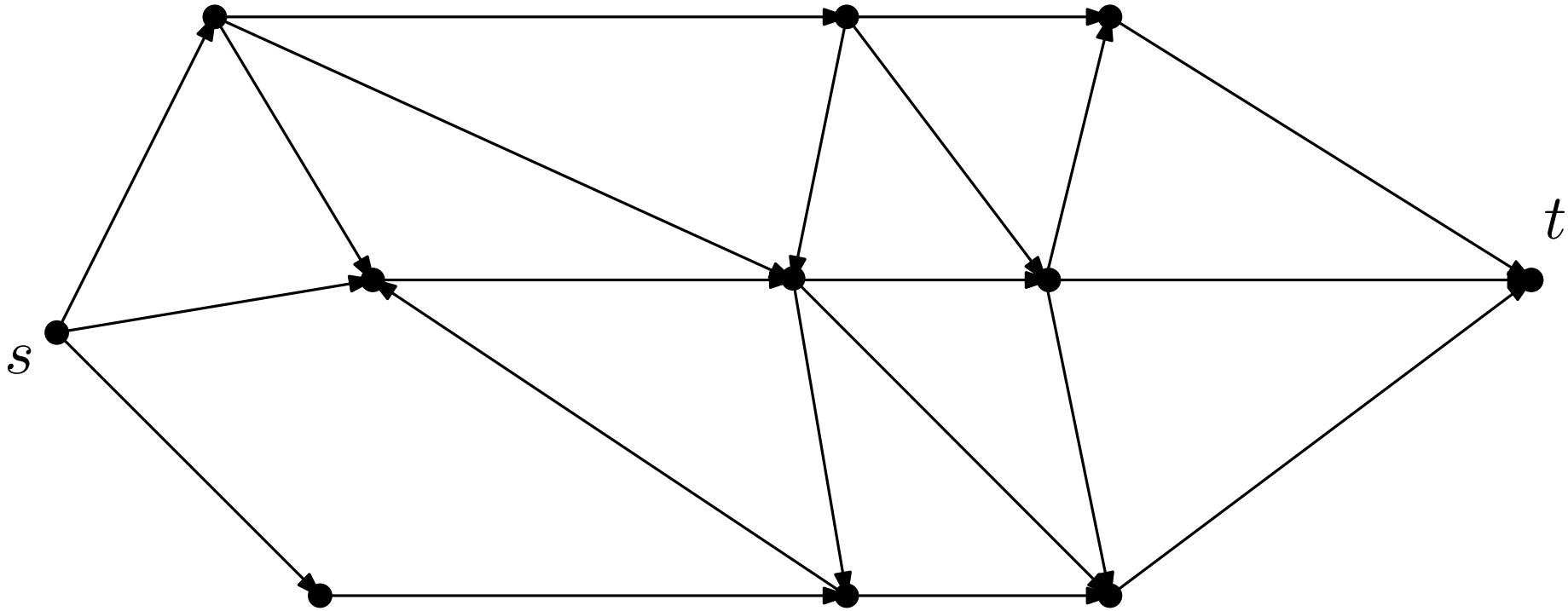

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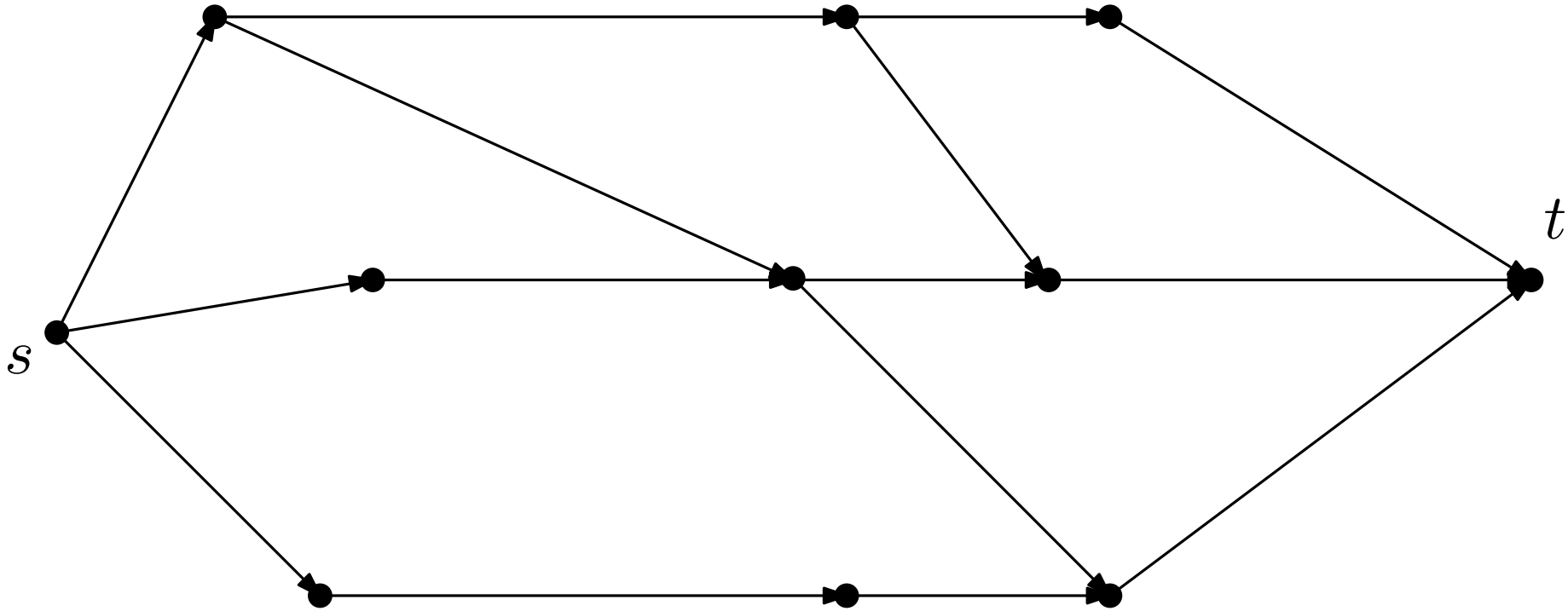

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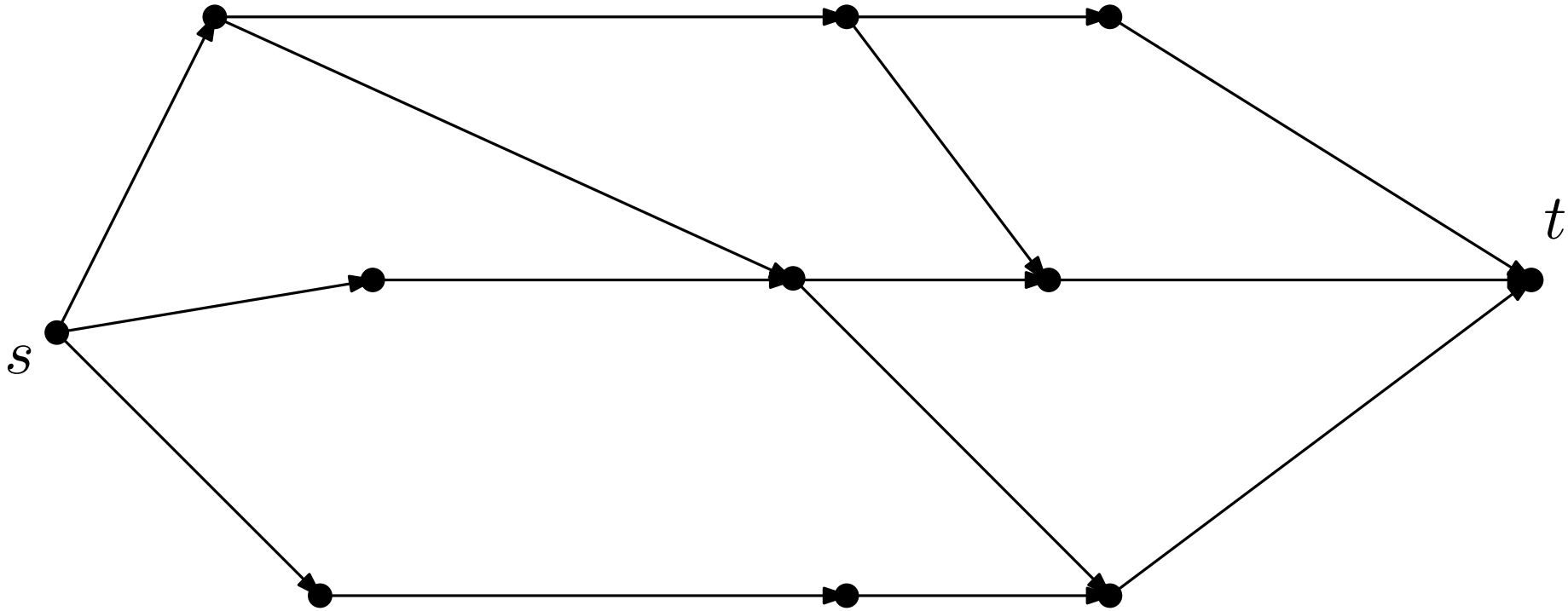

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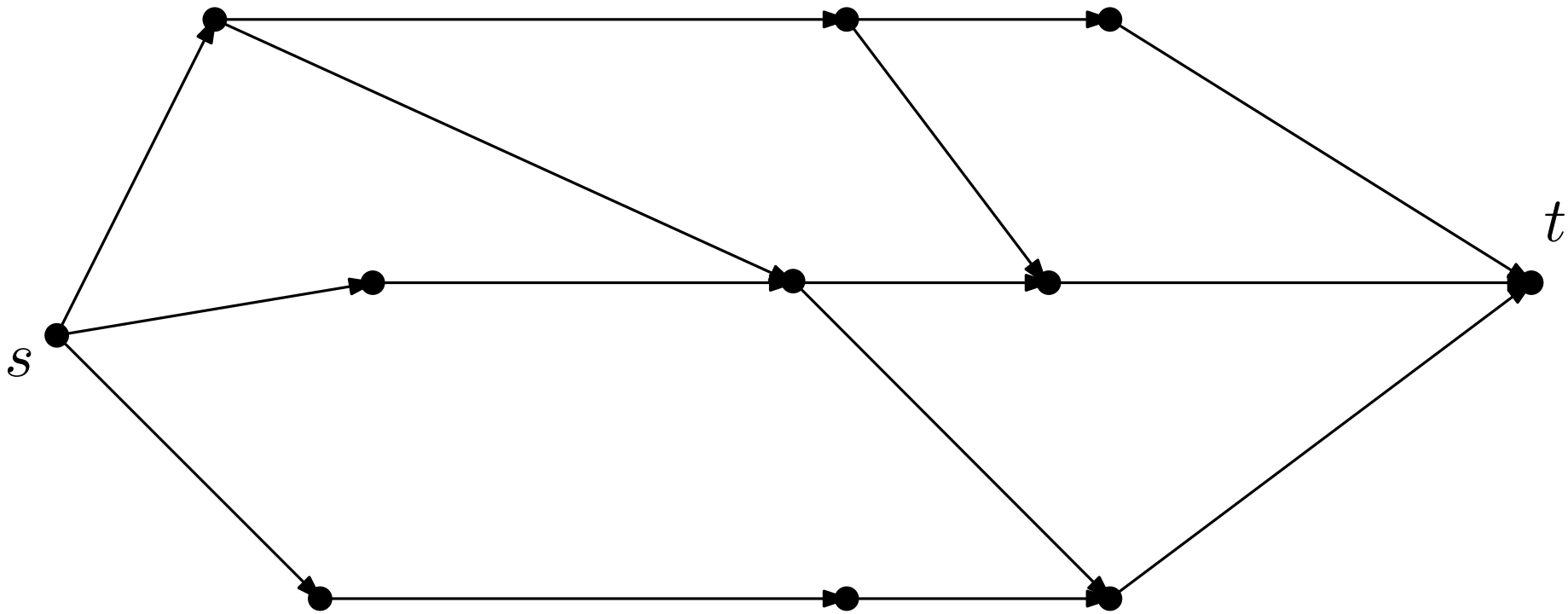

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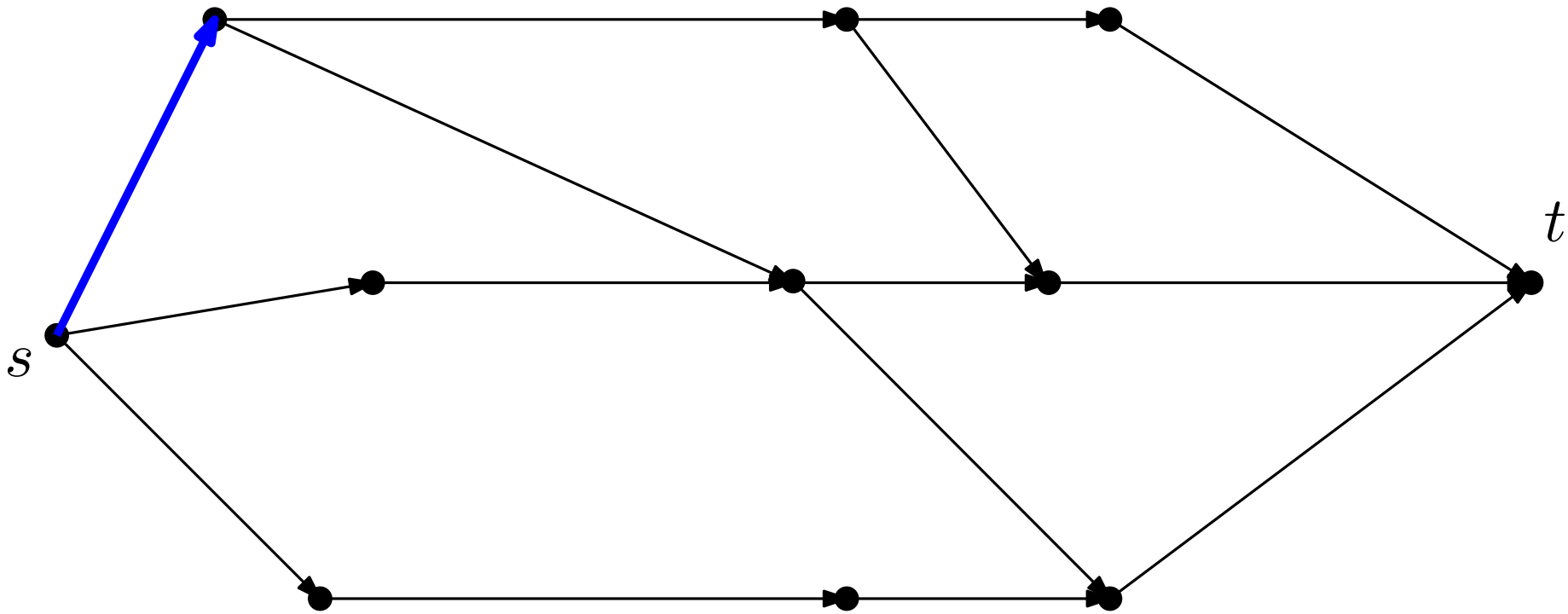

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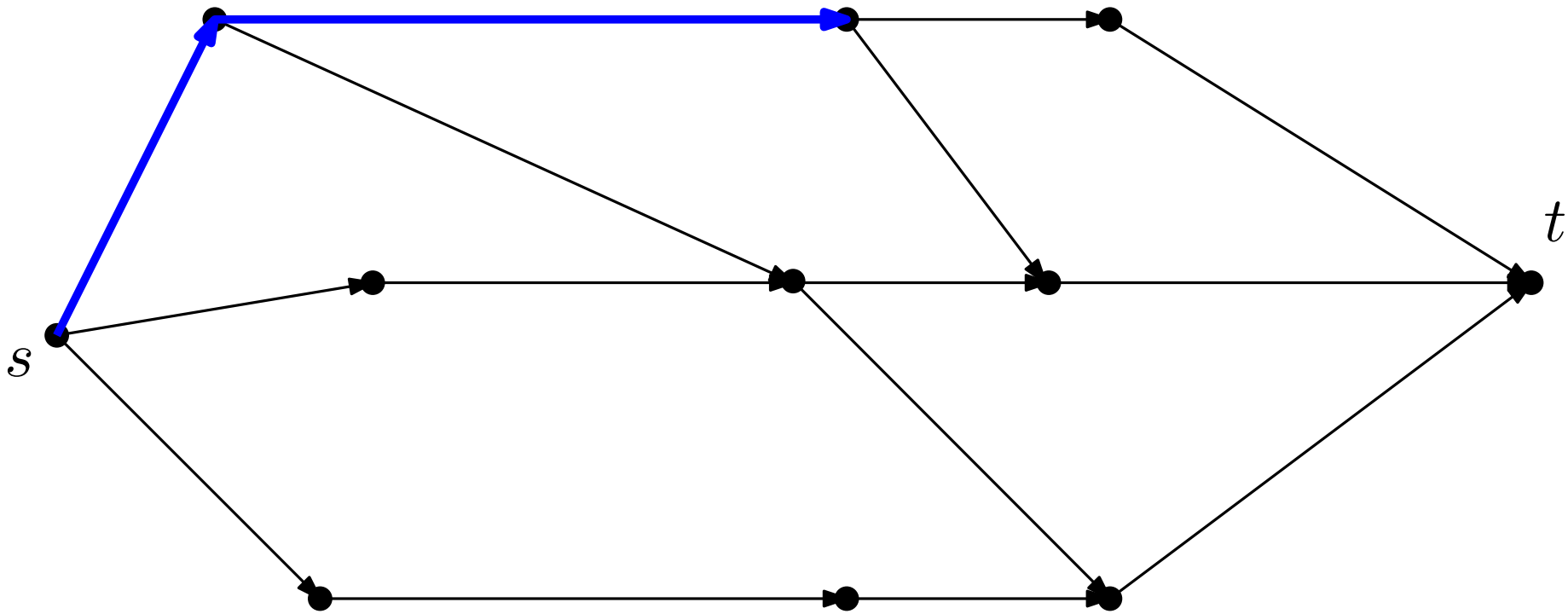

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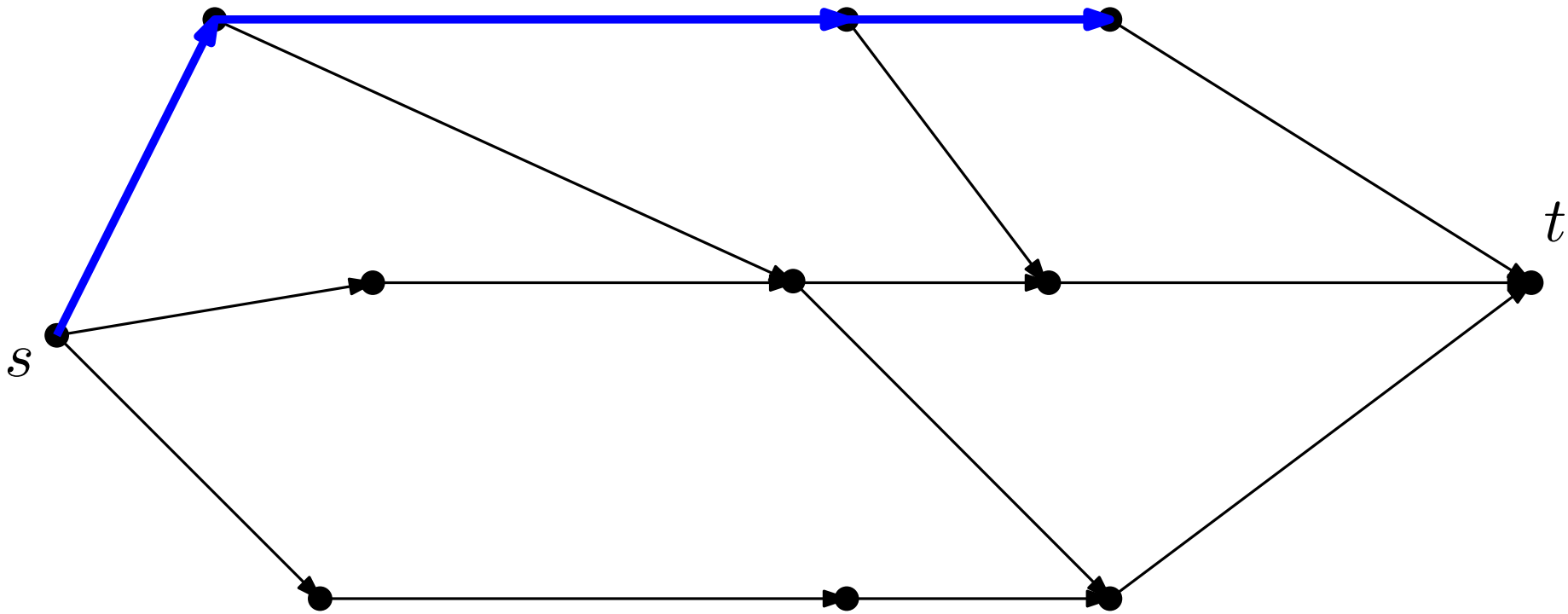

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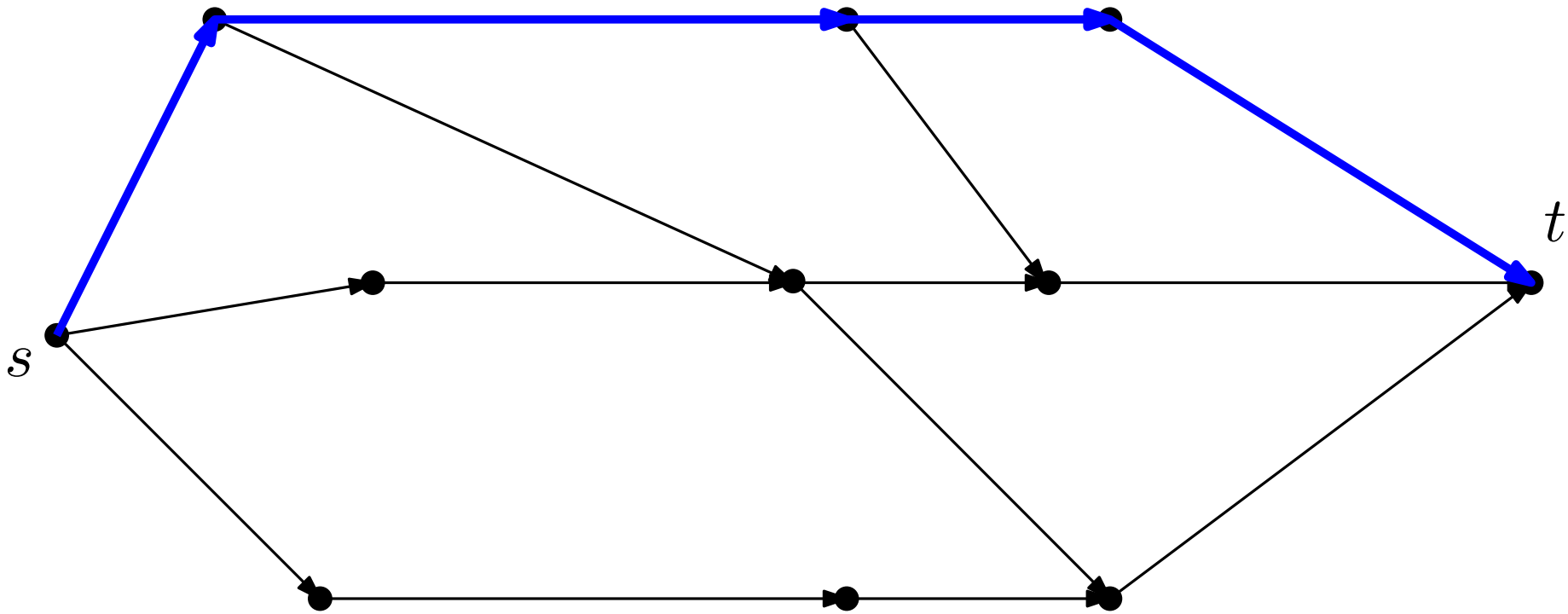

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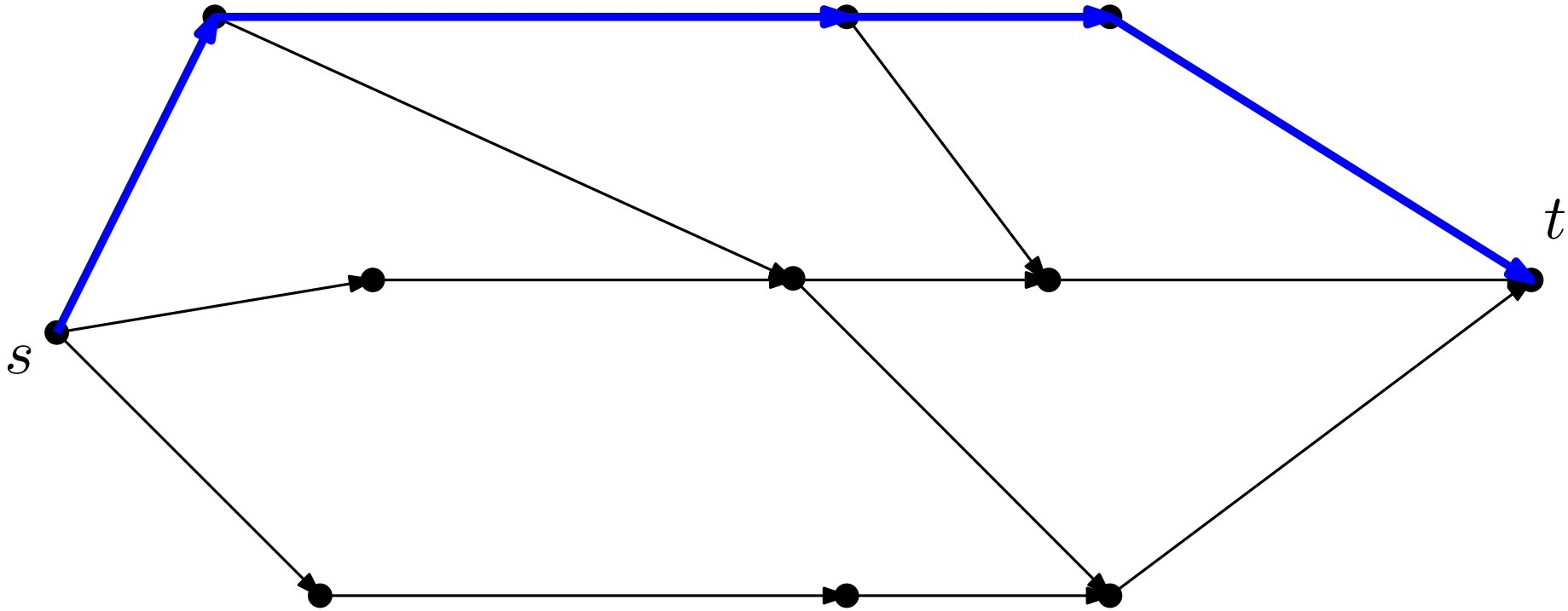

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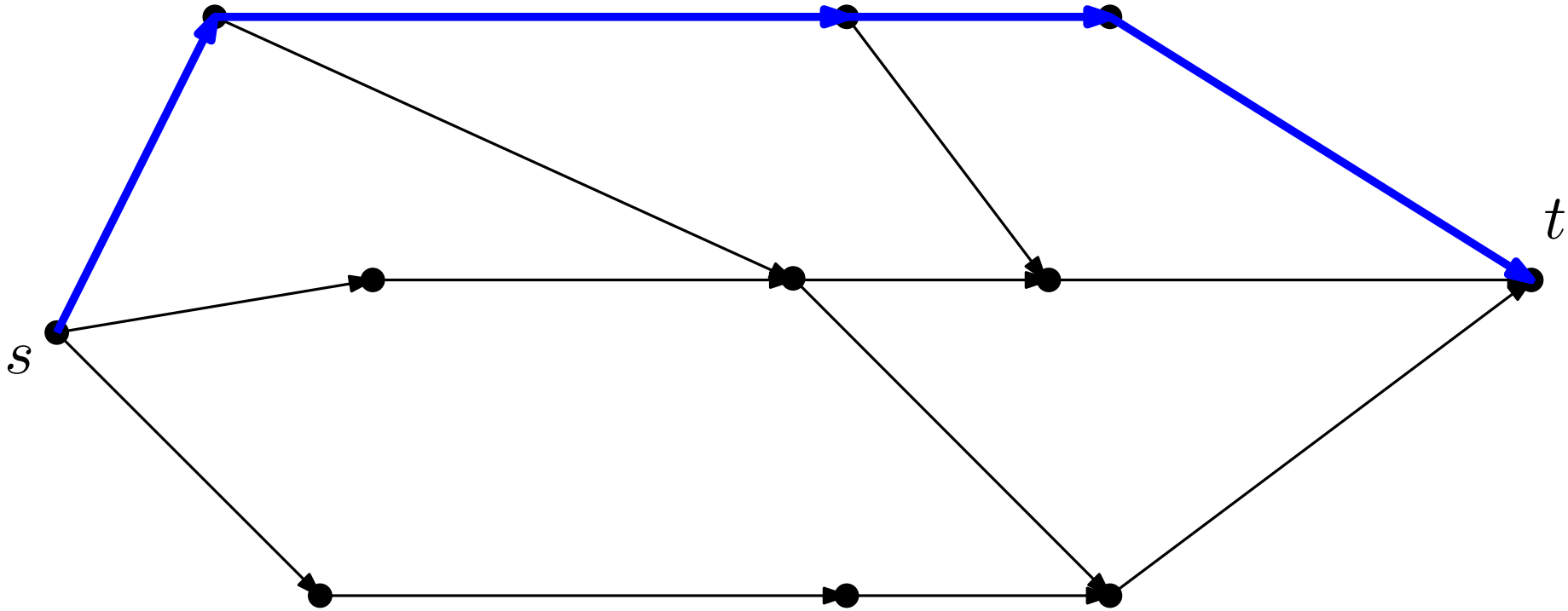

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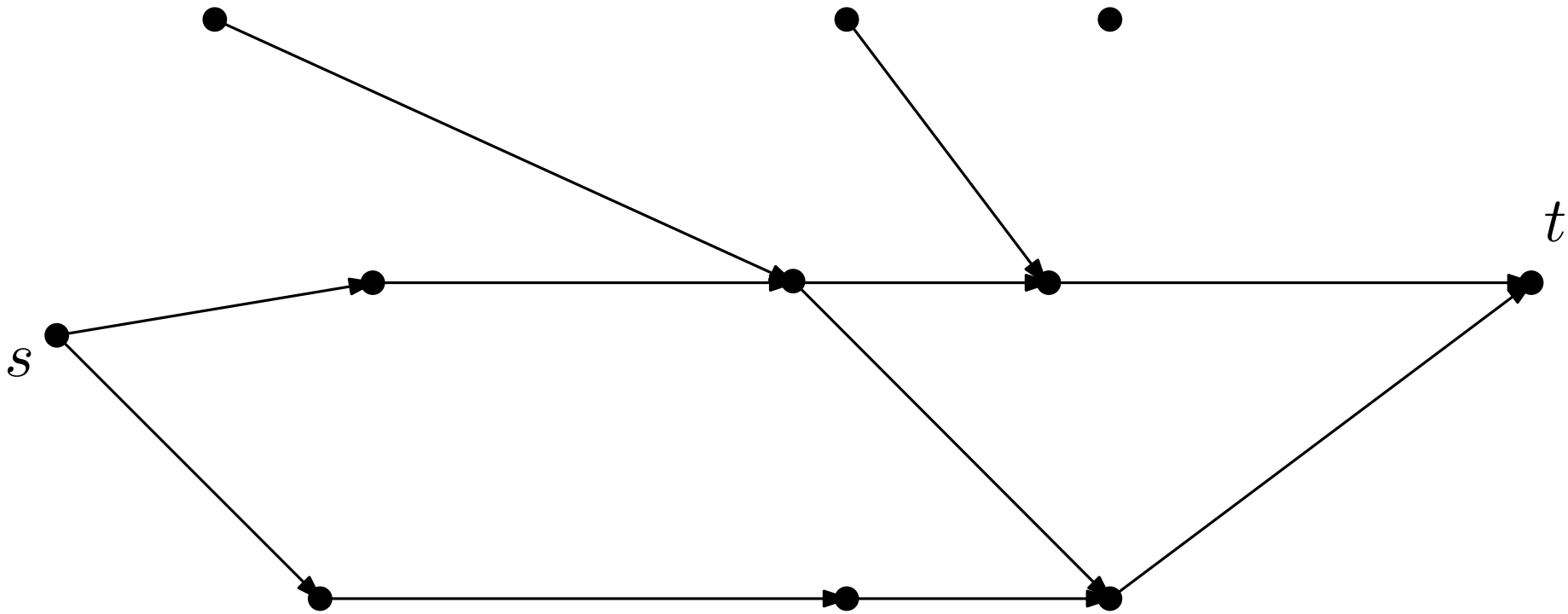

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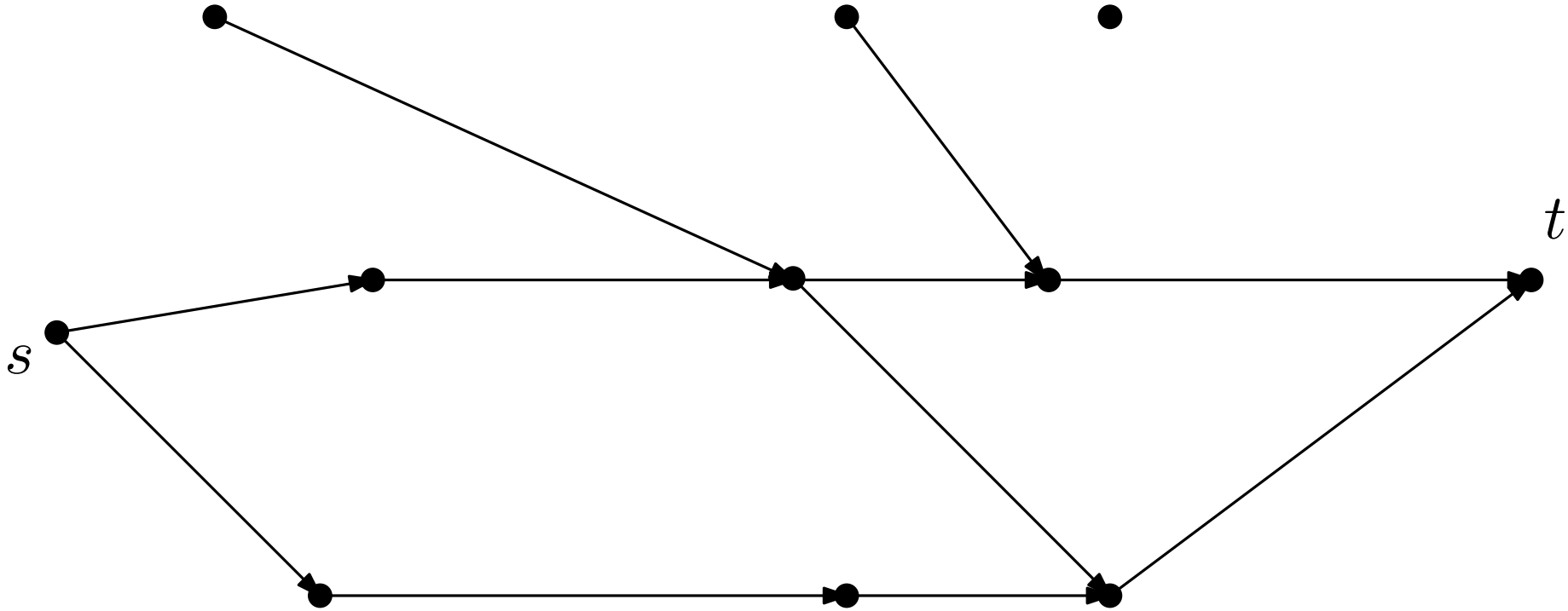

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Update capacities. This deletes k edges!

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There are at most $n - 2$ phases.

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Part III

Number of Phases in Unit Capacity Networks

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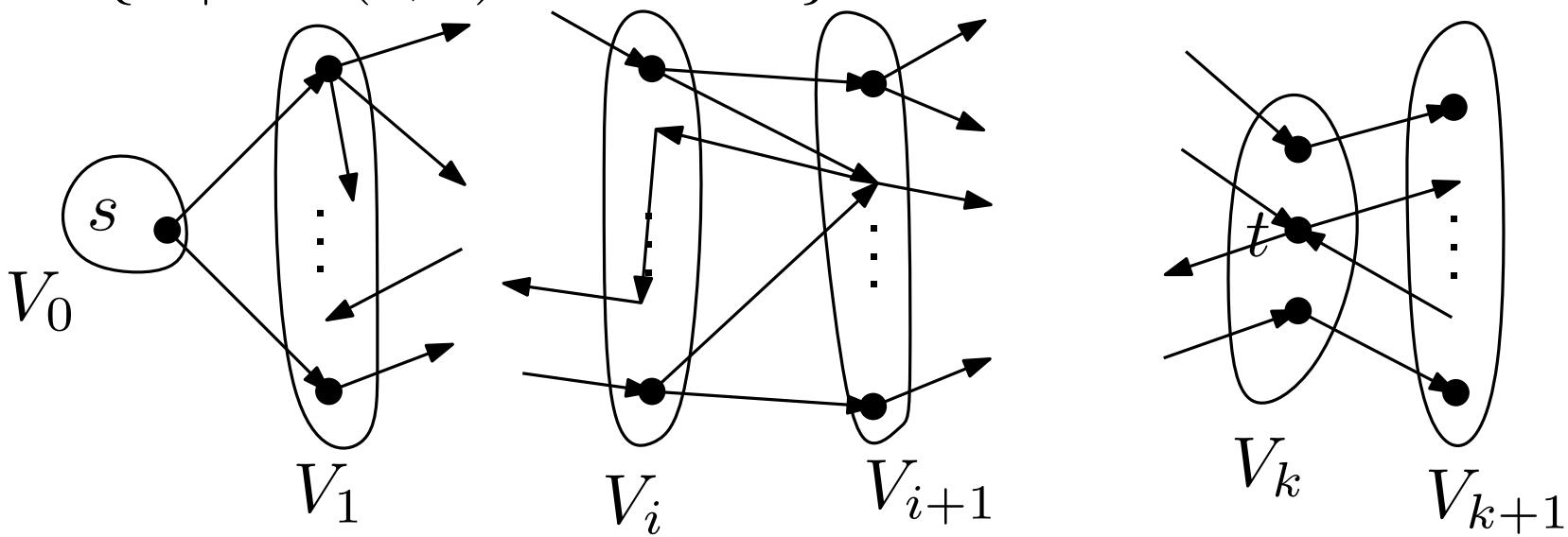

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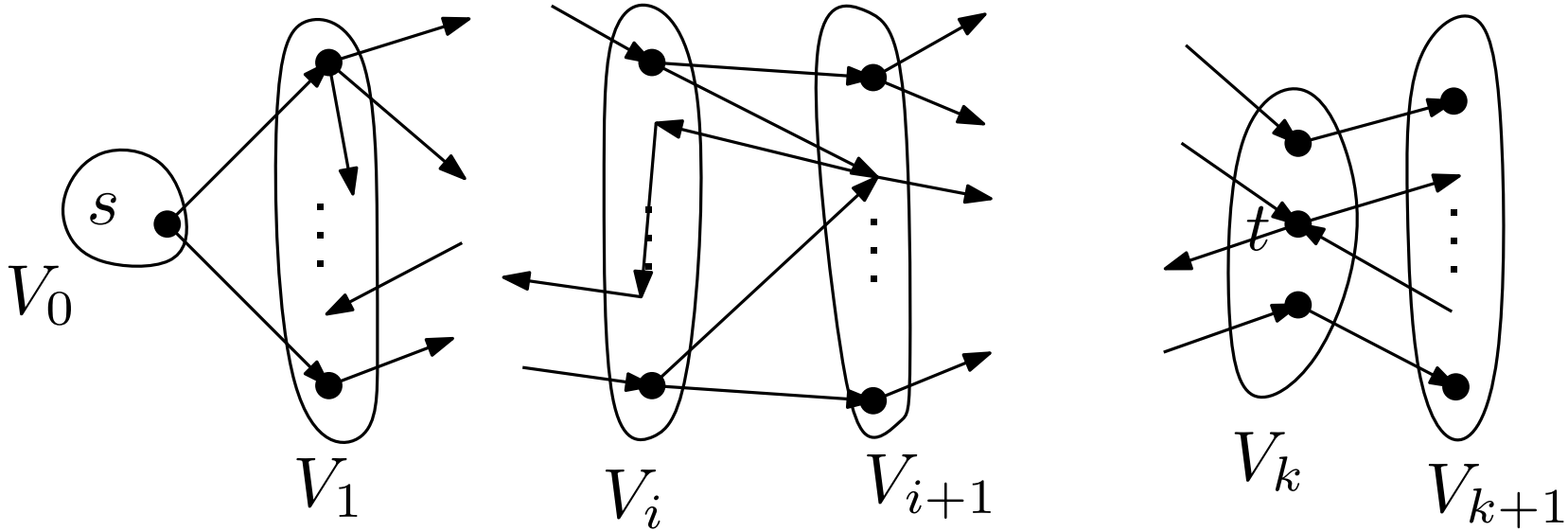

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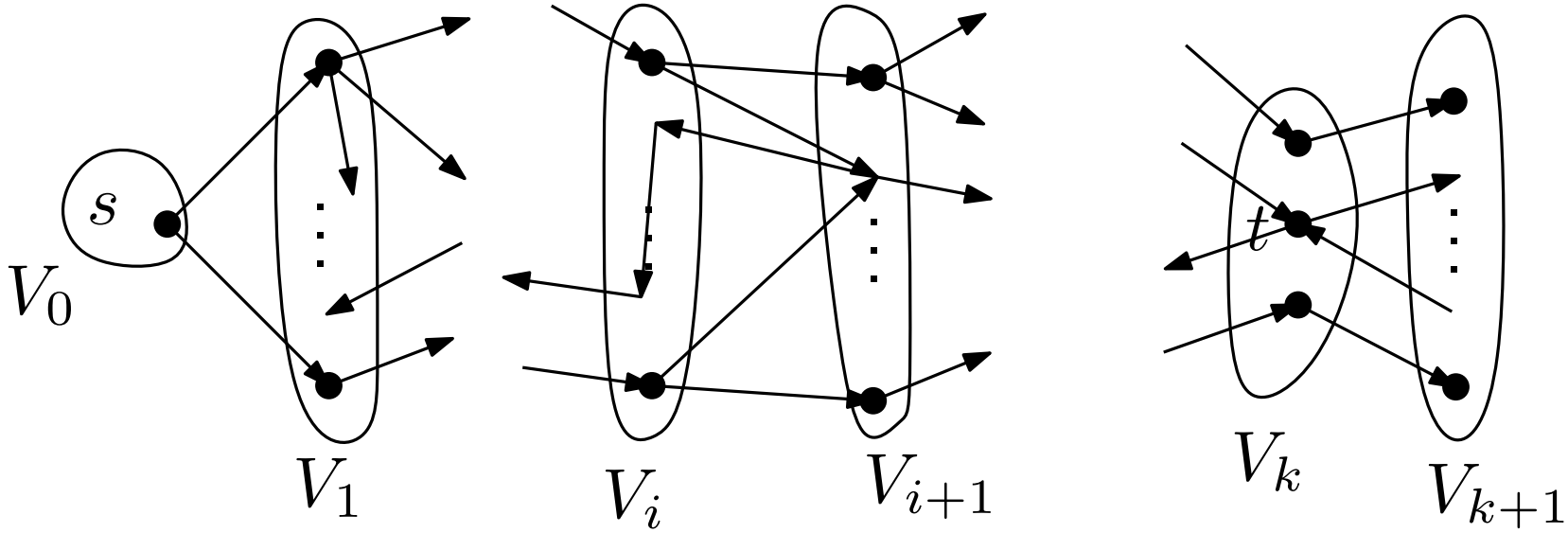

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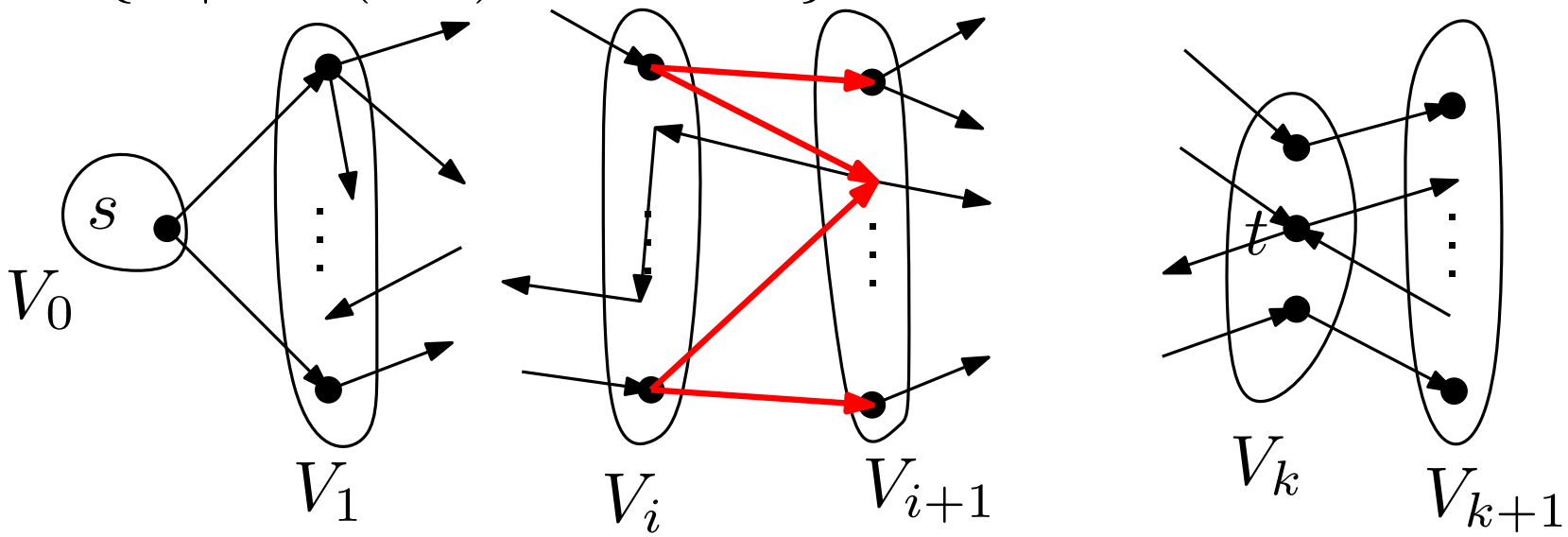

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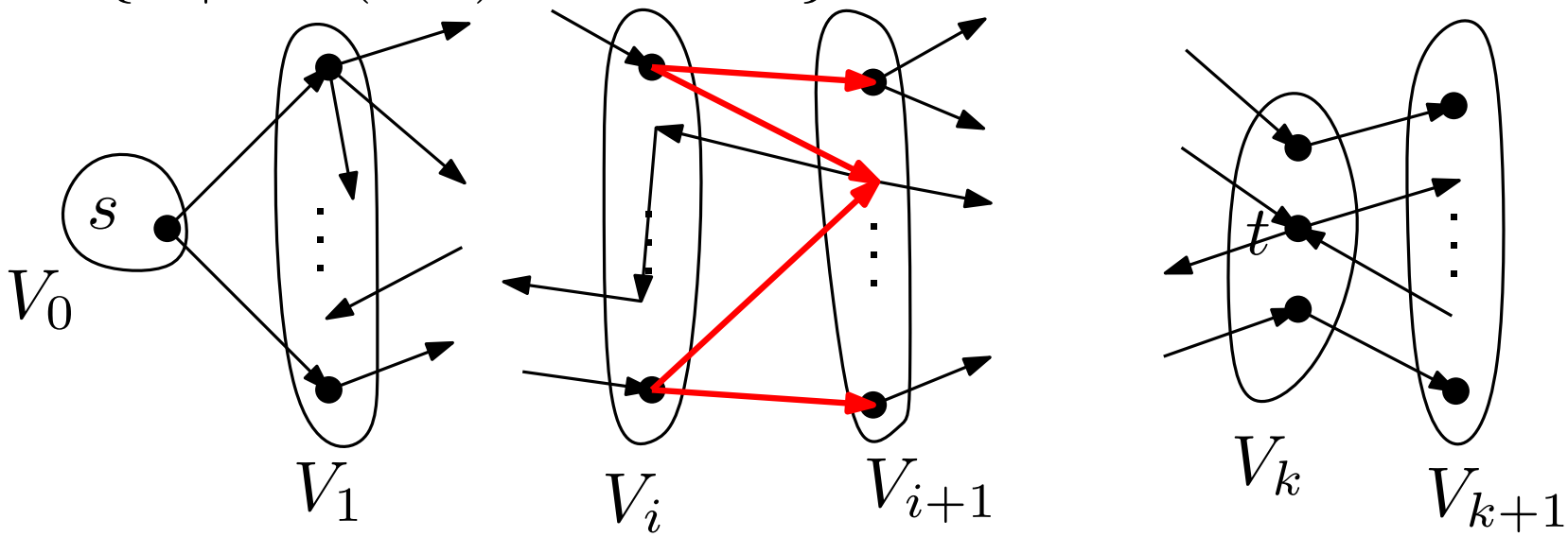

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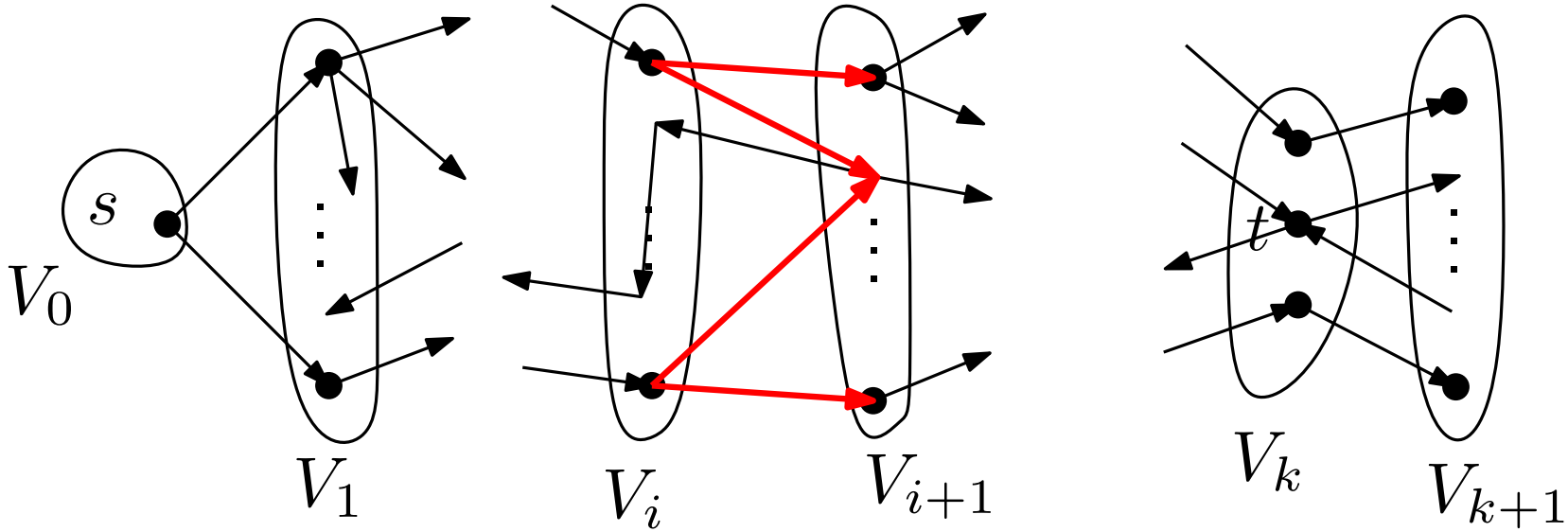

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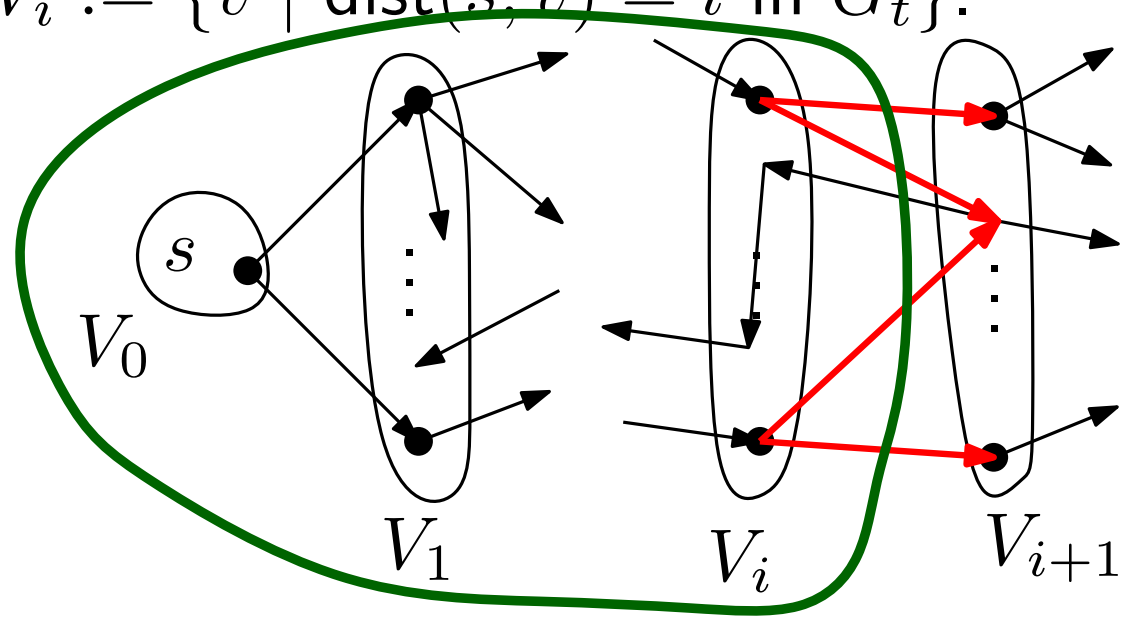

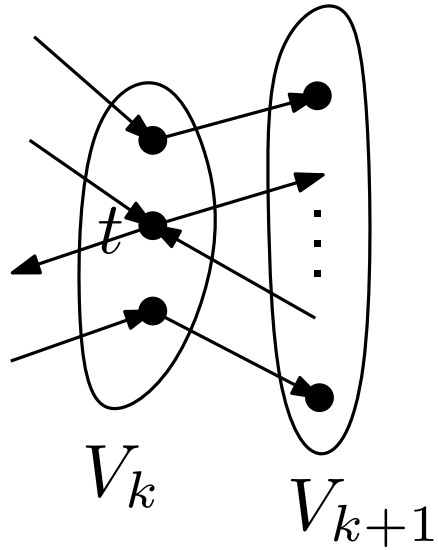

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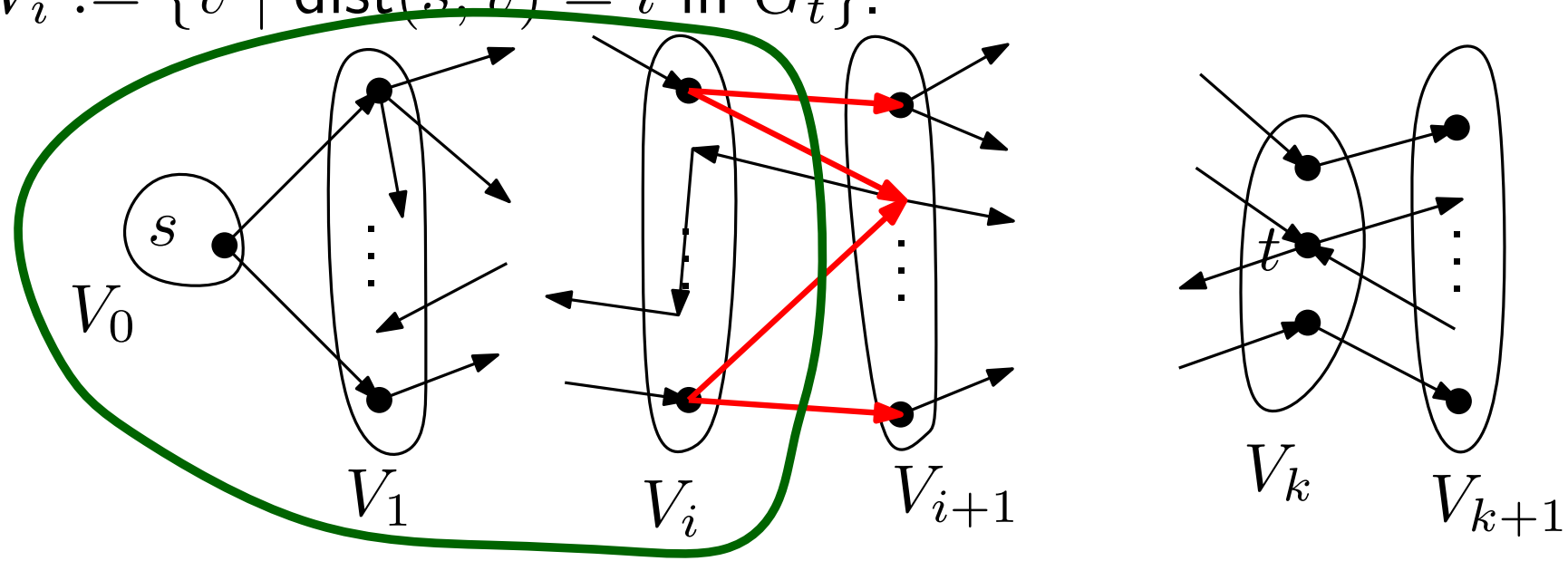

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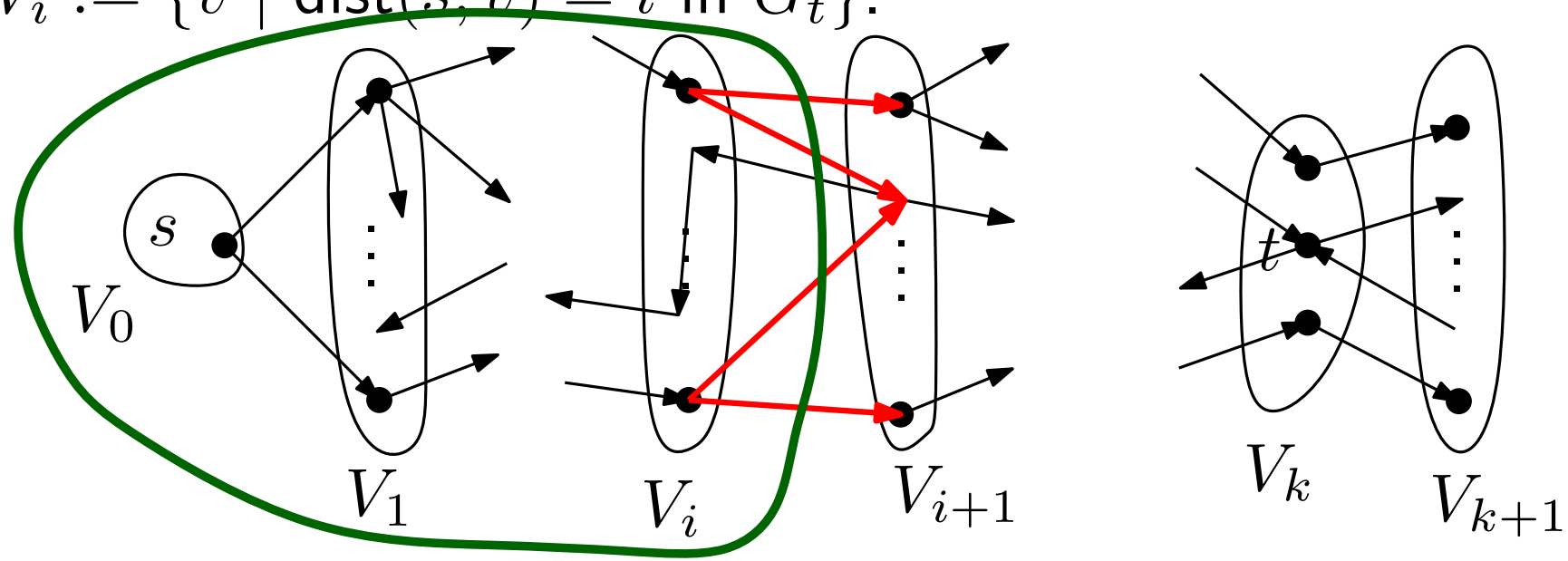

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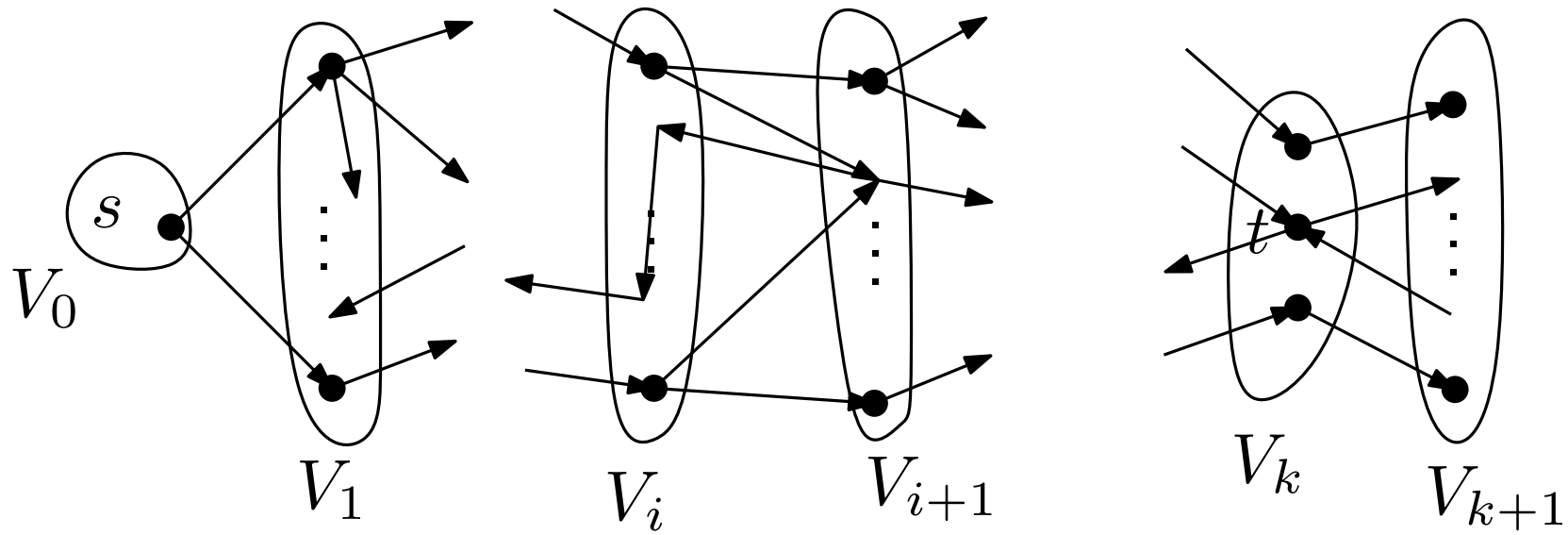

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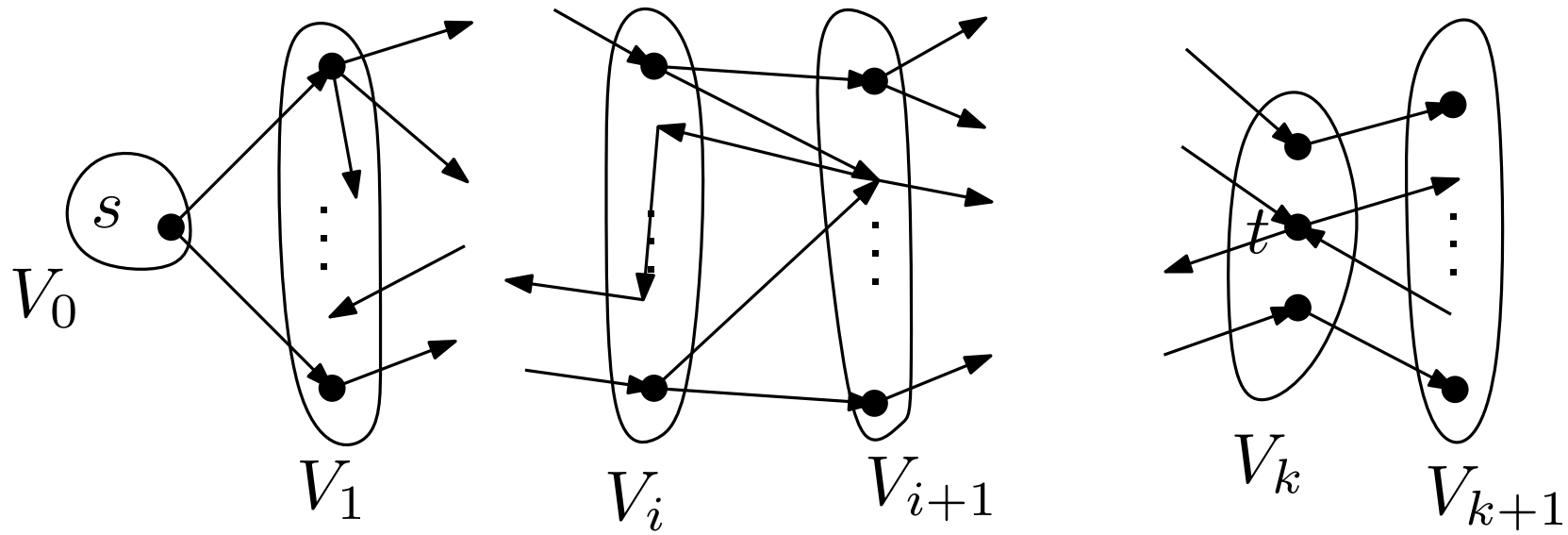

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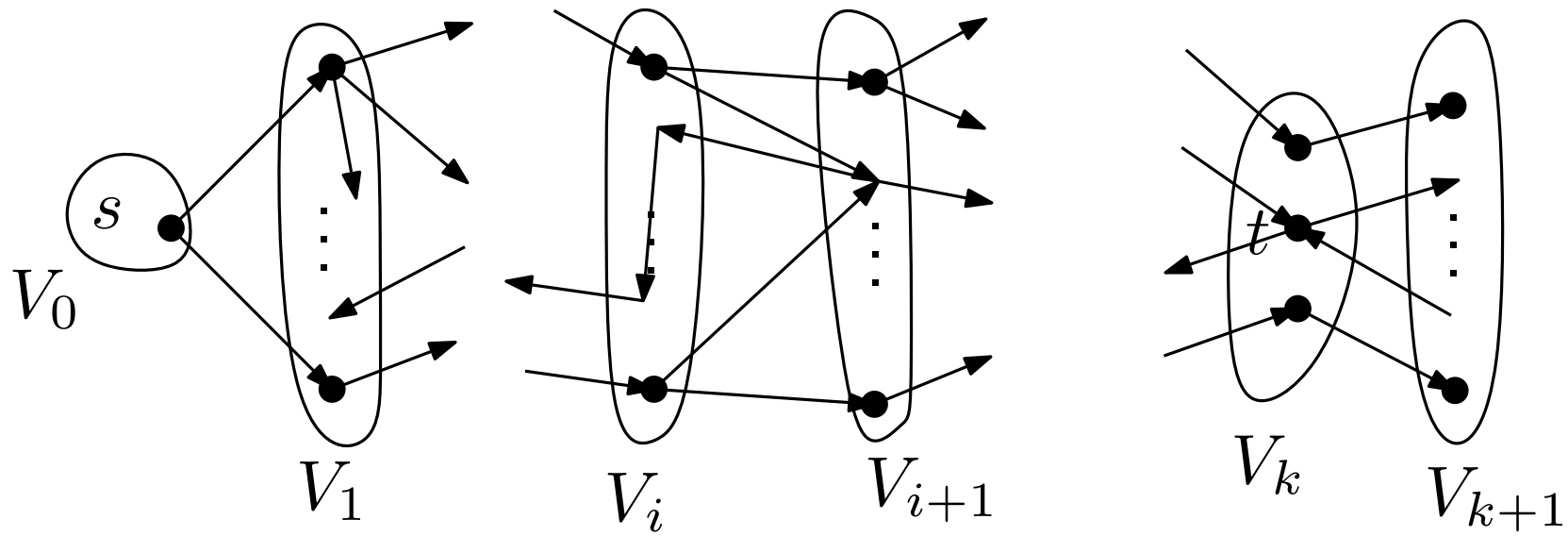

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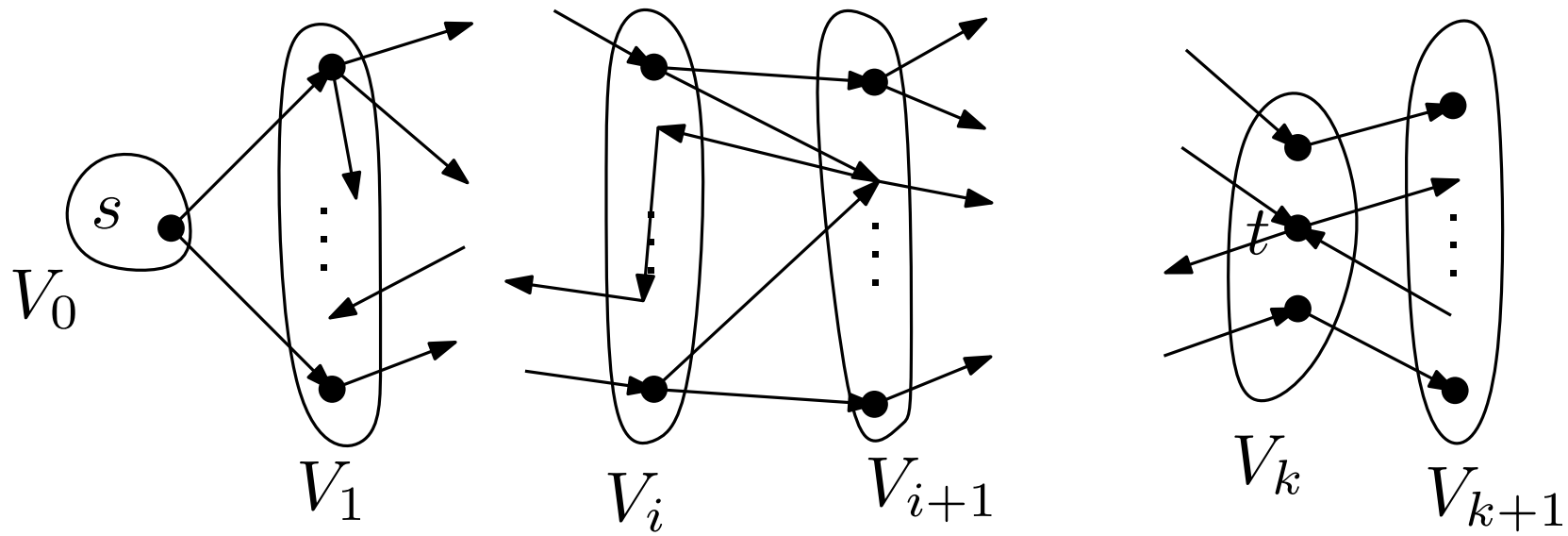

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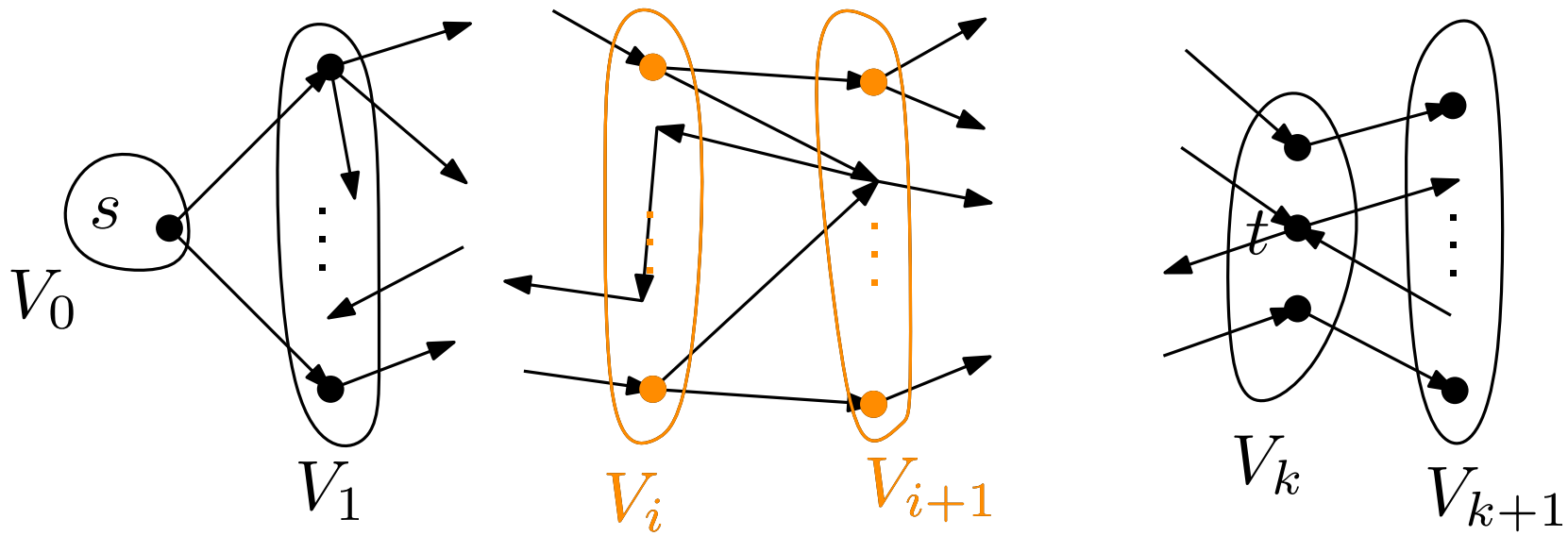

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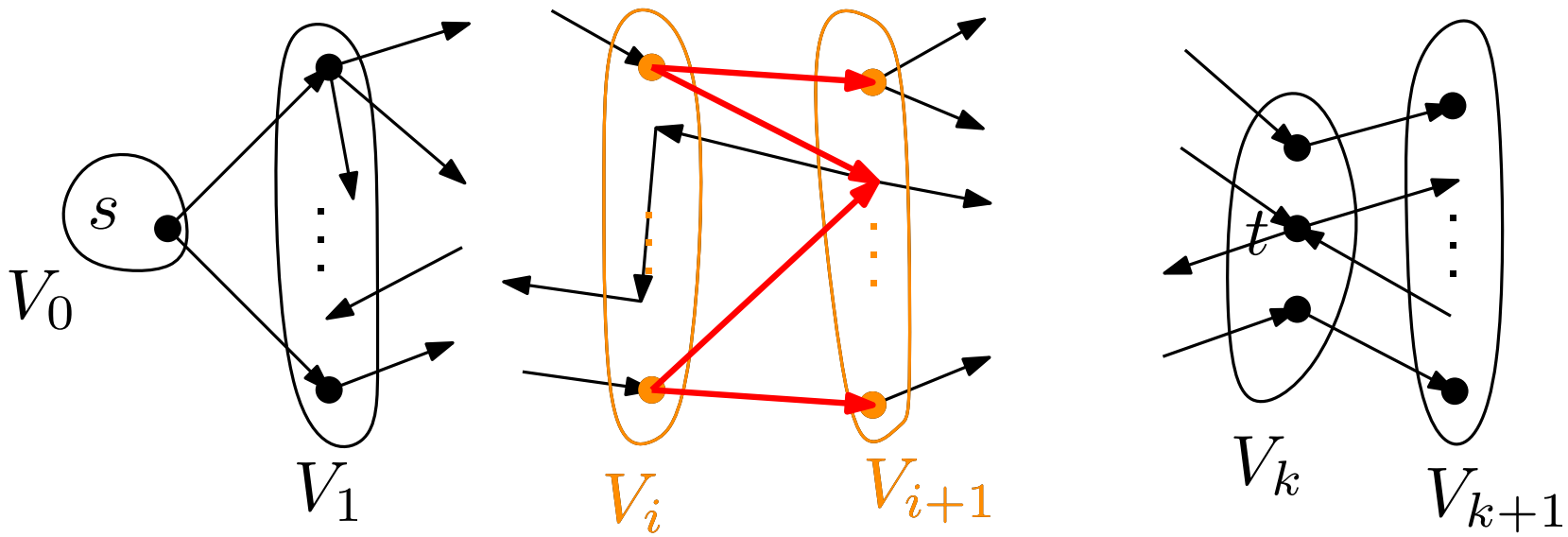

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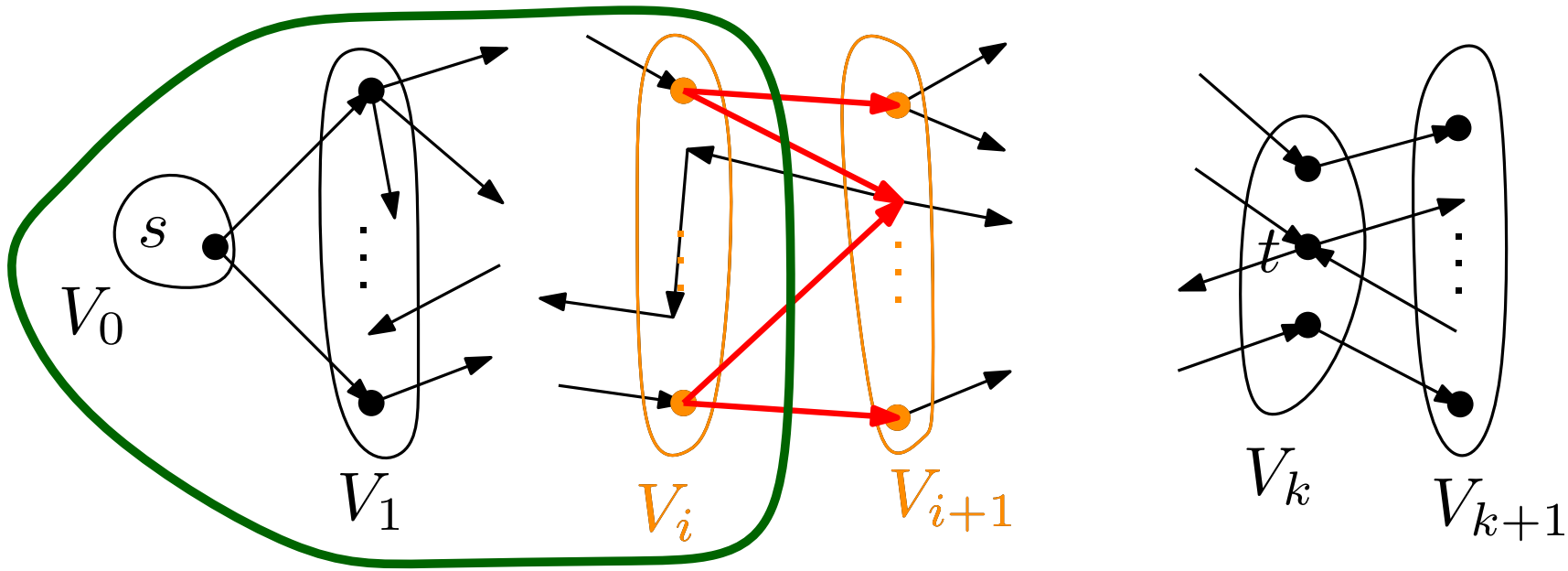

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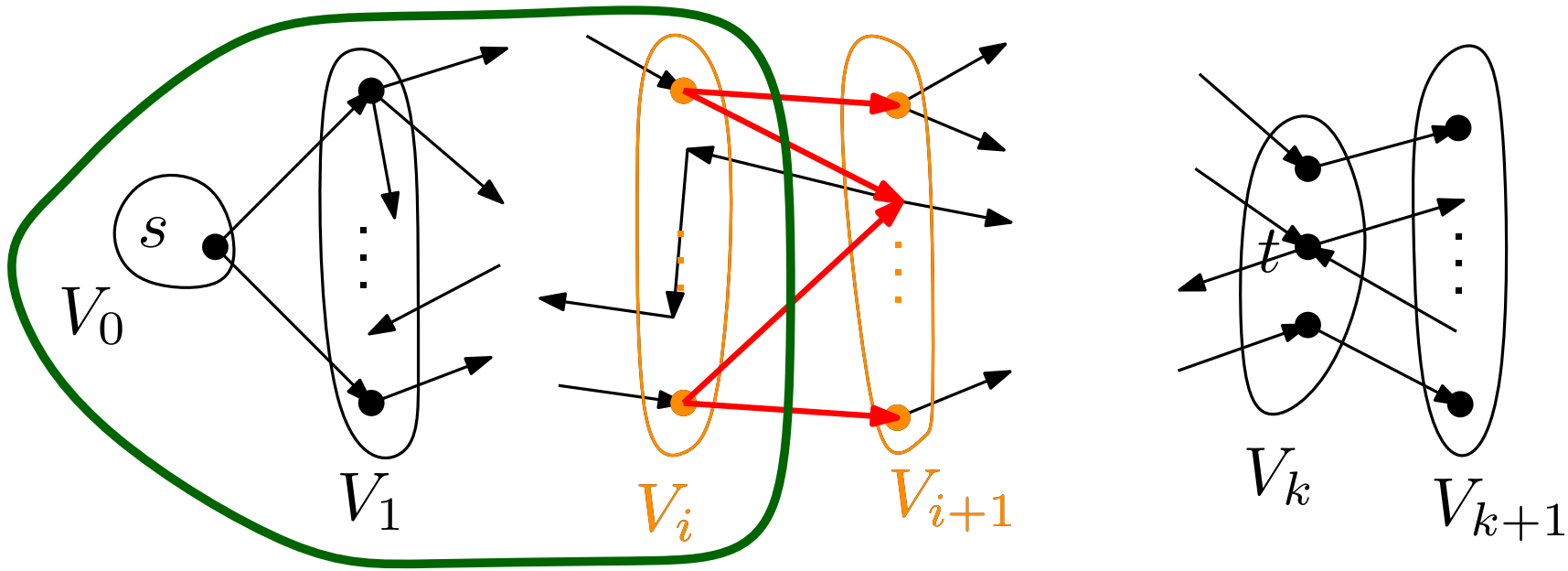

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Fewer than $\frac{n}{2n/k} = \frac{k}{2}$ sets V_i are big.

Thus, $V_0, \ldots, V_k$ contains two consecutive "small" sets V_i, V_{i+1}.

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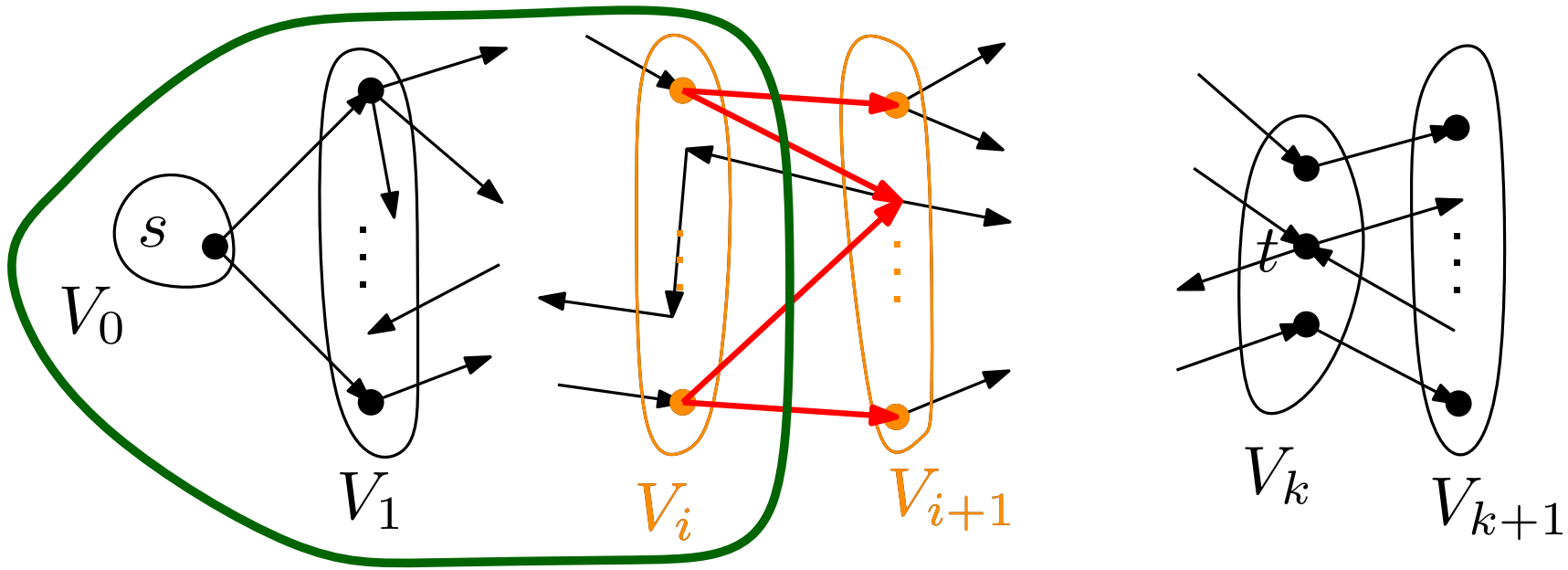

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$$\text{maxflow}(G_t) \leq \text{cap}(S, V \setminus S) \leq \frac{4n^2}{k^2} \leq \frac{4n^2}{t^2}.$$

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