Energy-Efficient Size Approximation of Radio Networks with no Collision Detection ⋆

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Abstract

Algorithms for radio networks are studied in two scenarios: (a) the number of active stations is known (or approximately known) (b) the number of active stations is unknown. In the second (more realistic) case it is much harder to design efficient algorithms. For this purpose, we design an efficient randomized algorithm for a single-hop radio network that approximately counts the number of its active stations. With probability better than $1-\frac{1}{n}$, this approximation is within a constant factor, the algorithm runs in poly-logarithmic time and is very energy efficient: its energy cost is $o(\log\log n)$. This improves the previous $O(\log n)$ bound (for energy). In particular, our algorithm can be applied to improve energy cost of known leader election and initialization protocols (without loss of time efficiency).

1 Introduction

Background In recent years mobile and wireless communication focuses a lot of attention. New technologies provide communication means, quite exact positioning and time synchronization via GPS systems, and quite reasonable computing resources in mobile devices. This provides grounds for new applications such as law enforcement, logistics, disaster-relief, and so on.

This development redefines the demands on communication algorithms. Communication through radio channels enables easy broadcasting and delivery of messages in a distributed system. However, new problems arise: collision of messages sent simultaneously produces noise, sending and receiving time has to be minimized due to energy consumption (pocket radio devices run on batteries!). Moreover, we cannot control which stations are in use, even cannot say how many of them are switched on! This chaotic behavior has also one advantage: once we are able to run an algorithm on such a network, then it is very robust.

The model A *radio network* (see e.g. [1, 3, 4, 9, 15, 18]), or *RN* for short, consists of an unknown number of computing devices called here *stations* communicating through a shared channel. This corresponds to hand-held devices running on batteries using a radio channel for communication. The devices are bulk-produced so we assume, as many authors, that the stations of a RN have no ID's or serial numbers.

We assume that the RN stations have local clocks synchronized globally (this is realistic due to GPS technology). Communication is possible in time slots determined by global time,

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called *steps*. During a single step any station may send and/or receive radio signals. If exactly one station sends a message, then it is available to all stations listening at this moment. That is, we consider here *single-hop* RN's. Many authors consider also the case in which several independent communication channels are available. Other generalization considered is that a network is defined by a graph, where nodes denote stations and a station v is reachable from the station u if and only if there is an edge (u, v) in the graph (so-called *multi-hop RN*, [1, 4, 5, 8]).

If more than one station sends during a step, then a *collision* occurs, the message is scrambled so that the stations receive noise. We consider here a weak no-collision-detection model, no-CD for short, in which the stations cannot even recognize that a collision has occured. It is quite often assumed that the station sending a message may simultaneously hear and thereby recognize that a collision has occured [2, 3, 7, 10, 14–16, 18]. This feature is quite strong algorithmically, but unavailable in some technologies (see [11] for results concerning the weaker model). Let us remark that the algorithm presented in this paper can be adopted to the weak model.

A radio network consists of an unknown number of stations that are switched on or *active*. An active station is *awake* in a step, when it is sending a message or listening, otherwise it is *sleeping*.

Complexity measures Most important complexity measures for RN's are *time* and *energy cost* [14–16]: time is the number of steps required for executing an algorithm, energy cost is the maximal number of steps at which a station of a RN is awake. This is motivated by the fact that sending and receiving messages are main sources of energy consumption ([6, 17]). For the sake of simplicity we neglect here differences between energy consumption of sending and listening.

RN's intensively use randomization. This is absolutely necessary, for instance, for symmetry breaking. In the case of a randomized algorithm energy cost e means that with high probability no station is awake for more than e steps. This is motivated by the fact that an accidental burst of energy consumption at a single station may exhaust its batteries and prevent the station to follow the algorithm. For this reason, a lot of recent research is focused on energy efficient RN algorithms (see e.g. [2, 5, 14–16]).

Known versus unknown number of stations The algorithms for RN's are designed under the following assumptions:

scenario I: the number of active stations is known;
scenario II: the number of active stations is unknown, but can be approximated within a constant factor;
scenario III: there is no information on the number of active stations.

Since many algorithms for RN are designed for known number of active stations, it is desirable to design an efficient procedure for converting scenario III into scenario I or at least II. Therefore we consider the following *size approximation problem:*

> given a RN with unknown number n of active stations. Find a number n' such that $\frac{1}{c}n \leq n' \leq c \cdot n$ for a certain constant c. At the end of the protocol each active station should be aware of n'.

In [2], a solution with logarithmic energy cost and $O(\log^2 n)$ time is given.

It is much harder to design algorithms in scenario I than in scenario III. For instance, an energy efficient solution to initialization problem in scenario I is proposed in in [15]: with probability at least $1-\frac{1}{n}$ its energy cost is $O(\log\log n)$ and execution time is $O(n)$. It can be easily generalized to the case of scenario II; finding an energy efficient solution for scenario III was left as open problem.

For another important problem, leader election, energy efficient algorithms have been designed in [14]. The authors present a randomized algorithm that for n stations elects a leader in scenario I in time $O(\log f)$ and energy $O(\log\log f + \frac{\log f}{\log n})$ with probability $1-1/f$ for any $f \geq 1$. Moreover, they present algorithms that work in scenario III and elect a leader within $O(\log n)$ energy cost and $O(\log^2 n)$ time with probability $1-\frac{1}{n}$.

A deterministic RN algorithm determining the number of active stations (for stations with distinct ID's from the set $\{1,\ldots,n\}$) is presented in [12, Theorem 2] (formulated as a solution for leader election):

Lemma 1 ([12]). *Consider a RN with at most n active stations with labels in the range $[1,n]$, each $t<n$ assigned to at most one active station. There is a deterministic protocol finding the station with the largest label and counting the number of active stations which runs in time $O(n)$ and has energy cost $O((\log n)^{\varepsilon})$ for any $\varepsilon>0$.*

2 New results

Our main result is an energy efficient solution to size approximation problem:

Theorem 1. *There is a randomized algorithm for weak no-CD RN such that with probability at least $1-\frac{1}{n}$ a number n_0 is found such that $\frac{1}{c}n_0 \leq n \leq cn_0$ (for some constant $c \geq 1$) within time $O(\log^{2+\varepsilon} n)$ with energy cost $O((\log\log n)^{\varepsilon})$ for any constant $\varepsilon>0$.*

An example for application of this result is that together with [15, Theorem 6.1] it solves an open problem from [15]:

Corollary 1. *There is an initialization protocol for a no-CD RN with n stations in scenario III that runs in time $O(n)$ and has energy cost $O(\log\log n)$ with probability at least $1-\frac{1}{n}$.*

It also yields an algorithm for leader election in scenario III with poly-logarithmic time and energy cost $O(\log\log n)$ (with probability $1-\frac{1}{n}$). This improves exponentially energy cost upon the algorithms from [14], without loss of time efficiency, when time achieved with high probability (that is, $1-1/n$) is considered.

3 Size approximation algorithm

3.1 Basic algorithm

Let us consider the following experiment. During a step each of n active stations sends a message with probability p. If exactly one station chooses to send, then we say that the step is *successful* and that the station that have sent a message *succeeds*. Note that the probability p_S that a step is successful, equals $np(1-p)^{n-1}$. The largest value of this expression is obtained for $p=1/n$; then $p_S \approx 1/e$. If we repeat the experiment independently in l steps, then the expected number of successful steps is approximately l/e. By Chernoff Bounds (see Appendix, inequalities (2),(3)), we may bound the probability that the number of successes is far from its expectation.

Using these observations we construct the first, energy inefficient, algorithm. Let d be a sufficiently large constant.

Basic Algorithm

```
for k = 1,2,...  run phase k:
  repeat d·k times:
      each station sends with probability 1/2^k and listens all
        the time counting the number of successful steps
  if the number of successful steps is close to d·k/e then n ← 2^k, halt
```

Using Chernoff Bounds one can show that the value of n found by the algorithm is not far from the number of active stations. However, this simple algorithm has large energy cost: about $\log n$ phases are to be executed, during each phase every active station is listening all the time. So each station is awake for $\Theta(\log^2 n)$ steps.

3.2 Improvements idea

The first change is that only the stations that send can listen at this moment. This reduces the energy cost of each station to the number of steps in which it sends. However, then no station knows the number of successful steps, but only the number of the successful steps during which it has been sending. To solve this problem we apply the algorithm from Lemma 1. For this algorithm each station has a "temporary" ID – the first i such that it has succeeded in step i during this phase. In this way, all successful stations learn the total number of successful steps (a slight adjustment is necessary since the same station may succeed more than once). In an extra step the station with the smallest "temporary" ID informs all stations about this number.

Still, every station has to listen at the end of each phase, and about $\log n$ phases are to be executed making energy cost $\Omega(\log n)$. A relatively simple remedy is to make such a step "all have to listen" only for the phases l such that $l = g(j)$ for $j \in \mathbb{N}$ where g is a function satisfying $g(j+1) \geq \lceil g(j)^{1+\varepsilon} \rceil$ for some $\varepsilon > 0$ (again, we apply Lemma 1, for determining the first successful phase among $g(j-1)+1,\ldots,g(j)$). This does not postpone getting the final result very much: we may expect that the number of phases is at most $m^{1+\varepsilon}$, where m is such that Basic Algorithm finishes its work at phase m, i.e. $m = \Theta(\log n)$. Now the number of the "obligatory listening steps" is reduced to the minimal l such that $g(l) \geq \log n$.

Even with these changes, we are unable to guarantee with high probability that every station is awake for $o(\log\log n)$ steps. Namely, we have to guarantee low energy cost due to sending messages. For technical reasons, we split the protocol in two parts. During the first part, we execute the simple algorithm described above for $k = 1,2,\ldots,k_0$, where k_0 is some constant. For the second part, we make a crucial change that every station sends at most once in the loop taken from Basic Algorithm. (However, some additional activities will be necessary for informing all stations about the number n found.) Then, it becomes "passive" and only listens to the results. The idea is that this modification does not change the number of successful steps at each phase substantially, because the number of stations "eliminated" in this way is very small with respect to n. However, we are faced with nasty technical details due to the fact that the steps are stochastically dependent. So, a careful argument is required to estimate the number of successes.

3.3 Description of the algorithm

Let $g(1) = 2$, $g(i+1) = \lceil (g(i))^{1+\varepsilon} \rceil$ for a constant $\varepsilon > 0$; let d and k_0 be large enough constants.

Algorithm ApproxSize(ε)**:**

(01) run k_0 phases of Basic Algorithm
(02) each station sets *status* $\leftarrow$ *fresh* and *temp2ID* $\leftarrow 0$
(03) `for` $k = k_0+1, k_0+2\ldots$ `do`
(04) each station sets *tempID* $\leftarrow 0$
(05) `for` $j = 1$ `to` $d \cdot k$ `do`
(06) each *fresh* station sends and listens with probability $1/2^k$
(07) each *fresh* station that sends a message sets *status* $\leftarrow$ *used*
(08) `if` a station have sent successfully `then` it sets *tempID* $\leftarrow j$
(09) using Lemma 1 for $n = dk$ and ε the stations with *tempID* $\neq 0$:
(10) compute N, the number of such stations and
(11) find a station with the smallest *tempID*
(12) `if` $N \geq dk/20$ `then`
(13) the station with the smallest *tempID* sets its *temp2ID* $\leftarrow k$
(14) `if` $k = g(l)$ for an $l \in \mathbb{N}$ `then`
(15) all stations with *temp2ID* $\neq 0$ elect a leader (with the smallest *temp2ID*) using Lemma 1 for $n = k$ and ε
(16) a leader (if elected) sends its *temp2ID* and all other stations listen
(17) `if` not(noise/silence) `then`
(18) every station sets $m \leftarrow 2^p$ after receiving p `and` halts

Let the loop consisting of lines (05)-(08) for $k = i$ be called *phase* i. Let *trial* j *of phase* i denote the jth execution of line (06) during phase i.

The following properties follow directly from the construction: if ApproxSize halts with $k = k''$, then the first value k' of k, for which the condition from line (12) has been satisfied, fulfills $\lceil (k')^{1+\varepsilon} \rceil \geq k''$. Also, after halting all active stations hold the same number $m = 2^{k'}$ assigned in line (18).

4 Analysis of the algorithm

4.1 Complexity analysis

Let n be the number of active stations. We say that the protocol succeeds for a number k', if the condition in (12) is satisfied for $k = k'$. Let k_s be the smallest number k for which the condition in (12) is satisfied.

Lemma 2 (Main Lemma). *For $k \leq \log n - 6$ algorithm ApproxSize succeeds with probability $O(\frac{1}{n^2})$. For $k \leq \lceil \log n \rceil$ it does not succeed with probability $O(\frac{1}{n^2})$.*

We postpone the proof of this result and yield corollaries which establish Theorem 1.

Corollary 2. *ApproxSize running on n active stations halts and outputs m such that $m = \Theta(n)$ with probability $1-q$, where $q = O(\log n/n^2)$.*

Proof. It follows from the fact that with high probability $\log n - 6 \leq k_s \leq \log n + 1$ and $m = 2^{k_s}$. ∎

Corollary 3. *ApproxSize(ε) running on n active stations halts within $O(\log^{2+2\varepsilon} n)$ steps for any $\varepsilon \geq 0$ with probability $1-q$, where $q = O(1/n^2)$.*

Proof. Assume that $\log n > k_0$. Our algorithm needs a constant time for steps (01)-(02) and with high probability it finishes the loop (03)-(18) for $k = k_f$ for some $k_f \leq (\log n)^{1+\varepsilon}$ (by Lemma 2). One execution of (05)-(08) takes time $O((\log n)^{1+\varepsilon})$. By Lemma 1, steps (09)-(11) take time $O((\log n)^{1+\varepsilon})$. The same applies to the line (15). So the loop (05)-(18) takes $O(\log^{1+\varepsilon} n)$ steps, and the whole protocol $O(\log^{2+2\varepsilon} n)$ steps. ∎

Corollary 4. *ApproxSize*(ε) *running on n active stations has energy cost* $O((\log\log n)^{\varepsilon})$ *with probability* $1-q$*, where* $q = O(1/n^2)$.

Proof. Assume that $\log n > k_0$. Every station needs a constant energy for lines (01)-(02) and with high probability it finishes the loop (03)-(18) for $k = k_f$ for some $k_f \leq (\log n)^{1+\varepsilon}$. In lines (05)-(08) each station sends (and listens) at most once (after the step in which the station was awake for sending and listening, it becomes *used*). Every station takes part in lines (09)-(13) at most once (in iteration in which it becomes *used*) and is awake $O((\log\log n)^{\varepsilon})$ times. Similarly, every station executes line (15) at most once (also $O((\log\log n)^{\varepsilon})$ energy). Finally, each station listens in line (16), but with high probability, this line is executed $O(\log\log(\log^{1+\varepsilon} n))$ times. ∎

Remark. If we put $g(l+1) = 2g(l)$ then we obtain energy bound $O(\log\log n)$ and time $O(\log^2 n)$ with probability bigger than $1 - 1/n$ (the same time bound as in the algorithm for leader election from [14], which requires logarithmic energy).

4.2 Proof of Main Lemma

Probability that a station becomes *used* is upper bounded by the expression

$$\sum_{k=k_0+1}^{\infty} dk/2^k \leq c\,,$$

for some small constant c. Hence the expected number of used stations does not exceed cn. By choosing k_0 sufficiently large we get $c \leq \frac{1}{4e}$. If we imagine that stations continue the lines (03)-(08) infinitely then the probabilities of being *used* are independent. Using Chernoff Bound (see Appendix) we get:

Corollary 5. *Probability that the number of used stations exceeds* $n/2$ *is at most* $2^{-n/2}$.

One may be tempted to apply the results on Poisson trials in order to estimate the expected value of N in line (10) of ApproxSize and deviations from the expected value. The main technical problem that arises here is that our modification (making the station *used* after sending) makes different iterations of the loop in the lines (06)-(08) stochastically dependent. Moreover, one success may improve success probabilities in subsequent steps, even if the increase is not substantial. Note that nice properties of sums of Poisson random variables are based on the fact that success probabilities do not depend on the history of computation!

This section is organized as follows: First we analyze success probabilities in line (06) provided that less than $n/2$ stations are *used*. We analyze separately the stages for $k \leq \log n - 6$ and for $k = \lceil \log n \rceil$. The core of the proof is showing a relationship between the number of successes in a line (06) during a phase and the sum of independent random variables.

Success probabilities

Proposition 1. *Assume that the number of unused stations before a step is at least* $n/2$*. Then for* $k = \lceil \log n \rceil$*, the probability that exactly one station sends at line* (06) *is at least* 0.1.

Proposition 2. *Assume that the number of unused stations before a step is at least $n/2$. Then for $k \le \log n - 6$ and some constant c', the probability that line (06) is successful is not greater than $p_k = \frac{n}{2^{k-1}}\left(1-\frac{1}{2^k}\right)^{n/2}$.*

The proofs of Propositions 1 and 2 are given in Appendix.

Sums of independent random variables Let $\mathbf{P}[A]$ denote probability of an event A. Let $p_k = \frac{n}{2^{k-1}}\left(1-\frac{1}{2^k}\right)^{n/2}$ for $k \le \log n - 6$, and $p_k = 0.1$ for $k = \lceil \log n \rceil$. We consider independent random variables $x_1, \ldots, x_{dk}$, where $x_i \in \{0,1\}$, $\mathbf{P}[x_i = 1] = p_k$ for $i \le dk$ (from the context it will be clear which k do we mean.)

Lemma 3. *Let $X = \sum_{i=1}^{dk} x_i$. Then for sufficiently large n:*

(a) $\mathbf{P}\left[X > \frac{dk}{20}\right] < \frac{1}{n^2}$ *for* $k \le \log n - 6$,
(b) $\mathbf{P}\left[X > \frac{dk}{20}\right] \ge 1 - \frac{1}{n^2}$ *for* $k = \lceil \log n \rceil$.

The proof of this lemma is tedious, but straightforward so we put it in Appendix.

Estimating the number of successes Let S_i^j denote the event "the number of used stations is less than $n/2$ immediately before trial j of phase i". Recall that by Corollary 5, $\mathbf{P}\left[S_i^j\right] \ge 1 - 2^{-n/2}$. First, we examine the number of successes during phase k, for $k \le \log n - 6$, under assumption that event S_k^1 holds. Let w_i be a random variable, such that $w_i = 1$ if the trial i at phase k is successful, and $w_i = 0$ otherwise. Let x_i be random variables defined as above. Note that, for every $U \subseteq S_k^i$, $\mathbf{P}[w_i = 1 \,|\, U] \le \mathbf{P}[x_i = 1]$ if $k \le \log n - 6$ (by Proposition 2) and $\mathbf{P}[w_i = 1 \,|\, U] \ge \mathbf{P}[x_i = 1]$ if $k = \lceil \log n \rceil$ (by Proposition 1).

Lemma 4. *Let $M = 2^{n/2}$, $k \le \log n - 6$. For each $c > 0$ holds:*

$$\mathbf{P}\left[\sum_{i=1}^{dk} w_i > c \,|\, S_k^1\right] \le \mathbf{P}\left[\sum_{i=1}^{dk} x_i > c\right] + \frac{2^{dk}-1}{M \cdot \mathbf{P}\left[S_k^1\right]}.$$

Proof. We prove the lemma in a slightly general form by inverse induction on j: let U be any event such that $U \subseteq S_k^j$, then

$$\mathbf{P}\left[\sum_{i=j}^{dk} w_i > c \,|\, U\right] \le \mathbf{P}\left[\sum_{i=j}^{dk} x_i > c\right] + \frac{2^{(dk-j)}-1}{M \cdot \mathbf{P}[U]}. \qquad (1)$$

The case $j = dk$ is given by Proposition 2. Now, for $j < dk$ we have

$$\mathbf{P}\left[\sum_{i=j}^{dk} w_i > c \,|\, U\right] = \mathbf{P}\left[\sum_{i=j}^{dk} w_i > c \wedge S_k^{j+1} \,|\, U\right] + \mathbf{P}\left[\sum_{i=j}^{dk} w_i > c \wedge \neg S_k^{j+1} \,|\, U\right].$$

Since the second term is bounded from above by

$$\mathbf{P}\left[\neg S_k^{j+1} \,|\, U\right] \le \frac{\mathbf{P}\left[\neg S_k^{j+1}\right]}{\mathbf{P}[U]} \le \frac{1}{M \cdot \mathbf{P}[U]},$$

we get

$$\mathbf{P}\left[\sum_{i=j}^{dk} w_i > c \,|\, U\right] \le \mathbf{P}\left[\sum_{i=j}^{dk} w_i > c \wedge S_k^{j+1} \,|\, U\right] + \frac{1}{M \cdot \mathbf{P}[U]}.$$

The first term on the left hand side of the last inequality equals

$$\mathbf{P}\left[\sum_{i=j+1}^{dk} w_i > c \wedge w_j = 0 \wedge S_k^{j+1} \,|\, U\right] + \mathbf{P}\left[\sum_{i=j+1}^{dk} w_i > c-1 \wedge w_j = 1 \wedge S_k^{j+1} \,|\, U\right].$$

Let the probabilities above be denoted by H_0 and H_1, respectively. We estimate H_0. Let W_0 denote the event $S_k^{j+1} \wedge w_j = 0 \wedge U$. Then , by induction hypothesis:

$$\begin{aligned} H_0 &= \mathbf{P}\left[\sum_{i=j+1}^{dk} w_i > c \,|\, W_0\right] \cdot \mathbf{P}[W_0 \,|\, U] \le \left(\mathbf{P}\left[\sum_{i=j+1}^{dk} x_i > c\right] + \frac{2^{dk-j-1}-1}{M \cdot \mathbf{P}[W_0]}\right) \cdot \frac{\mathbf{P}[W_0]}{\mathbf{P}[U]} = \\ &\left(\mathbf{P}\left[\sum_{i=j+1}^{dk} x_i > c\right] \cdot \frac{\mathbf{P}[W_0]}{\mathbf{P}[U]}\right) + \frac{2^{dk-j-1}-1}{M \cdot \mathbf{P}[U]} = \\ &\left(\mathbf{P}\left[\sum_{i=j+1}^{dk} x_i > c\right] \cdot \mathbf{P}\left[S_k^{j+1} \wedge w_j = 0 \,|\, U\right]\right) + \frac{2^{dk-j-1}-1}{M \cdot \mathbf{P}[U]} \le \\ &\left(\mathbf{P}\left[\sum_{i=j+1}^{dk} x_i > c\right] \cdot \mathbf{P}\left[w_j = 0 \,|\, U\right]\right) + \frac{2^{dk-j-1}-1}{M \cdot \mathbf{P}[U]}. \end{aligned}$$

Similarly we obtain

$$H_1 \le \left(\mathbf{P}\left[\sum_{i=j+1}^{dk} x_i > c-1\right] \cdot \mathbf{P}\left[w_j = 1 \,|\, U\right]\right) + \frac{2^{dk-j-1}-1}{M \cdot \mathbf{P}[U]}.$$

So we get

$$\begin{aligned} \mathbf{P}\left[\sum_{i=j}^{dk} w_i > c|U\right] \le &\left(\mathbf{P}\left[\sum_{i=j+1}^{dk} x_i > c\right] \cdot \mathbf{P}\left[w_j = 0|U\right]\right) + \\ &\left(\mathbf{P}\left[\sum_{i=j+1}^{dk} x_i > c-1\right] \cdot \mathbf{P}\left[w_j = 1|U\right]\right) + \frac{2(2^{dk-j-1}-1)+1}{M \cdot \mathbf{P}[U]}. \end{aligned}$$

The last term equals $\frac{2^{dk-j}-1}{M \cdot \mathbf{P}[U]}$. In order to estimate the sum of the first two terms note that by replacing $\mathbf{P}\left[w_j = 1|U\right]$ by any number $p \ge \mathbf{P}\left[w_j = 1|U\right]$ and $\mathbf{P}\left[w_j = 0|U\right]$ by $1-p$ we do not decrease the sum. Indeed, it follows from the fact that

$$\mathbf{P}\left[\sum_{i=j+1}^{dk} x_i > c-1\right] \ge \mathbf{P}\left[\sum_{i=j+1}^{dk} x_i > c\right].$$

By taking $p = \mathbf{P}[x_1 = 1]$, we get

$$\begin{aligned} \mathbf{P}\left[\sum_{i=j}^{dk} w_i > c|U\right] \le &\left(\mathbf{P}\left[\sum_{i=j+1}^{dk} x_i > c\right] \cdot \mathbf{P}[x_1 = 0]\right) + \\ &\left(\mathbf{P}\left[\sum_{i=j+1}^{dk} x_i > c-1\right] \cdot \mathbf{P}[x_1 = 1]\right) + \frac{2^{dk-j}-1}{M \cdot \mathbf{P}[U]} = \\ &\mathbf{P}\left[\sum_{i=j}^{dk} x_i > c\right] + \frac{2^{dk-j}-1}{M \cdot \mathbf{P}[U]}. \end{aligned}$$

This concludes the proof of inductive step for inequality (1). ∎

It follows from Lemma 4 that if there are at least $n/2$ unused stations at the beginning of phase $k \le \log n - 6$, then the probability that $\sum_{i=1}^{dk} w_i > c$ is bounded by

$$\mathbf{P}\left[\sum_{i=1}^{dk} x_i > c\right] + \frac{2^{dk}-1}{M \cdot \mathbf{P}\left[S_k^1\right]} \le \mathbf{P}\left[\sum_{i=1}^{dk} x_i > c\right] + \frac{2^{dk}-1}{M-1}$$

(we have used the fact that $\mathbf{P}\left[S_k^1\right] \ge 1 - 1/M$). For $c = dk/20$, we have $\mathbf{P}\left[\sum_{i=1}^{dk} x_i > c\right] \le n^{-2}$, so

$$\begin{aligned} &\mathbf{P}\left[\sum_{i=1}^{dk} w_i > \frac{dk}{20}\right] = \mathbf{P}\left[\sum_{i=1}^{dk} w_i > \frac{dk}{20} \wedge S_k^1\right] + \mathbf{P}\left[\sum_{i=1}^{dk} w_i > \frac{dk}{20} \wedge \neg S_k^1\right] \le \\ &\mathbf{P}\left[\sum_{i=1}^{dk} w_i > \frac{dk}{20} \,|\, S_k^1\right] \cdot \mathbf{P}\left[S_k^1\right] + 2^{-n/2} \le \frac{1}{n^2} + \frac{2^{d(\log n - 6)}-1}{2^{n/2}} + 2^{-n/2} = O\left(\frac{1}{n^2}\right). \end{aligned}$$

For $\kappa = \lceil \log n \rceil$, a similar estimation from below of $\mathbf{P}[\sum_{i=1}^{\kappa} w_i \leq c]$ can be obtained by considering the variables $v_i = 1 - w_i$ and $y_i = 1 - x_i$. Exactly as before we show that

$$\mathbf{P}[\sum_{i=1}^{\kappa} w_i \leq c] = \mathbf{P}[\sum_{i=1}^{\kappa} v_i > d\kappa - c] \leq \mathbf{P}[\sum_{i=1}^{\kappa} y_i > d\kappa - c] + \frac{2^{d\kappa}-1}{M-1} = \mathbf{P}[\sum_{i=1}^{\kappa} x_i \leq c] + \frac{2^{d\kappa}-1}{M-1}\,.$$

Since $\mathbf{P}\left[\sum_{i=1}^{dk} x_i \leq d\kappa/20\right] \leq n^{-2}$, so we get in a similar way that $\mathbf{P}\left[\sum_{i=1}^{dk} w_i < d\kappa/20\right] = O\left(\frac{1}{n^2}\right)$.

Conclusions

Our complexity analysis of the algorithm ApproxSize convince that its *asymptotic* behavior is nice. Although we did not care about values of constants a more detailed probabilistic analysis should show that the algorithm is quite practical.

References

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Appendix

Let $\mathbf{E}[Y]$ denote the expected value of a random variable Y.

5.1 Chernoff bounds

Let $X_1,\ldots,X_n$ be independent Poisson trials such that for $1 \le i \le n$, $\mathbf{P}[X_i = 1] = p_i$ and $\mathbf{P}[X_i = 0] = 1 - p_i$, and $X = X_1 + \ldots + X_n$. Then (see [13, pages 68, 70])

$$\mathbf{P}[X > (1+\delta)\mathbf{E}[X]] < \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mathbf{E}[X]} \quad \text{for} \quad \delta > 0\,, \tag{2}$$

$$\mathbf{P}[X < (1-\varepsilon)\mathbf{E}[X]] < e^{\frac{-\varepsilon^2\,\mathbf{E}[X]}{2}} \quad \text{for} \quad 0 < \varepsilon < 1\,. \tag{3}$$

5.2 Proof of Proposition 1

Let p denote probability of success at line (06). Let $\kappa = \lceil \log n \rceil$. It is easy to see that $p \ge \min\left\{\frac{l}{2^\kappa}\left(1-\frac{1}{2^\kappa}\right)^{l-1} \mid l = n/2,\ldots,n\right\}$. The last expression is minimized for $l = n/2$, thus

$$p \ge \tfrac{n}{2}\cdot\tfrac{1}{2^\kappa}\left(1-\tfrac{1}{2^\kappa}\right)^{n/2}\cdot\left(\tfrac{2^\kappa}{2^\kappa-1}\right) \ge \tfrac{n}{2\cdot(2^\kappa-1)}\cdot\left(\left(1-\tfrac{1}{n}\right)^n\right)^{1/2} \ge \tfrac{1}{4}\cdot\left(\tfrac{1}{4}\right)^{1/2} = 1/8.$$

■

5.3 Proof of Proposition 2

Let p denote probability of success at line (06). Observe that

$$p \le \max\left\{\tfrac{l}{2^k}\left(1-\tfrac{1}{2^k}\right)^{l-1} \mid l = n/2,\ldots,n\right\}.$$

Thus

$$p \le \tfrac{n}{2^k}\left(1-\tfrac{1}{2^k}\right)^{n/2-1} = \tfrac{n}{2^k}\left(1-\tfrac{1}{2^k}\right)^{n/2}\tfrac{2^k}{2^k-1} \le \tfrac{n}{2^{k-1}}\left(1-\tfrac{1}{2^k}\right)^{n/2}.$$

■

5.4 Proof of Lemma 3

Let $n = 2^l$ (where not necessarily $l \in \mathbb{N}$).

(a) Assume that $l - k \ge 6$. Let $p = \frac{n}{2^{k-1}}\left(1-\frac{1}{2^k}\right)^{n/2}$. Then

$$\mathbf{E}[X] = dk\cdot p = dk\cdot\tfrac{n}{2^{k-1}}\cdot\left(1-\tfrac{1}{2^k}\right)^{n/2} = dk\cdot 2^{l-k+1}\cdot\left(1-\tfrac{1}{2^k}\right)^{2^{l-1}}.$$

Let δ satisfy the equation: $(1+\delta)\cdot\mathbf{E}[X] = \frac{dk}{20}$, that is,

$$1+\delta = \tfrac{dk}{20\cdot\,\mathbf{E}[X]} = \tfrac{1}{20\cdot 2^{l-k+1}\cdot(1-1/2^k)^{2^{l-1}}}.$$

Now, observe that

$$\frac{e}{1+\delta} = 20e \cdot 2^{l-k+1} \cdot \left((1-\frac{1}{2^k})^{2^k}\right)^{2^{l-1}/2^k} \leq e^{1+\ln 20+(l-k+1)\ln 2-2^{l-k-1}} \leq e^{-2^{l-k-2}}.$$

(the last inequality follows from the fact that $1+\ln 20+(l-k+1)\ln 2 \leq \frac{1}{2}2^{l-k-1}$ for $l-k \geq 6$ and $l > k_0$ and the first from the fact that $e^{-1} \geq (1-1/z)^z$ for each $z \geq 2$). So $1+\delta \geq e^{2^{l-k-2}} \geq 3$. Therefore $\delta \geq 1$. By Chernoff Bound, inequality (2), we get

$$\mathbf{P}\left[X > \frac{dk}{20}\right] = \mathbf{P}\left[X > (1+\delta)\cdot\mathbf{E}\left[X\right]\right] < \left(\frac{e}{1+\delta}\right)^{\delta\cdot\,\mathbf{E}[X]} \cdot \left(\frac{1}{1+\delta}\right)^{\mathbf{E}[X]} < \left(\frac{e}{1+\delta}\right)^{\delta\cdot\,\mathbf{E}[X]},$$

We need to bound an expression $\delta \cdot \mathbf{E}\left[X\right]$ from below. Since $\delta \geq 1$, we get $\delta > \frac{1}{2}\cdot(\delta+1)$. Thus

$$\delta \cdot \mathbf{E}\left[X\right] \geq \frac{(1+\delta)\cdot\,\mathbf{E}[X]}{2} \geq \frac{dk}{40}.$$

Finally,

$$\mathbf{P}\left[X > \frac{dk}{20}\right] < \left(\frac{e}{1+\delta}\right)^{\delta\,\mathbf{E}[X]} < \left(e^{-2^{l-k-2}}\right)^{dk/40} = e^{-\frac{2^{l-k}}{160}\cdot dk}.$$

It is easy to check (for instance by considering the cases $k \geq l/2$ and $k < l/2$) that

$$\frac{2^{l-k}}{160}\cdot dk \geq 2l$$

(for l large enough, such that $l/2 > 1+\log l$). It follows that $\mathbf{P}\left[X > \frac{dk}{20}\right] < 1/n^2$.

(b) Let $k = \lceil \log n \rceil$. Then $k-l \in [0,1)$ and

$$\mathbf{E}\left[X\right] = 0.1\cdot dk.$$

Finally, using Chernoff bound, inequality (3), we get for $\varepsilon = 1/2$

$$\mathbf{P}\left[X \leq \frac{dk}{20}\right] \leq \mathbf{P}\left[X \leq (1-\varepsilon)\mathbf{E}\left[X\right]\right] \leq e^{\frac{-\varepsilon^2 dk\cdot 0.1}{2}} \leq 1/n^2$$

for almost all n.

∎