Inverse Kinematics

Serial link manipulators

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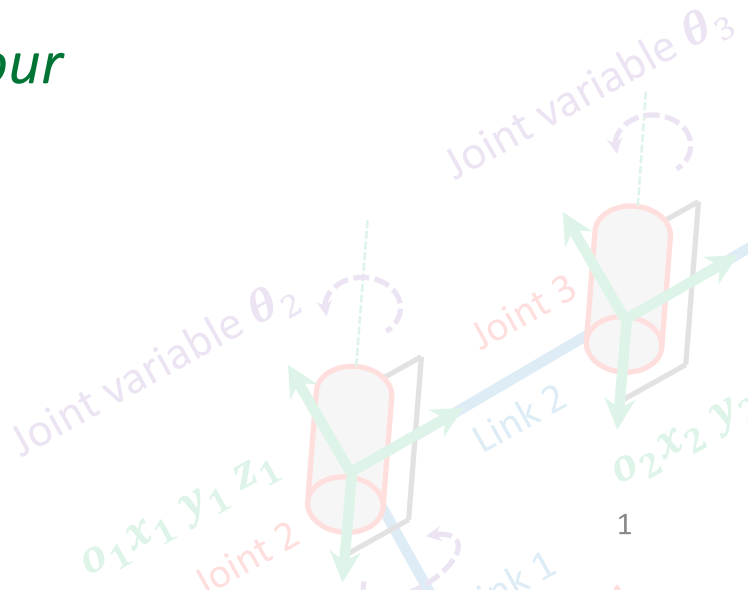


Inverse Kinematics

Given a desired position and orientation of the end-effector work out the joint variables which can bring the robot to the desired configuration.

Given $\mathbf{H} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{d} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$

and a manipulator with $\boldsymbol{n}$ joints

find the joint variables $\boldsymbol{q_1}, \dots \boldsymbol{q_n}$

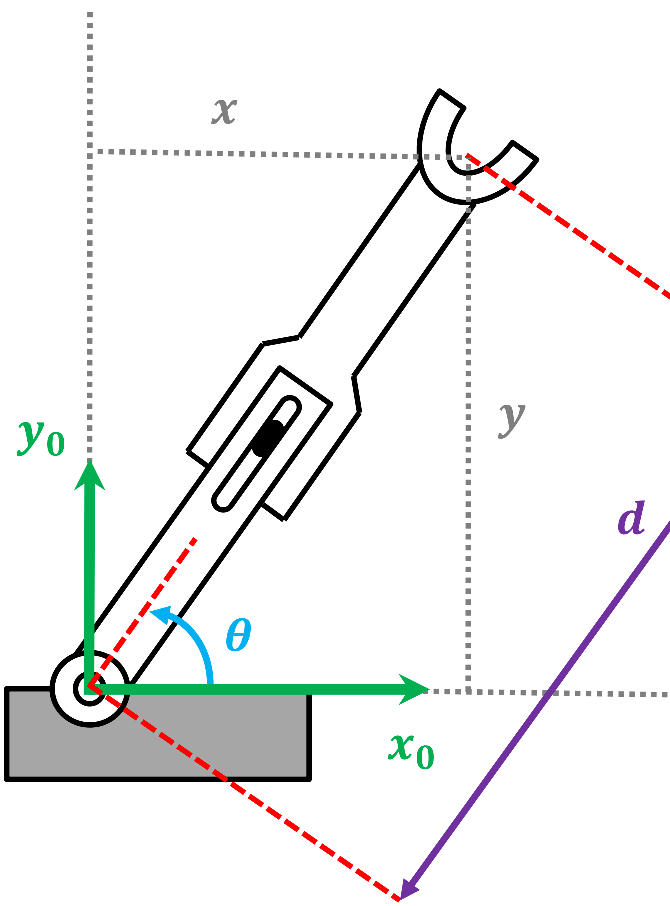
Such that:

$$\boldsymbol{T_6^0}\,(\,\boldsymbol{q_1}, \dots \boldsymbol{q_n}) = \mathbf{H}$$

12 nonlinear equations with *n* unknown

A Simple Example

Given a desired position of the end-effector work out the joint variables $q_i = (\theta, d)$



$$\theta = arctan2(\frac{y}{x})$$

To specify that it is in the first quadrant

$$d = \sqrt{(x^2 + y^2)}$$

Two equations with two unknown

Easy to solve

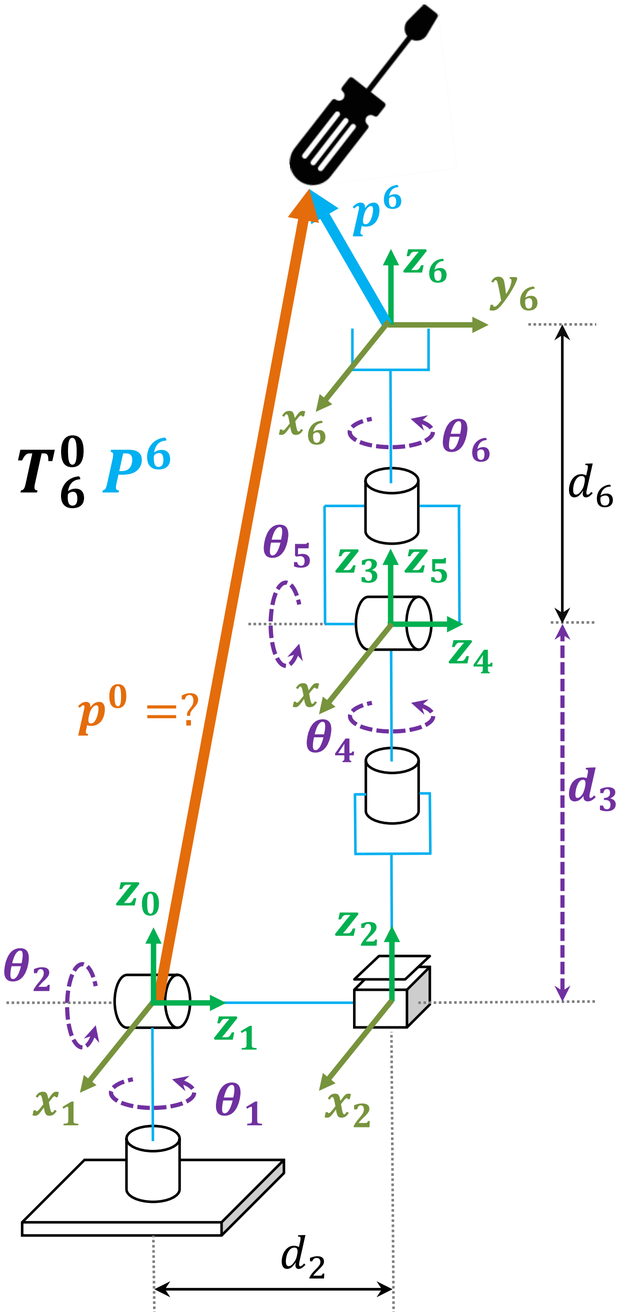
Stanford Arm

$\boldsymbol{a_i}$ is distance from $\boldsymbol{z_{i-1}}$ to $\boldsymbol{z_i}$ measured along $\boldsymbol{x_i}$.
$\boldsymbol{\alpha_i}$ is angle from $\boldsymbol{z_{i-1}}$ to $\boldsymbol{z_i}$ measured about $\boldsymbol{x_i}$.
$\boldsymbol{d_i}$ is distance from $\boldsymbol{x_{i-1}}$ to $\boldsymbol{x_i}$ measured along $\boldsymbol{z_{i-1}}$.
$\boldsymbol{\theta_i}$ is angle from $\boldsymbol{x_{i-1}}$ to $\boldsymbol{x_i}$ measured about $\boldsymbol{z_{i-1}}$.

i	a_i	α_i	d_i	θ_i
1	0	-90°	0	θ_1 *
2	0	$+90^\circ$	d_2	θ_2 *
3	0	0°	d_3 *	0
4	0	-90°	0	θ_4 *
5	0	$+90^\circ$	0	θ_5 *
6	0	0°	d_6	θ_6 *

$$T_6^0 = A_1A_2A_3A_4A_5A_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^0 = T_6^0\, P^6$$



Stanford Arm

i	a_i	α_i	d_i	θ_i
1	0	-90°	0	θ_1 *
2	0	$+90^\circ$	d_2	θ_2 *
3	0	0°	d_3 *	0
4	0	-90°	0	θ_4 *
5	0	$+90^\circ$	0	θ_5 *
6	0	0°	d_6	θ_6 *

Reminder: A_i

$$\begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Stanford Arm

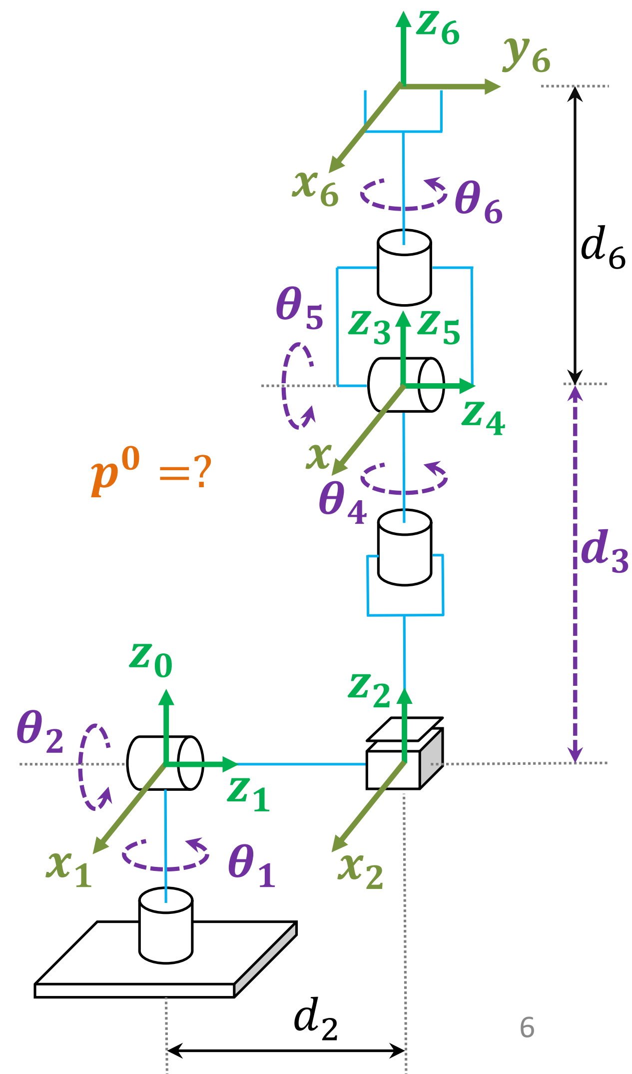
$$T_6^0 = A_1A_2A_3A_4A_5A_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$r_{11} = -c_6(c_5(s_1s_4 - c_1c_2c_4) + c_1s_2s_5) - s_6(c_4s_1 + c_1c_2s_4)$$
$$r_{21} = c_6(c_5(c_1s_4 + c_2c_4s_1) - s_1s_2s_5) + s_6(c_1c_4 - c_2s_1s_4)$$
$$r_{31} = s_2s_4s_6 - c_6(c_2s_5 + c_4c_5s_2)$$

$$r_{12} = s_6(c_5(s_1s_4 - c_1c_2c_4) + c_1s_2s_5) - c_6(c_4s_1 + c_1c_2s_4)$$
$$r_{22} = c_6(c_1c_4 - c_2s_1s_4) - s_6(c_5(c_1s_4 + c_2c_4s_1) - s_1s_2s_5)$$
$$r_{32} = s_6(c_2s_5 + c_4c_5s_2) + c_6s_2s_4$$

$$r_{13} = c_1c_5s_2 - s_5(s_1s_4 - c_1c_2c_4)$$
$$r_{23} = s_5(c_1s_4 + c_2c_4s_1) + c_5s_1s_2$$
$$r_{33} = c_2c_5 - c_4s_2s_5$$

$$d_x = d_3c_1s_2 - d_6(s_5(s_1s_4 - c_1c_2c_4) - c_1c_5s_2) - d_2s_1$$
$$d_y = d_2c_1 + d_6(s_5(c_1s_4 + c_2c_4s_1) + c_5s_1s_2) + d_3s_1s_2$$
$$d_z = d_6(c_2c_5 - c_4s_2s_5) + d_3c_2$$



Solvability

Given the numerical values $r_{11}, r_{12}, \dots\ r_{33}, dx, d_y, dz$
Find out $\theta_1, \theta_2, d_3, \theta_4, \theta_5, \theta_6$

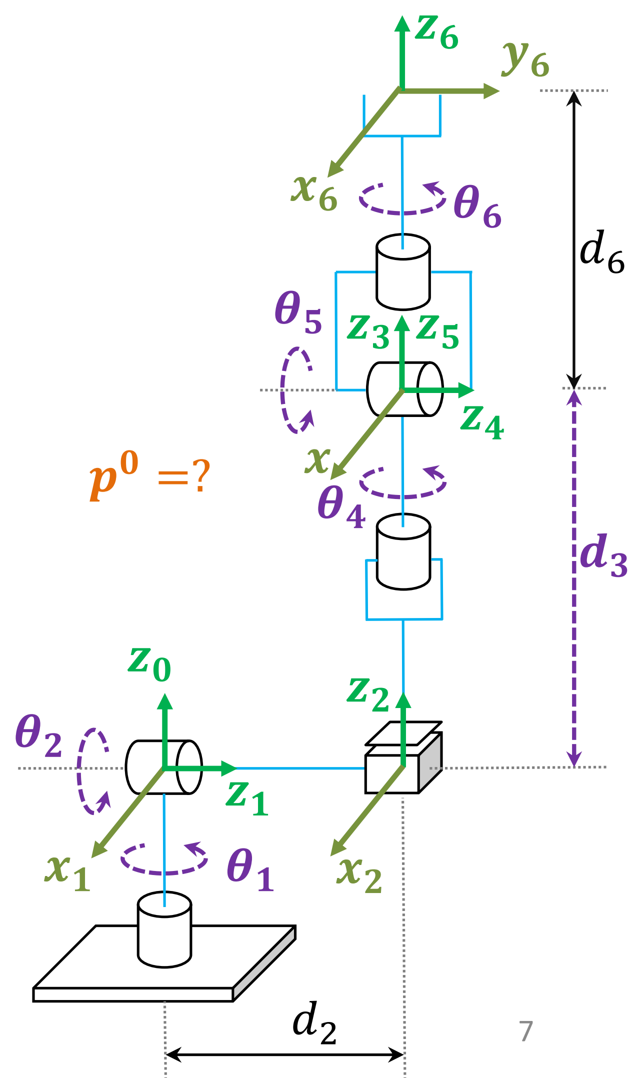
$$T_6^0 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

12 nonlinear equations and 6 unknowns

6 independent equations and 6 unknowns

Nonlinear
Transcendental

Difficult to solve



Solvability

We are concerned with:

- The existence of solutions
- Uniqueness/multiplicity of solutions
- Solution methods

The Workspace

Volume of space which can be reached by the end effector

Primary workspace $\boldsymbol{WS_1}$:
Set of all positions $\boldsymbol{p}$ that can be reached with at least one orientation $\boldsymbol{R}$

- No solution out of $\boldsymbol{WS_1}$.
- For $\boldsymbol{p} \in \boldsymbol{WS_1}$ there is at least one solution for a suitable $\boldsymbol{R}$

Secondary (dexterous) workspace $\boldsymbol{WS_2}$:
Set of all positions $\boldsymbol{p}$ that can be reached with any orientation.
Volume of space where the end effector can be arbitrarily oriented.

- For $\boldsymbol{p} \in \boldsymbol{WS_2}$ there is at least one solution for a any suitable $\boldsymbol{R}$

$$\boldsymbol{WS_2} \subseteq \boldsymbol{WS_1}$$

Singularity

A singularity is a configuration of a serial manipulator in which the joint parameters no longer completely define the position and orientation of the end-effector.

Singularities occur in configurations when joint axes align in a way that reduces the ability of the arm to position the end-effector.

For example when a serial manipulator is fully extended it is in what is known as the boundary singularity.

At a singularity the end-effector **loses** one or more degrees of freedom (instantaneously, the end-effector cannot move in these directions).

The Workspace For Planar RR Arm

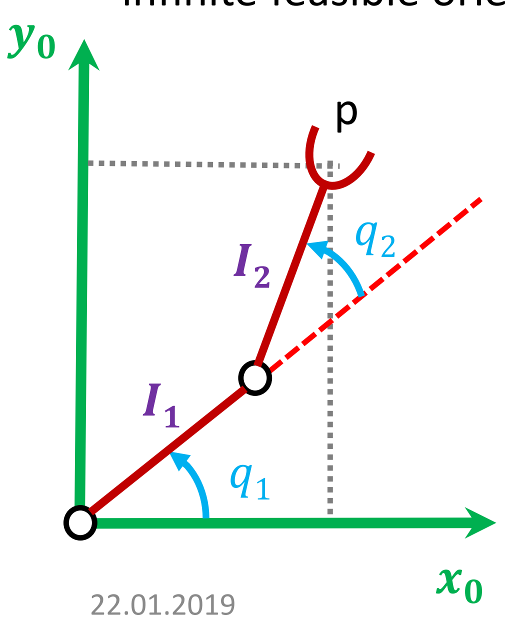
Workspace:

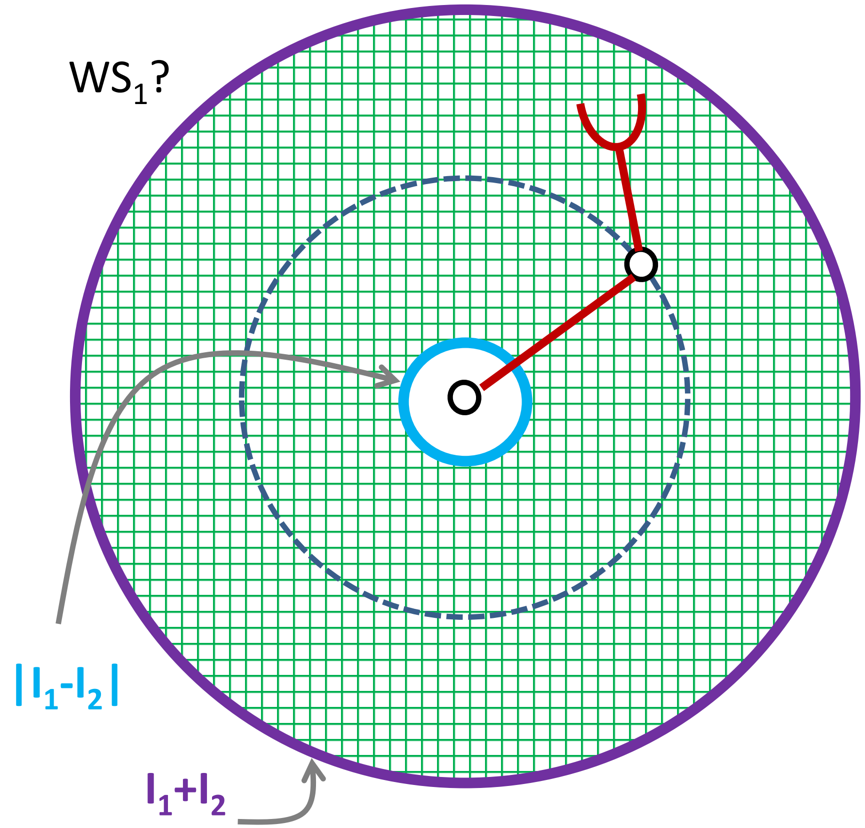
If $l_1 \neq l_2$

- $WS_1 = \{p \in R^2 : |l_1 - l_2| \leq \| p \| \leq l_1 + l_2 \}$
- $WS_2 = \emptyset$

If $l_1 = l_2 = \ell$

- $WS_1 = \{p \in R^2 : \| p \| \leq 2\ell \}$
- $WS_2 = \{p = 0\}$
  infinite feasible orientations at the origin





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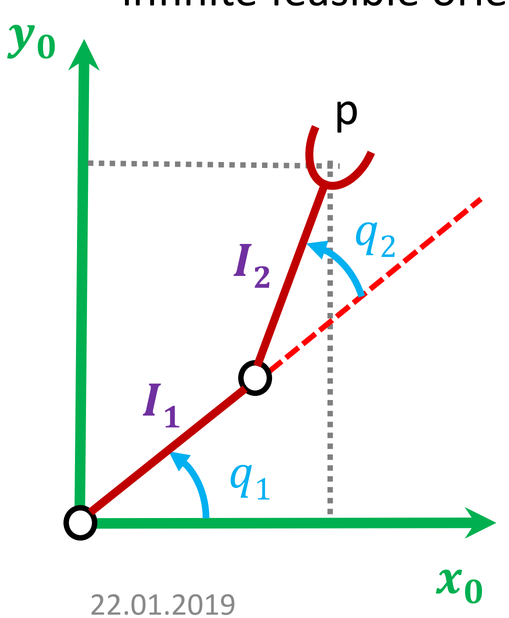
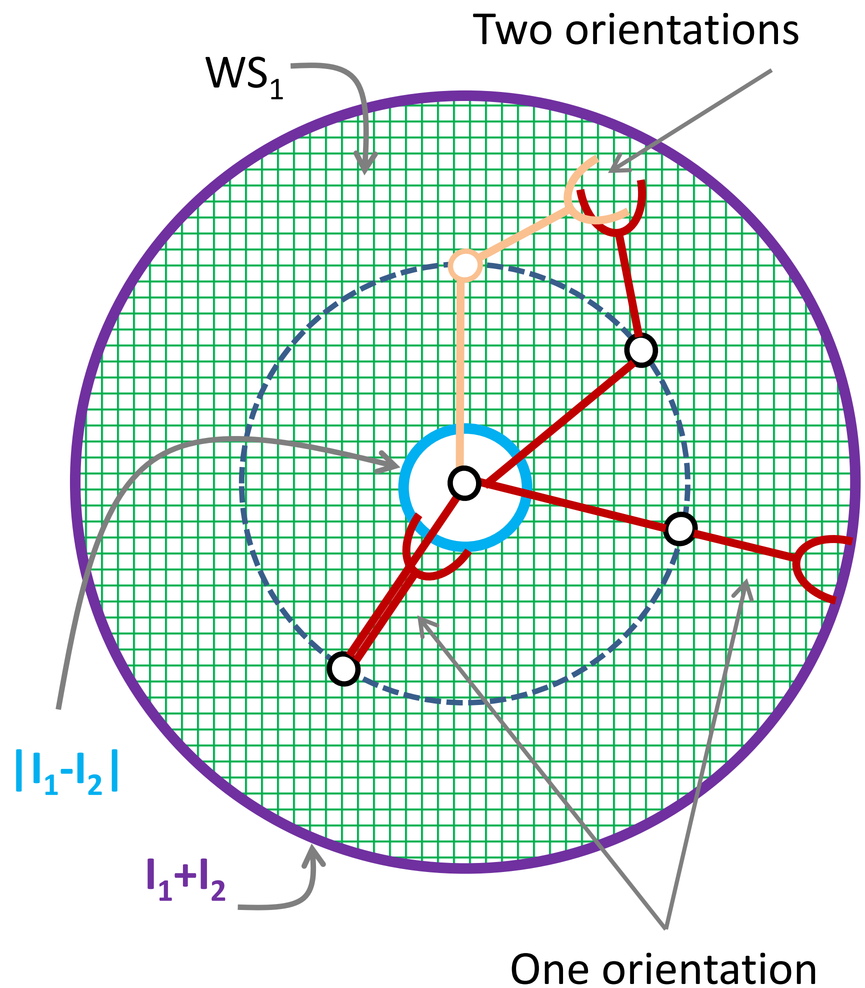
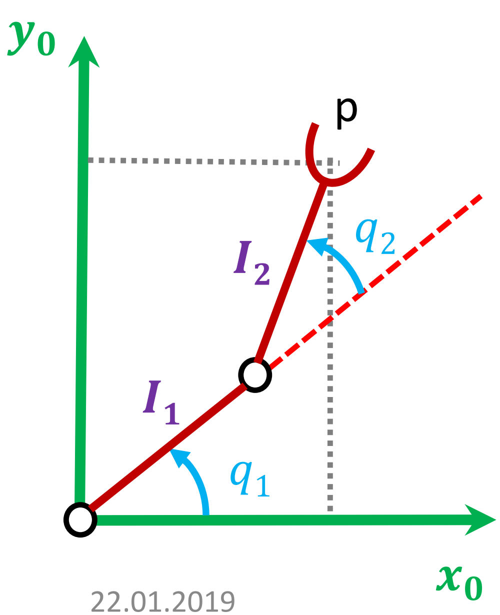


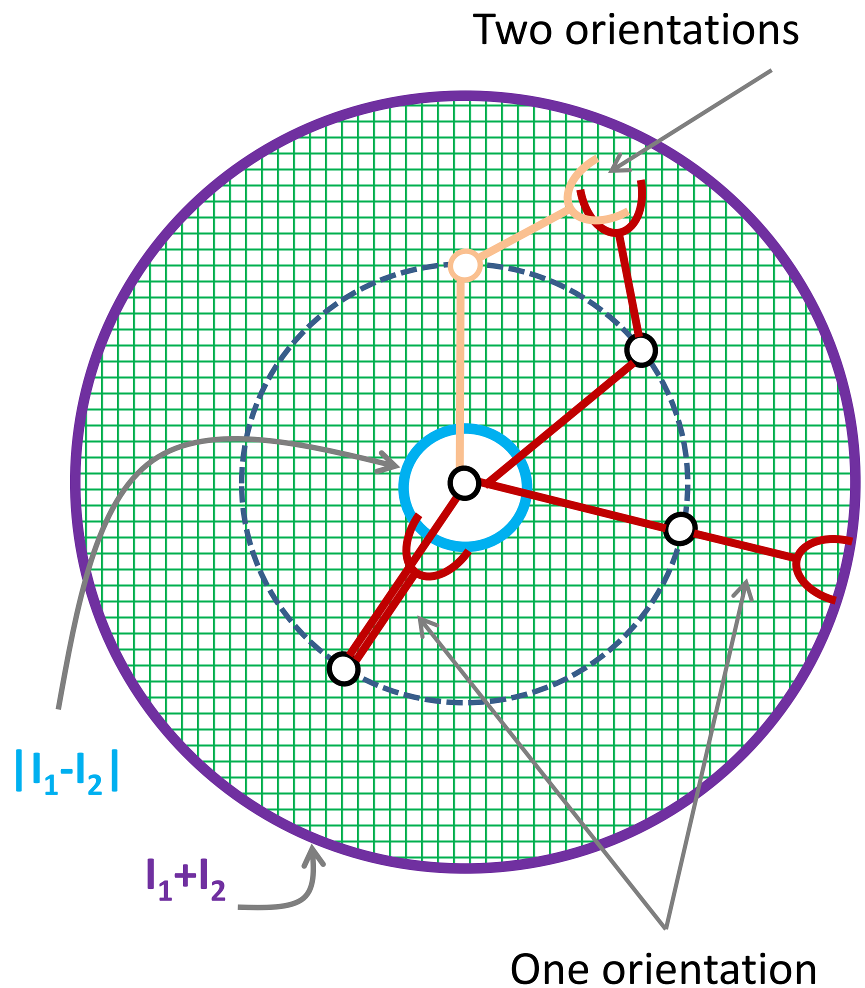


The Workspace For Planar RR Arm

Number of solution:
(End-effector positions)

- 2 solution in WS_1
- 1 solution on WS_1 limit.
- ∞ solutions in WS_2



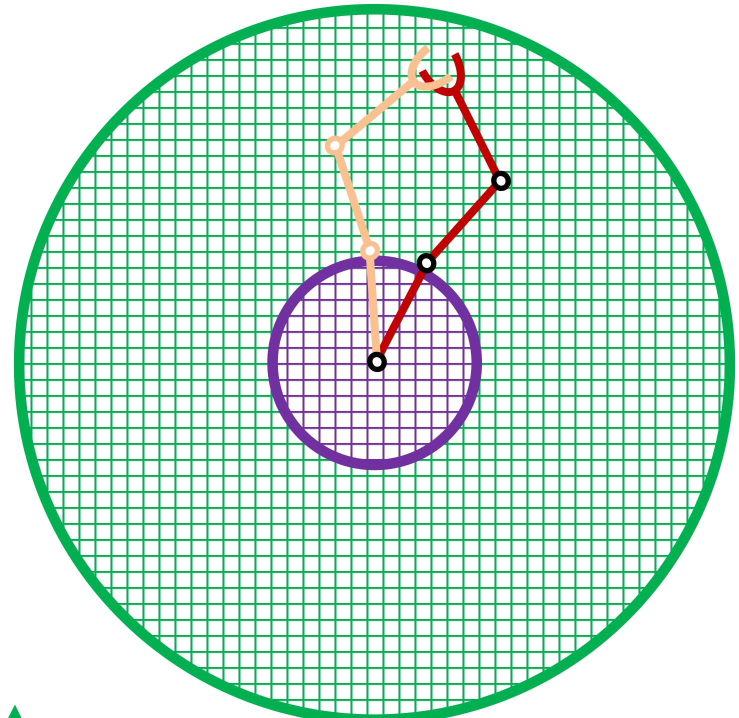


The Workspace For Planar 3R Arm

Workspace:

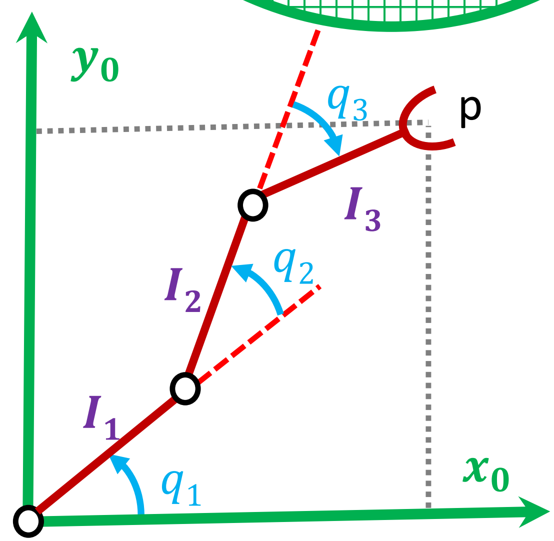
If ($l_1 = l_2 = l_3 = \ell$)

- $WS_1 = \{p \in R^2 : \| p \| \leq 3\,\ell\}$
- $WS_2 = \{p \in R^2 : \| p \| \leq \ell\}$

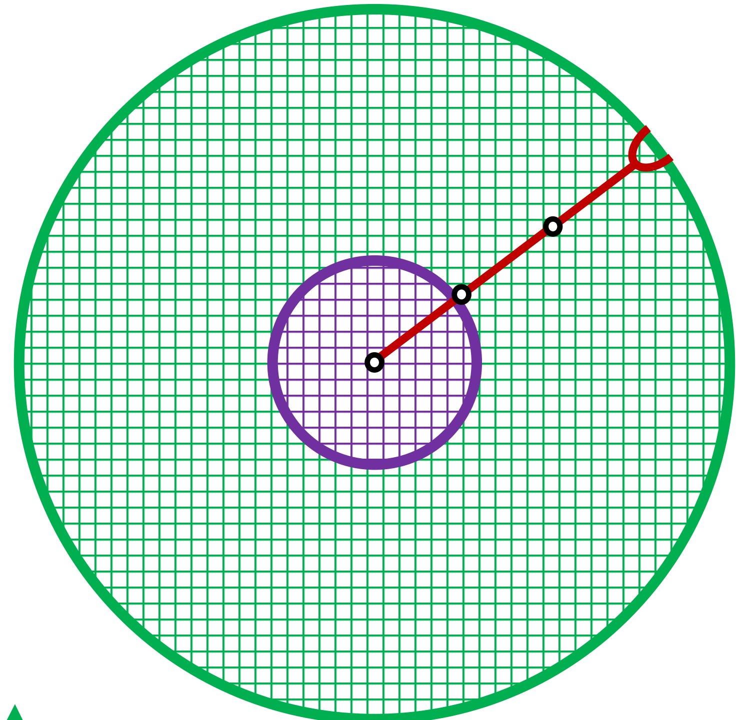


Number of solution:

(End-effector positions)



The Workspace For Planar 3R Arm



Workspace:

If ($l_1=l_2=l_3=\ell$)

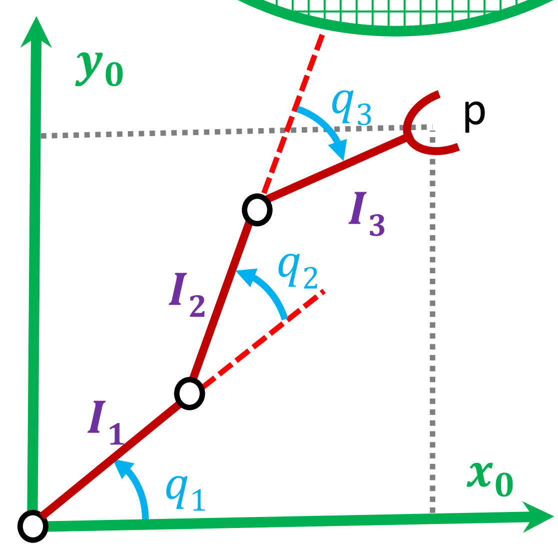
- $WS_1 = \{p \in R^2 : \| p \| \leq 3\,\ell\}$
- $WS_2 = \{p \in R^2 : \| p \| \leq \ell\}$

Number of solution:

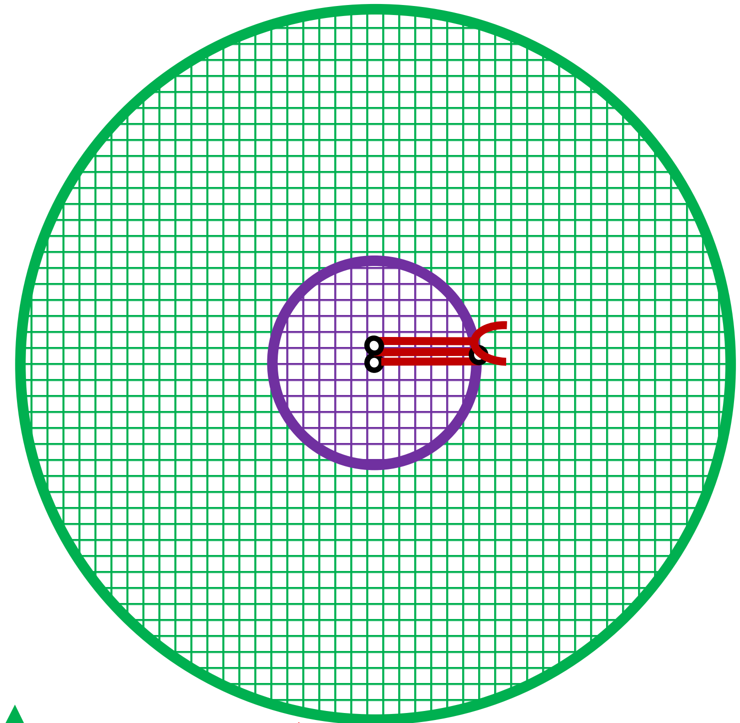
(End-effector positions)

In WS_1

- If $\| p \| = 3\ell$: one **singular** solution



The Workspace For Planar 3R Arm



Workspace:

If ($l_1=l_2=l_3=\ell$)

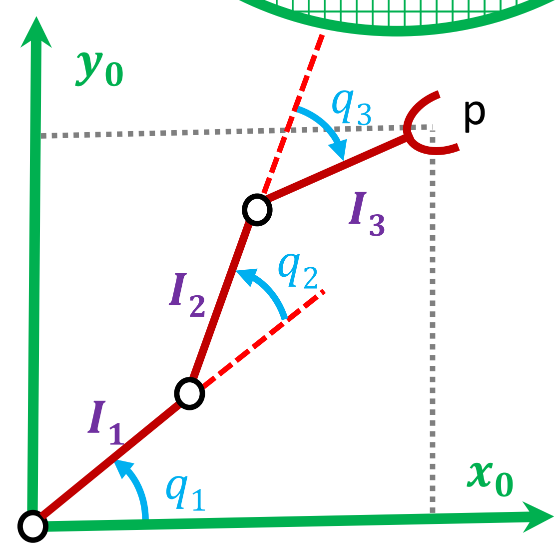
- $WS_1 = \{p \in R^2 : \| p \| \leq 3\ \ell\ \}$
- $WS_2 = \{p \in R^2 : \| p \| \leq\ \ell\ \}$

Number of solution:

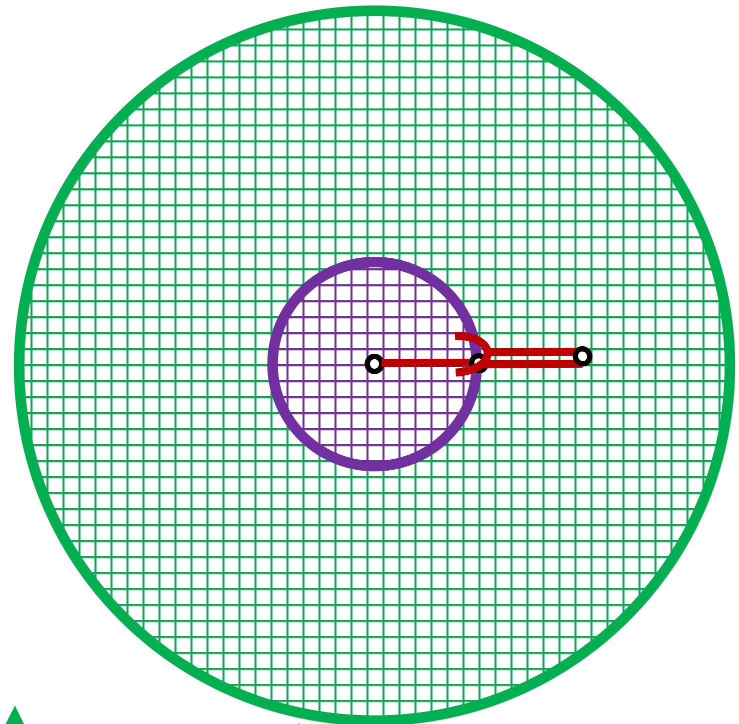
(End-effector positions)

In WS_1

- If $\| p \| = 3\ell$: one **singular** solution
- If $\| p \| = \ell$: ∞ solutions, with 3 **singular** solutions



The Workspace For Planar 3R Arm



Workspace:

If ($l_1=l_2=l_3=\ell$)

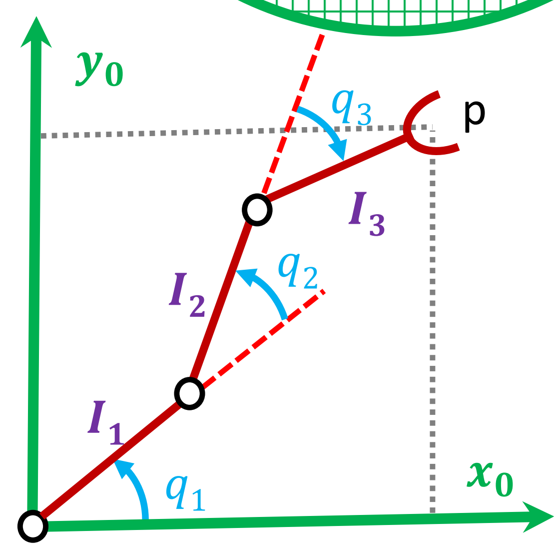
- $WS_1 = \{p \epsilon R^2 : \| p \| \leq 3\ \ell\ \}$
- $WS_2 = \{p \epsilon R^2 : \| p \| \leq \ \ell\ \}$

Number of solution:

(End-effector positions)

In WS_1

- If $\| p \| = 3\ell$: one **singular** solution
- If $\| p \| = \ell$: ∞ solutions, with 3 **singular** solutions



The Workspace For Planar 3R Arm

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If ($l_1=l_2=l_3=\ell$)

- $WS_1 = \{p \in R^2 : \| p \| \leq 3\ \ell\ \}$
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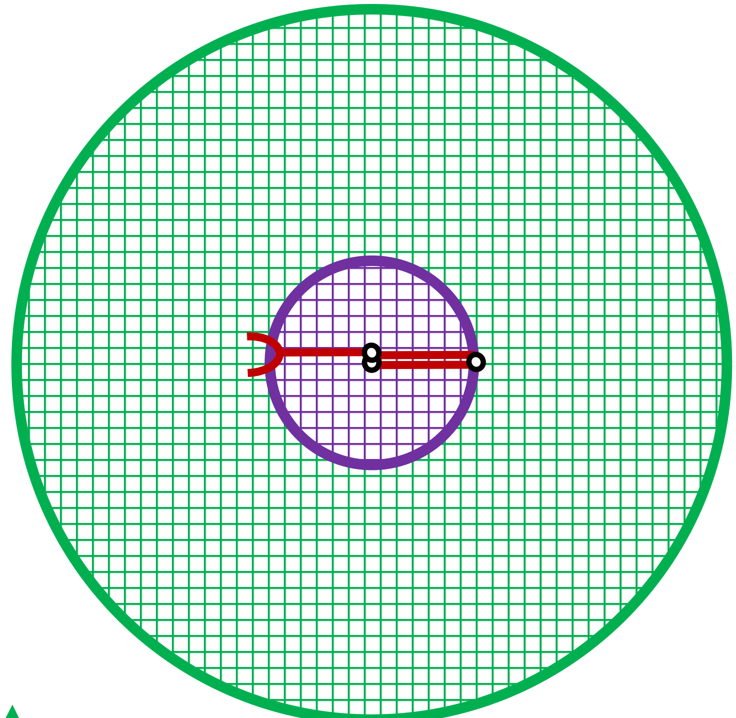

Number of solution:

(End-effector positions)

In WS_1

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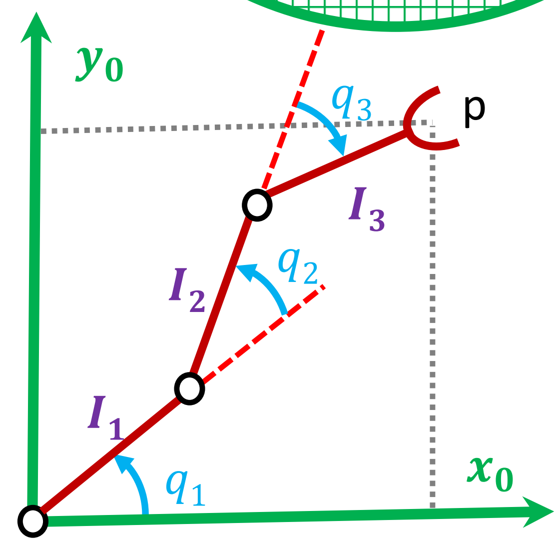

The Workspace For Planar 3R Arm

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If ($l_1=l_2=l_3=\ell$)

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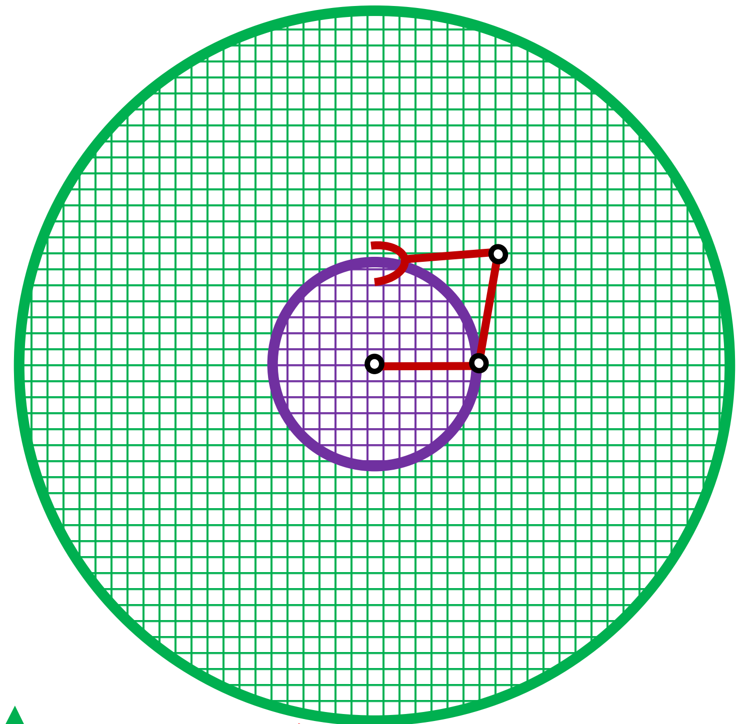
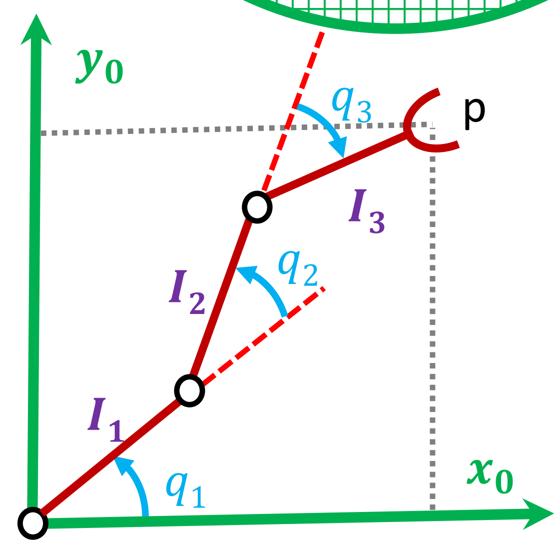


Number of solution:

(End-effector positions)

In WS_1

- If $\| p \| = 3\ell$: one **singular** solution
- If $\| p \| = \ell$: ∞ solutions, with 3 **singular** solutions
- Else ∞ **regular** solutions

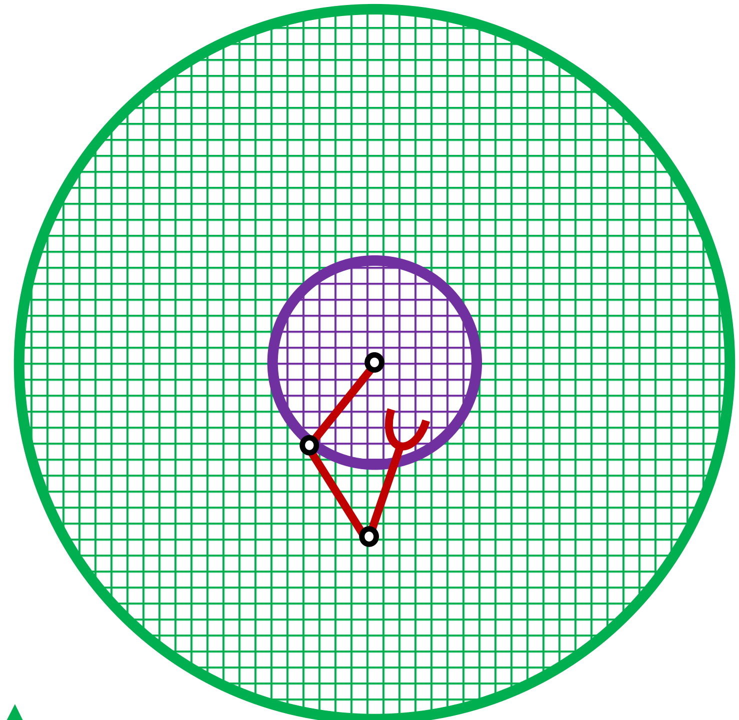


The Workspace For Planar 3R Arm

Workspace:

If ($l_1=l_2=l_3=\ell$)

- $WS_1 = \{p \epsilon R^2 : \| p \| \leq 3\ \ell\ \}$
- $WS_2 = \{p \epsilon R^2 : \| p \| \leq\ \ell\ \}$



Number of solution:

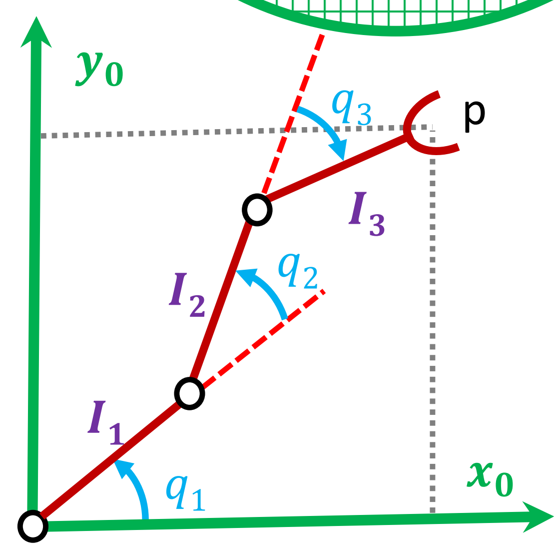
(End-effector positions)

In $\boldsymbol{WS_1}$

- If $\| p \| = 3\ell$: one **singular** solution
- If $\| p \| = \ell$: ∞ solutions, with 3 **singular** solutions
- Else ∞ **regular** solutions

In $\boldsymbol{WS_2}$

- ∞ **regular** solutions (**never singular**)

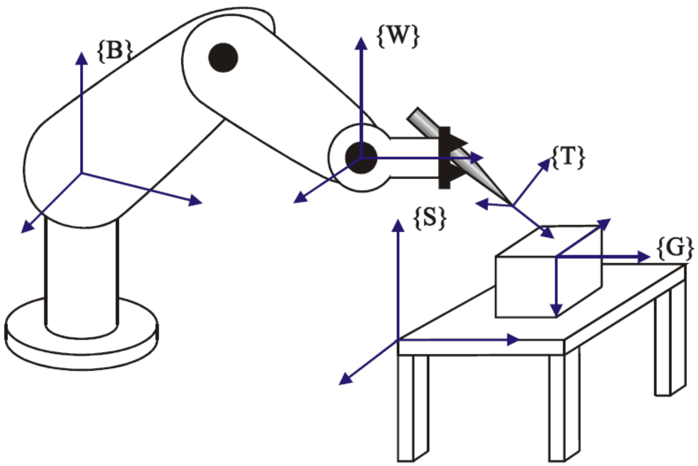


Tool-frame Transformation

Workspace also depends on the tool-frame transformation, because it is usually the tool-tip that is discussed when we speak about reachable points in space.

Generally, the tool-frame transformation is performed independently of the manipulator kinematics and inverse kinematics, so **we are often led to consider the workspace of the wrist frame, {W}**.

For a given end-effector, a tool frame, {T}, is defined.



Given a goal frame, {G}, the corresponding wrist frame {W} is calculated, and then we ask:

Does this desired position and orientation of {W} lie in the workspace?

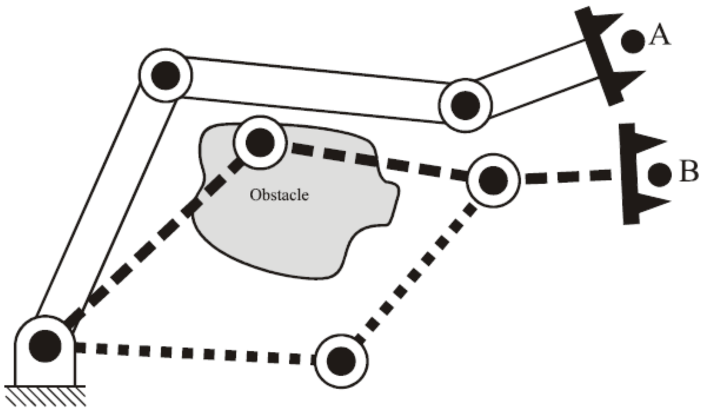
In this way, the workspace that we must concern ourselves with, is different from the one imagined by the user, who is concerned with the workspace of the end-effector (the {T} frame).

Multiple Solutions

A planar arm with three revolute joints has a large dexterous workspace in the plane, because any position in the interior of its workspace can be reached with any orientation.

The lower dashed lines indicate a second possible configuration in which the same end-effector position and orientation is achieved.

The fact, that a manipulator has multiple solutions, can cause problems, because the system has to be able to choose one. A very reasonable choice would be the closest solution.



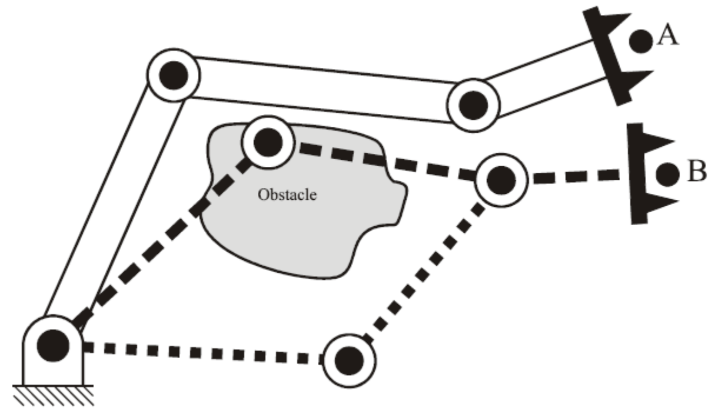
For example, if the manipulator is at point A and we want to move it to point B, a good choice would be the one **that minimizes the amount that each joint is required to move**.

Hence, in the absence of the **obstacle**, the upper dashed configuration would be chosen. This suggests that one input argument to our kinematic inverse procedure might be the present position of the manipulator. In this way, if there is a choice, our algorithm can choose the solution closest in joint-space.

Multiple Solutions

The notion of "close" might be defined in several ways. Typical robots could have three large links followed by three smaller, orienting links near the end-effector.
In this case, weights might be applied in the calculation of the closest solution so that the selection favors smaller joints rather than moving the large joints, when a choice exists.

The presence of obstacles might force a farther solution.
In general, **we need to be able to calculate all the possible solutions**.
Clearly, the presence of the obstacle implies that the lower dashed configuration is to be used to reach point B.

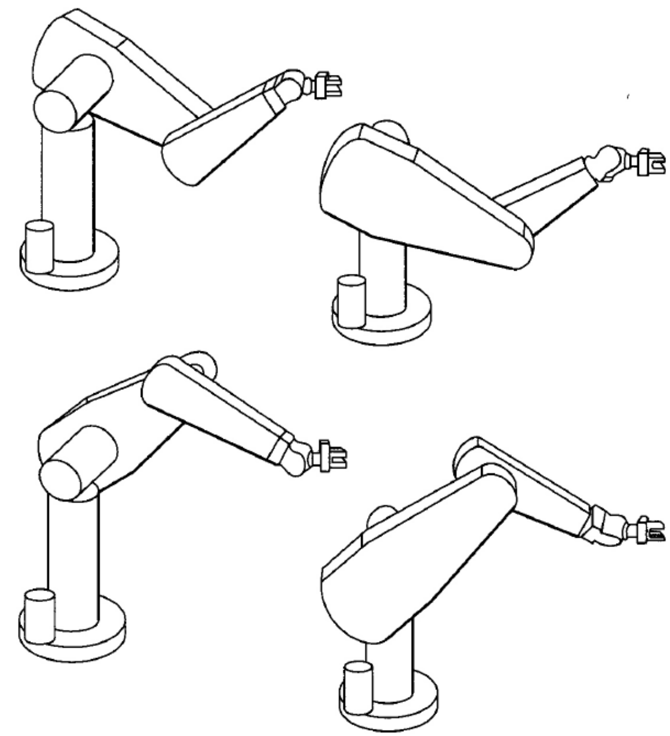


The number of solutions depends upon the number of joints in the manipulator but is also a function of the link parameters and the allowable ranges of motion of the joints.

Multiple Solutions

For example, the PUMA 560 can reach certain goals with up to eight different solutions. The figure shows four solutions; all place the hand with the same position and orientation. For each solution in the figure, there is another solution in which the last three joints "flip" to an alternate configuration:

$$\Theta_4' = \Theta_4 + 180°,$$
$$\Theta_5' = -\Theta_5,$$
$$\Theta_6' = \Theta_6 + 180°.$$



So, in total, for the PUMA 560, there can be eight solutions for a single goal.
Because of limits on joint ranges, some of these eight configurations are inaccessible.

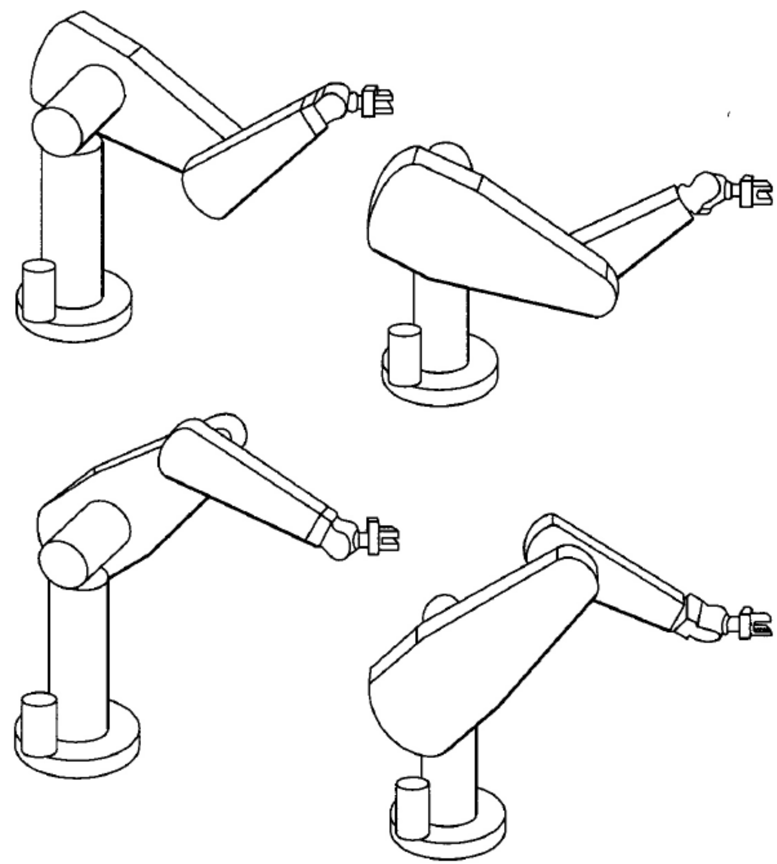
Multiple Solutions

In general, the **more** **nonzero link parameter** there are, the **more** ways there will be to reach a certain goal.

For example, consider a manipulator with six rotational joints like the PUMA 560.

The maximum number of solutions is related to how many of the link length parameters (a_i) are zero.

For a completely general rotary-joint manipulator with six degrees of freedom, there are up to sixteen solutions possible.



Method of Solution

A manipulator is **solvable** if the joint variables can be determined by an algorithm. The algorithm should find ***all*** possible solutions.

Solutions

- **Closed form solutions**
- **Numerical solutions**

Method of Solution

Unlike linear equations, there are no general algorithms that may be employed to solve a set of nonlinear equations.

In considering methods of solution, it will be wise to define what constitutes “the solution” of a **given** manipulator.

A manipulator will be considered solvable if the joint variables can be determined by an algorithm that allows one to determine all the sets of joint variables associated with a given position and orientation.

Method of Solution

We will split all proposed manipulator solution strategies into two broad classes:

Closed-form solutions and **numerical solutions**.

Because of their iterative nature, numerical solutions generally are much slower than closed-form solutions and do not assure to really find all solutions.
For most uses, we are not interested in the numerical approach to solve inverse kinematics.

Here, we will restrict our attention to closed-form solutions.
In this context, we search for solutions based on an analytic expression.

Numerical Solutions

Results in a numerical, iterative solution to system of equations, for example Newton/Raphson techniques.

Unknown number of operations to solve.

Only returns a single solution.

Accuracy is dictated by user.

Because of these reasons, this is much less desirable than a closed-form solution.

Can be applied to all robots.

Closed-form Solutions

Within the class of closed-form solutions, we distinguish:

algebraic approaches and **geometric approaches**.

A major recent result in kinematics is that, according to our definition of solvability, all systems with revolute and prismatic joints having a total of six degrees of freedom in a single series chain are solvable.

Closed-form Solutions

Robots, for which a closed-form solution exists, are characterized either by having several intersecting joint axes or by having many twist angles α_i be equal to 0 or +/-90°.

Hence, it is considered very important to **design a manipulator so that a closed-form solution exists**. Nowadays, virtually all industrial robots are designed sufficiently simple that a closed-form solution can be developed.

A sufficient condition, that a manipulator with six revolute joints has a closed-form solution, is that three neighboring joint axes intersect at a point. For example, the axes 4, 5, and 6 of the PUMA 560 intersect.

Closed-form Solutions

- Analytical solution to system of equations
- Can be solved in a fixed number of operations (therefore, computationally fast/known speed)
- Results all possible solutions to the manipulator kinematics
- Often difficult to find
- Most desirable for real-time control
- Most desirable overall

Methods of Solutions

In general the IK problem can be solved by various methods such as:

- Inverse Transform (Paul, 1981)
- Screw Algebra (Kohli 1975)
- Dual Matrices (Denavit 1956)
- Iterative (Uicker 1964)
- Geometric approach (Lee 1984)
- Decoupling of position and orientation (Pieper 1968)

Closed-form Solutions

We are interested in closed-form solutions:

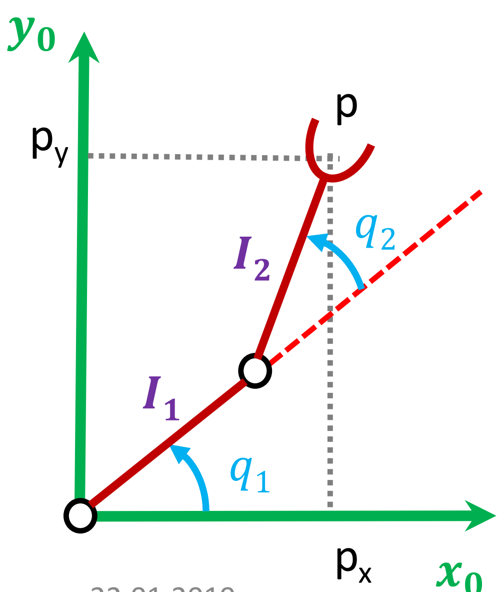
1. Algebraic methods

2. Geometric methods

Algebraic Solution

Forward kinematics:

$$p_x = I_1\, c_1 + I_2\, c_{12}$$
$$p_y = I_1\, s_1 + I_2\, s_{12}$$



Given p find q_1 and q_2.

Squaring and summing the forward kinematics equations:

$$P_x{}^2 + py^2 - (I_1{}^2 + I_2{}^2) = 2\, I_1 I_2 (c_1 c_{12} + s_1 s_{12})$$
$$= 2\, I_1\, I_2\, \mathrm{c}_2$$

Then:

$$c_2 = (px^2 + py^2 - I_1{}^2 - I_2{}^2)/2\, I_1\, I_2$$

$$s_2 = \pm\sqrt{1 - c_2{}^2}$$

Then:

$$q_2 = \operatorname{atan2}(s_2, c_2)$$

Algebraic Solution

Forward kinematics:

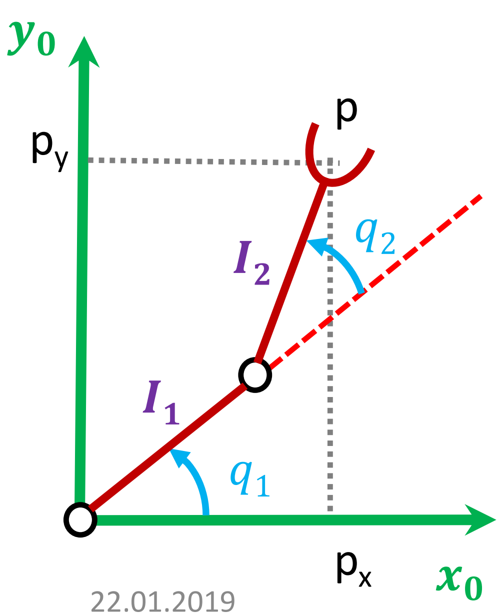
$$p_x = I_1\, c_1 + I_2\, c_{12}$$
$$p_y = I_1\, s_1 + I_2\, s_{12}$$

Given p find q_1 and q_2.

$$p_x = I_1\, c_1 + I_2\,(c_1 c_2 - s_1 s_2)$$
$$p_y = I_1\, s_1 + I_2\,(s_1 c_2 + c_1 s_2)$$

$$\begin{bmatrix} I_1 + I_2 c_2 & -I_2 s_2 \\ I_2 s_2 & I_1 + I_2 c_2 \end{bmatrix} \begin{bmatrix} c_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

$$q_1 = \operatorname{atan2}(s_1, c_1)$$

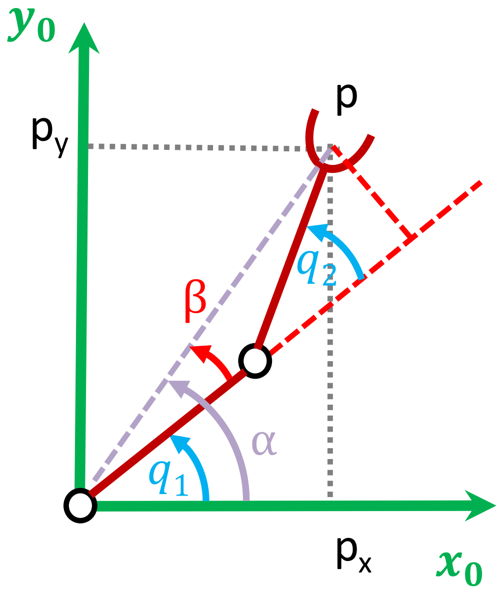


Geometric Solution for q_1

By geometric inspection:

$$q_1 = \alpha - \beta$$

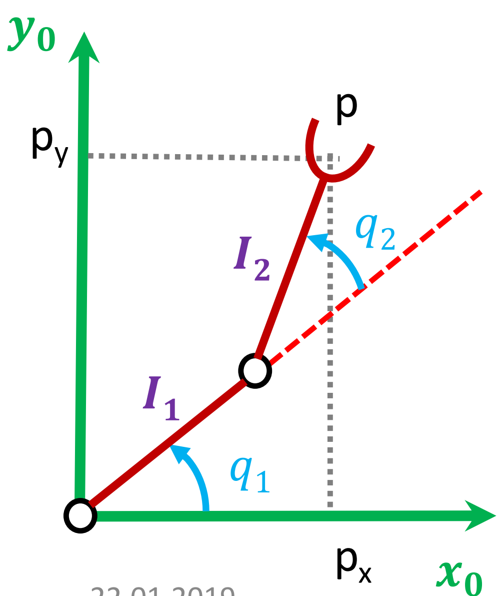
$$q_1 = \operatorname{atan2}(p_y, p_x) - \operatorname{atan2}(l_2 s_2, l_1 + l_2 c_2)$$



I.K. of Planar 2R Arm

$$c_2 = (px^2 + py^2 - I_1{}^2 - I_2{}^2)/2\, I_1\, I_2$$
$$s_2 = \pm\sqrt{1 - c_2{}^2}$$
$$q_2 = \operatorname{atan2}(s_2, c_2)$$

$$q_1 = \operatorname{atan2}(p_y, p_x) - \operatorname{atan2}(\, I_2\, s_2\,, I_1 + I_2\, c_2)$$

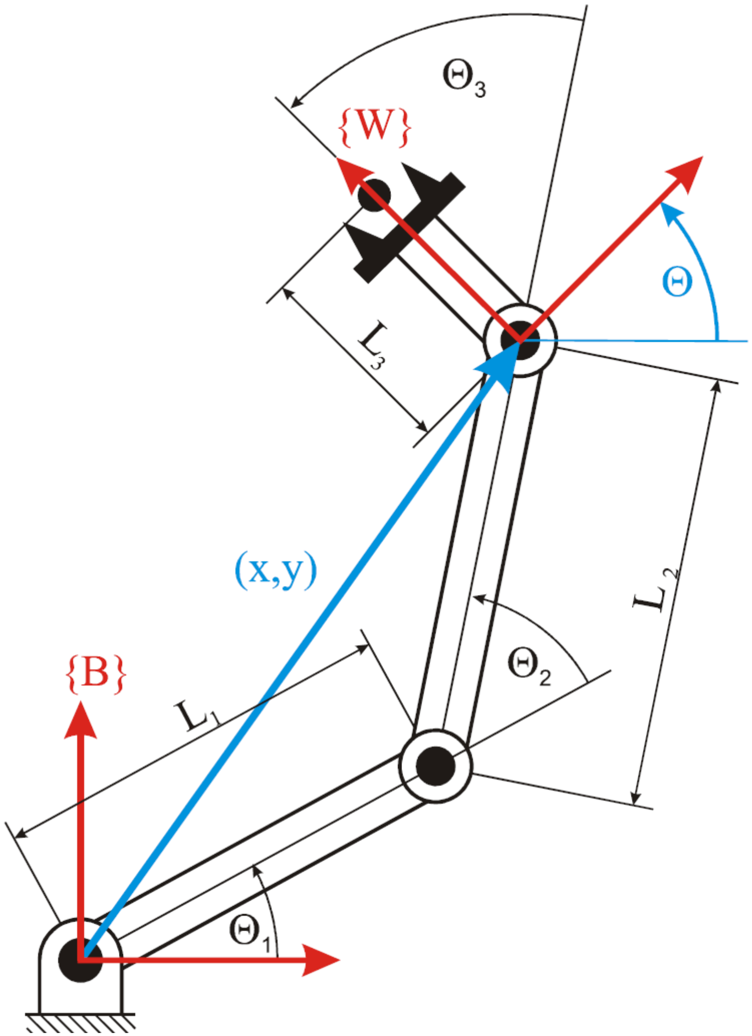


Given $l_1 = 10$, $l_2 = 9$, $p_x = 12$, $p_y = 12$
Work out q_1 and q_2.

Given $l_1 = 10$, $l_2 = 9$, $p_x = 18$, $p_y = 18$
Work out q_1 and q_2.

I.K. of Planar 3R Arm

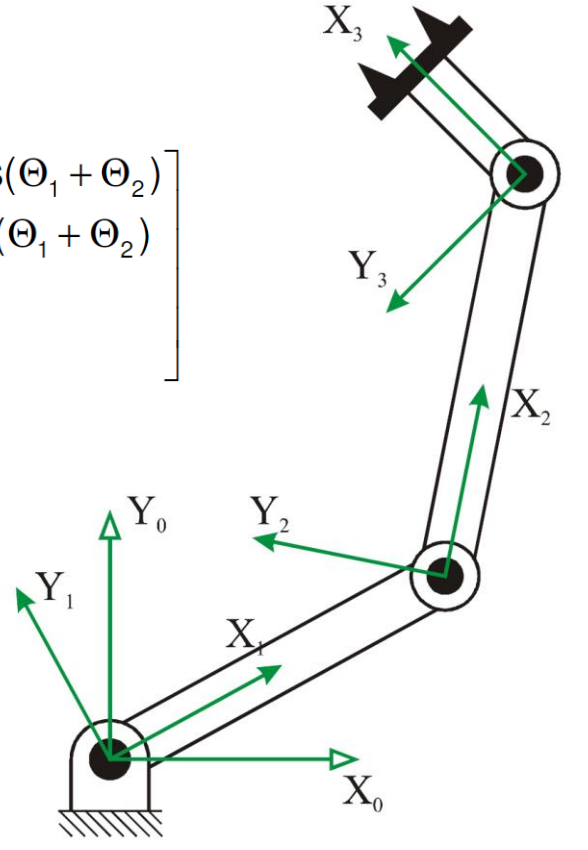
$$
{}^{B}_{W}T = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 & x \\ \sin\Theta & \cos\Theta & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



I.K. of Planar 3R Arm

We can derive kinematic equations:

$$
{}^{B}_{W}T = {}^{0}_{3}T = \begin{bmatrix} \cos(\Theta_1+\Theta_2+\Theta_3) & -\sin(\Theta_1+\Theta_2+\Theta_3) & 0 & L_1\cos\Theta_1 + L_2\cos(\Theta_1+\Theta_2) \\ \sin(\Theta_1+\Theta_2+\Theta_3) & \cos(\Theta_1+\Theta_2+\Theta_3) & 0 & L_1\sin\Theta_1 + L_2\sin(\Theta_1+\Theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



Happy New Year

I.K. of Planar 3R Arm

$$\begin{bmatrix} \cos\Theta & -\sin\Theta & 0 & x \\ \sin\Theta & \cos\Theta & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\Theta_1+\Theta_2+\Theta_3) & -\sin(\Theta_1+\Theta_2+\Theta_3) & 0 & L_1\cos\Theta_1 + L_2\cos(\Theta_1+\Theta_2) \\ \sin(\Theta_1+\Theta_2+\Theta_3) & \cos(\Theta_1+\Theta_2+\Theta_3) & 0 & L_1\sin\Theta_1 + L_2\sin(\Theta_1+\Theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A set of four nonlinear equations:

$$\cos\Theta = \cos(\Theta_1+\Theta_2+\Theta_3)$$
$$\sin\Theta = \sin(\Theta_1+\Theta_2+\Theta_3)$$
$$x = L_1\cos\Theta_1 + L_2\cos(\Theta_1+\Theta_2)$$
$$y = L_1\sin\Theta_1 + L_2\sin(\Theta_1+\Theta_2)$$

Given (Θ, x, y) find Θ_1 Θ_2 Θ_3?

I.K. of Planar 3R Arm

$$\cos\Theta = \cos(\Theta_1 + \Theta_2 + \Theta_3)$$
$$\sin\Theta = \sin(\Theta_1 + \Theta_2 + \Theta_3)$$
$$x = L_1\cos\Theta_1 + L_2\cos(\Theta_1 + \Theta_2)$$
$$y = L_1\sin\Theta_1 + L_2\sin(\Theta_1 + \Theta_2)$$

Reminder:

$$\cos(\Theta_1 + \Theta_2) = \cos\Theta_1\cos\Theta_2 - \sin\Theta_1\sin\Theta_2$$
$$\sin(\Theta_1 + \Theta_2) = \cos\Theta_1\sin\Theta_2 + \sin\Theta_1\cos\Theta_2$$

$$x^2 + y^2 = L_1^2 + L_2^2 + 2L_1L_2\{\cos\Theta_1\cos(\Theta_1 + \Theta_2) + \sin\Theta_1\sin(\Theta_1 + \Theta_2)\}$$

$$x^2 + y^2 = L_1^2 + L_2^2 + 2L_1L_2\cos\Theta_2$$

$$\cos\Theta_2 = \frac{x^2 + y^2 - L_1^2 - L_2^2}{2L_1L_2} \qquad \sin\Theta_2 = \pm\sqrt{1 - \cos^2\Theta_2}$$

$$\Theta_2 = A\tan 2(\sin\Theta_2, \cos\Theta_2)$$

I.K. of Planar 3R Arm

$$\cos\Theta = \cos(\Theta_1 + \Theta_2 + \Theta_3)$$
$$\sin\Theta = \sin(\Theta_1 + \Theta_2 + \Theta_3)$$
$$x = L_1\cos\Theta_1 + L_2\cos(\Theta_1 + \Theta_2)$$
$$y = L_1\sin\Theta_1 + L_2\sin(\Theta_1 + \Theta_2)$$

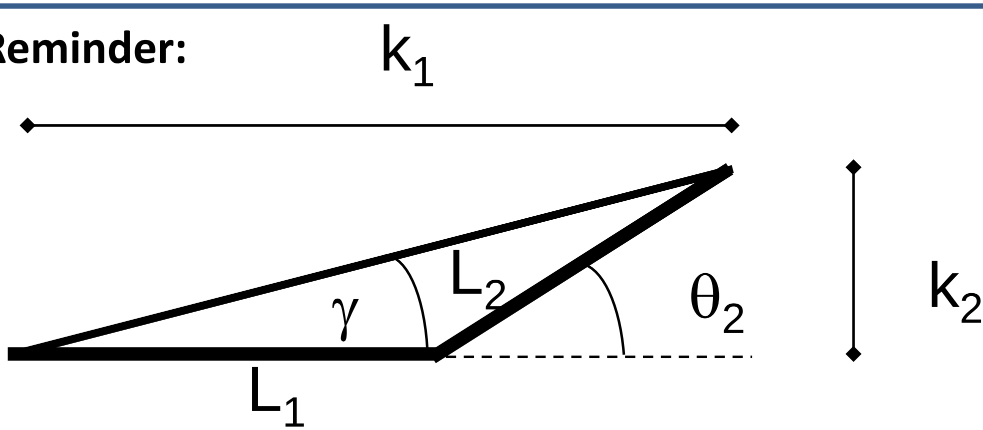
Reminder:

$$\cos(\Theta_1 + \Theta_2) = \cos\Theta_1\cos\Theta_2 - \sin\Theta_1\sin\Theta_2$$
$$\sin(\Theta_1 + \Theta_2) = \cos\Theta_1\sin\Theta_2 + \sin\Theta_1\cos\Theta_2$$

$$\Theta_2 = A\tan 2(\sin\Theta_2, \cos\Theta_2)$$

$$x = L_1\cos\Theta_1 + L_2\{\cos\Theta_1\cos\Theta_2 - \sin\Theta_1\sin\Theta_2\}$$
$$y = L_1\sin\Theta_1 + L_2\{\cos\Theta_1\sin\Theta_2 + \sin\Theta_1\cos\Theta_2\}$$

Reminder:



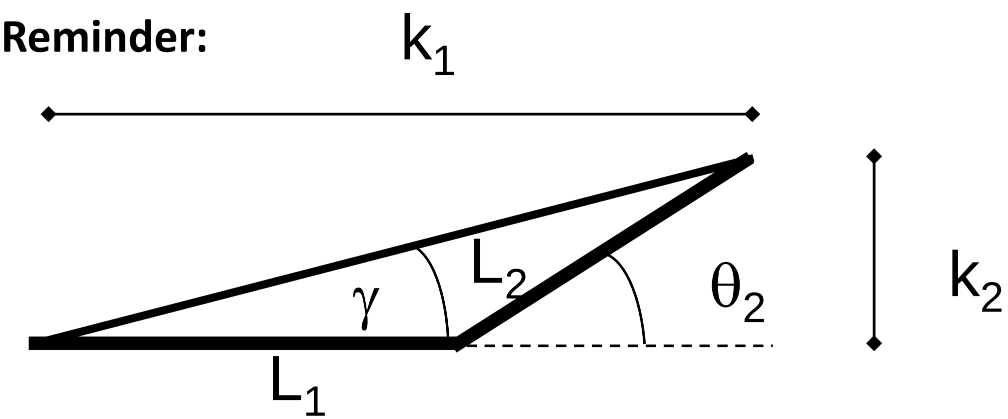
$$k_1 = L_1 + L_2\cos\Theta_2$$
$$k_2 = L_2\sin\Theta_2$$

I.K. of Planar 3R Arm

$$x = L_1 \cos\Theta_1 + L_2\{\cos\Theta_1 \cos\Theta_2 - \sin\Theta_1 \sin\Theta_2\}$$

$$y = L_1 \sin\Theta_1 + L_2\{\cos\Theta_1 \sin\Theta_2 + \sin\Theta_1 \cos\Theta_2\}$$

Reminder:



$$k_1 = L_1 + L_2 \cos\Theta_2$$

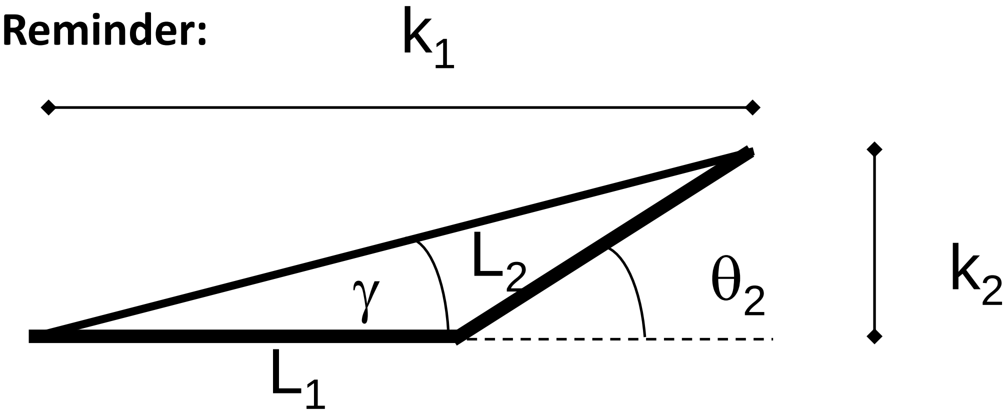
$$k_2 = L_2 \sin\Theta_2$$

$$x = k_1 \cos\Theta_1 - k_2 \sin\Theta_1$$

$$y = k_1 \sin\Theta_1 + k_2 \cos\Theta_1$$

I.K. of Planar 3R Arm

Reminder:



$$k_1 = L_1 + L_2 \cos\Theta_2$$

$$k_2 = L_2 \sin\Theta_2$$

$$x = k_1 \cos\Theta_1 - k_2 \sin\Theta_1$$

$$y = k_1 \sin\Theta_1 + k_2 \cos\Theta_1$$

To solve the equations we perform a change of variables:

$$r = \sqrt{k_1^2 + k_2^2} \qquad \gamma = A\tan 2(k_2, k_1)$$

Then:

$$k_1 = r\cos\gamma \qquad k_2 = r\sin\gamma$$

I.K. of Planar 3R Arm

$$x = k_1 \cos\Theta_1 - k_2 \sin\Theta_1$$

$$y = k_1 \sin\Theta_1 + k_2 \cos\Theta_1$$

To solve the equations we perform a change of variables:

$$r = \sqrt{k_1^2 + k_2^2} \qquad \gamma = A\tan2(k_2, k_1)$$

Then:

$$k_1 = r\cos\gamma \qquad k_2 = r\sin\gamma$$

$$\frac{x}{r} = \cos\gamma\cos\Theta_1 - \sin\gamma\sin\Theta_1 \qquad \frac{y}{r} = \cos\gamma\sin\Theta_1 + \sin\gamma\cos\Theta_1$$

$$\frac{x}{r} = \cos(\gamma + \Theta_1) \qquad \frac{y}{r} = \sin(\gamma + \Theta_1)$$

I.K. of Planar 3R Arm

$$\frac{x}{r} = \cos(\gamma + \Theta_1) \qquad \frac{y}{r} = \sin(\gamma + \Theta_1)$$

$$\gamma + \Theta_1 = A\tan 2\left(\frac{y}{r}, \frac{x}{r}\right) = A\tan 2(y, x)$$

$$\Theta_1 = A\tan 2(y, x) - A\tan 2(k_2, k_1)$$

I.K. of Planar 3R Arm

$$\cos\Theta = \cos(\Theta_1 + \Theta_2 + \Theta_3)$$
$$\sin\Theta = \sin(\Theta_1 + \Theta_2 + \Theta_3)$$

$$\Theta_1 + \Theta_2 + \Theta_3 = A\tan 2(\sin\Theta, \cos\Theta) = \Theta$$

I.K. of Planar 3R Arm

Geometric solution:

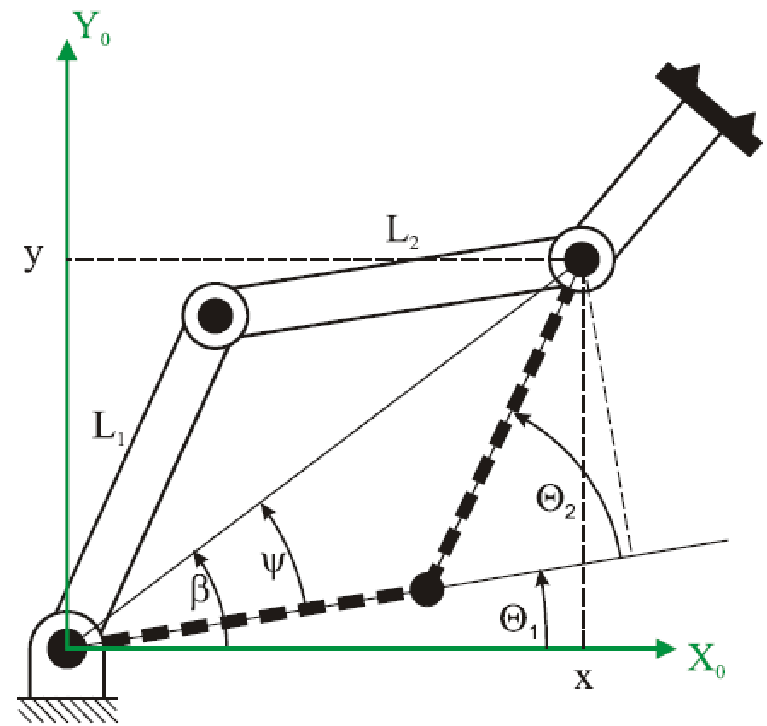
Reminder: Law Of Cosines

Sides a,b,c - Angles A,B,C
a Opposte A, b Opposite B, c Opposite C
$c^2 = a^2 + b^2 - 2ab\cos(C)$

$$x^2 + y^2 = L_1^2 + L_2^2 - 2L_1L_2\cos(180° - \Theta_2)$$

$$\cos(180° - \Theta_2) = -\cos\Theta_2$$

$$\cos\Theta_2 = \frac{x^2 + y^2 - L_1^2 - L_2^2}{2L_1L_2}$$



I.K. of Planar 3R Arm

Geometric solution:

Reminder: Law Of Cosines

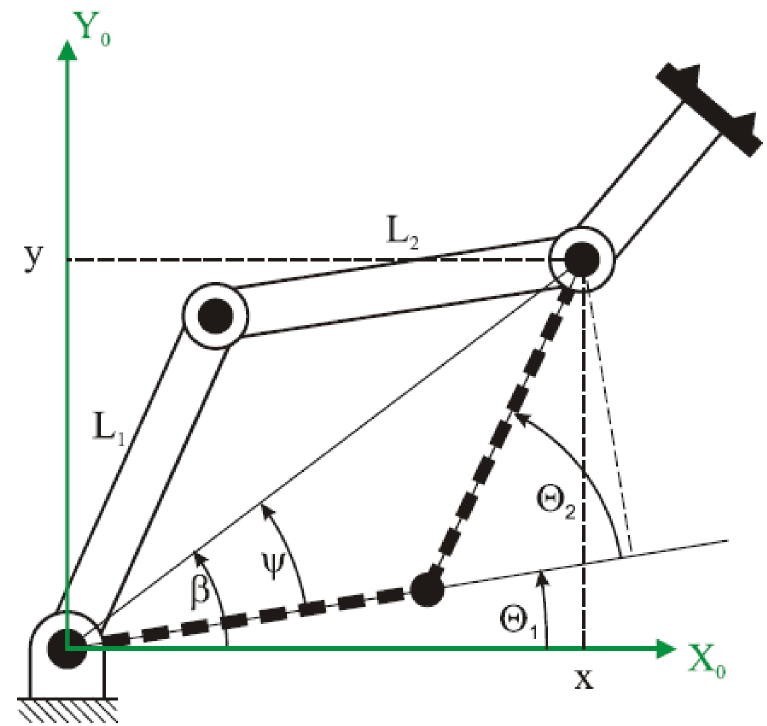
Sides a,b,c - Angles A,B,C
a Opposte A, b Opposite B, c Opposite C
$c^2 = a^2 + b^2 - 2ab\cos(C)$

$$\cos\psi = \frac{x^2 + y^2 + L_1^2 - L_2^2}{2L_1\sqrt{x^2 + y^2}}$$

$$\beta = A\tan 2(y, x)$$

$$\Theta_1 = \beta \pm \psi$$

$$\Theta_1 + \Theta_2 + \Theta_3 = \phi$$



Algebraic Solution

The arc cosine function does not behave well as its accuracy in determining the angle is dependent on the angle (cos(q)=cos(-q)).

When sin(q) approaches zero, division by sin(q) give inaccurate solutions.

Therefore an arc tangent function which is more consistent is used.

$$\Theta = \operatorname{atan2}(s, c)$$

Algebraic solution by reduction to polynomial

Transcendental equations are often difficult to solve.

Expression in terms of a single variable using the following substitutions:

$$u = \tan\frac{\Theta}{2}, \quad \cos\Theta = \frac{1-u^2}{1+u^2}, \quad \sin\Theta = \frac{2u}{1+u^2}$$

These substitutions convert transcendental equations into polynomial equations in u.

Algebraic solution by reduction to polynomial

Example: Convert the transcendental equation $a\cos\Theta + b\ \sin\Theta = c$ into a polynomial in the tangent of the half angle, and work out **Θ**.

$$a\frac{1-u^2}{1+u^2} + b\frac{2u}{1+u^2} = c$$

$$a\left(1-u^2\right) + 2bu = c\left(1+u^2\right)$$

$$(a+c)u^2 - 2bu + (c-a) = 0$$

Which is solved by the quadratic formula:

$$u_{1,2} = \frac{b \pm \sqrt{a^2+b^2-c^2}}{a+c}$$

Reminder: Substitutions

$$u = \tan\frac{\Theta}{2}, \quad \cos\Theta = \frac{1-u^2}{1+u^2}, \quad \sin\Theta = \frac{2u}{1+u^2}$$

$$\boxed{\Theta = 2\arctan\left(\frac{b \pm \sqrt{a^2+b^2-c^2}}{a+c}\right)}$$

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Quadratic equation

From Wikipedia, the free encyclopedia

This article is about single-variable quadratic equations and their solutions. For more general information about the single-variable case, see Quadratic function. For the case of more than one variable, see Quadric.

In elementary algebra, a **quadratic equation** (from the Latin *quadratus* for "square") is any equation having the form

$$ax^2 + bx + c = 0$$

where x represents an unknown, and a, b, and c

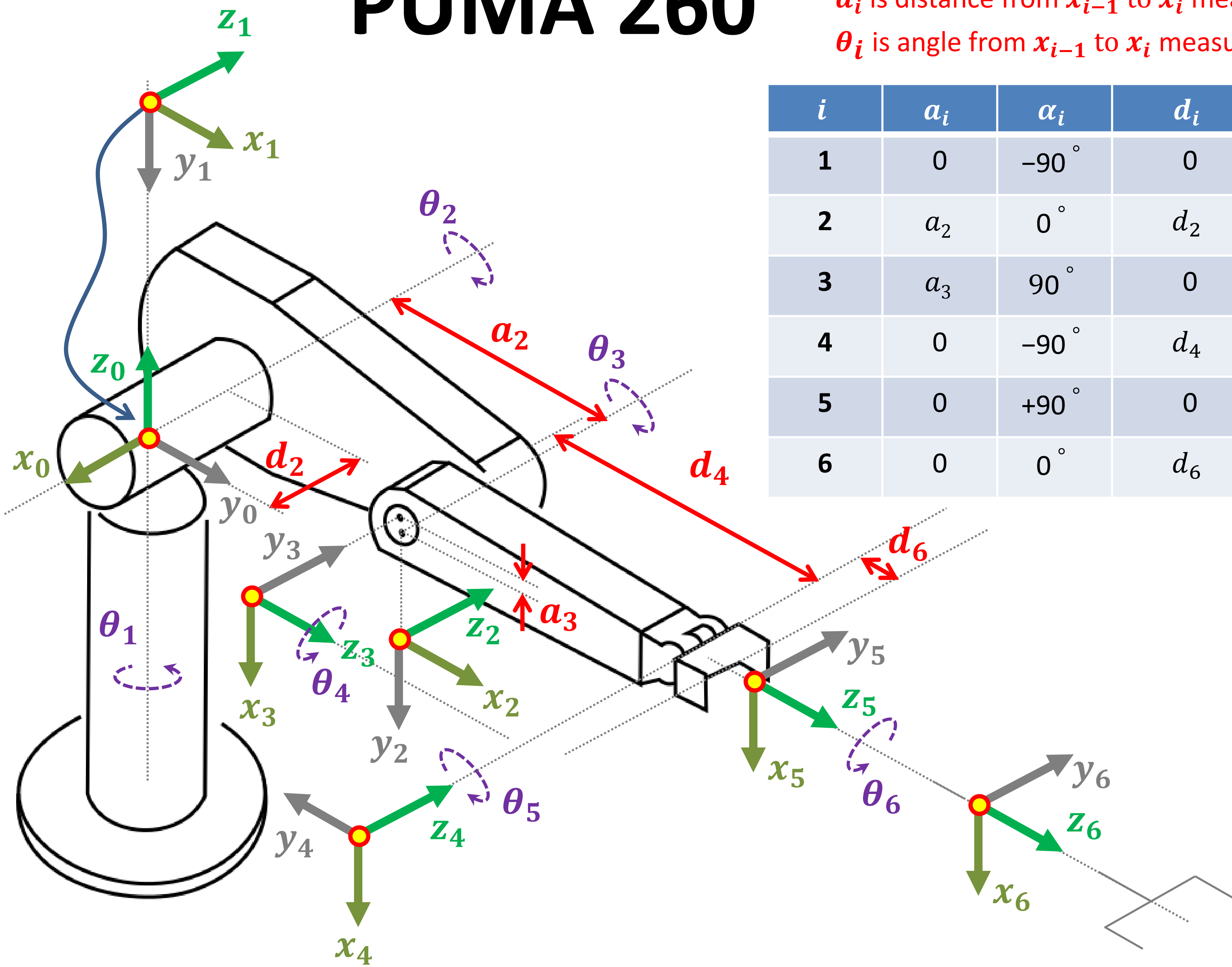
$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

The quadratic formula for the roots of the general quadratic equation

PUMA 260

a_i is distance from z_{i-1} to z_i measured along x_i.
α_i is angle from z_{i-1} to z_i measured about x_i.
d_i is distance from x_{i-1} to x_i measured along z_{i-1}.
θ_i is angle from x_{i-1} to x_i measured about z_{i-1}.

i	a_i	α_i	d_i	θ_i
1	0	-90°	0	θ_1 *
2	a_2	0°	d_2	θ_2 *
3	a_3	90°	0	θ_3 *
4	0	-90°	d_4	θ_4 *
5	0	$+90^\circ$	0	θ_5 *
6	0	0°	d_6	θ_6 *



PUMA 260

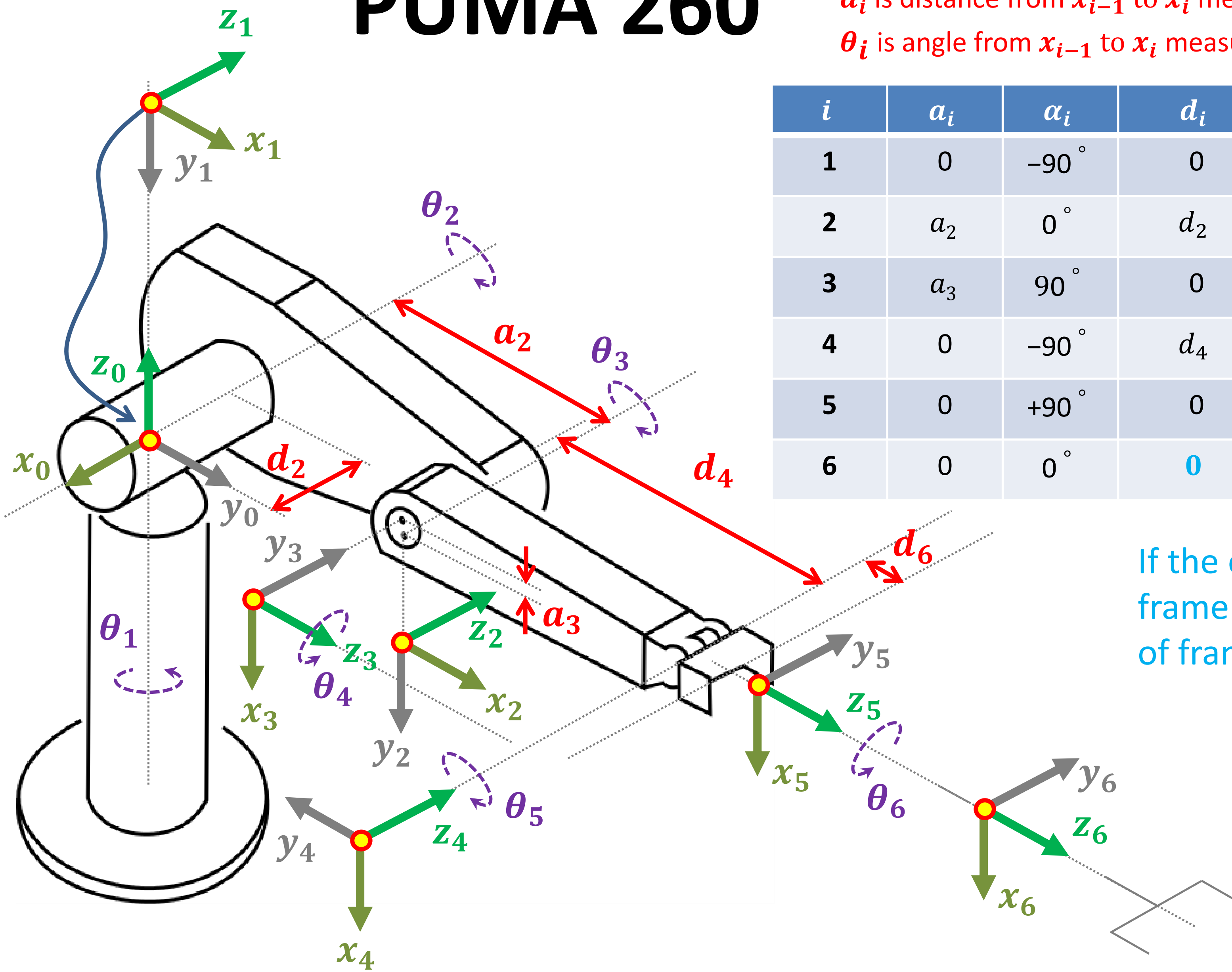
r11 = - s(6)*(c(4)*s(1) - s(4)*(c(1)*s(2)*s(3) - c(1)*c(2)*c(3))) - c(6)*(c(5)*(s(1)*s(4) + c(4)*(c(1)*s(2)*s(3) - c(1)*c(2)*c(3))) + s(5)*(c(1)*c(2)*s(3) + c(1)*c(3)*s(2)))
r12 = s(6)*(c(5)*(s(1)*s(4) + c(4)*(c(1)*s(2)*s(3) - c(1)*c(2)*c(3))) + s(5)*(c(1)*c(2)*s(3) + c(1)*c(3)*s(2))) - c(6)*(c(4)*s(1) - s(4)*(c(1)*s(2)*s(3) - c(1)*c(2)*c(3)))
r13 = c(5)*(c(1)*c(2)*s(3) + c(1)*c(3)*s(2)) - s(5)*(s(1)*s(4) + c(4)*(c(1)*s(2)*s(3) - c(1)*c(2)*c(3)))
dx = d4*s(2 + 3)*c(1) - d2*s(1) + a2*c(1)*c(2) + d6*s(2 + 3)*c(1)*c(5) + a3*c(1)*c(2)*c(3) - a3*c(1)*s(2)*s(3) - d6*s(1)*s(4)*s(5) + d6*c(1)*c(2)*c(3)*c(4)*s(5) –
d6*c(1)*c(4)*s(2)*s(3)*s(5)

r21 = s(6)*(c(1)*c(4) + s(4)*(s(1)*s(2)*s(3) - c(2)*c(3)*s(1))) + c(6)*(c(5)*(c(1)*s(4) - c(4)*(s(1)*s(2)*s(3) - c(2)*c(3)*s(1))) - s(5)*(c(2)*s(1)*s(3) + c(3)*s(1)*s(2)))
r22 = c(6)*(c(1)*c(4) + s(4)*(s(1)*s(2)*s(3) - c(2)*c(3)*s(1))) - s(6)*(c(5)*(c(1)*s(4) - c(4)*(s(1)*s(2)*s(3) - c(2)*c(3)*s(1))) - s(5)*(c(2)*s(1)*s(3) + c(3)*s(1)*s(2)))
r23 = s(5)*(c(1)*s(4) - c(4)*(s(1)*s(2)*s(3) - c(2)*c(3)*s(1))) + c(5)*(c(2)*s(1)*s(3) + c(3)*s(1)*s(2))
dy = d2*c(1) + d4*s(2 + 3)*s(1) + a2*c(2)*s(1) + d6*s(2 + 3)*c(5)*s(1) + a3*c(2)*c(3)*s(1) + d6*c(1)*s(4)*s(5) - a3*s(1)*s(2)*s(3) + d6*c(2)*c(3)*c(4)*s(1)*s(5) –
d6*c(4)*s(1)*s(2)*s(3)*s(5)

r31 = s(2 + 3)*s(4)*s(6) - c(6)*(c(2 + 3)*s(5) + s(2 + 3)*c(4)*c(5))
r32 = s(6)*(c(2 + 3)*s(5) + s(2 + 3)*c(4)*c(5)) + s(2 + 3)*c(6)*s(4)
r33 = c(2 + 3)*c(5) - s(2 + 3)*c(4)*s(5)
dz = d4*c(2 + 3) - a3*s(2 + 3) - a2*s(2) - (d6*s(2 + 3)*s(4 + 5))/2 + d6*c(2 + 3)*c(5) + (d6*s(4 - 5)*s(2 + 3))/2

PUMA 260

a_i is distance from z_{i-1} to z_i measured along x_i.
α_i is angle from z_{i-1} to z_i measured about x_i.
d_i is distance from x_{i-1} to x_i measured along z_{i-1}.
θ_i is angle from x_{i-1} to x_i measured about z_{i-1}.



i	a_i	α_i	d_i	θ_i
1	0	–90°	0	θ_1 *
2	a_2	0°	d_2	θ_2 *
3	a_3	90°	0	θ_3 *
4	0	–90°	d_4	θ_4 *
5	0	+90°	0	θ_5 *
6	0	0°	**0**	θ_6 *

If the origin of frame 6 is the origin of frame 5.

PUMA 260

r11 = - s(6)*(c(4)*s(1) - s(4)*(c(1)*s(2)*s(3) - c(1)*c(2)*c(3))) - c(6)*(c(5)*(s(1)*s(4) + c(4)*(c(1)*s(2)*s(3) - c(1)*c(2)*c(3))) + s(5)*(c(1)*c(2)*s(3) + c(1)*c(3)*s(2)))
r12 = s(6)*(c(5)*(s(1)*s(4) + c(4)*(c(1)*s(2)*s(3) - c(1)*c(2)*c(3))) + s(5)*(c(1)*c(2)*s(3) + c(1)*c(3)*s(2))) - c(6)*(c(4)*s(1) - s(4)*(c(1)*s(2)*s(3) - c(1)*c(2)*c(3)))
r13 = c(5)*(c(1)*c(2)*s(3) + c(1)*c(3)*s(2)) - s(5)*(s(1)*s(4) + c(4)*(c(1)*s(2)*s(3) - c(1)*c(2)*c(3)))
dx = d4*(c(1)*c(2)*s(3) + c(1)*c(3)*s(2)) - d2*s(1) + a2*c(1)*c(2) + a3*c(1)*c(2)*c(3) - a3*c(1)*s(2)*s(3)

r21 = s(6)*(c(1)*c(4) + s(4)*(s(1)*s(2)*s(3) - c(2)*c(3)*s(1))) + c(6)*(c(5)*(c(1)*s(4) - c(4)*(s(1)*s(2)*s(3) - c(2)*c(3)*s(1))) - s(5)*(c(2)*s(1)*s(3) + c(3)*s(1)*s(2)))
r22 = c(6)*(c(1)*c(4) + s(4)*(s(1)*s(2)*s(3) - c(2)*c(3)*s(1))) - s(6)*(c(5)*(c(1)*s(4) - c(4)*(s(1)*s(2)*s(3) - c(2)*c(3)*s(1))) - s(5)*(c(2)*s(1)*s(3) + c(3)*s(1)*s(2)))
r23 = s(5)*(c(1)*s(4) - c(4)*(s(1)*s(2)*s(3) - c(2)*c(3)*s(1))) + c(5)*(c(2)*s(1)*s(3) + c(3)*s(1)*s(2))
dy = d4*(c(2)*s(1)*s(3) + c(3)*s(1)*s(2)) + d2*c(1) + a2*c(2)*s(1) + a3*c(2)*c(3)*s(1) - a3*s(1)*s(2)*s(3)

r31 = s(2 + 3)*s(4)*s(6) - c(6)*(c(2 + 3)*s(5) + s(2 + 3)*c(4)*c(5))
r32 = s(6)*(c(2 + 3)*s(5) + s(2 + 3)*c(4)*c(5)) + s(2 + 3)*c(6)*s(4)
r33 = c(2 + 3)*c(5) - s(2 + 3)*c(4)*s(5)
dz = d4*c(2 + 3) - a3*s(2 + 3) - a2*s(2)

Algebraic Solution

Paul (1981) suggest pre multiplying the final transformation matrix by the inverse of its components successively and determine the unknown angle from the elements of the resultant matrix equation.

Algebraic Solution

$$T_6^0 = T_1^0 T_2^1 T_3^2 T_4^3 T_5^4 T_6^5$$

By multiplying both sides by $[T_1^0]^{-1}$:

$$[T_1^0]^{-1} T_6^0 = T_6^1$$

Then multiplying both sides by $[T_2^1]^{-1}$:

$$[T_2^1]^{-1} [T_1^0]^{-1} T_6^0 = T_6^2$$

Then multiplying both sides by $[T_3^2]^{-1}$:

$$[T_3^2]^{-1} [T_2^1]^{-1} [T_1^0]^{-1} T_6^0 = T_6^3$$

....

....

....

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Reminder: A_i

$$\begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	a_i	α_i	d_i	θ_i
1	0	-90°	0	θ_1 *
2	a_2	0°	d_2	θ_2 *
3	a_3	90°	0	θ_3 *
4	0	-90°	d_4	θ_4 *
5	0	$+90^\circ$	0	θ_5 *
6	0	0°	0	θ_6 *

Coordinate Transformation Matrices

$$\mathbf{A}_0^1 = \begin{bmatrix} C_1 & 0 & -S_1 & 0 \\ S_1 & 0 & C_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{A}_1^2 = \begin{bmatrix} C_2 & -S_2 & 0 & a_2 C_2 \\ S_2 & C_2 & 0 & a_2 S_2 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{A}_2^3 = \begin{bmatrix} C_3 & 0 & S_3 & a_3 C_3 \\ S_3 & 0 & -C_3 & a_3 S_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}_3^4 = \begin{bmatrix} C_4 & 0 & -S_4 & 0 \\ S_4 & 0 & C_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{A}_4^5 = \begin{bmatrix} C_5 & 0 & S_5 & 0 \\ S_5 & 0 & -C_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{A}_5^6 = \begin{bmatrix} C_6 & -S_6 & 0 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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The final transformation matrix:

r11 = - s(6)*(c(4)*s(1) - s(4)*(c(1)*s(2)*s(3) - c(1)*c(2)*c(3))) - c(6)*(c(5)*(s(1)*s(4) + c(4)*(c(1)*s(2)*s(3) - c(1)*c(2)*c(3))) + s(5)*(c(1)*c(2)*s(3) + c(1)*c(3)*s(2)))
r12 = s(6)*(c(5)*(s(1)*s(4) + c(4)*(c(1)*s(2)*s(3) - c(1)*c(2)*c(3))) + s(5)*(c(1)*c(2)*s(3) + c(1)*c(3)*s(2))) - c(6)*(c(4)*s(1) - s(4)*(c(1)*s(2)*s(3) - c(1)*c(2)*c(3)))
r13 = c(5)*(c(1)*c(2)*s(3) + c(1)*c(3)*s(2)) - s(5)*(s(1)*s(4) + c(4)*(c(1)*s(2)*s(3) - c(1)*c(2)*c(3)))
dx = d4*(c(1)*c(2)*s(3) + c(1)*c(3)*s(2)) - d2*s(1) + a2*c(1)*c(2) + a3*c(1)*c(2)*c(3) - a3*c(1)*s(2)*s(3)

r21 = s(6)*(c(1)*c(4) + s(4)*(s(1)*s(2)*s(3) - c(2)*c(3)*s(1))) + c(6)*(c(5)*(c(1)*s(4) - c(4)*(s(1)*s(2)*s(3) - c(2)*c(3)*s(1))) - s(5)*(c(2)*s(1)*s(3) + c(3)*s(1)*s(2)))
r22 = c(6)*(c(1)*c(4) + s(4)*(s(1)*s(2)*s(3) - c(2)*c(3)*s(1))) - s(6)*(c(5)*(c(1)*s(4) - c(4)*(s(1)*s(2)*s(3) - c(2)*c(3)*s(1))) - s(5)*(c(2)*s(1)*s(3) + c(3)*s(1)*s(2)))
r23 = s(5)*(c(1)*s(4) - c(4)*(s(1)*s(2)*s(3) - c(2)*c(3)*s(1))) + c(5)*(c(2)*s(1)*s(3) + c(3)*s(1)*s(2))
dy = d4*(c(2)*s(1)*s(3) + c(3)*s(1)*s(2)) + d2*c(1) + a2*c(2)*s(1) + a3*c(2)*c(3)*s(1) - a3*s(1)*s(2)*s(3)

r31 = s(2 + 3)*s(4)*s(6) - c(6)*(c(2 + 3)*s(5) + s(2 + 3)*c(4)*c(5))
r32 = s(6)*(c(2 + 3)*s(5) + s(2 + 3)*c(4)*c(5)) + s(2 + 3)*c(6)*s(4)
r33 = c(2 + 3)*c(5) - s(2 + 3)*c(4)*s(5)
dz = d4*c(2 + 3) - a3*s(2 + 3) - a2*s(2)

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inv(A1)*T06 =

```
[  r11*c(1) + r21*s(1),  r12*c(1) + r22*s(1),  r13*c(1) + r23*s(1),  dx*c(1) + dy*s(1)]
[                 -r31,                 -r32,                 -r33,               -dz]
[   r21*c(1) - r11*s(1),  r22*c(1) - r12*s(1),   r23*c(1) - r13*s(1),  dy*c(1) - dx*s(1)]
[                    0,                    0,                    0,                 1]
```

Work out t1.

A23456 =

(1,1): - c(6)*(s(2 + 3)*s(5) - c(2 + 3)*c(4)*c(5)) - c(2 + 3)*s(4)*s(6)
(1,2): s(6)*(s(2 + 3)*s(5) - c(2 + 3)*c(4)*c(5)) - c(2 + 3)*c(6)*s(4)
(1,3): s(2 + 3)*c(5) + c(2 + 3)*c(4)*s(5)
(1,4): a3*c(2 + 3) + d4*s(2 + 3) + a2*c(2)

(2,1): c(6)*(c(2 + 3)*s(5) + s(2 + 3)*c(4)*c(5)) - s(2 + 3)*s(4)*s(6)
(2,2): - s(6)*(c(2 + 3)*s(5) + s(2 + 3)*c(4)*c(5)) - s(2 + 3)*c(6)*s(4)
(2,3): s(2 + 3)*c(4)*s(5) - c(2 + 3)*c(5)
(2,4): a3*s(2 + 3) - d4*c(2 + 3) + a2*s(2)

(3,1): c(4)*s(6) + c(5)*c(6)*s(4)
(3,2): c(4)*c(6) - c(5)*s(4)*s(6)
(3,3): s(4)*s(5)
(3,4): **d2**

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To solve:

$$d2 = dy*\cos(\Theta_1) - dx*\sin(\Theta_1)$$

By the trigonometric substitutions: $dx = \rho \cos(\Phi)$, $dy = \rho \sin(\Phi)$

Where

$\rho = \sqrt{dx^2 + dy^2}$, $\Phi = \text{atan2}(dy,dx)$

We have therefore

$$d2 = \rho \sin(\Phi) \cos(\Theta_1) - \rho \cos(\Phi) \sin(\Theta_1)$$

$$\frac{d2}{\rho} = \sin(\Phi) \cos(\Theta_1) - \cos(\Phi) \sin(\Theta_1)$$

$$\frac{d2}{\rho} = \sin(\Phi - \Theta_1)$$

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$\frac{d_2}{\rho}$ = sin(Φ - Θ_1)

cos(Φ - Θ_1) = ±$\sqrt{1 - \frac{{d_2}^2}{\rho^2}}$

Φ - Θ_1 = atan2 ($\frac{d_2}{\rho}$, ±$\sqrt{1 - \frac{{d_2}^2}{\rho^2}}$)

Θ_1 = Φ - atan2 (d_2 , ±$\sqrt{{d_x}^2 + dy^2 \ - {d_2}^2}$)

Θ_1 = atan2($d_{y'}$ dx) - atan2 (d_2 , ±$\sqrt{{d_x}^2 + dy^2 \ - {d_2}^2}$)

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inv(A1)*T06 =

$$
\begin{bmatrix}
r11*c(1) + r21*s(1), & r12*c(1) + r22*s(1), & r13*c(1) + r23*s(1), & dx*c(1) + dy*s(1) \\
-r31, & -r32, & -r33, & -dz \\
r21*c(1) - r11*s(1), & r22*c(1) - r12*s(1), & r23*c(1) - r13*s(1), & dy*c(1) - dx*s(1) \\
0, & 0, & 0, & 1
\end{bmatrix}
$$

dx^2 + dy^2 + dz^2

A23456 =

(1,1): - c(6)*(s(2 + 3)*s(5) - c(2 + 3)*c(4)*c(5)) - c(2 + 3)*s(4)*s(6)
(1,2): s(6)*(s(2 + 3)*s(5) - c(2 + 3)*c(4)*c(5)) - c(2 + 3)*c(6)*s(4)
(1,3): s(2 + 3)*c(5) + c(2 + 3)*c(4)*s(5)
(1,4): **a3*c(2 + 3) + d4*s(2 + 3) + a2*c(2)**

(2,1): c(6)*(c(2 + 3)*s(5) + s(2 + 3)*c(4)*c(5)) - s(2 + 3)*s(4)*s(6)
(2,2): - s(6)*(c(2 + 3)*s(5) + s(2 + 3)*c(4)*c(5)) - s(2 + 3)*c(6)*s(4)
(2,3): s(2 + 3)*c(4)*s(5) - c(2 + 3)*c(5)
(2,4): **a3*s(2 + 3) - d4*c(2 + 3) + a2*s(2)**

(3,1): c(4)*s(6) + c(5)*c(6)*s(4)
(3,2): c(4)*c(6) - c(5)*s(4)*s(6)
(3,3): s(4)*s(5)
(3,4): **d2**

+A23456(1,4)^2 + A23456(2,4)^2 + A23456(3,4)^2

a2^2 + 2*cos(t3)*a2*a3 + 2*sin(t3)*a2*d4 + a3^2 + d2^2 + d4^2

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$$a_2^2 + 2*\cos(\Theta_3)*a_2*a_3 + 2*\sin(\Theta_3)*a_2*d_4 + a_3^2 + d_2^2 + d_4^2 = d_x^2 + d_y^2 + d_z^2$$

$$2*\cos(\Theta_3)*a_2*a_3 + 2*\sin(\Theta_3)*a_2*d_4 = d_x^2 + d_y^2 + d_z^2 - a_2^2 - a_3^2 - d_2^2 - d_4^2$$

$$\cos(\Theta_3)*a_3 + \sin(\Theta_3)*d_4 = \frac{d_x^2 + dy^2 + dz^2 - a_2^2 - a_3^2 - d_2^2 - d_4^2}{2*a_2}$$

$\cos(\Theta_3)*a_3 + \sin(\Theta_3)*d_4 = K$ where $K = \dfrac{d_x^2 + dy^2 + dz^2 - a_2^2 - a_3^2 - d_2^2 - d_4^2}{2*a_2}$

By the trigonometric substitutions: $a_3 = \rho\cos(\Phi)$, $d_4 = \rho\sin(\Phi)$

Where

$\rho = \sqrt{a_3^2 + d_4^2}$, $\Phi = \mathrm{atan2}(d_4, a_3)$

We have therefore

$$K = \rho\cos(\Phi)\cos(\Theta_3) + \rho\sin(\Phi)\sin(\Theta_3)$$

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$K = \rho \cos(\Phi) \cos(\Theta_3) + \rho \sin(\Phi) \sin(\Theta_3)$

$\frac{K}{\rho} = \cos(\Phi) \cos(\Theta_3) + \sin(\Phi) \sin(\Theta_3)$

$\frac{K}{\rho} = \cos(\Phi - \Theta_3)$

$\sin(\Phi - \Theta_3) = \pm\sqrt{1 - \frac{K^2}{\rho^2}}$

$\Phi - \Theta_3 = \text{atan2}\left(\pm\sqrt{1 - \frac{K^2}{\rho^2}}, \frac{K}{\rho}\right)$

$\Theta_3 = \Phi - \text{atan2}\left(\pm\sqrt{a_3{}^2 + d42 - K2}, K\right)$

$\Theta_3 = \text{atan2}(d_4, a_3) - \text{atan2}\left(\pm\sqrt{a_3{}^2 + d42 \; \bar{} \; K^2}, K\right)$

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inv(A2)*inv(A1)*T06 =

(1,1): r11*c(1)*c(2) - r31*s(2) + r21*c(2)*s(1)
(1,2): r12*c(1)*c(2) - r32*s(2) + r22*c(2)*s(1)
(1,3): r13*c(1)*c(2) - r33*s(2) + r23*c(2)*s(1)
(1,4): dx*c(1)*c(2) - dz*s(2) - a2 + dy*c(2)*s(1)

(2,1): - r31*c(2) - r11*c(1)*s(2) - r21*s(1)*s(2)
(2,2): - r32*c(2) - r12*c(1)*s(2) - r22*s(1)*s(2)
(2,3): - r33*c(2) - r13*c(1)*s(2) - r23*s(1)*s(2)
(2,4): - dz*c(2) - dx*c(1)*s(2) - dy*s(1)*s(2)

(3,1): r21*c(1) - r11*s(1)
(3,2): r22*c(1) - r12*s(1)
(3,3): r23*c(1) - r13*s(1)
(3,4): dy*c(1) - d2 - dx*s(1)

A3456 =

(1,1): - c(6)*(s(3)*s(5) - c(3)*c(4)*c(5)) - c(3)*s(4)*s(6)
(1,2): s(6)*(s(3)*s(5) - c(3)*c(4)*c(5)) - c(3)*c(6)*s(4)
(1,3): c(5)*s(3) + c(3)*c(4)*s(5)
(1,4): a3*c(3) + d4*s(3)

(2,1): c(6)*(c(3)*s(5) + c(4)*c(5)*s(3)) - s(3)*s(4)*s(6)
(2,2): - s(6)*(c(3)*s(5) + c(4)*c(5)*s(3)) - c(6)*s(3)*s(4)
(2,3): c(4)*s(3)*s(5) - c(3)*c(5)
(2,4): a3*s(3) - d4*c(3)

(3,1): c(4)*s(6) + c(5)*c(6)*s(4)
(3,2): c(4)*c(6) - c(5)*s(4)*s(6)
(3,3): s(4)*s(5)
(3,4): 0

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inv(A3)*inv(A2)*inv(A1)*T06=

(1,1): r11*c(1)*c(2)*c(3) - r31*c(3)*s(2) - r31*c(2)*s(3) + r21*c(2)*c(3)*s(1) - r11*c(1)*s(2)*s(3) - r21*s(1)*s(2)*s(3)
(1,2): r12*c(1)*c(2)*c(3) - r32*c(3)*s(2) - r32*c(2)*s(3) + r22*c(2)*c(3)*s(1) - r12*c(1)*s(2)*s(3) - r22*s(1)*s(2)*s(3)
(1,3): r13*c(1)*c(2)*c(3) - r33*c(3)*s(2) - r33*c(2)*s(3) + r23*c(2)*c(3)*s(1) - r13*c(1)*s(2)*s(3) - r23*s(1)*s(2)*s(3)
(1,4): **dx*c(1)*c(2)*c(3) - a2*c(3) - dz*c(2)*s(3) - dz*c(3)*s(2) - a3 + dy*c(2)*c(3)*s(1) - dx*c(1)*s(2)*s(3) - dy*s(1)*s(2)*s(3)**

(2,1): r21*c(1) - r11*s(1)
(2,2): r22*c(1) - r12*s(1)
(2,3): r23*c(1) - r13*s(1)
(2,4): dy*c(1) - d2 - dx*s(1)

(3,1): r31*c(2)*c(3) - r31*s(2)*s(3) + r11*c(1)*c(2)*s(3) + r11*c(1)*c(3)*s(2) + r21*c(2)*s(1)*s(3) + r21*c(3)*s(1)*s(2)
(3,2): r32*c(2)*c(3) - r32*s(2)*s(3) + r12*c(1)*c(2)*s(3) + r12*c(1)*c(3)*s(2) + r22*c(2)*s(1)*s(3) + r22*c(3)*s(1)*s(2)
(3,3): r33*c(2)*c(3) - r33*s(2)*s(3) + r13*c(1)*c(2)*s(3) + r13*c(1)*c(3)*s(2) + r23*c(2)*s(1)*s(3) + r23*c(3)*s(1)*s(2)
(3,4): **dz*c(2)*c(3) - a2*s(3) - dz*s(2)*s(3) + dx*c(1)*c(2)*s(3) + dx*c(1)*c(3)*s(2) + dy*c(2)*s(1)*s(3) + dy*c(3)*s(1)*s(2)**

A456 =

(1,1): c(4)*c(5)*c(6) - s(4)*s(6)
(1,2): - c(6)*s(4) - c(4)*c(5)*s(6)
(1,3): c(4)*s(5)
(1,4): **0**

(2,1): c(4)*s(6) + c(5)*c(6)*s(4)
(2,2): c(4)*c(6) - c(5)*s(4)*s(6)
(2,3): s(4)*s(5)
(2,4): 0

(3,1): -c(6)*s(5)
(3,2): s(5)*s(6)
(3,3): c(5)
(3,4): **d4**

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Reminder:

$$\cos(\Theta_1 + \Theta_2) = \cos\Theta_1 \cos\Theta_2 - \sin\Theta_1 \sin\Theta_2$$
$$\sin(\Theta_1 + \Theta_2) = \cos\Theta_1 \sin\Theta_2 + \sin\Theta_1 \cos\Theta_2$$

dx*c(1)*c(2)*c(3) - a2*c(3) - dz*c(2)*s(3) - dz*c(3)*s(2) - a3 + dy*c(2)*c(3)*s(1) - dx*c(1)*s(2)*s(3) - dy*s(1)*s(2)*s(3) = 0

dz*c(2)*c(3) - a2*s(3) - dz*s(2)*s(3) + dx*c(1)*c(2)*s(3) + dx*c(1)*c(3)*s(2) + dy*c(2)*s(1)*s(3) + dy*c(3)*s(1)*s(2) = d4

dx*c(1)*c(2)*c(3) - a2*c(3) - dz*c(2)*s(3) - dz*c(3)*s(2) - a3 + dy*c(2)*c(3)*s(1) - dx*c(1)*s(2)*s(3) - dy*s(1)*s(2)*s(3) = 0

dz*c(2)*c(3) - a2*s(3) - dz*s(2)*s(3) + dx*c(1)*c(2)*s(3) + dx*c(1)*c(3)*s(2) + dy*c(2)*s(3)*s(1) + dy*c(3)*s(2)*s(1) = d4

dx*c(1)*c(2+3) - a2*c(3) - dz*s(2+3) - a3 + dy*s(1)*c(2+3) = 0

dz*c(2+3) - a2*s(3) + dx*c(1)*s(2+3) + dy* s(1) *s(2+3) = d4

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dx*c(1)*c(2+3) - a2*c(3) - dz*s(2+3) - a3 + dy*s(1)*c(2+3) = 0

dz*c(2+3) - a2*s(3) + dx*c(1)*s(2+3) + dy* s(1) *s(2+3) = d4

(dx*c(1) + dy*s(1))*c(2+3) - dz*s(2+3) = a2*c(3) + a3
dz*c(2+3) + (dx*c(1)+ dy* s(1))*s(2+3) = a2*s(3) + d4

$$\begin{bmatrix} dx*c(1) + dy*s(1) & -dz \\ dz & dx*c(1)+ dy* s(1) \end{bmatrix} \begin{bmatrix} c(2+3) \\ s(2+3) \end{bmatrix} = \begin{bmatrix} a2*c(3) + a3 \\ a2*s(3) + d4 \end{bmatrix}$$

$$\Theta_2+\Theta_3 = \text{atan2}(\sin(\Theta_2+\Theta_3), \cos(\Theta_2+\Theta_3))$$

$$\Theta_2 = \text{atan2}(\sin(\Theta_2+\Theta_3), \cos(\Theta_2+\Theta_3)) - \Theta_3$$

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inv(A3)*inv(A2)*inv(A1)*T06=

(1,1): r11*c(1)*c(2)*c(3) - r31*c(3)*s(2) - r31*c(2)*s(3) + r21*c(2)*c(3)*s(1) - r11*c(1)*s(2)*s(3) - r21*s(1)*s(2)*s(3)
(1,2): r12*c(1)*c(2)*c(3) - r32*c(3)*s(2) - r32*c(2)*s(3) + r22*c(2)*c(3)*s(1) - r12*c(1)*s(2)*s(3) - r22*s(1)*s(2)*s(3)
(1,3): r13*c(1)*c(2)*c(3) - r33*c(3)*s(2) - r33*c(2)*s(3) + r23*c(2)*c(3)*s(1) - r13*c(1)*s(2)*s(3) - r23*s(1)*s(2)*s(3)
(1,4): **dx*c(1)*c(2)*c(3) - a2*c(3) - dz*c(2)*s(3) - dz*c(3)*s(2) - a3 + dy*c(2)*c(3)*s(1) - dx*c(1)*s(2)*s(3) - dy*s(1)*s(2)*s(3)**

(2,1): r21*c(1) - r11*s(1)
(2,2): r22*c(1) - r12*s(1)
(2,3): r23*c(1) - r13*s(1)
(2,4): dy*c(1) - d2 - dx*s(1)

(3,1): r31*c(2)*c(3) - r31*s(2)*s(3) + r11*c(1)*c(2)*s(3) + r11*c(1)*c(3)*s(2) + r21*c(2)*s(1)*s(3) + r21*c(3)*s(1)*s(2)
(3,2): r32*c(2)*c(3) - r32*s(2)*s(3) + r12*c(1)*c(2)*s(3) + r12*c(1)*c(3)*s(2) + r22*c(2)*s(1)*s(3) + r22*c(3)*s(1)*s(2)
(3,3): r33*c(2)*c(3) - r33*s(2)*s(3) + r13*c(1)*c(2)*s(3) + r13*c(1)*c(3)*s(2) + r23*c(2)*s(1)*s(3) + r23*c(3)*s(1)*s(2)
(3,4): **dz*c(2)*c(3) - a2*s(3) - dz*s(2)*s(3) + dx*c(1)*c(2)*s(3) + dx*c(1)*c(3)*s(2) + dy*c(2)*s(1)*s(3) + dy*c(3)*s(1)*s(2)**

A456 =

(1,1): c(4)*c(5)*c(6) - s(4)*s(6)
(1,2): - c(6)*s(4) - c(4)*c(5)*s(6)
(1,3): c(4)*s(5)
(1,4): **0**

(2,1): c(4)*s(6) + c(5)*c(6)*s(4)
(2,2): c(4)*c(6) - c(5)*s(4)*s(6)
(2,3): s(4)*s(5)
(2,4): 0

(3,1): -c(6)*s(5)
(3,2): s(5)*s(6)
(3,3): c(5)
(3,4): **d4**

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inv(A3)*inv(A2)*inv(A1)*T06=

(1,1): r11*c(1)*c(2)*c(3) - r31*c(3)*s(2) - r31*c(2)*s(3) + r21*c(2)*c(3)*s(1) - r11*c(1)*s(2)*s(3) - r21*s(1)*s(2)*s(3)
(1,2): r12*c(1)*c(2)*c(3) - r32*c(3)*s(2) - r32*c(2)*s(3) + r22*c(2)*c(3)*s(1) - r12*c(1)*s(2)*s(3) - r22*s(1)*s(2)*s(3)
(1,3): r13*c(1)*c(2)*c(3) - r33*c(3)*s(2) - r33*c(2)*s(3) + r23*c(2)*c(3)*s(1) - r13*c(1)*s(2)*s(3) - r23*s(1)*s(2)*s(3)
(1,4): **dx*c(1)*c(2)*c(3) - a2*c(3) - dz*c(2)*s(3) - dz*c(3)*s(2) - a3 + dy*c(2)*c(3)*s(1) - dx*c(1)*s(2)*s(3) - dy*s(1)*s(2)*s(3)**

(2,1): r21*c(1) - r11*s(1)
(2,2): r22*c(1) - r12*s(1)
(2,3): r23*c(1) - r13*s(1)
(2,4): dy*c(1) - d2 - dx*s(1)

(3,1): r31*c(2)*c(3) - r31*s(2)*s(3) + r11*c(1)*c(2)*s(3) + r11*c(1)*c(3)*s(2) + r21*c(2)*s(1)*s(3) + r21*c(3)*s(1)*s(2)
(3,2): r32*c(2)*c(3) - r32*s(2)*s(3) + r12*c(1)*c(2)*s(3) + r12*c(1)*c(3)*s(2) + r22*c(2)*s(1)*s(3) + r22*c(3)*s(1)*s(2)
(3,3): r33*c(2)*c(3) - r33*s(2)*s(3) + r13*c(1)*c(2)*s(3) + r13*c(1)*c(3)*s(2) + r23*c(2)*s(1)*s(3) + r23*c(3)*s(1)*s(2)
(3,4): **dz*c(2)*c(3) - a2*s(3) - dz*s(2)*s(3) + dx*c(1)*c(2)*s(3) + dx*c(1)*c(3)*s(2) + dy*c(2)*s(1)*s(3) + dy*c(3)*s(1)*s(2)**

A456 =

(1,1): c(4)*c(5)*c(6) - s(4)*s(6)
(1,2): - c(6)*s(4) - c(4)*c(5)*s(6)
(1,3): c(4)*s(5)
(1,4): **0**

(2,1): c(4)*s(6) + c(5)*c(6)*s(4)
(2,2): c(4)*c(6) - c(5)*s(4)*s(6)
(2,3): s(4)*s(5)
(2,4): 0

(3,1): -c(6)*s(5)
(3,2): s(5)*s(6)
(3,3): c(5)
(3,4): **d4**

PUMA 260

If Θ_5 non zero:

$$\Theta_4 = \text{atan2}(\ r23*c(1) - r13*s(1)\ ,\ r13*c(1)*c(2+3) - r33*s(2+3) + r23*c(2+3)*s(1)\ )$$

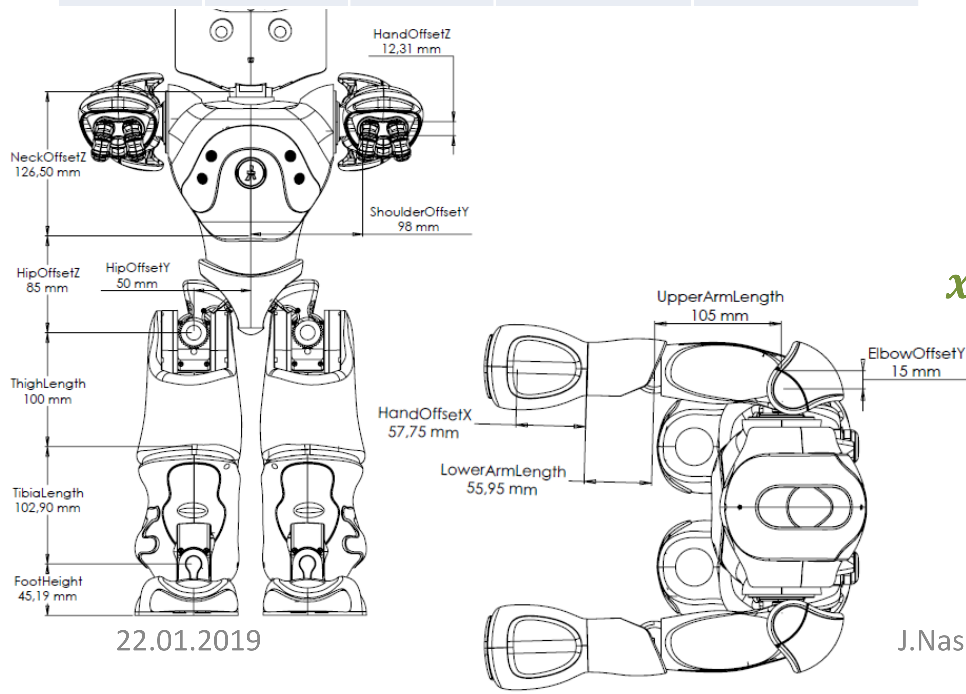
When Θ_5 = 0, the manipulator is in a singular configuration.
Joint 4 and 6 cause the same motion of the end-effector.

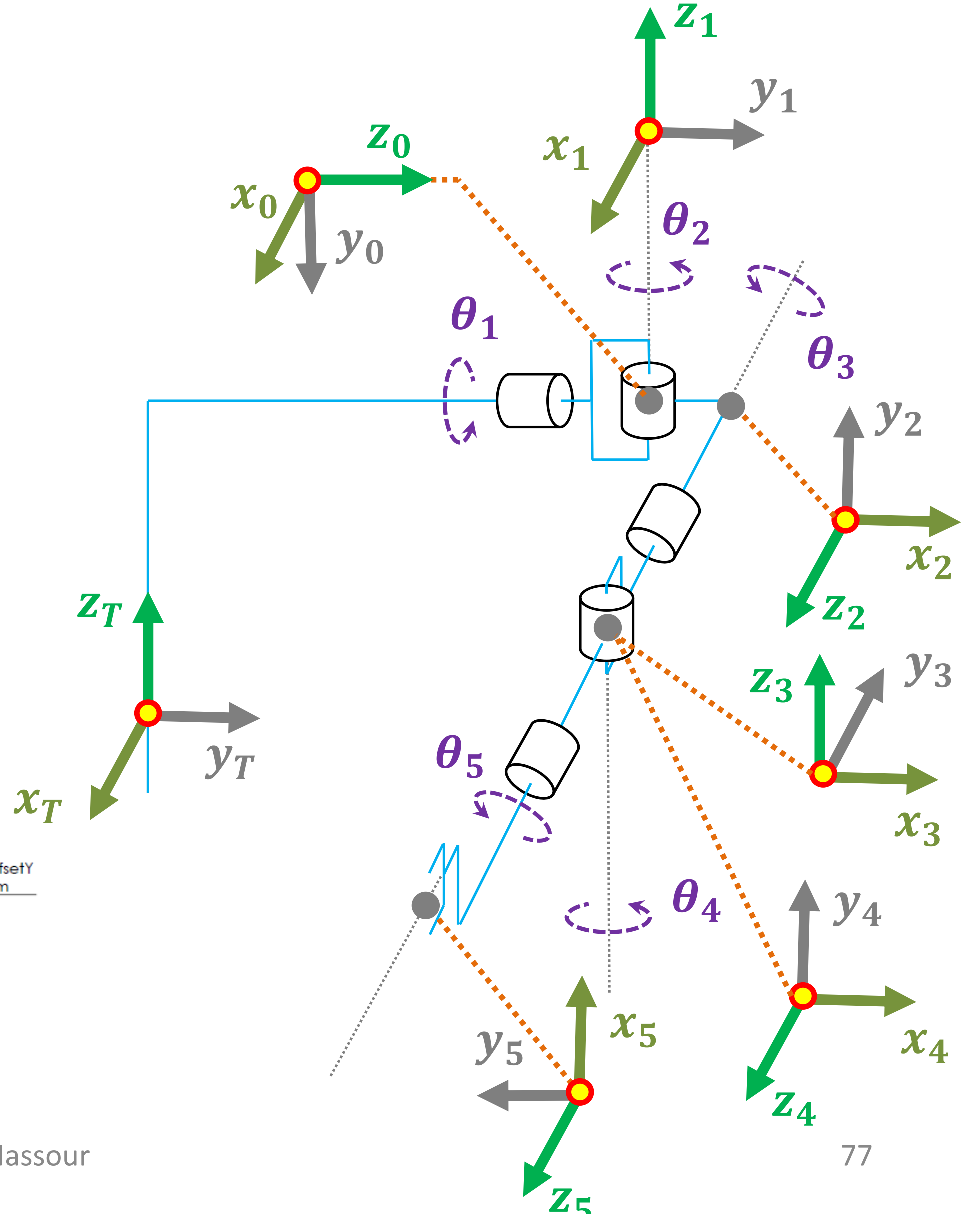
Homework: Work out Θ_5 and Θ_6.

NAO Left Arm

a_i is distance from z_{i-1} to z_i measured along x_i.
α_i is angle from z_{i-1} to z_i measured about x_i.
d_i is distance from x_{i-1} to x_i measured along z_{i-1}.
θ_i is angle from x_{i-1} to x_i measured about z_{i-1}.

i	a_i	α_i	d_i	θ_i
0	T_0^{BASE} ((			
1				
2				
3				
4				
5				



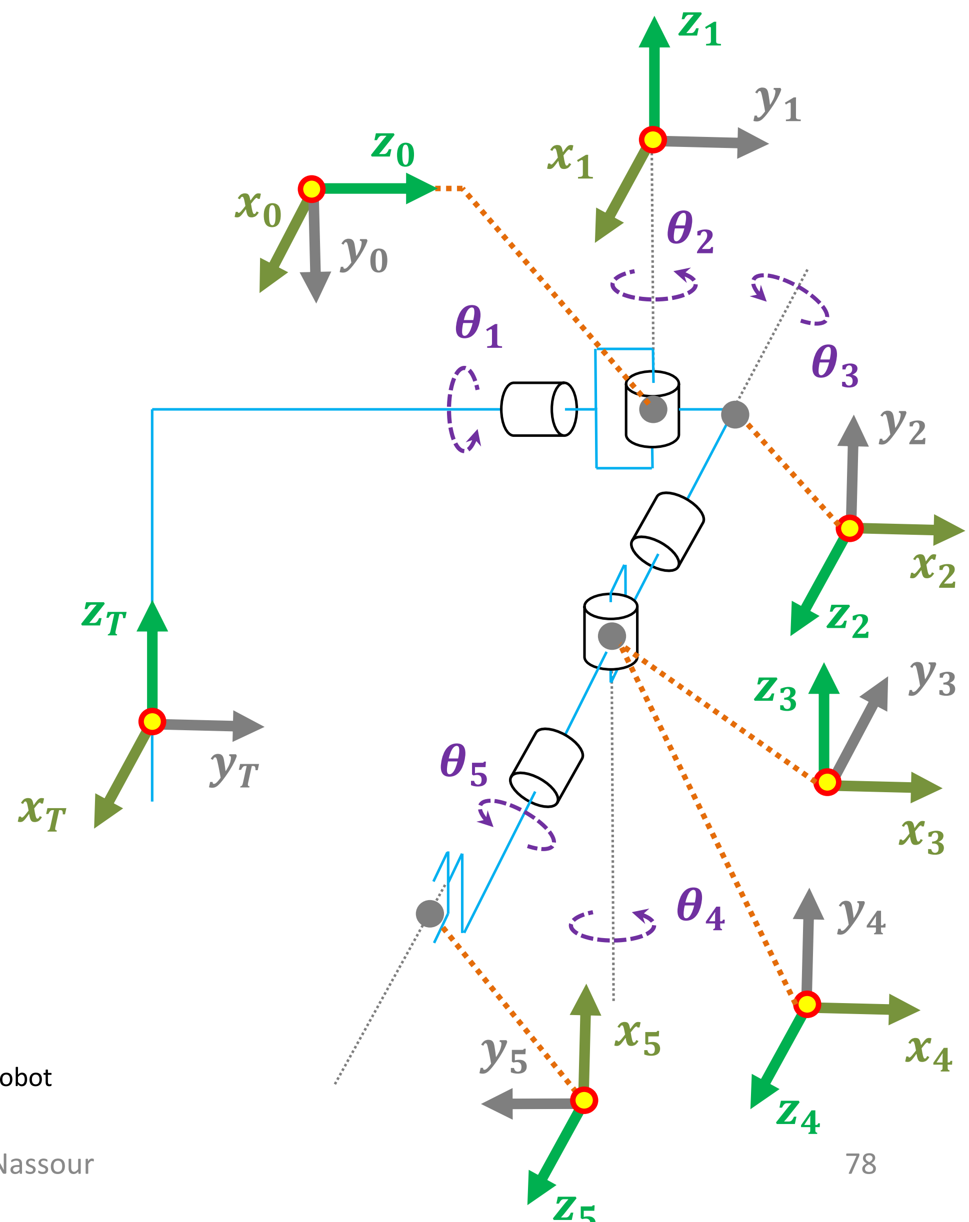


NAO Left Arm

i	a_i	α_i	d_i	θ_i
0	$T^{BASE}_{0}(0, ShoulderOffsetY, ShoulderOffsetZ)$			
1	0	90°	0	θ_1 *
2	a_2	90°	0	$\left(\frac{\pi}{2}\right) + \theta_2$ *
3	0	-90°	d_3	θ_3 *
4	0	$+90^\circ$	0	θ_4 *
5	a_5	0°	d_5	$\left(\frac{\pi}{2}\right) + \theta_5$ *

```
a2 = ElbowOffsetY;
y0 = ShoulderOffsetY;
z0 = ShoulderOffsetZ;
d3 = UpperArmLength;
ye = HandOffsetZ;
ze = LowerArmLength + HandOffsetX;
```

$$T^{BASE}_{5} = ?$$



Kofinas, N.; Orfanoudakis, E. & Lagoudakis, M.
Complete Analytical Forward and Inverse Kinematics for the NAO Humanoid Robot
Journal of Intelligent & Robotic Systems, Springer Netherlands, 2014, 1-14

NAO Left Arm

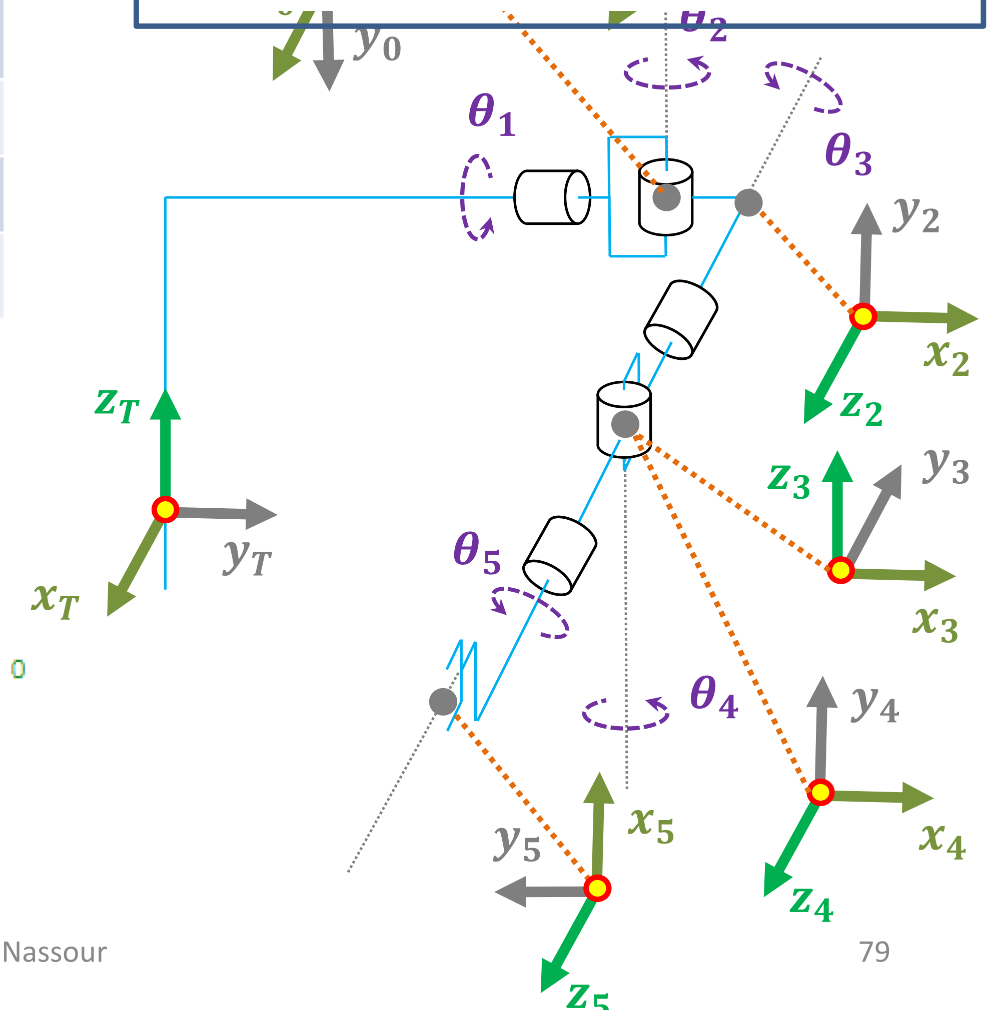
i	a_i	α_i	d_i	θ_i
0	$T^{BASE}_{0}\,(0, ShoulderOffsetY, ShoulderOffsetZ)$			
1	0	90°	0	θ_1 *
2	a_2	90°	0	$\left(\frac{\pi}{2}\right)+\theta_2$ *
3	0	-90°	d_3	θ_3 *
4	0	$+90^\circ$	0	θ_4 *
5	a_5	0°	d_5	$\left(\frac{\pi}{2}\right)+\theta_5$ *

```
a2 = ElbowOffsetY;
y0 = ShoulderOffsetY;
z0 = ShoulderOffsetZ;
d3 = UpperArmLength;
ye = HandOffsetZ;
ze = LowerArmLength + HandOffsetX;
```

$$T^{BASE}_{5} = ?$$

Reminder: A_i

$$\begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$




```
% translation from the base frame to the frame 0

A0=[          1,     0,    0,    0;...
              0,     0,    1,    y0;...
              0,    -1,    0,    z0;...
              0,     0,    0,    1 ];
```

NAO Left Arm

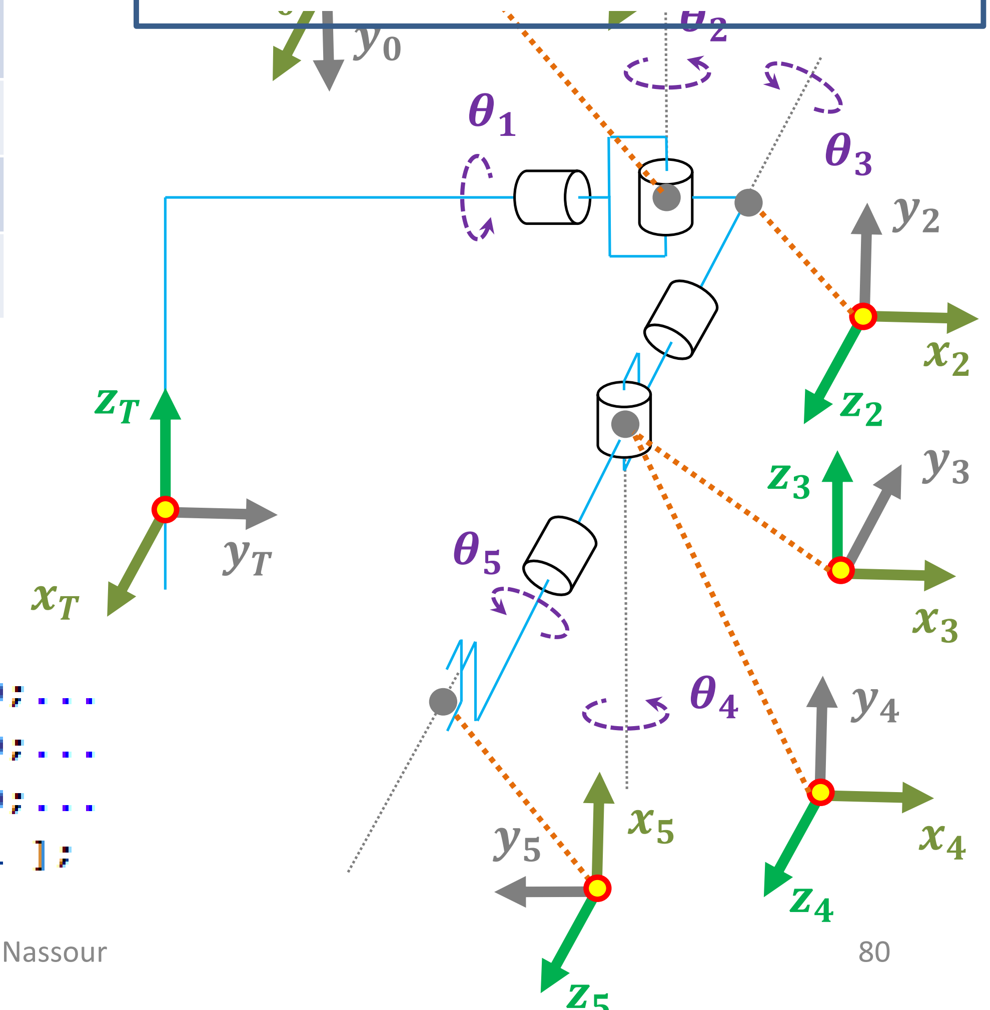
i	a_i	α_i	d_i	θ_i
0	$T^{BASE}{}_0\,(0, ShoulderOffsetY, ShoulderOffsetZ)$			
1	0	90 °	0	θ_1 *
2	a_2	90 °	0	$\left(\frac{\pi}{2}\right) + \theta_2$ *
3	0	–90 °	d_3	θ_3 *
4	0	+90 °	0	θ_4 *
5	a_5	0 °	d_5	$\left(\frac{\pi}{2}\right) + \theta_5$ *

```
a2 = ElbowOffsetY;
y0 = ShoulderOffsetY;
z0 = ShoulderOffsetZ;
d3 = UpperArmLength;
ye = HandOffsetZ;
ze = LowerArmLength + HandOffsetX;
```

$$T^{BASE}_{5} = ?$$

Reminder: A_i

$$\begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$




```
A1=[     cos(t1),      0,    sin(t1),     0;...
         sin(t1),      0,   -cos(t1),     0;...
               0,      1,          0,     0;...
               0,      0,          0,     1 ];
```

NAO Left Arm

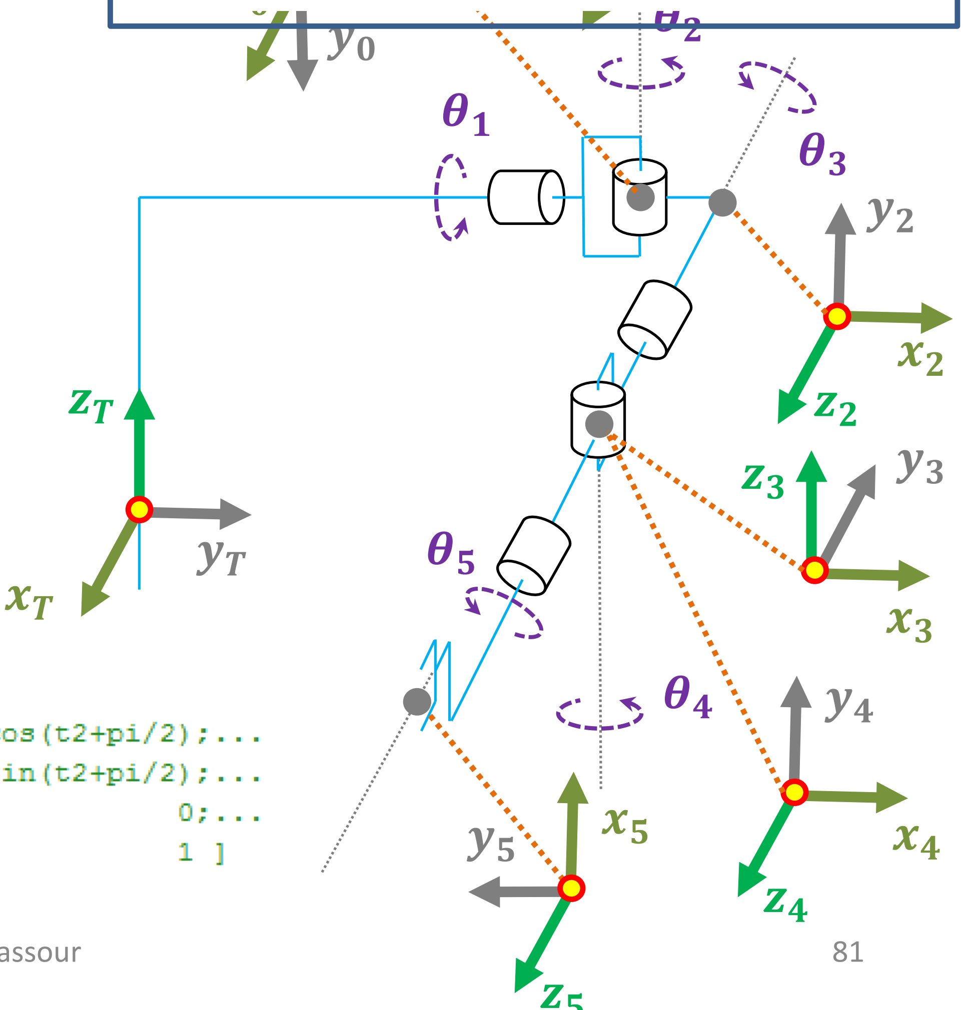
i	a_i	α_i	d_i	θ_i
0	$T^{BASE}_0\,(0, ShoulderOffsetY, ShoulderOffsetZ)$			
1	0	90°	0	θ_1 *
2	a_2	90°	0	$\left(\frac{\pi}{2}\right) + \theta_2$ *
3	0	-90°	d_3	θ_3 *
4	0	$+90^\circ$	0	θ_4 *
5	a_5	0°	d_5	$\left(\frac{\pi}{2}\right) + \theta_5$ *

```
a2 = ElbowOffsetY;
y0 = ShoulderOffsetY;
z0 = ShoulderOffsetZ;
d3 = UpperArmLength;
ye = HandOffsetZ;
ze = LowerArmLength + HandOffsetX;
```

$$T^{BASE}_{5} = ?$$

Reminder: A_i

$$\begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$




```
%A2=[    cos(t2+pi/2),        0,  sin(t2+pi/2),  a2*cos(t2+pi/2);...
%        sin(t2+pi/2),        0, -cos(t2+pi/2),  a2*sin(t2+pi/2);...
%                   0,        1,             0,                0;...
%                   0,        0,             0,                1 ]
```

NAO Left Arm

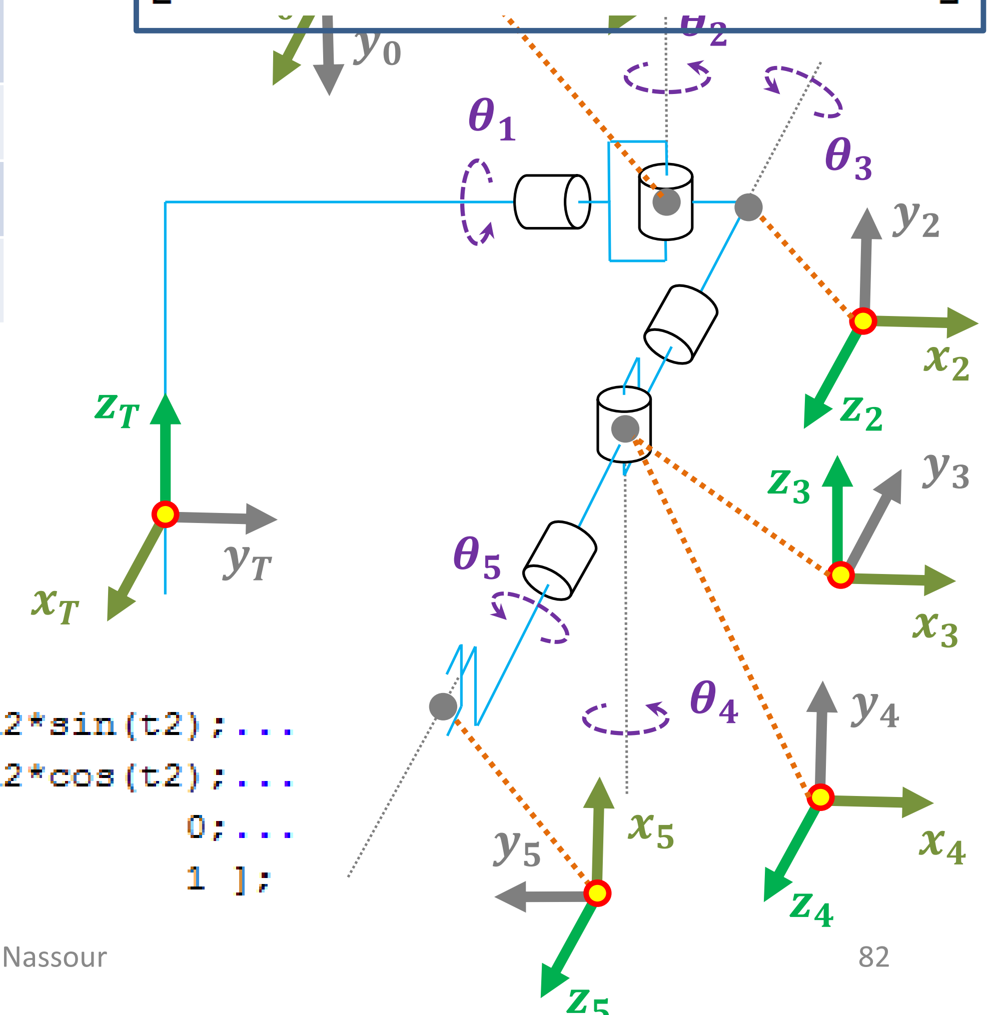
i	a_i	α_i	d_i	θ_i
0	$T^{BASE}_{0}(0, ShoulderOffsetY, ShoulderOffsetZ)$			
1	0	90°	0	$\boldsymbol{\theta_1}$ *
2	a_2	90°	0	$\left(\frac{\pi}{2}\right) + \boldsymbol{\theta_2}$ *
3	0	-90°	d_3	$\boldsymbol{\theta_3}$ *
4	0	$+90^\circ$	0	$\boldsymbol{\theta_4}$ *
5	a_5	0°	d_5	$\left(\frac{\pi}{2}\right) + \boldsymbol{\theta_5}$ *

Reminder: A_i

$$\begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
a2 = ElbowOffsetY;
y0 = ShoulderOffsetY;
z0 = ShoulderOffsetZ;
d3 = UpperArmLength;
ye = HandOffsetZ;
ze = LowerArmLength + HandOffsetX;
```

$$T^{BASE}_{5} = ?$$




```
A2=[   -sin(t2),        0,   cos(t2), -a2*sin(t2);...
        cos(t2),        0,   sin(t2),  a2*cos(t2);...
              0,        1,         0,           0;...
              0,        0,         0,           1 ];
```

NAO Left Arm

i	a_i	α_i	d_i	θ_i
0	$T^{BASE}_0\,(0, ShoulderOffsetY, ShoulderOffsetZ)$			
1	0	90 °	0	θ_1 *
2	a_2	90 °	0	$\left(\frac{\pi}{2}\right)+\theta_2$ *
3	0	–90 °	d_3	θ_3 *
4	0	+90 °	0	θ_4 *
5	a_5	0 °	d_5	$\left(\frac{\pi}{2}\right)+\theta_5$ *

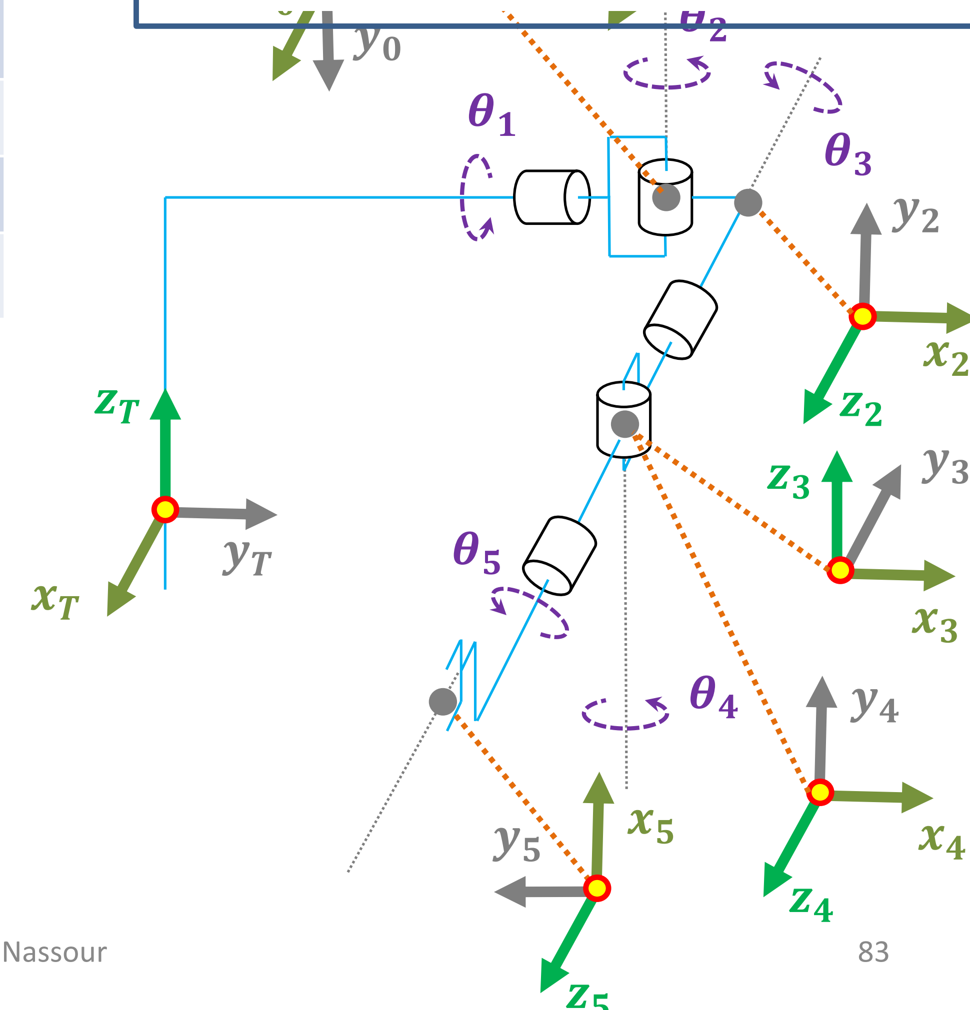
```
a2 = ElbowOffsetY;
y0 = ShoulderOffsetY;
z0 = ShoulderOffsetZ;
d3 = UpperArmLength;
ye = HandOffsetZ;
ze = LowerArmLength + HandOffsetX;
```

$$T^{BASE}_{5} = ?$$

```
A3=[    cos(t3),    0,  -sin(t3),    0;...
        sin(t3),    0,   cos(t3),    0;...
              0,   -1,         0,   d3;...
              0,    0,         0,    1 ];
```

Reminder: A_i

$$\begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



NAO Left Arm

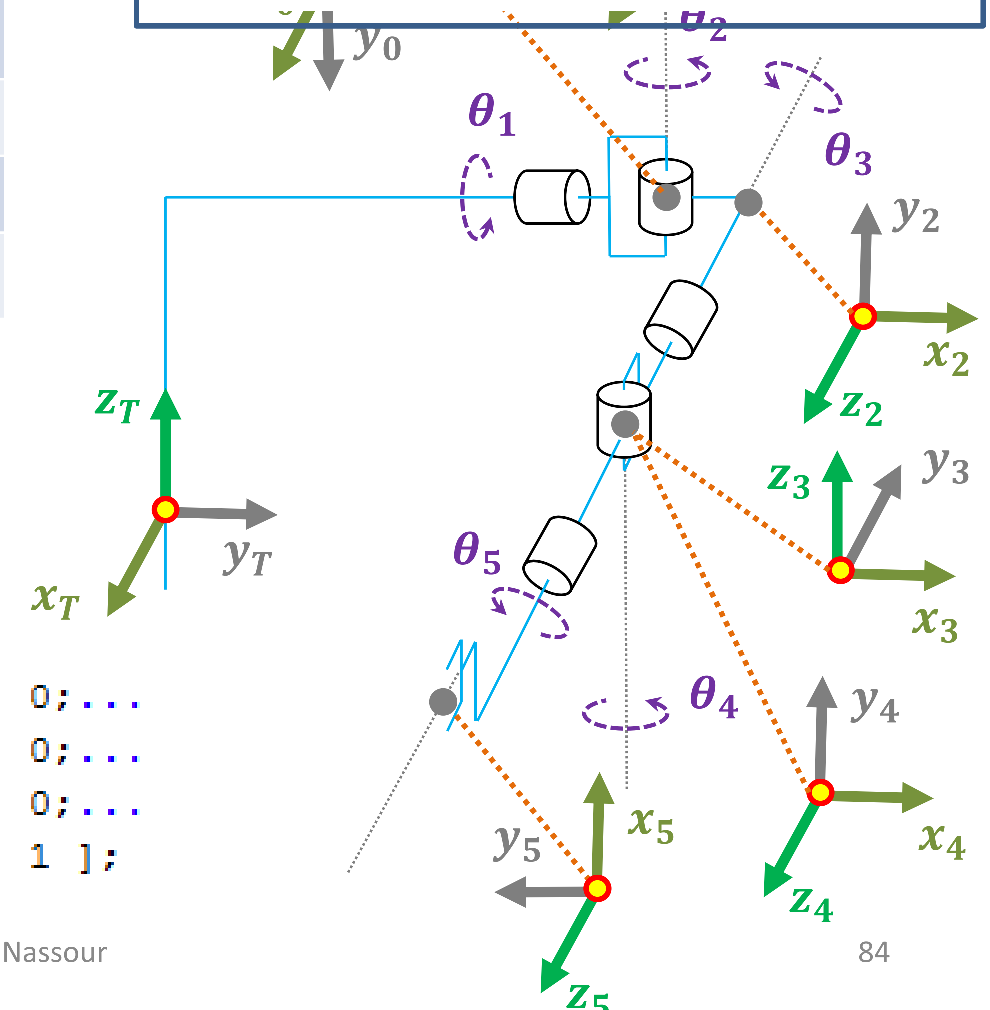
i	a_i	α_i	d_i	θ_i
0	$T^{BASE}{}_{0}\,(0, ShoulderOffsetY, ShoulderOffsetZ)$			
1	0	90°	0	θ_1 *
2	a_2	90°	0	$\left(\frac{\pi}{2}\right) + \theta_2$ *
3	0	-90°	d_3	θ_3 *
4	0	$+90^\circ$	0	θ_4 *
5	a_5	0°	d_5	$\left(\frac{\pi}{2}\right) + \theta_5$ *

```
a2 = ElbowOffsetY;
y0 = ShoulderOffsetY;
z0 = ShoulderOffsetZ;
d3 = UpperArmLength;
ye = HandOffsetZ;
ze = LowerArmLength + HandOffsetX;
```

$$T_5^{BASE} = ?$$

Reminder: A_i

$$\begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$




```
A4=[    cos(t4),    0,   sin(t4),   0;...
        sin(t4),    0,  -cos(t4),   0;...
              0,    1,         0,   0;...
              0,    0,         0,   1 ];
```

NAO Left Arm

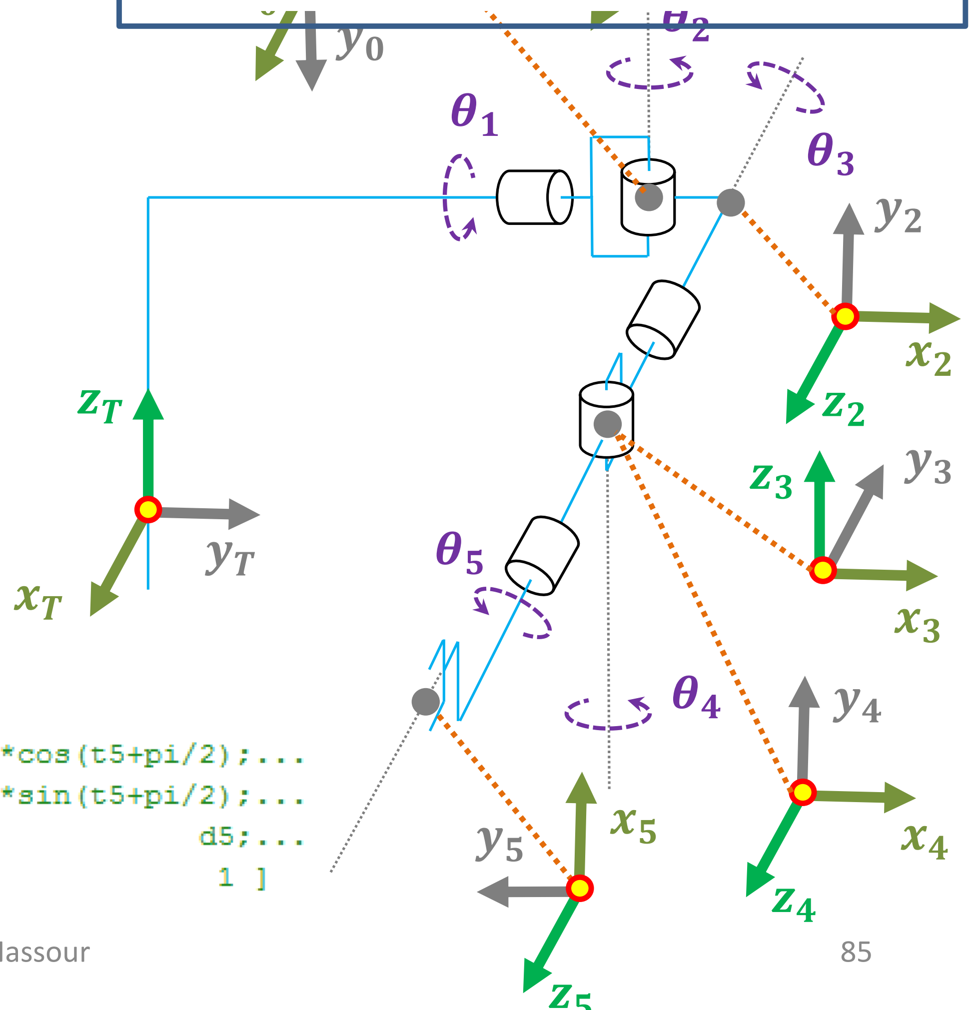
i	a_i	α_i	d_i	θ_i
0	$T^{BASE}_0(0, ShoulderOffsetY, ShoulderOffsetZ)$			
1	0	90°	0	θ_1 *
2	a_2	90°	0	$\left(\frac{\pi}{2}\right) + \theta_2$ *
3	0	−90°	d_3	θ_3 *
4	0	+90°	0	θ_4 *
5	a_5	0°	d_5	$\left(\frac{\pi}{2}\right) + \theta_5$ *

```
a2 = ElbowOffsetY;
y0 = ShoulderOffsetY;
z0 = ShoulderOffsetZ;
d3 = UpperArmLength;
ye = HandOffsetZ;
ze = LowerArmLength + HandOffsetX;
```

$$T^{BASE}_{5} = ?$$

Reminder: A_i

$$\begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$




```
%A5=[    cos(t5+pi/2),        -sin(t5+pi/2), 0,  a5*cos(t5+pi/2);...
%        sin(t5+pi/2),         cos(t5+pi/2), 0,  a5*sin(t5+pi/2);...
%                   0,                    0, 1,               d5;...
%                   0,                    0, 0,                1 ]
```

NAO Left Arm

i	a_i	α_i	d_i	θ_i
0	$T^{BASE}{}_0\,(0, ShoulderOffsetY, ShoulderOffsetZ)$			
1	0	90°	0	θ_1 *
2	a_2	90°	0	$\left(\frac{\pi}{2}\right) + \theta_2$ *
3	0	-90°	d_3	θ_3 *
4	0	$+90^\circ$	0	θ_4 *
5	a_5	0°	d_5	$\left(\frac{\pi}{2}\right) + \theta_5$ *

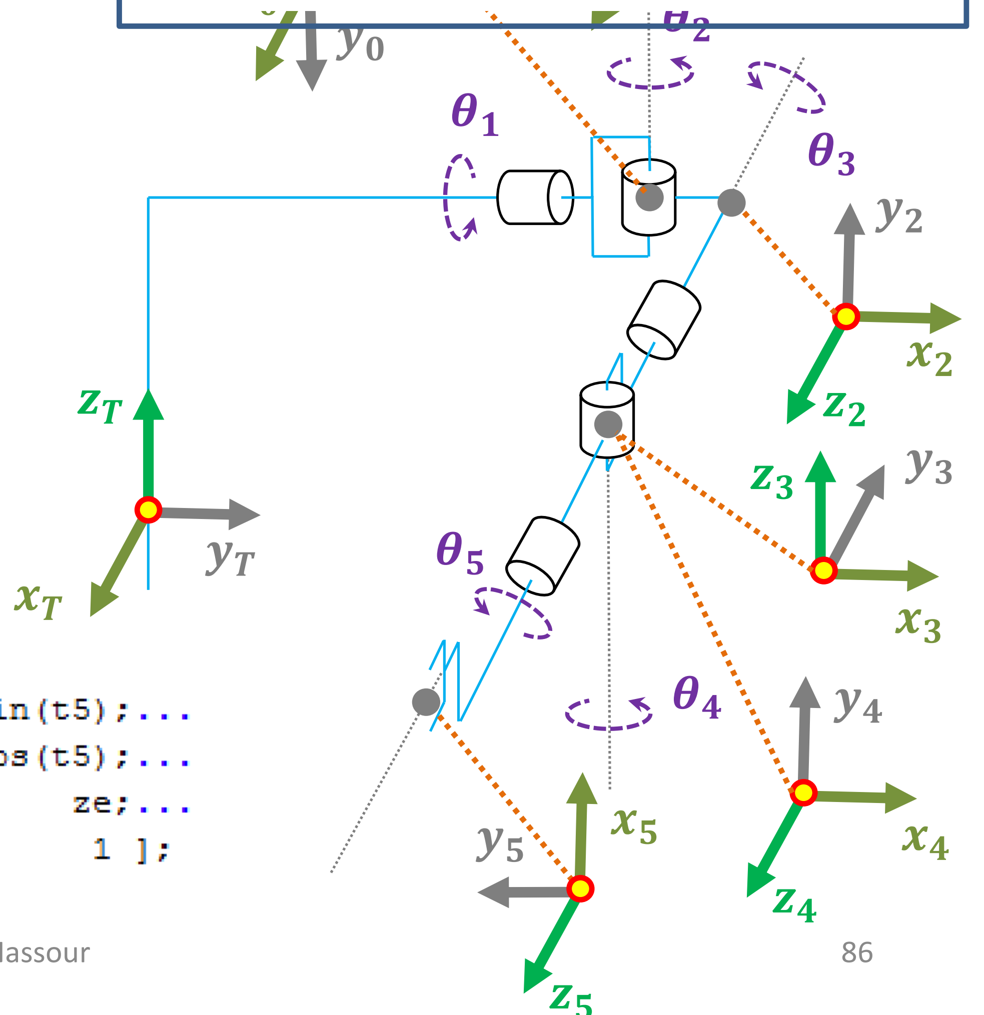
```
a2 = ElbowOffsetY;
y0 = ShoulderOffsetY;
z0 = ShoulderOffsetZ;
d3 = UpperArmLength;
ye = HandOffsetZ;
ze = LowerArmLength + HandOffsetX;
```

$$T^{BASE}_{5} = ?$$

Reminder: A_i

$$\begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
A5=[   -sin(t5),     -cos(t5),   0,   -ye*sin(t5);...
        cos(t5),     -sin(t5),   0,    ye*cos(t5);...
              0,            0,   1,            ze;...
              0,            0,   0,             1 ];
```

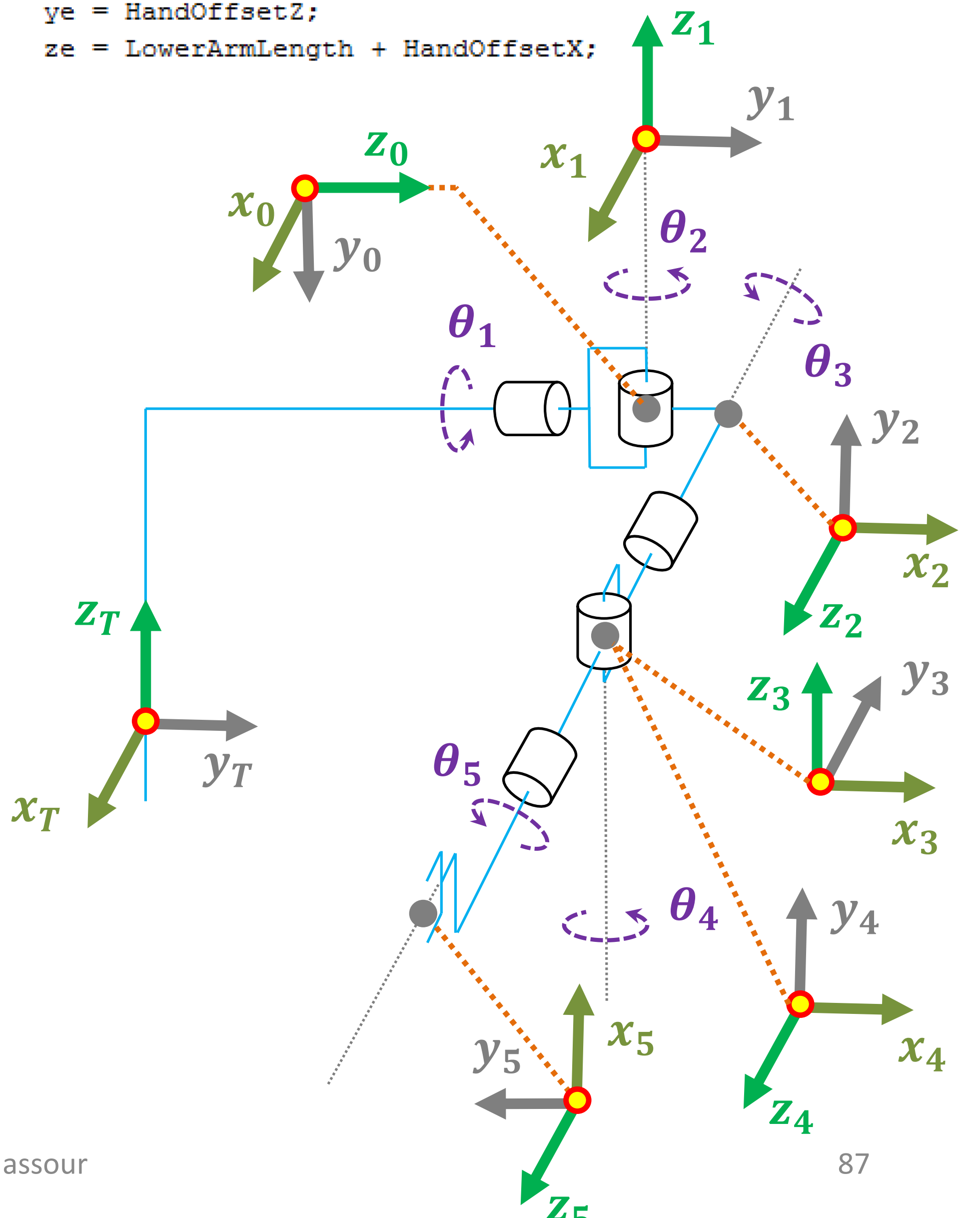


NAO Left Arm

i	a_i	α_i	d_i	θ_i
0	$T^{BASE}_{0}(0, ShoulderOffsetY, ShoulderOffsetZ)$			
1	0	90°	0	θ_1 *
2	a_2	90°	0	$\left(\frac{\pi}{2}\right) + \theta_2$ *
3	0	-90°	d_3	θ_3 *
4	0	$+90^\circ$	0	θ_4 *
5	a_5	0°	d_5	$\left(\frac{\pi}{2}\right) + \theta_5$ *

$$T^{BASE}_{5} = ?$$

```
a2 = ElbowOffsetY;
y0 = ShoulderOffsetY;
z0 = ShoulderOffsetZ;
d3 = UpperArmLength;
ye = HandOffsetZ;
ze = LowerArmLength + HandOffsetX;
```



$$T_{5}^{BASE} = ?$$

A012345 =

r11 = cos(t5)*(cos(t3)*sin(t1) + cos(t1)*sin(t2)*sin(t3)) - sin(t5)*(cos(t4)*(sin(t1)*sin(t3) - cos(t1)*cos(t3)*sin(t2)) - cos(t1)*cos(t2)*sin(t4))
r12 = - cos(t5)*(cos(t4)*(sin(t1)*sin(t3) - cos(t1)*cos(t3)*sin(t2)) - cos(t1)*cos(t2)*sin(t4)) - sin(t5)*(cos(t3)*sin(t1) + cos(t1)*sin(t2)*sin(t3))
r13 = sin(t4)*(sin(t1)*sin(t3) - cos(t1)*cos(t3)*sin(t2)) + cos(t1)*cos(t2)*cos(t4)
dx = ze*(sin(t4)*(sin(t1)*sin(t3) - cos(t1)*cos(t3)*sin(t2)) + cos(t1)*cos(t2)*cos(t4)) + ye*cos(t5)*(cos(t3)*sin(t1) + cos(t1)*sin(t2)*sin(t3)) + d3*cos(t1)*cos(t2) - a2*cos(t1)*sin(t2) - ye*sin(t5)*(cos(t4)*(sin(t1)*sin(t3) - cos(t1)*cos(t3)*sin(t2)) - cos(t1)*cos(t2)*sin(t4))

r21 = sin(t5)*(sin(t2)*sin(t4) - cos(t2)*cos(t3)*cos(t4)) - cos(t2)*cos(t5)*sin(t3)
r22 = cos(t5)*(sin(t2)*sin(t4) - cos(t2)*cos(t3)*cos(t4)) + cos(t2)*sin(t3)*sin(t5)
r23 = cos(t4)*sin(t2) + cos(t2)*cos(t3)*sin(t4)
dy = y0 + ze*(cos(t4)*sin(t2) + cos(t2)*cos(t3)*sin(t4)) + a2*cos(t2) + d3*sin(t2) + ye*sin(t5)*(sin(t2)*sin(t4) - cos(t2)*cos(t3)*cos(t4)) - ye*cos(t2)*cos(t5)*sin(t3)

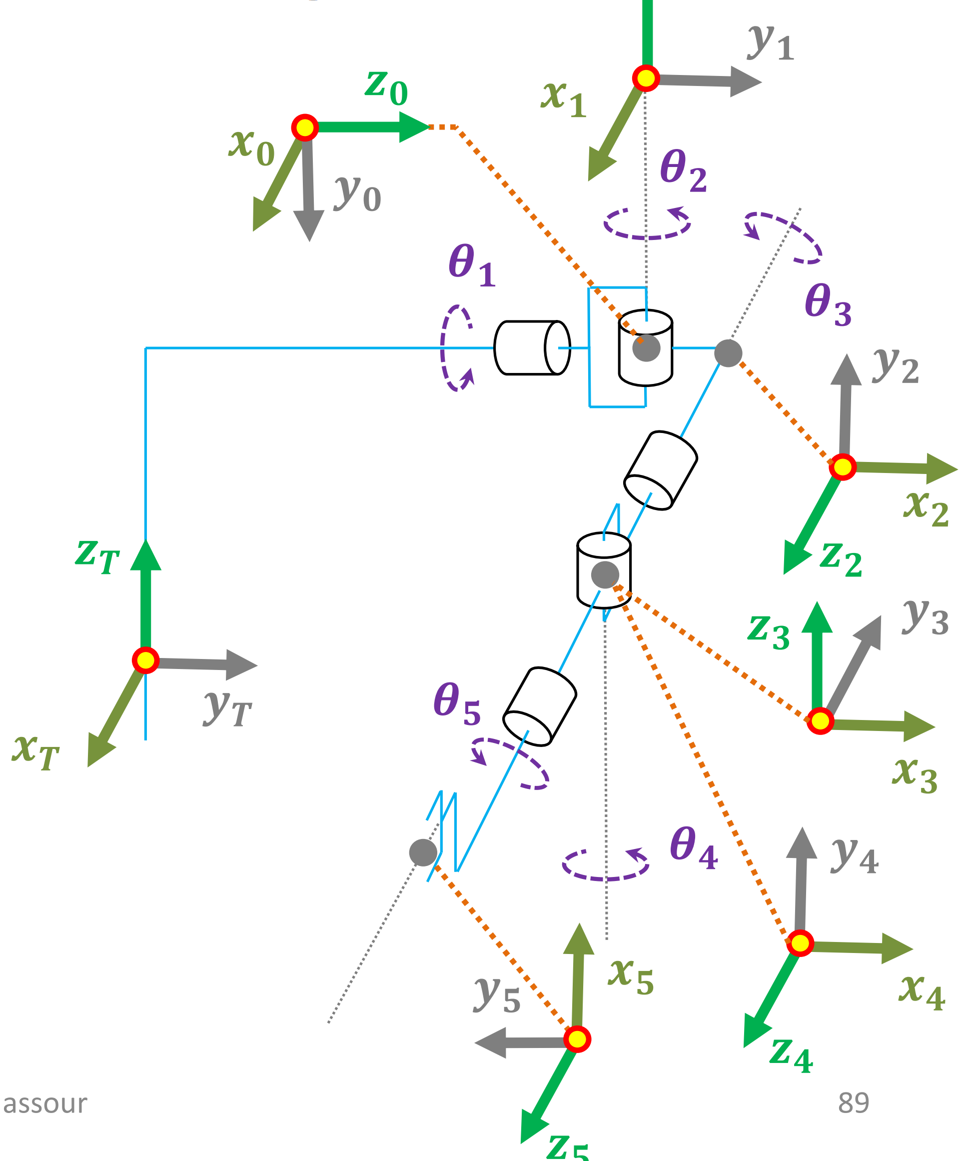
r31 = cos(t5)*(cos(t1)*cos(t3) - sin(t1)*sin(t2)*sin(t3)) - sin(t5)*(cos(t4)*(cos(t1)*sin(t3) + cos(t3)*sin(t1)*sin(t2)) + cos(t2)*sin(t1)*sin(t4))
r32 = - cos(t5)*(cos(t4)*(cos(t1)*sin(t3) + cos(t3)*sin(t1)*sin(t2)) + cos(t2)*sin(t1)*sin(t4)) - sin(t5)*(cos(t1)*cos(t3) - sin(t1)*sin(t2)*sin(t3))
r33 = sin(t4)*(cos(t1)*sin(t3) + cos(t3)*sin(t1)*sin(t2)) - cos(t2)*cos(t4)*sin(t1)
dz = z0 + ze*(sin(t4)*(cos(t1)*sin(t3) + cos(t3)*sin(t1)*sin(t2)) - cos(t2)*cos(t4)*sin(t1)) + ye*cos(t5)*(cos(t1)*cos(t3) - sin(t1)*sin(t2)*sin(t3)) - d3*cos(t2)*sin(t1) + a2*sin(t1)*sin(t2) - ye*sin(t5)*(cos(t4)*(cos(t1)*sin(t3) + cos(t3)*sin(t1)*sin(t2)) + cos(t2)*sin(t1)*sin(t4))

NAO Left Arm

i	a_i	α_i	d_i	θ_i
0	$T^{BASE}_0(0, ShoulderOffsetY, ShoulderOffsetZ)$			
1	0	90 °	0	θ_1 *
2	a_2	90 °	0	$\left(\frac{\pi}{2}\right) + \theta_2$ *
3	0	−90 °	d_3	θ_3 *
4	0	+90 °	0	θ_4 *
5	a_5	0 °	d_5	$\left(\frac{\pi}{2}\right) + \theta_5$ *

```
a2 = ElbowOffsetY;
y0 = ShoulderOffsetY;
z0 = ShoulderOffsetZ;
d3 = UpperArmLength;
ye = HandOffsetZ;
ze = LowerArmLength + HandOffsetX;
```

ShoulderOffsetY =98.00;
ElbowOffsetY =15.00;
UpperArmLength =105.00;
LowerArmLength =55.95;
ShoulderOffsetZ =100.00;
HandOffsetX =57.75;
HandOffsetZ =12.31;



Verification:

```
t1= 0;
t2= 0;
t3= 0;
t4= 0;
t5= 0;
```

```
A012345 =

         0         0    1.0000  218.7000
         0   -1.0000         0  113.0000
    1.0000         0         0  112.3100
         0         0         0    1.0000
```

NAO Left Arm

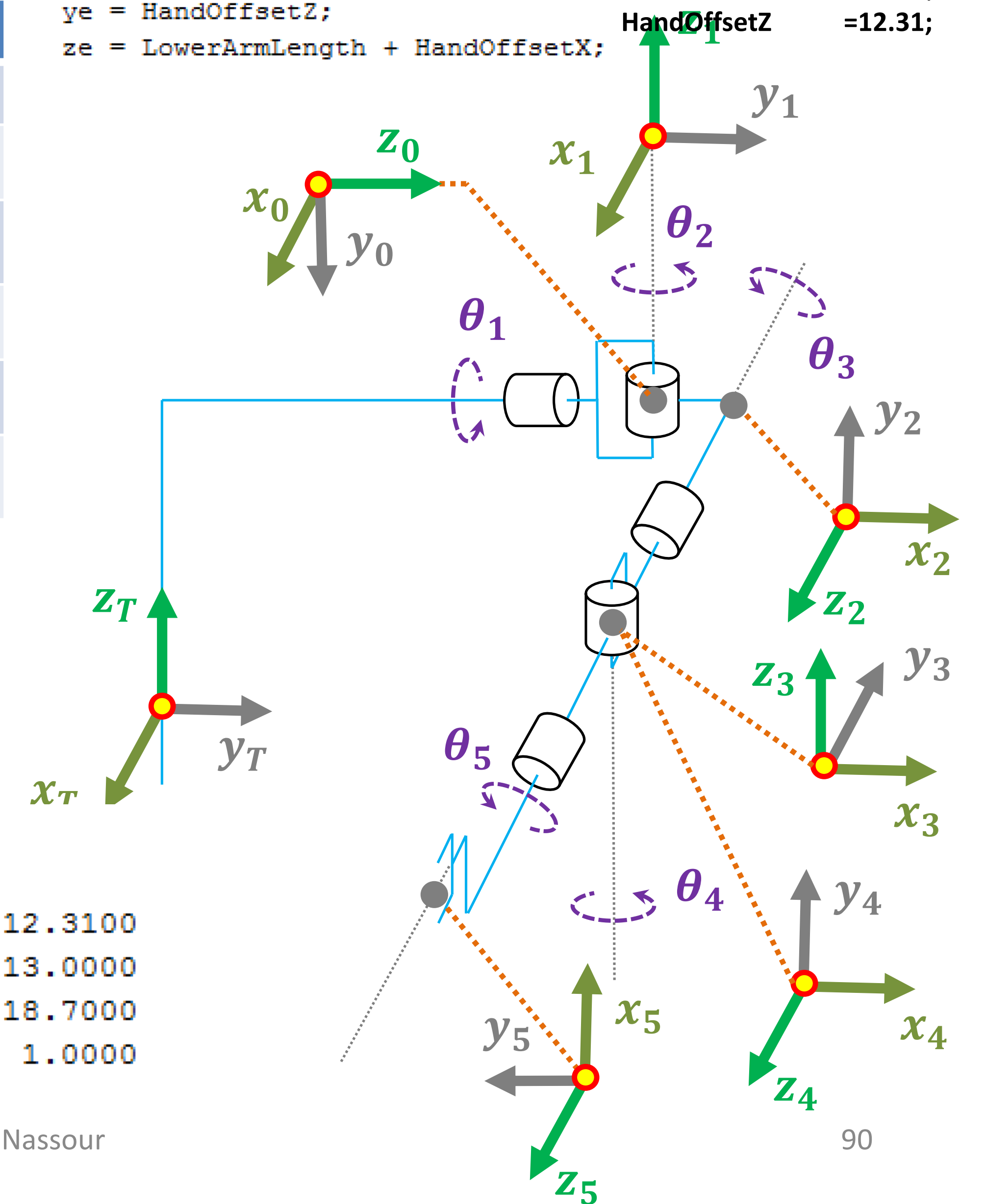
i	a_i	α_i	d_i	θ_i
0	$T^{BASE}_{\ 0}\,(0, ShoulderOffsetY, ShoulderOffsetZ)$			
1	0	90 °	0	$\boldsymbol{\theta_1}$ *
2	a_2	90 °	0	$\left(\frac{\pi}{2}\right) + \boldsymbol{\theta_2}$ *
3	0	–90 °	d_3	$\boldsymbol{\theta_3}$ *
4	0	+90 °	0	$\boldsymbol{\theta_4}$ *
5	a_5	0 °	d_5	$\left(\frac{\pi}{2}\right) + \boldsymbol{\theta_5}$ *

```
a2 = ElbowOffsetY;
y0 = ShoulderOffsetY;
z0 = ShoulderOffsetZ;
d3 = UpperArmLength;
ye = HandOffsetZ;
ze = LowerArmLength + HandOffsetX;
```

ShoulderOffsetY =98.00;
ElbowOffsetY =15.00;
UpperArmLength =105.00;
LowerArmLength =55.95;
ShoulderOffsetZ =100.00;
HandOffsetX =57.75;
HandOffsetZ =12.31;

Verification:

```
t1= pi/2;
t2= 0;
t3= 0;
t4= 0;
t5= 0;   A012345 =

     1.0000         0    0.0000   12.3100
          0   -1.0000         0  113.0000
     0.0000         0   -1.0000 -118.7000
          0         0         0    1.0000
```

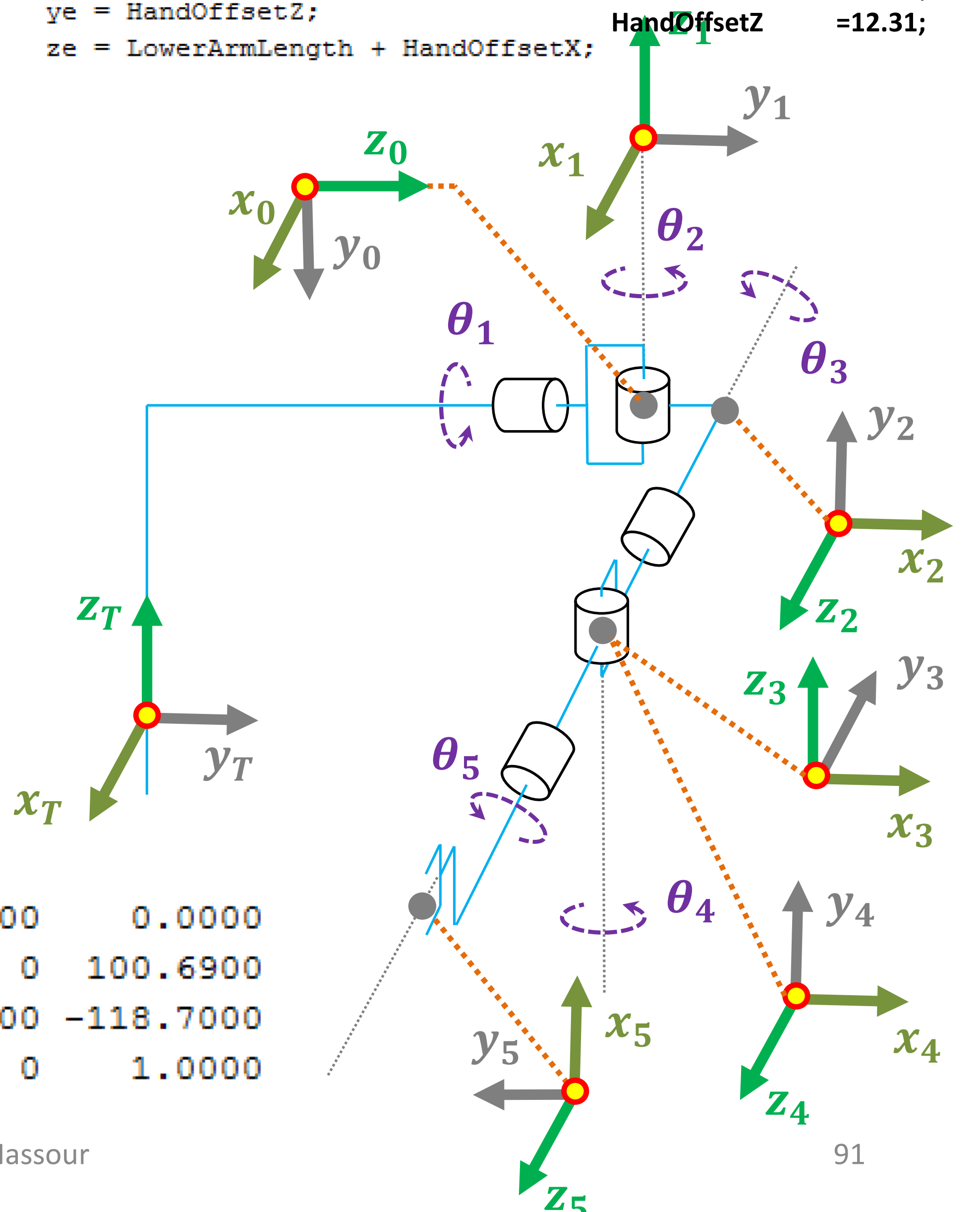


NAO Left Arm

i	a_i	α_i	d_i	θ_i
0	$T^{BASE}_0(0, ShoulderOffsetY, ShoulderOffsetZ)$			
1	0	90°	0	θ_1 *
2	a_2	90°	0	$\left(\frac{\pi}{2}\right) + \theta_2$ *
3	0	-90°	d_3	θ_3 *
4	0	$+90^\circ$	0	θ_4 *
5	a_5	0°	d_5	$\left(\frac{\pi}{2}\right) + \theta_5$ *

```
a2 = ElbowOffsetY;
y0 = ShoulderOffsetY;
z0 = ShoulderOffsetZ;
d3 = UpperArmLength;
ye = HandOffsetZ;
ze = LowerArmLength + HandOffsetX;
```

ShoulderOffsetY	=98.00;
ElbowOffsetY	=15.00;
UpperArmLength	=105.00;
LowerArmLength	=55.95;
ShoulderOffsetZ	=100.00;
HandOffsetX	=57.75;
HandOffsetZ	=12.31;



Verification:

```
t1= pi/2;
t2= 0;
t3= 0;
t4= 0;
t5= pi/2;
```

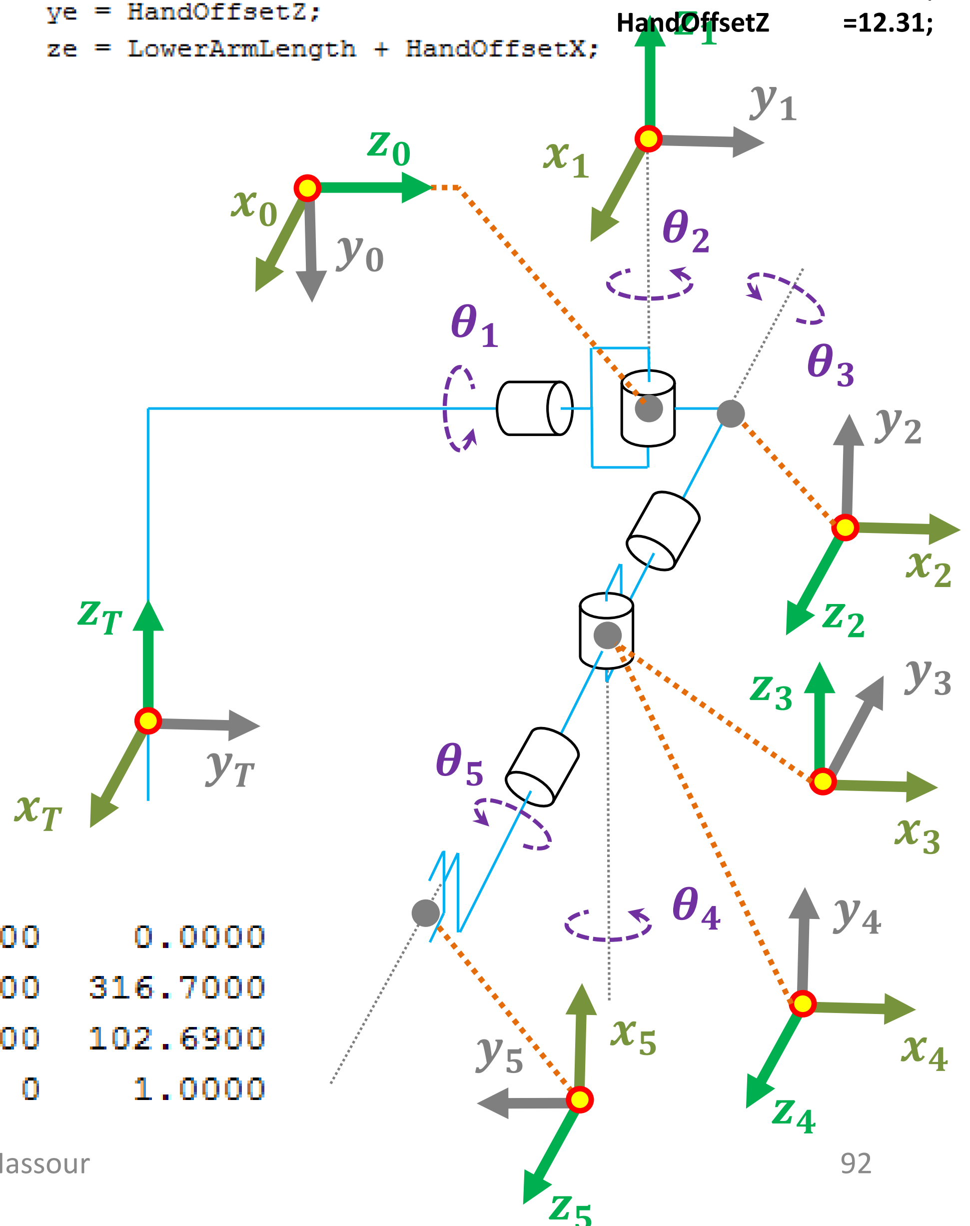
```
A012345 =

    0.0000   -1.0000    0.0000    0.0000
   -1.0000   -0.0000         0  100.6900
    0.0000   -0.0000   -1.0000 -118.7000
         0         0         0    1.0000
```

NAO Left Arm

i	a_i	α_i	d_i	θ_i
0	$T^{BASE}_{0}(0, ShoulderOffsetY, ShoulderOffsetZ)$			
1	0	90 °	0	$\boldsymbol{\theta_1}$ *
2	a_2	90 °	0	$\left(\frac{\pi}{2}\right) + \boldsymbol{\theta_2}$ *
3	0	−90 °	d_3	$\boldsymbol{\theta_3}$ *
4	0	+90 °	0	$\boldsymbol{\theta_4}$ *
5	a_5	0 °	d_5	$\left(\frac{\pi}{2}\right) + \boldsymbol{\theta_5}$ *

```
a2 = ElbowOffsetY;
y0 = ShoulderOffsetY;
z0 = ShoulderOffsetZ;
d3 = UpperArmLength;
ye = HandOffsetZ;
ze = LowerArmLength + HandOffsetX;
```

ShoulderOffsetY =98.00;
ElbowOffsetY =15.00;
UpperArmLength =105.00;
LowerArmLength =55.95;
ShoulderOffsetZ =100.00;
HandOffsetX =57.75;
HandOffsetZ =12.31;



Verification:

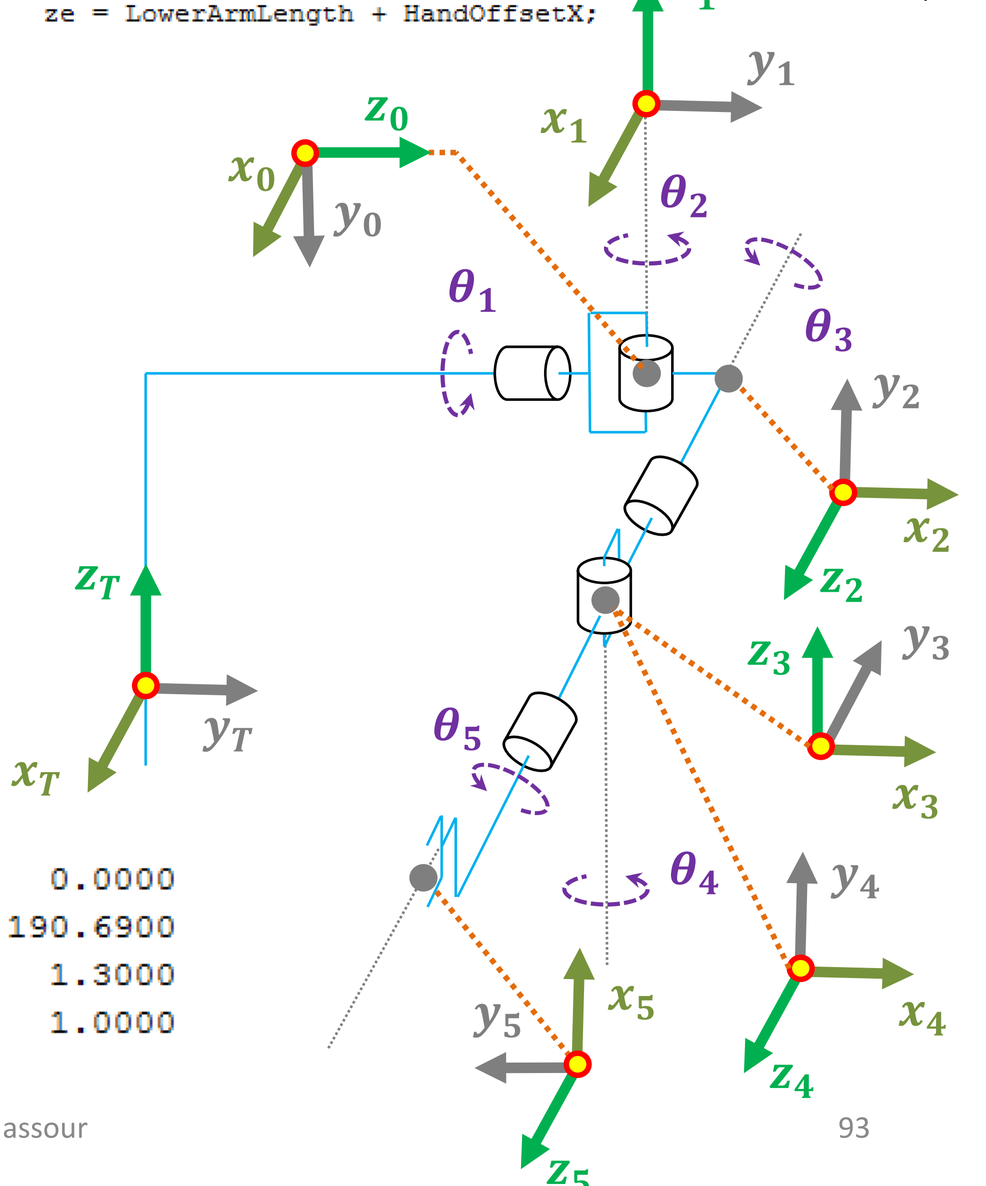
```
t1= pi/2;
t2= pi/2;
t3= 0;
t4= 0;
t5= pi/2;

A012345 =

    0.0000   -1.0000    0.0000    0.0000
   -0.0000   -0.0000    1.0000  316.7000
   -1.0000   -0.0000   -0.0000  102.6900
         0         0         0    1.0000
```

NAO Left Arm

```
a2 = ElbowOffsetY;
y0 = ShoulderOffsetY;
z0 = ShoulderOffsetZ;
d3 = UpperArmLength;
ye = HandOffsetZ;
ze = LowerArmLength + HandOffsetX;
```

ShoulderOffsetY =98.00;
ElbowOffsetY =15.00;
UpperArmLength =105.00;
LowerArmLength =55.95;
ShoulderOffsetZ =100.00;
HandOffsetX =57.75;
HandOffsetZ =12.31;

i	a_i	α_i	d_i	θ_i
0	$T^{BASE}_{0}(0, ShoulderOffsetY, ShoulderOffsetZ)$			
1	0	90°	0	θ_1 *
2	a_2	90°	0	$\left(\frac{\pi}{2}\right) + \theta_2$ *
3	0	−90°	d_3	θ_3 *
4	0	+90°	0	θ_4 *
5	a_5	0°	d_5	$\left(\frac{\pi}{2}\right) + \theta_5$ *



Verification:

```
t1= pi/2;
t2= pi/2;
t3= 0;
t4= -pi/2;
t5= pi/2;
```

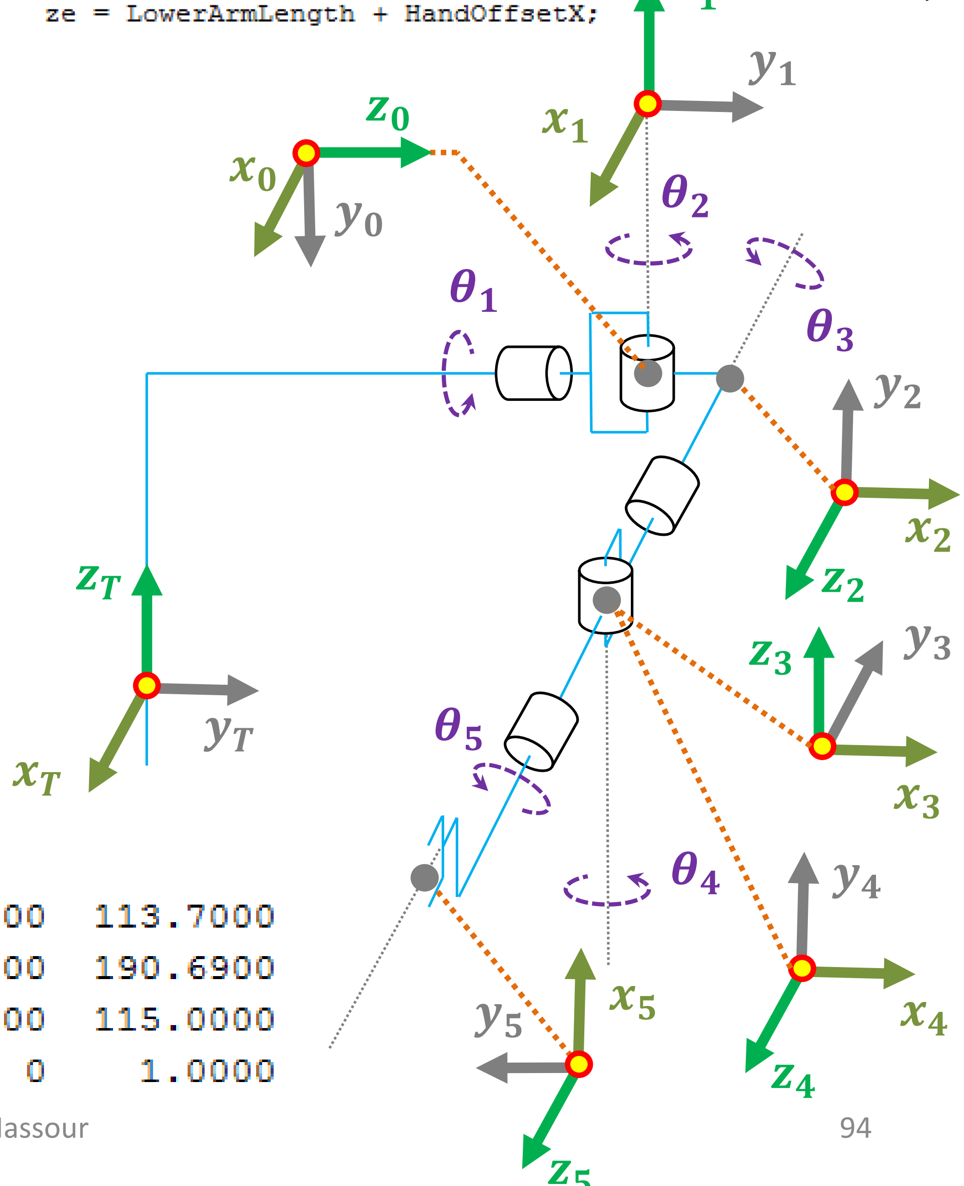
```
A012345 =

    0.0000   -1.0000    0.0000    0.0000
   -1.0000   -0.0000         0  190.6900
    0.0000   -0.0000   -1.0000    1.3000
         0         0         0    1.0000
```

NAO Left Arm

i	a_i	α_i	d_i	θ_i
0	$T^{BASE}_{0}(0, ShoulderOffsetY, ShoulderOffsetZ)$			
1	0	90°	0	θ_1 *
2	a_2	90°	0	$\left(\frac{\pi}{2}\right) + \theta_2$ *
3	0	-90°	d_3	θ_3 *
4	0	$+90^\circ$	0	θ_4 *
5	a_5	0°	d_5	$\left(\frac{\pi}{2}\right) + \theta_5$ *

```
a2 = ElbowOffsetY;
y0 = ShoulderOffsetY;
z0 = ShoulderOffsetZ;
d3 = UpperArmLength;
ye = HandOffsetZ;
ze = LowerArmLength + HandOffsetX;
```

ShoulderOffsetY =98.00;
ElbowOffsetY =15.00;
UpperArmLength =105.00;
LowerArmLength =55.95;
ShoulderOffsetZ =100.00;
HandOffsetX =57.75;
HandOffsetZ =12.31;



Verification:

```
t1= pi/2;
t2= pi/2;
t3= -pi/2;
t4= -pi/2;
t5= pi/2;
```

```
A012345 =

    0.0000    0.0000    1.0000  113.7000
   -1.0000   -0.0000    0.0000  190.6900
    0.0000   -1.0000   -0.0000  115.0000
         0         0         0    1.0000
```

NAO Left Arm I.K.

Given T, find q_i

```
T=[ r11, r12, r13, dx;...
    r21, r22, r23, dy;...
    r31, r32, r33, dz;...
      0,   0,   0,  1 ];
```

Tpr = inv(A0)*T*inv(A5)

[- r12*cos(t5) - r11*sin(t5), r11*cos(t5) - r12*sin(t5), **r13**, **dx - r11*ye - r13*ze**]
[r32*cos(t5) + r31*sin(t5), r32*sin(t5) - r31*cos(t5), **-r33**, **z0 - dz + r31*ye + r33*ze**]
[- r22*cos(t5) - r21*sin(t5), r21*cos(t5) - r22*sin(t5), **r23**, **dy - y0 - r21*ye - r23*ze**]
[0, 0, 0, 1]

NAO Left Arm I.K.

Given T, find q_i

```
T=[ r11, r12, r13, dx;...
    r21, r22, r23, dy;...
    r31, r32, r33, dz;...
      0,   0,   0,  1 ];
```

A1234 = A1*A2*A3*A4

```
[   cos(t4)*(sin(t1)*sin(t3) - cos(t1)*cos(t3)*sin(t2)) - cos(t1)*cos(t2)*sin(t4), cos(t3)*sin(t1) + cos(t1)*sin(t2)*sin(t3), sin(t4)*(sin(t1)*sin(t3) - cos(t1)*cos(t3)*sin(t2)) + cos(t1)*cos(t2)*cos(t4),  cos(t1)*(d3*cos(t2) - a2*sin(t2))]
[ - cos(t4)*(cos(t1)*sin(t3) + cos(t3)*sin(t1)*sin(t2)) - cos(t2)*sin(t1)*sin(t4), sin(t1)*sin(t2)*sin(t3) - cos(t1)*cos(t3),  cos(t2)*cos(t4)*sin(t1) - sin(t4)*(cos(t1)*sin(t3) + cos(t3)*sin(t1)*sin(t2)),  sin(t1)*(d3*cos(t2) - a2*sin(t2))]
[                              cos(t2)*cos(t3)*cos(t4) - sin(t2)*sin(t4),                  -cos(t2)*sin(t3),                            cos(t4)*sin(t2) + cos(t2)*cos(t3)*sin(t4),            a2*cos(t2) + d3*sin(t2)]
[                                                                        0,                                 0,                                                                        0,                                  1]
```

(1,4): cos(t1)*(d3*cos(t2) - a2*sin(t2))
(2,4): sin(t1)*(d3*cos(t2) - a2*sin(t2))

t1 = atan(Tpr(2,4) / Tpr(1,4))

NAO Left Arm I.K.

Given T, find q_i

```
T=[ r11, r12, r13, dx;...
    r21, r22, r23, dy;...
    r31, r32, r33, dz;...
      0,   0,   0,  1 ];
```

Tpr2 = inv(A1)*Tpr

```
[   r32*cos(t5)*sin(t1) - r11*cos(t1)*sin(t5) - r12*cos(t1)*cos(t5) + r31*sin(t1)*sin(t5), r11*cos(t1)*cos(t5) - r12*cos(t1)*sin(t5) - r31*cos(t5)*sin(t1) + r32*sin(t1)*sin(t5),  r13*cos(t1) - r33*sin(t1),   sin(t1)*(z0 - dz + r31*ye + r33*ze) - cos(t1)*(r11*ye - dx + r13*ze)]
[                                                      - r22*cos(t5) - r21*sin(t5),                                                      r21*cos(t5) - r22*sin(t5),                       r23,                                                       dy - y0 - r21*ye - r23*ze]
[  - r32*cos(t1)*cos(t5) - r12*cos(t5)*sin(t1) - r31*cos(t1)*sin(t5) - r11*sin(t1)*sin(t5), r31*cos(t1)*cos(t5) + r11*cos(t5)*sin(t1) - r32*cos(t1)*sin(t5) - r12*sin(t1)*sin(t5), r33*cos(t1) + r13*sin(t1), - sin(t1)*(r11*ye - dx + r13*ze) - cos(t1)*(z0 - dz + r31*ye + r33*ze)]
[                                                                                0,                                                                              0,                         0,                                                                       1]
```

(1,4): **sin(t1)*(z0 - dz + r31*ye + r33*ze) - cos(t1)*(r11*ye - dx + r13*ze)**

(2,4): **dy - y0 - r21*ye - r23*ze**

NAO Left Arm I.K.

Given T, find q_i

```
T=[ r11, r12, r13, dx;...
    r21, r22, r23, dy;...
    r31, r32, r33, dz;...
      0,   0,   0,  1 ];
```

A234 = A2*A3*A4

[- cos(t2)*sin(t4) - cos(t3)*cos(t4)*sin(t2),	sin(t2)*sin(t3),	cos(t2)*cos(t4) - cos(t3)*sin(t2)*sin(t4),	**d3*cos(t2) - a2*sin(t2)**]
[cos(t2)*cos(t3)*cos(t4) - sin(t2)*sin(t4),	-cos(t2)*sin(t3),	cos(t4)*sin(t2) + cos(t2)*cos(t3)*sin(t4),	**a2*cos(t2) + d3*sin(t2)**]
[cos(t4)*sin(t3),	cos(t3),	sin(t3)*sin(t4),	0]
[0,	0,	0,	1]

(1,4): d3*cos(t2) - a2*sin(t2)

(2,4): a2*cos(t2) + d3*sin(t2)

NAO Left Arm I.K.

Given T, find q_i

```
T=[ r11, r12, r13, dx;...
    r21, r22, r23, dy;...
    r31, r32, r33, dz;...
      0,   0,   0,  1 ];
```

A234 = A2*A3*A4

```
[ - cos(t2)*sin(t4) - cos(t3)*cos(t4)*sin(t2),  sin(t2)*sin(t3),   cos(t2)*cos(t4) - cos(t3)*sin(t2)*sin(t4),  d3*cos(t2) - a2*sin(t2)]
[   cos(t2)*cos(t3)*cos(t4) - sin(t2)*sin(t4), -cos(t2)*sin(t3),   cos(t4)*sin(t2) + cos(t2)*cos(t3)*sin(t4),  a2*cos(t2) + d3*sin(t2)]
[                             cos(t4)*sin(t3),          cos(t3),                             sin(t3)*sin(t4),                        0]
[                                           0,                0,                                           0,                        1]
```

(1,4): d3*cos(t2) - a2*sin(t2)

(2,4): a2*cos(t2) + d3*sin(t2)

A234(1,4)*d3 + A234(2,4)*a2 = cos(t2)*(a2^2 + d3^2)

NAO Left Arm I.K.

Given T, find q_i

```
T=[ r11, r12, r13, dx;...
    r21, r22, r23, dy;...
    r31, r32, r33, dz;...
      0,   0,   0,  1 ];
```

A234 = A2*A3*A4

```
[ - cos(t2)*sin(t4) - cos(t3)*cos(t4)*sin(t2),  sin(t2)*sin(t3),   cos(t2)*cos(t4) - cos(t3)*sin(t2)*sin(t4),  d3*cos(t2) - a2*sin(t2)]
[   cos(t2)*cos(t3)*cos(t4) - sin(t2)*sin(t4), -cos(t2)*sin(t3),   cos(t4)*sin(t2) + cos(t2)*cos(t3)*sin(t4),  a2*cos(t2) + d3*sin(t2)]
[                          cos(t4)*sin(t3),          cos(t3),                          sin(t3)*sin(t4),                        0]
[                                        0,                0,                                        0,                        1]
```

A234(1,4)*d3 + A234(2,4)*a2 = cos(t2)*(a2^2 + d3^2)

t2 = ±acos((d3*Tpr2(1,4) + a2*Tpr2(2,4)) / (d3^2 + a2^2))

NAO Left Arm I.K.

Given T, find q_i

```
T=[ r11, r12, r13, dx;...
    r21, r22, r23, dy;...
    r31, r32, r33, dz;...
      0,   0,   0,  1 ];
```

Tpr3 = inv(A2)*Tpr2

```
[ r12*cos(t1)*cos(t5)*sin(t2) - r21*cos(t2)*sin(t5) - r22*cos(t2)*cos(t5) + r11*cos(t1)*sin(t2)*sin(t5) - r32*cos(t5)*sin(t1)*sin(t2) - r31*sin(t1)*sin(t2)*sin(t5),  r21*cos(t2)*cos(t5) - r22*cos(t2)*sin(t5) - r11*cos(t1)*cos(t5)*sin(t2) + r12*cos(t1)*sin(t2)*sin(t5) + r31*cos(t5)*sin(t1)*sin(t2) - r32*sin(t1)*sin(t2)*sin(t5),  r23*cos(t2) - r13*cos(t1)*sin(t2) + r33*sin(t1)*sin(t2), - a2 - sin(t2)*(sin(t1)*(z0 - dz + r31*ye + r33*ze) - cos(t1)*(r11*ye - dx + r13*ze)) - cos(t2)*(y0 - dy + r21*ye + r23*ze)]
[ - r32*cos(t1)*cos(t5) - r12*cos(t5)*sin(t1) - r31*cos(t1)*sin(t5) - r11*sin(t1)*sin(t5),  r31*cos(t1)*cos(t5) + r11*cos(t5)*sin(t1) - r32*cos(t1)*sin(t5) - r12*sin(t1)*sin(t5),  r33*cos(t1) + r13*sin(t1),  - sin(t1)*(r11*ye - dx + r13*ze) - cos(t1)*(z0 - dz + r31*ye + r33*ze)]
[ r32*cos(t2)*cos(t5)*sin(t1) - r21*sin(t2)*sin(t5) - r12*cos(t1)*cos(t2)*cos(t5) - r11*cos(t1)*cos(t2)*sin(t5) - r22*cos(t5)*sin(t2) + r31*cos(t2)*sin(t1)*sin(t5), r21*cos(t5)*sin(t2) - r22*sin(t2)*sin(t5) + r11*cos(t1)*cos(t2)*cos(t5) - r12*cos(t1)*cos(t2)*sin(t5) - r31*cos(t2)*cos(t5)*sin(t1) + r32*cos(t2)*sin(t1)*sin(t5), r23*sin(t2) + r13*cos(t1)*cos(t2) - r33*cos(t2)*sin(t1),  cos(t2)*(sin(t1)*(z0 - dz + r31*ye + r33*ze) - cos(t1)*(r11*ye - dx + r13*ze)) - sin(t2)*(y0 - dy + r21*ye + r23*ze)]
[ 0, 0, 0, 1]
```

(1,3): r23*cos(t2) - r13*cos(t1)*sin(t2) + r33*sin(t1)*sin(t2)
(2,3): r33*cos(t1) + r13*sin(t1)

NAO Left Arm I.K.

Given T, find q_i

```
T=[ r11, r12, r13, dx;...
    r21, r22, r23, dy;...
    r31, r32, r33, dz;...
      0,   0,   0,  1 ];
```

A34 = A3*A4

[cos(t3)*cos(t4),	-sin(t3),	**cos(t3)*sin(t4)**,	0]
[cos(t4)*sin(t3),	cos(t3),	**sin(t3)*sin(t4)**,	0]
[-sin(t4),	0,	cos(t4),	d3]
[0,	0,	0,	1]

t3 = atan(Tpr3(2,3) / Tpr3(1,3))

NAO Left Arm I.K.

Given T, find q_i

```
T=[ r11, r12, r13, dx;...
    r21, r22, r23, dy;...
    r31, r32, r33, dz;...
      0,   0,   0,  1 ];
```

Tpr4 = inv(A3)*Tpr3

```
[ r12*cos(t1)*cos(t3)*cos(t5)*sin(t2) - r21*cos(t2)*cos(t3)*sin(t5) - r32*cos(t1)*cos(t5)*sin(t3) - r12*cos(t5)*sin(t1)*sin(t3) - r31*cos(t1)*sin(t3)*sin(t5) - r11*sin(t1)*sin(t3)*sin(t5) - r22*cos(t2)*cos(t3)*cos(t5) + r11*cos(t1)*cos(t3)*sin(t2)*sin(t5) - r32*cos(t3)*cos(t5)*sin(t1)*sin(t2) - r31*cos(t3)*sin(t1)*sin(t2)*sin(t5), r21*cos(t2)*cos(t3)*cos(t5) - r22*cos(t2)*cos(t3)*sin(t5) + r31*cos(t1)*cos(t5)*sin(t3) + r11*cos(t5)*sin(t1)*sin(t3) - r32*cos(t1)*sin(t3)*sin(t5) - r12*sin(t1)*sin(t3)*sin(t5) - r11*cos(t1)*cos(t3)*cos(t5)*sin(t2) + r12*cos(t1)*cos(t3)*sin(t2)*sin(t5) + r31*cos(t3)*cos(t5)*sin(t1)*sin(t2) - r32*cos(t3)*sin(t1)*sin(t2)*sin(t5), r23*cos(t2)*cos(t3) + r33*cos(t1)*sin(t3) + r13*sin(t1)*sin(t3) - r13*cos(t1)*cos(t3)*sin(t2) + r33*cos(t3)*sin(t1)*sin(t2), - sin(t3)*(sin(t1)*(r11*ye - dx + r13*ze) + cos(t1)*(z0 - dz + r31*ye + r33*ze)) - cos(t3)*(a2
+ sin(t2)*(sin(t1)*(z0 - dz + r31*ye + r33*ze) - cos(t1)*(r11*ye - dx + r13*ze)) + cos(t2)*(y0 - dy + r21*ye + r23*ze))]
[ r22*cos(t5)*sin(t2) + r21*sin(t2)*sin(t5) + r12*cos(t1)*cos(t2)*cos(t5) + r11*cos(t1)*cos(t2)*sin(t5) - r32*cos(t2)*cos(t5)*sin(t1) - r31*cos(t2)*sin(t1)*sin(t5), r22*sin(t2)*sin(t5) - r21*cos(t5)*sin(t2) - r11*cos(t1)*cos(t2)*cos(t5) + r12*cos(t1)*cos(t2)*sin(t5) + r31*cos(t2)*cos(t5)*sin(t1) - r32*cos(t2)*sin(t1)*sin(t5), r33*cos(t2)*sin(t1) - r13*cos(t1)*cos(t2) - r23*sin(t2), d3 - cos(t2)*(sin(t1)*(z0 - dz + r31*ye + r33*ze) - cos(t1)*(r11*ye - dx + r13*ze)) + sin(t2)*(y0 - dy + r21*ye + r23*ze)]
[ r22*cos(t2)*cos(t5)*sin(t3) - r12*cos(t3)*cos(t5)*sin(t1) - r32*cos(t1)*cos(t3)*cos(t5) - r31*cos(t1)*cos(t3)*sin(t5) - r11*cos(t3)*sin(t1)*sin(t5) + r21*cos(t2)*sin(t3)*sin(t5) - r12*cos(t1)*cos(t5)*sin(t2)*sin(t3) - r11*cos(t1)*sin(t2)*sin(t3)*sin(t5) + r32*cos(t5)*sin(t1)*sin(t2)*sin(t3) + r31*sin(t1)*sin(t2)*sin(t3)*sin(t5), r31*cos(t1)*cos(t3)*cos(t5) + r11*cos(t3)*cos(t5)*sin(t1) - r21*cos(t2)*cos(t5)*sin(t3) - r32*cos(t1)*cos(t3)*sin(t5) - r12*cos(t3)*sin(t1)*sin(t5) + r22*cos(t2)*sin(t3)*sin(t5) + r11*cos(t1)*cos(t5)*sin(t2)*sin(t3) - r12*cos(t1)*sin(t2)*sin(t3)*sin(t5) - r31*cos(t5)*sin(t1)*sin(t2)*sin(t3) + r32*sin(t1)*sin(t2)*sin(t3)*sin(t5), r33*cos(t1)*cos(t3) + r13*cos(t3)*sin(t1) - r23*cos(t2)*sin(t3) + r13*cos(t1)*sin(t2)*sin(t3) - r33*sin(t1)*sin(t2)*sin(t3), sin(t3)*(a2 + sin(t2)*(sin(t1)*(z0 - dz + r31*ye + r33*ze) - cos(t1)*(r11*ye - dx + r13*ze)) +
cos(t2)*(y0 - dy + r21*ye + r23*ze)) - cos(t3)*(sin(t1)*(r11*ye - dx + r13*ze) + cos(t1)*(z0 - dz + r31*ye + r33*ze))]
[ 0, 0, 0, 1]
```

(1,3): r23*cos(t2)*cos(t3) + r33*cos(t1)*sin(t3) + r13*sin(t1)*sin(t3) - r13*cos(t1)*cos(t3)*sin(t2) + r33*cos(t3)*sin(t1)*sin(t2)

(2,3): r33*cos(t2)*sin(t1) - r13*cos(t1)*cos(t2) - r23*sin(t2)

NAO Left Arm I.K.

Given T, find q_i

```
T=[ r11, r12, r13, dx;...
    r21, r22, r23, dy;...
    r31, r32, r33, dz;...
      0,   0,   0,  1 ];
```

A4 =

```
[ cos(t4),  0,   sin(t4), 0]
[ sin(t4),  0,  -cos(t4), 0]
[       0,  1,         0, 0]
[       0,  0,         0, 1]
```

(1,3): sin(t4)
(2,3): -cos(t4)

t4 = atan(Tpr4(1,3) / -Tpr4(2,3))

NAO Left Arm I.K.

Given T, find q_i

```
T=[ r11, r12, r13, dx;...
    r21, r22, r23, dy;...
    r31, r32, r33, dz;...
      0,   0,   0,  1 ];
```

TTpr = inv(A4)*inv(A3)*inv(A2)*inv(A1)*inv(A0)*A012345;

```
[ -sin(t5),  -cos(t5), 0, -ye*sin(t5)]
[  cos(t5),  -sin(t5), 0,  ye*cos(t5)]
[        0,         0, 1,          ze]
[        0,         0, 0,           1]
```

(1,1): -sin(t5)
(2,1): cos(t5)

t5 = atan(-TTpr(1,1) / TTpr(2,1))

Let's do the verification phase!

I.K. by Kinematic Decoupling

Difficult to find always a close form solution for robot with six degrees of freedom.

If the last three joints are intersecting (e.g, Spherical Wrists) it is possible to decouple the inverse kinematics problem into two simpler problem:

- Inverse position kinematics
- Inverse orientation kinematics

I.K. by Kinematic Decoupling

For a 6-DOF manipulator with a spherical wrist:

- Finding the position of the intersection of the wrist axes (the wrist center).
- Finding the orientation of the wrist.

I.K. by Kinematic Decoupling

We can express the general transformation matrix as:

$$\begin{aligned} R_6^0(q_1, \ldots, q_6) &= R \\ o_6^0(q_1, \ldots, q_6) &= o \end{aligned}$$

where ***o*** and ***R*** are the desired position and orientation of the tool frame expressed with respect to the base frame.

With the assumption of a spherical wrist the last three joints will not have an influence on the wrist center $\boldsymbol{o_c}$.

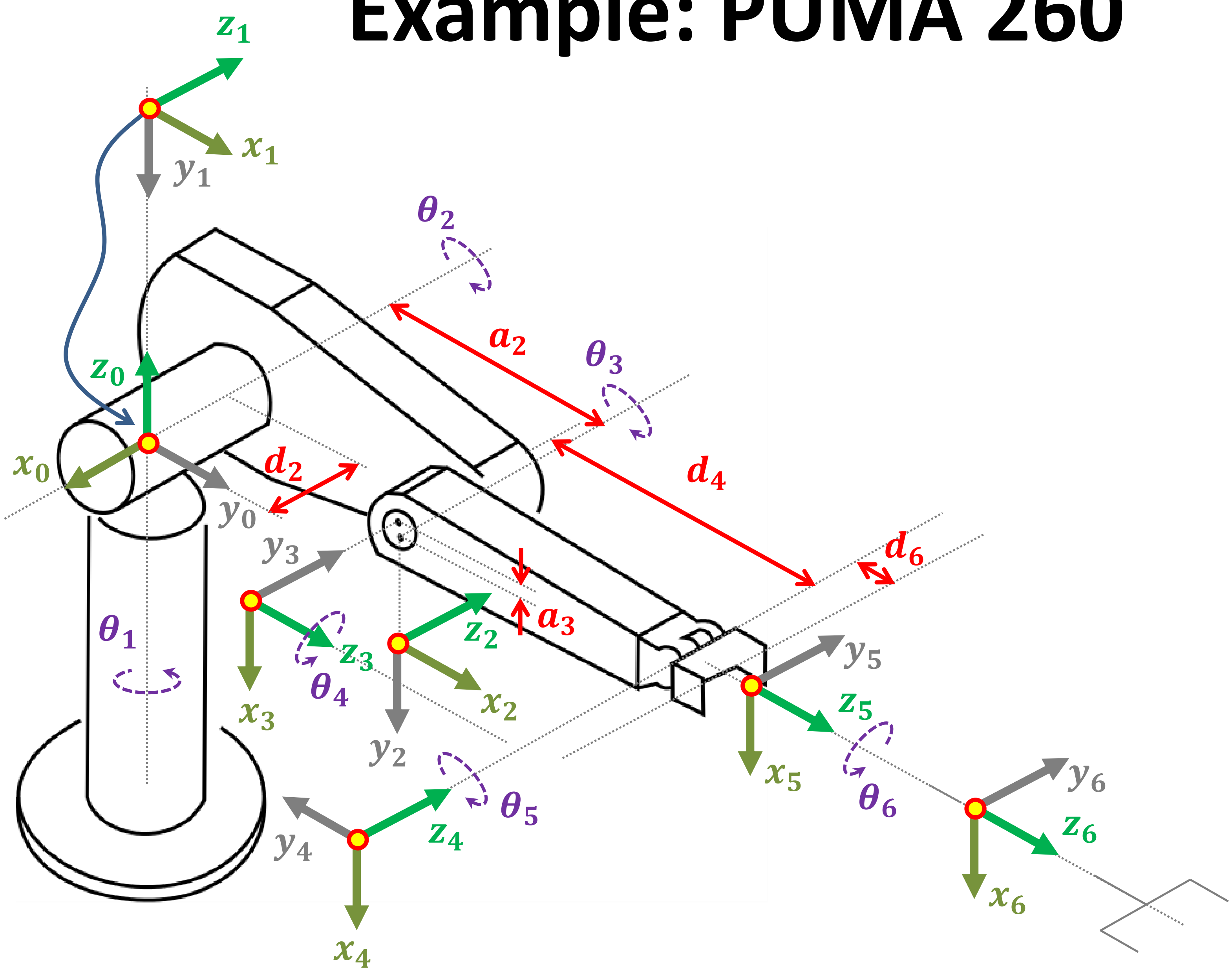
The position of the wrist center is thus a function of only the first three joint variables $\boldsymbol{q_1}$, $\boldsymbol{q_2}$, and $\boldsymbol{q_3}$.

I.K. by Kinematic Decoupling

The origin of the tool frame is obtained by a translation of distance $\boldsymbol{d_6}$ along $\boldsymbol{z_5}$ from $\boldsymbol{o_c}$.

$$o = o_c^0 + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Example: PUMA 260



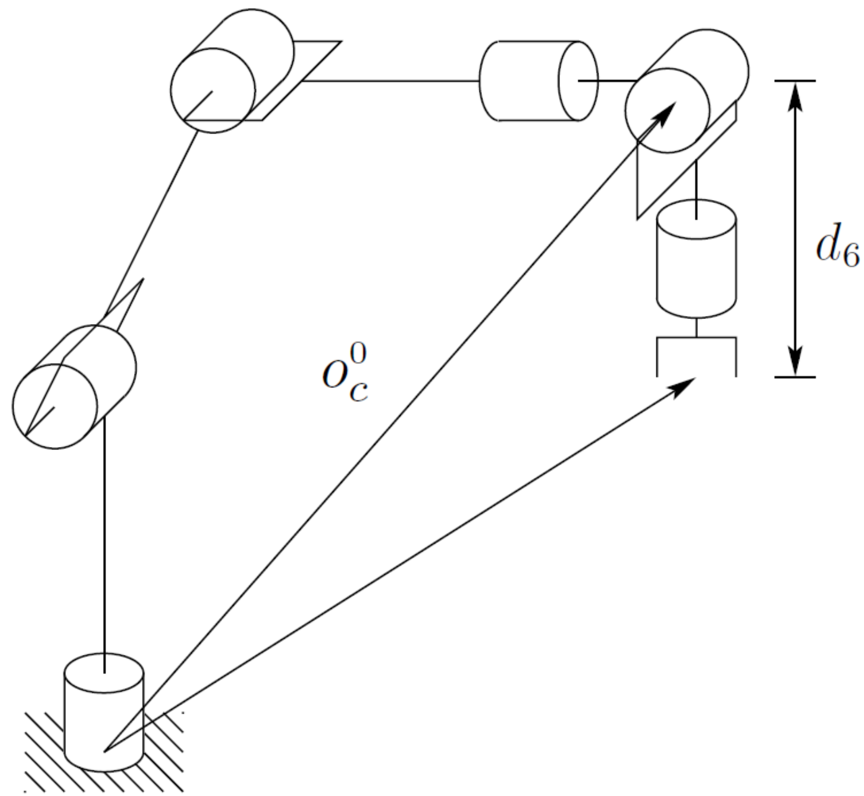
I.K. by Kinematic Decoupling

The origin of the tool frame is obtained by a translation of distance $\boldsymbol{d_6}$ along $\boldsymbol{z_5}$ from $\boldsymbol{o_c}$.

$$o = o_c^0 + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$o_c^0 = o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$

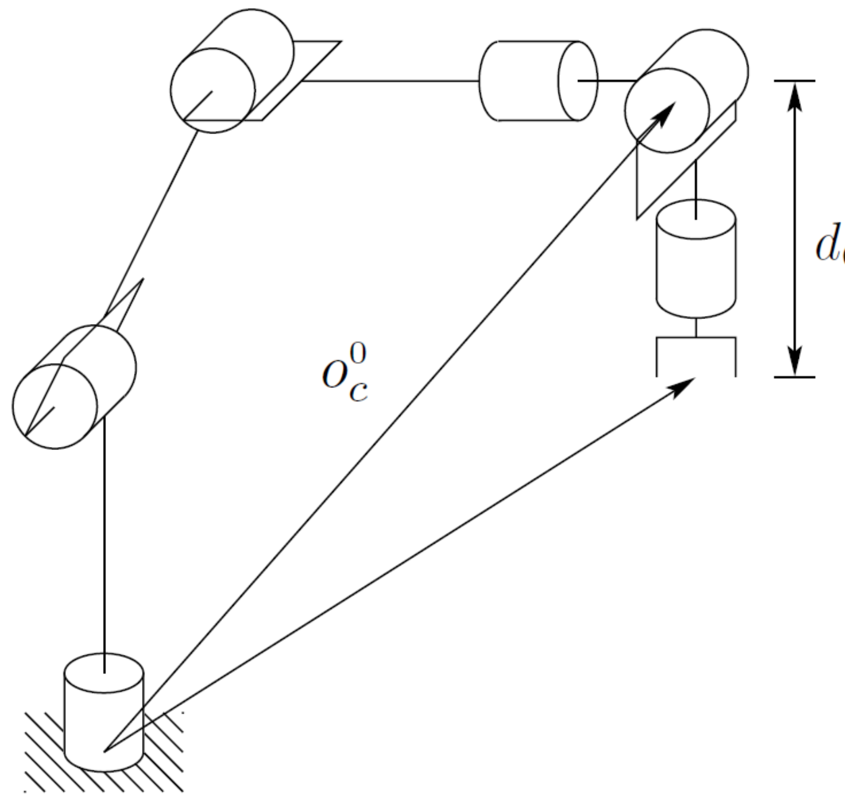


We may find the values of the first three joint variables $\boldsymbol{q_1}$, $\boldsymbol{q_2}$, and $\boldsymbol{q_3}$.

I.K. by Kinematic Decoupling

$$R = R_3^0 R_6^3$$
$$R_6^3 = (R_3^0)^{-1} R = (R_3^0)^T R$$

The final three joint angles can then be found as a set of Euler angles.



Questions?