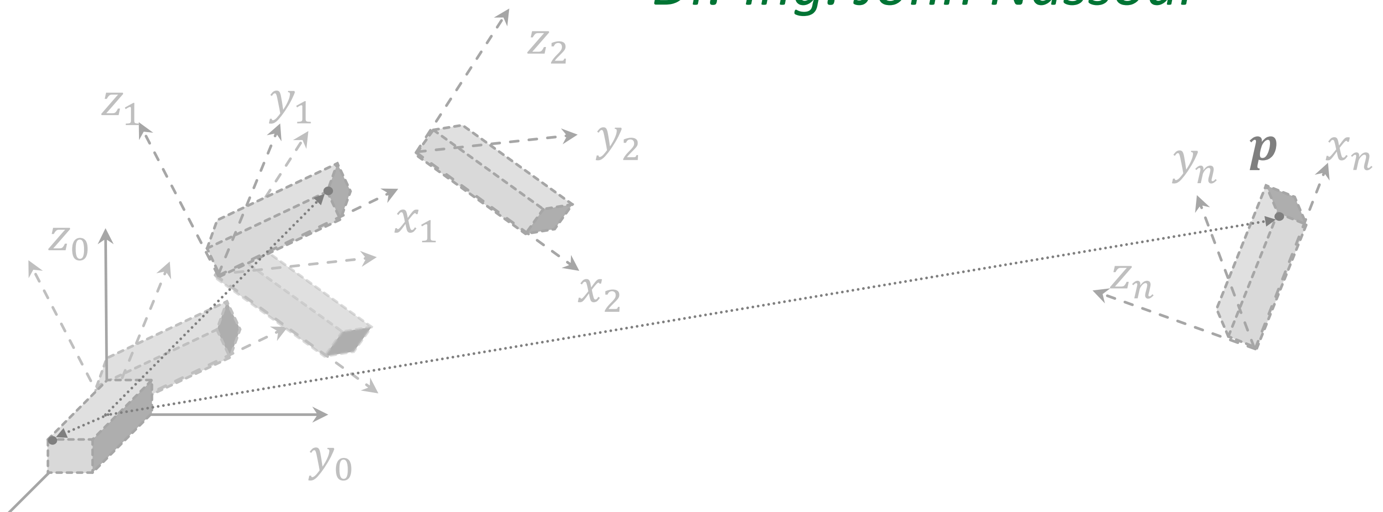
Artificial Intelligence & Neuro Cognitive Systems
Fakultät für Informatik

Transformation Matrices

Dr.-Ing. John Nassour



Suggested literature

- Robot Modeling and Control
- Robotics: Modelling, Planning and Control

Motivation

A large part of robot kinematics is concerned with the establishment of various coordinate systems to represent the positions and orientations of rigid objects, and with transformations among these coordinate systems.

Indeed, the **geometry of three-dimensional space** and **of rigid motions** plays a central role in all aspects of robotic manipulation.

A rigid motion is the action of taking an object and moving it to a different location without altering its shape or size

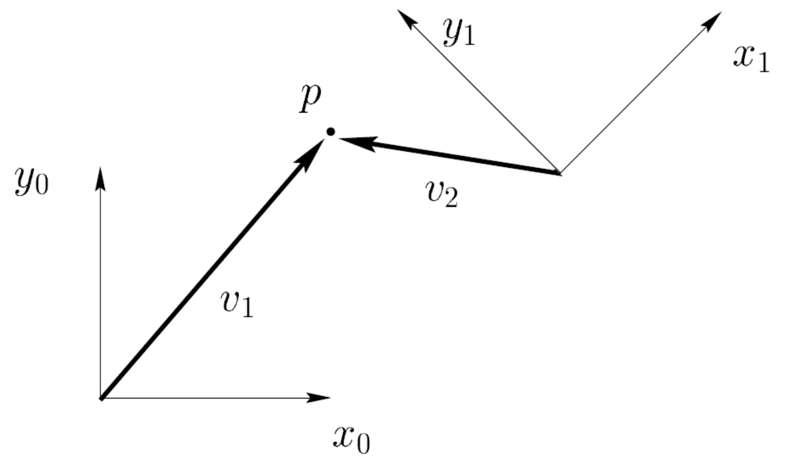
Transformation

The operations of **ROTATION** and **TRANSLATION**.

Introduce the notion of **HOMOGENEOUS TRANSFORMATIONS** (combining the operations of rotation and translation into a single matrix multiplication).

Representing Positions

The coordinate vectors that represent the location of the point $\boldsymbol{p}$ in space with respect to coordinate frames $\boldsymbol{o_0\ x_0\ y_0}$ and $\boldsymbol{o_1\ x_1\ y_1}$, respectively are:

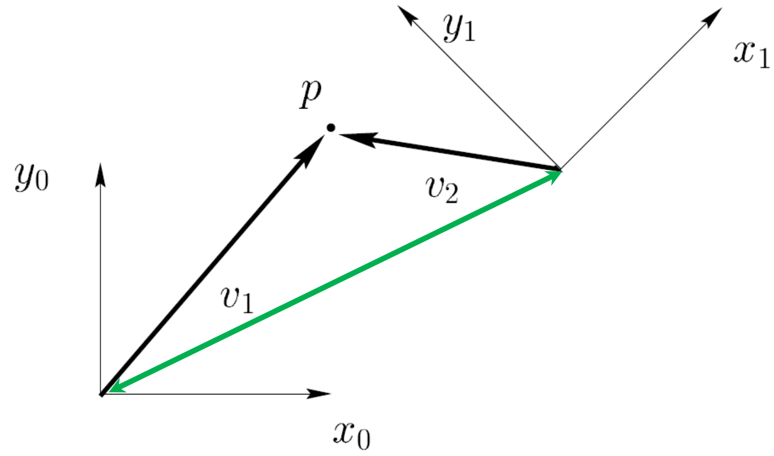


Two coordinate frames, a point p, and two vectors v_1 and v_2.

$$p^0 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \qquad p^1 = \begin{bmatrix} -2.8 \\ 4.2 \end{bmatrix}$$

Representing Positions

Lets assign coordinates that represent the position of the origin of one coordinate system (frame) with respect to another.

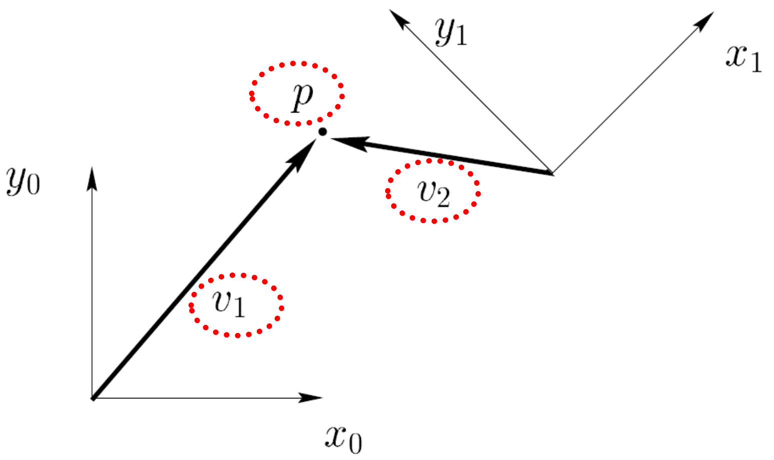


Two coordinate frames, a point p, and two vectors v_1 and v_2.

$$o_1^0 = \begin{bmatrix} 10 \\ 5 \end{bmatrix}, \qquad o_0^1 = \begin{bmatrix} -10.6 \\ 3.5 \end{bmatrix}$$

Representing Positions

What is the difference between the geometric entity called $\boldsymbol{p}$ and any particular coordinate vector $\boldsymbol{v}$ that is assigned to represent $\boldsymbol{p}$?



Two coordinate frames, a point p, and two vectors v_1 and v_2.

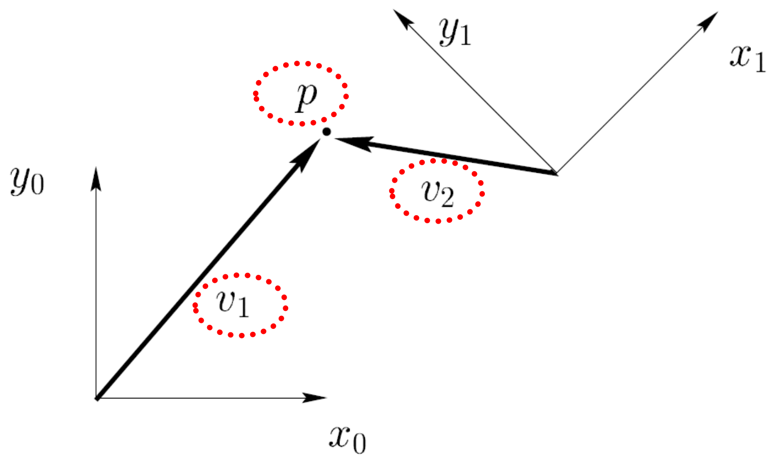
$\boldsymbol{p}$ is independent of the choice of coordinate systems.
$\boldsymbol{v}$ depends on the choice of coordinate frames.

Representing Positions

A point corresponds to a specific location in space.

A vector specifies a direction and a magnitude (e.g. displacements or forces).

The point $\boldsymbol{p}$ is **not equivalent** to the vector $\boldsymbol{v_1}$, the displacement from the origin $\boldsymbol{o_0}$ to the point $\boldsymbol{p}$ is given by the vector $\boldsymbol{v_1}$.



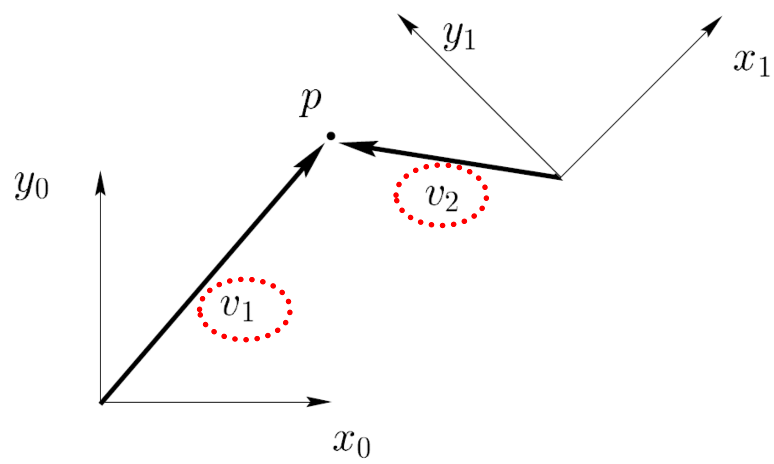
Two coordinate frames, a point p, and two vectors v_1 and v_2.

We will use the term **vector** to refer to what are sometimes called **free vectors**, i.e., ***vectors that are not constrained to be located at a particular point in space***.

Representing Positions

When **assigning coordinates to vectors**, we use the same notational convention that we used when assigning coordinates to points.

Thus, $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are geometric entities that are invariant with respect to the choice of coordinate systems, but the representation by coordinates of these vectors depends directly on the choice of reference coordinate frame.



Two coordinate frames, a point p, and two vectors v_1 and v_2.

$$v_1^0 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \qquad v_1^1 = \begin{bmatrix} 7.77 \\ 0.8 \end{bmatrix}, \qquad v_2^0 = \begin{bmatrix} -5.1 \\ 1 \end{bmatrix}, \qquad v_2^1 = \begin{bmatrix} -2.89 \\ 4.2 \end{bmatrix}$$

Coordinate Convention

In order to perform **algebraic manipulations** using coordinates, it is essential that all coordinate vectors **be defined with respect to the same coordinate frame**.

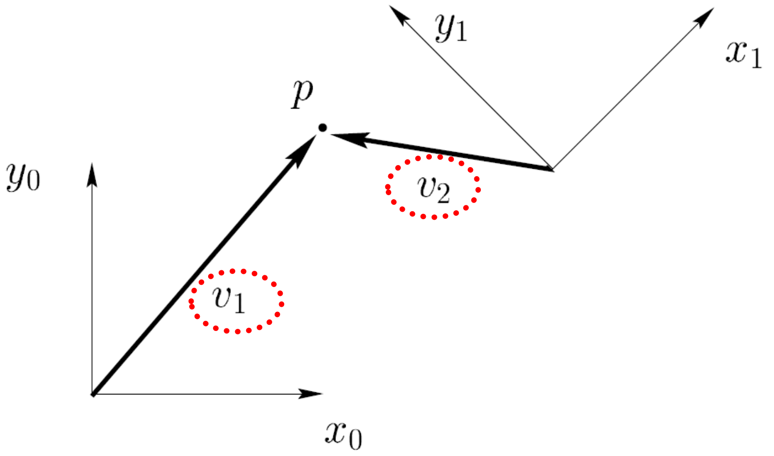
In the case of free vectors, it is enough that they be defined with respect to **parallel** coordinate frames.

Coordinate Convention

An expression of the form:

$$v_1^0 + v_2^1$$

is not defined since the frames $\boldsymbol{o_0\ x_0\ y_0}$ and $\boldsymbol{o_1\ x_1\ y_1}$ are not parallel.



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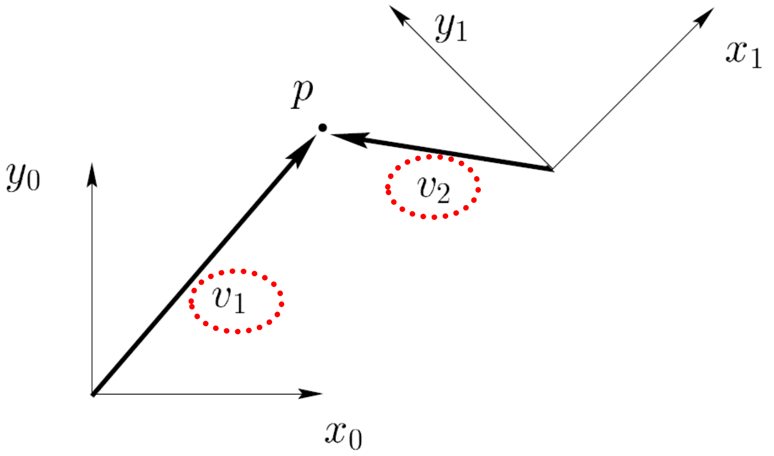
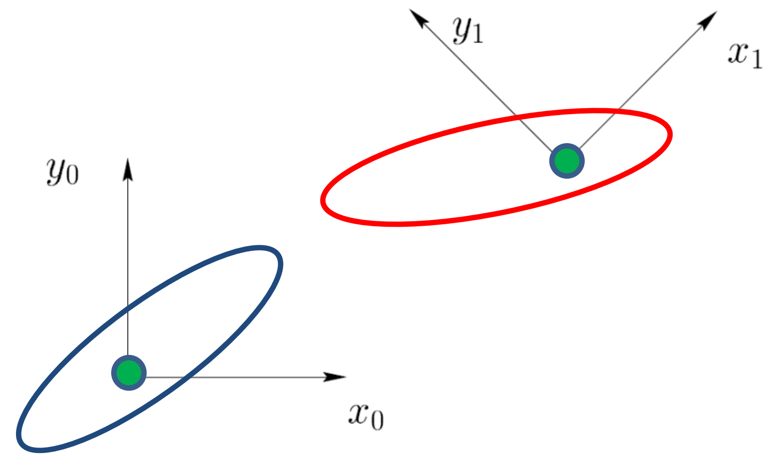


Two coordinate frames, a point p, and two vectors v_1 and v_2.

Thus, we see a clear need, not only for a representation system that allows points to be expressed with respect to various coordinate systems, but also for a **mechanism that allows us to transform the coordinates of points** that are expressed in one coordinate system into the appropriate coordinates with respect to some other coordinate frame.

Representing Rotations

In order to represent the relative position and orientation of one rigid body with respect to another, we will **rigidly** attach coordinate frames to each body, and then specify the geometric relationships between these coordinate frames.



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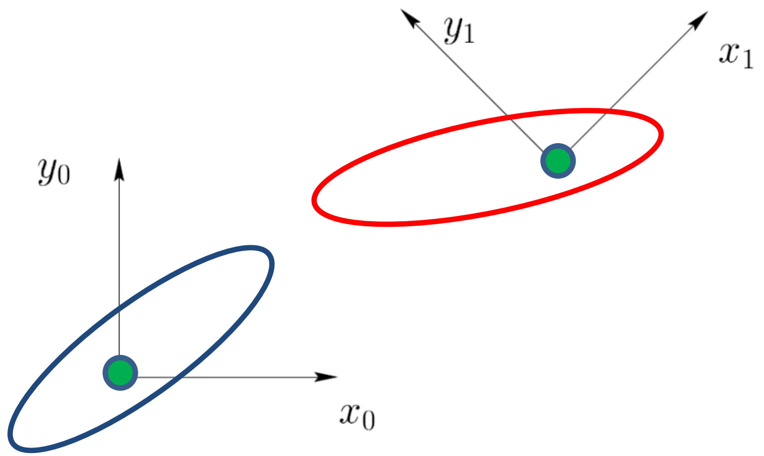

$$o_1^0 = \begin{bmatrix} ? \\ ? \end{bmatrix}, \qquad o_0^1 = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

Representing Rotations

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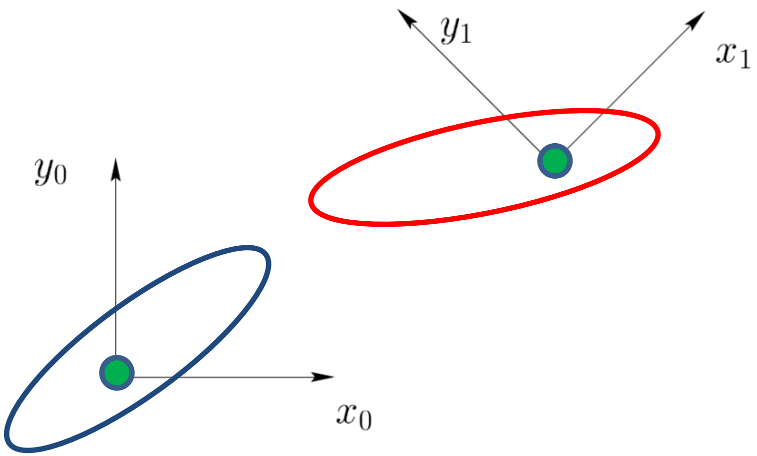
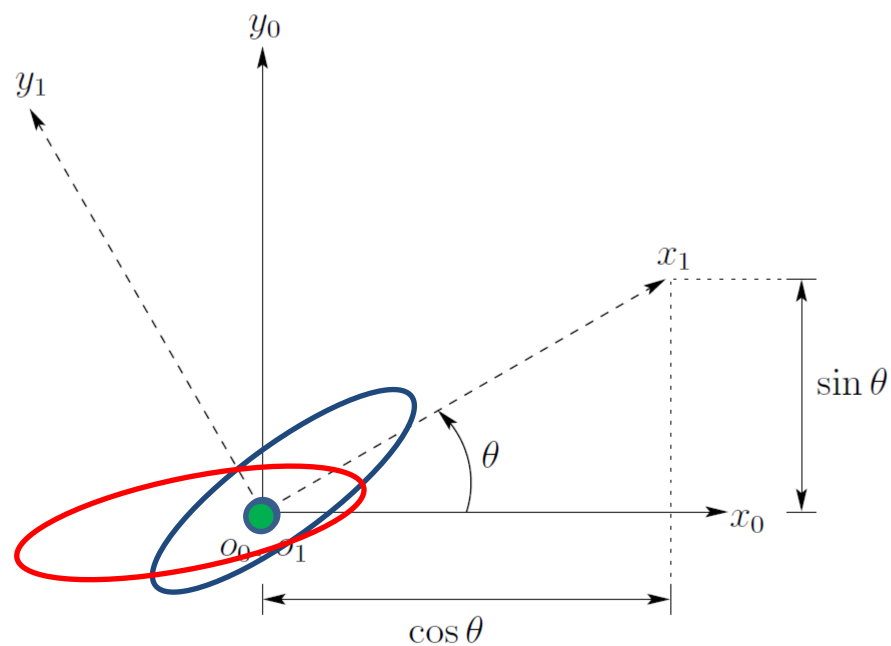


$$o_1^0 = \begin{bmatrix} 10 \\ 5 \end{bmatrix}, \qquad o_0^1 = \begin{bmatrix} -10.6 \\ 3.5 \end{bmatrix}$$

How to describe the orientation of one coordinate frame relative to another frame?

Rotation In The Plane



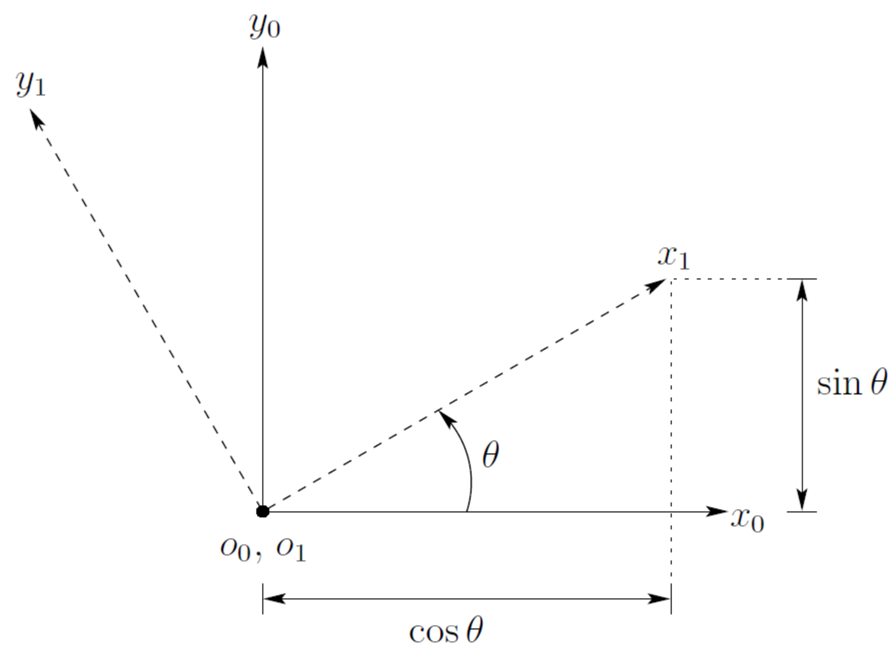
Coordinate frame $o_1x_1y_1$ is oriented at an angle θ with respect to $o_0x_0y_0$.

Rotation In The Plane

Fram $o_1\, x_1\, y_1$ is obtained by rotating frame $o_0\, x_0\, y_0$ by an angle θ.

The coordinate vectors for the axes of frame $o_1\, x_1\, y_1$ with respect to coordinate frame $o_0\, x_0\, y_0$ are described by a rotation matrix:

$$R_1^0 = [x_1^0 | y_1^0]$$



Coordinate frame $o_1 x_1 y_1$ is oriented at an angle θ with respect to $o_0 x_0 y_0$.

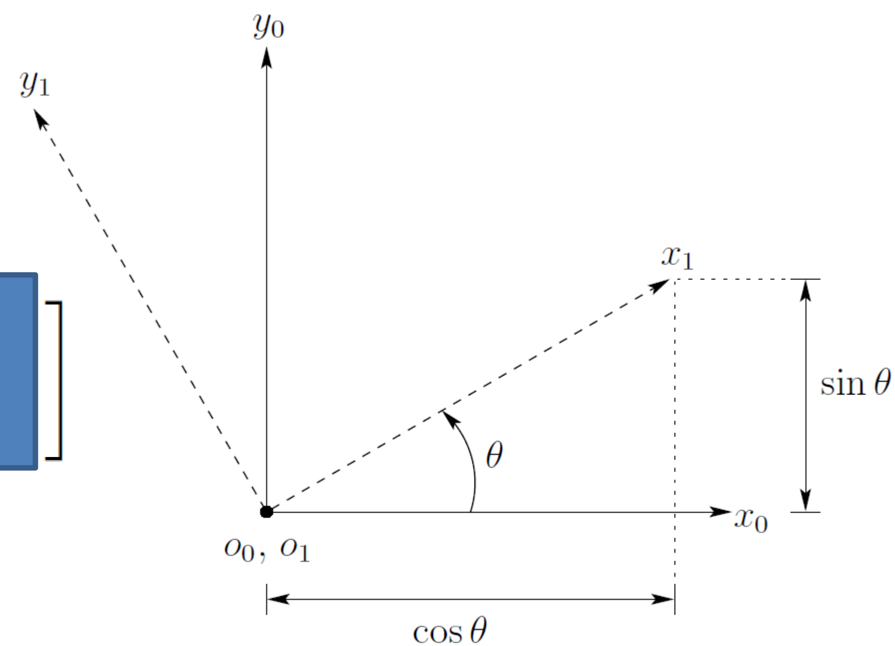
wherex_1^0 and y_1^0 are the coordinates in frame $o_0\, x_0\, y_0$ of unit vectors $x_{\ 1}$ and $y_{\ 1}$, respectively.

Rotation In The Plane

$$R_1^0 = [x_1^0 | y_1^0]$$

$$x_1^0 = \left[\quad \right] \qquad y_1^0 = \left[\quad \right]$$

$$R_1^0 = \left[\quad \right]$$



Coordinate frame $o_1x_1y_1$ is oriented at an angle θ with respect to $o_0x_0y_0$.

$\boldsymbol{R_1^0}$ is a matrix whose column vectors are the coordinates of the (unit vectors along the) axes of frame $\boldsymbol{o_1\ x_1\ y_1}$ expressed relative to frame $\boldsymbol{o_0\ x_0\ y_0}$.

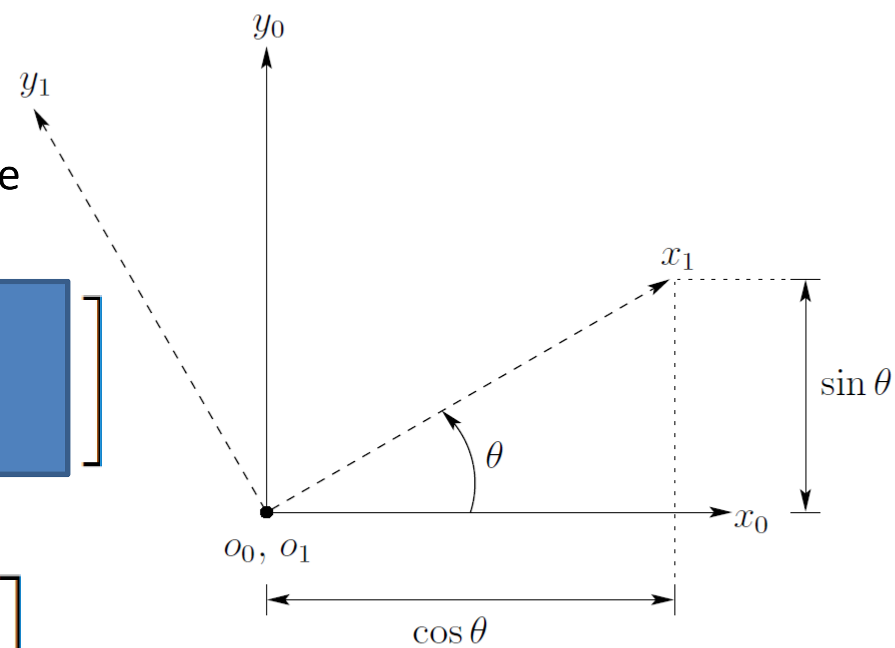
Rotation In The Plane

$$R_1^0 = [x_1^0 | y_1^0]$$

The dot product of **two unit vectors** gives the projection of one onto the other

$$x_1^0 = \begin{bmatrix} \quad \end{bmatrix} \qquad y_1^0 = \begin{bmatrix} \quad \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} \quad \end{bmatrix}$$



Coordinate frame $o_1x_1y_1$ is oriented at an angle θ with respect to $o_0x_0y_0$.

$\boldsymbol{R_1^0}$ describes the orientation of frame $\boldsymbol{o_1\ x_1\ y_1}$ with respect to the frame $\boldsymbol{o_0\ x_0\ y_0}$.

$$\boldsymbol{R_0^1} = ?$$

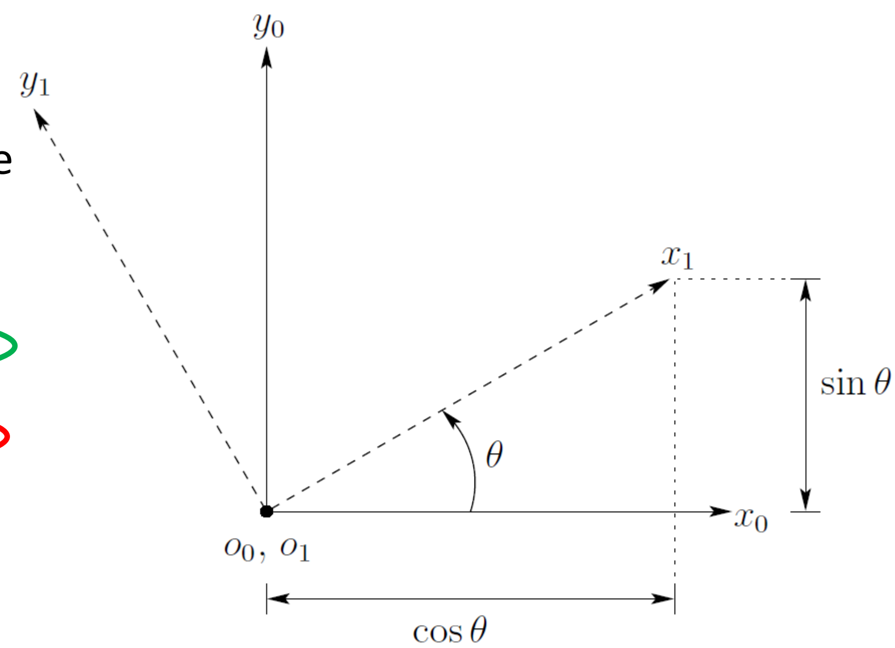
Rotation In The Plane

The orientation of frame $o_0\, x_0\, y_0$ with respect to the frame $o_1\, x_1\, y_1$.

The dot product of two unit vectors gives the projection of one onto the other

$$R_0^1 = \begin{bmatrix} x_0 \cdot x_1 & y_0 \cdot x_1 \\ x_0 \cdot y_1 & y_0 \cdot y_1 \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 \end{bmatrix}$$



Coordinate frame $o_1x_1y_1$ is oriented at an angle θ with respect to $o_0x_0y_0$.

Since the inner product is commutative

$$x_i \cdot y_j = y_j \cdot x_i$$

$$R_0^1 = (R_1^0)^T$$

The transpose

Rotations In Three Dimensions

Each axis of the frame $o_1x_1y_1z_1$ is projected onto coordinate frame $o_0x_0y_0z_0$.

The resulting rotation matrix is given by:

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

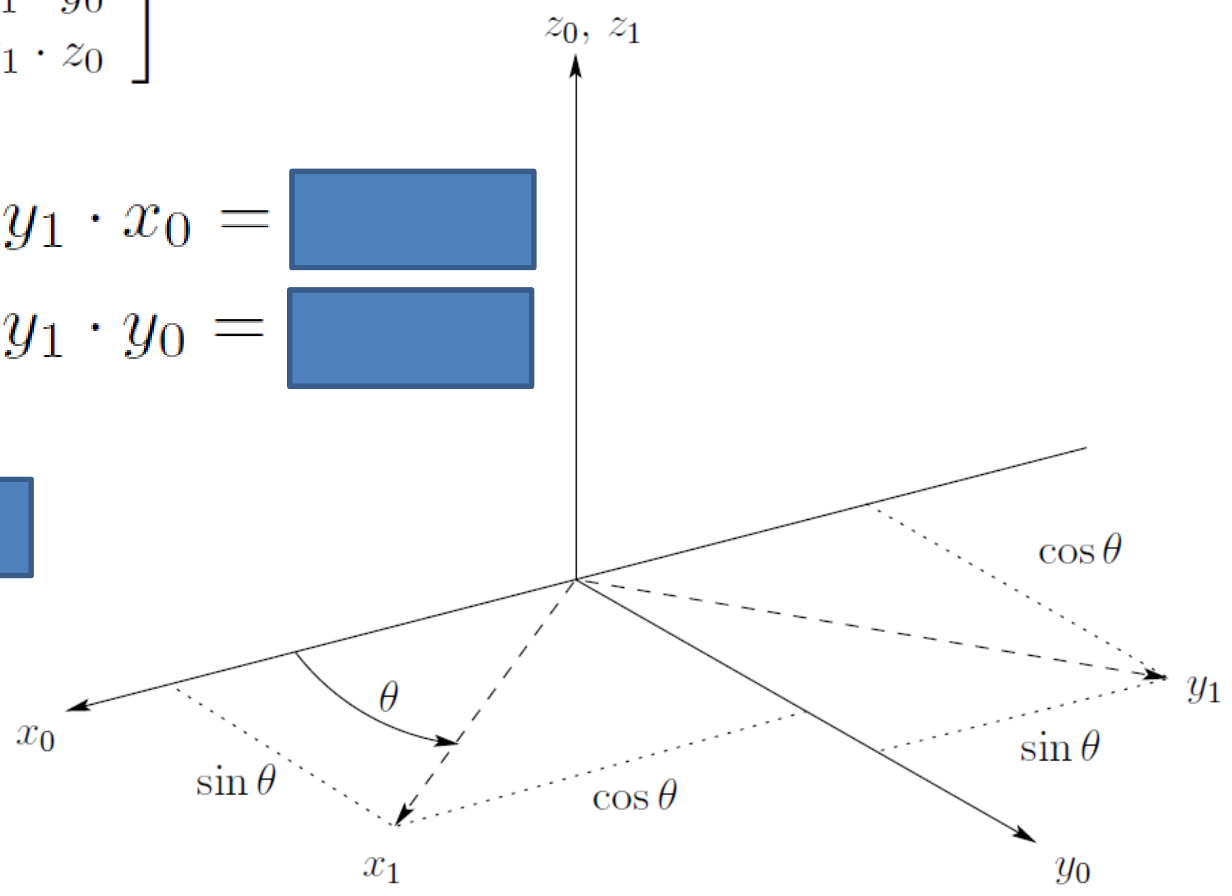
Rotation About z_0 By An Angle θ

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

$x_1 \cdot x_0 =$ ▭ $\quad y_1 \cdot x_0 =$ ▭

$x_1 \cdot y_0 =$ ▭ $\quad y_1 \cdot y_0 =$ ▭

$z_0 \cdot z_1 =$ ▭

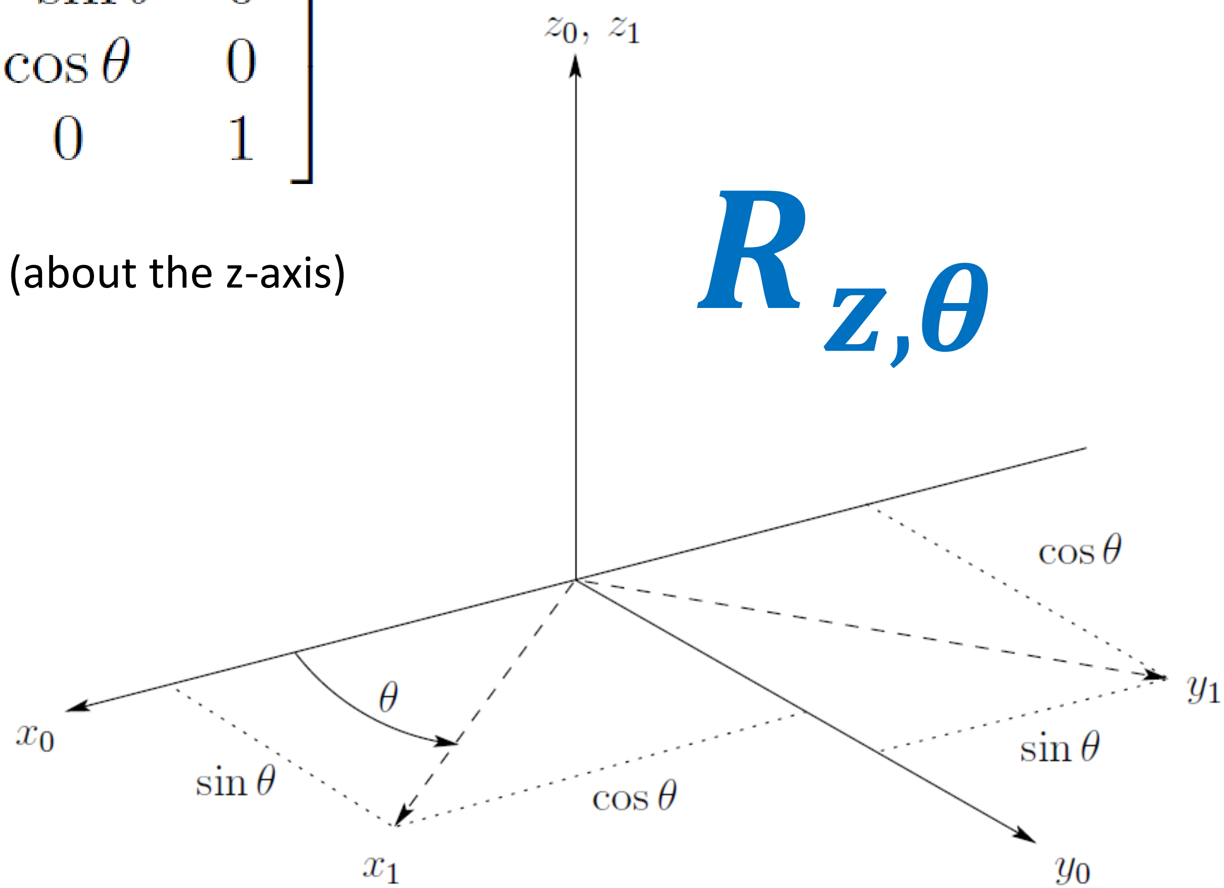


Rotation About z_0 By An Angle θ

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Called a basic rotation matrix (about the z-axis)

$$R_{z,\theta}$$



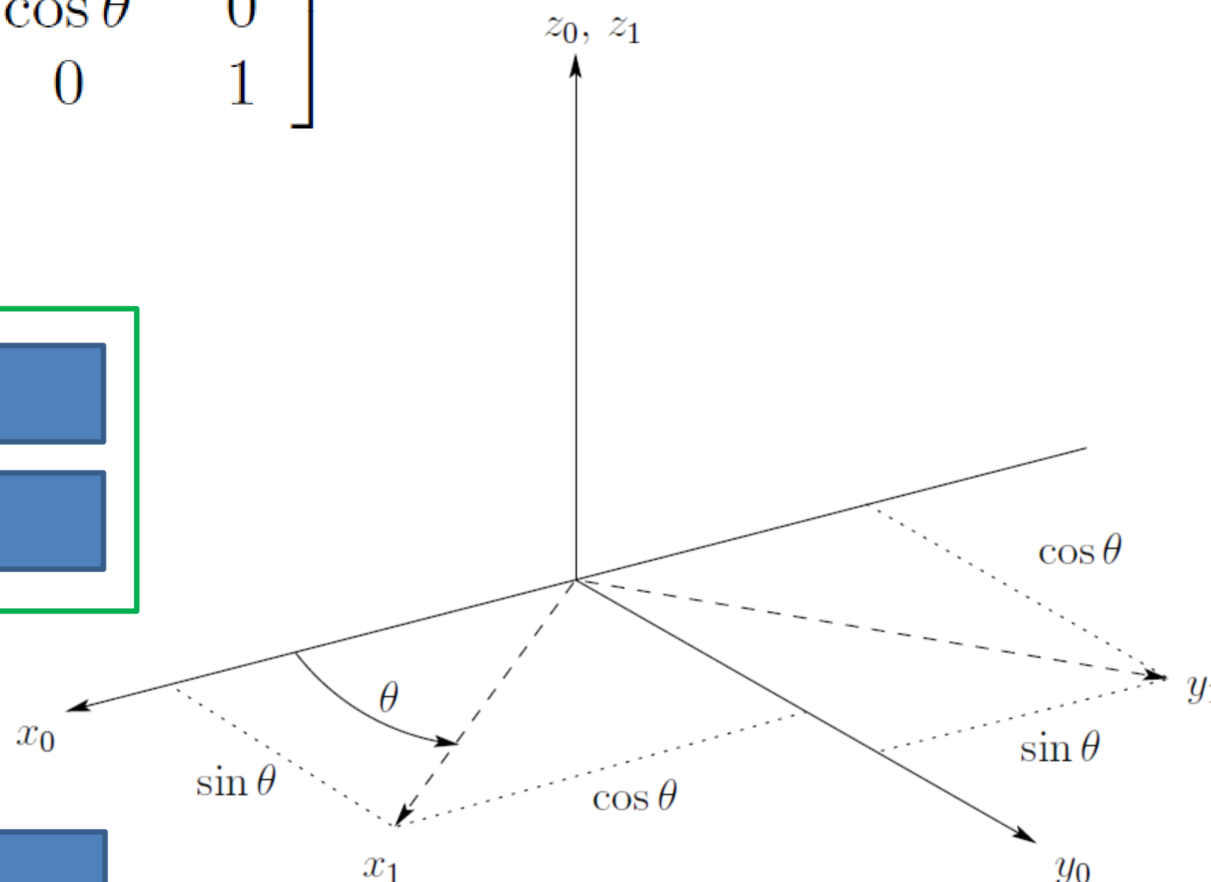
Basic Rotation Matrix About The Z-axis

$$R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{z,0} = \square$$

$$R_{z,\theta}R_{z,\phi} = \square$$

$$\left(R_{z,\theta}\right)^{-1} = \square$$



Basic Rotation Matrix About The X-axis

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

$$\boldsymbol{R_{x,\theta}}$$

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Basic Rotation Matrix About The Y-axis

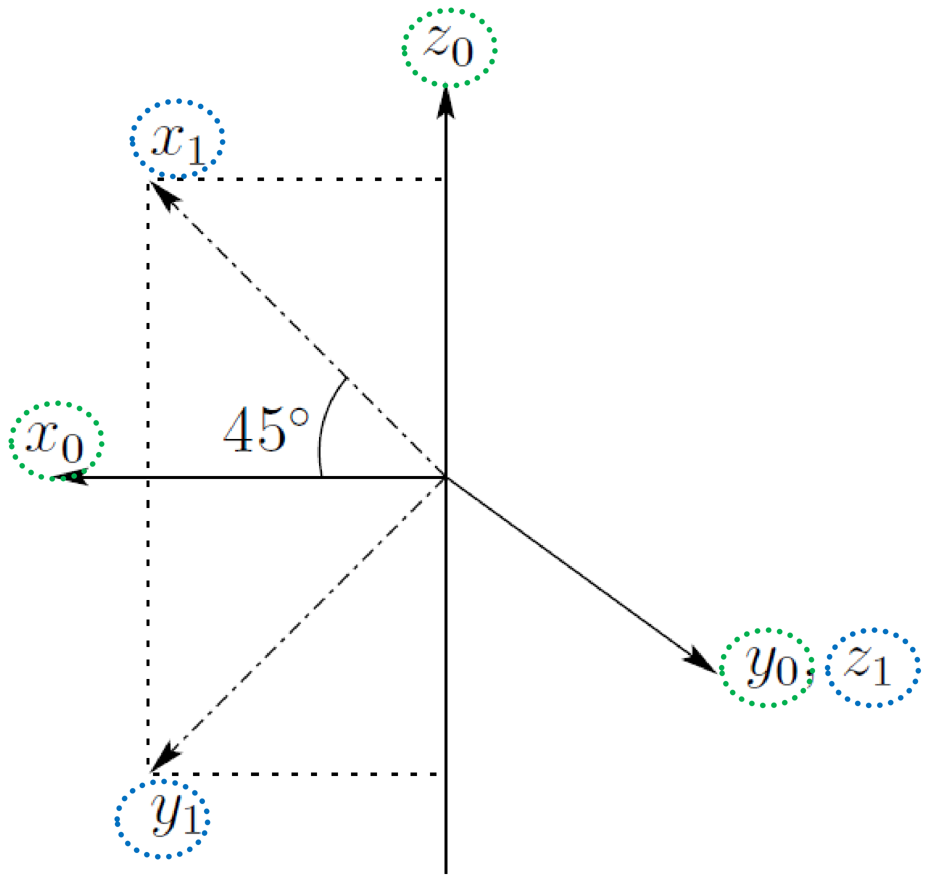
$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

$\boldsymbol{R_{y,\theta}}$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Example

Find the description of frame $o_1x_1y_1z_1$ with respect to the frame $o_0x_0y_0z_0$.



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$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

The coordinates of x_1 are

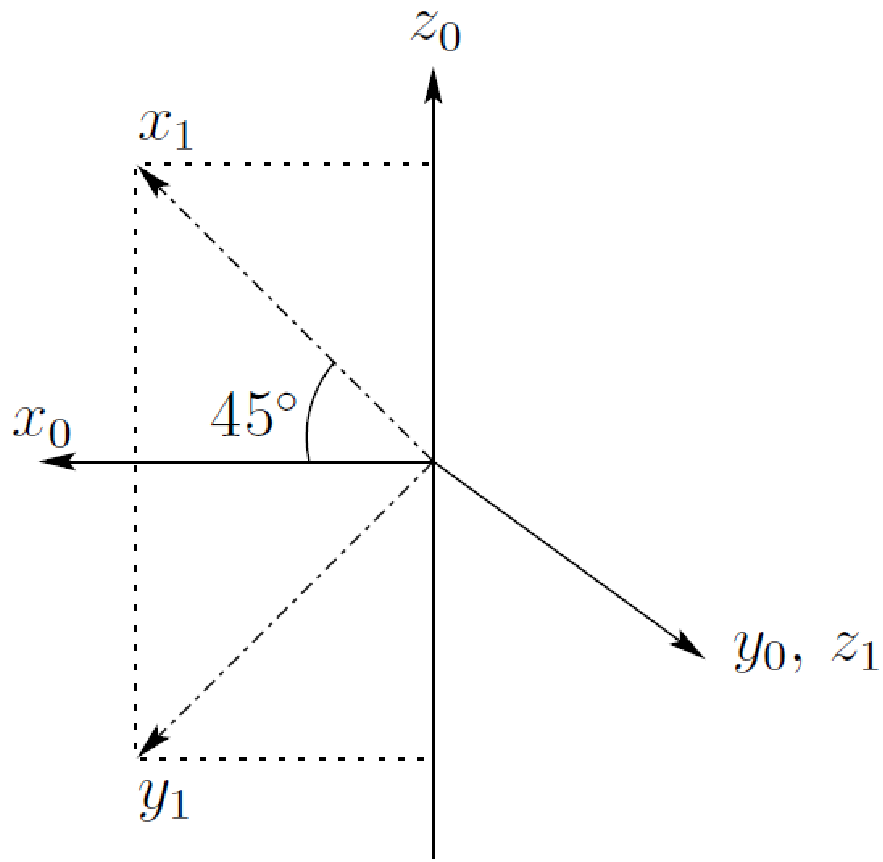
$$\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)^T$$

The coordinates of y_1 are

$$\left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)^T$$

The coordinates of z_1 are

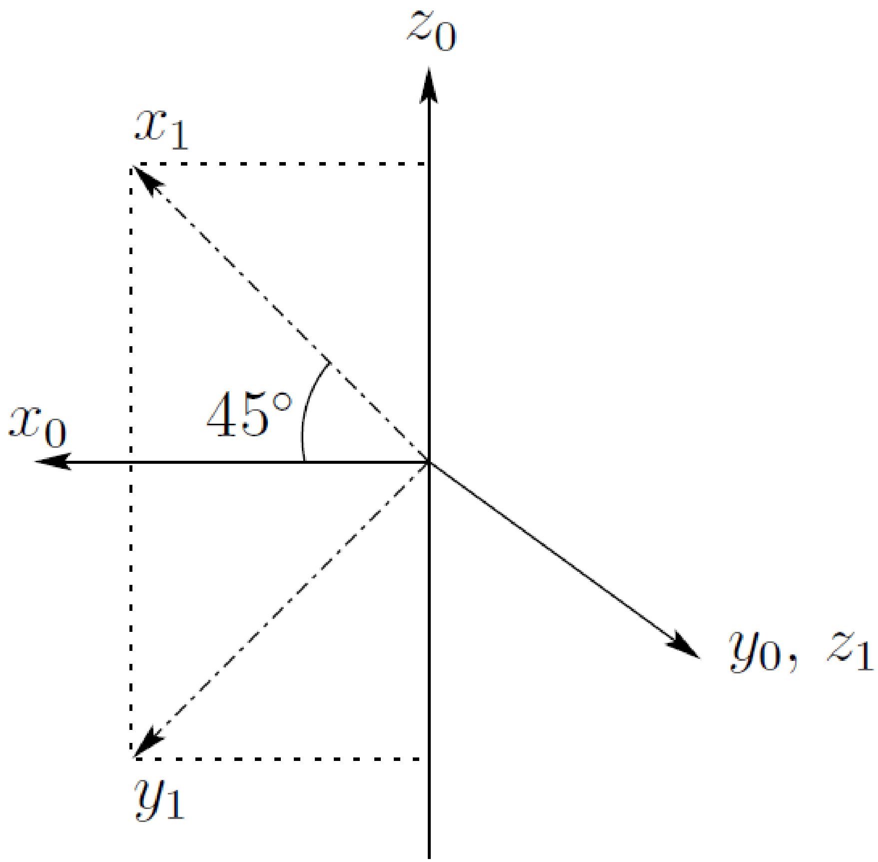
$$(0, 1, 0)^T$$



Example

Find the description of frame $o_1x_1y_1z_1$ with respect to the frame $o_0x_0y_0z_0$.

$$R_1^0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}$$



The coordinates of x_1 are

$$\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)^T$$

The coordinates of y_1 are

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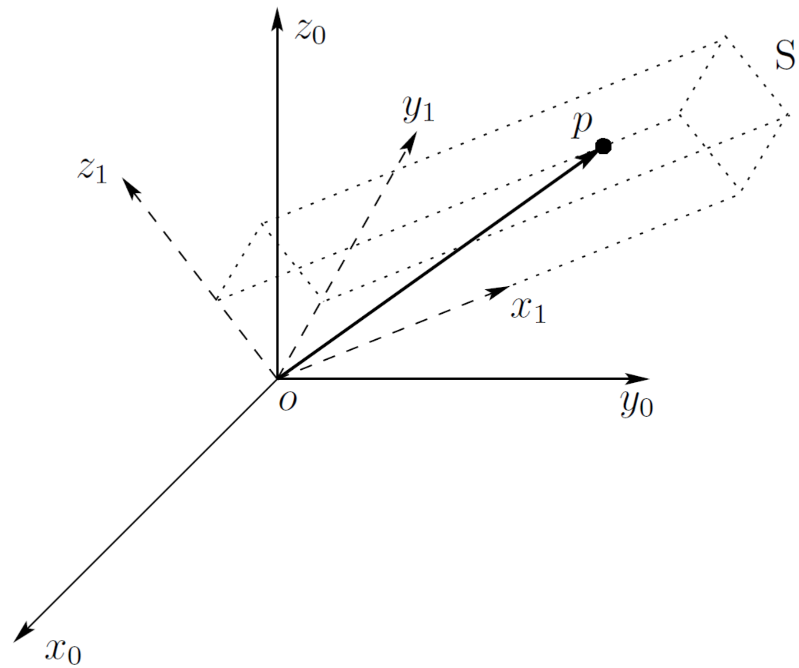
The coordinates of z_1 are

$$(0, 1, 0)^T$$

Rotational Transformations

$\boldsymbol{S}$ is a rigid object to which a coordinate $\boldsymbol{frame\ 1}$ is attached.
Given $\boldsymbol{p^1}$ of the point $\boldsymbol{p}$, determine the coordinates of $\boldsymbol{p}$ relative to a fixed reference $\boldsymbol{frame\ 0}$.



$$p^1 = (u, v, w)^T$$
$$p = ux_1 + vy_1 + wz_1$$

The projection of the point $\boldsymbol{p}$ onto the coordinate axes of the $\boldsymbol{frame\ 0}$:

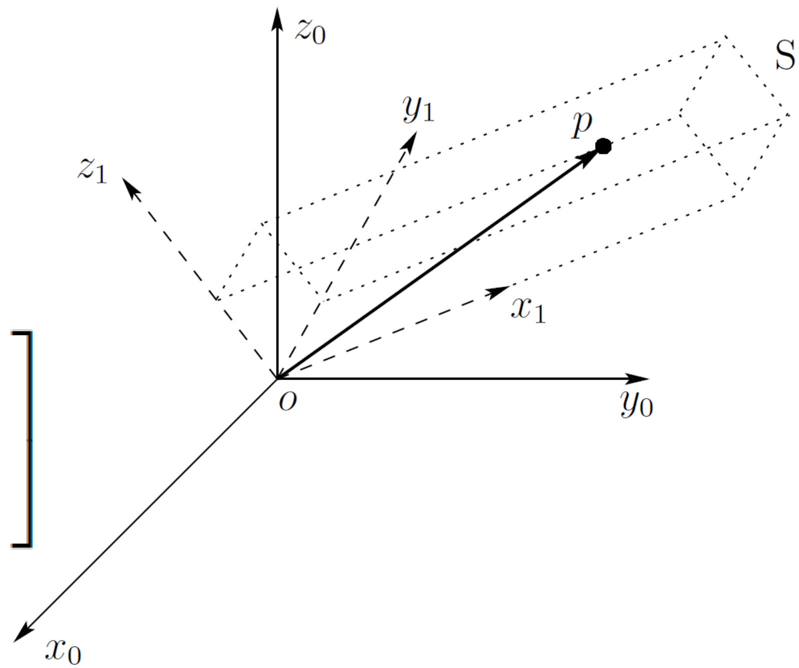
$$p^0 = \begin{bmatrix} p \cdot x_0 \\ p \cdot y_0 \\ p \cdot z_0 \end{bmatrix}$$

Rotational Transformations

$$p^0 = \begin{bmatrix} (ux_1 + vy_1 + wz_1) \cdot x_0 \\ (ux_1 + vy_1 + wz_1) \cdot y_0 \\ (ux_1 + vy_1 + wz_1) \cdot z_0 \end{bmatrix}$$

$=$

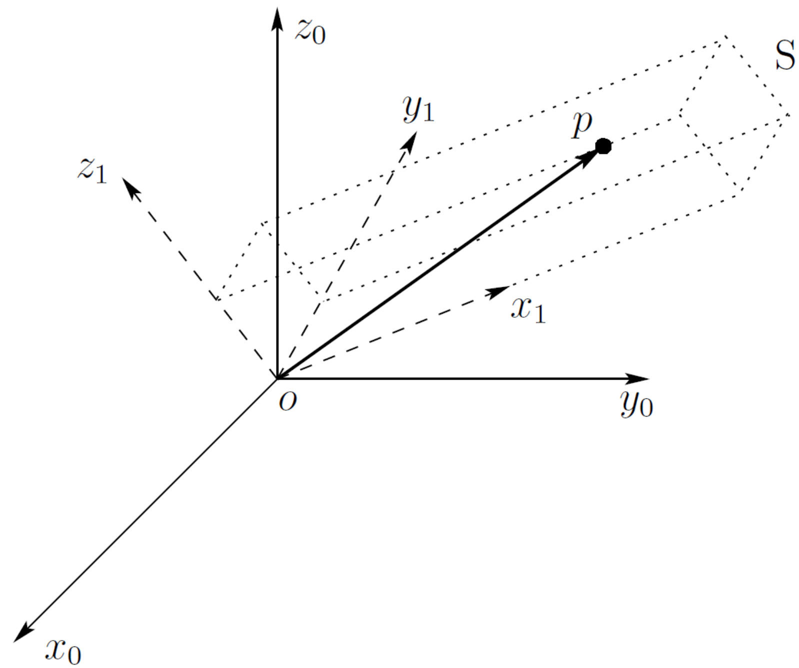
$=$



$$p^0 = R_1^0 p^1$$

Rotational Transformations

$$p^0 = R_1^0 p^1$$

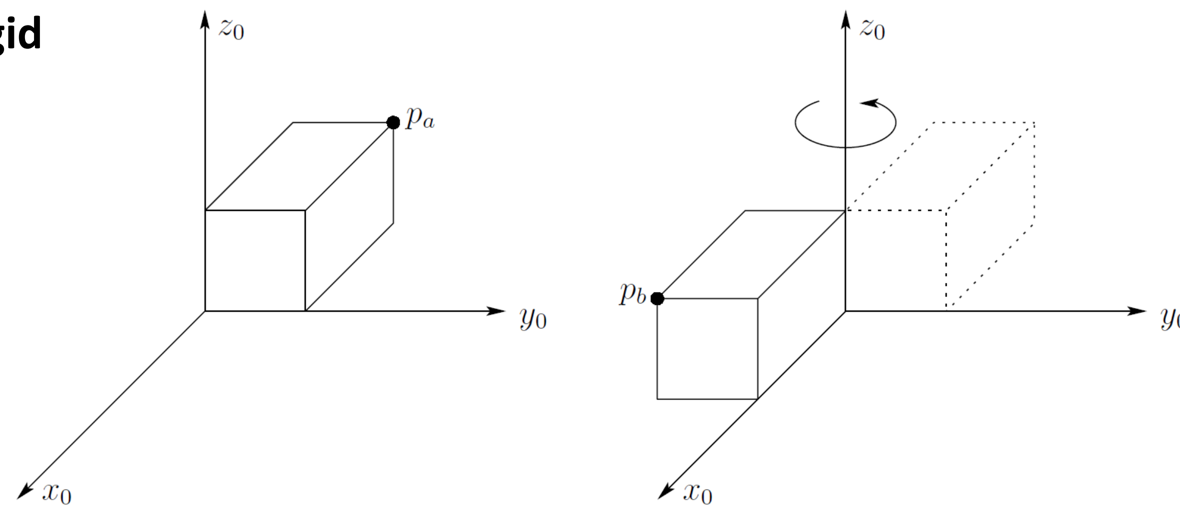


Thus, the rotation matrix $\boldsymbol{R_1^0}$ can be used not only to represent the orientation of coordinate frame $\boldsymbol{o_1\, x_1\, y_1\, z_1}$ with respect to frame $\boldsymbol{o_0\, x_0\, y_0\, z_0}$, but also to transform the coordinates of a point from one frame to another.

If a given point is expressed relative to $\boldsymbol{o_1\, x_1\, y_1\, z_1}$ by coordinates $\boldsymbol{p^1}$, then $\boldsymbol{R_1^0\, p^1}$ represents the same point expressed relative to the frame $\boldsymbol{o_0\, x_0\, y_0\, z_0}$.

Rotational Transformations

Rotation matrices to represent **rigid motions**



(a) (b)

The block in (b) is obtained by rotating the block in (a) by π about z_0.

It is possible to derive the coordinates for $\boldsymbol{p_b}$ given only the coordinates for $\boldsymbol{p_a}$ and the rotation matrix that corresponds to the rotation about $\boldsymbol{z_0}$.

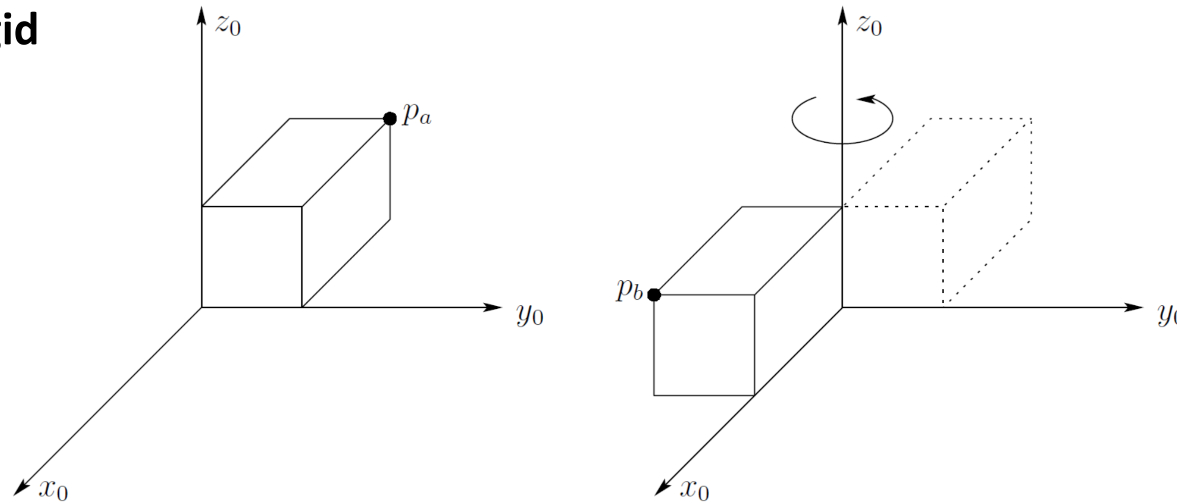
Suppose that a coordinate frame $\boldsymbol{o_0\ x_0\ y_0\ z_0}$ is rigidly attached to the block. After the rotation by $\boldsymbol{\pi}$, the block's coordinate frame, which is rigidly attached to the block, is also rotated by $\boldsymbol{\pi}$.

Rotational Transformations

Rotation matrices to represent **rigid motions**

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1^0 = R_{z,\pi} = \begin{bmatrix} \quad \end{bmatrix}$$



(a) (b)

The block in (b) is obtained by rotating the block in (a) by π about z_0.

The coordinates of $\boldsymbol{p_b}$ with respect to the reference frame $\boldsymbol{o_0\ x_0\ y_0\ z_0}$:

$$p_b^0 = R_{z,\pi} p_a^0$$

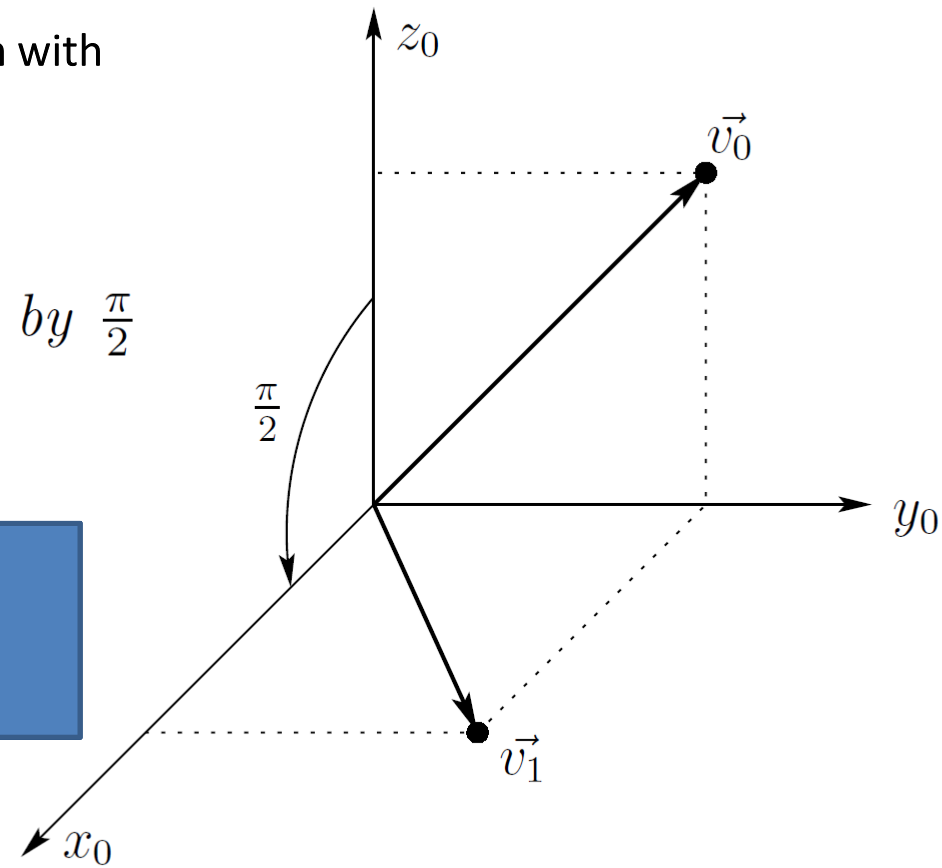
Rotational Transformations

Rotation matrices to represent **vector rotation** with respect to a coordinate frame.

$$v^0 = (0,1,1)^T \ \textit{is rotated about } y_0 \textit{ by } \frac{\pi}{2}$$

$$v_1^0 =$$

$$=$$



Rotating a vector about axis y_0.

Reminder:

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Summary: Rotation Matrix

1. It represents a coordinate transformation relating the coordinates of a point p in two different frames.

2. It gives the orientation of a transformed coordinate frame with respect to a fixed coordinate frame.

3. It is an **operator** taking a vector and rotating it to a new vector in the same coordinate system.

Similarity Transformations

The matrix representation of a general linear transformation is transformed from one frame to another using a so-called **similarity transformation**.

For example, if $\boldsymbol{A}$ is the matrix representation of a given linear transformation in $\boldsymbol{o_0}\ \boldsymbol{x_0}\ \boldsymbol{y_0}\ \boldsymbol{z_0}$ and $\boldsymbol{B}$ is the representation of the same linear transformation in $\boldsymbol{o_1}\ \boldsymbol{x_1}\ \boldsymbol{y_1}\ \boldsymbol{z_1}$ then $\boldsymbol{A}$ and $\boldsymbol{B}$ are related as:

$$B = (R_1^0)^{-1} A R_1^0$$

where $\boldsymbol{R_1^0}$ is the coordinate transformation between frames $\boldsymbol{o_1}\ \boldsymbol{x_1}\ \boldsymbol{y_1}\ \boldsymbol{z_1}$ and $\boldsymbol{o_0}\ \boldsymbol{x_0}\ \boldsymbol{y_0}\ \boldsymbol{z_0}$. In particular, if $\boldsymbol{A}$ itself is a rotation, then so is $\boldsymbol{B}$, and thus the use of similarity transformations allows us to express the same rotation easily with respect to different frames.

Example

$$B = (R_1^0)^{-1} A R_1^0$$

Suppose frames $\boldsymbol{o_0}\ \boldsymbol{x_0}\ \boldsymbol{y_0}\ \boldsymbol{z_0}$ and $\boldsymbol{o_1}\ \boldsymbol{x_1}\ \boldsymbol{y_1}\ \boldsymbol{z_1}$ are related by the rotation

$$R_1^0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

If $\boldsymbol{A} = \boldsymbol{R_z}$ relative to the frame $\boldsymbol{o_0}\ \boldsymbol{x_0}\ \boldsymbol{y_0}\ \boldsymbol{z_0}$, then, relative to frame $\boldsymbol{o_1}\ \boldsymbol{x_1}\ \boldsymbol{y_1}\ \boldsymbol{z_1}$ we have

$$B \quad = \quad (R_1^0)^{-1} A^0 R_1^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & s_\theta \\ 0 & -s_\theta & c_\theta \end{bmatrix}$$

$\boldsymbol{B}$ is a rotation about the $\boldsymbol{z0 - axis}$ but expressed relative to the frame $\boldsymbol{o_1}\ \boldsymbol{x_1}\ \boldsymbol{y_1}\ \boldsymbol{z_1}$.

Rotation With Respect To The Current Frame

The matrix $\boldsymbol{R_1^0}$ represents a rotational transformation between the frames $\boldsymbol{o_0\ x_0\ y_0\ z_0}$ and $\boldsymbol{o_1\ x_1\ y_1\ z_1}$.

Suppose we now add a third coordinate frame $\boldsymbol{o_2\ x_2\ y_2\ z_2}$ related to the frames $\boldsymbol{o_0\ x_0\ y_0\ z_0}$ and $\boldsymbol{o_1\ x_1\ y_1\ z_1}$ by rotational transformations.
A given point $\boldsymbol{p}$ can then be represented by coordinates specified with respect to any of these three frames: $\boldsymbol{p^0}$, $\boldsymbol{p^1}$ and $\boldsymbol{p^2}$.

The relationship among these representations of $\boldsymbol{p}$ is:

$$\begin{aligned} p^0 &= R_1^0 p^1 \\ p^1 &= R_2^1 p^2 \\ p^0 &= R_2^0 p^2 \end{aligned}$$

$$p^0 = R_1^0 R_2^1 p^2$$

$$R_2^0 = R_1^0 R_2^1$$

where each $\boldsymbol{R_j^i}$ is a rotation matrix

Composition Law for Rotational Transformations

In order to transform the coordinates of a point $\boldsymbol{p}$ from its representation $\boldsymbol{p}^2$ in the frame $\boldsymbol{o}_2\,\boldsymbol{x}_2\,\boldsymbol{y}_2\,\boldsymbol{z}_2$ to its representation $\boldsymbol{p}^0$ in the frame $\boldsymbol{o}_0\,\boldsymbol{x}_0\,\boldsymbol{y}_0\,\boldsymbol{z}_0$, we may first transform to its coordinates $\boldsymbol{p}^1$ in the frame $\boldsymbol{o}_1\,\boldsymbol{x}_1\,\boldsymbol{y}_1\,\boldsymbol{z}_1$ using $\boldsymbol{R}_2^1$ and then transform $\boldsymbol{p}^1$ to $\boldsymbol{p}^0$ using $\boldsymbol{R}_1^0$.

$$
\begin{aligned}
p^0 &= R_1^0 p^1 \\
p^1 &= R_2^1 p^2 \\
p^0 &= R_2^0 p^2
\end{aligned}
$$

$$p^0 = R_1^0 R_2^1 p^2$$

$$R_2^0 = R_1^0 R_2^1$$

Composition Law for Rotational Transformations

$$R_2^0 = R_1^0 R_2^1$$

Suppose initially that all three of the coordinate frames are coincide.

We first rotate the frame $\boldsymbol{o_2\ x_2\ y_2\ z_2}$ relative to $\boldsymbol{o_0\ x_0\ y_0\ z_0}$ according to the transformation $\boldsymbol{R_1^0}$.

Then, with the frames $\boldsymbol{o_1\ x_1\ y_1\ z_1}$ and $\boldsymbol{o_2\ x_2\ y_2\ z_2}$ coincident, we rotate $\boldsymbol{o_2\ x_2\ y_2\ z_2}$ relative to $\boldsymbol{o_1\ x_1\ y_1\ z_1}$ according to the transformation $\boldsymbol{R_2^1}$.

In each case we call the frame relative to which the rotation occurs the **current frame**.

Coincident: lie exactly on top of each other

Example

Suppose a rotation matrix R represents

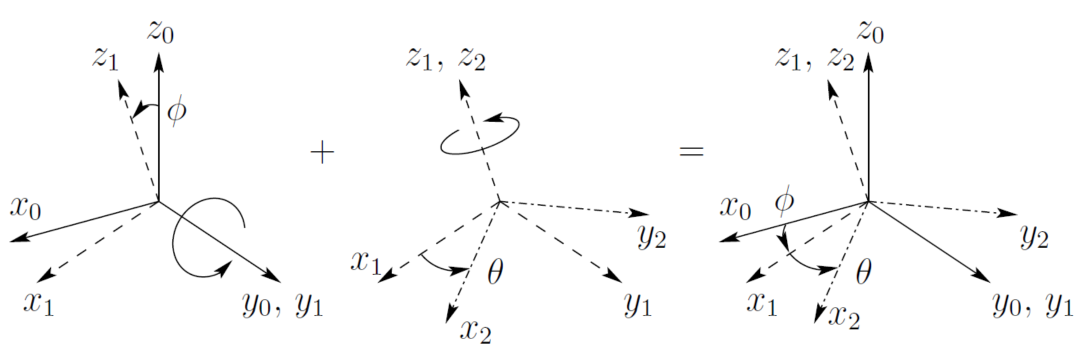
- a rotation of angle $\boldsymbol{\phi}$ about the current $\boldsymbol{y-axis}$ followed by
- a rotation of angle $\boldsymbol{\theta}$ about the current $\boldsymbol{z-axis}$.

$$R \quad =$$

$$R_{x,\theta} \quad = \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_{y,\theta} \quad = \quad \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_{z,\theta} \quad = \quad \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Example

Suppose a rotation matrix R represents

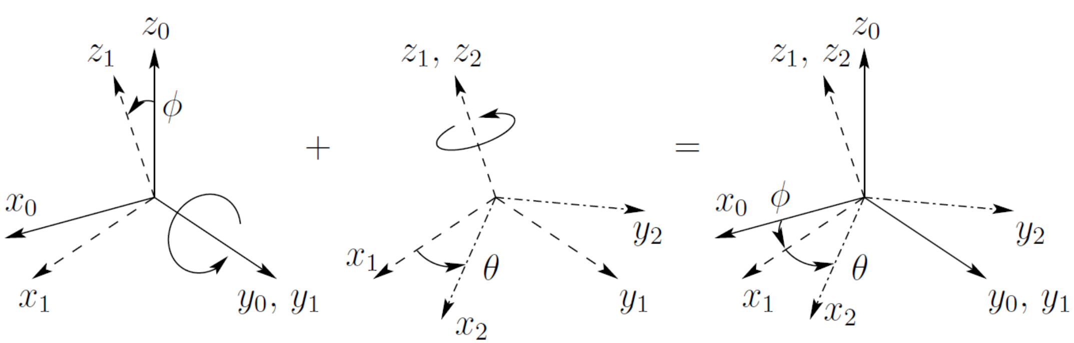
- a rotation of angle $\boldsymbol{\phi}$ about the current $\boldsymbol{y - axis}$ followed by
- a rotation of angle $\boldsymbol{\theta}$ about the current $\boldsymbol{z - axis}$.

$$R = R_{y,\phi} R_{z,\theta}$$

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Example

Suppose a rotation matrix R represents

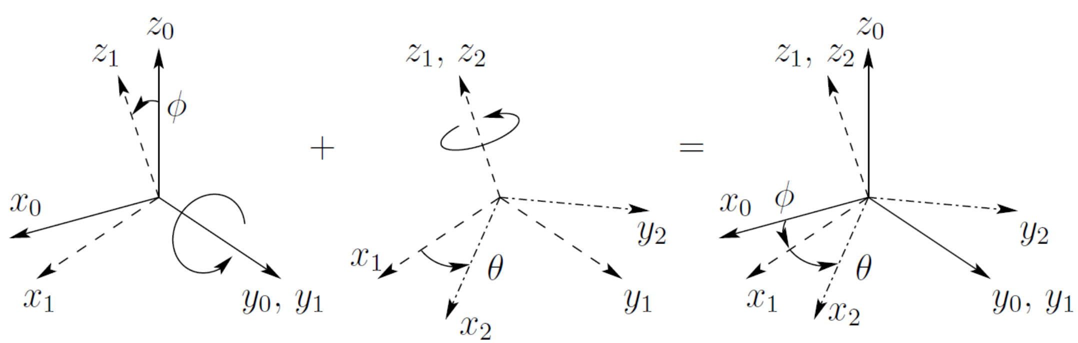
- a rotation of angle $\boldsymbol{\phi}$ about the current $\boldsymbol{y-axis}$ followed by
- a rotation of angle $\boldsymbol{\theta}$ about the current $\boldsymbol{z-axis}$.

$$\begin{aligned} R &= R_{y,\phi}R_{z,\theta} \\ &= \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix} \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Example

Suppose a rotation matrix R represents

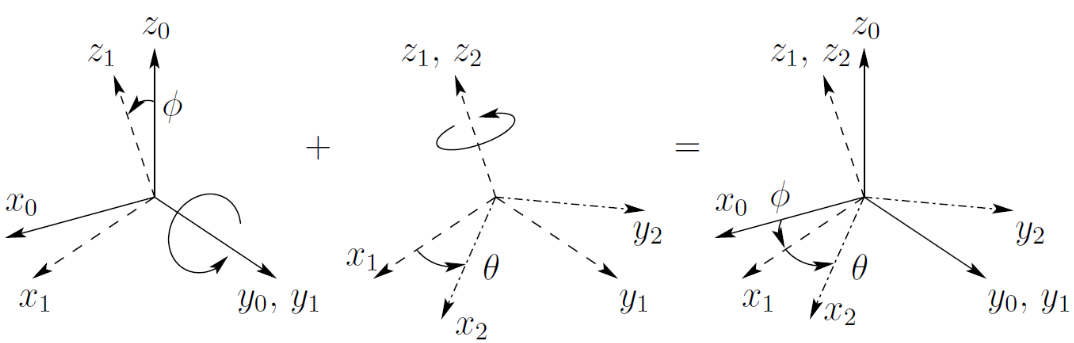
- a rotation of angle $\boldsymbol{\phi}$ about the current $\boldsymbol{y-axis}$ followed by
- a rotation of angle $\boldsymbol{\theta}$ about the current $\boldsymbol{z-axis}$.

$$\begin{aligned} R &= R_{y,\phi}R_{z,\theta} \\ &= \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix} \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_\phi c_\theta & -c_\phi s_\theta & s_\phi \\ s_\theta & c_\theta & 0 \\ -s_\phi c_\theta & s_\phi s_\theta & c_\phi \end{bmatrix} \end{aligned}$$

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Example

Suppose a rotation matrix R represents

- a rotation of angle $\boldsymbol{\theta}$ about the current $\boldsymbol{z-axis}$ followed by
- a rotation of angle $\boldsymbol{\phi}$ about the current $\boldsymbol{y-axis}$

$$R' =$$

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

Suppose a rotation matrix R represents
- a rotation of angle $\boldsymbol{\theta}$ about the current $\boldsymbol{z-axis}$ followed by
- a rotation of angle $\boldsymbol{\phi}$ about the current $\boldsymbol{y-axis}$

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} R' &= R_{z,\theta} R_{y,\phi} \\ &= \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix} \\ &= \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \end{aligned}$$

Example

Suppose a rotation matrix R represents

- a rotation of angle $\boldsymbol{\theta}$ about the current $\boldsymbol{z-axis}$ followed by
- a rotation of angle $\boldsymbol{\phi}$ about the current $\boldsymbol{y-axis}$

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} R' &= R_{z,\theta} R_{y,\phi} \\ &= \begin{bmatrix} c_\theta & -s_\phi & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix} \\ &= \begin{bmatrix} c_\theta c_\phi & -s_\theta & c_\theta s_\phi \\ s_\theta c_\phi & c_\theta & s_\theta s_\phi \\ -s_\phi & 0 & c_\phi \end{bmatrix} \end{aligned}$$

Rotational transformations do not commute

$$R \neq R'$$

Rotation With Respect To The Fixed Frame

Performing a sequence of rotations, each about a given fixed coordinate frame, rather than about successive current frames.

For example we may wish to perform a rotation about $\boldsymbol{x_0}$ followed by a rotation about $\boldsymbol{y_0}$ (and not $\boldsymbol{y_1}$!). We will refer to $\boldsymbol{o_0}\ \boldsymbol{x_0}\ \boldsymbol{y_0}\ \boldsymbol{z_0}$ as the **fixed frame**. In this case the composition law given before is not valid.

$$R_2^0 = R_1^0 R_2^1$$

The composition law that was obtained by multiplying the successive rotation matrices in the reverse order from that given by is not valid.

Rotation with Respect to the Fixed Frame

Suppose we have two frames $\boldsymbol{o_0\ x_0\ y_0\ z_0}$ and $\boldsymbol{o_1\ x_1\ y_1\ z_1}$ related by the rotational transformation $\boldsymbol{R_1^0}$.

If $\boldsymbol{R}$ represents a rotation relative to $\boldsymbol{o_0\ x_0\ y_0\ z_0}$, the representation for $\boldsymbol{R}$ in the current frame $\boldsymbol{o_1\ x_1\ y_1\ z_1}$ is given by:

$$(R_1^0)^{-1} R R_1^0$$

With applying the composition law for rotations about the current axis:

$$R_2^0 \quad = \quad R_1^0 \left[(R_1^0)^{-1} R R_1^0 \right] = R R_1^0$$

Reminder:

Similarity Transformations

$$B = (R_1^0)^{-1} A R_1^0$$

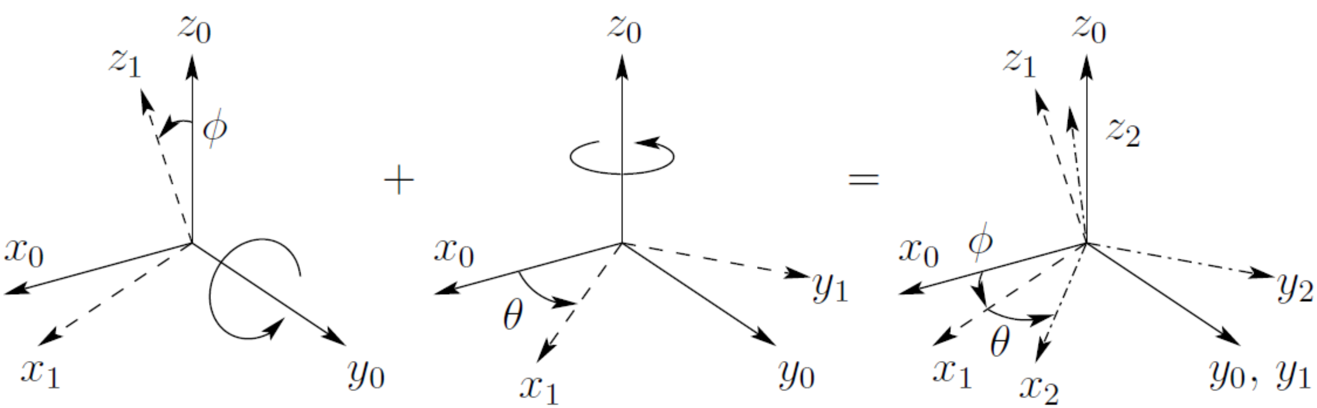
composition law for rotations about the current axis

$$R_2^0 = R_1^0 R_2^1$$

Example

Suppose a rotation matrix R represents

- a rotation of angle $\boldsymbol{\phi}$ about $\boldsymbol{y_0 - axis}$ followed by
- a rotation of angle $\boldsymbol{\theta}$ about the fixed $\boldsymbol{z_0 - axis}$



The **second** rotation about the fixed axis is given by

$$R_{y,-\phi} R_{z,\theta} R_{y,\phi}$$

which is the basic rotation about **the z-axis** expressed relative to the frame $\boldsymbol{o_1\ x_1\ y_1\ z_1}$ using a similarity transformation.

Reminder:

Similarity Transformations

$$B = (R_1^0)^{-1} A R_1^0$$

composition law for rotations about the current axis

$$R_2^0 = R_1^0 R_2^1$$

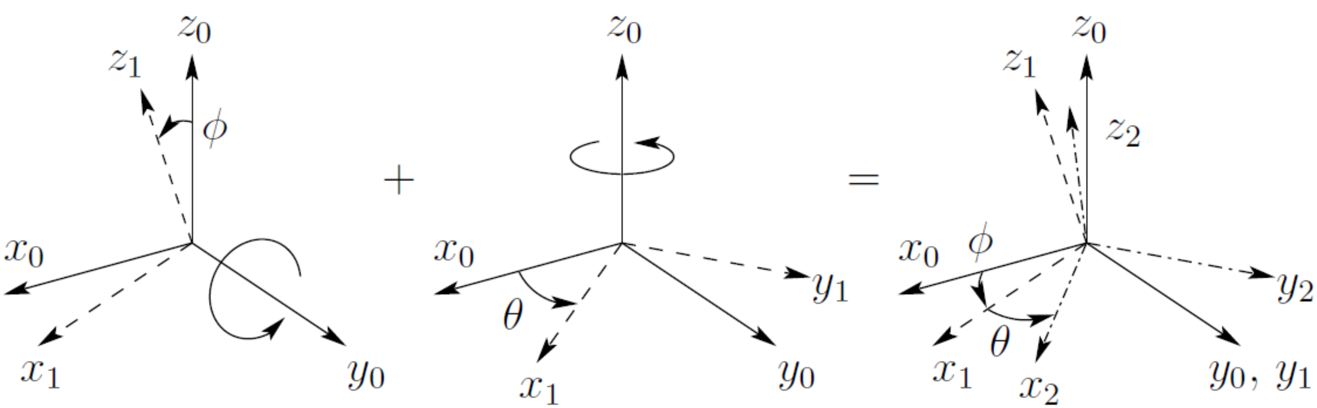
composition law for rotations about the fixed axis

$$\begin{aligned} R_2^0 &= R_1^0 \left[(R_1^0)^{-1} R R_1^0 \right] \\ &= R R_1^0 \end{aligned}$$

Example

Suppose a rotation matrix R represents

- a rotation of angle $\boldsymbol{\phi}$ about $\boldsymbol{y_0 - axis}$ followed by
- a rotation of angle $\boldsymbol{\theta}$ about the fixed $\boldsymbol{z_0 - axis}$



Therefore, the composition rule for rotational transformations

$$\begin{aligned} p^0 &= \square\, p^1 \\ &= \square\, p^2 \\ &= \square\, p^2 \end{aligned}$$

Reminder:

Similarity Transformations

$$B = (R_1^0)^{-1} A R_1^0$$

composition law for rotations about the current axis

$$R_2^0 = R_1^0 R_2^1$$

composition law for rotations about the fixed axis

$$\begin{aligned} R_2^0 &= R_1^0 \left[(R_1^0)^{-1} R R_1^0 \right] \\ &= R R_1^0 \end{aligned}$$

Example

Suppose a rotation matrix R represents

- a rotation of angle $\boldsymbol{\phi}$ about $\boldsymbol{y_0} - \boldsymbol{axis}$ followed by
- a rotation of angle$\boldsymbol{\theta}$about the fixed $\boldsymbol{z_0}-\boldsymbol{axis}$

$$
\begin{aligned}
p^0 &= R_{y,\phi} p^1 \\
&= R_{y,\phi} \left[R_{y,-\phi} R_{z,\theta} R_{y,\phi} \right] p^2 \\
&= R_{z,\theta} R_{y,\phi} p^2
\end{aligned}
$$

Example

Suppose a rotation matrix R represents

- a rotation of angle $\boldsymbol{\phi}$ about $\boldsymbol{y_0} - \boldsymbol{axis}$ followed by
- a rotation of angle $\boldsymbol{\theta}$ about the fixed $\boldsymbol{z_0} - \boldsymbol{axis}$

$$\begin{aligned} p^0 &= R_{y,\phi} p^1 \\ &= R_{y,\phi} \left[R_{y,-\phi} R_{z,\theta} R_{y,\phi} \right] p^2 \\ &= R_{z,\theta} R_{y,\phi} p^2 \end{aligned}$$

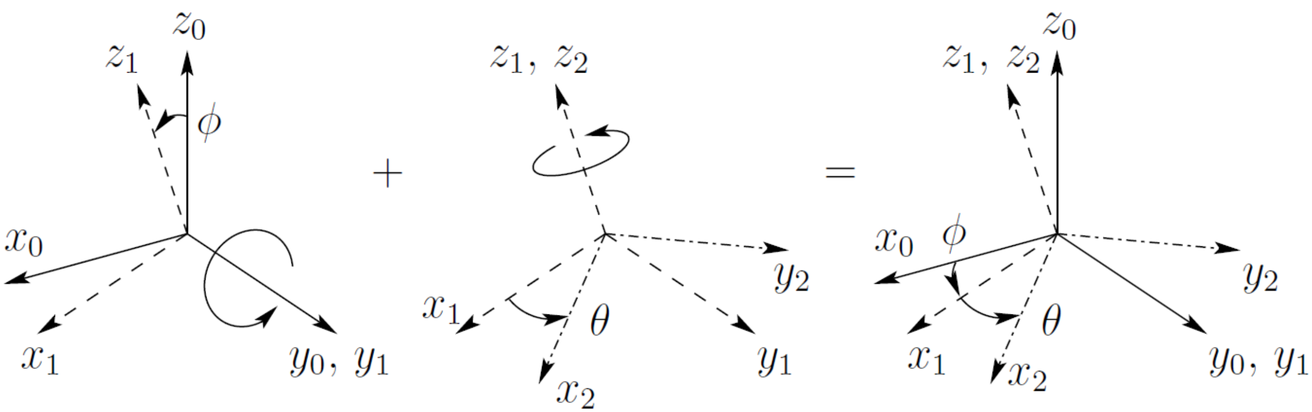
Suppose a rotation matrix R represents

- a rotation of angle $\boldsymbol{\phi}$ about the current $\boldsymbol{y} - \boldsymbol{axis}$ followed by
- a rotation of angle $\boldsymbol{\theta}$ about the current $\boldsymbol{z} - \boldsymbol{axis}$.

$$R = R_{y,\phi} R_{z,\theta}$$

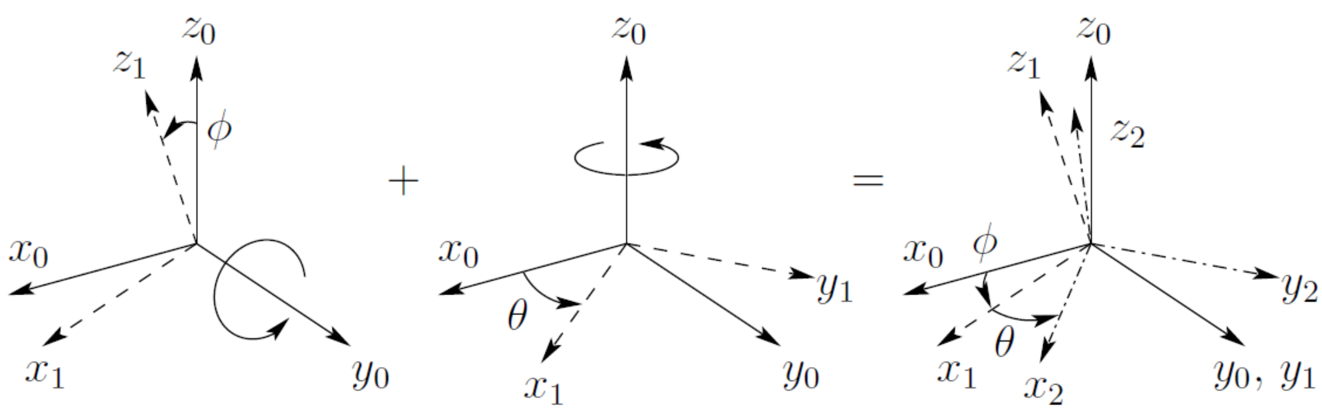
Summary

To note that we obtain the same basic rotation matrices, but in the reverse order.



Rotation with Respect to the Current Frame

$$R_2^0 = R_1^0 R_2^1$$



Rotation with Respect to the Fixed Frame

$$R_2^0 = R_2^1 R_1^0$$

$$R_2^0 = R \, R_1^0$$

Rules for Composition of Rotational Transformations

We can summarize the rule of composition of rotational transformations by:

Given a fixed frame $\boldsymbol{o_0\, x_0\, y_0\, z_0}$ a current frame $\boldsymbol{o_1\, x_1\, y_1\, z_1}$, together with rotation matrix $\boldsymbol{R_1^0}$ relating them, if a third frame $\boldsymbol{o_2\, x_2\, y_2\, z_2}$ is obtained by a rotation $\boldsymbol{R}$ performed relative to the current frame then post-multiply $\boldsymbol{R_1^0}$ by $\boldsymbol{R = R_2^1}$ to obtain

$$R_2^0 = R_1^0 R_2^1$$

If the second rotation is to be performed relative to the fixed frame then it is both confusing and inappropriate to use the notation $\boldsymbol{R_2^1}$ to represent this rotation. Therefore, if we represent the rotation by $\boldsymbol{R}$, we pre-multiply $\boldsymbol{R_1^0}$ by $\boldsymbol{R}$ to obtain

$$R_2^0 = R\, R_1^0$$

In each case $\boldsymbol{R_2^0}$ represents the transformation between the frames $\boldsymbol{o_0\, x_0\, y_0\, z_0}$ and $\boldsymbol{o_2\, x_2\, y_2\, z_2}$.

Example

Find **R** for the following sequence of basic rotations:

1. A rotation of Θ about the current x-axis
2. A rotation of ϕ about the current z-axis
3. A rotation of α about the fixed z-axis
4. A rotation of β about the current y-axis
5. A rotation of δ about the fixed x-axis

$$R =$$

Reminder:

Rotation with Respect to the Current Frame

$$R_2^0 = R_1^0 R_2^1$$

Rotation with Respect to the Fixed Frame

$$R_2^0 = R\, R_1^0$$

Example

Find **R** for the following sequence of basic rotations:

1. A rotation of Θ about the current x-axis
2. A rotation of ϕ about the current z-axis
3. A rotation of α about the fixed z-axis
4. A rotation of β about the current y-axis
5. A rotation of δ about the fixed x-axis

$$R = \qquad R_{x,\theta}$$

Reminder:

Rotation with Respect to the Current Frame

$$R_2^0 = R_1^0 R_2^1$$

Rotation with Respect to the Fixed Frame

$$R_2^0 = R\, R_1^0$$

Example

Find **R** for the following sequence of basic rotations:

1. A rotation of Θ about the current x-axis
2. A rotation of ϕ about the current z-axis
3. A rotation of α about the fixed z-axis
4. A rotation of β about the current y-axis
5. A rotation of δ about the fixed x-axis

$$R = \qquad R_{x,\theta} R_{z,\phi}$$

Reminder:

Rotation with Respect to the Current Frame

$$R_2^0 = R_1^0 R_2^1$$

Rotation with Respect to the Fixed Frame

$$R_2^0 = R\, R_1^0$$

Example

Find **R** for the following sequence of basic rotations:

1. A rotation of Θ about the current x-axis
2. A rotation of ϕ about the current z-axis
3. A rotation of α about the fixed z-axis
4. A rotation of β about the current y-axis
5. A rotation of δ about the fixed x-axis

$$R = \quad R_{z,\alpha} R_{x,\theta} R_{z,\phi}$$

Reminder:

Rotation with Respect to the Current Frame

$$R_2^0 = R_1^0 R_2^1$$

Rotation with Respect to the Fixed Frame

$$R_2^0 = R\, R_1^0$$

Example

Find **R** for the following sequence of basic rotations:

1. A rotation of Θ about the current x-axis
2. A rotation of ϕ about the current z-axis
3. A rotation of α about the fixed z-axis
4. A rotation of β about the current y-axis
5. A rotation of δ about the fixed x-axis

$$R = \qquad R_{z,\alpha} R_{x,\theta} R_{z,\phi} R_{y,\beta}$$

Reminder:

Rotation with Respect to the Current Frame

$$R_2^0 = R_1^0 R_2^1$$

Rotation with Respect to the Fixed Frame

$$R_2^0 = R\, R_1^0$$

Example

Find **R** for the following sequence of basic rotations:

1. A rotation of Θ about the current x-axis
2. A rotation of ϕ about the current z-axis
3. A rotation of α about the fixed z-axis
4. A rotation of β about the current y-axis
5. A rotation of δ about the fixed x-axis

$$R = R_{x,\delta} R_{z,\alpha} R_{x,\theta} R_{z,\phi} R_{y,\beta}$$

Reminder:

Rotation with Respect to the Current Frame

$$R_2^0 = R_1^0 R_2^1$$

Rotation with Respect to the Fixed Frame

$$R_2^0 = R\, R_1^0$$

Example

Find **R** for the following sequence of basic rotations:

1. A rotation of δ about the fixed x-axis
2. A rotation of β about the current y-axis
3. A rotation of α about the fixed z-axis
4. A rotation of ϕ about the current z-axis
5. A rotation of Θ about the current x-axis

Reminder:

Rotation with Respect to the Current Frame

$$R_2^0 = R_1^0 R_2^1$$

Rotation with Respect to the Fixed Frame

$$R_2^0 = R\, R_1^0$$

Reminder:

Rotations in Three Dimensions

Each axis of the frame $o_1\ x_1\ y_1\ z_1$ is projected onto the coordinate frame $o_0\ x_0\ y_0\ z_0$.

The resulting rotation matrix is given by

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

The nine elements r_{ij} in a general rotational transformation **R** are not independent quantities.

$$R \epsilon\, SO(3)$$

Where SO**(n)** denotes the **Special Orthogonal** group of order **n**.

Recommendation to read:
Special Orthogonal group.
General Orthogonal Group

Rotations In Three Dimensions

For any $R \epsilon\ SO(n)$ The following properties hold

- $R^T = R^{-1}\ \epsilon\ SO(n)$
- The columns and the rows of R are mutually orthogonal
- Each column and each row of R is a unit vector
- $\det R\ =\ 1$ (the determinant)

Where SO(n) denotes the Special Orthogonal group of order n.

Example for $\boldsymbol{R^T = R^{-1}\ \epsilon\ SO(2)}$:

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T$$

Parameterizations Of Rotations

The nine elements $\boldsymbol{r_{ij}}$ in a general rotational transformation R are not independent quantities.

$$R \epsilon\ SO(3)$$

Where SO(n) denotes the **Special Orthogonal** group of order n.

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

As each column of R is a unit vector, then we can write:

$$\sum_i r_{ij}^2 = 1, \quad j \in \{1,2,3\}$$

3 Equations

As the columns of R are mutually orthogonal, then we can write:

$$r_{1i}r_{1j} + r_{2i}r_{2j} + r_{3i}r_{3j} = 0, \quad i \neq j$$

3 Equations

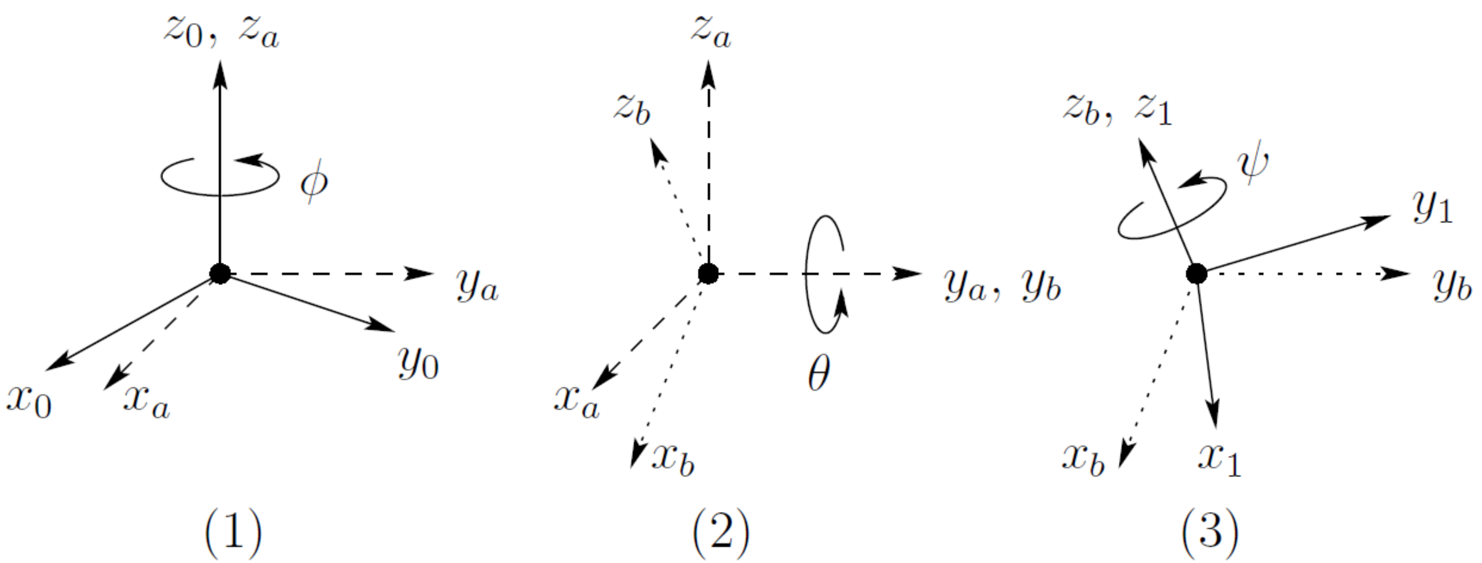
Together, these constraints define **six independent equations with nine unknowns**, which implies that there are **three free variables**.

Parameterizations of Rotations

We present three ways in which an arbitrary rotation can be represented using only three independent quantities:

- **Euler Angles** representation
- **Roll-Pitch-Yaw** representation
- **Axis/Angle** representation

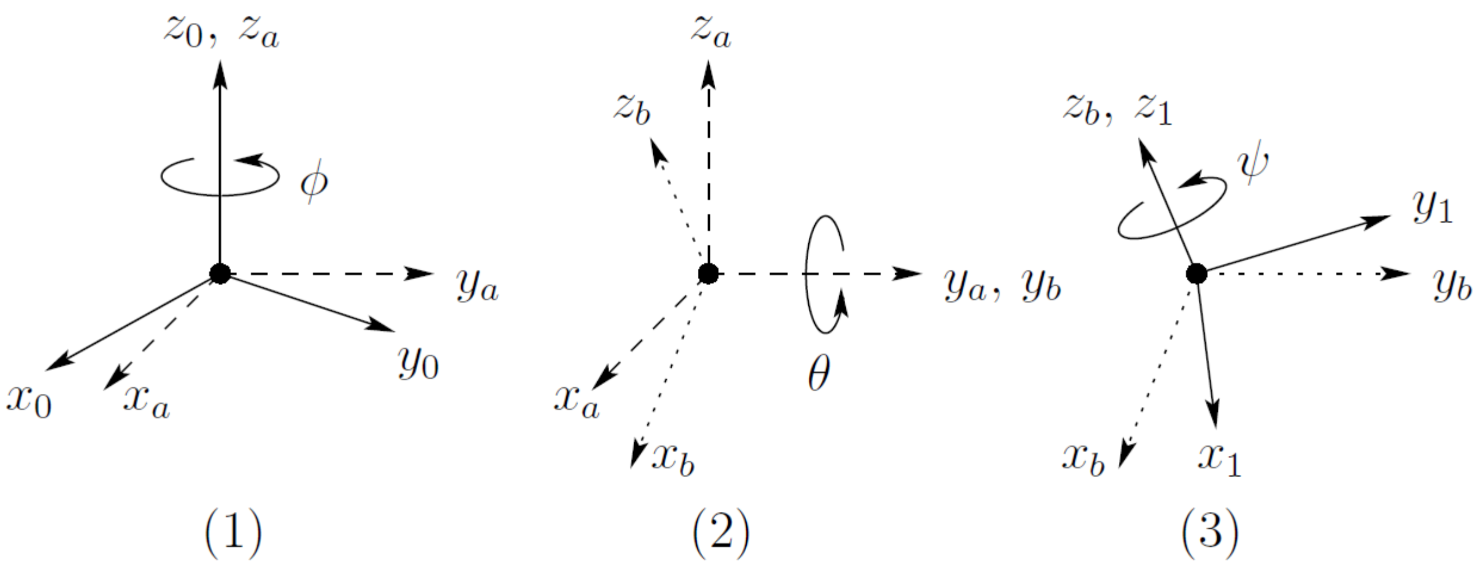
Euler Angles Representation



We can specify the orientation of the frame $\boldsymbol{o_1\ x_1\ y_1\ z_1}$ relative to the frame $\boldsymbol{o_0\ x_0\ y_0\ z_0}$ by three angles $(\boldsymbol{\phi}, \boldsymbol{\theta}, \boldsymbol{\psi})$, known as Euler Angles, and obtained by three successive rotations as follows:

1. rotation about the ***z-axis*** by the angle $\boldsymbol{\phi}$
2. rotation about the ***current y–axis*** by the angle $\boldsymbol{\theta}$
3. rotation about the ***current z–axis*** by the angle $\boldsymbol{\psi}$

Euler Angles Representation



ZYZ-Euler Angle Transformation

$$
\begin{aligned}
R_{ZYZ} &= R_{z,\phi} R_{y,\theta} R_{z,\psi} \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}
\end{aligned}
$$

Mathway

Basic Math | Pre-Algebra | Algebra | Geometry | Trigonometry | Precalculus | Calculus | Statistics | Finite M

Problem

Graph

Worksheet

Glossary

$$\begin{bmatrix} \cos(p) & -\sin(p) & 0 \\ \sin(p) & \cos(p) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(t) & 0 & \sin(t) \\ 0 & 1 & 0 \\ -\sin(t) & 0 & \cos(t) \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

Simplify

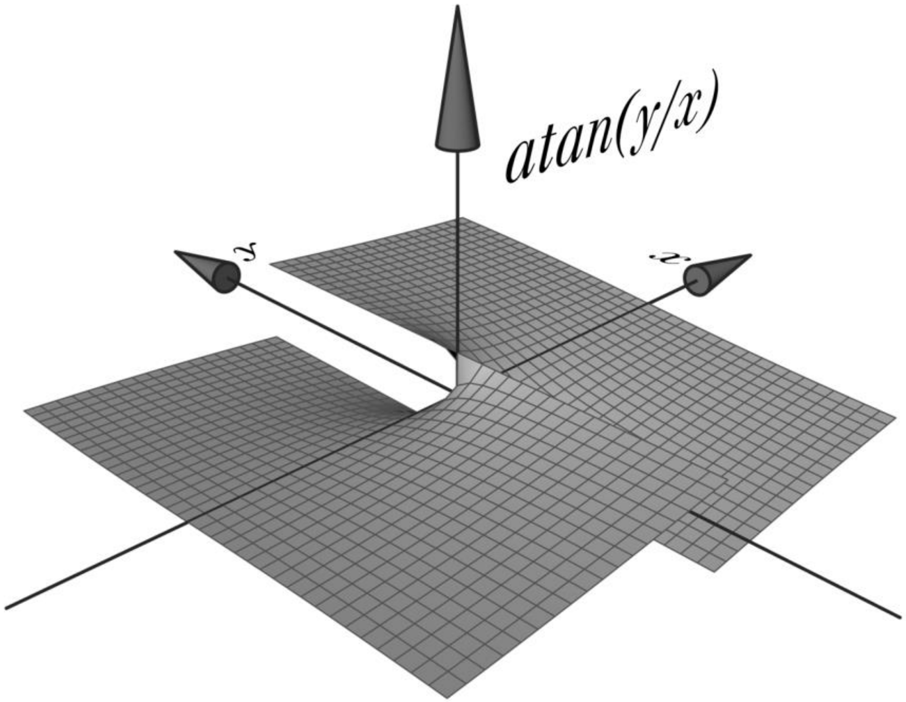
Answer

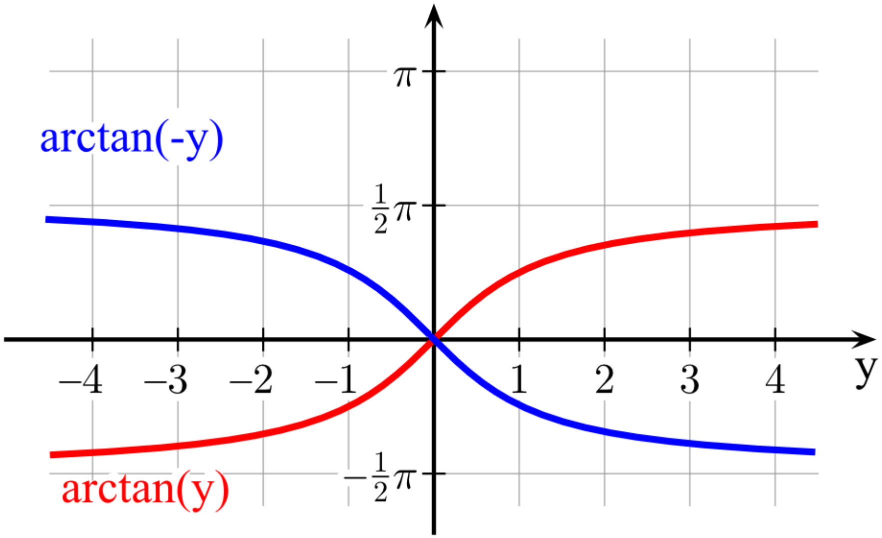
Answer

$$\begin{bmatrix} \cos(p)\cos(t) & -\sin(p)\cos(a)+\cos(p)\sin(t)\sin(a) & \sin(p)\sin(a)+\cos(p)\sin(t)\cos(a) \\ \sin(p)\cos(t) & \cos(p)\cos(a)+\sin(p)\sin(t)\sin(a) & -\cos(p)\sin(a)+\sin(p)\sin(t)\cos(a) \\ -\sin(t) & \cos(t)\sin(a) & \cos(t)\cos(a) \end{bmatrix}$$ ✓

Reminder:

Trigonometry (Atan vs. Atan2)

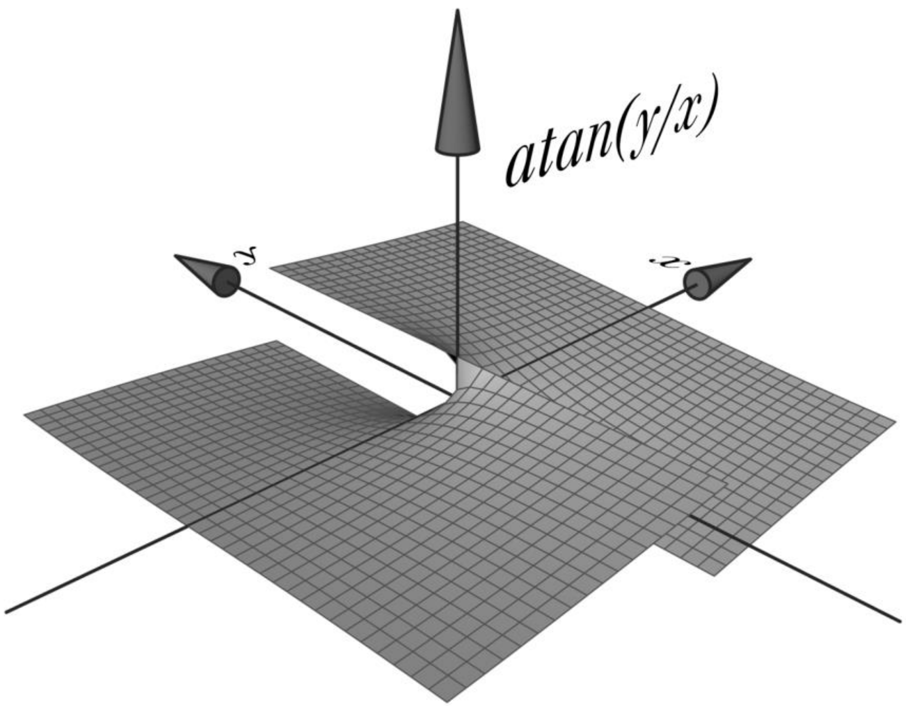


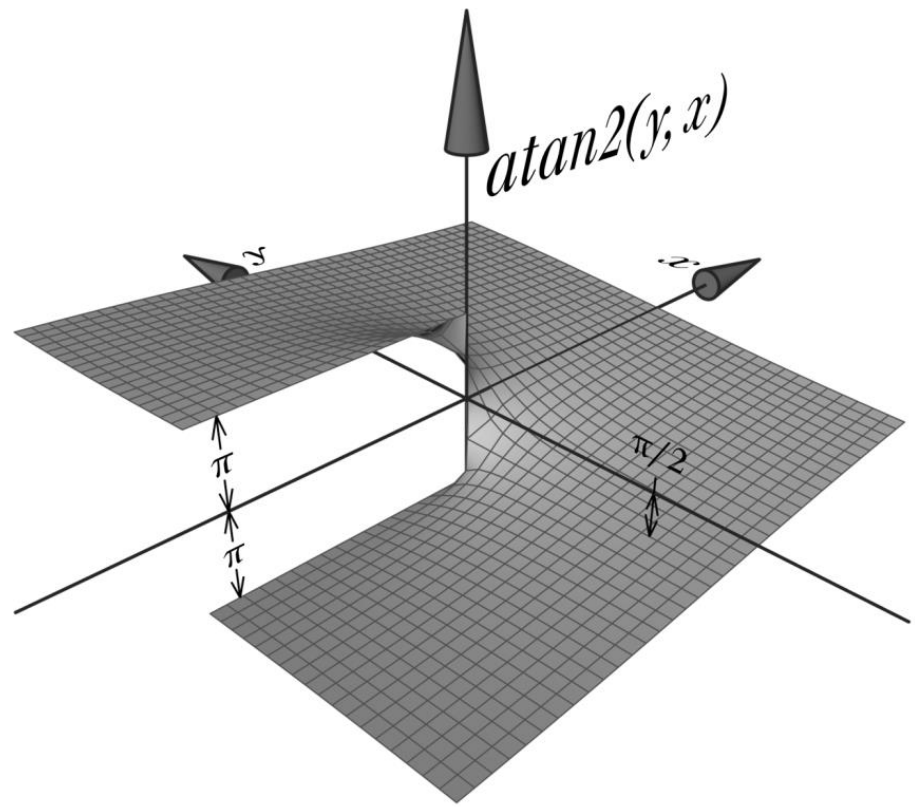


Reminder:

Trigonometry (Atan vs. Atan2)

$$\operatorname{atan2}(y,x)=\begin{cases}\arctan\frac{y}{x} & x>0\\ \arctan\frac{y}{x}+\pi & y\geq 0, x<0\\ \arctan\frac{y}{x}-\pi & y<0, x<0\\ +\frac{\pi}{2} & y>0, x=0\\ -\frac{\pi}{2} & y<0, x=0\\ \text{undefined} & y=0, x=0\end{cases}$$

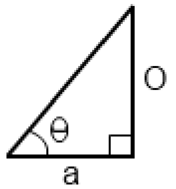




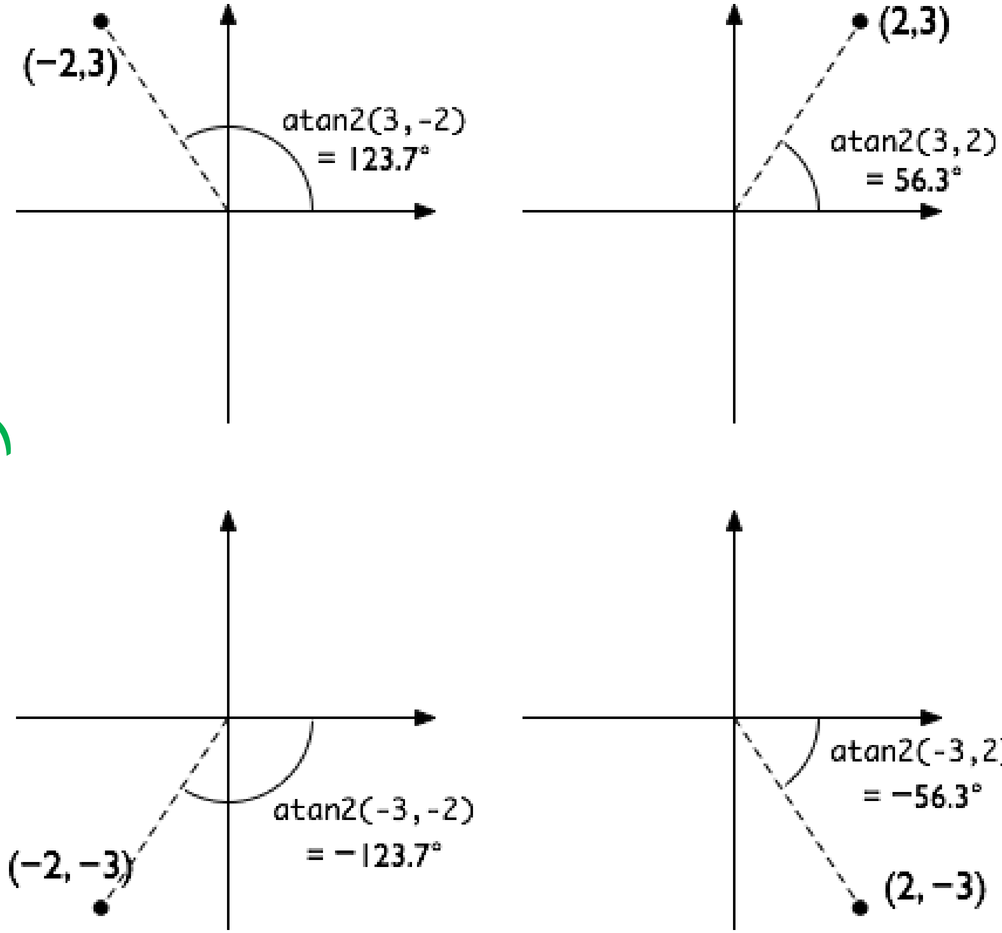
Reminder:

Trigonometry (Atan vs. Atan2)

tan(angle) = opposite/adjacent
atan(opposite/adjacent) = angle



atan2(opposite , adjacent)



Euler Angles Representation

Given a matrix $R \epsilon\ SO(3)$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Determine a set of Euler angles $\boldsymbol{\phi}, \boldsymbol{\theta}$, and $\boldsymbol{\psi}$ so that $\boldsymbol{R} = \boldsymbol{R_{ZYZ}}$

$$R_{ZYZ} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

If $\boldsymbol{r_{13} \neq 0}$ and $\boldsymbol{r_{23} \neq 0}$,
it follow that: $\boldsymbol{\phi = Atan2(r_{23}, r_{13})}$
where the function $Atan2(y, x)$ computes the arctangent of the ration y/x.
Then squaring the summing of the elements (1,3) and (2,3) and using the element (3,3) yields:

$$\boldsymbol{\theta = Atan2(+\sqrt{r_{13}^2 + r_{23}^2}, r_{33})} \quad \text{or} \quad \boldsymbol{\theta = Atan2(-\sqrt{r_{13}^2 + r_{23}^2}, r_{33})}$$

If we consider the first choice then $\boldsymbol{sin(\theta) > 0}$ then: $\boldsymbol{\psi = Atan2(r_{32}, -r_{31})}$

If we consider the second choice then $\boldsymbol{sin(\theta) < 0}$ then: $\boldsymbol{\psi = Atan2(-r_{32}, r_{31})}$
and $\boldsymbol{\phi = Atan2(-r_{23}, -r_{13})}$

Euler Angles Representation

Given a matrix $R \epsilon\, SO(3)$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Determine a set of Euler angles $\boldsymbol{\phi}$, $\boldsymbol{\theta}$, and $\boldsymbol{\psi}$ so that $\boldsymbol{R} = \boldsymbol{R_{ZYZ}}$

$$R_{ZYZ} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

If $\boldsymbol{r_{13} = r_{23} = 0}$, then the fact that R is orthogonal implies that $\boldsymbol{r_{33} = \pm 1}$ and that $\boldsymbol{r_{31} = r_{32} = 0}$ thus R has the form:

$$R = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

If $\boldsymbol{r_{33} = +1}$ then $c_\theta = 1$ and $s_\theta = 0$, so that $\theta = 0$.

$$\begin{bmatrix} c_\phi c_\psi - s_\phi s_\psi & -c_\phi s_\psi - s_\phi c_\psi & 0 \\ s_\phi c_\psi + c_\phi s_\psi & -s_\phi s_\psi + c_\phi c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{\phi+\psi} & -s_{\phi+\psi} & 0 \\ s_{\phi+\psi} & c_{\phi+\psi} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the sum $\boldsymbol{\phi + \psi}$ can be determined as ***$\boldsymbol{\phi + \psi}$ = Atan2($\boldsymbol{r_{21}, r_{11}}$) = Atan2($\boldsymbol{-r_{12}, r_{22}}$)***
There is infinity of solutions.

Euler Angles Representation

Given a matrix $R \epsilon\ SO(3)$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Determine a set of Euler angles $\boldsymbol{\phi}, \boldsymbol{\theta}$, and $\boldsymbol{\psi}$ so that $\boldsymbol{R} = \boldsymbol{R_{ZYZ}}$

$$R_{ZYZ} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

If $\boldsymbol{r_{13}} = \boldsymbol{r_{23}} = \boldsymbol{0}$, then the fact that R is orthogonal implies that $\boldsymbol{r_{33}} = \pm\boldsymbol{1}$ and that $\boldsymbol{r_{31}} = \boldsymbol{r_{32}} = \boldsymbol{0}$ thus R has the form:

$$R = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

If $\boldsymbol{r_{33}} = \boldsymbol{-1}$ then $c_\theta = -1$ and $s_\theta = 0$, so that $\theta = \pi$.

$$\begin{bmatrix} -c_{\phi-\psi} & -s_{\phi-\psi} & 0 \\ s_{\phi-\psi} & c_{\phi-\psi} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

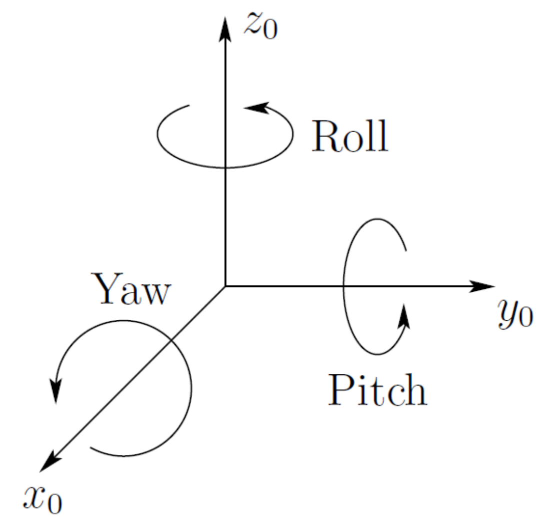
Thus, the $\boldsymbol{\phi} - \boldsymbol{\psi}$ can be determined as $\boldsymbol{\phi} - \boldsymbol{\psi}$ ***= Atan2(***$-\boldsymbol{r_{12}}, -\boldsymbol{r_{11}}$***) = Atan2(***$\boldsymbol{r_{21}}, \boldsymbol{r_{22}}$***)***
As before there is infinity of solutions.

Yaw-Pitch-Roll Representation

A rotation matrix $\boldsymbol{R}$ can also be described as a product of successive rotations about the principal coordinate axes $\boldsymbol{o_0\ x_0\ y_0\ z_0}$ taken in a specific order. These rotations define the **roll**, **pitch**, and **yaw** angles, which we shall also denote $(\boldsymbol{\phi}, \boldsymbol{\theta}, \boldsymbol{\psi})$

We specify the order in three successive rotations as follows:

1. Yaw rotation about $\boldsymbol{x_0}$ $-$***axis*** by the angle $\boldsymbol{\psi}$
2. Pitch rotation about $\boldsymbol{y_0}$ $-$ ***axis*** by the angle $\boldsymbol{\theta}$
3. Roll rotation about $\boldsymbol{z_0}$ $-$ ***axis*** by the angle $\boldsymbol{\phi}$



Since the successive rotations are relative to the **fixed frame**, the resulting transformation matrix is given by:

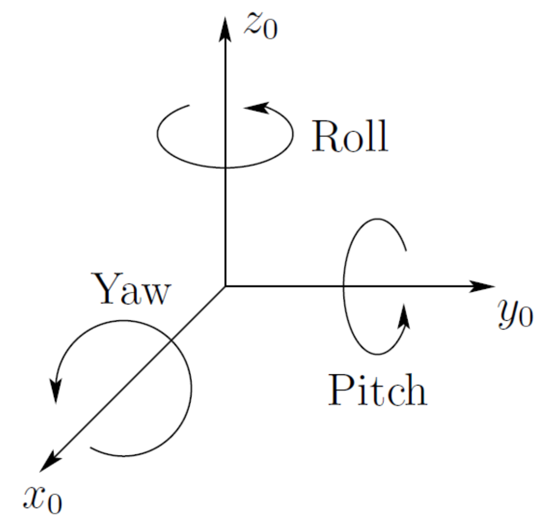
$$R_{XYZ} \quad =$$

Yaw-Pitch-Roll Representation

A rotation matrix $\boldsymbol{R}$ can also be described as a product of successive rotations about the principal coordinate axes $\boldsymbol{o_0}\ \boldsymbol{x_0}\ \boldsymbol{y_0}\ \boldsymbol{z_0}$ taken in a specific order. These rotations define the **roll**, **pitch**, and **yaw** angles, which we shall also denote $(\boldsymbol{\phi}, \boldsymbol{\theta}, \boldsymbol{\psi})$

We specify the order in three successive rotations as follows:

1. Yaw rotation about $\boldsymbol{x_0}$ −***axis*** by the angle $\boldsymbol{\psi}$
2. Pitch rotation about $\boldsymbol{y_0}$ − ***axis*** by the angle $\boldsymbol{\theta}$
3. Roll rotation about $\boldsymbol{z_0}$ − ***axis*** by the angle $\boldsymbol{\phi}$



Since the successive rotations are relative to the **fixed frame**, the resulting transformation matrix is given by:

$$R_{XYZ} = R_{z,\phi} R_{y,\theta} R_{x,\psi}$$

Yaw-Pitch-Roll Representation

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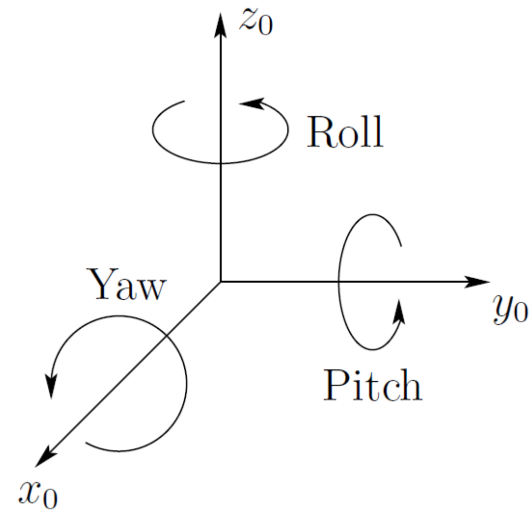
1. Yaw rotation about $\boldsymbol{x_0}$ $-$***axis*** by the angle $\boldsymbol{\psi}$
2. Pitch rotation about $\boldsymbol{y_0}$ $-$ ***axis*** by the angle $\boldsymbol{\theta}$
3. Roll rotation about $\boldsymbol{z_0}$ $-$ ***axis*** by the angle $\boldsymbol{\phi}$



Since the successive rotations are relative to the **fixed frame**, the resulting transformation matrix is given by:

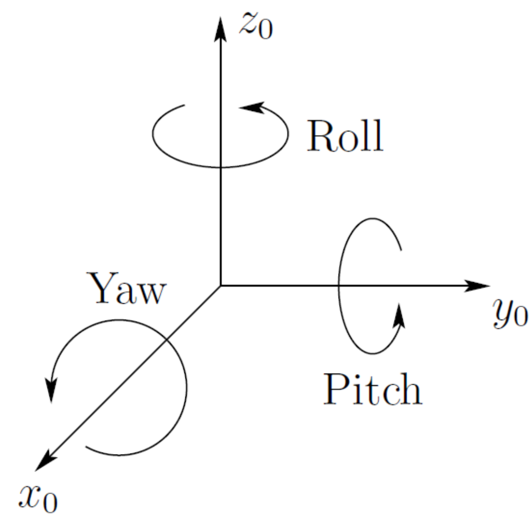
$$
\begin{aligned}
R_{XYZ} &= R_{z,\phi}R_{y,\theta}R_{x,\psi} \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix}
\end{aligned}
$$

Yaw-Pitch-Roll Representation

A rotation matrix $\boldsymbol{R}$ can also be described as a product of successive rotations about the principal coordinate axes $\boldsymbol{o_0}\ \boldsymbol{x_0}\ \boldsymbol{y_0}\ \boldsymbol{z_0}$ taken in a specific order. These rotations define the **roll**, **pitch**, and **yaw** angles, which we shall also denote $(\boldsymbol{\phi}, \boldsymbol{\theta}, \boldsymbol{\psi})$

We specify the order in three successive rotations as follows:

1. Yaw rotation about $\boldsymbol{x_0}$ $-$***axis*** by the angle $\boldsymbol{\psi}$
2. Pitch rotation about $\boldsymbol{y_0}$ $-$ ***axis*** by the angle $\boldsymbol{\theta}$
3. Roll rotation about $\boldsymbol{z_0}$ $-$ ***axis*** by the angle $\boldsymbol{\phi}$



Since the successive rotations are relative to the **fixed frame**, the resulting transformation matrix is given by:

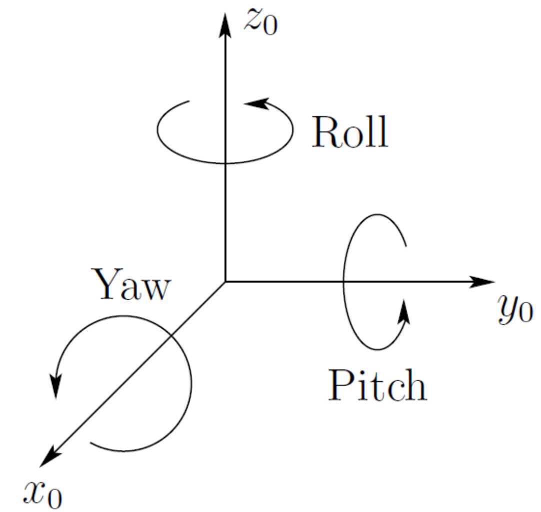
$$
\begin{aligned}
R_{XYZ} &= R_{z,\phi} R_{y,\theta} R_{x,\psi} \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}
\end{aligned}
$$

Yaw-Pitch-Roll Representation

A rotation matrix $\boldsymbol{R}$ can also be described as a product of successive rotations about the principal coordinate axes $\boldsymbol{o_0}\ \boldsymbol{x_0}\ \boldsymbol{y_0}\ \boldsymbol{z_0}$ taken in a specific order. These rotations define the **roll**, **pitch**, and **yaw** angles, which we shall also denote $(\boldsymbol{\phi}, \boldsymbol{\theta}, \boldsymbol{\psi})$

We specify the order in three successive rotations as follows:

1. Yaw rotation about $\boldsymbol{x_0}$ −***axis*** by the angle $\boldsymbol{\psi}$
2. Pitch rotation about $\boldsymbol{y_0}$ − ***axis*** by the angle $\boldsymbol{\theta}$
3. Roll rotation about $\boldsymbol{z_0}$ − ***axis*** by the angle $\boldsymbol{\phi}$



Since the successive rotations are relative to the **fixed frame**, the resulting transformation matrix is given by:

$$R_{XYZ} = R_{z,\phi} R_{y,\theta} R_{x,\psi}$$

Yaw-Pitch-Roll Representation

A rotation matrix $\boldsymbol{R}$ can also be described as a product of successive rotations about the principal coordinate axes $\boldsymbol{o_0}\ \boldsymbol{x_0}\ \boldsymbol{y_0}\ \boldsymbol{z_0}$ taken in a specific order. These rotations define the **roll**, **pitch**, and **yaw** angles, which we shall also denote $(\boldsymbol{\phi}, \boldsymbol{\theta}, \boldsymbol{\psi})$

We specify the order in three successive rotations as follows:

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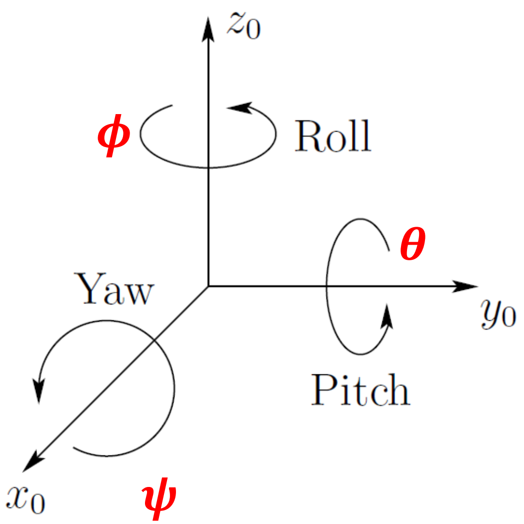
2. Pitch rotation about $\boldsymbol{y_0}$ − ***axis*** by the angle $\boldsymbol{\theta}$
3. Roll rotation about $\boldsymbol{z_0}$ − ***axis*** by the angle $\boldsymbol{\phi}$



Since the successive rotations are relative to the **fixed frame**, the resulting transformation matrix is given by:

$$R_{XYZ} = R_{z,\phi} R_{y,\theta} R_{x,\psi}$$

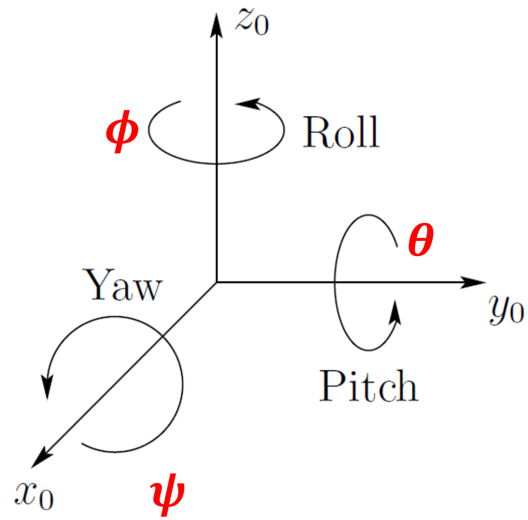
Instead of **yaw-pitch-roll** relative to the **fixed** frames we could also interpret the above transformation as **roll-pitch-yaw**, in that order, each taken with respect to the **current** frame.
The end result is the same matrix.

Yaw-Pitch-Roll Representation

Find the inverse solution to a given rotation matrix R.

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$R_{XYZ} = \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}$$



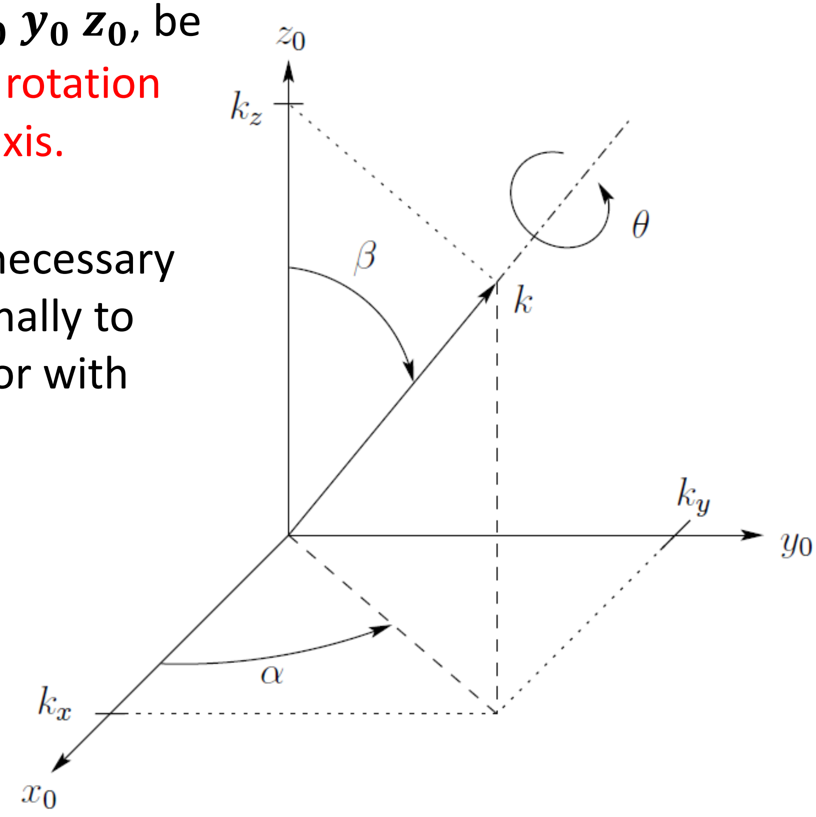
Determine a set of **Roll-Pitch-Yaw** angles $\boldsymbol{\phi}$, $\boldsymbol{\theta}$, and $\boldsymbol{\psi}$ so that $\boldsymbol{R} = \boldsymbol{R}_{XYZ}$

Axis/Angle Representation

Rotations are not always performed about the principal coordinate axes. We are often interested in a rotation about an arbitrary axis in space. This provides both a convenient way to describe rotations, and an alternative parameterization for rotation matrices.

Let $\boldsymbol{k} = [\boldsymbol{k_x}, \boldsymbol{k_y}, \boldsymbol{k_z}]^T$, expressed in the frame $\boldsymbol{o_0}\ \boldsymbol{x_0}\ \boldsymbol{y_0}\ \boldsymbol{z_0}$, be a unit vector defining an axis. We wish to derive the rotation matrix $\boldsymbol{R_{k,\theta}}$ representing a rotation of $\boldsymbol{\theta}$ about this axis.

A possible solution is to rotate first $\boldsymbol{k}$ by the angles necessary to align it with $\boldsymbol{z}$, then to rotate by $\boldsymbol{\theta}$ about $\boldsymbol{z}$, and finally to rotate by the angels necessary to align the unit vector with the initial direction.



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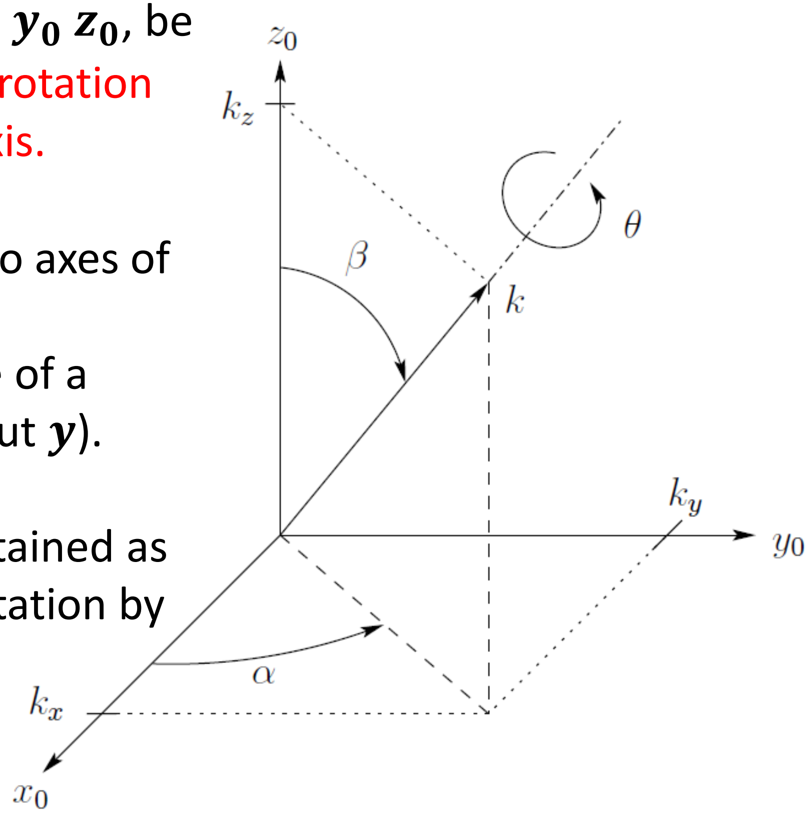


The sequence of rotations to be made with respect to axes of **fixed frame** is the following:

- Align $\boldsymbol{k}$ with $\boldsymbol{z}$ (which is obtained as the sequence of a rotation by $-\boldsymbol{\alpha}$ about $\boldsymbol{z}$ and a rotation of $-\boldsymbol{\beta}$ about $\boldsymbol{y}$).
- Rotate by $\boldsymbol{\theta}$ about $\boldsymbol{z}$.
- Realign with the initial direction of $\boldsymbol{k}$, which is obtained as the sequence of a rotation by $\boldsymbol{\beta}$ about $\boldsymbol{y}$ and a rotation by $\boldsymbol{\alpha}$ about $\boldsymbol{z}$.

$$\boldsymbol{R_{k,\theta}} = \boldsymbol{R_{z,\alpha}}\,\boldsymbol{R_{y,\beta}}\boldsymbol{R_{z,\theta}}\boldsymbol{R_{y,-\beta}}\boldsymbol{R_{z,-\alpha}}$$

Axis/Angle Representation

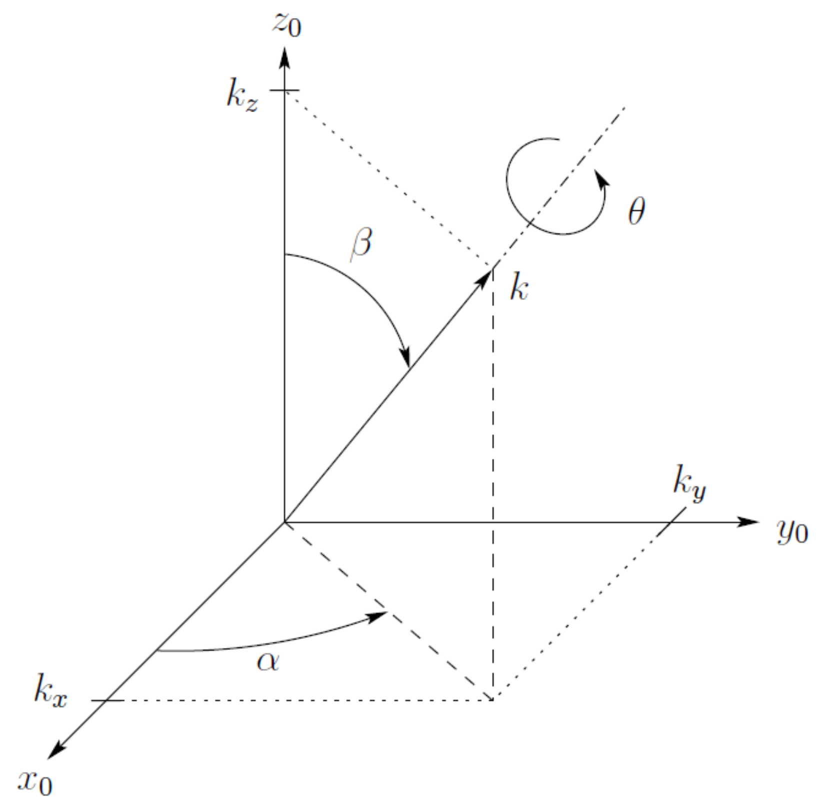
$$R_{k,\theta} = R_{z,\alpha} R_{y,\beta} R_{z,\theta} R_{y,-\beta} R_{z,-\alpha}$$

$$\sin\alpha = \frac{k_y}{\sqrt{k_x^2 + k_y^2}}$$

$$\cos\alpha = \frac{k_x}{\sqrt{k_x^2 + k_y^2}}$$

$$\sin\beta = \sqrt{k_x^2 + k_y^2}$$

$$\cos\beta = k_z$$



Axis/Angle Representation

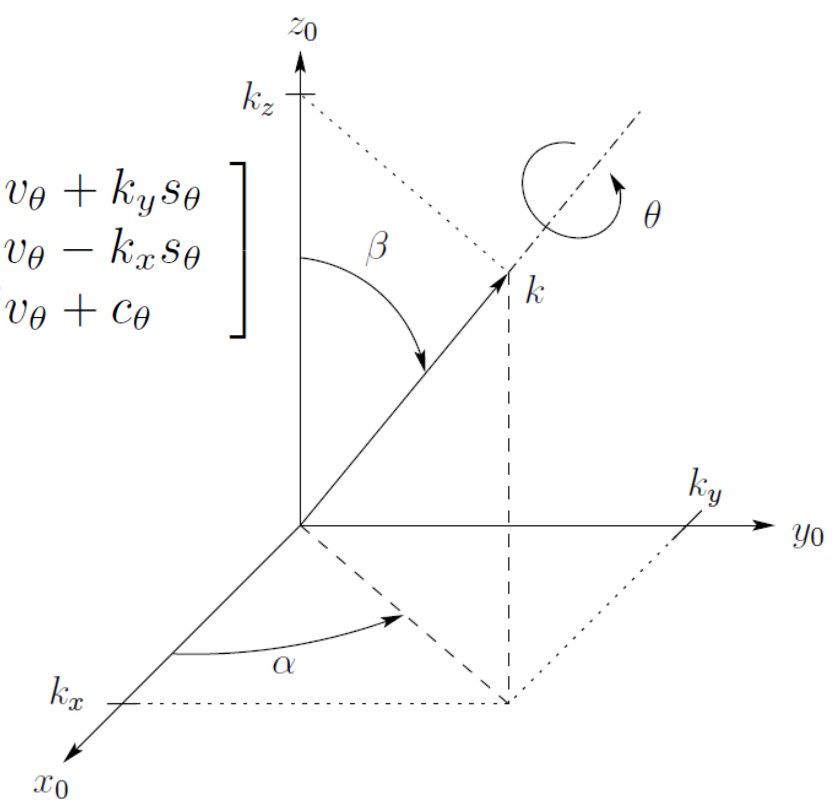
Rotations are not always performed about the principal coordinate axes. We are often interested in a rotation about an arbitrary axis in space. This provides both a convenient way to describe rotations, and an alternative parameterization for rotation matrices.

$$\boldsymbol{R}_{k,\theta} = \boldsymbol{R}_{z,\alpha}\,\boldsymbol{R}_{y,\beta}\boldsymbol{R}_{z,\theta}\boldsymbol{R}_{y,-\beta}\boldsymbol{R}_{z,-\alpha}$$

$$R_{k,\theta} = \left[\begin{array}{c|c|c} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{array}\right]$$

$$v_\theta = \text{ vers } \theta = 1 - c_\theta .$$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$



Axis/Angle Representation

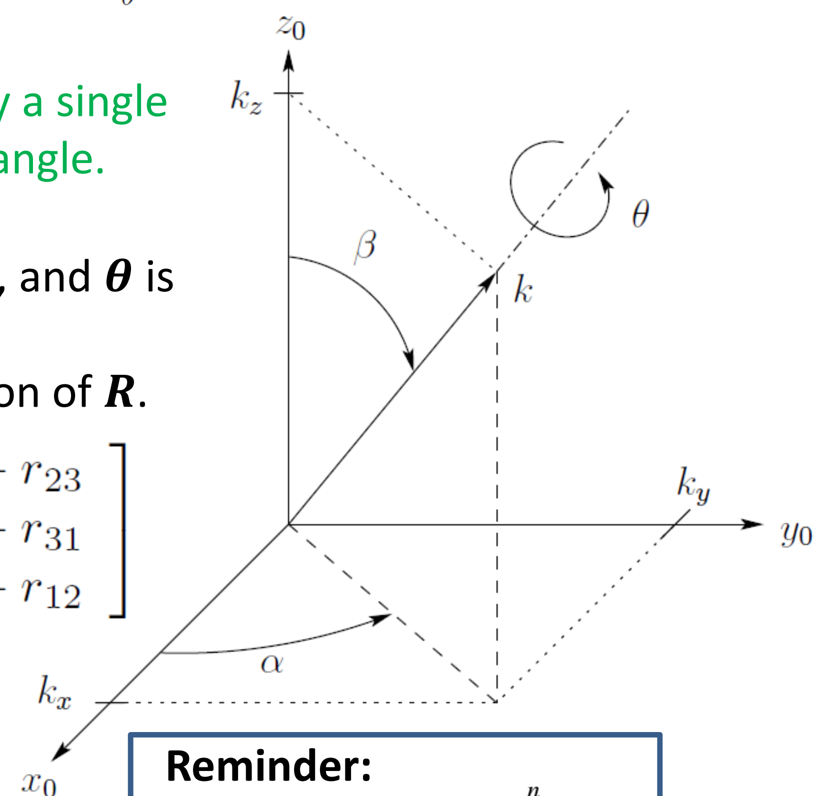
$$R_{k,\theta} = \left[\begin{array}{c|c|c} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{array}\right] \qquad R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$v_\theta = \text{vers}\,\theta = 1 - c_\theta.$$

Any rotation matrix $R \epsilon\, SO(3)$ can be represented by a single rotation about a suitable axis in space by a suitable angle.

$$\boldsymbol{R} = \boldsymbol{R_{k,\theta}}$$

where $\boldsymbol{k}$ is a unit vector defining the axis of rotation, and $\boldsymbol{\theta}$ is the angle of rotation about $\boldsymbol{k}$.
The matrix $\boldsymbol{R_{k,\theta}}$ is called the axis/angle representation of $\boldsymbol{R}$.



Given $\boldsymbol{R}$ find $\boldsymbol{\theta}$ and $\boldsymbol{k}$:

$$k = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{Tr(R) - 1}{2}\right) \\ &= \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)\end{aligned}$$

Reminder:

$$trace(\boldsymbol{A}) = tr(\boldsymbol{A}) = \sum_{i=1}^{n} a_{ii}$$

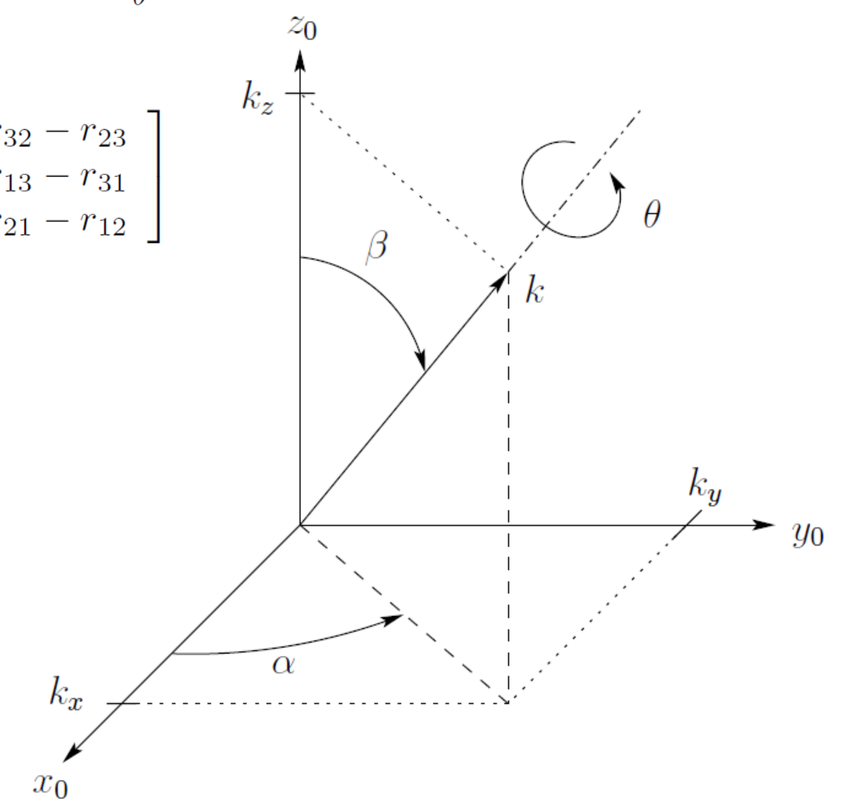
Axis/Angle Representation

$$R_{k,\theta} = \left[\begin{array}{c|c|c} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{array} \right]$$

$$v_\theta = \text{vers}\,\theta = 1 - c_\theta.$$

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{Tr(R) - 1}{2}\right) \\ &= \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right) \end{aligned}$$

$$k = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$



The axis/angle representation is not unique since a rotation of $-\boldsymbol{\theta}$ about $-\boldsymbol{k}$ is the same as a rotation of $\boldsymbol{\theta}$ about $\boldsymbol{k}$.

$$\boldsymbol{R_{k,\theta} = R_{-k,-\theta}}$$

If $\boldsymbol{\theta = 0}$ then $\boldsymbol{R}$ is the identity matrix and the axis of rotation is undefined.

Example

Suppose $\boldsymbol{R}$ is generated by a rotation of 90° about z_0 followed by a rotation of 30° about y_0 followed by a rotation of 60° about x_0. Find the axis/angle representation of $\boldsymbol{R}$

$$R = R_{x,60}R_{y,30}R_{z,90}$$

Reminder:
The axis/angle representation of $\boldsymbol{R}$

$$\theta = \cos^{-1}\left(\frac{Tr(R)-1}{2}\right) = \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right)$$

$$k = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}$$

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Suppose $\boldsymbol{R}$ is generated by a rotation of 90° about z_0 followed by a rotation of 30° about y_0 followed by a rotation of 60° about y_0. Find the axis/angle representation of $\boldsymbol{R}$

$$\begin{aligned} R &= R_{x,60}R_{y,30}R_{z,90} \\ &= \begin{bmatrix} 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\ \frac{\sqrt{3}}{2} & \frac{1}{4} & \frac{\sqrt{3}}{4} \end{bmatrix} \end{aligned}$$

$$Tr(R) =$$

$$\theta =$$

$$k =$$

Reminder:
The axis/angle representation of $\boldsymbol{R}$

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\end{aligned}
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\end{aligned}
$$

$$Tr(R) = 0$$

$$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = 120^{\circ}$$

$$k =$$

Reminder:
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\end{aligned}
$$

$$Tr(R) = 0$$

$$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = 120^{\circ}$$

$$k = \left(\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{3}} - \frac{1}{2}, \frac{1}{2\sqrt{3}} + \frac{1}{2}\right)^T$$

Reminder:
The axis/angle representation of $\boldsymbol{R}$

$$
\begin{aligned}
\theta &= \cos^{-1}\left(\frac{Tr(R)-1}{2}\right) \\
&= \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right) \\
k &= \frac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}
\end{aligned}
$$

Axis/Angle Representation

The above axis/angle representation characterizes a given rotation by four quantities, namely the three components of the equivalent axis $\boldsymbol{k}$ and the equivalent angle $\boldsymbol{\theta}$.

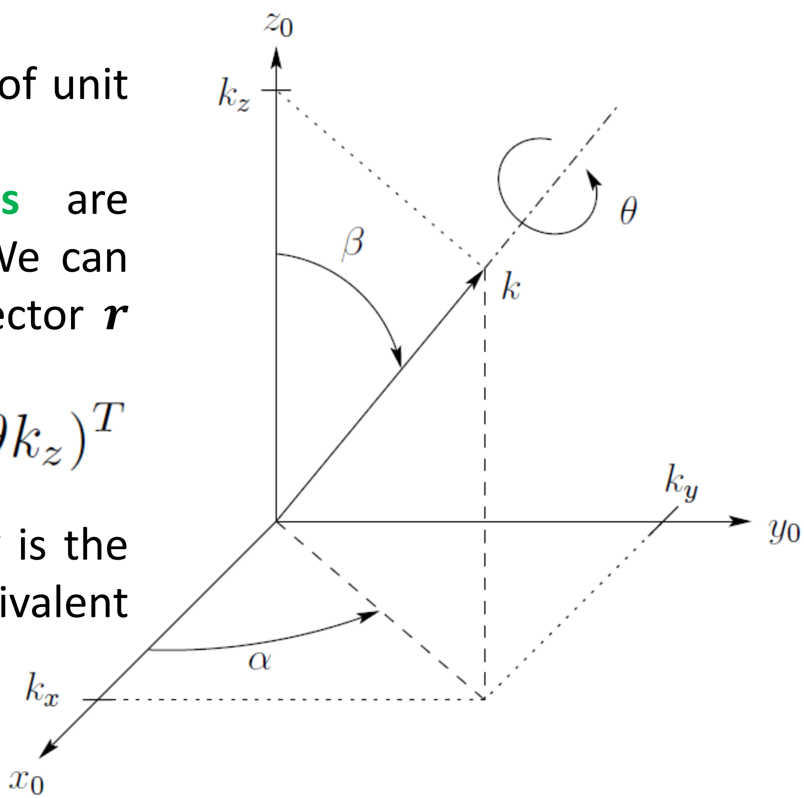
However, since the equivalent axis $\boldsymbol{k}$ is given as a unit vector only two of its components are independent.

The third is constrained by the condition that $\boldsymbol{k}$ is of unit length.

Therefore, only **three independent quantities** are required in this representation of a rotation $\boldsymbol{R}$. We can represent the equivalent axis/angle by a single vector $\boldsymbol{r}$ as:

$$r \quad = \quad (r_x, r_y, r_z)^T = (\theta k_x, \theta k_y, \theta k_z)^T$$

since $\boldsymbol{k}$ is a unit vector, the length of the vector $\boldsymbol{r}$ is the equivalent angle $\boldsymbol{\theta}$ and the direction of $\boldsymbol{r}$ is the equivalent axis $\boldsymbol{k}$.



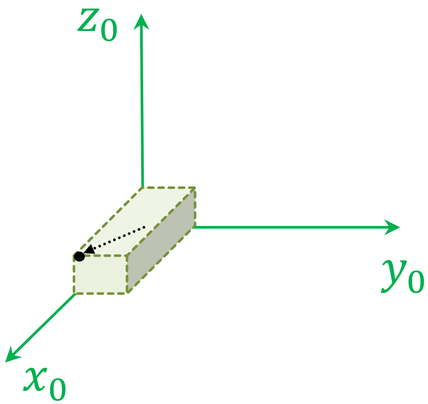
Rigid Motions

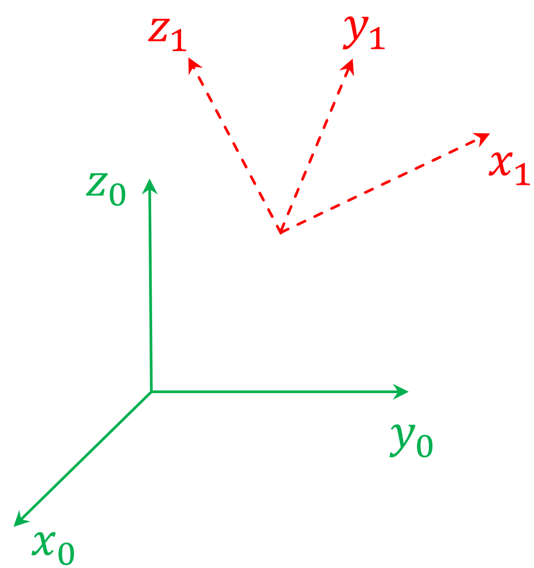
A rigid motion is a pure translation together with a pure rotation.

A rigid motion is an ordered pair $(\boldsymbol{d}, \boldsymbol{R})$ where $\boldsymbol{d} \in \mathbb{R}^{\mathbf{3}}$ and $\boldsymbol{R} \in \boldsymbol{SO}(\mathbf{3})$. The group of all rigid motions is known as the **Special Euclidean Group** and is denoted by $\boldsymbol{SE}(\mathbf{3})$. We see then that $\boldsymbol{SE}(\mathbf{3}) = \mathbb{R}^{\mathbf{3}} \times \boldsymbol{SO}(\mathbf{3})$.

One Rigid Motion

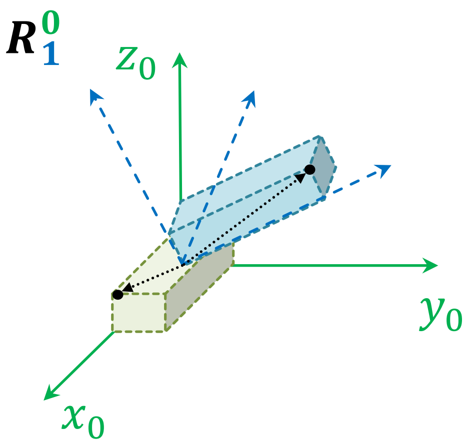
If frame $\boldsymbol{o_1\, x_1\, y_1\, z_1}$ is obtained from frame $\boldsymbol{o_0\, x_0\, y_0\, z_0}$ by first applying a rotation specified by $\boldsymbol{R_1^0}$

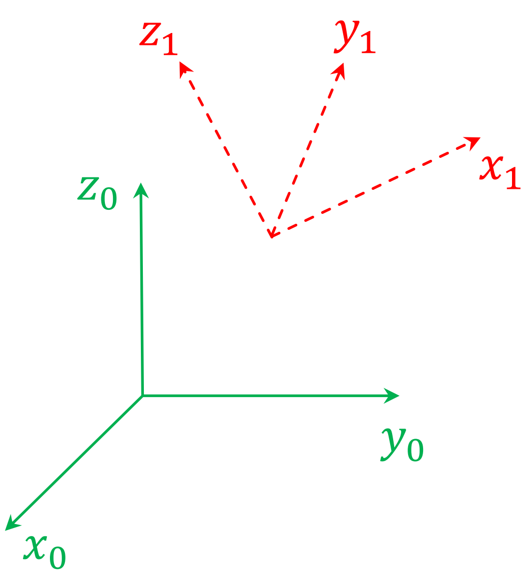




One Rigid Motion

If frame $o_1\ x_1\ y_1\ z_1$ is obtained from frame $o_0\ x_0\ y_0\ z_0$ by first applying a rotation specified by R_1^0 followed by a translation given (with respect to $o_0\ x_0\ y_0\ z_0$) by d_1^0





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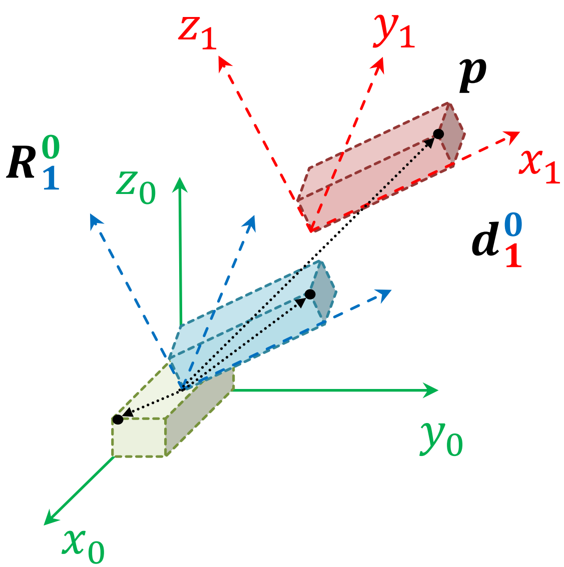
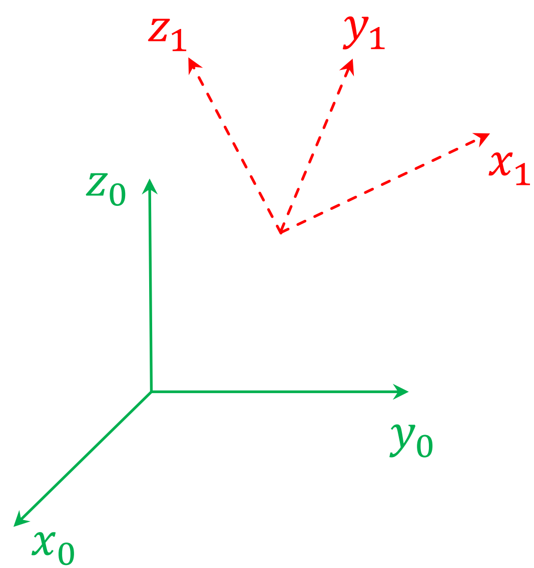
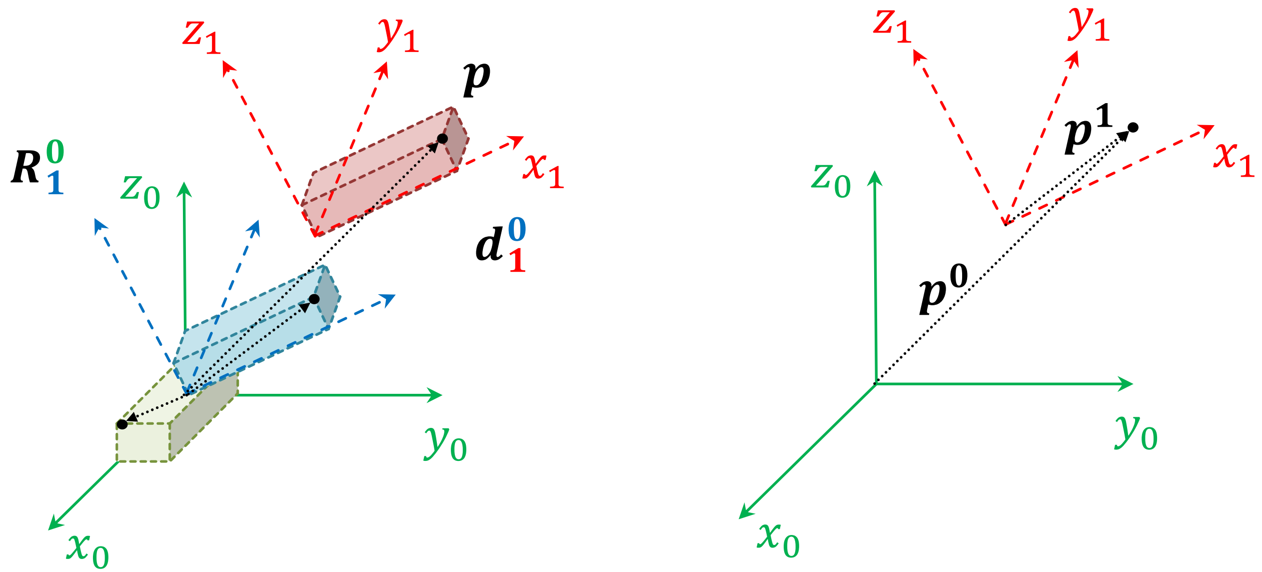




One Rigid Motion

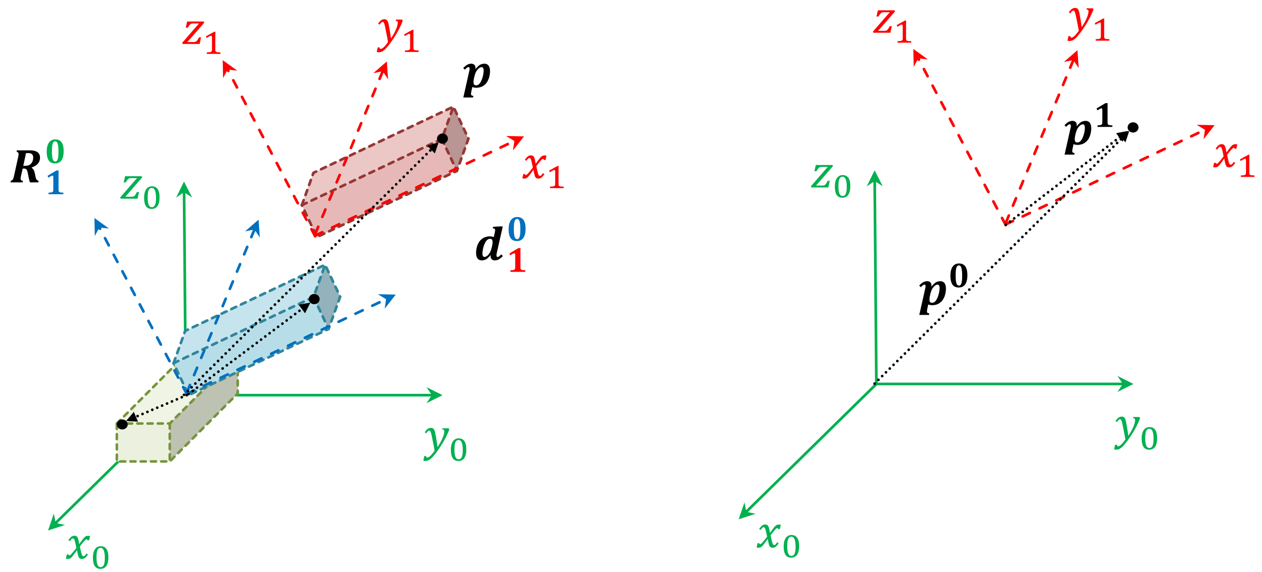
If frame $\boldsymbol{o_1\, x_1\, y_1\, z_1}$ is obtained from frame $\boldsymbol{o_0\, x_0\, y_0\, z_0}$ by first applying a rotation specified by $\boldsymbol{R_1^0}$ followed by a translation given (with respect to $\boldsymbol{o_0\, x_0\, y_0\, z_0}$) by $\boldsymbol{d_1^0}$, then the coordinates $\boldsymbol{p^0}$ are given by:



One Rigid Motion

If frame $o_1\, x_1\, y_1\, z_1$ is obtained from frame $o_0\, x_0\, y_0\, z_0$ by first applying a rotation specified by R_1^0 followed by a translation given (with respect to $o_0\, x_0\, y_0\, z_0$) by d_1^0 , then the coordinates p^0 are given by:

$$p^0 = R_1^0\; p^1 + d_1^0$$



One Rigid Motion

If frame $o_1\, x_1\, y_1\, z_1$ is obtained from frame $o_0\, x_0\, y_0\, z_0$ by first applying a rotation specified by R_1^0 followed by a translation given (with respect to $o_0\, x_0\, y_0\, z_0$) by d_1^0 , then the coordinates p^0 are given by:

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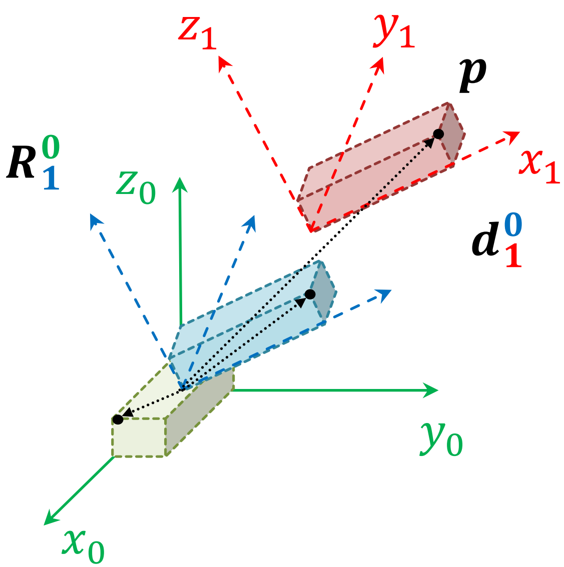
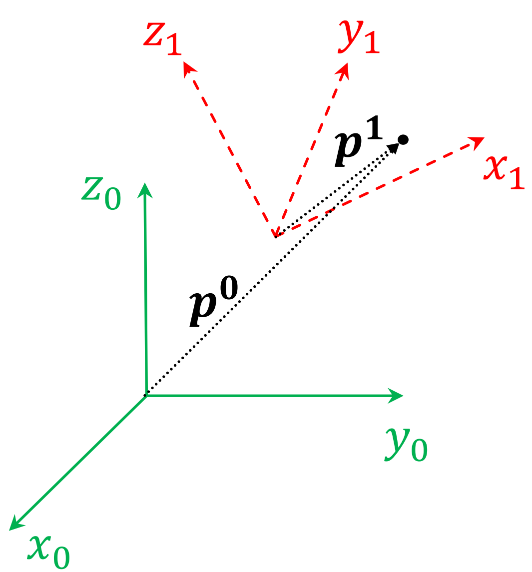
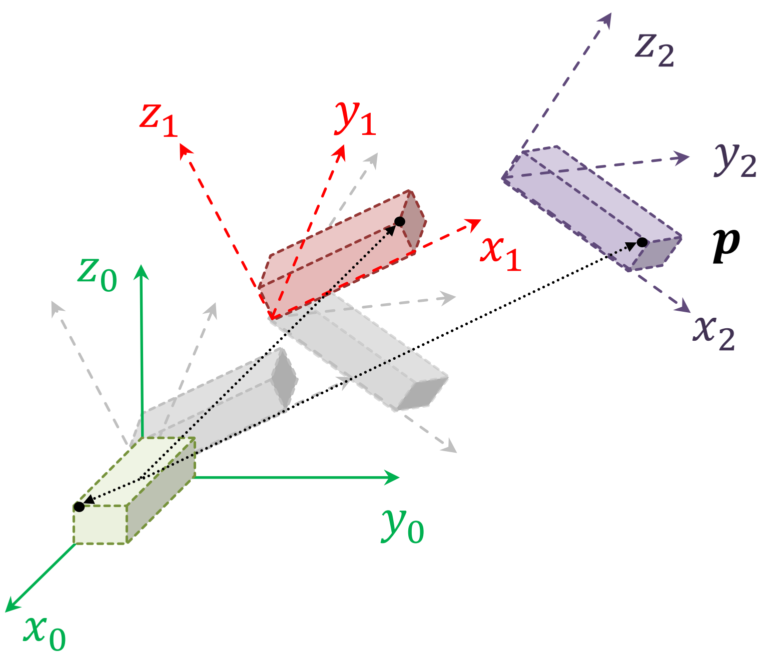




Two Rigid Motions

If frame $o_2\, x_2\, y_2\, z_2$ is obtained from frame $o_1\, x_1\, y_1\, z_1$ by first applying a rotation specified by R_2^1 followed by a translation given (with respect to $o_1\, x_1\, y_1\, z_1$) by d_2^1 .
If frame $o_1\, x_1\, y_1\, z_1$ is obtained from frame $o_0\, x_0\, y_0\, z_0$ by first applying a rotation specified by R_1^0 followed by a translation given (with respect to $o_0\, x_0\, y_0\, z_0$) by d_1^0 , find the coordinates p^0.



For the first rigid motion:

$$p^0 = R_1^0\, p^1 + d_1^0$$

For the second rigid motion:

$$p^1 = R_2^1\, p^2 + d_2^1$$

Both rigid motions can be described as one rigid motion:

$$p^0 = R_2^0\, p^2 + d_2^0$$

$$p^0 = R_1^0\, R_2^1\, p^2 + R_1^0\, d_2^1 + d_1^0$$

Two Rigid Motions

R_2^0 The orientation transformations can simply be multiplied together.

d_2^0 The translation transformation is the sum of:

- d_1^0 the vector from the origin o_0 to the origin o_1 expressed with respect to $o_0\ x_0\ y_0\ z_0$.
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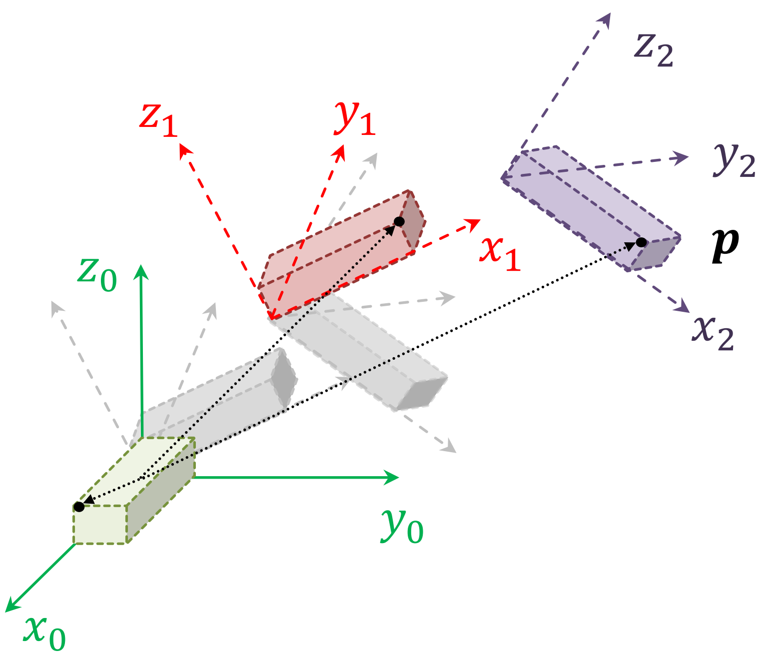
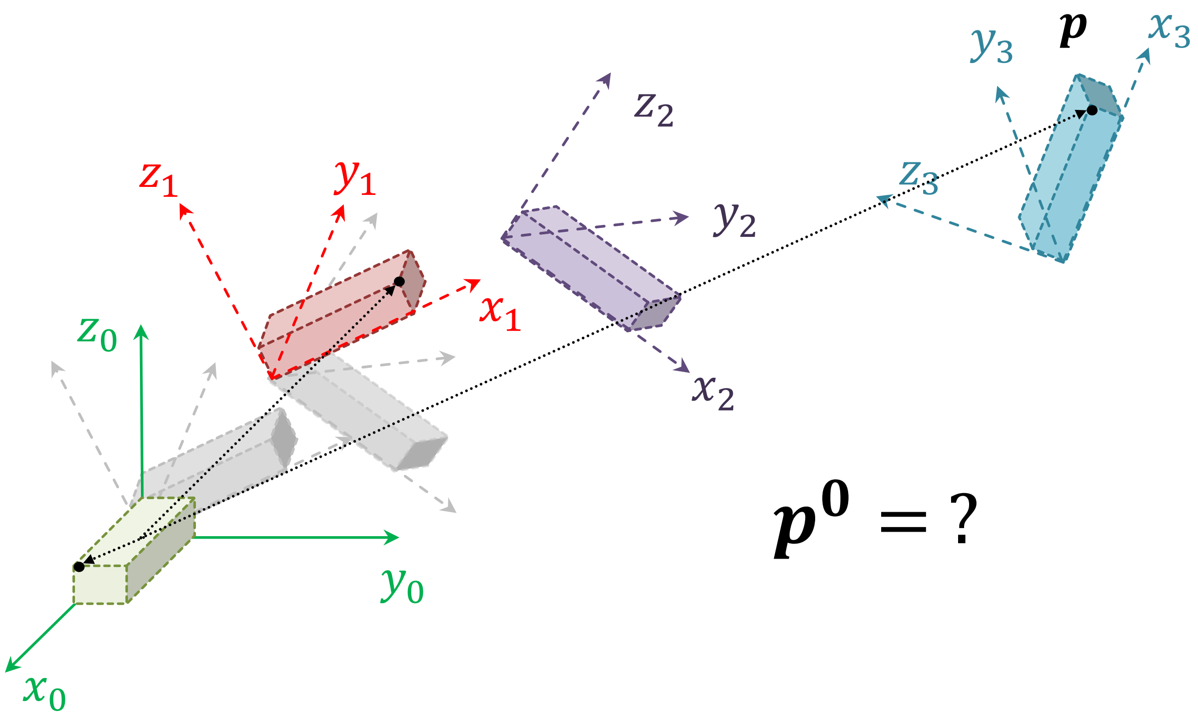


$$p^0 = R_1^0\ R_2^1\ p^2 + R_1^0\ d_2^1 + d_1^0$$

Three Rigid Motions

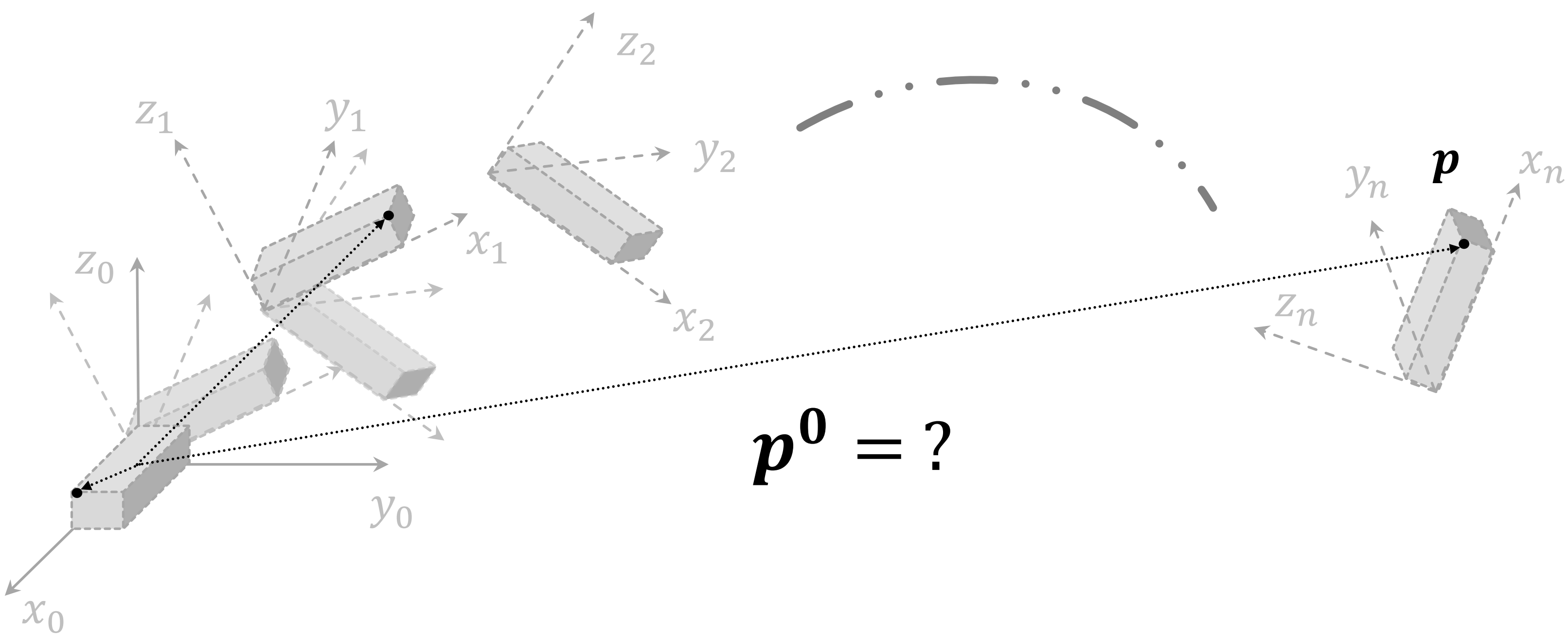


$$p^0 = ?$$

Homogeneous Transformations

A long sequence of rigid motions, find $\boldsymbol{p}^{\mathbf{0}}$.

$$\boldsymbol{p}^{\mathbf{0}} = \boldsymbol{R}_{\boldsymbol{n}}^{\mathbf{0}}\ \boldsymbol{p}^{\boldsymbol{n}} + \boldsymbol{d}_{\boldsymbol{n}}^{\mathbf{0}}$$

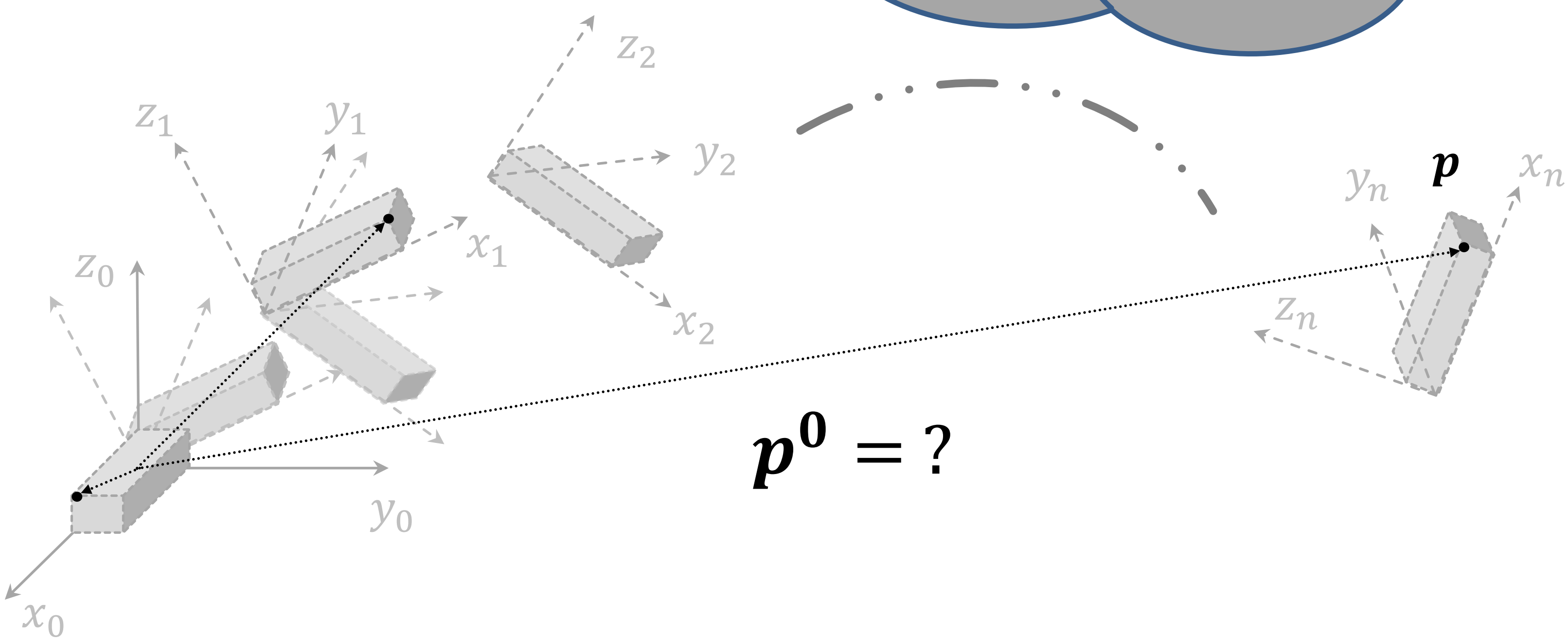


Homogeneous Transformations

A long sequence of rigid motions, find $\boldsymbol{p^0}$.

Represent rigid motions in matrix so that composition of rigid motions can be reduced to matrix multiplication as was the case for composition of rotations

$$\boldsymbol{p^0} = \boldsymbol{R_n^0}\ \boldsymbol{p^n} + \boldsymbol{d_n^0}$$



Homogeneous Transformations

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Represent rigid motions in matrix so that composition of rigid motions can be reduced to matrix multiplication as was the case for composition of rotations

$$\boldsymbol{p}^{\mathbf{0}} = \boldsymbol{R}_{n}^{\mathbf{0}}\ \boldsymbol{p}^{n} + \boldsymbol{d}_{n}^{\mathbf{0}}$$

$$\mathbf{H} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{d} \\ & \end{bmatrix};\ \boldsymbol{d} \in \mathbb{R}^{\mathbf{3}}, \boldsymbol{R} \in \boldsymbol{SO}(\mathbf{3})$$

Homogeneous Transformations

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Transformation matrices of the form **H** are called **homogeneous transformations**.

A **homogeneous transformation** is therefore a matrix representation of a rigid motion.

Homogeneous Transformations

A long sequence of rigid motions, find $\boldsymbol{p}^{\mathbf{0}}$.

$$\boldsymbol{p}^{\mathbf{0}} = \boldsymbol{R}_n^{\mathbf{0}}\ \boldsymbol{p}^{\boldsymbol{n}} + \boldsymbol{d}_n^{\mathbf{0}}$$

Represent rigid motions in matrix so that composition of rigid motions can be reduced to matrix multiplication as was the case for composition of rotations

$$\mathbf{H} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{d} \\ \mathbf{0} & \mathbf{1} \end{bmatrix};\ \boldsymbol{d} \in \mathbb{R}^{3}, \boldsymbol{R} \in \boldsymbol{SO}(3)$$

The inverse transformation $\boldsymbol{H}^{-1}$ is given by

$$\mathbf{H}^{-1} = \begin{bmatrix} \boldsymbol{R}^{T} & -\boldsymbol{R}^{T}\boldsymbol{d} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Ex. :Two Rigid Motions

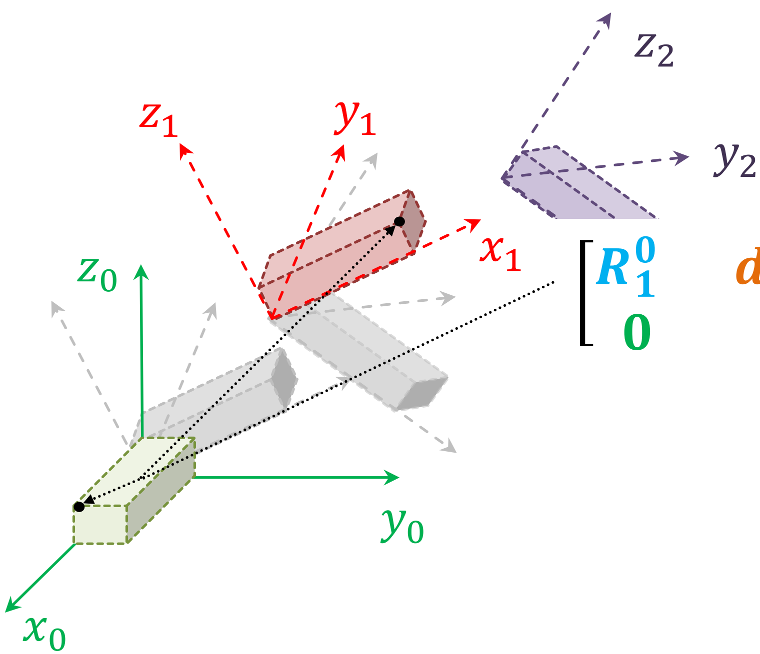
R_2^0 The orientation transformations can simply be multiplied together.

d_2^0 The translation transformation is the sum of:

- d_1^0 the vector from the origin o_0 to the origin o_1 expressed with respect to $o_0\ x_0\ y_0\ z_0$.
- $R_1^0\ d_2^1$ the vector from o_1 to o_2 expressed in the orientation of the coordinate system $o_0\ x_0\ y_0\ z_0$.

$$p^0 = R_1^0\ R_2^1\ p^2 + R_1^0\ d_2^1 + d_1^0$$

$$\mathbf{H} = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix};\ d \in \mathbb{R}^3, R \in SO(3)$$



$$\begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

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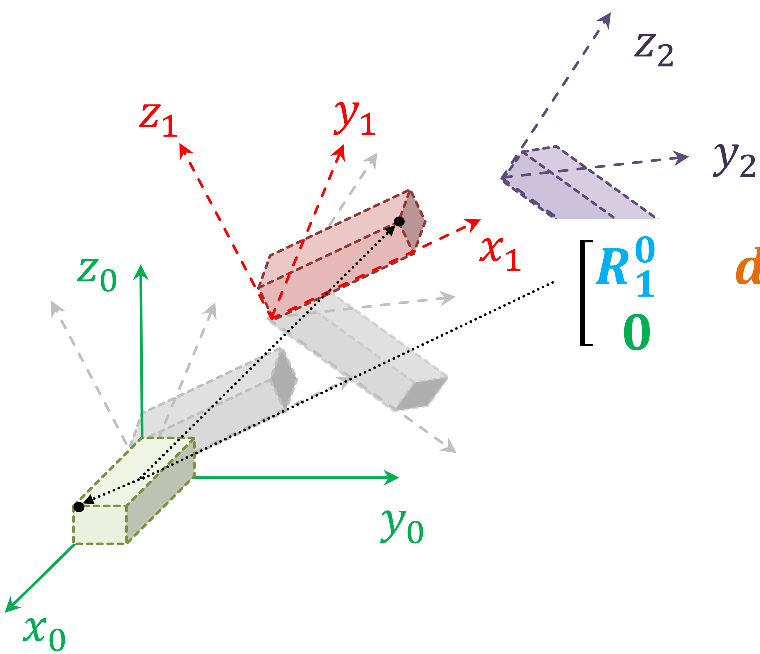

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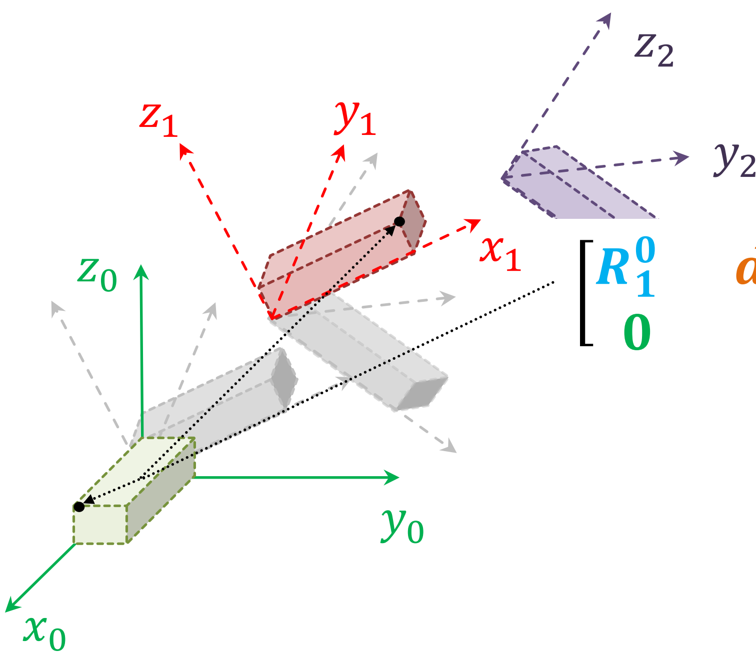
$$p^0 = R_1^0\ R_2^1\ p^2 + R_1^0\ d_2^1 + d_1^0$$

$$\mathbf{H} = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix};\ d \in \mathbb{R}^3, R \in SO(3)$$



$$\begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1^0\ R_2^1 & R_1^0\ d_2^1 + d_1^0 \\ 0 & 1 \end{bmatrix}$$

Ex. :Two Rigid Motions

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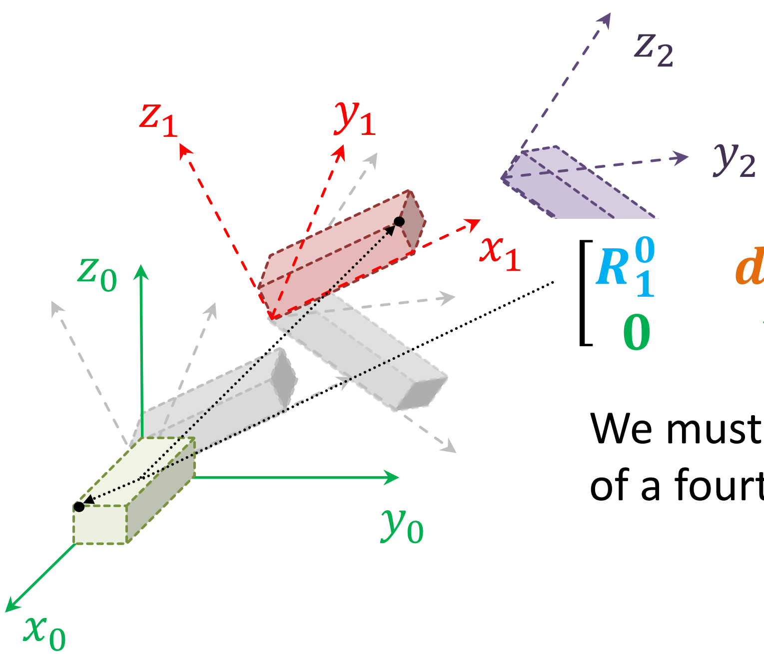
$$p^0 = R_1^0\ R_2^1\ p^2 + R_1^0\ d_2^1 + d_1^0$$

$$\mathbf{H} = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix};\ d \in \mathbb{R}^3, R \in SO(3)$$



$$\begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1^0\ R_2^1 & R_1^0\ d_2^1 + d_1^0 \\ 0 & 1 \end{bmatrix}$$

We must augment the vectors p^0, p^1 and p^2 by the addition of a fourth component of 1:

$$P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix}, P^1 = \begin{bmatrix} p^1 \\ 1 \end{bmatrix}, P^2 = \begin{bmatrix} p^2 \\ 1 \end{bmatrix}$$

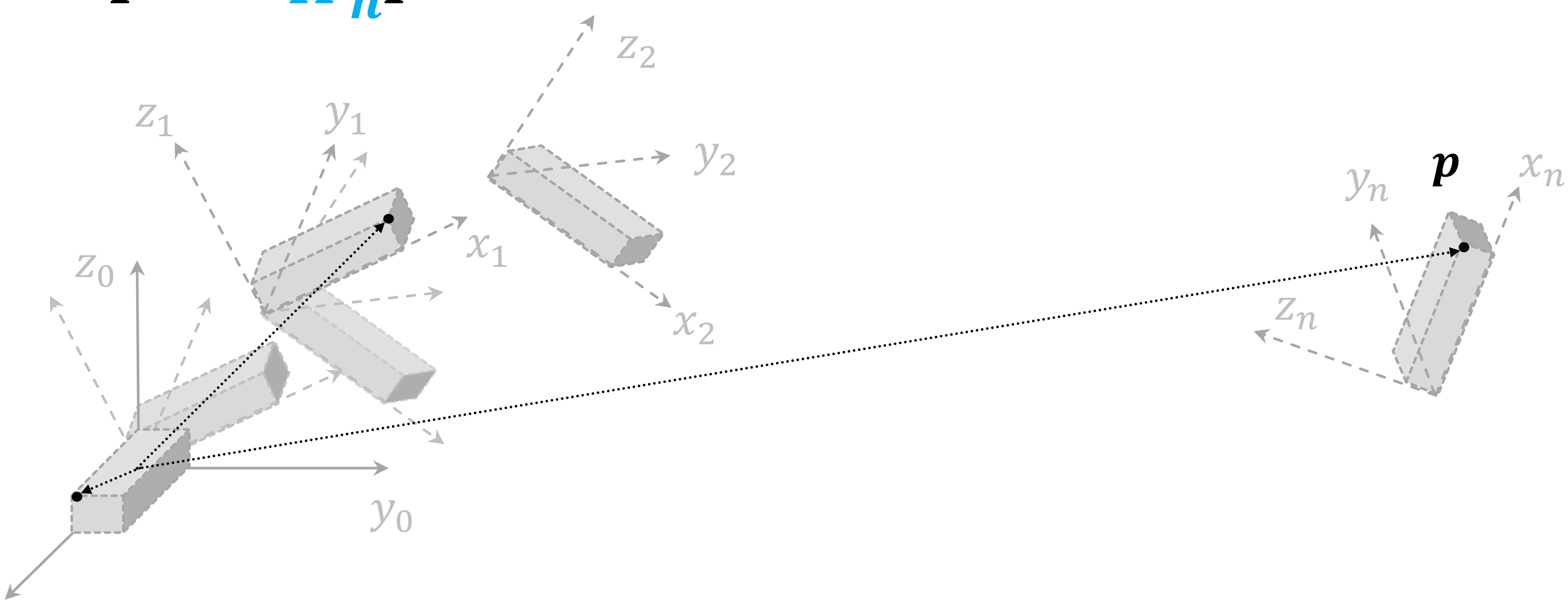
Homogeneous Transformations

$$P^0 = H_1^0 P^1$$
$$P^0 = H_2^0 P^2$$

.....

$$P^0 = H_n^0 P^n$$

$$P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix}$$



Basic Homogeneous Transformations

$$\text{Trans}_{x,a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad Rot_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Trans}_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad Rot_{y,\beta} = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Trans}_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad Rot_{z,\gamma} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Transformations

$$H_1^0 = \begin{bmatrix} n_x & s_x & a_x & d_x \\ n_y & s_y & a_y & d_y \\ n_z & s_z & a_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n & s & a & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\boldsymbol{n}$ is a vector representing the direction of $\boldsymbol{x_1}$ in the $\boldsymbol{o_0}$ $\boldsymbol{x_0}$ $\boldsymbol{y_0}$ $\boldsymbol{z_0}$ system

$\boldsymbol{s}$ is a vector representing the direction of $\boldsymbol{y_1}$ in the $\boldsymbol{o_0}$ $\boldsymbol{x_0}$ $\boldsymbol{y_0}$ $\boldsymbol{z_0}$ system

$\boldsymbol{a}$ is a vector representing the direction of $\boldsymbol{z_1}$ in the $\boldsymbol{o_0}$ $\boldsymbol{x_0}$ $\boldsymbol{y_0}$ $\boldsymbol{z_0}$ system

Composition Rule For Homogeneous Transformations

Given a homogeneous transformation $\boldsymbol{H_1^0}$ relating two frames, if a second rigid motion, represented by $\boldsymbol{H}$ is performed relative to the **current frame**, then:

$$H_2^0 = H_1^0 H$$

whereas if the second rigid motion is performed relative to the **fixed frame**, then:

$$H_2^0 = H H_1^0$$

Example

Find $\boldsymbol{H}$ for the following sequence of

1. a rotation by $\boldsymbol{\alpha}$ about the current $\boldsymbol{x-axis}$, followed by
2. a translation of $\boldsymbol{b}$ units along the current $\boldsymbol{x-axis}$, followed by
3. a translation of $\boldsymbol{d}$ units along the current $\boldsymbol{z-axis}$, followed by
4. a rotation by angle $\boldsymbol{\Theta}$ about the current $\boldsymbol{z-axis}$

$$H =$$

Reminder:

Transformation with respect to the **current** frame

$$\boldsymbol{H_2^0 = H_1^0\, H}$$

Transformation with respect to the **fixed** frame

$$\boldsymbol{H_2^0 = H\, H_1^0}$$

Example

Find $\boldsymbol{H}$ for the following sequence of

1. a rotation by $\boldsymbol{\alpha}$ about the current $\boldsymbol{x-axis}$, followed by
2. a translation of $\boldsymbol{b}$ units along the current $\boldsymbol{x-axis}$, followed by
3. a translation of $\boldsymbol{d}$ units along the current $\boldsymbol{z-axis}$, followed by
4. a rotation by angle $\boldsymbol{\Theta}$ about the current $\boldsymbol{z-axis}$

$$H = Rot_{x,\alpha}$$

Reminder:

Transformation with respect to the **current** frame

$$\boldsymbol{H_2^0 = H_1^0\, H}$$

Transformation with respect to the **fixed** frame

$$\boldsymbol{H_2^0 = H\, H_1^0}$$

Example

Find $\boldsymbol{H}$ for the following sequence of

1. a rotation by $\boldsymbol{\alpha}$ about the current $\boldsymbol{x-axis}$, followed by
2. a translation of $\boldsymbol{b}$ units along the current $\boldsymbol{x-axis}$, followed by
3. a translation of $\boldsymbol{d}$ units along the current $\boldsymbol{z-axis}$, followed by
4. a rotation by angle $\boldsymbol{\Theta}$ about the current $\boldsymbol{z-axis}$

$$H = Rot_{x,\alpha} Trans_{x,b}$$

Reminder:

Transformation with respect to the **current** frame

$$\boldsymbol{H_2^0 = H_1^0 H}$$

Transformation with respect to the **fixed** frame

$$\boldsymbol{H_2^0 = H H_1^0}$$

Example

Find $\boldsymbol{H}$ for the following sequence of

1. a rotation by $\boldsymbol{\alpha}$ about the current $\boldsymbol{x-axis}$, followed by
2. a translation of $\boldsymbol{b}$ units along the current $\boldsymbol{x-axis}$, followed by
3. a translation of $\boldsymbol{d}$ units along the current $\boldsymbol{z-axis}$, followed by
4. a rotation by angle $\boldsymbol{\Theta}$ about the current $\boldsymbol{z-axis}$

$$H = Rot_{x,\alpha}\, Trans_{x,b}\, Trans_{z,d}$$

Reminder:

Transformation with respect to the **current** frame

$$\boldsymbol{H_2^0 = H_1^0\, H}$$

Transformation with respect to the **fixed** frame

$$\boldsymbol{H_2^0 = H\, H_1^0}$$

Example

Find $\boldsymbol{H}$ for the following sequence of

1. a rotation by $\boldsymbol{\alpha}$ about the current $\boldsymbol{x-axis}$, followed by
2. a translation of $\boldsymbol{b}$ units along the current $\boldsymbol{x-axis}$, followed by
3. a translation of $\boldsymbol{d}$ units along the current $\boldsymbol{z-axis}$, followed by
4. a rotation by angle $\boldsymbol{\Theta}$ about the current $\boldsymbol{z-axis}$

$$H = Rot_{x,\alpha} Trans_{x,b} Trans_{z,d} Rot_{z,\theta}$$

Reminder:

Transformation with respect to the **current** frame

$$\boldsymbol{H_2^0 = H_1^0 H}$$

Transformation with respect to the **fixed** frame

$$\boldsymbol{H_2^0 = H H_1^0}$$

Example

Find $\boldsymbol{H}$ for the following sequence of

1. a rotation by $\boldsymbol{\alpha}$ about the current $\boldsymbol{x-axis}$, followed by
2. a translation of $\boldsymbol{b}$ units along the current $\boldsymbol{x-axis}$, followed by
3. a translation of $\boldsymbol{d}$ units along the current $\boldsymbol{z-axis}$, followed by
4. a rotation by angle $\boldsymbol{\Theta}$ about the current $\boldsymbol{z-axis}$

$$H = Rot_{x,\alpha} Trans_{x,b} Trans_{z,d} Rot_{z,\theta}$$

$$= \begin{bmatrix} c_\theta & -s_\theta & 0 & b \\ c_\alpha s_\theta & c_\alpha c_\theta & -s_\alpha & -ds_\alpha \\ s_\alpha s_\theta & s_\alpha c_\theta & c_\alpha & dc_\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Reminder:

Transformation with respect to the **current** frame

$$\boldsymbol{H}_2^0 = \boldsymbol{H}_1^0 \boldsymbol{H}$$

Transformation with respect to the **fixed** frame

$$\boldsymbol{H}_2^0 = \boldsymbol{H} \boldsymbol{H}_1^0$$

Example

Find the homogeneous transformations $\boldsymbol{H_1^0}$, $\boldsymbol{H_2^0}$, $\boldsymbol{H_2^1}$ representing the transformations among the three frames Shown. Show that $\boldsymbol{H_2^0} = \boldsymbol{H_1^0}\,\boldsymbol{H_2^1}$.