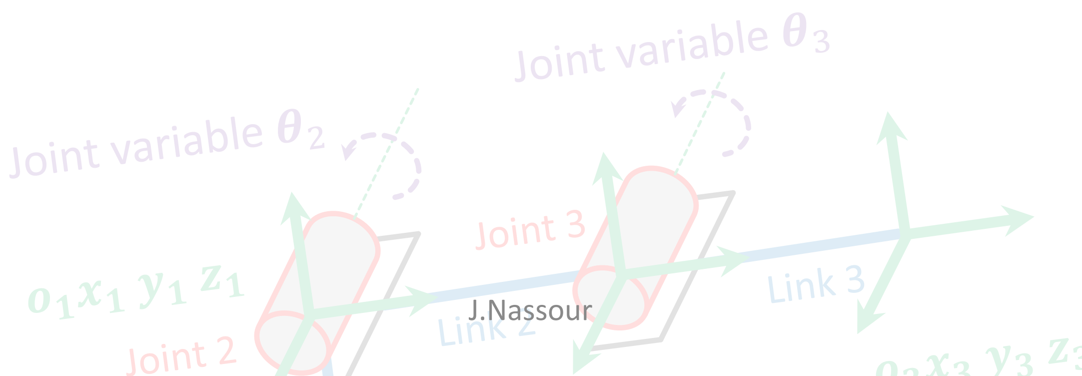
Velocity Kinematics

The Jacobian

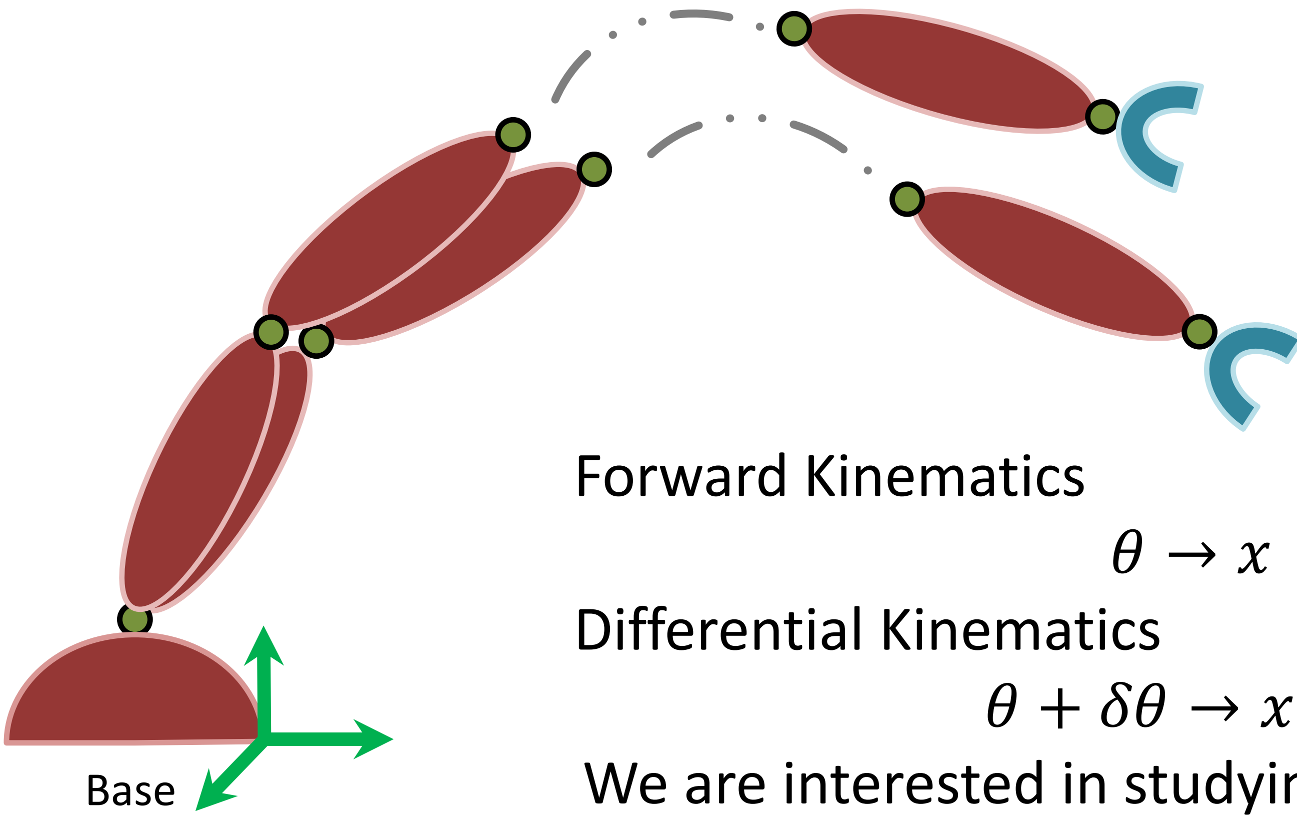
Dr.-Ing. John Nassour



Motivation

- Positions are not enough when commanding motors.
- Velocities are needed for better interaction.
- How fast the end-effector move given joints velocities?
- How fast each joint needs to move in order to guarantee a desired end-effector velocity.

Differential Motion



Forward Kinematics

$$\theta \rightarrow x$$

Differential Kinematics

$$\theta + \delta\theta \rightarrow x + \delta x$$

We are interested in studying the relationship:

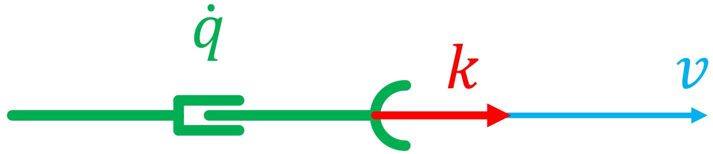
$$\delta\theta \leftrightarrow \delta x$$

Linear and angular velocities

Joint's Velocity

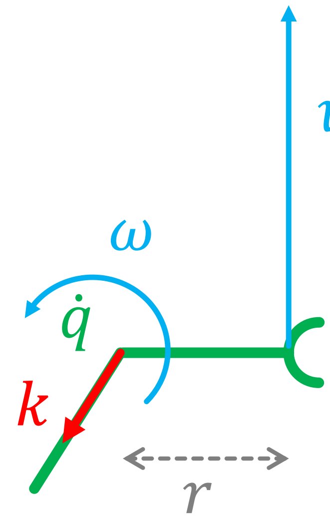
Prismatic Joint:

$$v = \dot{q}k$$
$$\omega = 0$$



Revolute Joint :

$$v = \dot{q}k \times r$$
$$\omega = \dot{q}k$$

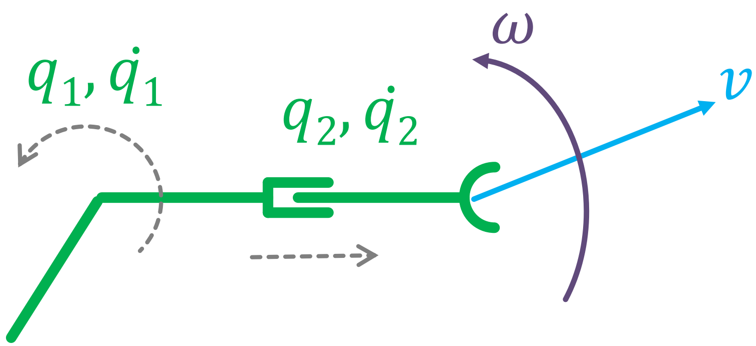


k is the unit vector.

Joint's Velocity

With more than one joint, the end effector velocities are a function of joint velocity and position:

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = f(\dot{q}_1, \dot{q}_2, q_1, q_2)$$



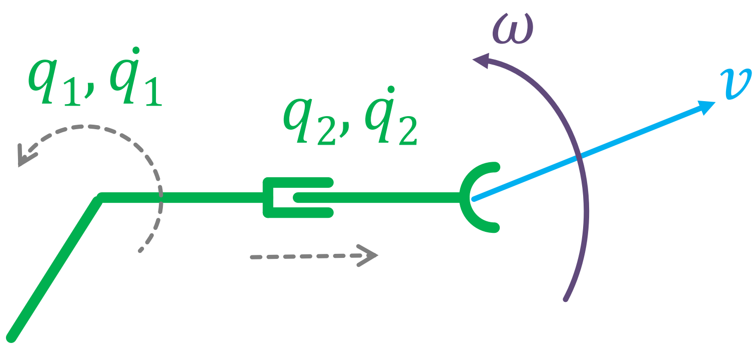
For any number of joints:

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = f(\dot{q}, q)$$

The Jacobian

The Jacobian is a matrix that is a function of joint position, that linearly relates joint velocity to tool point velocity.

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = \mathcal{J}(q_1, q_2) \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$



For the linear velocity:

$$v = \mathcal{J}_v \, \dot{q} \iff \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} \\ \mathcal{J}_{21} & \mathcal{J}_{22} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

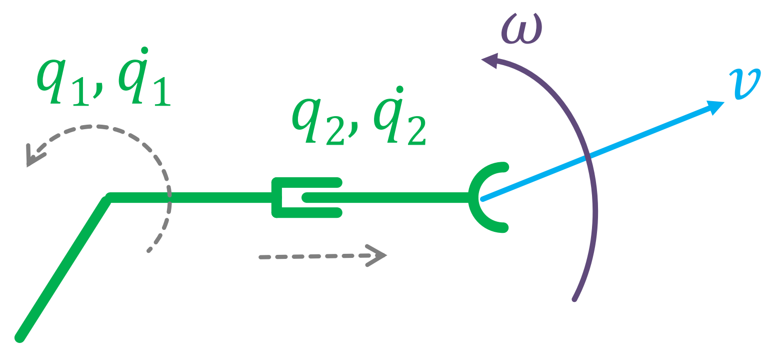
The Jacobian

The elements of the Jacobian $\mathcal{J}_{ij}$ can be obtained by partial differentiation of the forward kinematic equations:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} \\ \mathcal{J}_{21} & \mathcal{J}_{22} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

$$\frac{dx}{dt} = \frac{\partial x}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial x}{\partial q_2}\frac{dq_2}{dt}$$

$$\frac{dy}{dt} = \frac{\partial y}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial y}{\partial q_2}\frac{dq_2}{dt}$$



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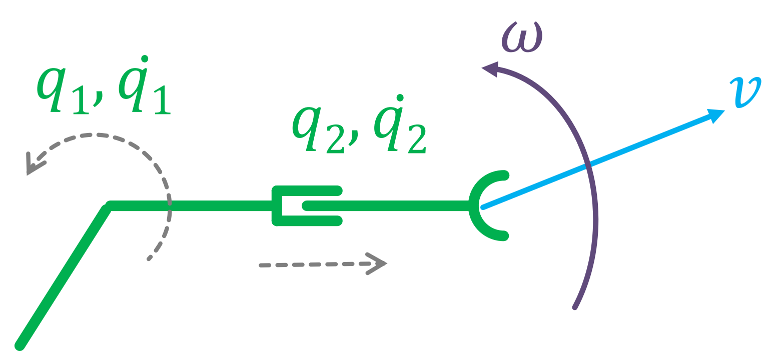
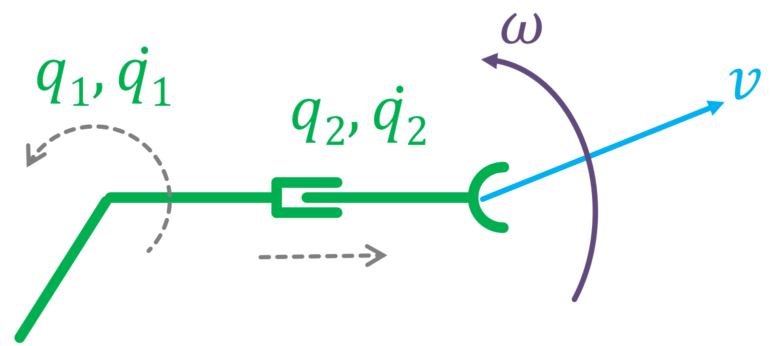


The Jacobian

In this example the forward kinematics are given by:

$$x = q_2 \cos(q_1)$$
$$y = q_2 \sin(q_1)$$

Find the elements of the Jacobian $\mathcal{J}_{ij}$.



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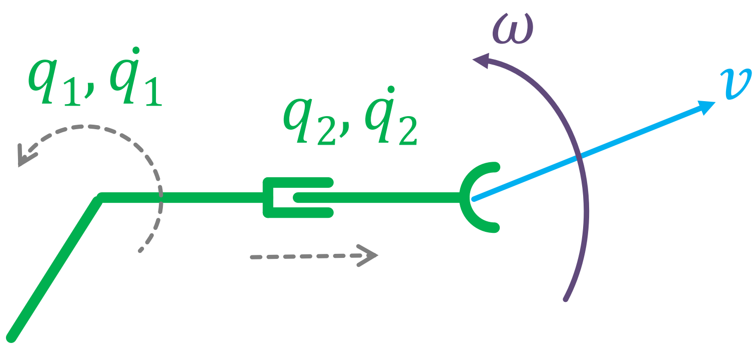

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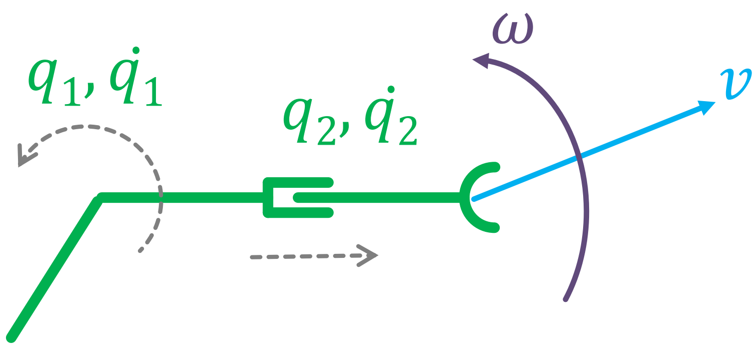

$$\mathcal{J}_{11} = \frac{\partial x}{\partial q_1} = -q_2 \sin(q_1)$$

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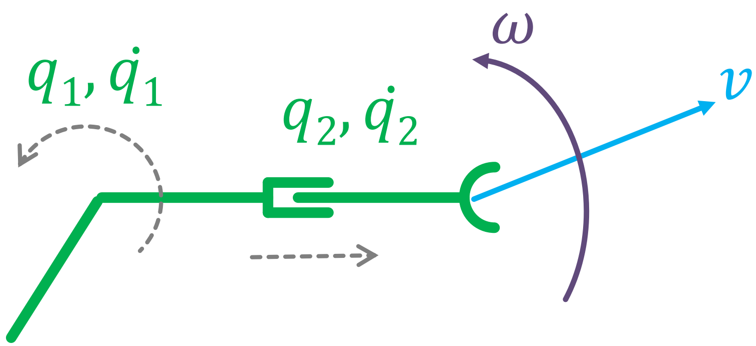

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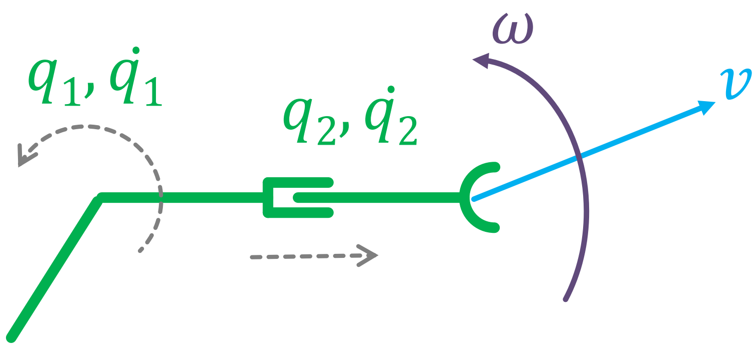

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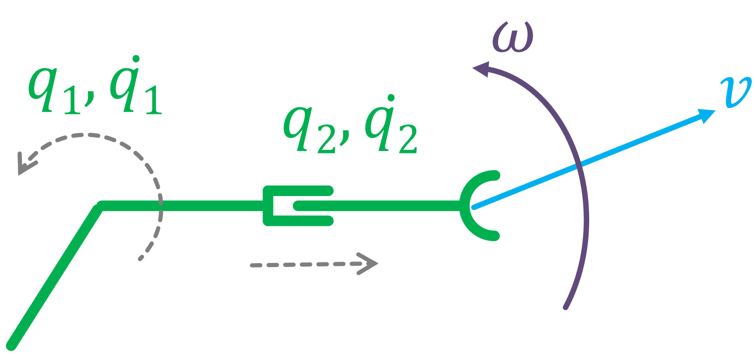

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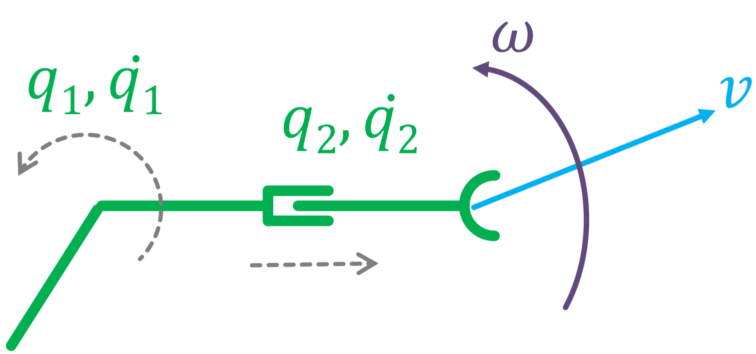

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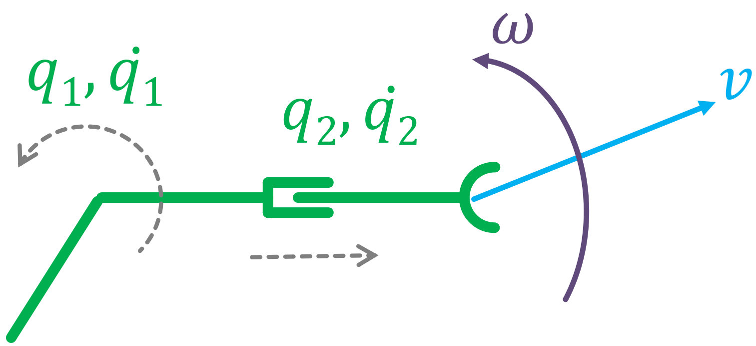

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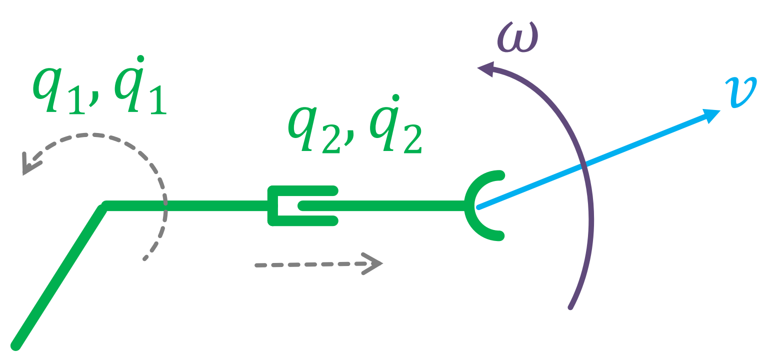

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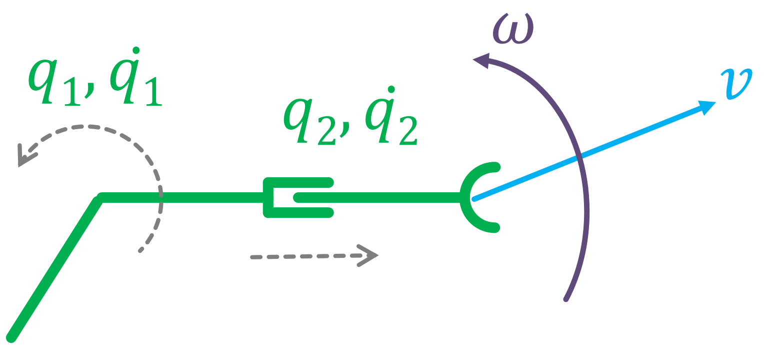

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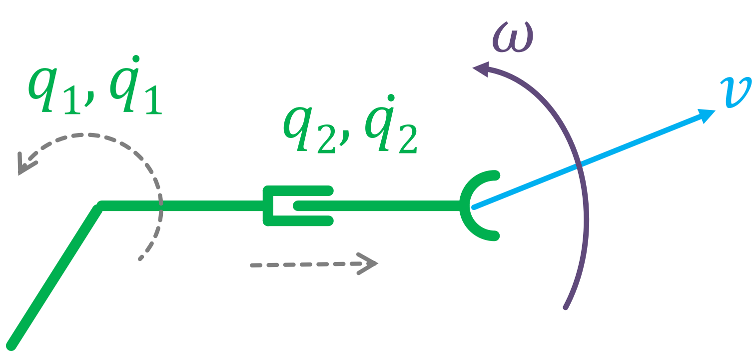
This is the linear velocity Jacobian.

The Jacobian

The angular velocity Jacobian:

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = \mathcal{J}(q_1, q_2) \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$



For the angular velocity:

$$\omega = \mathcal{J}_\omega \dot{q}$$

$$\omega = [\mathcal{J}_1 \quad \mathcal{J}_2] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

In this example: $\mathcal{J}_1 = \quad , \ \mathcal{J}_2 =$

The Jacobian

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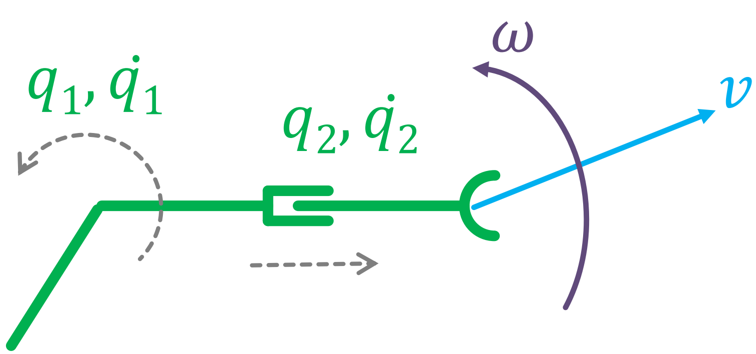

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$$\omega = \mathcal{J}_\omega \dot{q}$$

$$\omega = [\mathcal{J}_1 \quad \mathcal{J}_2] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

In this example: $\mathcal{J}_1 = 1$, $\mathcal{J}_2 =$

The Jacobian

The angular velocity Jacobian:

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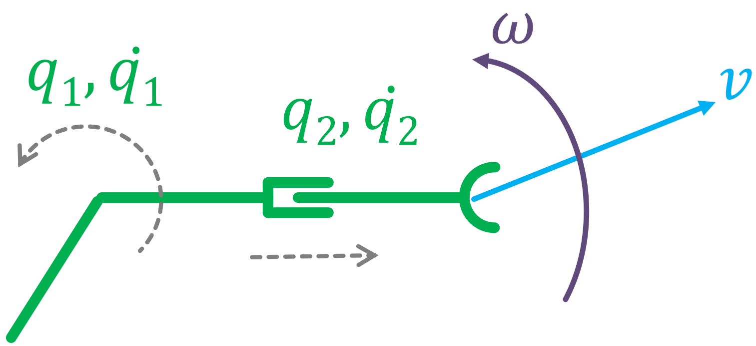


For the angular velocity:

$$\omega = \mathcal{J}_\omega \dot{q}$$

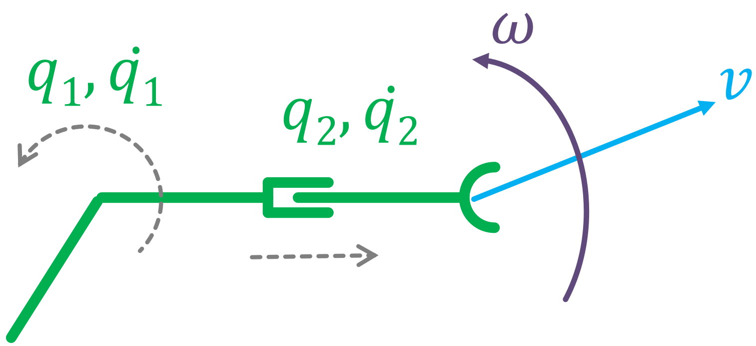
$$\omega = \begin{bmatrix} \mathcal{J}_1 & \mathcal{J}_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

In this example: $\mathcal{J}_1 = 1\,,\ \mathcal{J}_2 = 0$

Full Manipulator Jacobian

By combining the angular velocity Jacobian and the linear velocity Jacobian:

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = \mathcal{J}(q_1, q_2) \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$



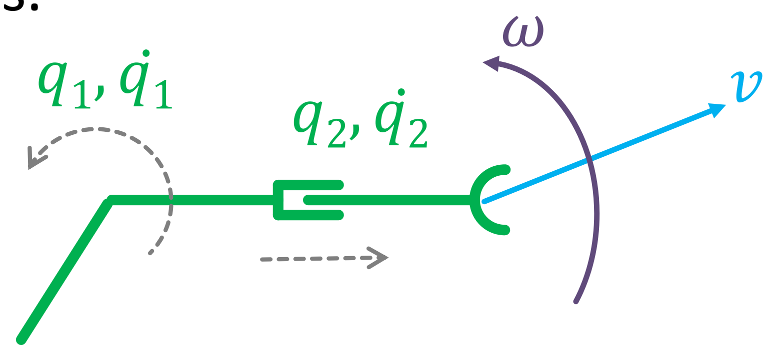
$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \omega \end{bmatrix} = \begin{bmatrix} -q_2 \sin(q_1) & \cos(q_1) \\ q_2 \cos(q_1) & \sin(q_1) \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

The full Jacobian is an *n×m* matrix where n is the number of joints, and m is the number of variables describing motion.

Full Manipulator Jacobian

Work out the linear and the angular velocities, with joint 2 extended to 0.5 m. The arm points in the x direction. Joint 1 is rotating at 2 rad/s and joint 2 is extending at 1 m/s.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \omega \end{bmatrix} = \begin{bmatrix} -q_2 \sin(q_1) & \cos(q_1) \\ q_2 \cos(q_1) & \sin(q_1) \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$



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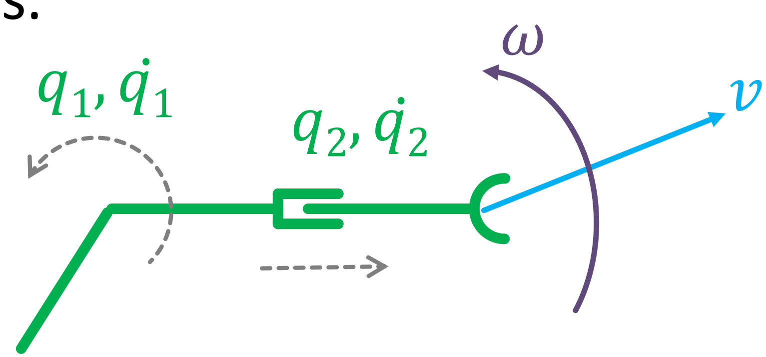


$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \omega \end{bmatrix} = \begin{bmatrix} -q_2 \sin(q_1) & \cos(q_1) \\ q_2 \cos(q_1) & \sin(q_1) \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \omega \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$\dot{x}$=1 m/s ; $\dot{y}$=1 m/s ; ω=2 rad/s

Inverting The Jacobian

To determine the joint velocities for a given end effector velocity, we need to invert the Jacobian:

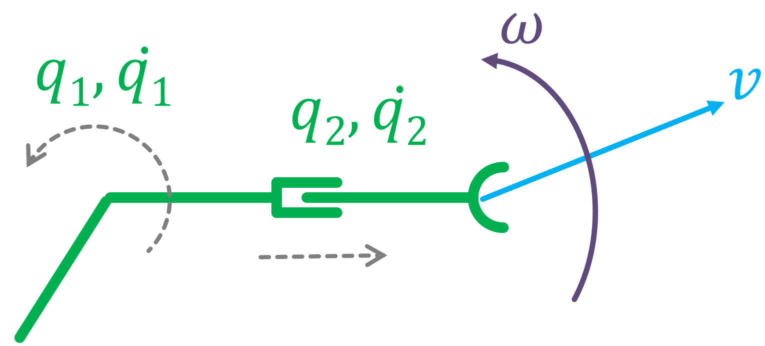
$$\begin{bmatrix} v \\ \omega \end{bmatrix} = \mathcal{J}(q)\dot{q}$$

$$\dot{q} = \mathcal{J}^{-1}(q)\begin{bmatrix} v \\ \omega \end{bmatrix}$$

Inverting The Jacobian

Find the joint velocities $(\dot{q}_1, \dot{q}_2)$ in terms of the end effector velocity $(\dot{x}, \dot{y})$.

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}-q_2\sin(q_1) & \cos(q_1)\\ q_2\cos(q_1) & \sin(q_1)\end{bmatrix}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}$$



$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} = \mathcal{J}^{-1}(q)\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}$$

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}^{-1} = \frac{1}{\underbrace{ad-bc}_{\text{determinant}}}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

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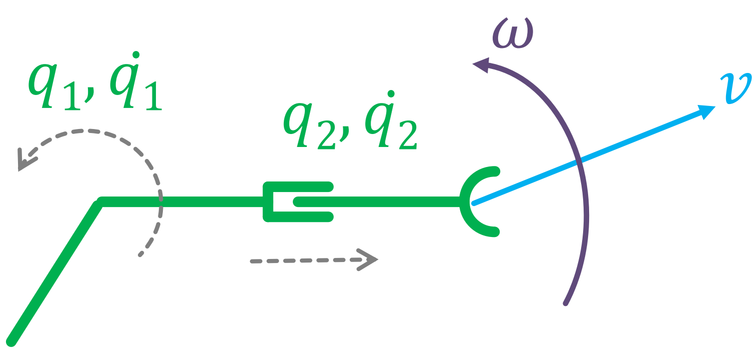

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \frac{1}{\quad} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

determinant

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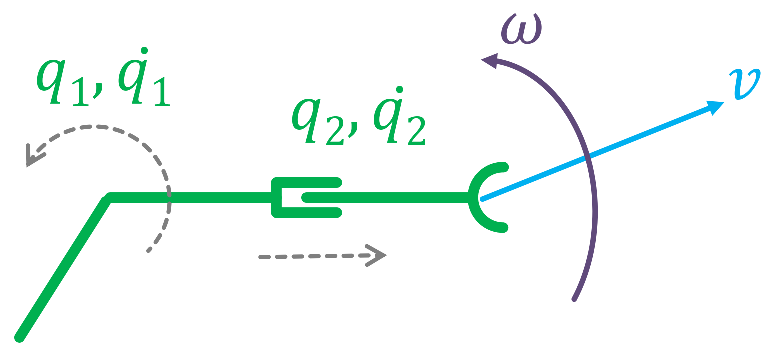

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \frac{1}{-q_2} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

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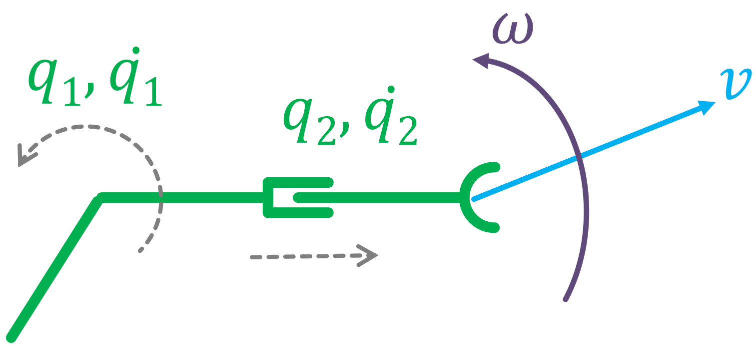

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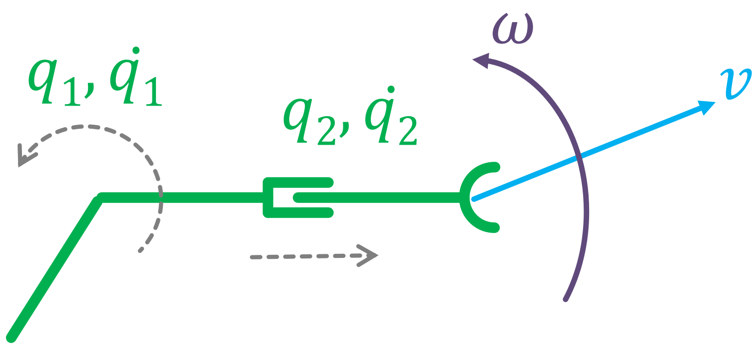

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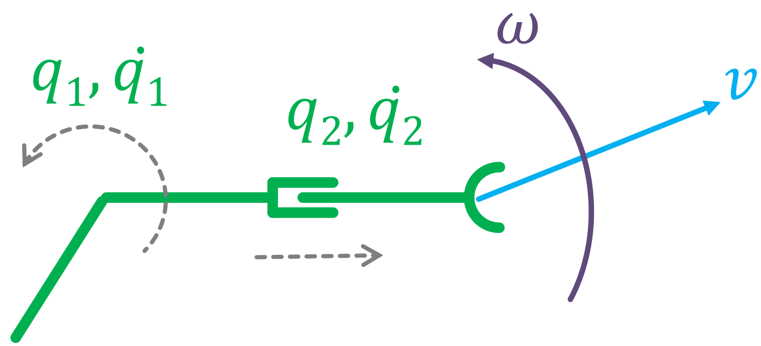
$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -q_2 \sin(q_1) & \cos(q_1) \\ q_2 \cos(q_1) & \sin(q_1) \end{bmatrix}^{-1} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$



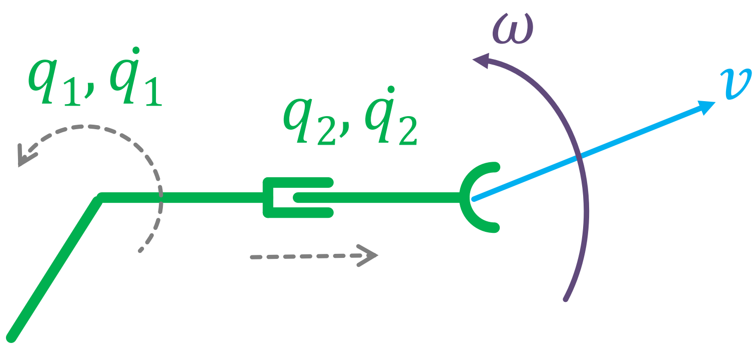
$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \frac{1}{-q_2} \begin{bmatrix} \sin(q_1) & \\ -q_2 \cos(q_1) & -q_2 \sin(q_1) \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\underbrace{ad-bc}_{\text{determinant}}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Inverting The Jacobian

Find the joint velocities $(\dot{q}_1, \dot{q}_2)$ in terms of the end effector velocity $(\dot{x}, \dot{y})$.

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -q_2 \sin(q_1) & \cos(q_1) \\ q_2 \cos(q_1) & \sin(q_1) \end{bmatrix}^{-1} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$



$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \frac{1}{-q_2} \begin{bmatrix} \sin(q_1) & -\cos(q_1) \\ -q_2 \cos(q_1) & -q_2 \sin(q_1) \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

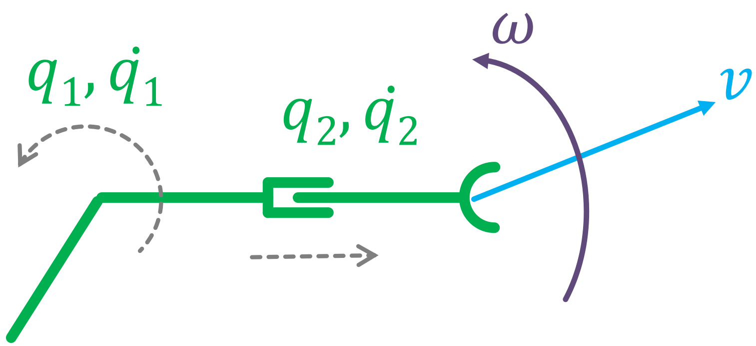
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

determinant

Inverting The Jacobian

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \frac{1}{-q_2} \begin{bmatrix} \sin(q_1) & -\cos(q_1) \\ -q_2\cos(q_1) & -q_2\sin(q_1) \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

The example arm points in the x Direction, with joint 2 extended to 0.5 m.



Find the joint velocities to move the end effector such that:

$\dot{x}$=1 m/s ; $\dot{y}$=1 m/s

$\dot{q}_1$= 2 rad/s ; $\dot{q}_2$= 1 m/s

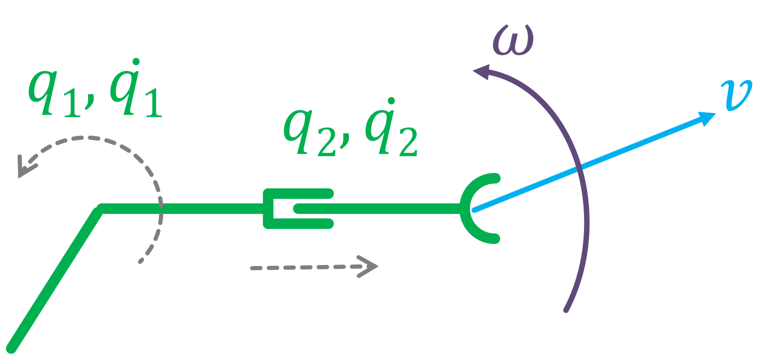
Singularities

The effect of the determinant in the inverse Jacobian example:

$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} = \boxed{\frac{1}{-q_2}} \begin{bmatrix}\sin(q_1) & -\cos(q_1)\\ -q_2\cos(q_1) & -q_2\sin(q_1)\end{bmatrix} \begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}$$

Whenever q_2= 0 m, there are no valid joint velocity solutions.

Limited end effector velocities give unlimited joint velocities.



Singularities

If the determinant of a square Jacobian is zero, the manipulator cannot be controlled.

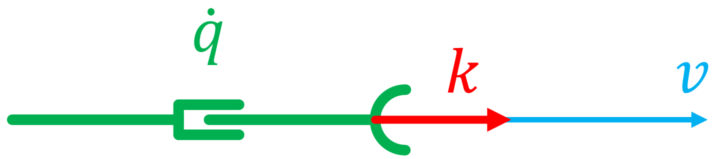
- It is useful to observe the determinant of the Jacobian as the robot moves to avoid singularities.

- Avoid configuration where the determinant approaches zero.

Joint's Velocity

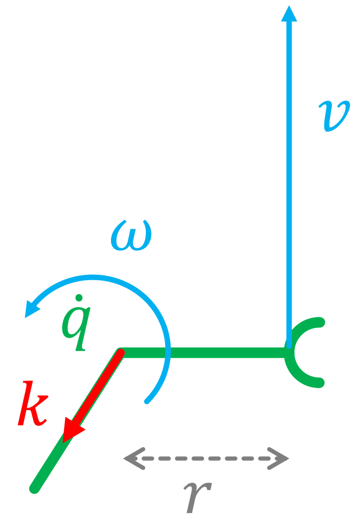
Prismatic Joint:

$$v = \dot{q}k$$
$$\boldsymbol{\omega} = \mathbf{0}$$



Revolute Joint :

$$v = \dot{q}k \times r$$
$$\boldsymbol{\omega} = \dot{\boldsymbol{q}}\boldsymbol{k}$$



k is the unit vector.

Angular Velocity

Prismatic joint gives $\omega = 0$ and revolute joint gives $\omega = \dot{q}k$

The general Jacobian for the angular velocity:

$\mathcal{J}_\omega = [\rho_1 z_0 \quad \rho_2 z_1 \quad \cdots \quad \rho_n z_{n-1}]$

- ρ_i is 1 if the joint is revolute and 0 if the joint is prismatic.
- z_i is the direction of the z axis of the i^th coordinate frame with respect to the base frame.
- z_i is the first three elements of third column of the general transformation matrix.

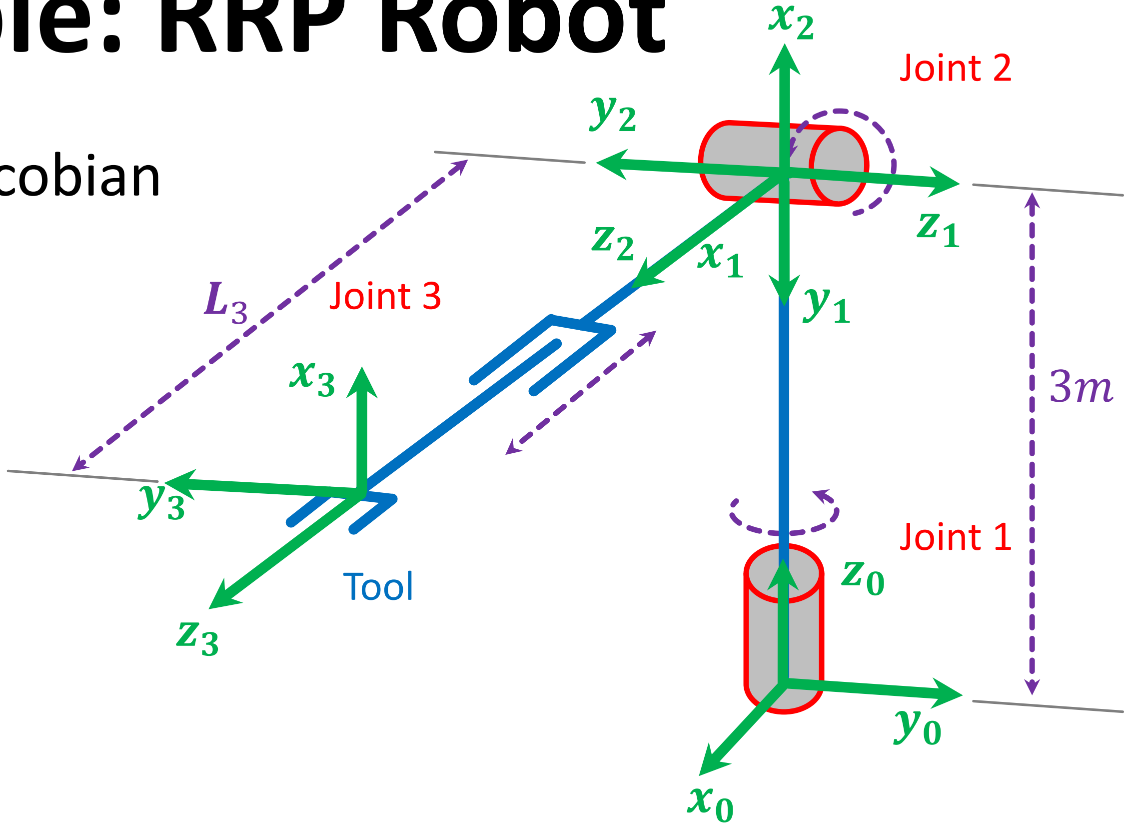
$$A_1.A_2 \ldots A_i = T^0_i = \begin{bmatrix} x_x & y_x & z_x & o_x \\ x_y & y_y & z_y & o_y \\ x_z & y_z & z_z & o_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: RRP Robot

Find the angular velocity Jacobian for the arm RRP.

$\mathcal{J}_\omega = [\rho_1 z_0 \quad \rho_2 z_1 \quad \cdots \quad \rho_n z_{n-1}]$

$\rho_1 = ?\,,\ \rho_2 = ?,\ \rho_3 = ?$



$$T_1^0 = A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_2^0 = A_1 A_2 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & 0 \\ s_1 c_2 & -c_1 & -s_1 s_2 & 0 \\ -s_2 & 0 & -c_2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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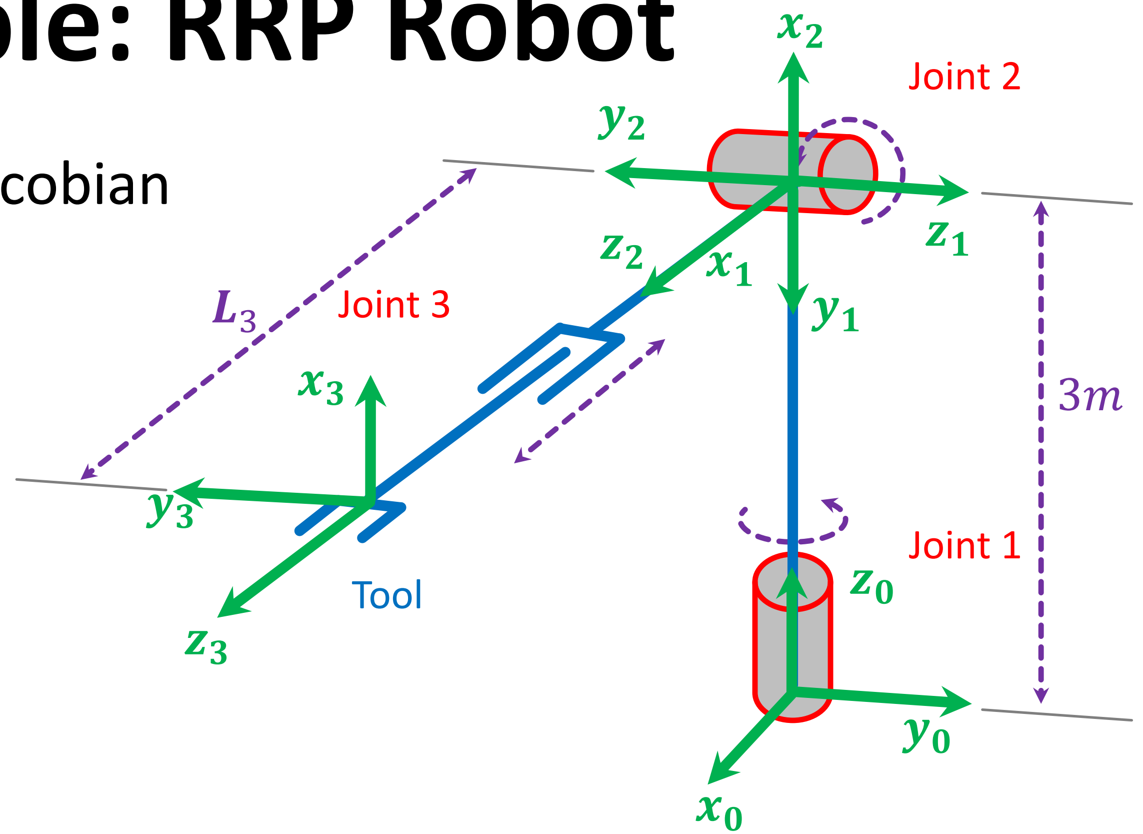
$$\rho_1 = 1\,,\ \rho_2 = 1,\ \rho_3 = 0$$



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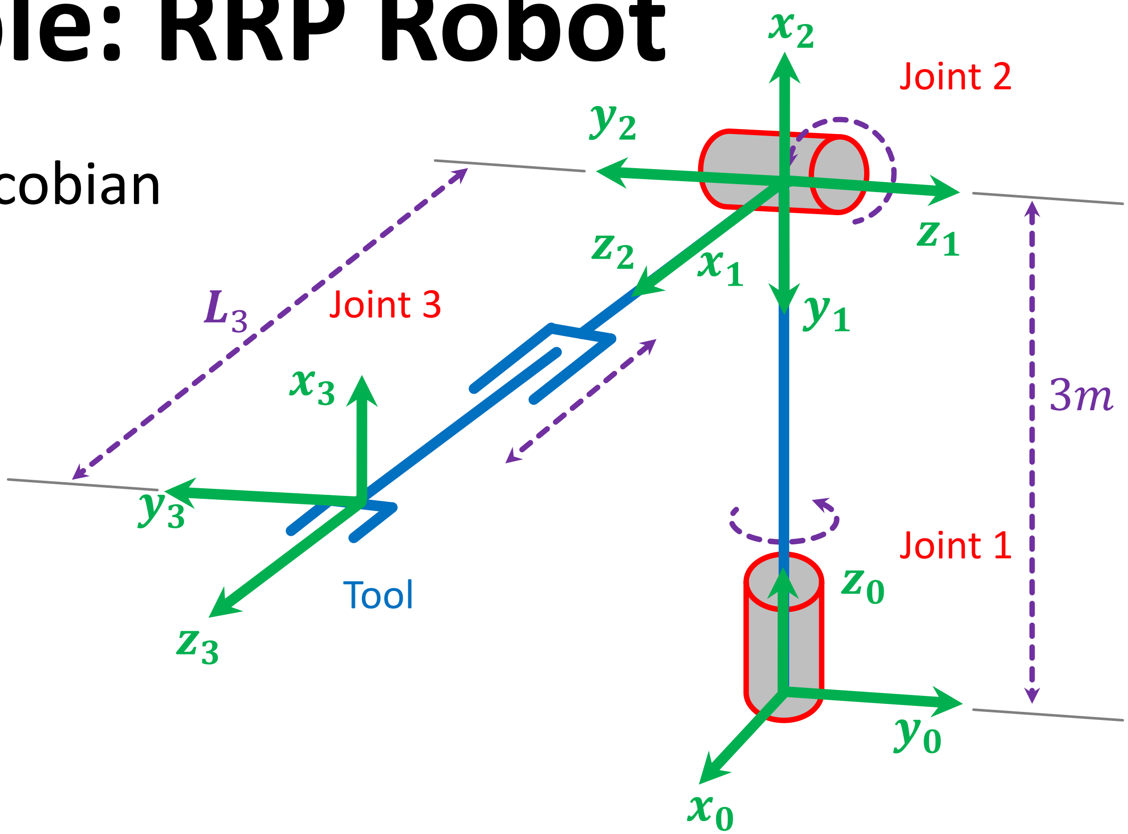
$\rho_1 = 1\,,\ \rho_2 = 1,\ \rho_3 = 0$

$z_0 =$



$$T^0_1 = A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T^0_2 = A_1 A_2 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & 0 \\ s_1 c_2 & -c_1 & -s_1 s_2 & 0 \\ -s_2 & 0 & -c_2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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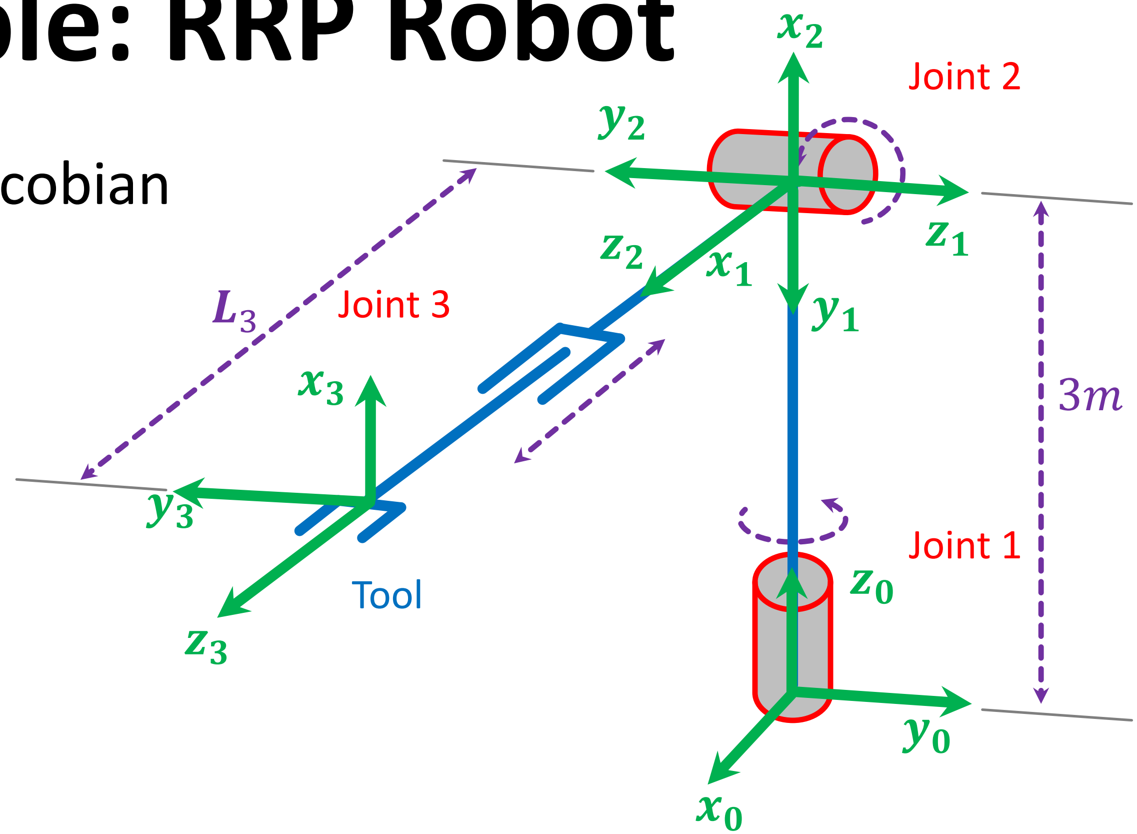
$\rho_1 = 1\,,\ \rho_2 = 1,\ \rho_3 = 0$

$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$



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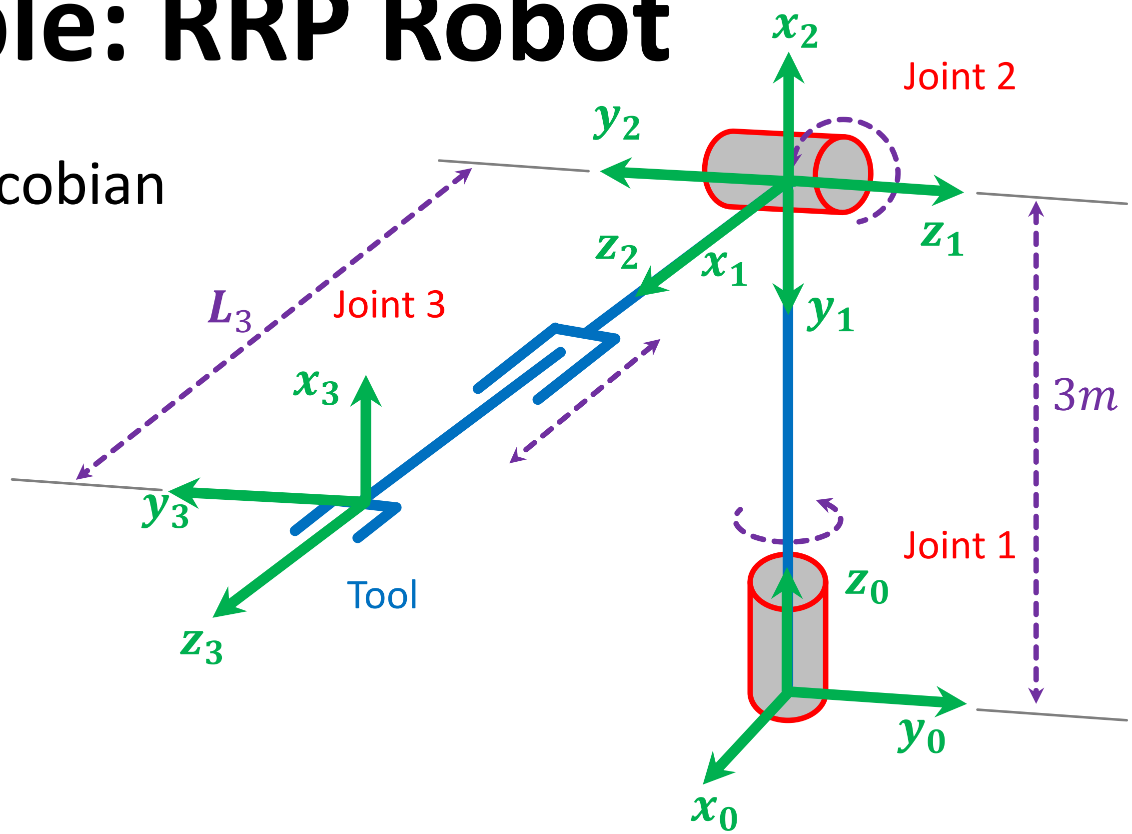
$\rho_1 = 1\,,\ \rho_2 = 1,\ \rho_3 = 0$

$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, z_1 =$



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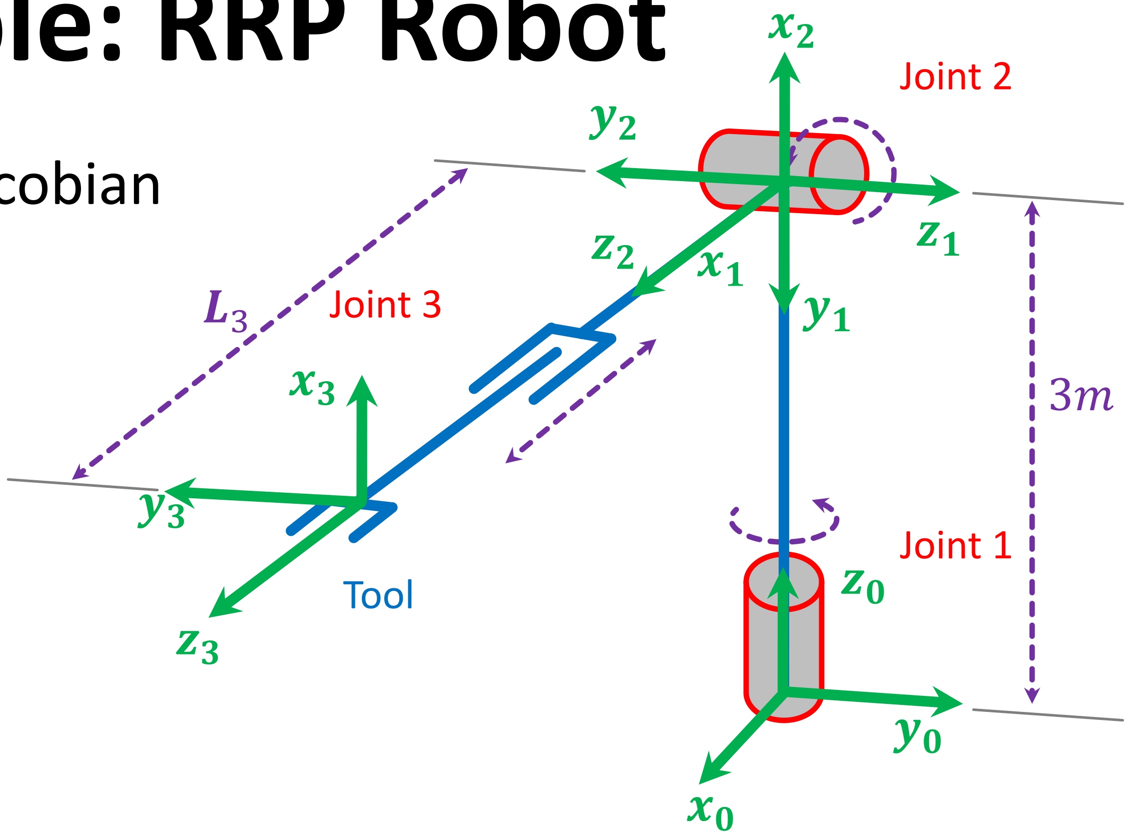

$$\mathcal{J}_\omega = [\rho_1 z_0 \quad \rho_2 z_1 \quad \cdots \quad \rho_n z_{n-1}]$$

$$\rho_1 = 1\,, \ \rho_2 = 1, \ \rho_3 = 0$$

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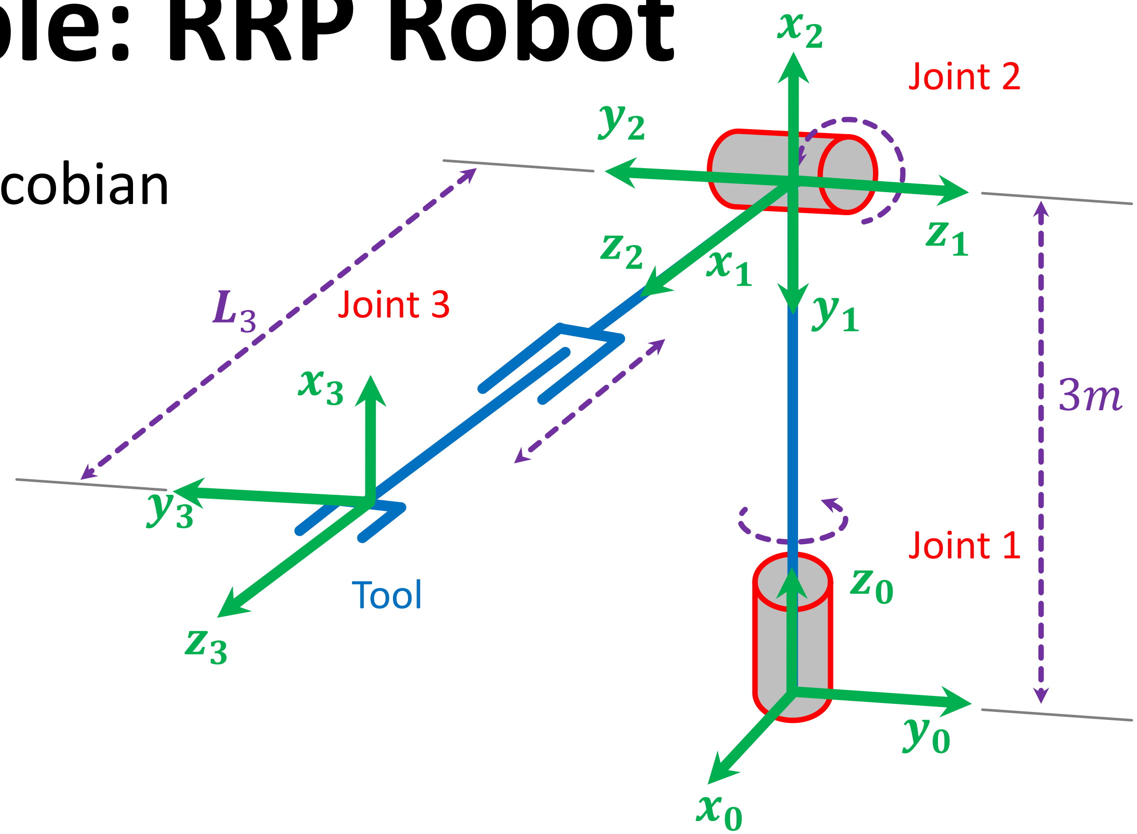
$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ z_1 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix},\ z_2 =$



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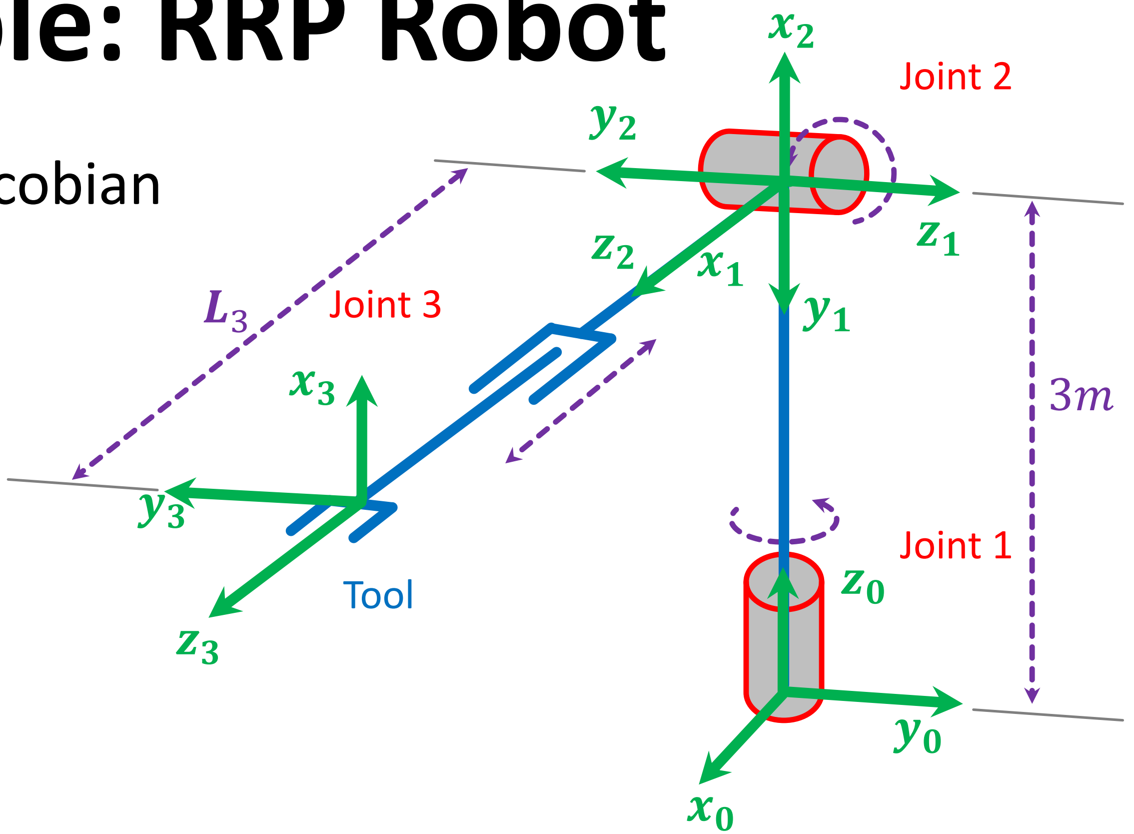

$$\mathcal{J}_\omega = [\rho_1 z_0 \quad \rho_2 z_1 \quad \cdots \quad \rho_n z_{n-1}]$$

$$\rho_1 = 1\,,\ \rho_2 = 1,\ \rho_3 = 0$$

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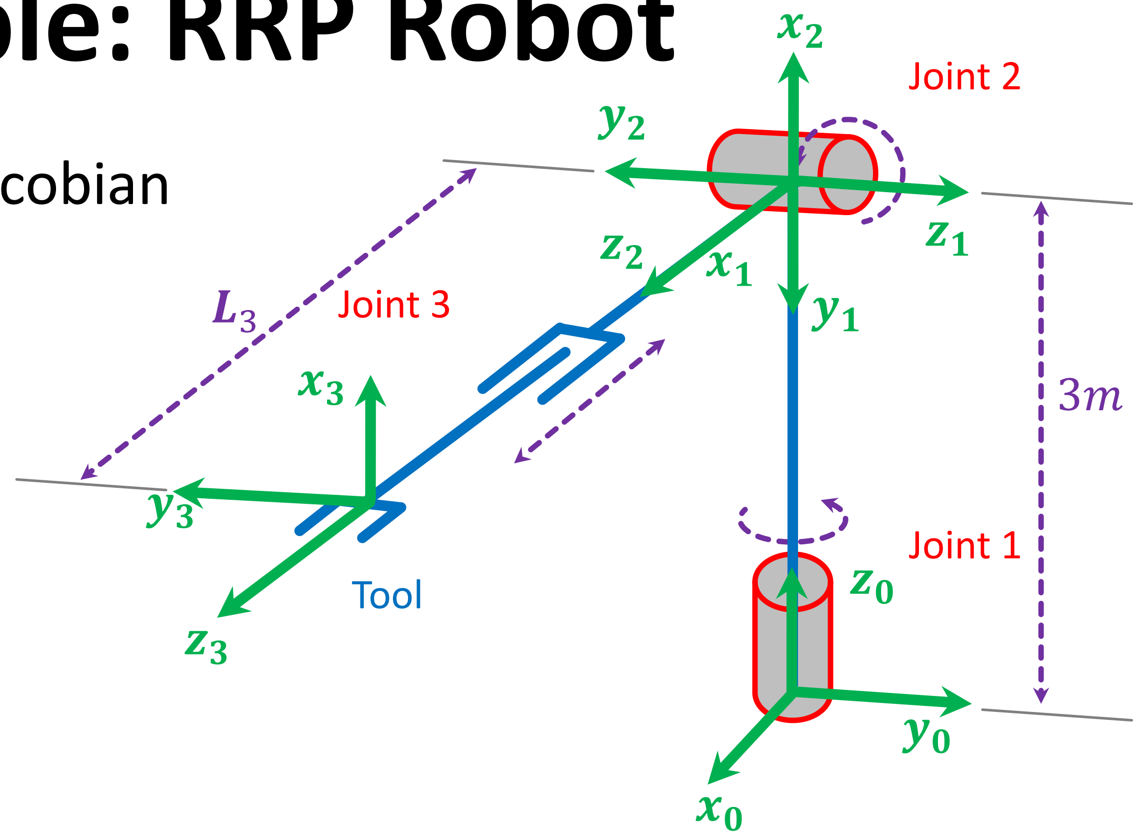

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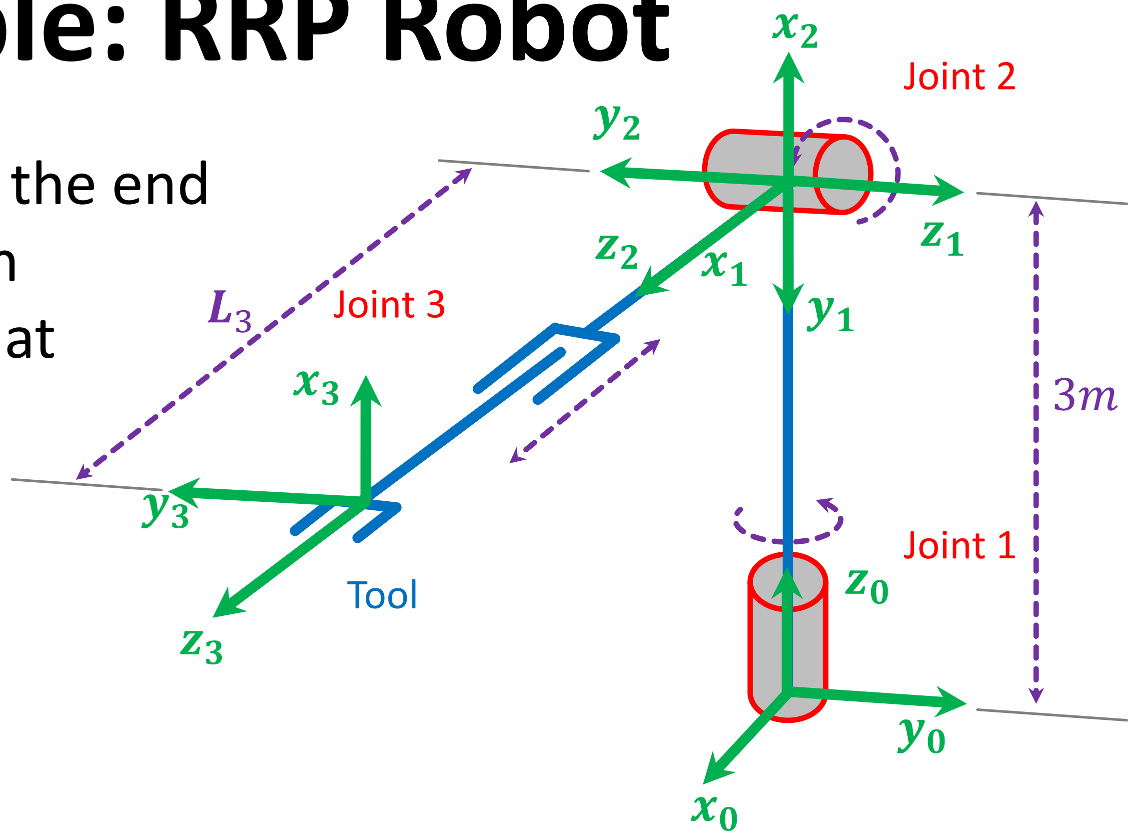
$$\rho_1 = 1 \,, \; \rho_2 = 1, \; \rho_3 = 0$$

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; z_1 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix}, \; z_2 = \begin{bmatrix} -c_1 s_2 \\ -s_1 s_2 \\ -c_2 \end{bmatrix}$$

$$\mathcal{J}_\omega = \begin{bmatrix} 0 & -s_1 & 0 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Example: RRP Robot

Find the angular velocity of the end effector in the configuration shown with joint 1 rotating at 2 rad/s, joint 2 rotating at -1 rad/s and joint 3 fixed At 2 m.



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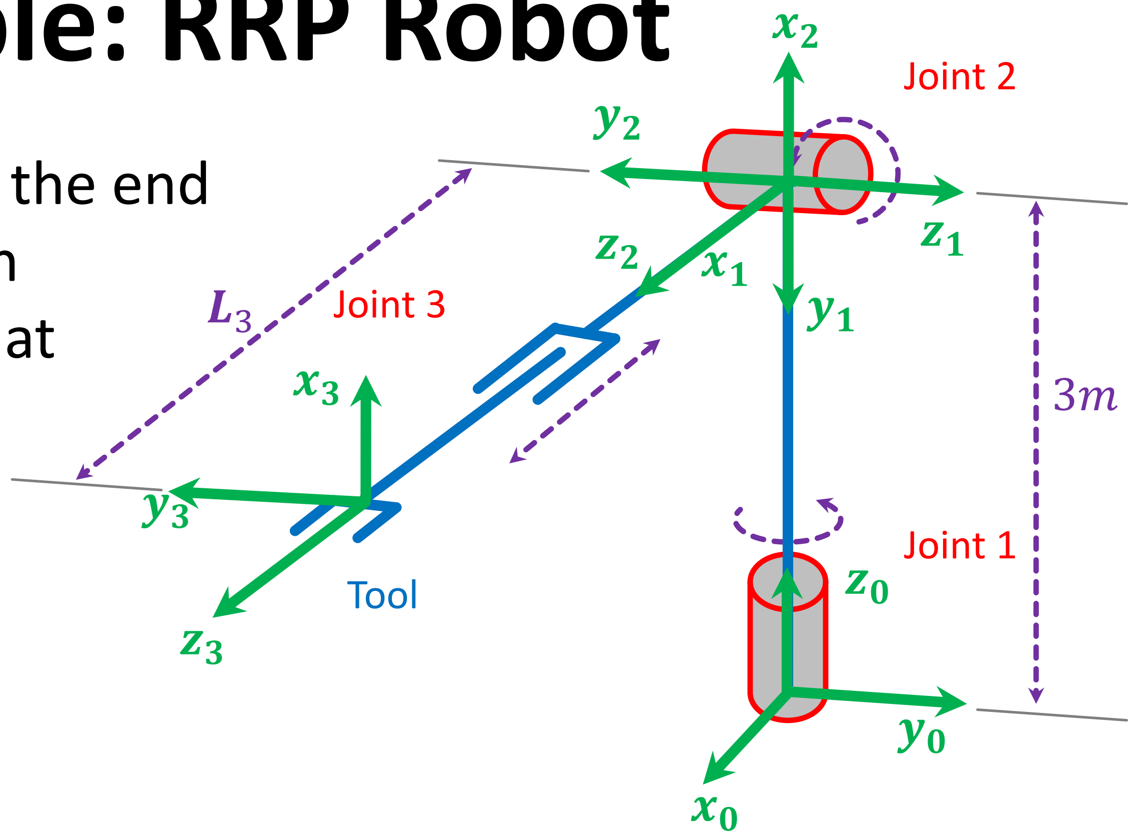

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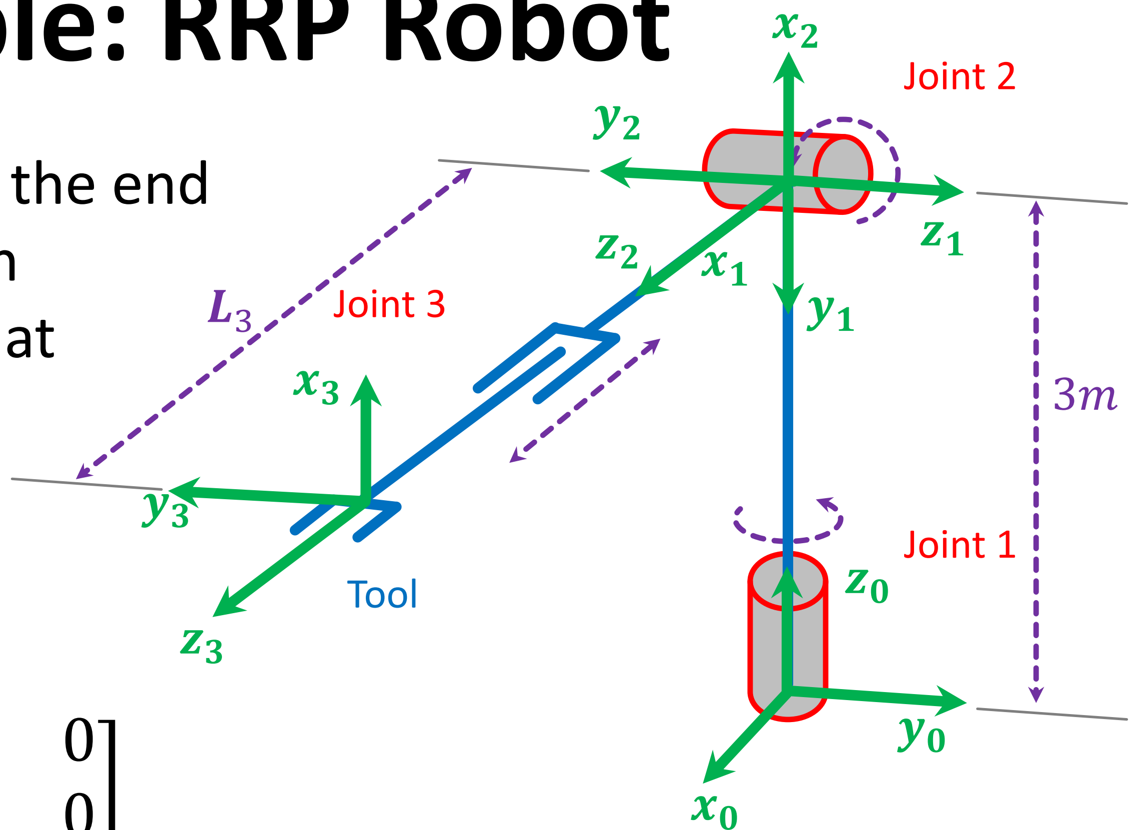

$$\mathcal{J}_\omega = \begin{bmatrix} 0 & -s_1 & 0 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

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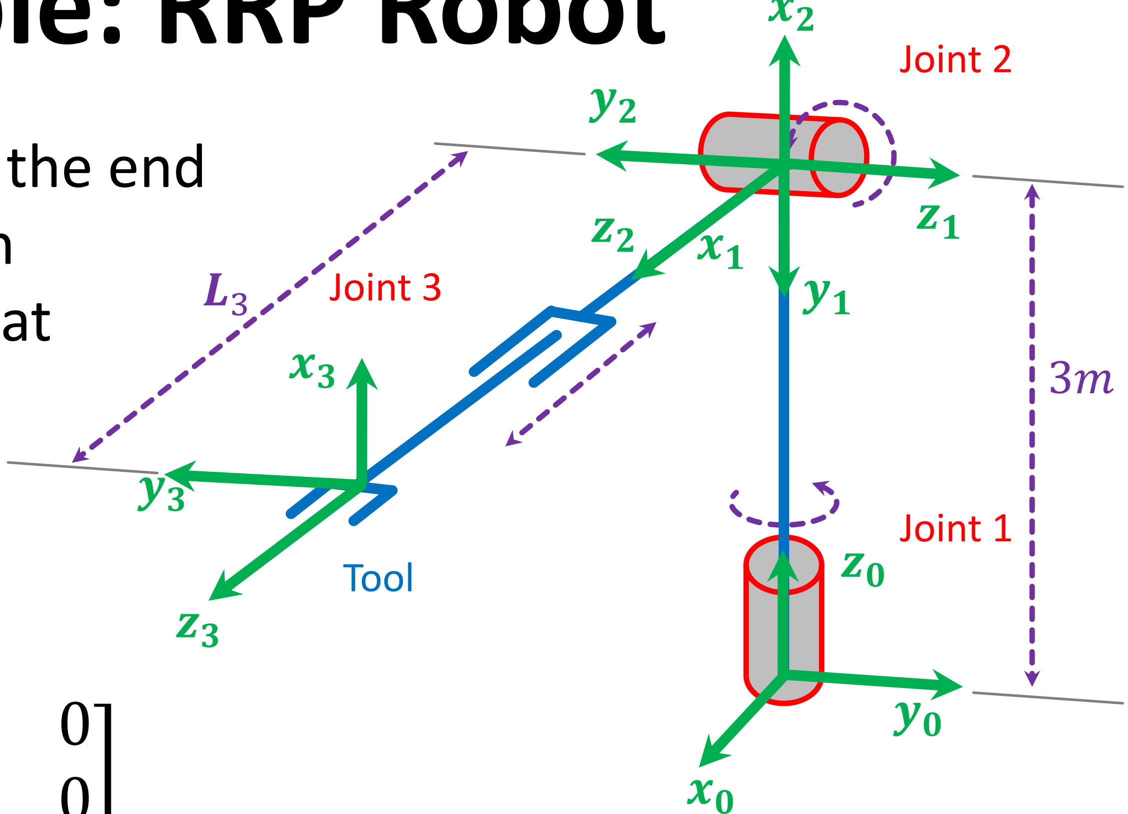

$$\mathcal{J}_\omega = \begin{bmatrix} 0 & -s_1 & 0 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

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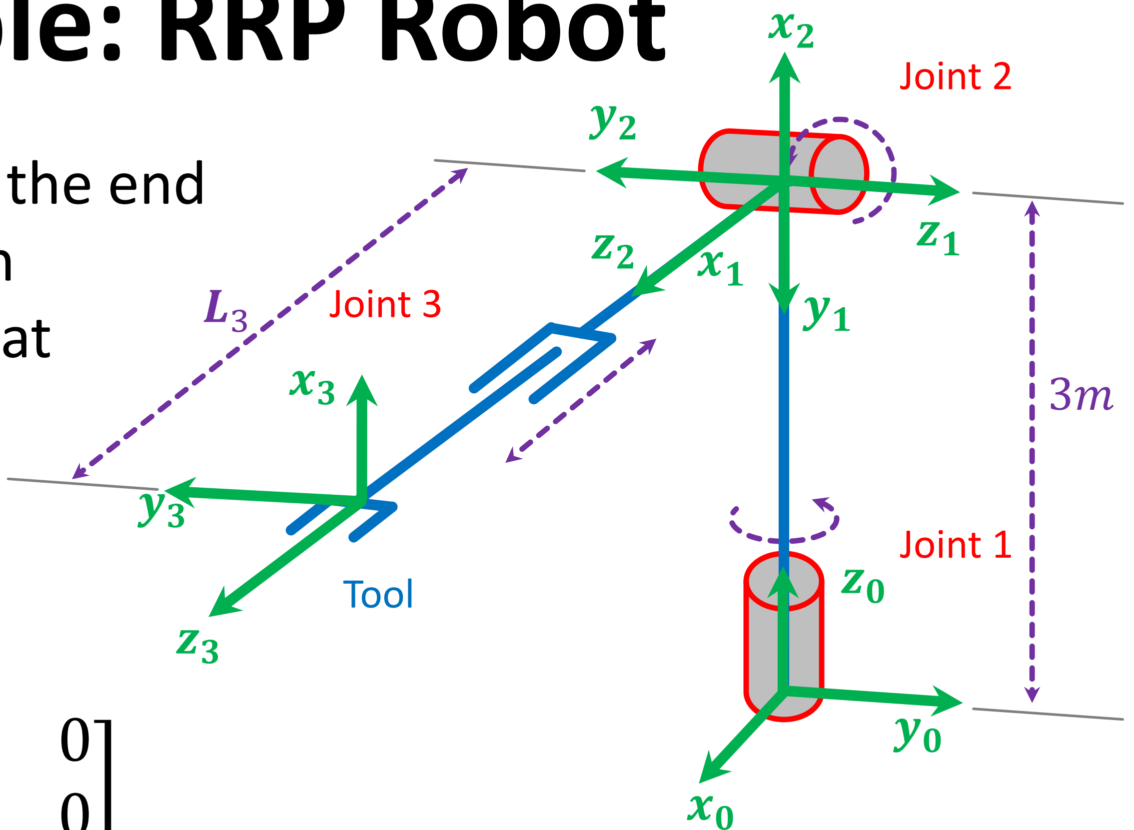

$$\mathcal{J}_\omega = \begin{bmatrix} 0 & -s_1 & 0 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

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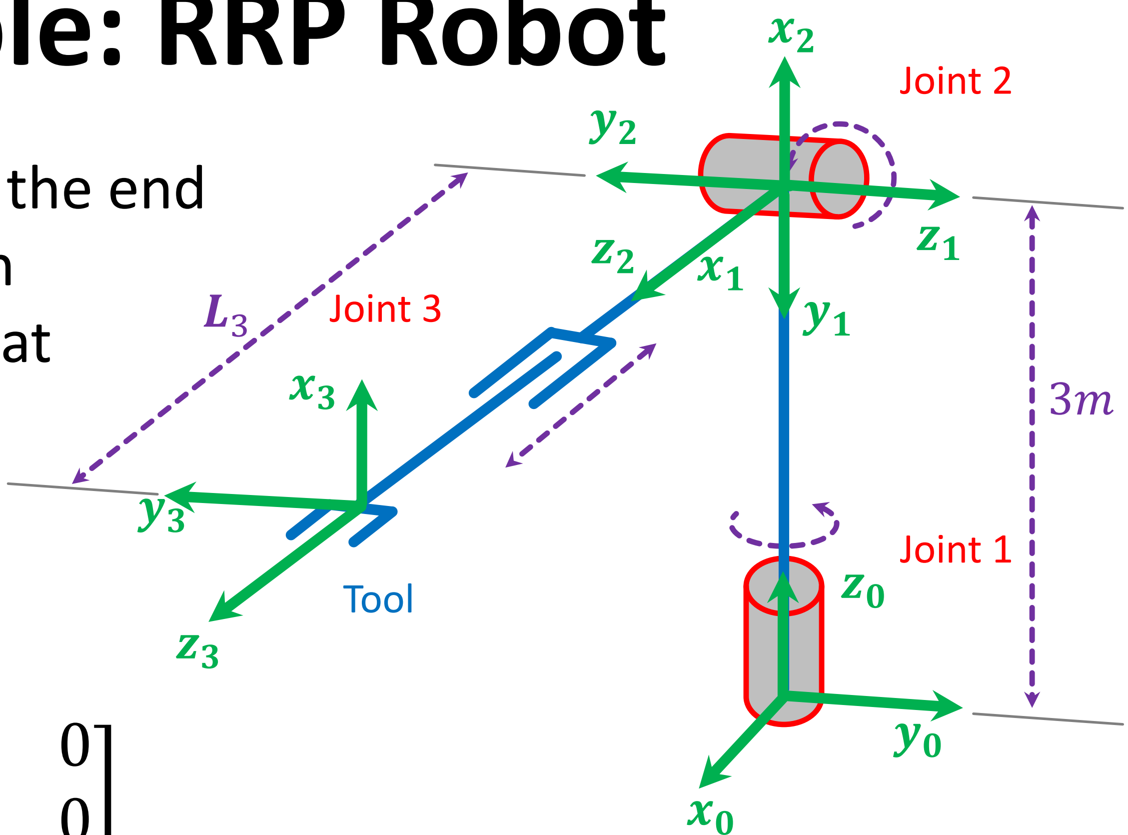
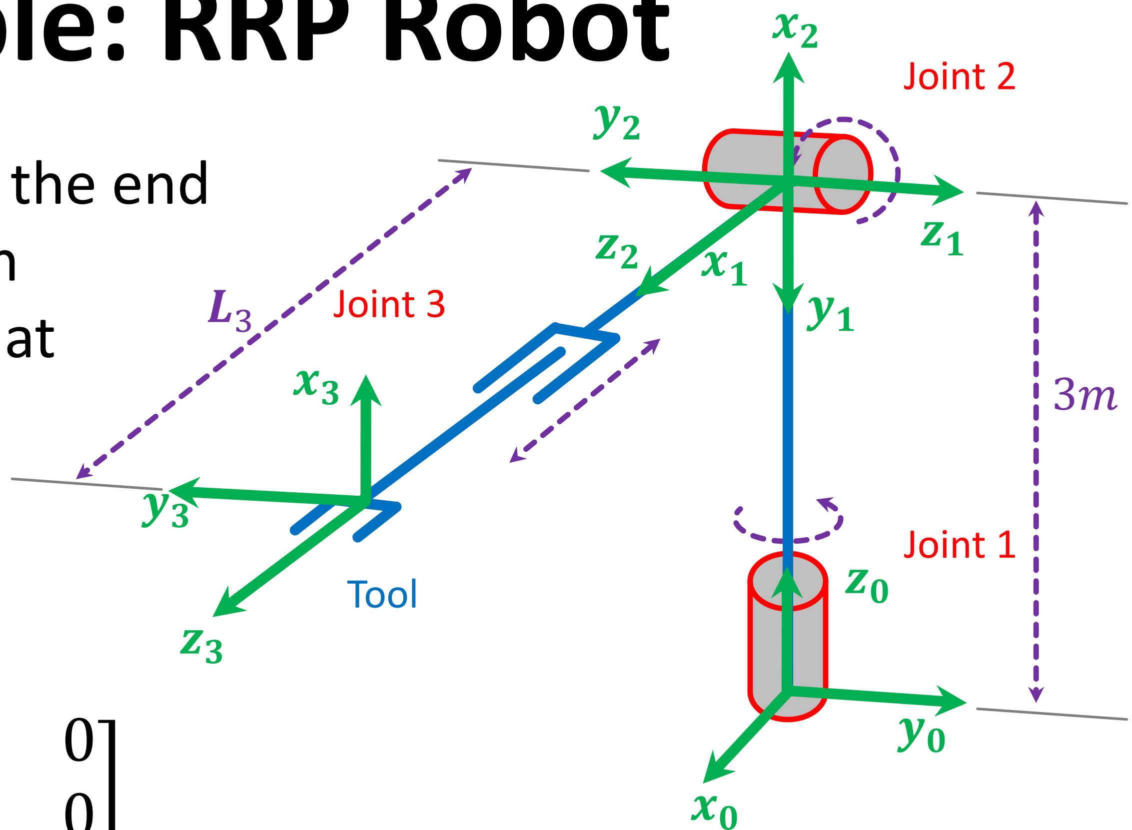


$$\mathcal{J}_\omega = \begin{bmatrix} 0 & -s_1 & 0 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \ rad/s$$

Example: RRP Robot



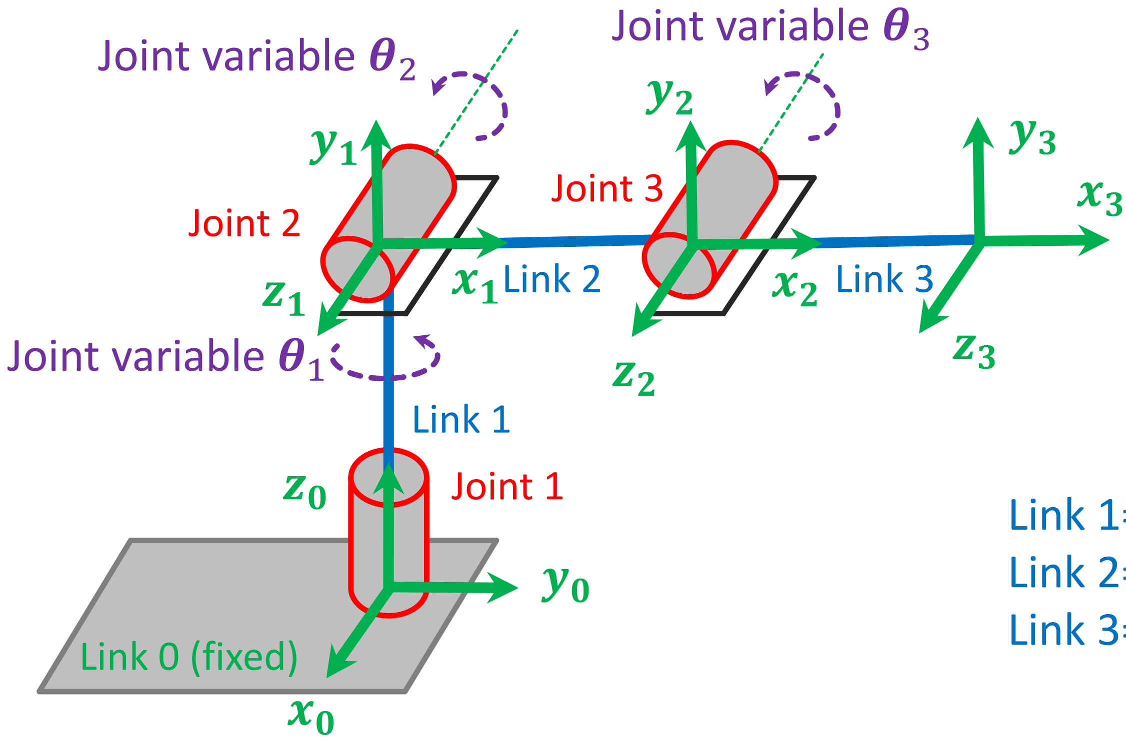
Find the angular velocity of the end effector in the configuration shown with joint 1 rotating at 2 rad/s, joint 2 rotating at -1 rad/s and joint 3 fixed At 2 m.

$$\mathcal{J}_\omega = \begin{bmatrix} 0 & -s_1 & 0 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \; rad/s$$

The end effector is rotating 0 rad/s around x axis, -1 rad/s around y axis, and 2 rad/s around z axis.

Example: RRR Robot



Link 1= 2 m
Link 2= 2 m
Link 3= 2 m

Find the angular velocity of the end effector in the configuration shown with joint 1 rotating at 2 rad/s, joint 2 rotating at -1 rad/s and joint 3 rotating at 2 rad/s.

"This is a typical example that can be discussed in the oral exam"

Linear Velocity Jacobian

The velocity of the end effector for an n link manipulator is the race of change of the origin of the end effector frame with respect to the base frame.

$$\dot{o}_n^0 = \sum_{i=1}^{n} \frac{\partial o_n^0}{\partial q_i} \dot{q}_i$$

The Origin o_i^0

The origin of the i^{th} reference frame

$$T_i^0 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & o_x \\ r_{21} & r_{22} & r_{23} & o_y \\ r_{31} & r_{32} & r_{33} & o_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Linear Velocity Jacobian

$$\dot{o}_n^0 = \sum_{i=1}^{n} \frac{\partial o_n^0}{\partial q_i} \dot{q}_i$$

The contribution of each joint in the linear velocity of the end effector.

$$\mathcal{J}_{v_i} = \frac{\partial o_n^0}{\partial q_i}$$

Each column in the Jacobian is the rate of changes of the end effector in the base reference frame with respect to the rate of change of a joint variable q_i.

The i^{th} column shows the movement of the end effector caused by $\dot{q}_i$.

Prismatic Joint

Displacement along the axis of actuation z_{i-1}:

$$\dot{o}_n^0 = \dot{d}_i z_{i-1}^0$$

End effector Velocity

$$\boxed{v} = \boxed{z_{i-1}^0}\ \boxed{\dot{q}_i}$$

Joint Velocity

$$\boxed{\boldsymbol{\mathcal{J}_{v_i} = z_{i-1}^0}}$$

Revolute Joint

Displacement around the axis of actuation z_{i-1}:

$$\dot{o}_n^0 = \omega \times r$$

$$\dot{o}_n^0 = \dot{q}_i \; z_{i-1}^0 \times (o_n - o_{i-1})$$

End effector Velocity

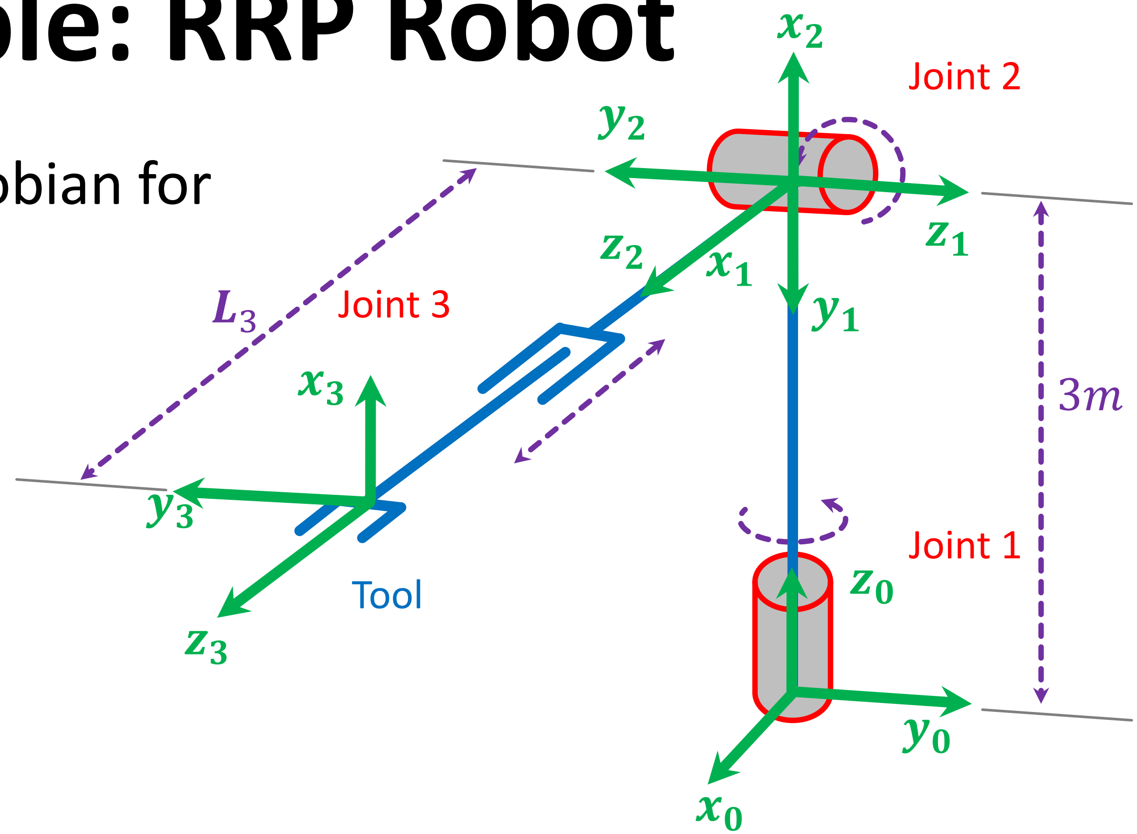
Joint Velocity

$$\boxed{v} = \boxed{z_{i-1}^0 \times (o_n - o_{i-1})} \; \boxed{\dot{q}_i}$$

$$\boxed{\boldsymbol{\mathcal{J}_{v_i} = z_{i-1}^0 \times (o_n - o_{i-1})}}$$

Example: RRP Robot

Find the linear velocity Jacobian for the arm RRP.



$$T_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_2^0 = \begin{bmatrix} c_1c_2 & s_1 & -c_1s_2 & 0 \\ s_1c_2 & -c_1 & -s_1s_2 & 0 \\ -s_2 & 0 & -c_2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_3^0 = \begin{bmatrix} c_1c_2 & s_1 & -c_1s_2 & -L_3c_1s_2 \\ s_1c_2 & -c_1 & -s_1s_2 & -L_3s_1s_2 \\ -s_2 & 0 & -c_2 & 3-L_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: RRP Robot

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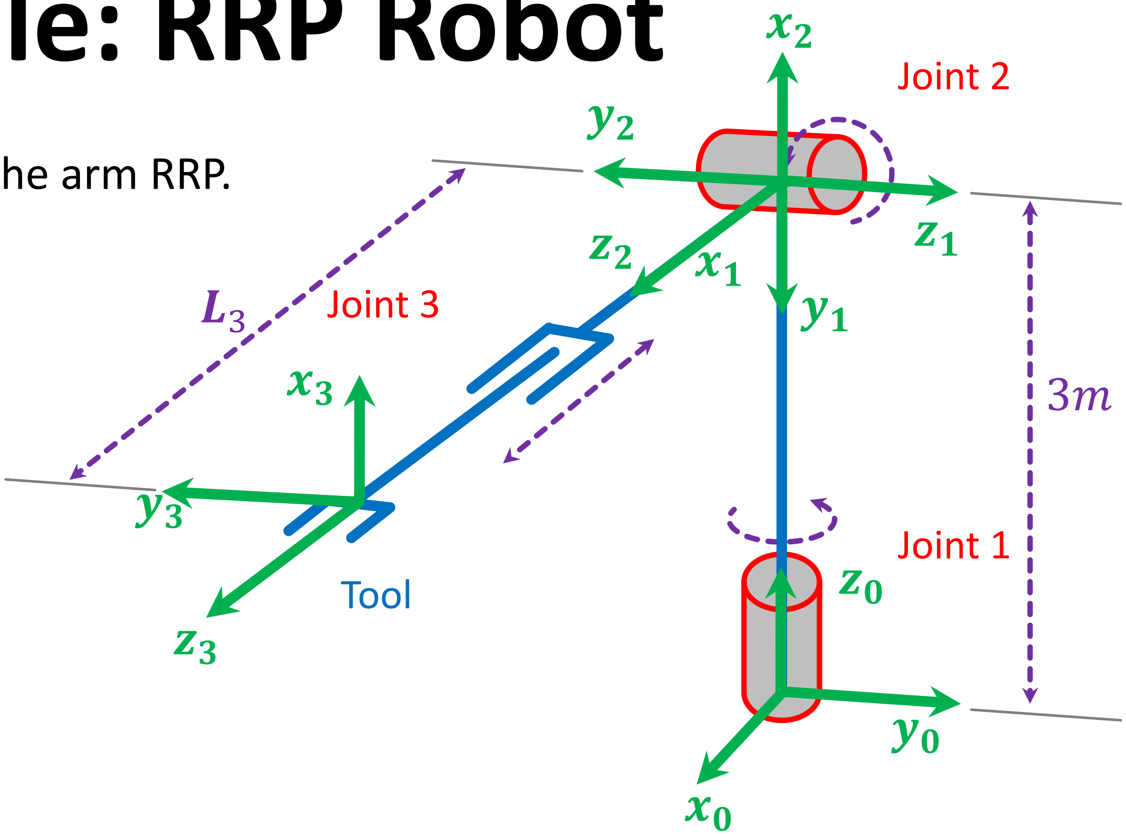

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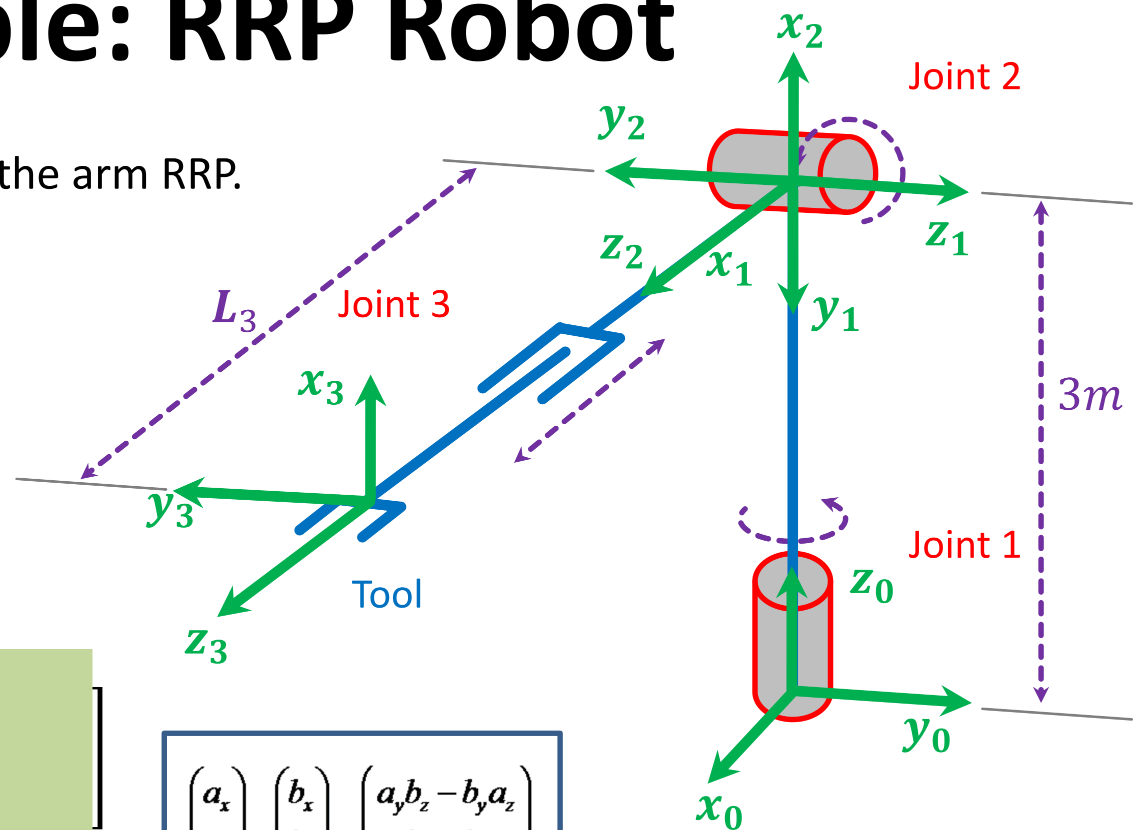

$$\mathcal{J}_{v_1} = \square \times \square$$

$$\mathcal{J}_{v_1} = \left[\square\right] \times \left[\square\right] = \left[\square\right]$$

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \times \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a_y b_z - b_y a_z \\ a_z b_x - b_z a_x \\ a_x b_y - b_x a_y \end{pmatrix}$$

$$T_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_2^0 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & 0 \\ s_1 c_2 & -c_1 & -s_1 s_2 & 0 \\ -s_2 & 0 & -c_2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_3^0 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & -L_3 c_1 s_2 \\ s_1 c_2 & -c_1 & -s_1 s_2 & -L_3 s_1 s_2 \\ -s_2 & 0 & -c_2 & 3 - L_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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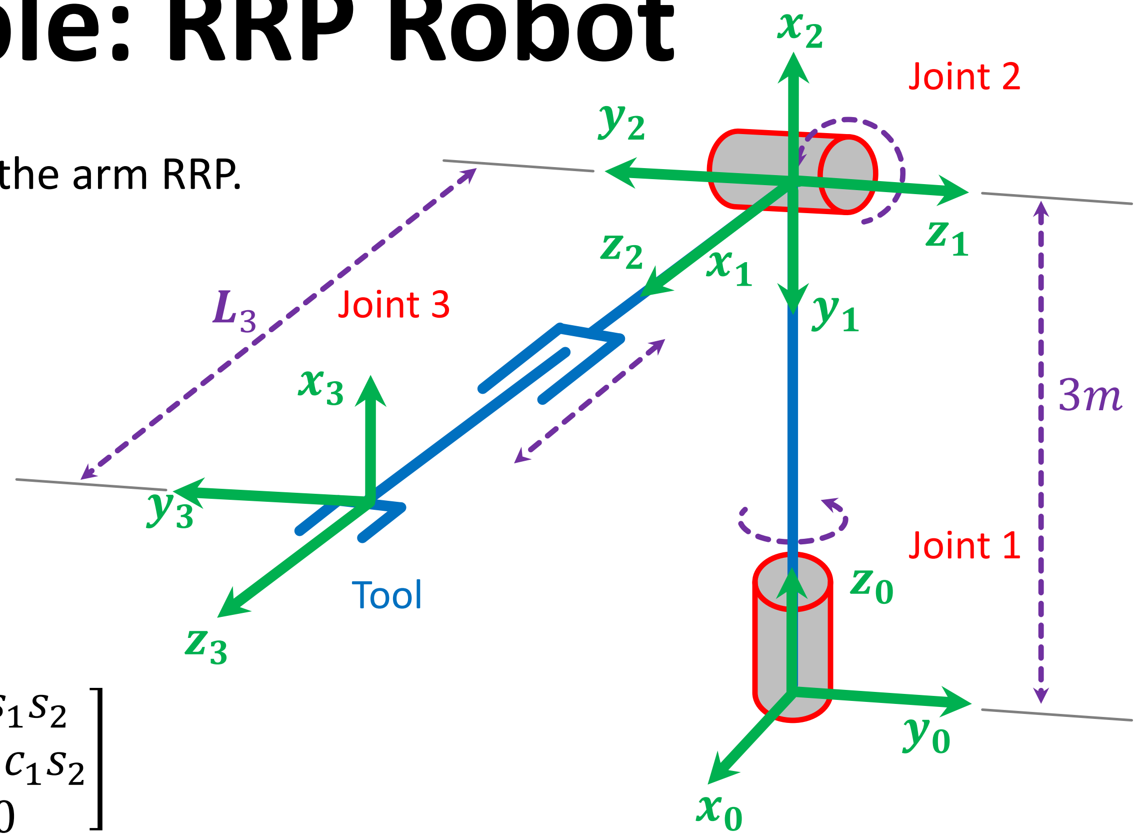

$$\mathcal{J}_{v_1} = z_0^0 \times (o_3 - o_0)$$

$$\mathcal{J}_{v_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -L_3 c_1 s_2 - 0 \\ -L_3 s_1 s_2 - 0 \\ 3 - L_3 c_2 - 0 \end{bmatrix} = \begin{bmatrix} L_3 s_1 s_2 \\ -L_3 c_1 s_2 \\ 0 \end{bmatrix}$$

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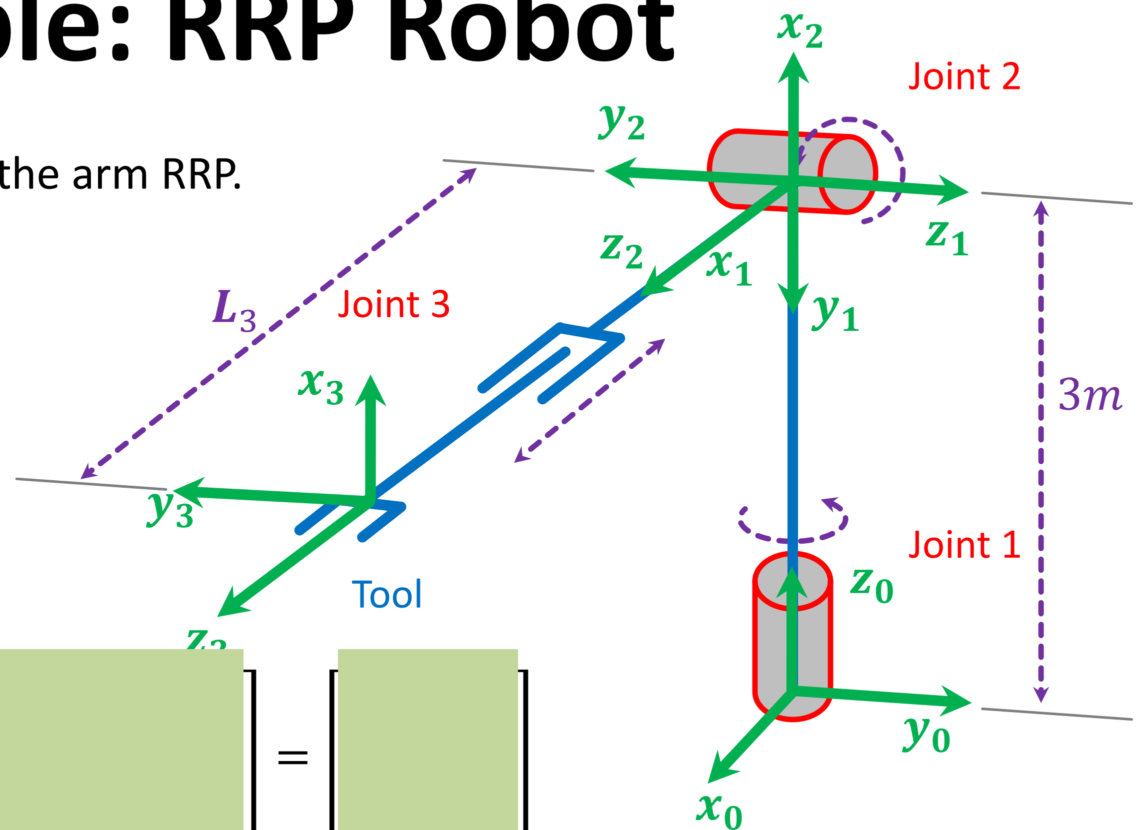

$$\mathcal{J}_{v_2} = \square \times \square$$

$$\mathcal{J}_{v_2} = \left[\ \square\ \right] \times \left[\ \square\ \right] = \left[\ \square\ \right] = \left[\ \square\ \right]$$

$$T_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_2^0 = \begin{bmatrix} c_1c_2 & s_1 & -c_1s_2 & 0 \\ s_1c_2 & -c_1 & -s_1s_2 & 0 \\ -s_2 & 0 & -c_2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_3^0 = \begin{bmatrix} c_1c_2 & s_1 & -c_1s_2 & -L_3c_1s_2 \\ s_1c_2 & -c_1 & -s_1s_2 & -L_3s_1s_2 \\ -s_2 & 0 & -c_2 & 3-L_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Find the linear velocity Jacobian for the arm RRP.

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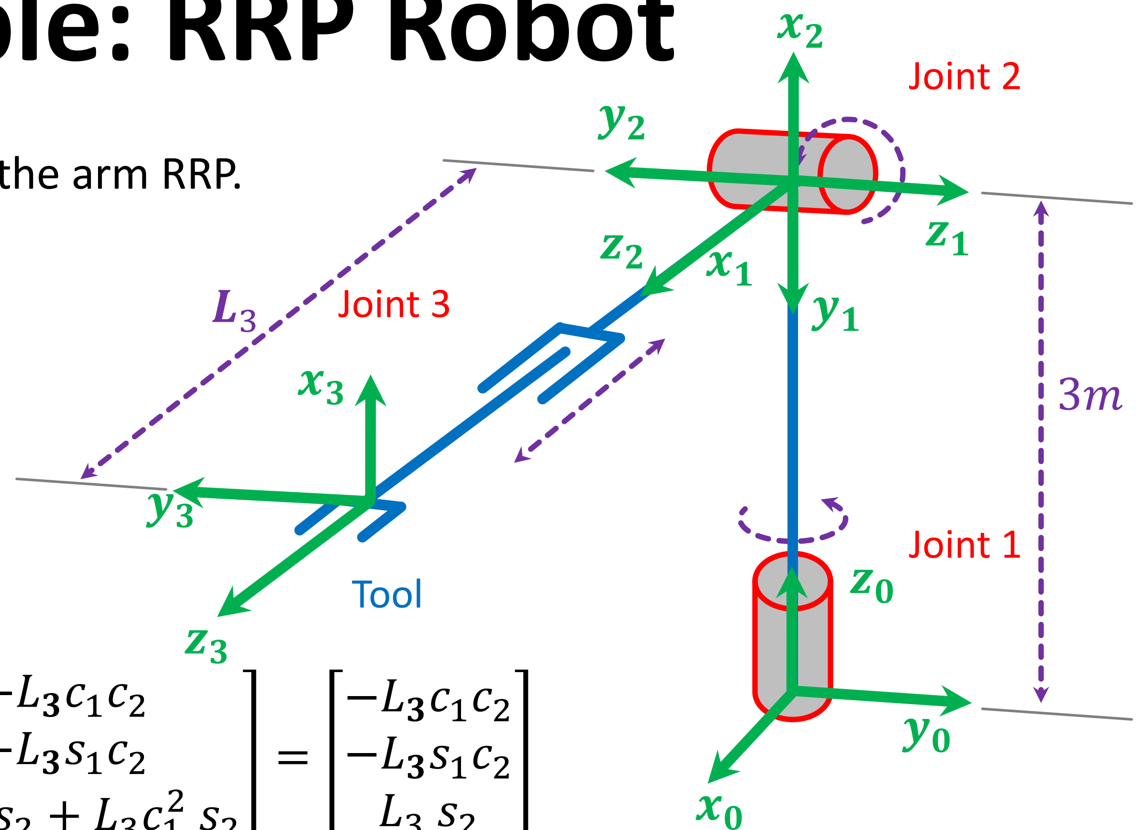

$$\mathcal{J}_{v_2} = z_1^0 \times (o_3 - o_1)$$

$$\mathcal{J}_{v_2} = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -L_3 c_1 s_2 \\ -L_3 s_1 s_2 \\ -L_3 c_2 \end{bmatrix} = \begin{bmatrix} -L_3 c_1 c_2 \\ -L_3 s_1 c_2 \\ L_3 s_1^2\, s_2 + L_3 c_1^2\, s_2 \end{bmatrix} = \begin{bmatrix} -L_3 c_1 c_2 \\ -L_3 s_1 c_2 \\ L_3\, s_2 \end{bmatrix}$$

$$T_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_2^0 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & 0 \\ s_1 c_2 & -c_1 & -s_1 s_2 & 0 \\ -s_2 & 0 & -c_2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_3^0 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & -L_3 c_1 s_2 \\ s_1 c_2 & -c_1 & -s_1 s_2 & -L_3 s_1 s_2 \\ -s_2 & 0 & -c_2 & 3 - L_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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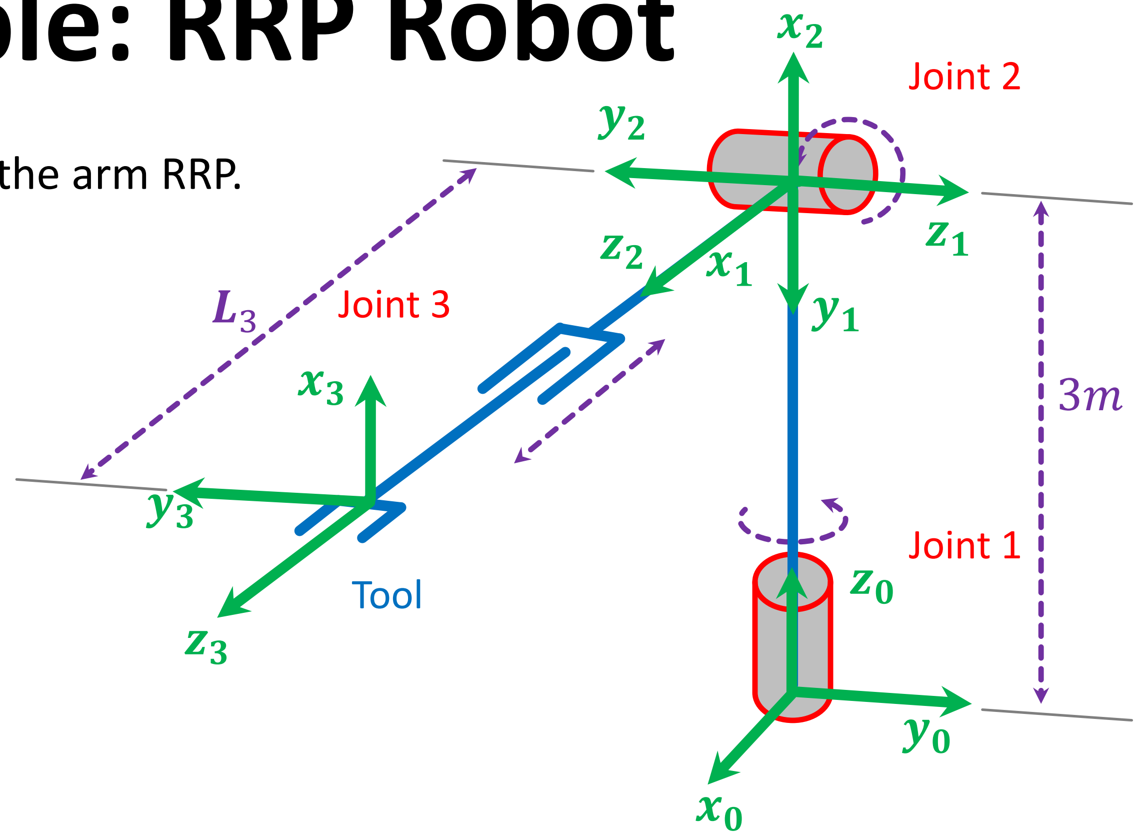

$\mathcal{J}_{v_3} =$

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$$T_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_2^0 = \begin{bmatrix} c_1c_2 & s_1 & -c_1s_2 & 0 \\ s_1c_2 & -c_1 & -s_1s_2 & 0 \\ -s_2 & 0 & -c_2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_3^0 = \begin{bmatrix} c_1c_2 & s_1 & -c_1s_2 & -L_3c_1s_2 \\ s_1c_2 & -c_1 & -s_1s_2 & -L_3s_1s_2 \\ -s_2 & 0 & -c_2 & 3-L_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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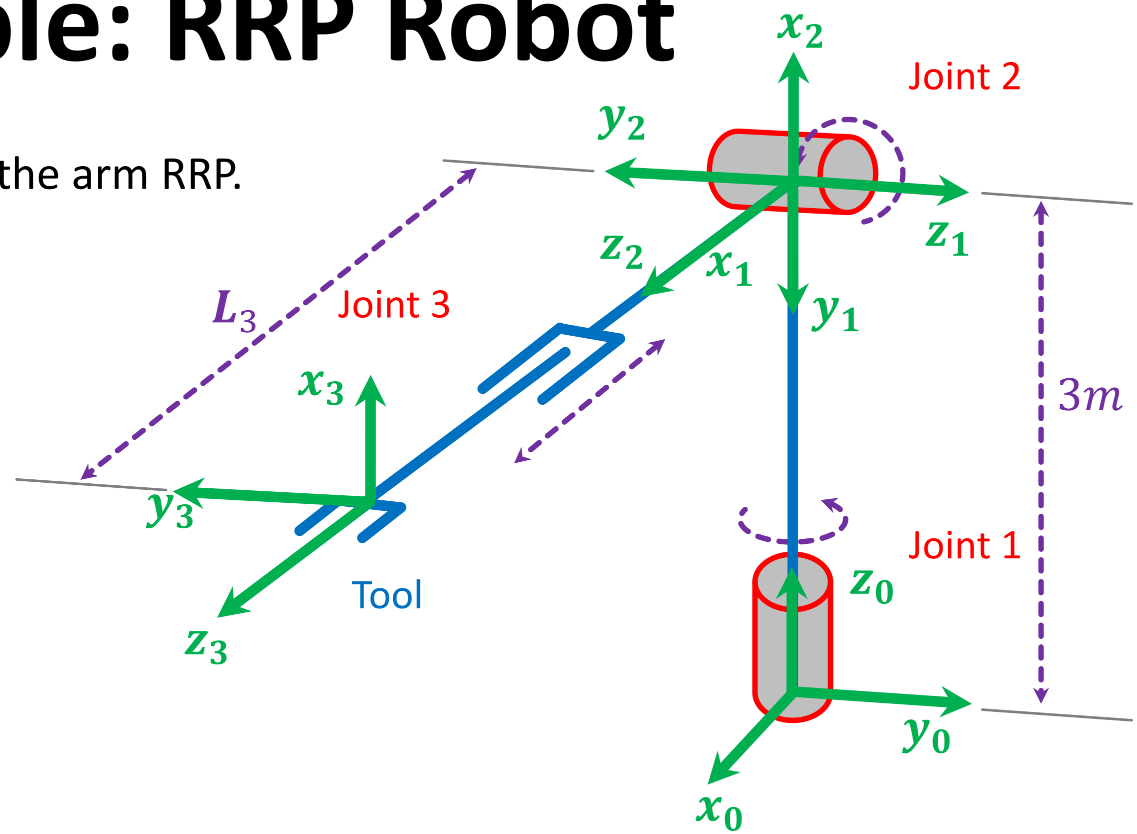

$$\mathcal{J}_{v_3} = z_2^{\,0}$$

$$\mathcal{J}_{v_3} = \begin{bmatrix} -c_1 s_2 \\ -s_1 s_2 \\ -c_2 \end{bmatrix}$$

$$T_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_2^0 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & 0 \\ s_1 c_2 & -c_1 & -s_1 s_2 & 0 \\ -s_2 & 0 & -c_2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_3^0 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & -L_3 c_1 s_2 \\ s_1 c_2 & -c_1 & -s_1 s_2 & -L_3 s_1 s_2 \\ -s_2 & 0 & -c_2 & 3 - L_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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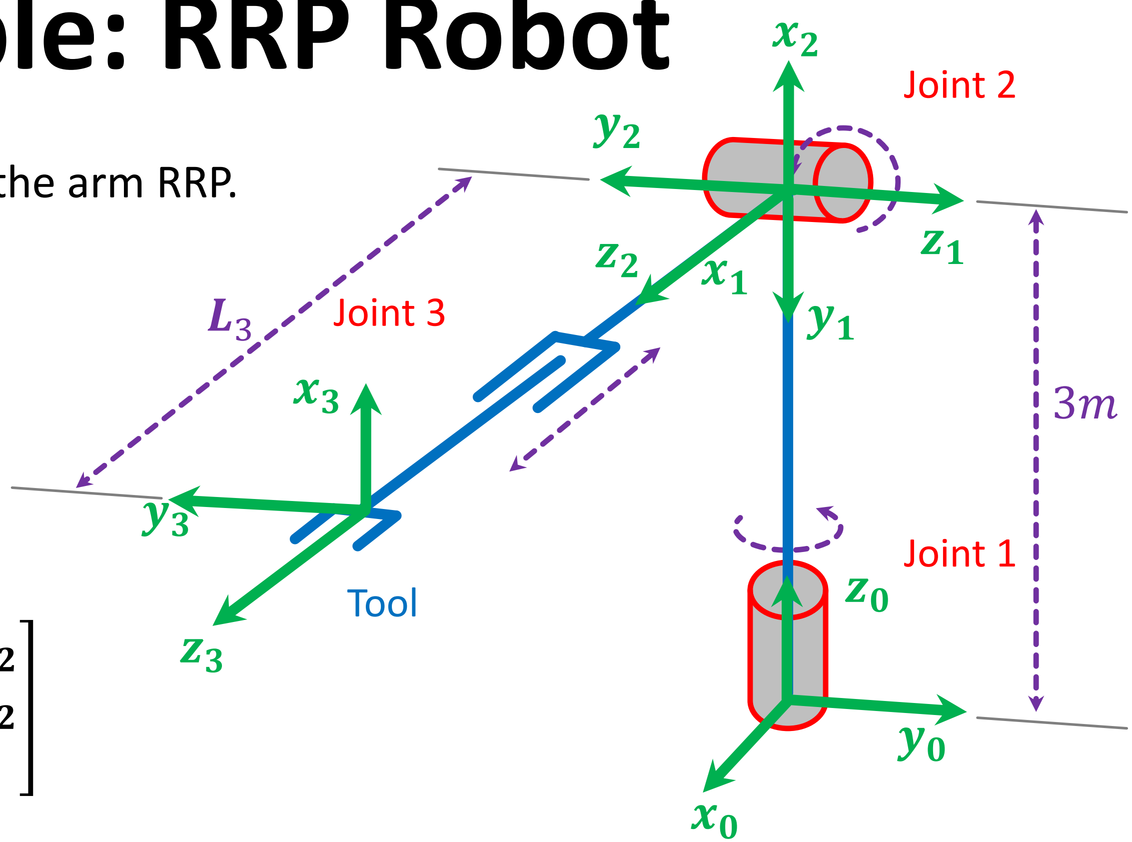

$$\mathcal{J}_v = \begin{bmatrix} L_3 s_1 s_2 & -L_3 c_1 c_2 & -c_1 s_2 \\ -L_3 c_1 s_2 & -L_3 s_1 c_2 & -s_1 s_2 \\ 0 & L_3\, s_2 & -c_2 \end{bmatrix}$$

$$T\,_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T\,_2^0 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & 0 \\ s_1 c_2 & -c_1 & -s_1 s_2 & 0 \\ -s_2 & 0 & -c_2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_3^0 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & -L_3 c_1 s_2 \\ s_1 c_2 & -c_1 & -s_1 s_2 & -L_3 s_1 s_2 \\ -s_2 & 0 & -c_2 & 3 - L_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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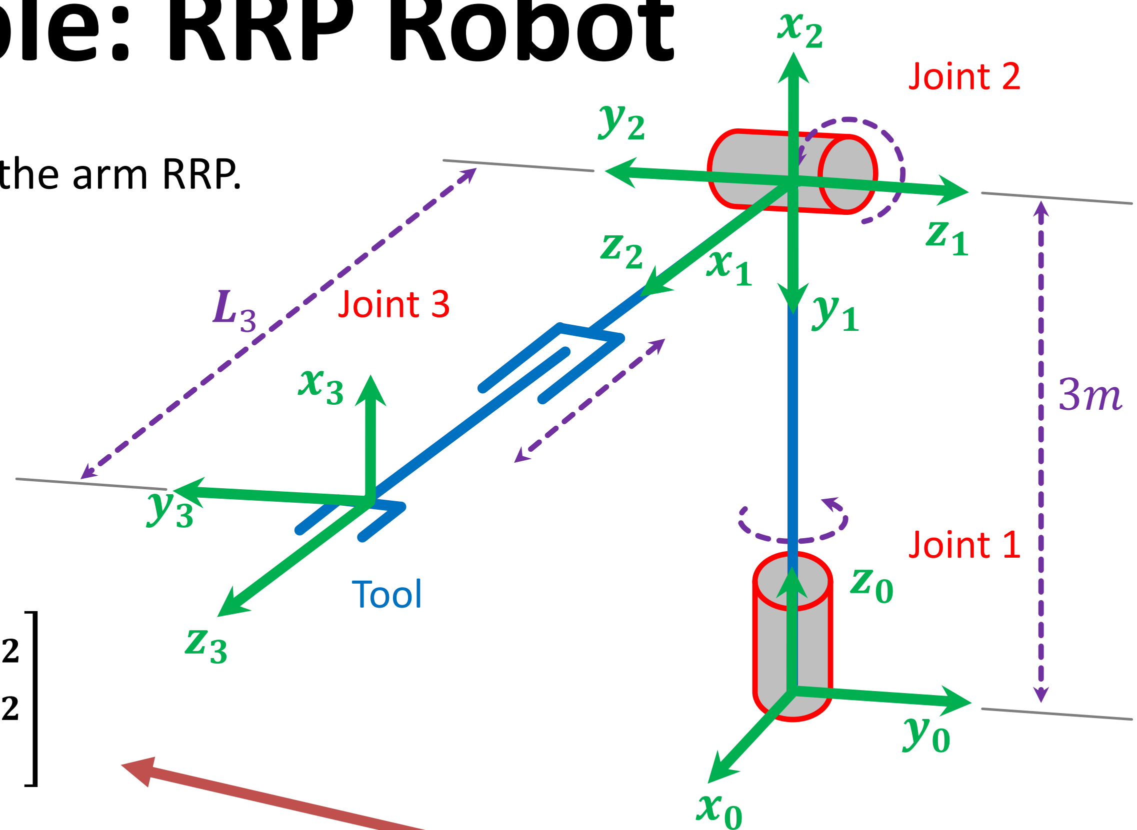
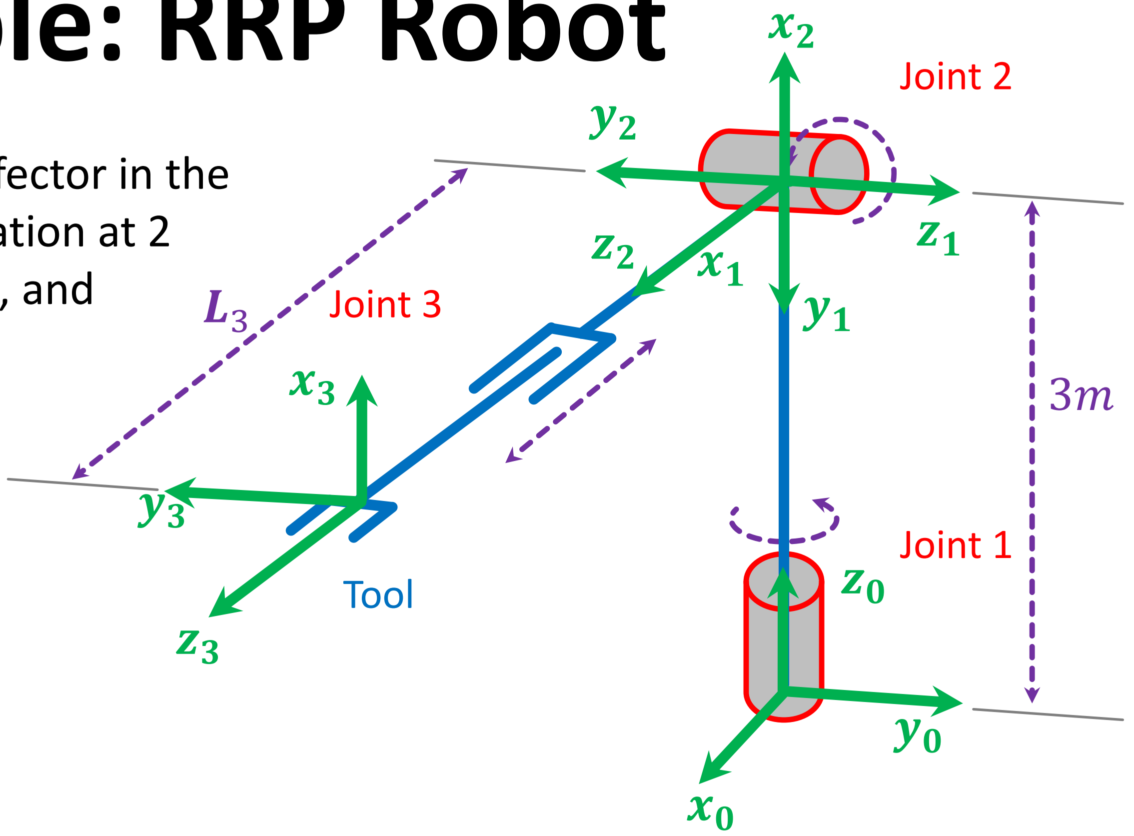


$$\boldsymbol{\mathcal{J}_v} = \begin{bmatrix} \boldsymbol{L_3 s_1 s_2} & \boldsymbol{-L_3 c_1 c_2} & \boldsymbol{-c_1 s_2} \\ \boldsymbol{-L_3 c_1 s_2} & \boldsymbol{-L_3 s_1 c_2} & \boldsymbol{-s_1 s_2} \\ \boldsymbol{0} & \boldsymbol{L_3\ s_2} & \boldsymbol{-c_2} \end{bmatrix}$$

$$T_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_2^0 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & 0 \\ s_1 c_2 & -c_1 & -s_1 s_2 & 0 \\ -s_2 & 0 & -c_2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_3^0 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & \boldsymbol{-L_3 c_1 s_2} \\ s_1 c_2 & -c_1 & -s_1 s_2 & \boldsymbol{-L_3 s_1 s_2} \\ -s_2 & 0 & -c_2 & \boldsymbol{3 - L_3 c_2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: RRP Robot

Find the linear velocity of the end effector in the configuration shown with joint 1 rotation at 2 rad/sec, joint 2 rotating at -1 rad/sec, and joint 3 fixed at 2 m.



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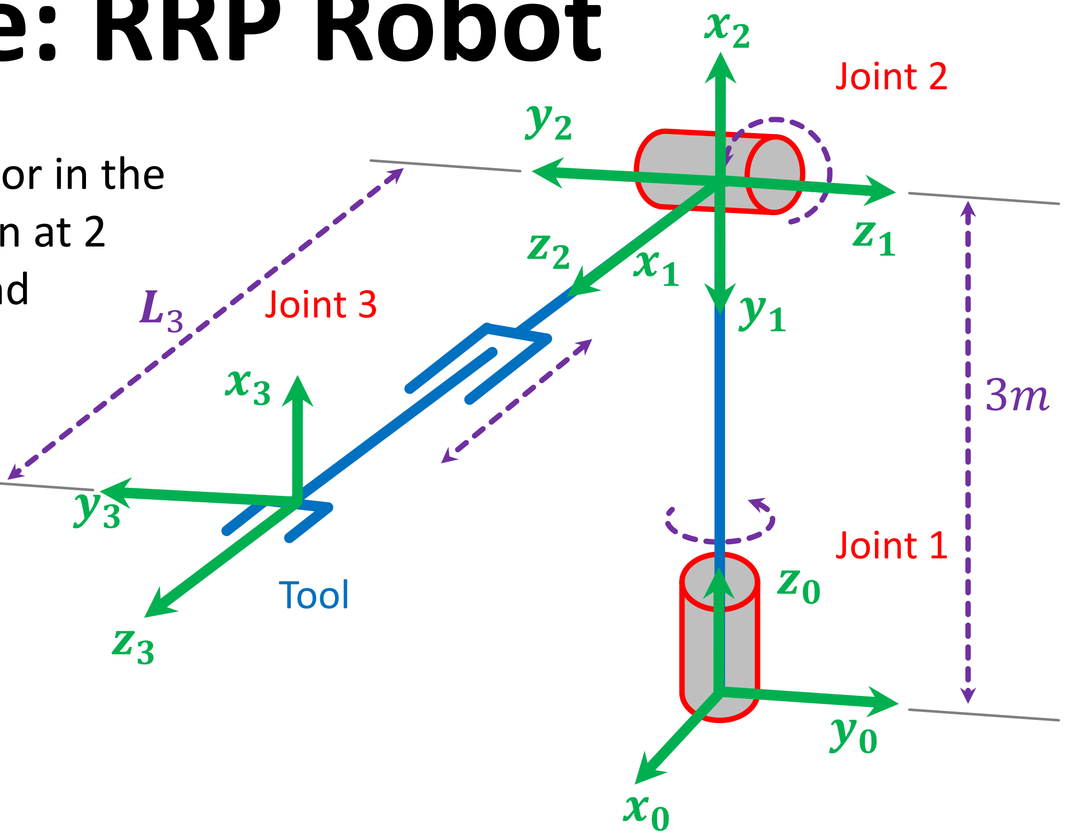
$\theta_1 =$, $\theta_2 =$, $L_3 =$

$\mathcal{J}_v = \begin{bmatrix} \quad \end{bmatrix}$

$v =$ $=$



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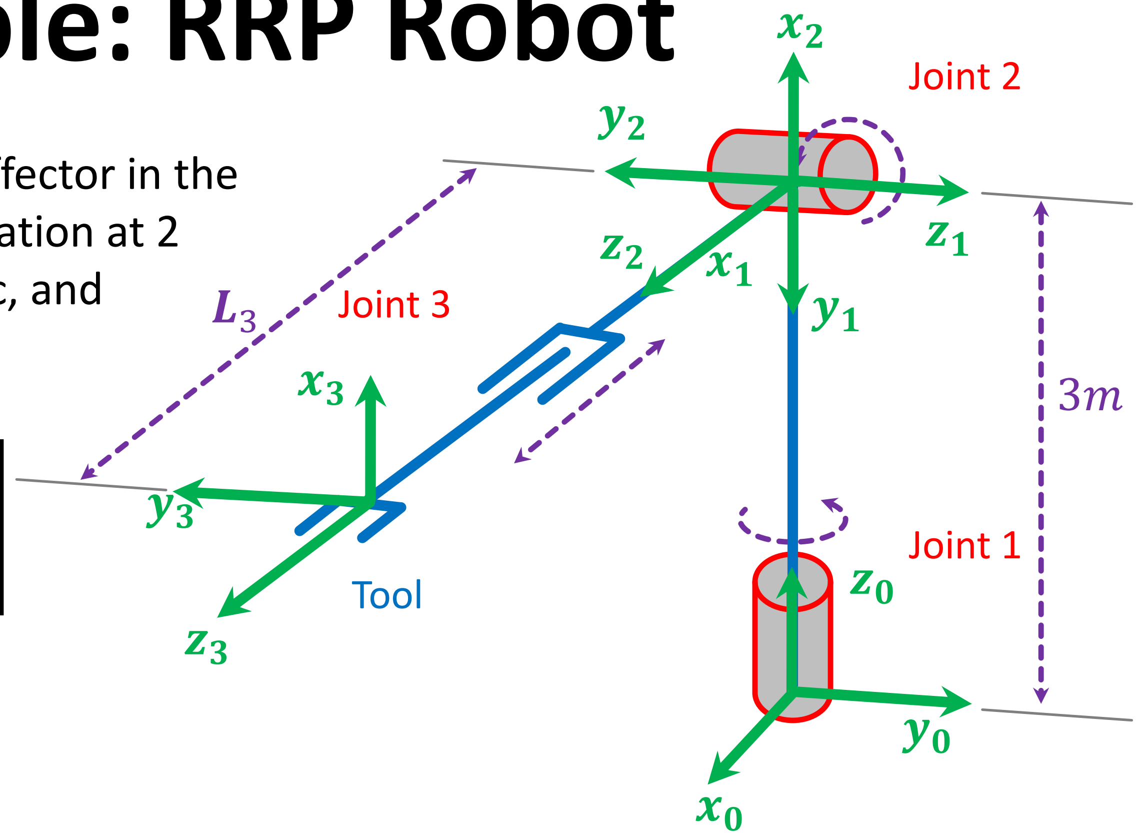


$$\mathcal{J}_v = \begin{bmatrix} L_3 s_1 s_2 & -L_3 c_1 c_2 & -c_1 s_2 \\ -L_3 c_1 s_2 & -L_3 s_1 c_2 & -s_1 s_2 \\ 0 & L_3\ s_2 & -c_2 \end{bmatrix}$$

$$\theta_1 = 0^{\circ}, \theta_2 = -90^{\circ}, L_3 = 2\ m$$

$$\mathcal{J}_v = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

$$v = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} \quad m/s$$

Inverting The Jacobian

- Analytical inverse (more DOF more Complexity)
- Numerical inverse

Reminder

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\underbrace{ad-bc}_{\text{determinant}}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= aei + bfg + cdh - ceg - bdi - afh.$$

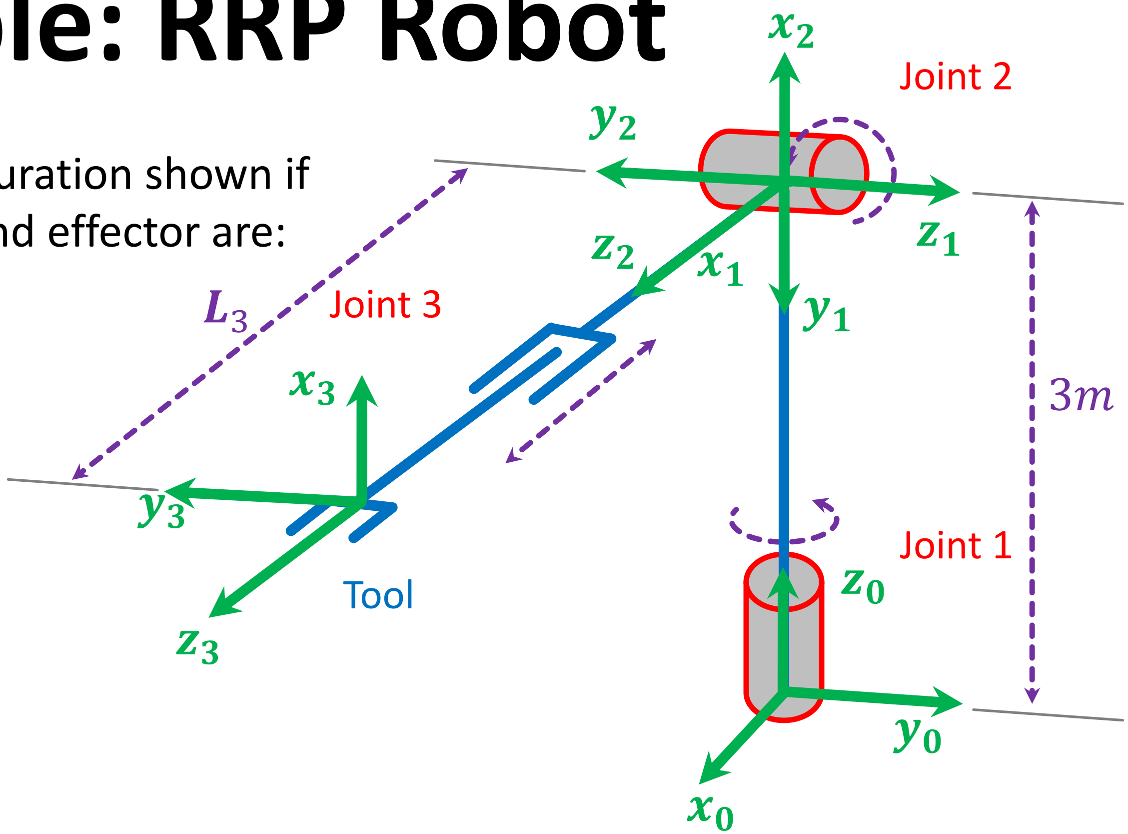
$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{\begin{bmatrix} ei - fh & hc - ib & bf - ce \\ gf - di & ai - gc & dc - af \\ dh - ge & gb - ah & ae - db \end{bmatrix}}{|\mathbf{A}|}$$

$$= \frac{\begin{bmatrix} ei - fh & hc - ib & bf - ce \\ gf - di & ai - gc & dc - af \\ dh - ge & gb - ah & ae - db \end{bmatrix}}{aei + bfg + cdh - gec - hfa - idb}$$

Example: RRP Robot

Find the joint velocities in the configuration shown if the desired linear velocities of the end effector are:
0 m/sec on x axis
4 m/sec on y axis
2 m/sec on z axis



$$\mathcal{J}_v = \begin{bmatrix} L_3 s_1 s_2 & -L_3 c_1 c_2 & -c_1 s_2 \\ -L_3 c_1 s_2 & -L_3 s_1 c_2 & -s_1 s_2 \\ 0 & L_3\, s_2 & -c_2 \end{bmatrix}$$

$$\theta_1 = 0^\circ, \theta_2 = -90^\circ, L_3 = 2\,m$$

$$\mathcal{J}_v = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

$$\mathcal{J}_v{}^{-1} = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & -0.5 \\ 1 & 0 & 0 \end{bmatrix}$$

Example: RRP Robot

Find the joint velocities in the configuration shown if the desired linear velocities of the end effector are:
0 m/sec on x axis
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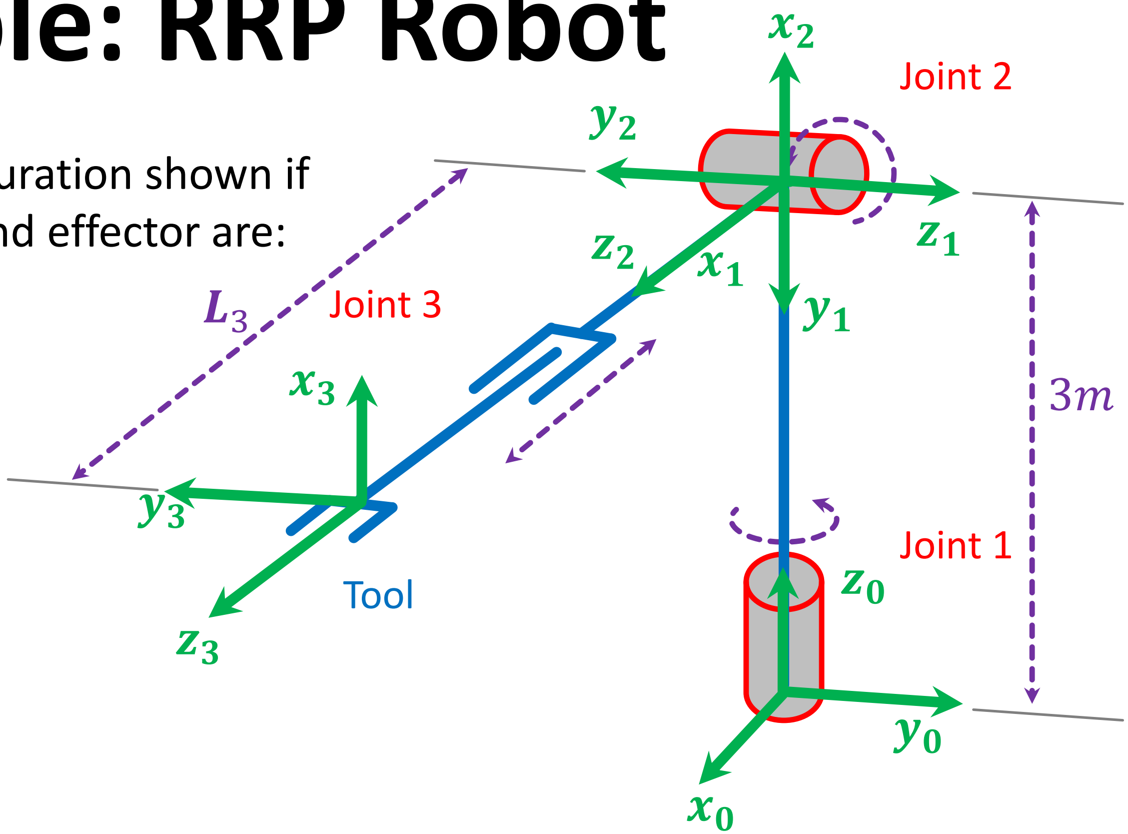

$$\mathcal{J}_v^{-1} = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & -0.5 \\ 1 & 0 & 0 \end{bmatrix}$$

$\dot{q} =$ [ ] = [ ]

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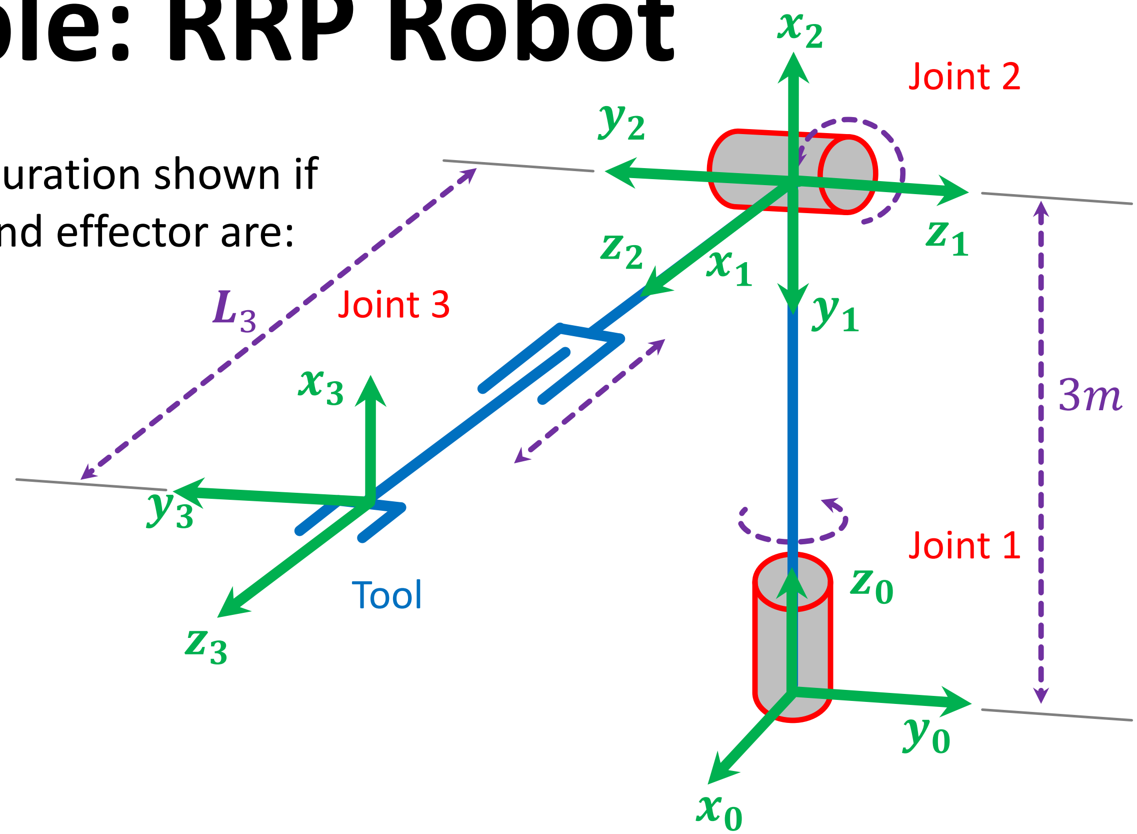


$$\mathcal{J}_v^{\ -1} = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & -0.5 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\dot{q} = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & -0.5 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Example: NAO

Throwing in 2D only with ***ShoulderPitch*** and ***ElbowRoll***.

Example: RR Robot

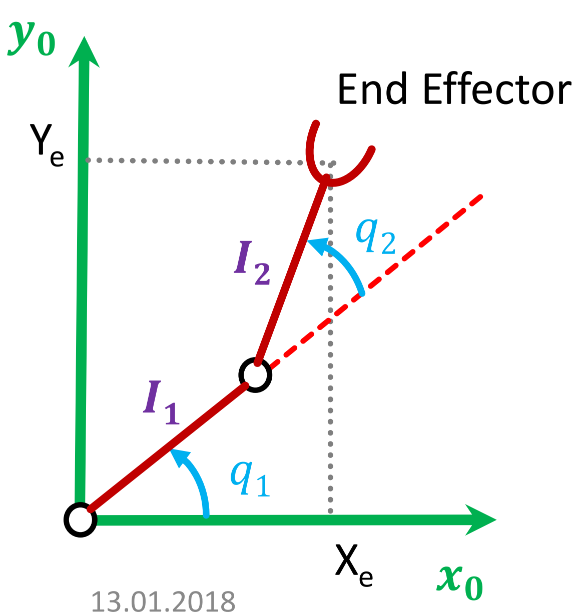
Forward kinematics:

$$X_e = I_1\, c_1 \,+\, I_2\, c_{12}$$
$$Y_e = I_1\, s_1 \,+\, I_2\, s_{12}$$

Find the end-effector velocities in function of joint velocities.

The total derivatives of the kinematics equations:

$$dX_e = \frac{\partial X_e(q_1, q_2)}{\partial q_1} dq_1 + \frac{\partial X_e(q_1, q_2)}{\partial q_2} dq_2$$
$$dY_e = \frac{\partial Y_e(q_1, q_2)}{\partial q_1} dq_1 + \frac{\partial Y_e(q_1, q_2)}{\partial q_2} dq_2$$



The Jacobian matrix represents the differential relationship between the joint displacement and the resulting end effector motion.

It comprises the partial derivatives of $X_e(q_1, q_2)$ and $Y_e(q_1, q_2)$ with respect to the joint displacements q_1 and q_2.

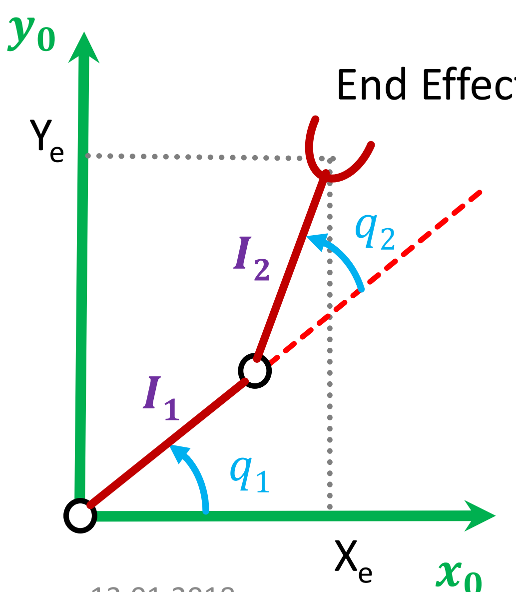
Example: RR Robot

Forward kinematics:

$$X_e = I_1\, c_1 + I_2\, c_{12}$$
$$Y_e = I_1\, s_1 + I_2\, s_{12}$$

$\mathcal{J}_v =$ [ ]

$\mathcal{J}_v =$ [ ]

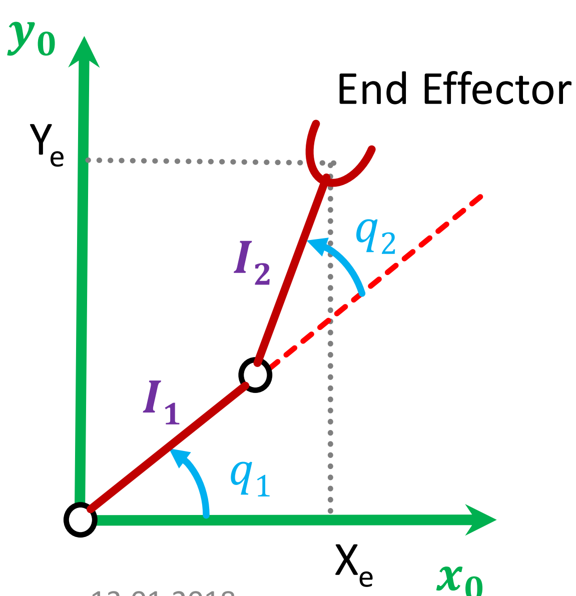


Example: RR Robot

Forward kinematics:

$$X_e = l_1 c_1 + l_2 c_{12}$$
$$Y_e = l_1 s_1 + l_2 s_{12}$$

$$\mathcal{J}_v = \begin{bmatrix} \dfrac{\partial X_e(q_1,q_2)}{\partial q_1} & \dfrac{\partial X_e(q_1,q_2)}{\partial q_2} \\ \dfrac{\partial Y_e(q_1,q_2)}{\partial q_1} & \dfrac{\partial Y_e(q_1,q_2)}{\partial q_2} \end{bmatrix}$$

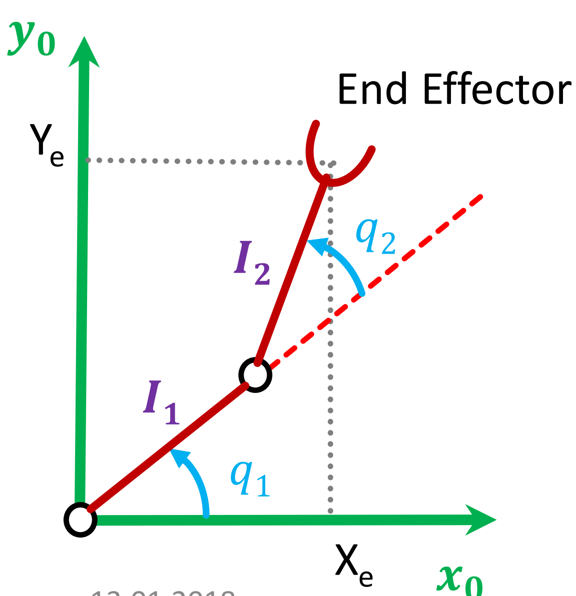


$$\mathcal{J}_v = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$

Example: RR Robot

Forward kinematics:

$$X_e = l_1 c_1 + l_2 c_{12}$$
$$Y_e = l_1 s_1 + l_2 s_{12}$$



$$\mathcal{J}_v = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$

We can divide the 2-by-2 Jacobian into two column vectors:

$$\mathcal{J}_v = (\mathcal{J}_1, \mathcal{J}_2), \qquad \mathcal{J}_1, \mathcal{J}_2 \in \mathfrak{R}^{2 \times 1}$$

We can then write the resulting end-effector velocity vector:

$$V_e = \mathcal{J}_1 . \dot{q}_1 + \mathcal{J}_2 . \dot{q}_2$$

Each column vector of the Jacobian matrix represents the end –effector velocity generated by the corresponding joint moving when all other joints are not moving.

Example: RR Robot

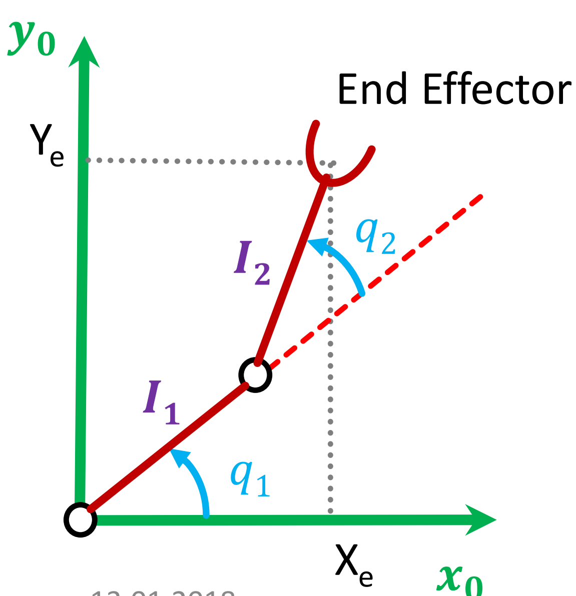
Forward kinematics:

$$X_e = I_1\, c_1 + I_2\, c_{12}$$
$$Y_e = I_1\, s_1 + I_2\, s_{12}$$

$$\mathcal{J}_v = \begin{bmatrix} -I_1 s_1 - I_2 s_{12} & -I_2 s_{12} \\ I_1 c_1 + I_2 c_{12} & I_2 c_{12} \end{bmatrix}$$

$\mathcal{J}_1 =$

$\mathcal{J}_2 =$



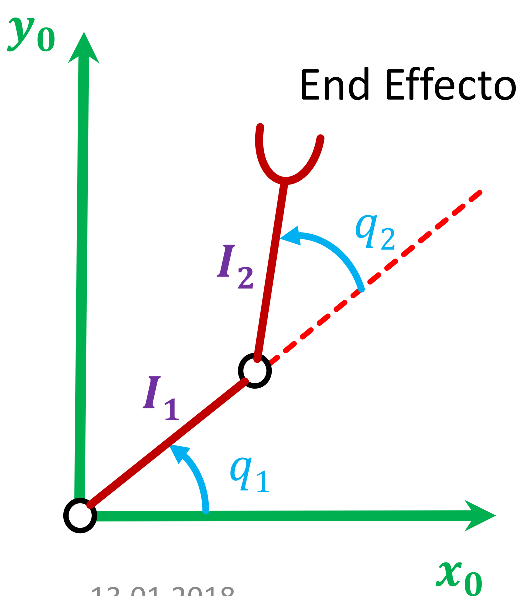
Example: RR Robot

Forward kinematics:

$$X_e = I_1\, c_1 \; + \; I_2\, c_{12}$$
$$Y_e = I_1\, s_1 \; + \; I_2\, s_{12}$$

$$\mathcal{J}_1 = \begin{bmatrix} -I_1 s_1 - I_2 s_{12} \\ I_1 c_1 + I_2 c_{12} \end{bmatrix}, \mathcal{J}_2 = \begin{bmatrix} -I_2 s_{12} \\ I_2 c_{12} \end{bmatrix}$$

Illustrate the column vector of the Jacobian in the space at the end-effector point.



Example: RR Robot

Forward kinematics:

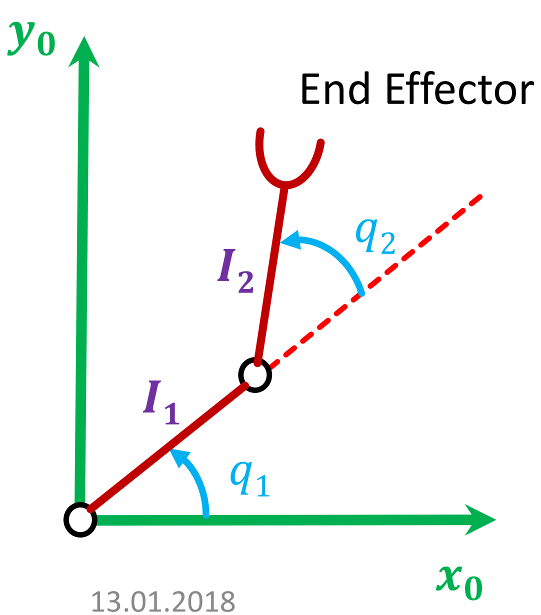
$$X_e = l_1 c_1 + l_2 c_{12}$$
$$Y_e = l_1 s_1 + l_2 s_{12}$$

$$\mathcal{J}_1 = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} \end{bmatrix}, \mathcal{J}_2 = \begin{bmatrix} -l_2 s_{12} \\ l_2 c_{12} \end{bmatrix}$$

Illustrate the column vector of the Jacobian in the space at the end-effector point.

$\mathcal{J}_2$ points in the direction perpendicular to link 2.

While $\mathcal{J}_1$ is not perpendicular to link 1 but is perpendicular to the vector (X_e,Y_e).
This is because $\mathcal{J}_1$ represent the endpoint velocity caused by joint 1 when joint 2 is not rotating.
In other word, link 1 and 2 are rigidly connected, becoming a single rigid body of length (X_e,Y_e) and $\mathcal{J}_1$ is the tip velocity of this body.



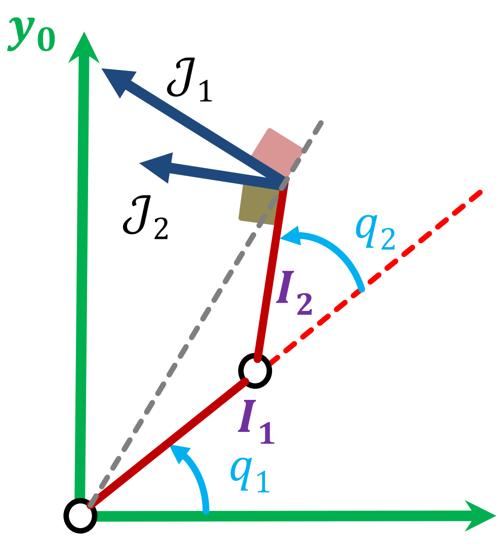
Example: RR Robot

Forward kinematics:

$$X_e = l_1 c_1 + l_2 c_{12}$$
$$Y_e = l_1 s_1 + l_2 s_{12}$$

$$\mathcal{J}_1 = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} \end{bmatrix}, \mathcal{J}_2 = \begin{bmatrix} -l_2 s_{12} \\ l_2 c_{12} \end{bmatrix}$$

Illustrate the column vector of the Jacobian in the space at the end-effector point.



$\mathcal{J}_2$ points in the direction perpendicular to link 2.

While $\mathcal{J}_1$ is not perpendicular to link 1 but is perpendicular to the vector (X_e, Y_e).
This is because $\mathcal{J}_1$ represent the endpoint velocity caused by joint 1 when joint 2 is not rotating. In other word, link 1 and 2 are rigidly connected, becoming a single rigid body of length (X_e, Y_e) and $\mathcal{J}_1$ is the tip velocity of this body.

Example: RR Robot

Forward kinematics:

$$X_e = l_1 c_1 + l_2 c_{12}$$
$$Y_e = l_1 s_1 + l_2 s_{12}$$

$$\mathcal{J}_1 = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} \end{bmatrix}, \mathcal{J}_2 = \begin{bmatrix} -l_2 s_{12} \\ l_2 c_{12} \end{bmatrix}$$

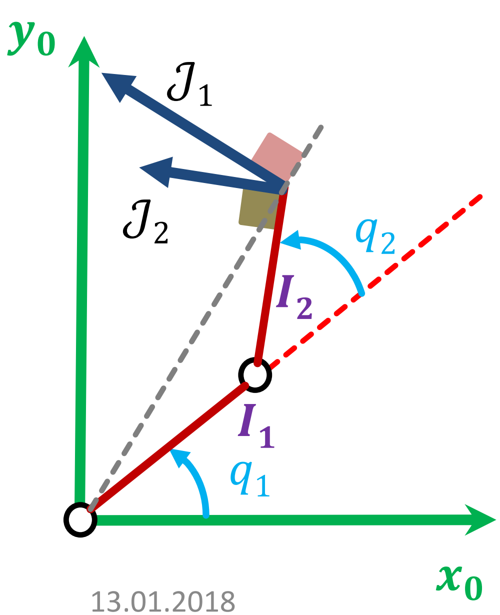
If the two Jacobian column vectors **are aligned**, the end-effector can not be moved in an arbitrary direction.

This may happen for particular arm configurations when the two links are fully contracted or extracted.

These arm configurations are referred to as singular configurations.

ACCORDINGLY, the Jacobian matrix become singular at these positions.

Find out the singular configurations…

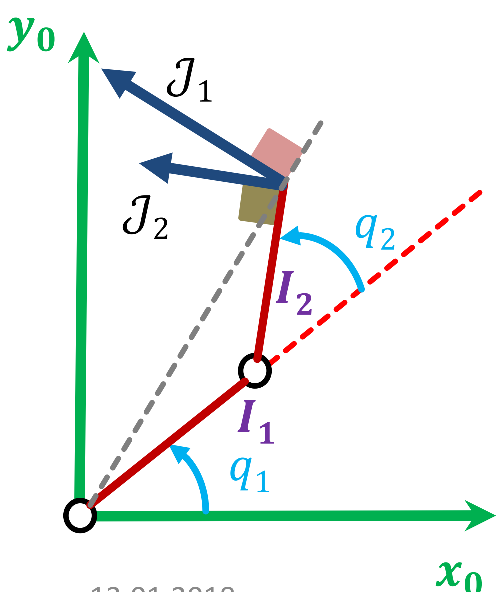


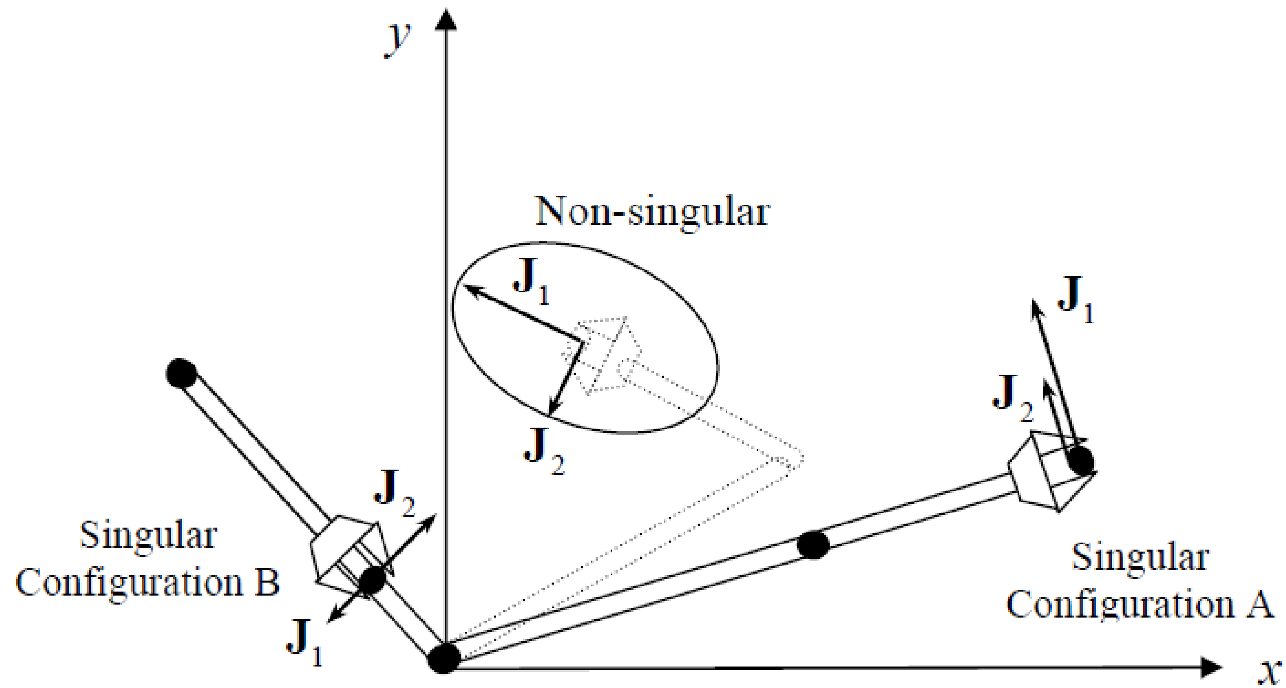
Example: RR Robot

Forward kinematics:

$$\mathcal{J}_1 = \begin{bmatrix} -I_1 s_1 - I_2 s_{12} \\ I_1 c_1 + I_2 c_{12} \end{bmatrix}, \mathcal{J}_2 = \begin{bmatrix} -I_2 s_{12} \\ I_2 c_{12} \end{bmatrix}$$

$$X_e = I_1\, c_1 + I_2\, c_{12}$$
$$Y_e = I_1\, s_1 + I_2\, s_{12}$$





Example: RR Robot

Forward kinematics:

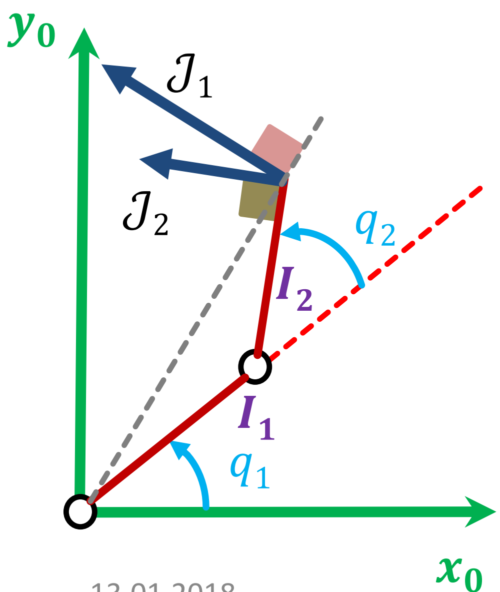
$$X_e = I_1\, c_1 + I_2\, c_{12}$$
$$Y_e = I_1\, s_1 + I_2\, s_{12}$$

$$\mathcal{J}_v = \begin{bmatrix} -I_1 s_1 - I_2 s_{12} & -I_2 s_{12} \\ I_1 c_1 + I_2 c_{12} & I_2 c_{12} \end{bmatrix}$$

The Jacobian reflects the singular configurations.

When joint 2 is 0 or 180 degrees:

$$det(\mathcal{J}_v) = \det\left(\quad\right) =$$



$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

determinant

Example: RR Robot

Forward kinematics:

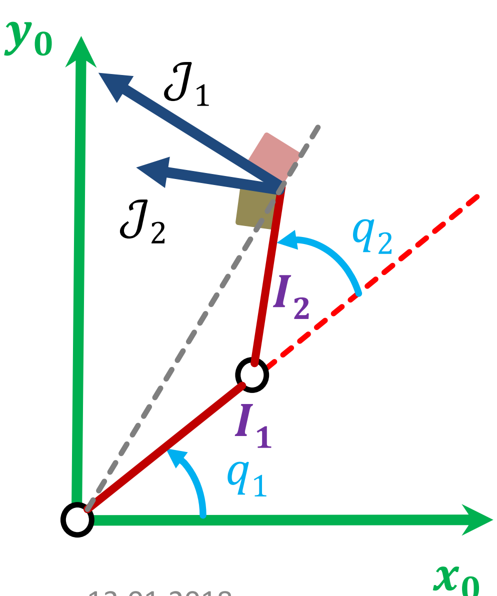
$$X_e = I_1\, c_1 + I_2\, c_{12}$$
$$Y_e = I_1\, s_1 + I_2\, s_{12}$$

$$\mathcal{J}_v = \begin{bmatrix} -I_1 s_1 - I_2 s_{12} & -I_2 s_{12} \\ I_1 c_1 + I_2 c_{12} & I_2 c_{12} \end{bmatrix}$$

The Jacobian reflects the singular configurations.

When joint 2 is 0 or 180 degrees:

$$det(\mathcal{J}_v) = \det\left(\begin{bmatrix} -(I_1 \pm I_2) s_1 & \mp I_2 s_1 \\ (I_1 \pm I_2) c_1 & \pm I_2 c_1 \end{bmatrix}\right) = 0$$



$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\underbrace{ad-bc}_{\text{determinant}}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

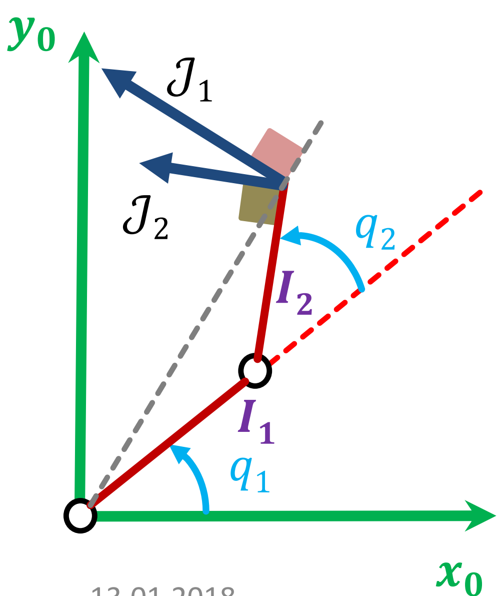
Example: RR Robot

Work out the joint velocities $\dot{q}$ ($\dot{q}_1$, $\dot{q}_2$) in terms of the end effector velocity $V_e(V_x, V_y)$.

If the arm configuration is not singular, this can be obtained by taking the inverse of the Jacobian matrix:

$$\dot{q} = J^{-1}.Ve$$

Note that the differential kinematics problem has **a unique solution** as long as the Jacobian is non-singular.

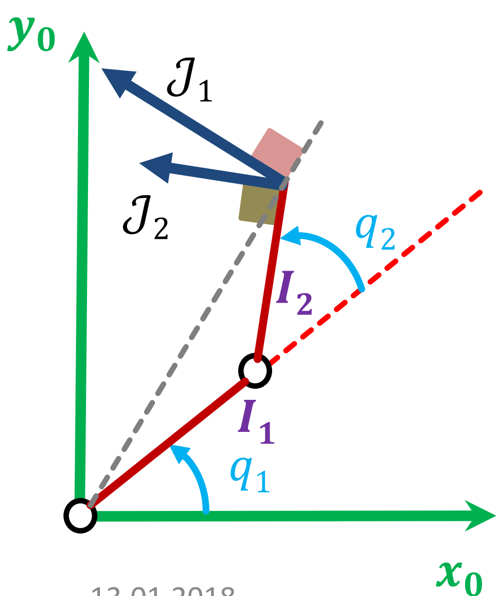


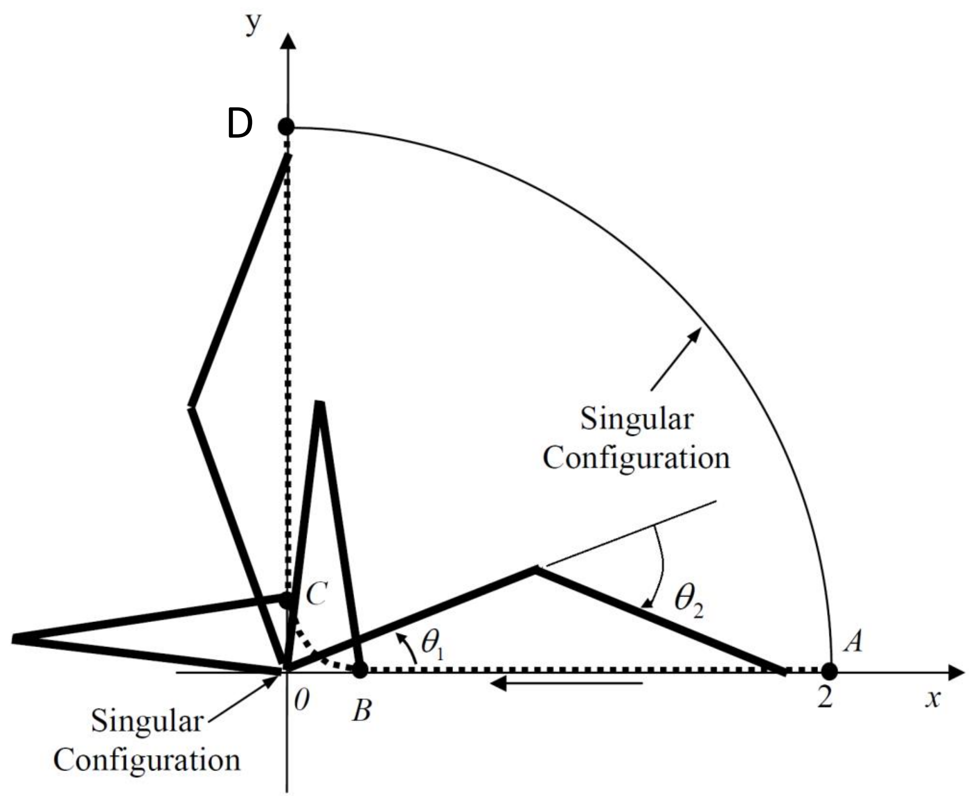
Since the elements of the Jacobian matrix are function of joint displacements, the inverse Jacobian varies depending on the arm configuration.
This means that **although the desired end-effector velocity is constant, the joint velocities are not.**

Example: RR Robot

We want to move the endpoint of the robot at a constant speed along a path starting at point “A” on the x-axis, (+2.0, 0), go around the origin through point “B” (+ε, 0) and “C” (0, +ε), and reach the final point “D” (0, +2.0) on the y-axis. Consider $l_1 = l_2$.

Work out joints velocities along this path.





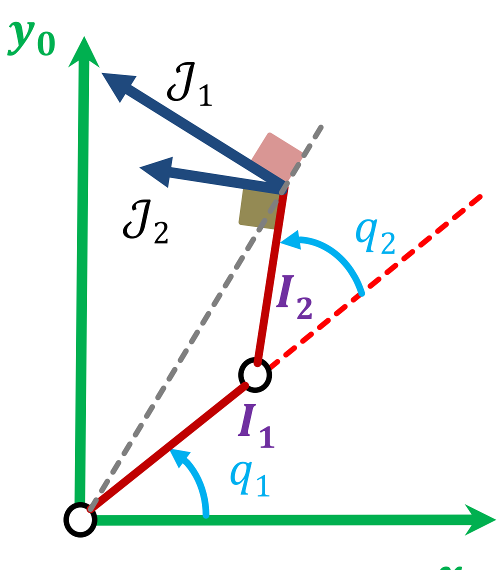
Example: RR Robot

We want to move the endpoint of the robot at a constant speed along a path starting at point “A” on the x-axis, (+2.0, 0), go around the origin through point “B” (+ε, 0) and “C” (0, +ε), and reach the final point “D” (0, +2.0) on the y-axis. Consider $I_1 = I_2$.

Work out joints velocities along this path.

The Jacobian is:

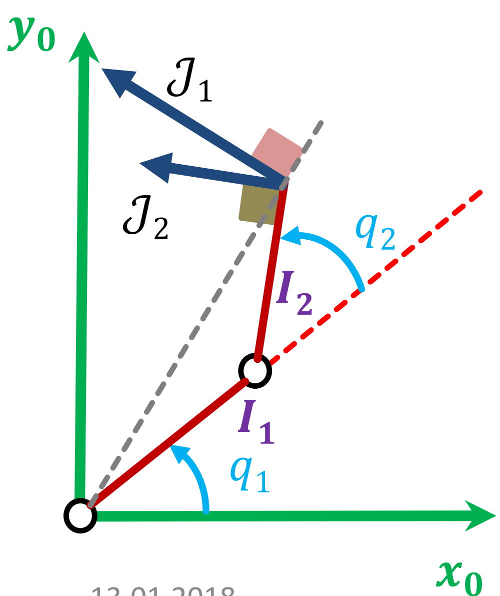
$$\mathcal{J}_v = \begin{bmatrix} -I_1 s_1 - I_2 s_{12} & -I_2 s_{12} \\ I_1 c_1 + I_2 c_{12} & I_2 c_{12} \end{bmatrix}$$



Example: RR Robot

We want to move the endpoint of the robot at a constant speed along a path starting at point “A” on the x-axis, (+2.0, 0), go around the origin through point “B” (+ε, 0) and “C” (0, +ε), and reach the final point “D” (0, +2.0) on the y-axis. Consider $l_1 = l_2$.

Work out joints velocities along this path.



The Jacobian is:

$$\mathcal{J}_v = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$

The inverse of the Jacobian is:

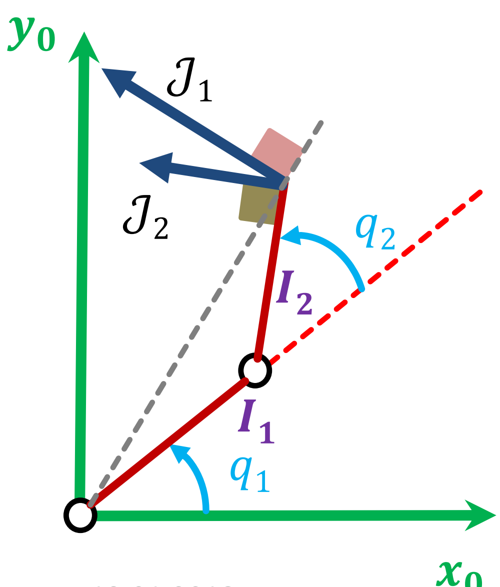
$$\mathcal{J}_v^{-1} = \frac{1}{\quad} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\underbrace{ad-bc}_{\text{determinant}}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example: RR Robot

We want to move the endpoint of the robot at a constant speed along a path starting at point “A” on the x-axis, (+2.0, 0), go around the origin through point “B” (+ε, 0) and “C” (0, +ε), and reach the final point “D” (0, +2.0) on the y-axis. Consider $I_1 = I_2$.

Work out joints velocities along this path.



The inverse of the Jacobian is:

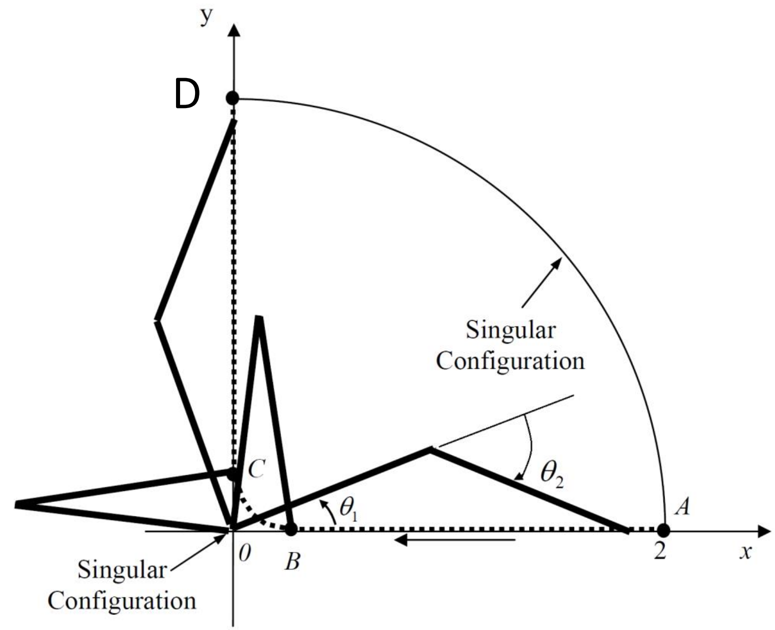
$$\mathcal{J}_v^{-1} = \frac{1}{I_1 I_2 s_2}\begin{bmatrix} I_2 c_{12} & I_2 s_{12} \\ -I_1 c_1 - I_2 c_{12} & -I_1 s_1 - I_2 s_{12} \end{bmatrix}$$

$\dot{q}_1 =$

$\dot{q}_2 =$

TwoLinksManipulator2.m

```
% two links arm
clc;
clear
%%%%%%%%%%%%%%%
% A to B
t1_AB = 0.01 : 0.02 : ((pi/2.0)-0.01);
t2_AB = -2*t1_AB;

Vx_AB = -1.0*ones(size(t1_AB)) ; % [m/s]
Vy_AB =  0.0*ones(size(t1_AB)) ; % [m/s]
%%%%%%%%%%%%%
t1 = [t1_AB ]; t2 = [t2_AB ];
Vx = [Vx_AB ]; Vy = [Vy_AB ];

t1D = (Vx.*cos(t1+t2)+Vy.*sin(t1+t2))./sin(t2);
t2D = -(Vx.*(cos(t1)+cos(t1+t2))+Vy.*(sin(t1)+sin(t1+t2)))./sin(t2

plot(t1D); hold on; plot(t2D);
%%%%%%%%%%%%%
% C to D
t1_CD = pi-0.01 : -0.02 : ((pi/2.0)+0.01);
t2_CD = t1_CD - pi/2.0 ;

Vx_CD =  0.0*ones(size(t1_AB)) ; % [m/s]
Vy_CD =  1.0*ones(size(t1_AB)) ; % [m/s]
%%%%%%%%%%%%%
t1 = [t1_CD ]; t2 = [t2_CD ];

Vx = [Vx_CD ]; Vy = [Vy_CD ];

t1D = (Vx.*cos(t1+t2)+Vy.*sin(t1+t2))./sin(t2);
t2D = -(Vx.*(cos(t1)+cos(t1+t2))+Vy.*(sin(t1)+sin(t1+t2)))./sin(t2);

plot(t1D); hold on; plot(t2D);
%%%%%%%%%%%%%
```

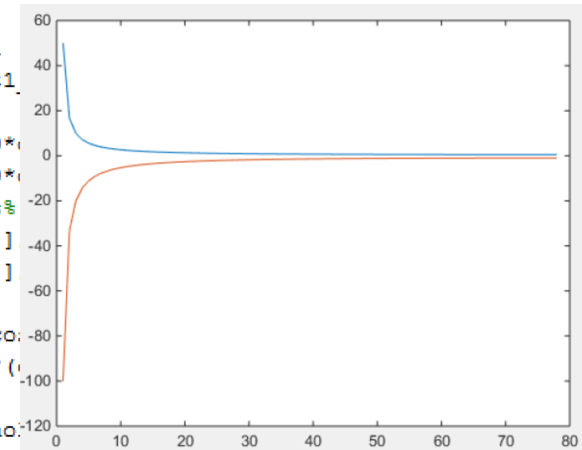


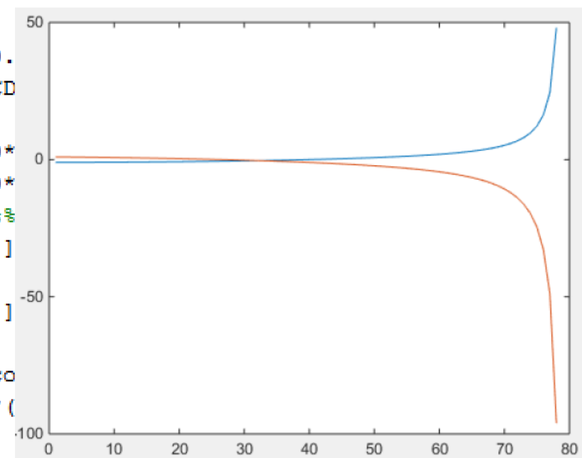
$$\dot{q}_1 = \frac{I_2 c_{12}.Vx + I_2 s_{12}.Vy}{I_1 I_2 s_2}$$

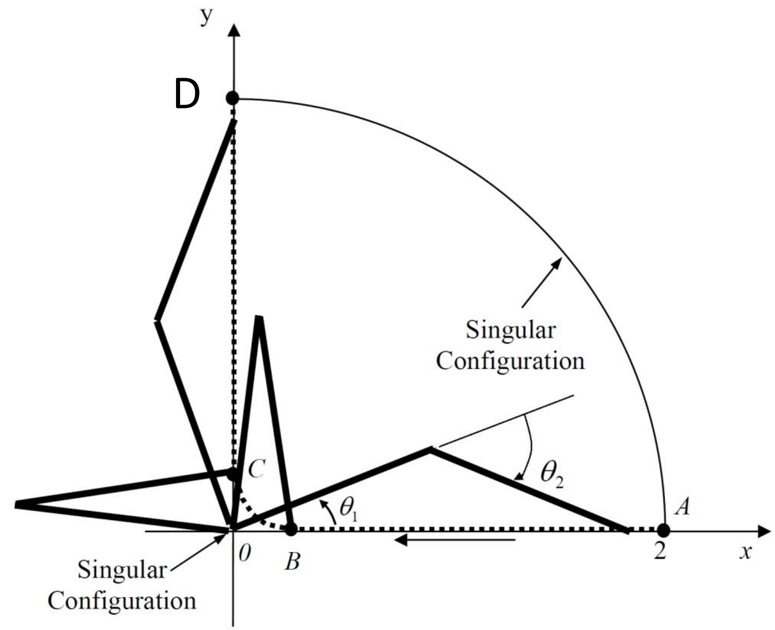
$$\dot{q}_2 = \frac{[-I_1 c_1 - I_2 c_{12}].Vx + [-I_1 s_1 - I_2 s_{12}].Vy}{I_1 I_2 s_2}$$

TwoLinksManipulator2.m

```
% two links arm
clc;
clear
%%%%%%%%%%%%%%%%
% A to B
t1_AB = 0.01
t2_AB = -2*t1_

Vx_AB = -1.0*
Vy_AB =  0.0*
%%%%%%%%%%%%%%
t1 = [t1_AB ]
Vx = [Vx_AB ]

t1D = (Vx.*co
t2D = -(Vx.*(                                t2)))./sin(t2

plot(t1D); ho
%%%%%%%%%%%%%%
% C to D
t1_CD = pi-0.
t2_CD = t1_CD

Vx_CD =  0.0*
Vy_CD =  1.0*
%%%%%%%%%%%%%%
t1 = [t1_CD ]

Vx = [Vx_CD ]

t1D = (Vx.*co
t2D = -(Vx.*(                                t2)))./sin(t2);

plot(t1D); hold on; plot(t2D);
%%%%%%%%%%%%%%
```







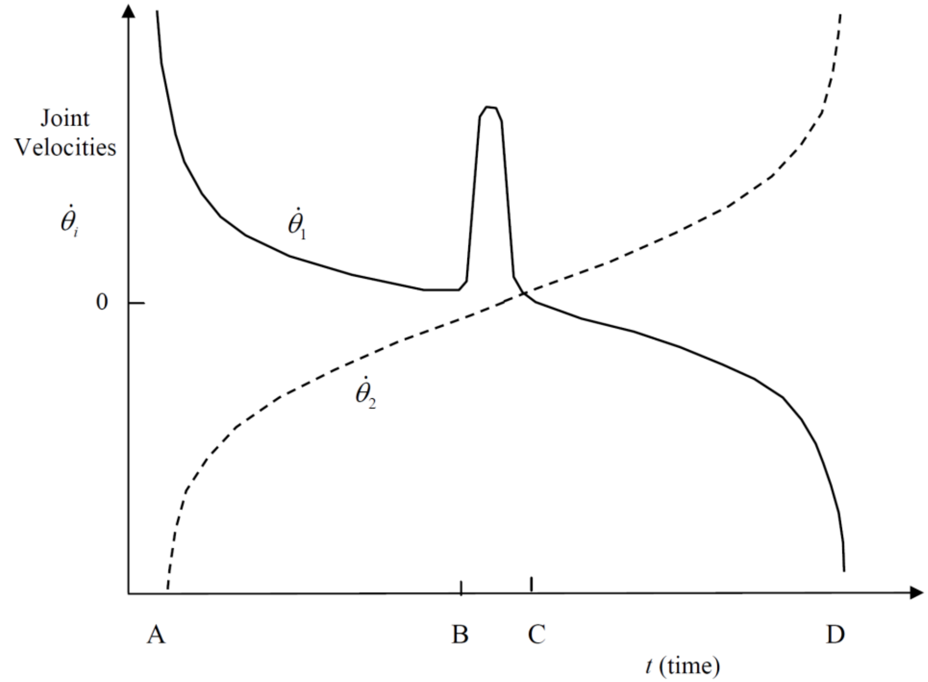
$$\dot{q}_1 = \frac{l_2 c_{12}.Vx + l_2 s_{12}.Vy}{l_1 l_2 s_2}$$

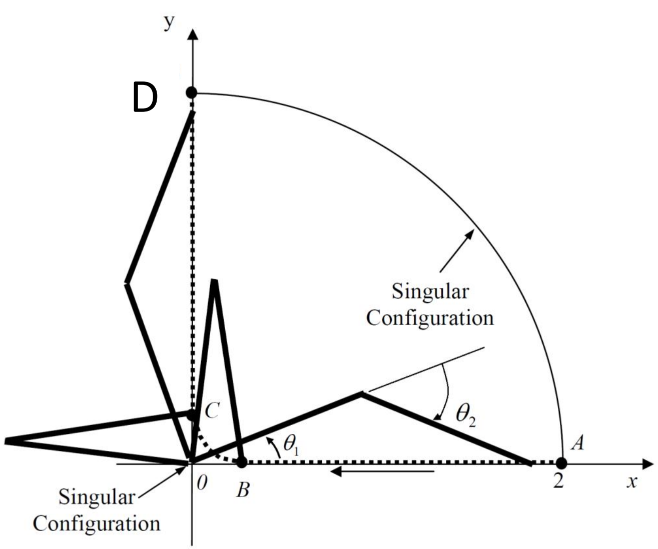
$$\dot{q}_2 = \frac{[-l_1 c_1 - l_2 c_{12}].Vx + [-l_1 s_1 - l_2 s_{12}].Vy}{l_1 l_2 s_2}$$

Example: RR Robot

$$\dot{q}_1 = \frac{I_2 c_{12}.Vx + I_2 s_{12}.Vy}{I_1 I_2 s_2}$$

$$\dot{q}_2 = \frac{[-I_1 c_1 - I_2 c_{12}].Vx + [-I_1 s_1 - I_2 s_{12}].Vy}{I_1 I_2 s_2}$$



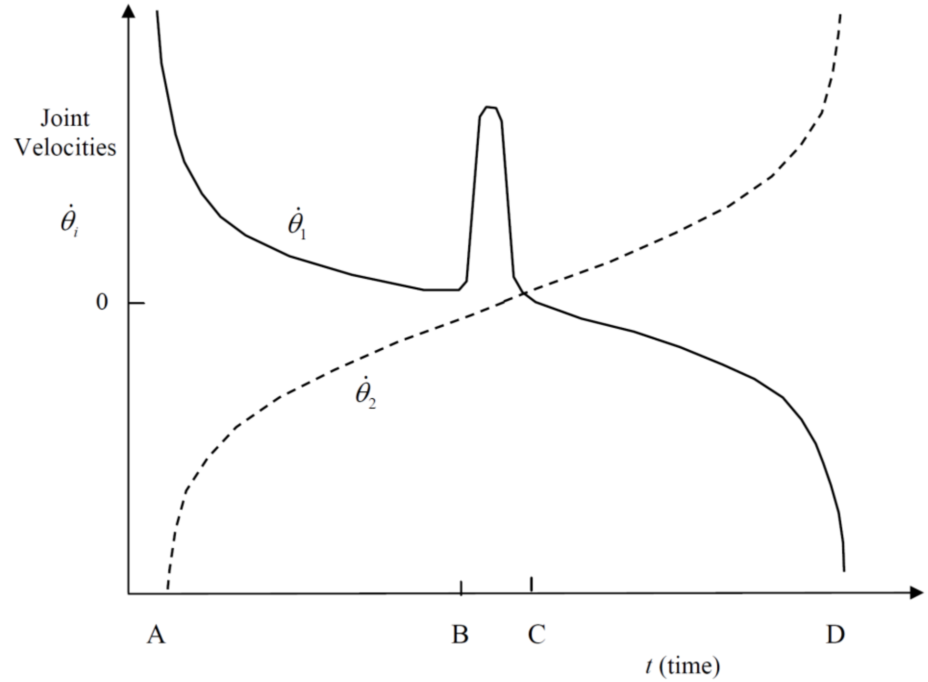


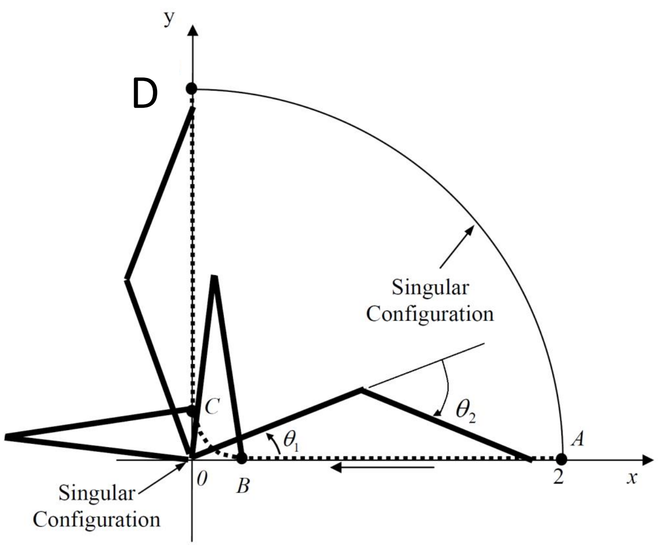
Note that the joint velocities are extremely large near the initial and the final points, and are unbounded at points A and D. These are the arm singular configurations q_2=0.

Example: RR Robot

$$\dot{q}_1 = \frac{I_2 c_{12}.Vx + I_2 s_{12}.Vy}{I_1 I_2 s_2}$$

$$\dot{q}_2 = \frac{[-I_1 c_1 - I_2 c_{12}].Vx + [-I_1 s_1 - I_2 s_{12}].Vy}{I_1 I_2 s_2}$$



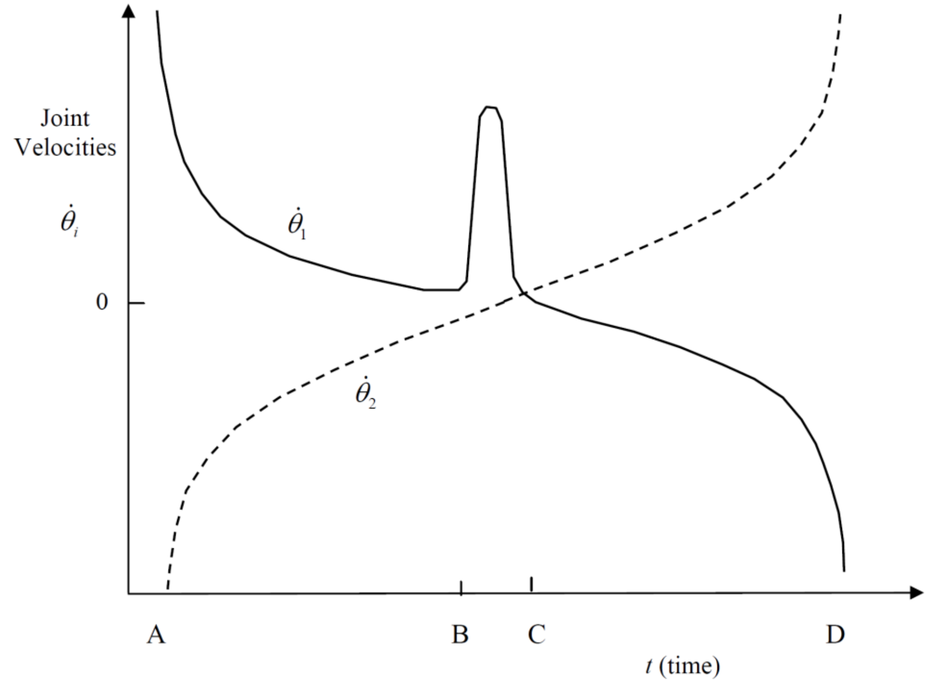


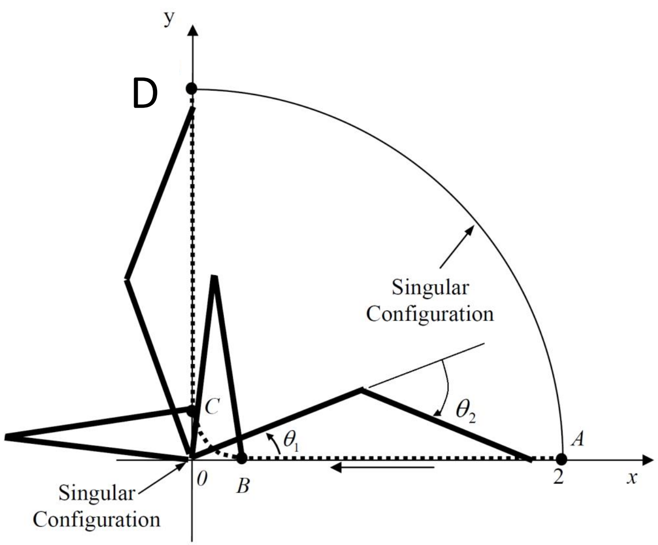
As the end-effector gets close to the origin, the velocity of the first joint becomes very large in order to quickly turn the arm around from point B to C. At these configurations, the second joint is almost -180 degrees, meaning that the arm is near singularity.

Example: RR Robot

$$\dot{q}_1 = \frac{l_2 c_{12}.Vx + l_2 s_{12}.Vy}{l_1 l_2 s_2}$$

$$\dot{q}_2 = \frac{[-l_1 c_1 - l_2 c_{12}].Vx + [-l_1 s_1 - l_2 s_{12}].Vy}{l_1 l_2 s_2}$$





This result agrees with the singularity condition using the determinant of the Jacobian:

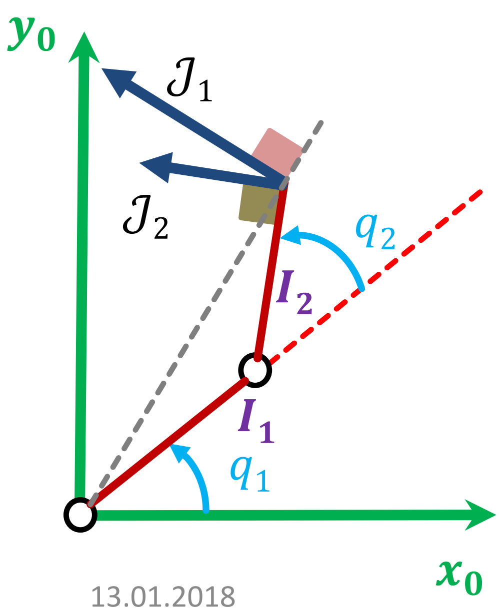
$$det(\mathcal{J}_v) = \sin(q_2) = 0 \quad \text{for } q_2 = k\pi, \; k = 0, \pm1, \pm2, \ldots$$

Example: RR Robot

Using the Jacobian, analyse the arm behaviour at the singular points. Consider (l_1=l_2=1).

The Jacobian is:

$$\mathcal{J}_v = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}, \mathcal{J}_1 = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} \end{bmatrix}, \mathcal{J}_2 = \begin{bmatrix} -l_2 s_{12} \\ l_2 c_{12} \end{bmatrix}$$



For q_2=0:

$$\mathcal{J}_1 = \begin{bmatrix} -2s_1 \\ 2c_1 \end{bmatrix}, \mathcal{J}_2 = \begin{bmatrix} -s_1 \\ c_1 \end{bmatrix}$$

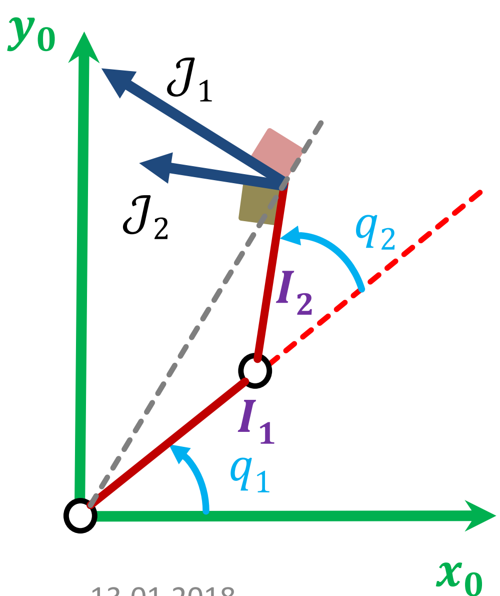
The Jacobian column vectors reduce to the ones in the same direction.
Note that no endpoint velocity can be generated in the direction perpendicular to the aligned arm links (singular configuration A and D).

Example: RR Robot

Using the Jacobian, analyse the arm behaviour at the singular points. Consider (l_1=l_2=1).

The Jacobian is:

$$\mathcal{J}_v = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}, \mathcal{J}_1 = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} \end{bmatrix}, \mathcal{J}_2 = \begin{bmatrix} -l_2 s_{12} \\ l_2 c_{12} \end{bmatrix}$$



For $q_2=\pi$:

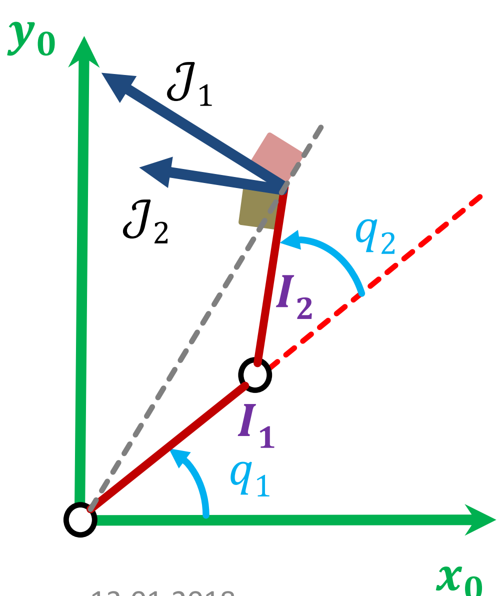
$$\mathcal{J}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathcal{J}_2 = \begin{bmatrix} s_1 \\ -c_1 \end{bmatrix}$$

The first joint cannot generate any endpoint velocity, since the arm is fully contracted (singular configuration B).

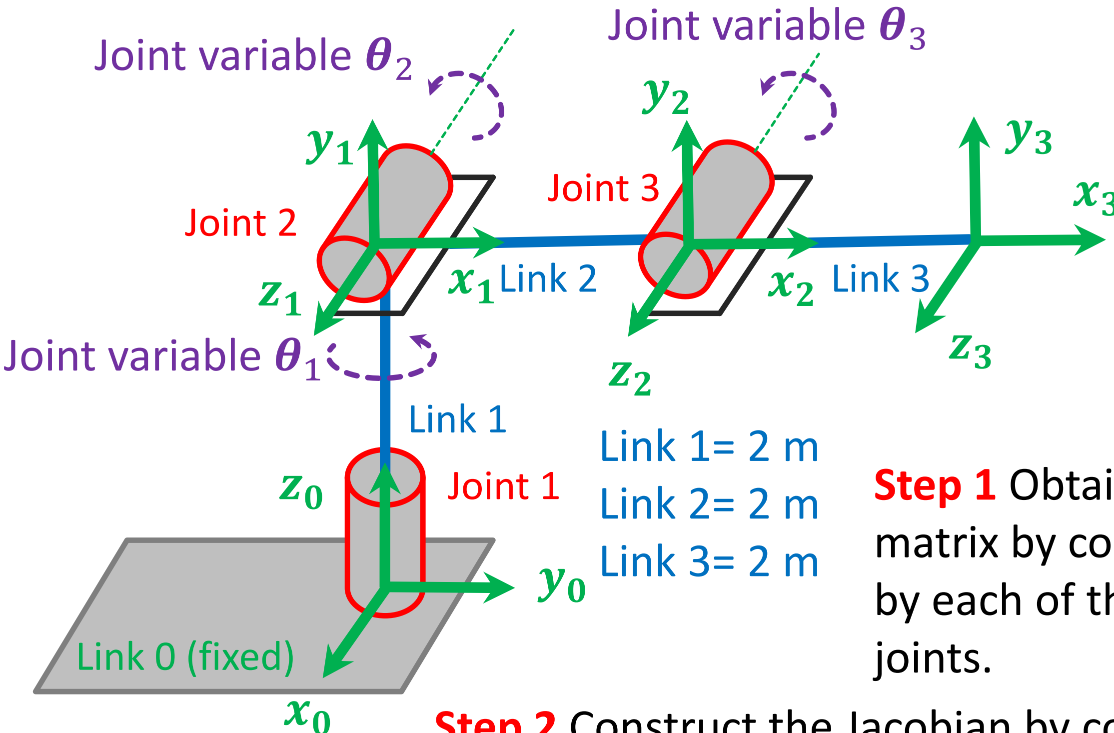
Example: RR Robot

Using the Jacobian, analyse the arm behaviour at the singular points. Consider (l_1=l_2=1).

At the singular configuration, there is at least one direction is which the robot cannot generate a non-zero velocity at the end effector.



Example: RRR Robot



The robot has three revolute joints that allow the endpoint to move in the three dimensional space. However, this robot mechanism has singular points inside the workspace. Analyze the singularity, following the procedure below.

Step 1 Obtain each column vector of the Jacobian matrix by considering the endpoint velocity created by each of the joints while immobilizing the other joints.

Step 2 Construct the Jacobian by concatenating the column vectors, and set the determinant of the Jacobian to zero for singularity: det J =0.

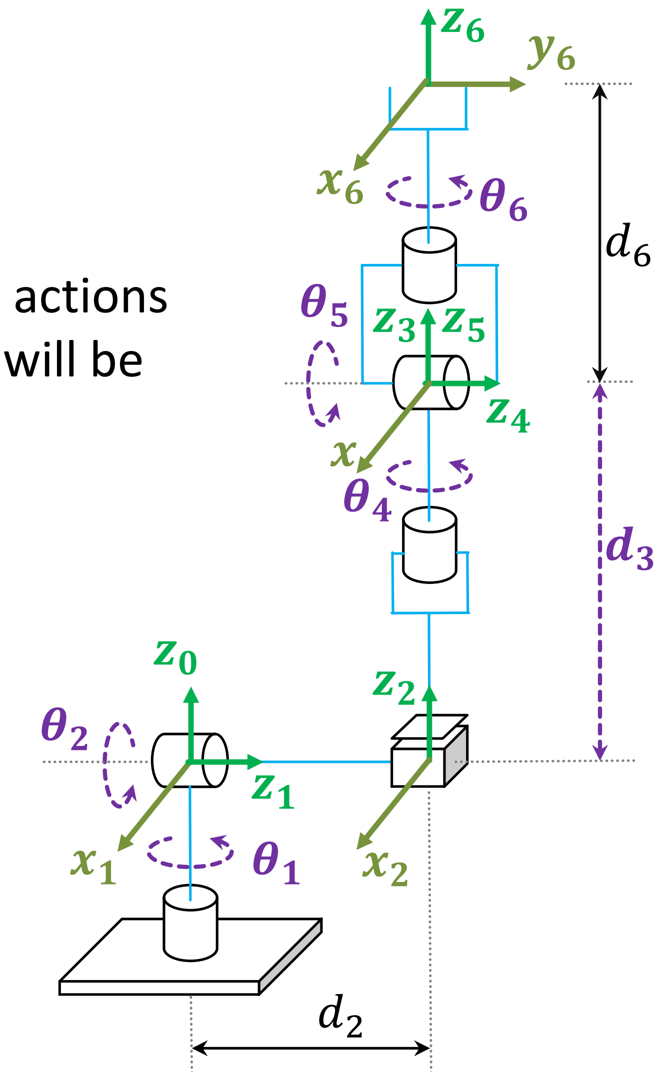
Step 3 Find the joint angles that make det J =0.

Step 4 Show the arm posture that is singular. Show where in the workspace it becomes singular. For each singular configuration, also show in which direction the endpoint cannot have a non-zero velocity.

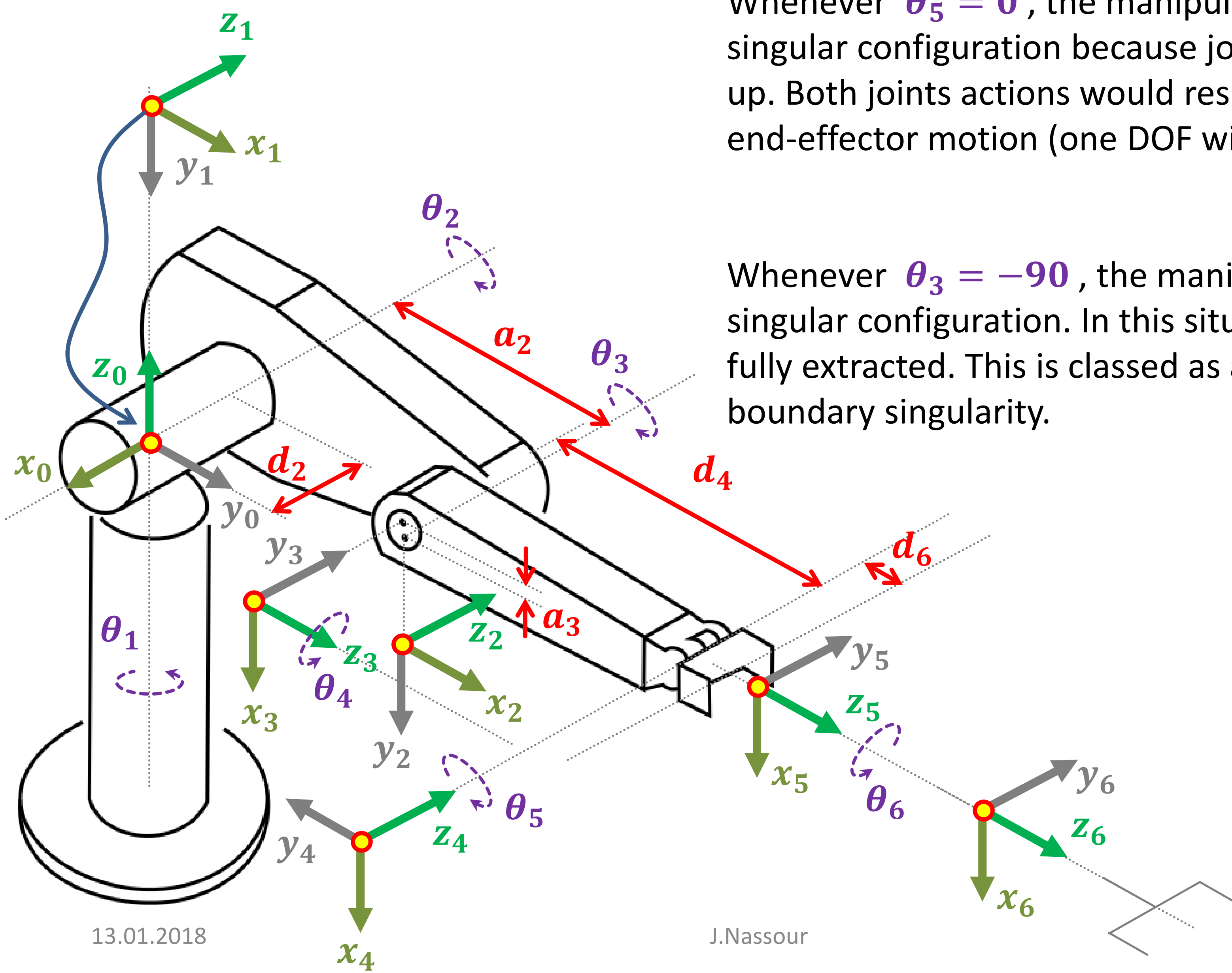
Stanford Arm

Give one example of singularity that can occur.

Whenever $\boldsymbol{\theta_5 = 0}$, the manipulator is in a singular configuration because joint 4 and 6 line up. Both joints actions would results the same end-effector motion (one DOF will be lost).



PUMA 260

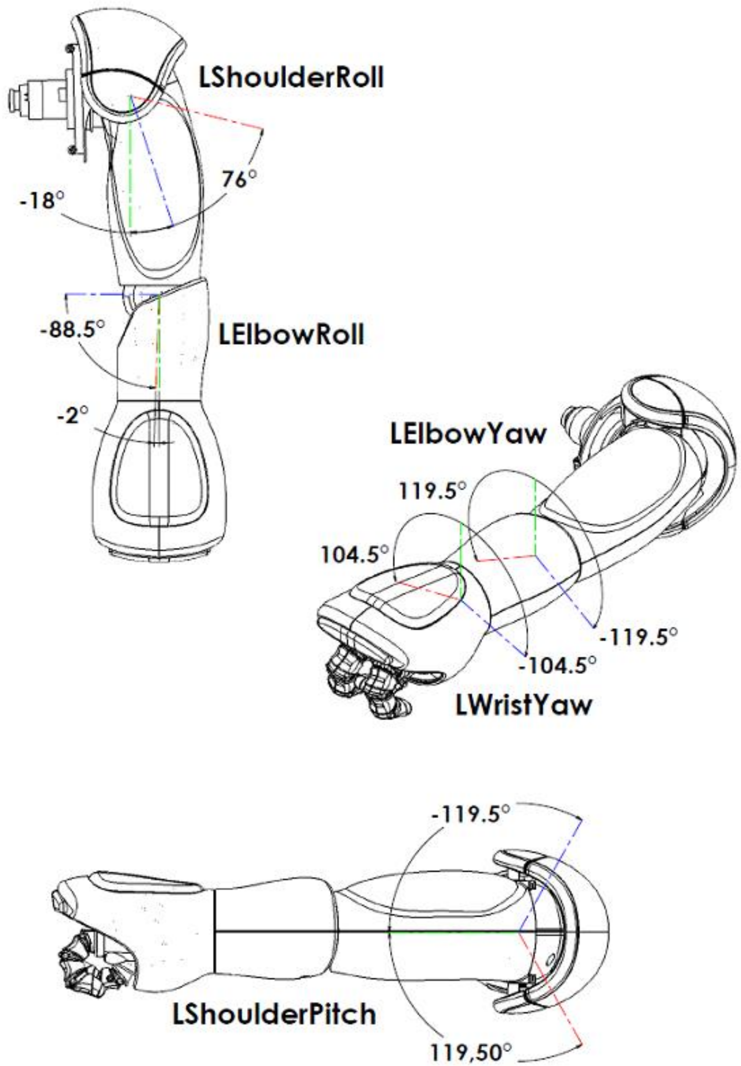


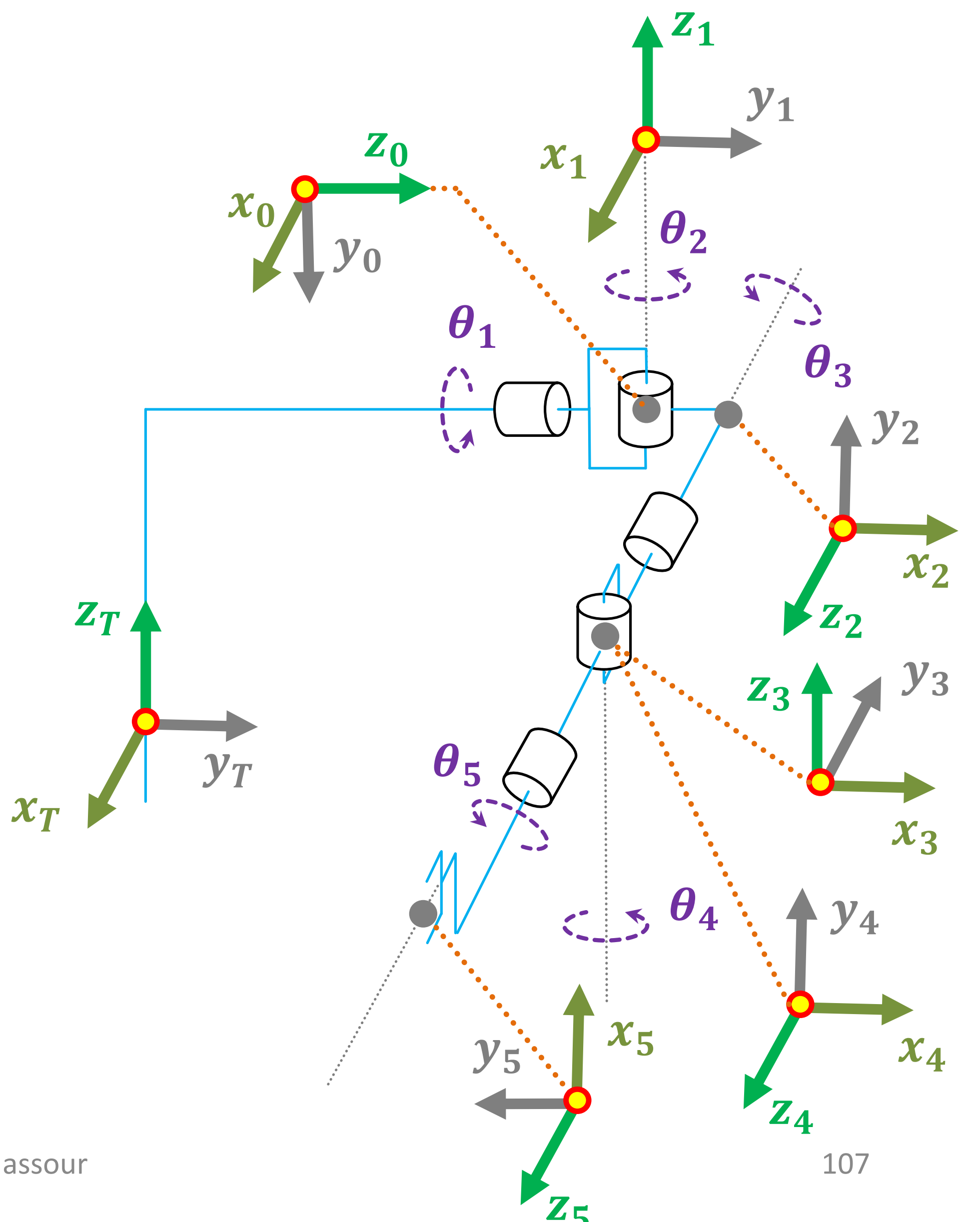
Give two examples of singularities that can occur.

Whenever $\theta_5 = 0$, the manipulator is in a singular configuration because joint 4 and 6 line up. Both joints actions would results the same end-effector motion (one DOF will be lost).

Whenever $\theta_3 = -90$, the manipulator is in a singular configuration. In this situation, the arm is fully extracted. This is classed as a workspace boundary singularity.

NAO Left Arm


LShoulderRoll
76°
-18°
-88.5°
LElbowRoll
-2°
LElbowYaw
119.5°
104.5°
-119.5°
-104.5°
LWristYaw
-119.5°
LShoulderPitch
119,50°


z_1
y_1
x_1
z_0
x_0
y_0
θ_2
θ_1
θ_3
y_2
x_2
z_2
z_T
z_3
y_3
x_3
y_T
x_T
θ_5
θ_4
y_4
x_4
z_4
x_5
y_5
z_5

NAO Right Arm