Artificial Intelligence & Neuro Cognitive Systems
Fakultät für Informatik

# Robot Dynamics

*Dr.-Ing. John Nassour*

# Introduction

- Dynamics concerns the motion of bodies
- Includes
  - Kinematics – study of motion without reference to the force that cause it
  - Kinetics – relates these forces to the motion
- Dynamic behaviour of a robot: rate of change of arm configuration in relation to the torques exerted by the actuators.

# Forward vs. Inverse

## Forward dynamics

Given vector of joint torques, work out the resulting manipulator motion.

$$\tau(t) = \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix} \rightarrow q(t) = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

## Inverse Dynamics

Given a vector of manipulator positions, velocities and acceleration. Find the required vector of joint torques.

$$q(t), \dot{q}(t), \ddot{q}(t) \rightarrow \tau(t)$$

# Torques

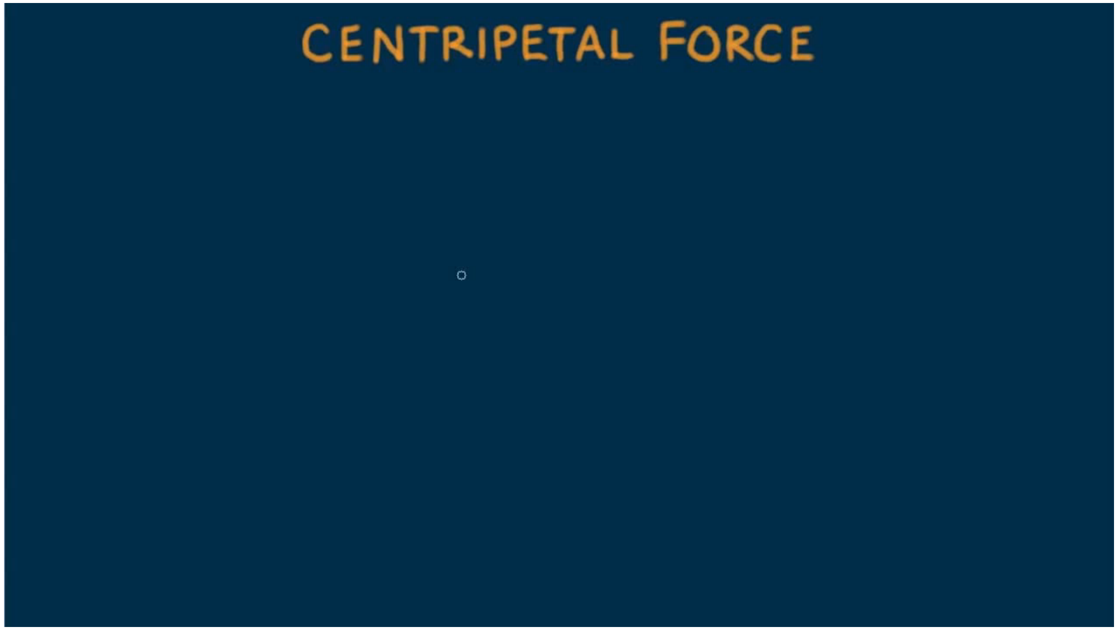
The actuator has to balance torques from 4 different sources:

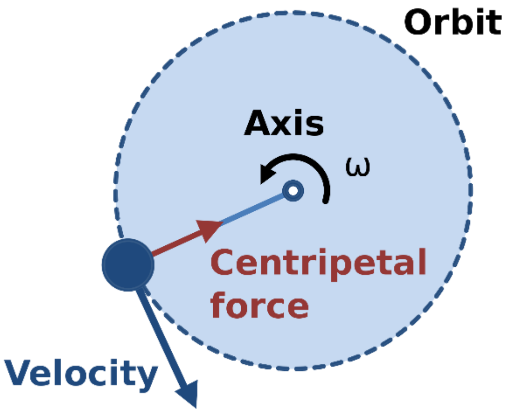
- Dynamic torques (caused by the motion)
  - Inertial (promotional to joint acceleration, according to Newton’s law)
  - Centripetal (promotional to square of joint velocity, direction toward the centre of circular motion)
  - Coriolis (vertical forces, interaction of two rotating links)

- Static torques (caused by friction)
- Gravity torques (caused by gravity)
- External torques (exerted on the end effector, caused by the task)

# Centripetal Force

A **centripetal force** is a **force** that makes a body follow a curved path. Its direction is always orthogonal to the velocity of the body and towards the fixed point of the instantaneous center of curvature of the path.



# Centrifugal Force

# The Coriolis Effect

# Manipulator Dynamics

Mathematical equations describing the dynamic behaviour of the manipulator.

Why needed?

Simulation of the manipulator

Design a suitable controller

Evaluate the robot structure

Joint torques $\Longleftrightarrow$ Robot motion, i.e. acceleration, velocity, position

# Approaches to Dynamic Modeling

**Newton-Euler Formulation**

Balance of forces/torques

Dynamic equations in numeric/recursive form.

**Lagrange Formulation**

Energy-Based approach

Dynamic equations in symbolic/closed form

**Other Formulations**

# Joint Space Dynamics

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{V}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{\Gamma}$$

$\boldsymbol{q}$: Generalized joint coordinates

$\boldsymbol{M}(\boldsymbol{q})$: Mass Matrix – Kinetic Energy Matrix

$\boldsymbol{V}(\boldsymbol{q},\dot{\boldsymbol{q}})$: Centrifugal and Coriolis forces

$\boldsymbol{G}(\boldsymbol{q})$: Gravity force

$\boldsymbol{\Gamma}$: Generalized forces

# Newton-Euler Formulation

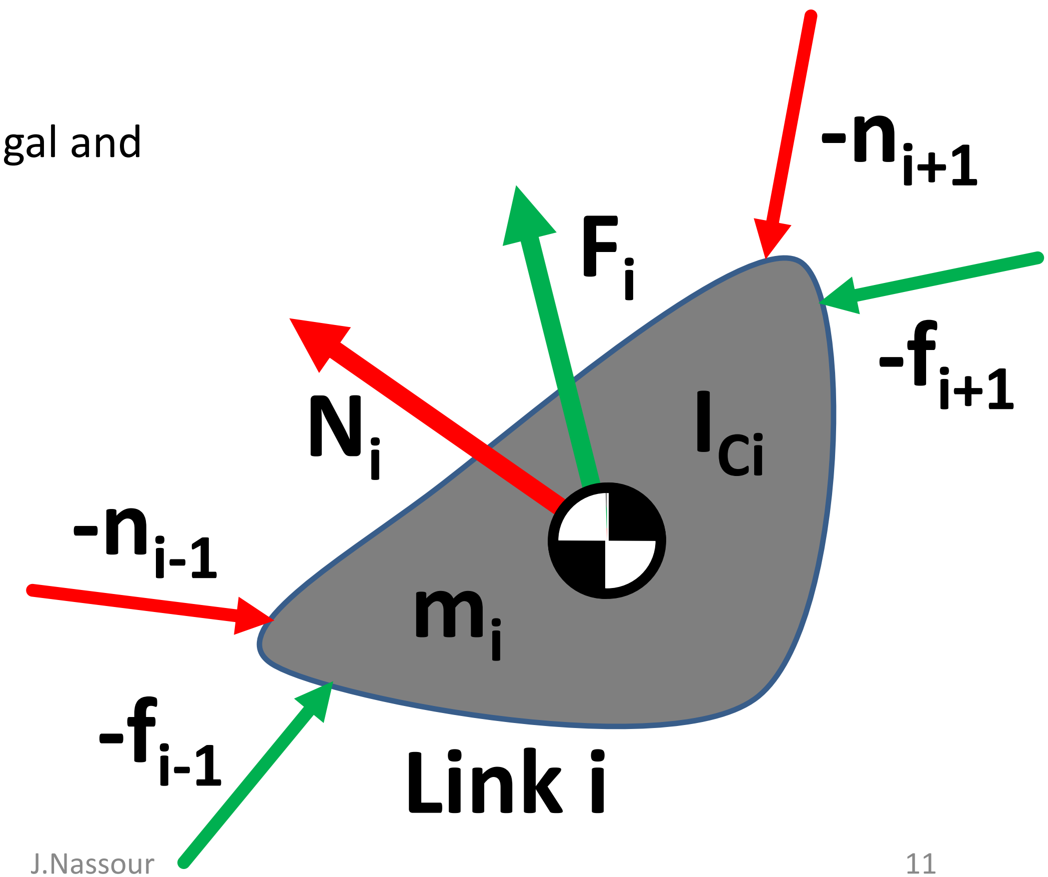
In static equilibrium $\mathbf{F_i}$ and $\mathbf{N_i}$ are equal to $\mathbf{0}$.

**Newton**:
Linear motion
$\boldsymbol{m_i \dot{v}_{ci} = F_i}$

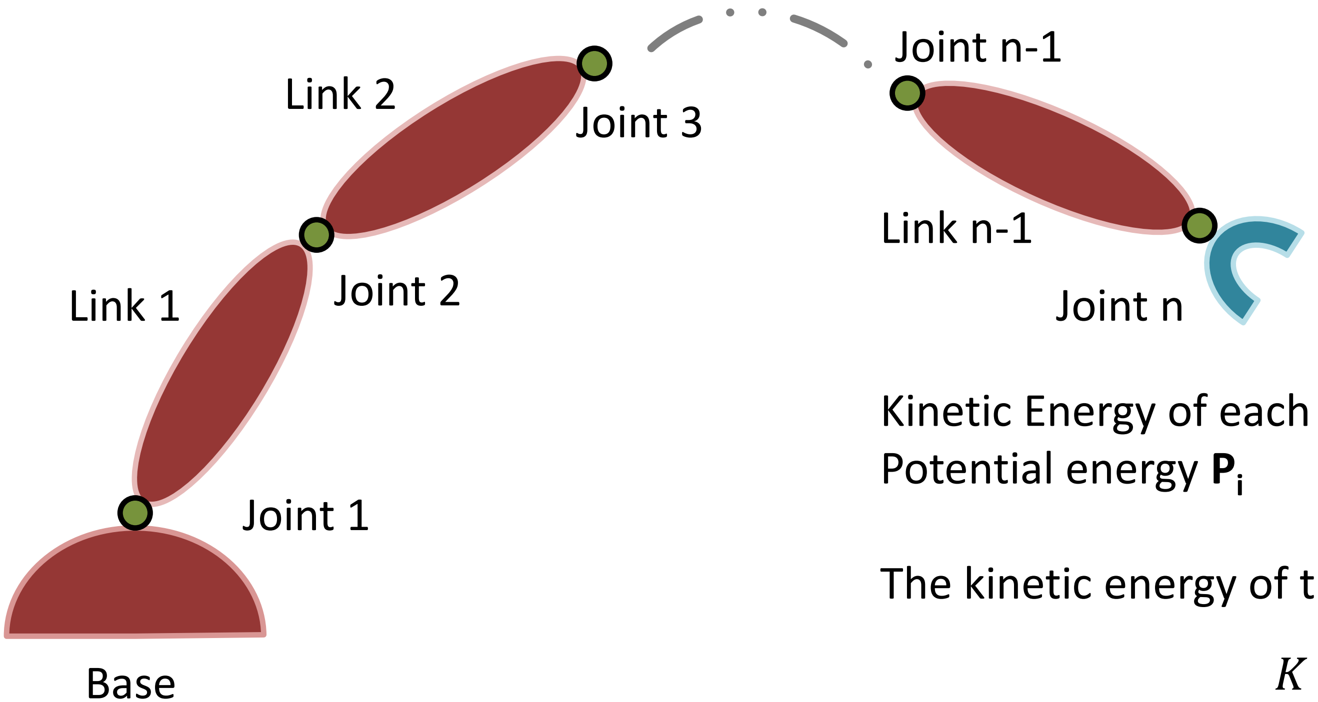
**Euler:**
Angular Moment = inertia. Acc. + Centrifugal and Coriolis forces.
$\boldsymbol{N_i = I_{Ci}\dot{\omega}_i + \omega_i \times I_{Ci}\omega_i}$

By projecting on each axis we eliminate the internal forces acting on the links, we find therefor the torques applied on each joint axis.

$$\tau_i = \begin{cases} {n_i}^T . z_i & \text{Revolute} \\ {f_i}^T . z_i & \text{Prismatic} \end{cases}$$



# Lagrange Formulation




Kinetic Energy of each link $\mathbf{K_i}$
Potential energy $\mathbf{P_i}$

The kinetic energy of the robot:

$$K = \sum_i K_i$$

$$K = \frac{1}{2}\dot{q}^T.M.\dot{q}$$

By computing M, we can use it in:

$$\boldsymbol{M\ddot{q} + V + G = \tau}$$

# Lagrange Formulation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_i$$

Lagrange function is defined

$$L = K - P$$

$K$: Total kinetic energy of the robot.

$P$: Total potential energy of the robot.

$q_i$: Joint variable of i-th joint.

$\dot{q}_i$: first time derivative of $q_i$

$\tau_i$: Generalized force (torque) at i-th joint.

# Center of Mass

COM is the point on a body that moves in the same way that a single particle subject to the same external force will move.

$$r_{cm} = \frac{1}{M}\int r.dm = \frac{1}{M}\sum_{1}^{i} m_i r_i$$

$r_{cm}$ is the location of the centre of mass.
$M$ is the total mass of the object
$r$ is the location of the reference frame.
$dm$ is the differential element of mass at point $r$.

# Inertia

The tendency of a body to remain in a state of rest or uniform motion.

Inertial frame: the frame in which this state is measured (a frame at rest or moving with constant velocity).

Non-Inertial frame: a frame that is accelerating with respect to the inertial frame.

# Moment of Inertia

Moment of inertia is the rotational inertia of a body with respect to the axis of rotation. Moment of inertia depends on the axis and on the manner in which the mass is distributed.

It is equal to the sum of the product between mass of particles and the square of their distance to the axis of rotation.

$$I = \sum_{1}^{i} m_i {r_i}^2$$

Where $m_i$ is the mass of particle $i$, and $r_i$ is the distance to the particle $i$.

# Moment of Inertia

For a rigid body with a continuous distribution of mass, summation becomes integral over the whole body:

$$I = \int r^2 dm$$

**Example**: Inertia of a cylinder (length $\boldsymbol{l}$, mass $\boldsymbol{M}$):

Consider the cylinder made up of an infinite number of cylinders thickness $\boldsymbol{dr}$, at radius $\boldsymbol{r}$, of mass $\boldsymbol{dm}$, and volume $\boldsymbol{dV}$.

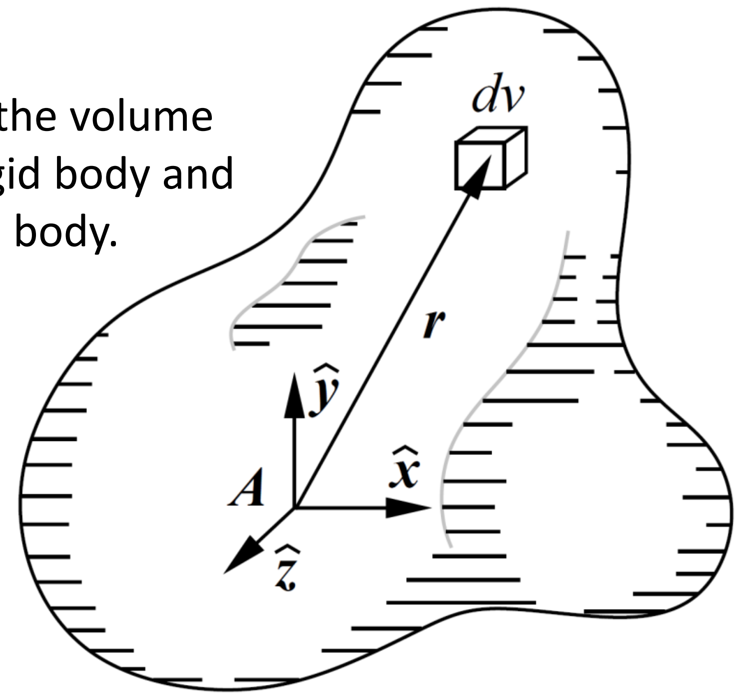
$$dm = \rho . dV = 2\pi l \rho r dr$$

$$I = \int r^2 dm = 2\pi l \int \rho r^3 dr = 2\pi l \rho \frac{{r_2}^4 - {r_1}^4}{4} = M \frac{{r_2}^2 + {r_1}^2}{4}$$

# Inertia Tensor

A rigid body, free to move in space, has an infinite number of possible rotation axis.

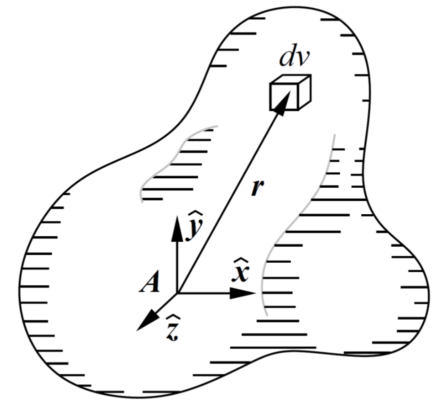
The inertia matrix ( Inertia tensor) is the integral over the volume of all the vectors $\boldsymbol{r}$ locating all the points $\boldsymbol{dv}$ on the rigid body and scaled by the density of all the masses $\boldsymbol{dm}$ of the rigid body.




For a rigid body rotating about a fixed point, which is not the centre of the mass, the inertia tensor is:

$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

# Inertia Tensor




$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

Mass moments of inertia (diagonal terms):

$$I_{xx} = \iiint (y^2 + z^2)\, \rho \, dv$$

$$I_{yy} = \iiint (x^2 + z^2)\, \rho \, dv$$

$$I_{zz} = \iiint (x^2 + y^2)\, \rho \, dv$$

Mass products of inertia (off-diagonal terms):

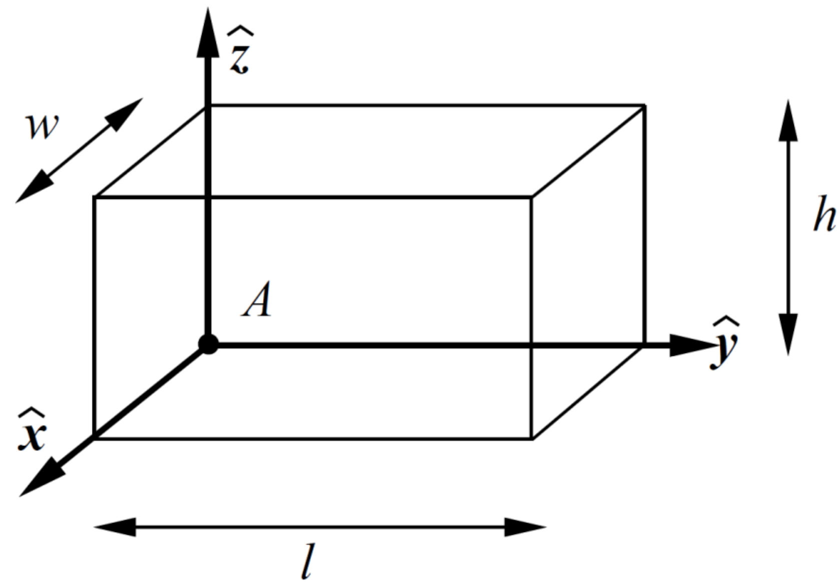
$$I_{xy} = \iiint xy \, \rho \, dv$$

$$I_{xz} = \iiint xz \, \rho \, dv$$

$$I_{yz} = \iiint yz \, \rho \, dv$$

# Example

Find the inertia tensor of a rectangle rotating about a fixed point **A**.




$$I_{xx} = \iiint (y^2 + z^2)\,\rho\, dv$$

$$I_{xx} = \int_0^h \int_0^l \int_0^w (y^2 + z^2)\,\rho\, dxdydz$$

$$I_{xx} = \int_0^h \int_0^l (y^2 + z^2)\, w\,\rho\, dydz$$

$$I_{xx} = \int_0^h [\left(\frac{y^3}{3} + z^2 y\right)]_0^l\, w\,\rho\, dz$$

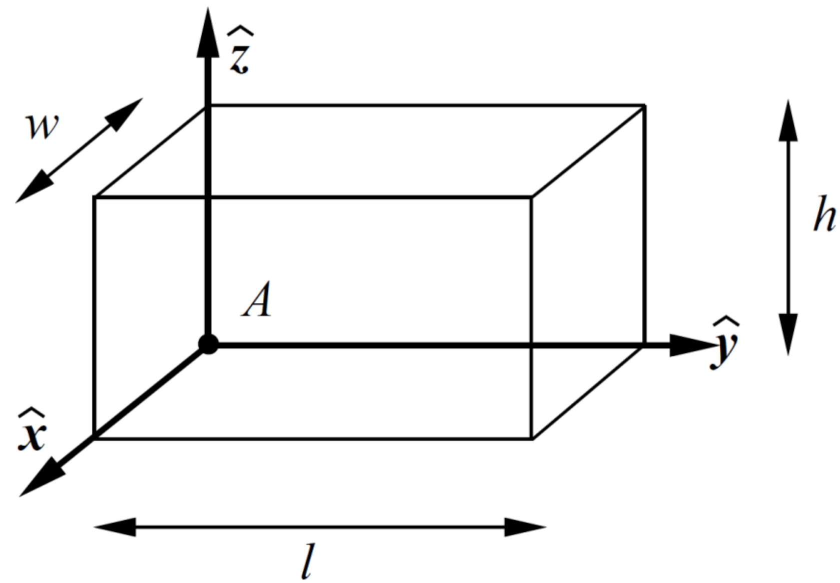
$$I_{xx} = \int_0^h \left(\frac{l^3}{3} + z^2 l\right) w\,\rho\, dz$$

$$I_{xx} = \left(\frac{l^3 z}{3} + \frac{z^3 l}{3}\right)\Bigg|_0^h w\,\rho$$

$$\boldsymbol{I_{xx} = \left(\frac{l^3 h}{3} + \frac{h^3 l}{3}\right) w\,\rho}$$

# Example

Find the inertia tensor of a rectangle rotating about a fixed point **A**.




$$\boldsymbol{I_{xx}} = \left(\frac{\boldsymbol{l^3 h}}{\boldsymbol{3}} + \frac{\boldsymbol{h^3 l}}{\boldsymbol{3}}\right)\boldsymbol{w\ \rho}$$

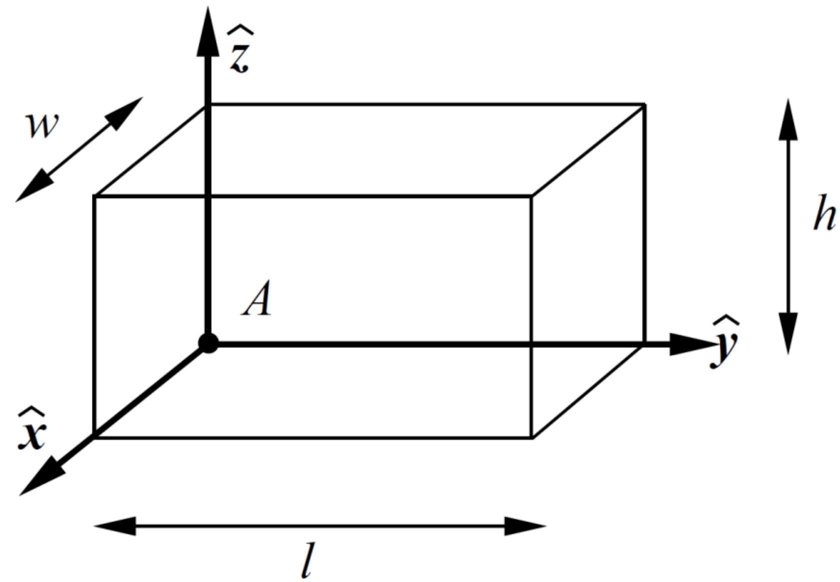
Since the mass of the rectangle:

$$m = (wlh)\rho$$

$$\boldsymbol{I_{xx}} = \frac{\boldsymbol{m}}{\boldsymbol{3}}(\boldsymbol{l^2} + \boldsymbol{h^2})$$

# Example

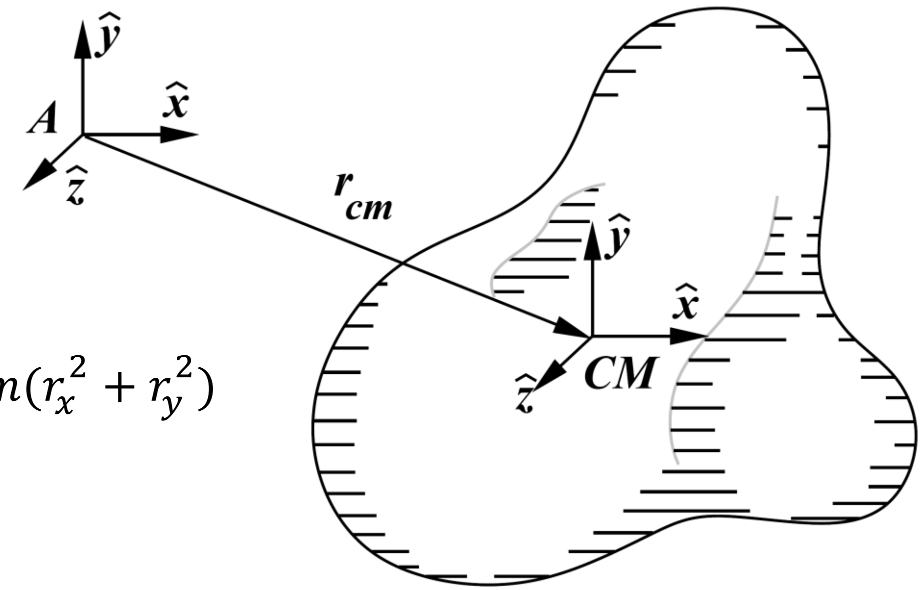
Find the inertia tensor of a rectangle rotating about a fixed point **A**.




$$
{}^{A}I = \begin{bmatrix} \frac{m}{3}(l^2+h^2) & \frac{m}{4}wl & \frac{m}{4}hw \\ \frac{m}{4}wl & \frac{m}{3}(w^2+h^2) & \frac{m}{4}hl \\ \frac{m}{4}hw & \frac{m}{4}hl & \frac{m}{3}(l^2+w^2) \end{bmatrix}
$$

# Translation of Inertia Tensor

Parallel axis theorem.




Moments of inertia:

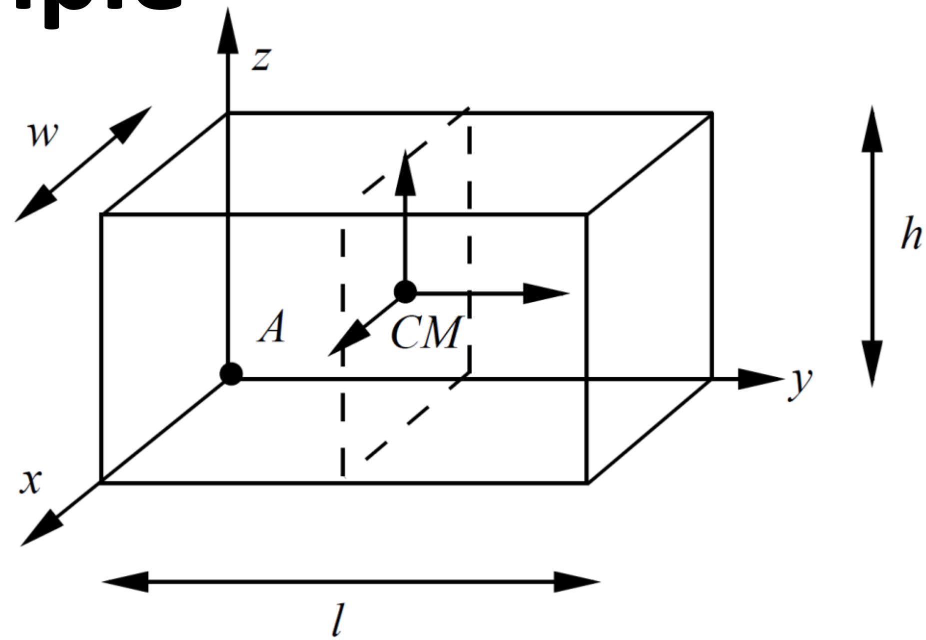
$$^{A}I_{zz} = {}^{CM}I_{zz} + m(r_x^2 + r_y^2)$$

Product of inertia:

$$^{A}I_{xy} = {}^{CM}I_{xy} + m(r_x\, r_y)$$

# Example

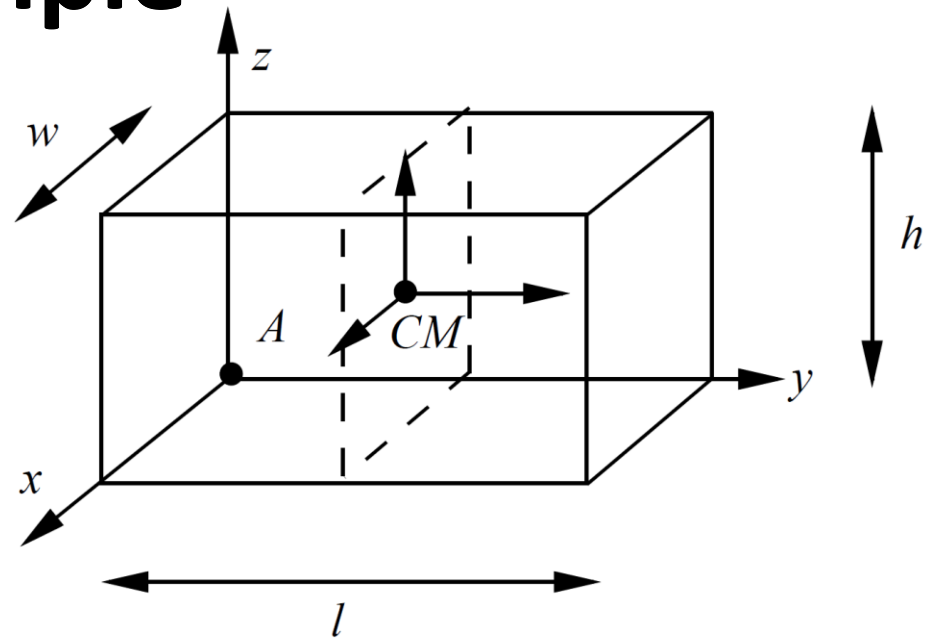
Find the inertia tensor of a rectangle rotating about the COM.




$$^{CM}I_{zz} = \ ^{A}I_{zz} - m(r_x^2 + r_y^2)$$

# Example

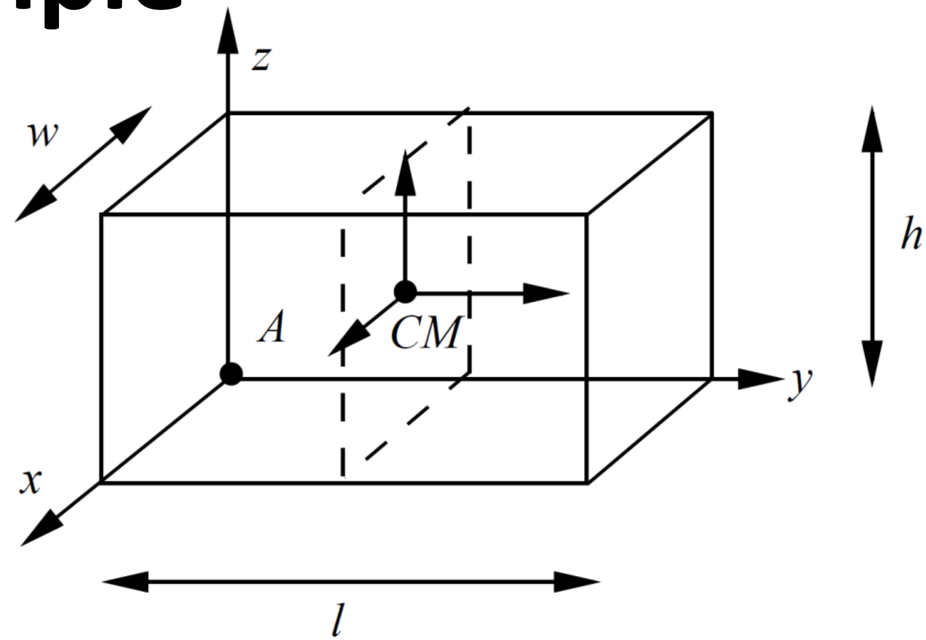
Find the inertia tensor of a rectangle rotating about the COM.




$$
\begin{aligned}
{}^{CM}I_{zz} &= {}^{A}I_{zz} - m(r_x^2 + r_y^2) \\
&= \frac{m}{3}(l^2 + w^2) - \frac{m}{4}(l^2 + w^2) \\
&= \frac{m}{12}(l^2 + w^2)
\end{aligned}
$$

# Example

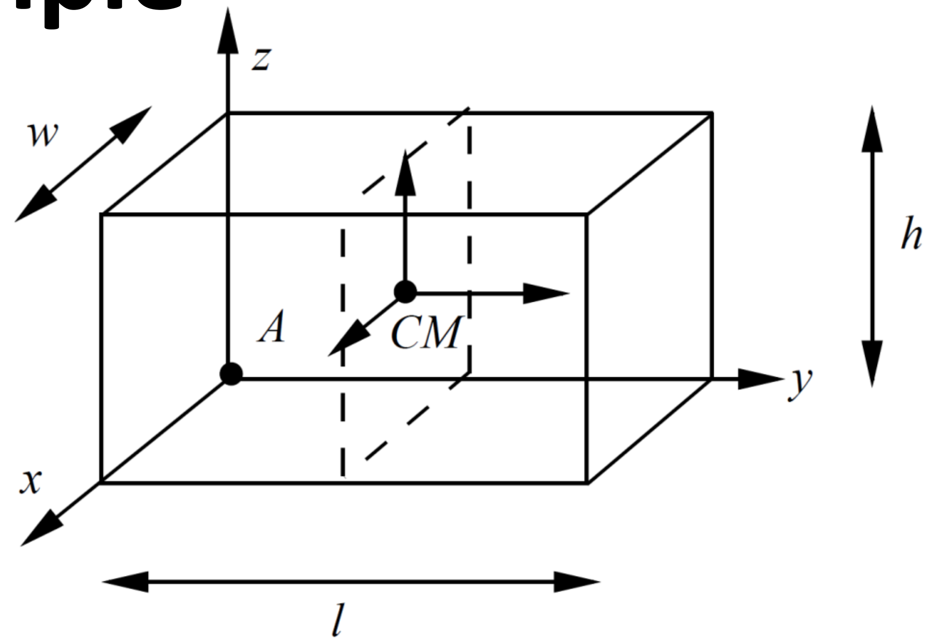
Find the inertia tensor of a rectangle rotating about the COM.




$$^{CM}I_{xy} =$$

# Example

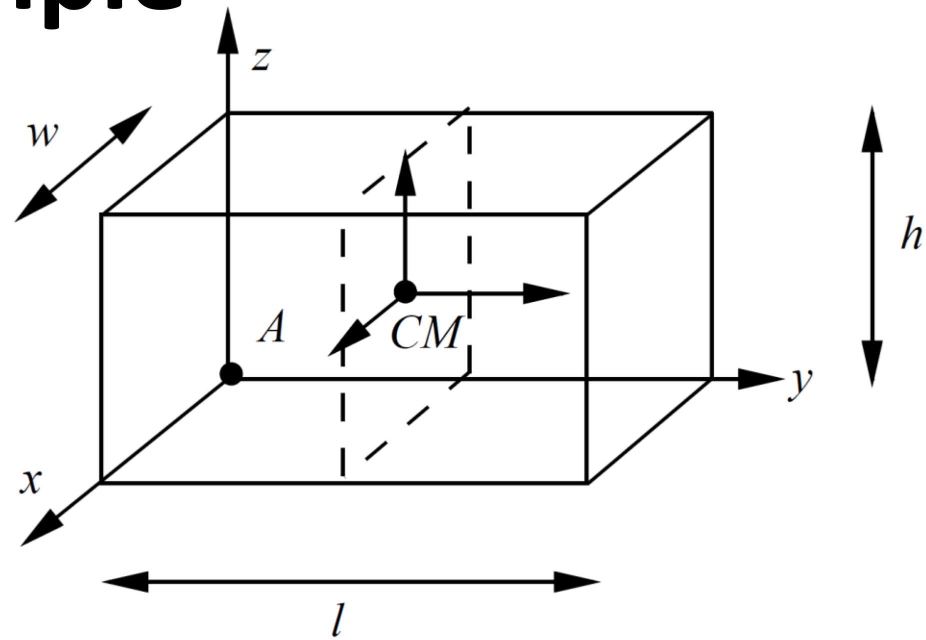
Find the inertia tensor of a rectangle rotating about the COM.




$$
\begin{aligned}
{}^{CM}I_{xy} &= \ {}^{A}I_{xy} - m(r_x\, r_y) \\
&= \frac{m}{4}(wl) - \frac{m}{4}(wl) = 0
\end{aligned}
$$

# Example

Find the inertia tensor of a rectangle rotating about the COM.




$$
{}^{CM}I = \frac{m}{12}\begin{bmatrix} (l^2+h^2) & 0 & 0 \\ 0 & (w^2+h^2) & 0 \\ 0 & 0 & (l^2+w^2) \end{bmatrix}
$$

# Example

$$ {}^{CM}I = \frac{m}{12}\begin{bmatrix} (l^2+h^2) & 0 & 0 \\ 0 & (w^2+h^2) & 0 \\ 0 & 0 & (l^2+w^2) \end{bmatrix} $$

$$ {}^{A}I = \begin{bmatrix} \frac{m}{3}(l^2+h^2) & \frac{m}{4}wl & \frac{m}{4}hw \\ \frac{m}{4}wl & \frac{m}{3}(w^2+h^2) & \frac{m}{4}hl \\ \frac{m}{4}hw & \frac{m}{4}hl & \frac{m}{3}(l^2+w^2) \end{bmatrix} $$

**Moving the axes of rotation to the centre of mass results in a diagonal inertia tensor.**

# Lagrange Formulation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \tau$$

Lagrange function is defined

$$L = K - P$$

$K$: Total kinetic energy of the robot.

$P$: Total potential energy of the robot.

$q$ : Joint variable.

$\dot{q}$ : first time derivative of $q$

$\tau$ : Generalized force (torque).

# Lagrange Formulation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \tau$$

Lagrange function is defined

$$L = K - P$$

Since $P = P(q)$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q} + \frac{\partial P}{\partial q} = \tau$$

**Inertial forces** **Gravity vector (gradient of potential energy)**

# Lagrange Formulation

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q} = \tau - G(q), \qquad G(q) = \frac{\partial P}{\partial q}$$

Inertial forces

This equation can be written in the following from:

**Joint Space Dynamics**

$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \Gamma$$

$$M(q)\ddot{q} + V(q,\dot{q}) = \Gamma - G(q)$$

# Lagrange Formulation

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q} = \tau - G$$

$$K = \frac{1}{2}\dot{q}^T M(q)\dot{q}$$

$$\frac{\partial K}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i}\left[\frac{1}{2}\dot{q}^T M(q)\dot{q}\right] = M(q)\dot{q}$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) = \frac{d}{dt}(M\dot{q}) = M\ddot{q} + \dot{M}\dot{q}$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q} = M\ddot{q} + \dot{M}\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \frac{\partial M}{\partial q_1}\dot{q} \\ \vdots \\ \dot{q}^T \frac{\partial M}{\partial qn}\dot{q} \end{bmatrix} = M\ddot{q} + V(q,\dot{q})$$

# Lagrange Formulation

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q} = M\ddot{q} + \dot{M}\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \dfrac{\partial M}{\partial q_1}\dot{q} \\ \vdots \\ \dot{q}^T \dfrac{\partial M}{\partial qn}\dot{q} \end{bmatrix} = M\ddot{q} + V(q,\dot{q})$$

Centrifugal and Coriolis forces

If $\dot{q} = 0, V(q,\dot{q}) = 0$

If $M \;=\; CONSTANT,, V(q,\dot{q}) = 0$

We need to compute **M** from the kinetic energy equations then we have the dynamics of the robot.

# Dynamic Equations

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q} = M\ddot{q} + \dot{M}\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \dfrac{\partial M}{\partial q_1} \dot{q} \\ \vdots \\ \dot{q}^T \dfrac{\partial M}{\partial qn} \dot{q} \end{bmatrix} = M\ddot{q} + V(q, \dot{q})$$

$$M(q)\ddot{q} + V(q, \dot{q}) + G(q) = \tau$$

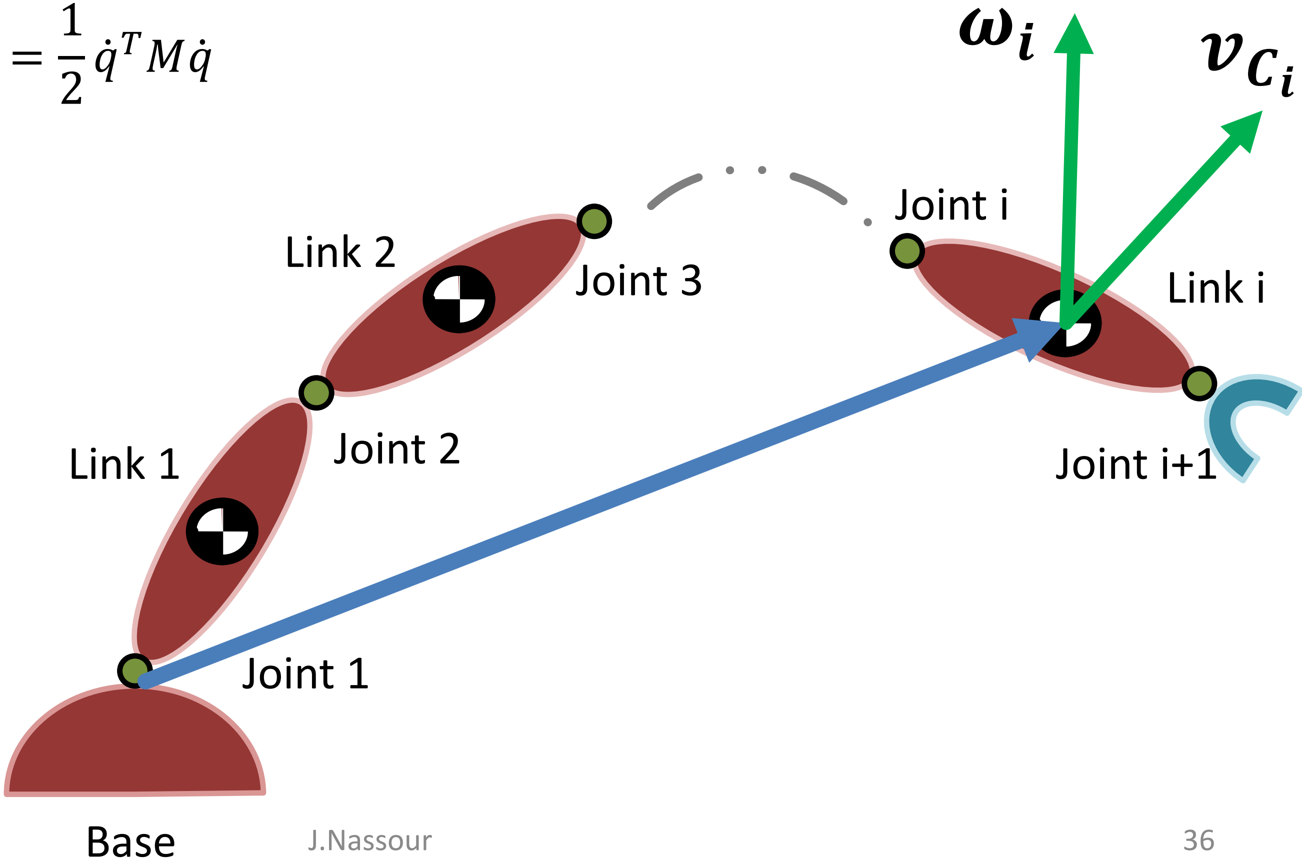
$$K = \frac{1}{2}\dot{q}^T M(q)\dot{q} \Longrightarrow M(q)$$

$$M(q) \Longrightarrow V(q, \dot{q})$$

# Dynamic Equations

Total kinetic energy

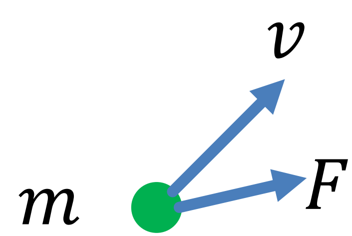
$$K = \sum K_{Link\ i} = \frac{1}{2}\dot{q}^T M \dot{q}$$



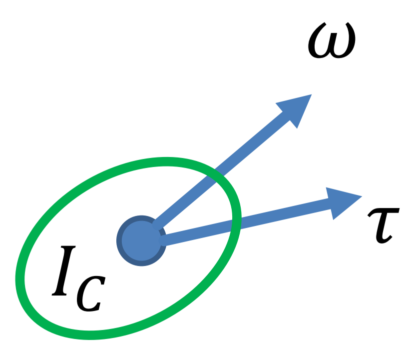
# Kinetic Energy

Work done by external forces to bring the system from rest to its current state.

$$K = \frac{1}{2} m v^2$$




$$K = \frac{1}{2} \omega^T I_C \, \omega$$




$I_C$ is a matrix
$\omega$ is a vector

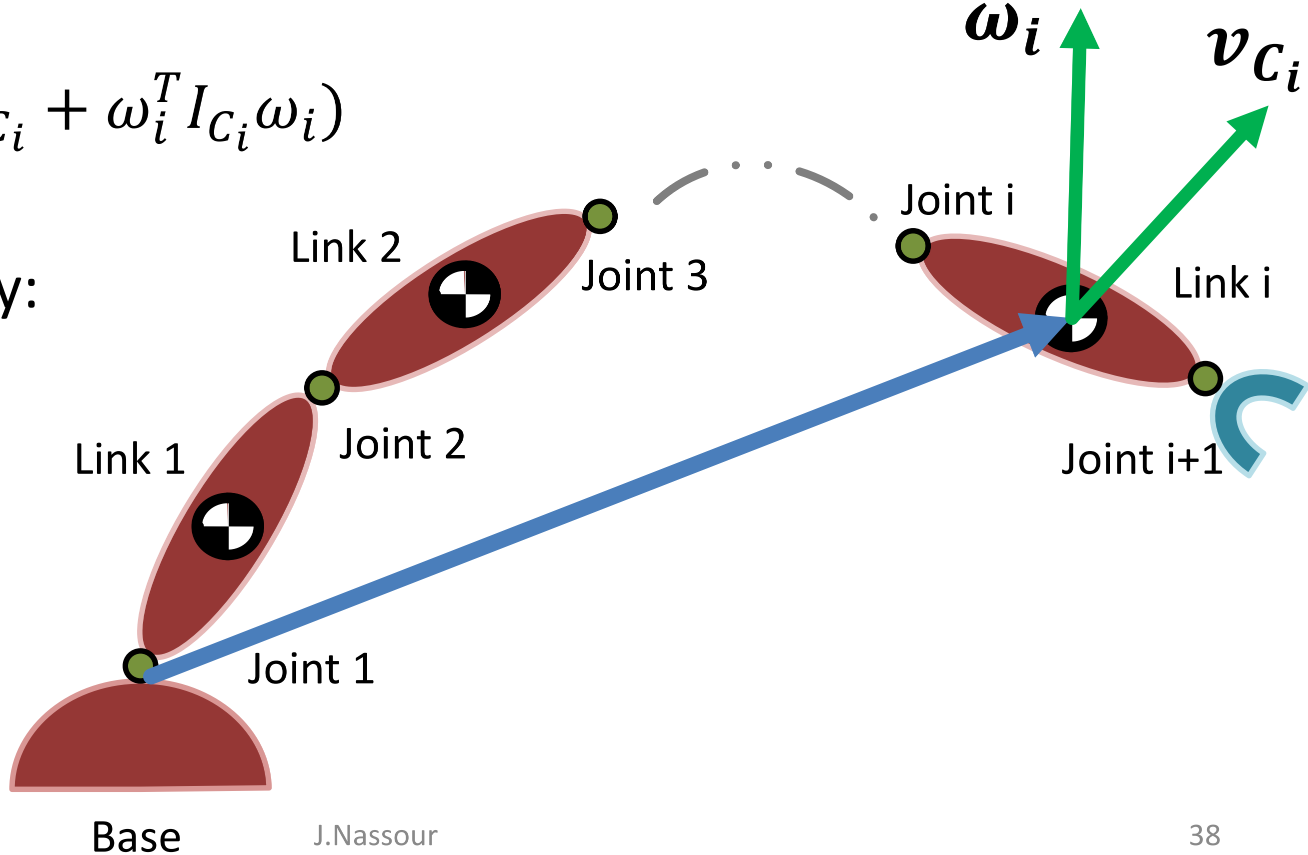
# Dynamic Equations

The kinetic energy for link i:

$$K_i = \frac{1}{2}(m_i v_{C_i}^T v_{C_i} + \omega_i^T I_{C_i} \omega_i)$$

Total kinetic energy:

$$K = \sum_{i=1}^{n} K_i$$



# Dynamic Equations

Kinetic energy in quadratic form of generalized velocities:

$$K = \frac{1}{2}\dot{q}^T M \dot{q}$$

Then we can write:

$$\frac{1}{2}\dot{q}^T M \dot{q} = \frac{1}{2}\sum_{i=1}^{n}(m_i v_{C_i}^T v_{C_i} + \omega_i^T I_{C_i} \omega_i)$$

**How extract the mass matrix $M$?**

# Dynamic Equations

Kinetic energy in quadratic form of generalized velocities:

$$K = \frac{1}{2}\dot{q}^T M \dot{q}$$

Then we can write:

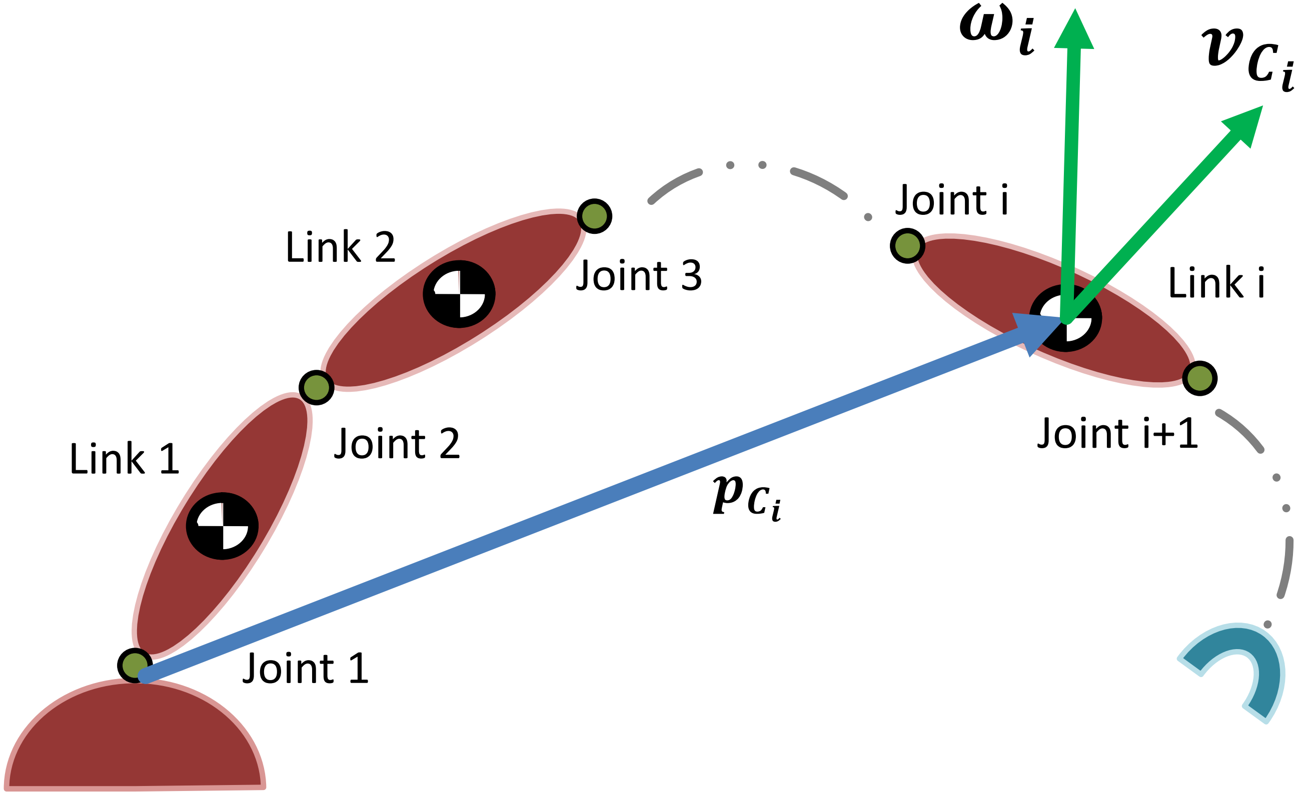
$$\frac{1}{2}\dot{q}^T M \dot{q} = \frac{1}{2}\sum_{i=1}^{n}(m_i v_{C_i}^T v_{C_i} + \omega_i^T I_{C_i}\omega_i)$$

**How extract the mass matrix $M$?**

$$v_{C_i} = \mathcal{J}_{v_i}\dot{q}\ , \qquad \omega_i = \mathcal{J}_{\omega_i}\dot{q}$$

# Dynamic Equations

$$v_{C_i} = \mathcal{J}_{v_i}\dot{q}$$
$$\omega_i = \mathcal{J}_{\omega_i}\dot{q}$$



# Dynamic Equations

$$\frac{1}{2}\dot{q}^T M\dot{q} = \frac{1}{2}\sum_{i=1}^{n}(m_i v_{C_i}^T v_{C_i} + \omega_i^T I_{C_i}\omega_i)$$

$$v_{C_i} = \mathcal{J}_{v_i}\dot{q}\ , \qquad\qquad \omega_i = \mathcal{J}_{\omega_i}\dot{q}$$

$$\frac{1}{2}\dot{q}^T M\dot{q} = \frac{1}{2}\sum_{i=1}^{n}(m_i\dot{q}^T {\mathcal{J}_{v_i}}^T\mathcal{J}_{v_i}\dot{q} + \dot{q}^T {\mathcal{J}_{\omega_i}}^T I_{C_i}\mathcal{J}_{\omega_i}\dot{q})$$

$$\frac{1}{2}\dot{q}^T M\dot{q} = \frac{1}{2}\dot{q}^T[\sum_{i=1}^{n}(m_i {\mathcal{J}_{v_i}}^T\mathcal{J}_{v_i} + {\mathcal{J}_{\omega_i}}^T I_{C_i}\mathcal{J}_{\omega_i})]\ \dot{q}$$

# Dynamic Equations

$$\boldsymbol{M} = \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{n}} (\boldsymbol{m_i} \boldsymbol{\mathcal{J}_{v_i}}^{\boldsymbol{T}} \boldsymbol{\mathcal{J}_{v_i}} + \boldsymbol{\mathcal{J}_{\omega_i}}^{\boldsymbol{T}} \boldsymbol{I_{C_i}} \boldsymbol{\mathcal{J}_{\omega_i}})$$

The mass matrix is the sum of Jacobians transpose Jacobians scaled by the mass properties ($m_i$, $I_{C_i}$).

If the robot has 1 DOF, M = …

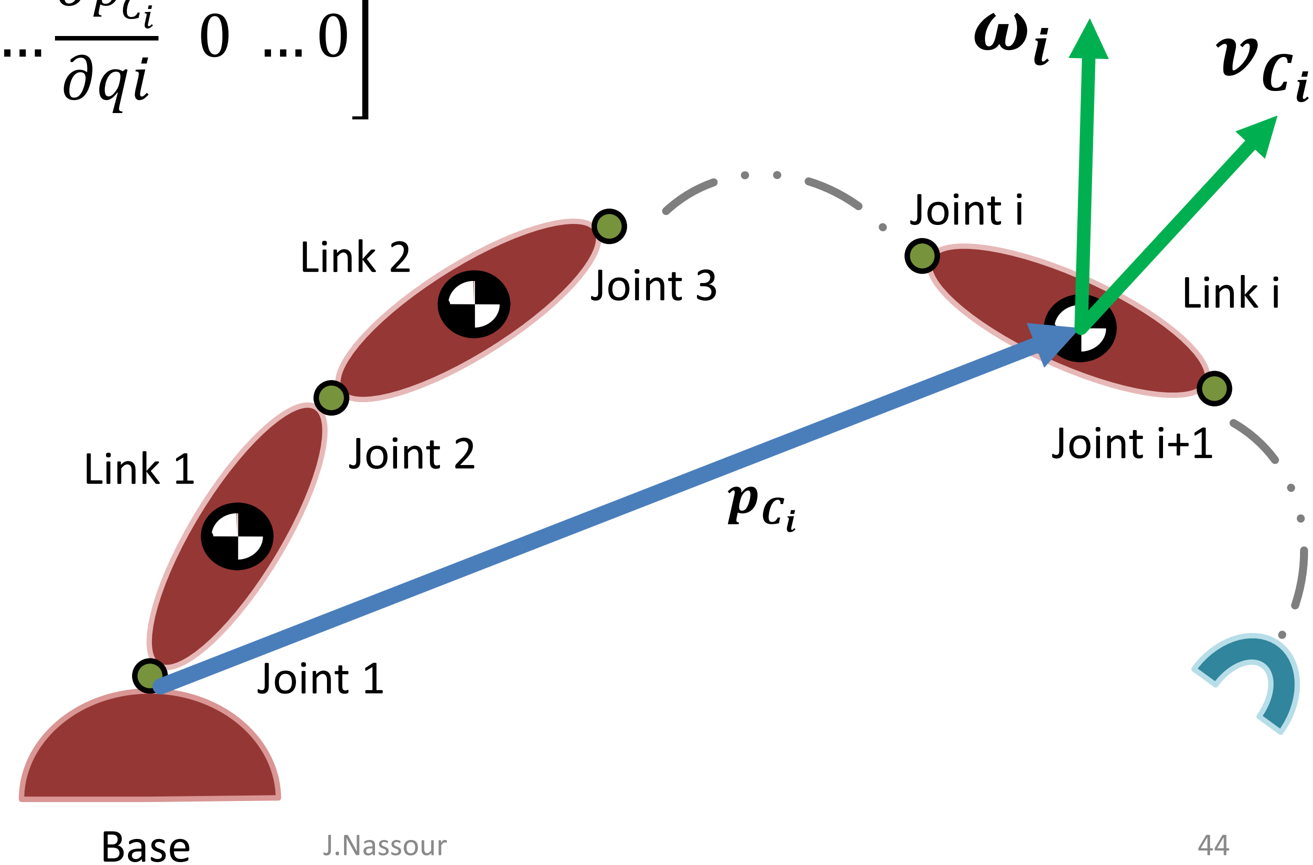
With multiple links, the Jacobian matrices of all links contribute in the mass matrix.

Each link has an impact on the total mass matrix.

# Dynamic Equations

$$M = \sum_{i=1}^{n} (m_i \mathcal{J}_{v_i}{}^T \mathcal{J}_{v_i} + \mathcal{J}_{\omega_i}{}^T I_{C_i} \mathcal{J}_{\omega_i})$$

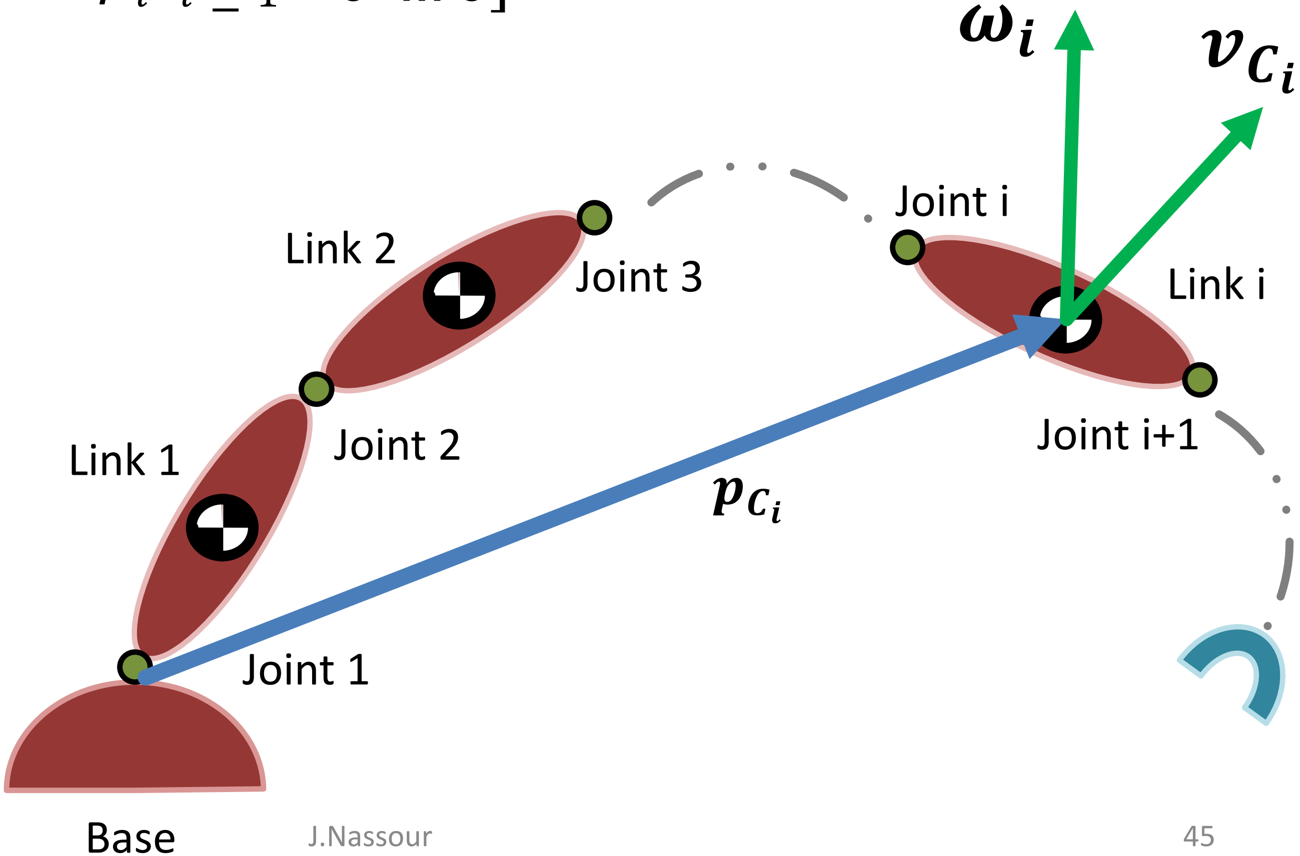
$$\mathcal{J}_{v_i} = \left[ \frac{\partial p_{C_i}}{\partial q_1} \quad \frac{\partial p_{C_i}}{\partial q_2} \quad \dots \frac{\partial p_{C_i}}{\partial qi} \quad 0 \quad \dots 0 \right]$$



# Dynamic Equations

$$M = \sum_{i=1}^{n} (m_i \mathcal{J}_{v_i}{}^T \mathcal{J}_{v_i} + \mathcal{J}_{\omega_i}{}^T I_{C_i} \mathcal{J}_{\omega_i})$$

$$\mathcal{J}_{\omega_i} = [\rho_1 z_0 \quad \rho_2 z_1 \quad \ldots \quad \rho_i z_{i-1} \quad 0 \ \ldots 0]$$



# Mass Matrix

$$M(q) = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix}$$

$m_{11}$ represents the inertia of the arm perceived at joint 1.
$m_{11} = f(q_{2:n})$
$m_{22} = f(q_{3:n})$
……
$m_{nn} = constant$

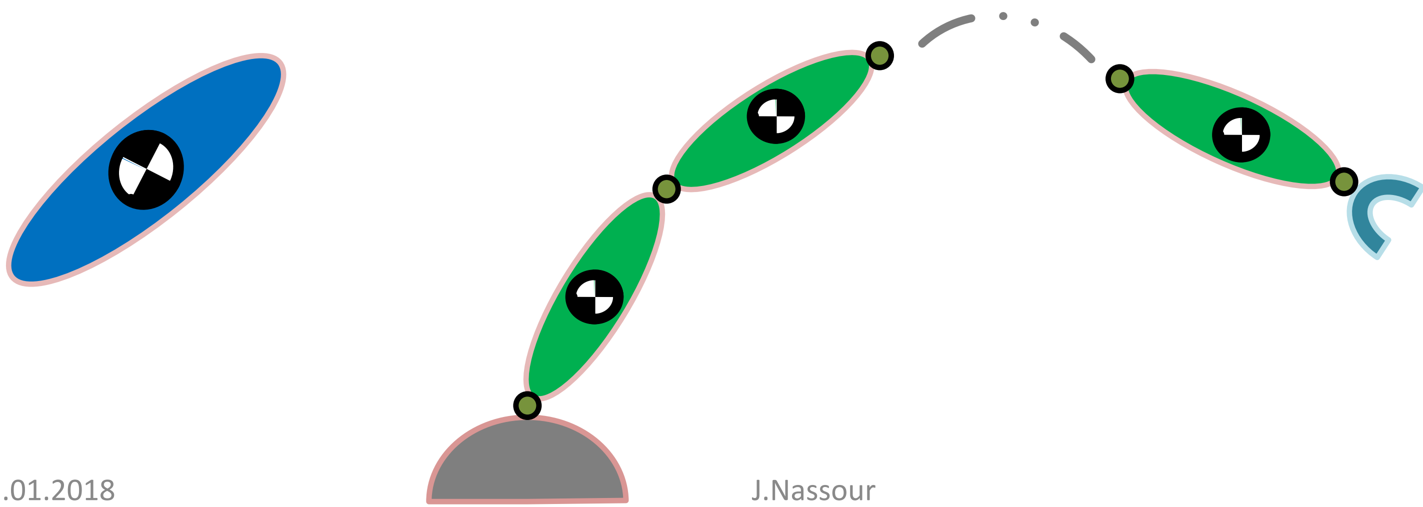
# Mass Matrix

$$M(q) = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix}$$

$m_{12}$ represents the coupling between the acceleration of joint 2 on joint 1.

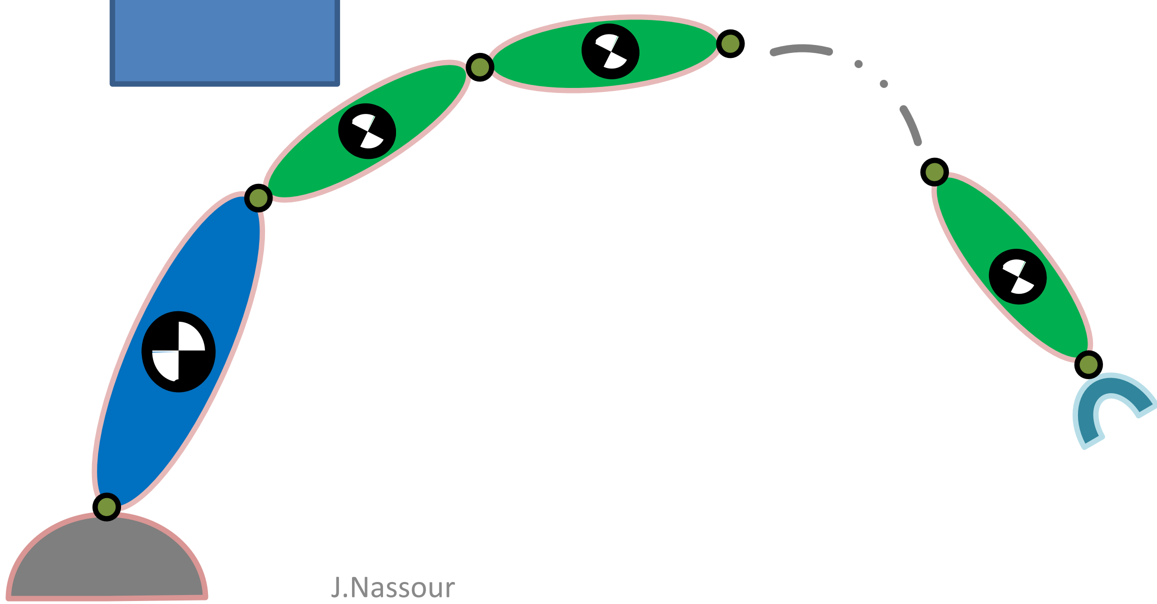
# Mass Matrix

$$M(q) = \begin{bmatrix} & & & \\ & m_{22} & \cdots & m_{2n} \\ & \vdots & \vdots & \vdots \\ & m_{n2} & \cdots & m_{nn} \end{bmatrix}$$



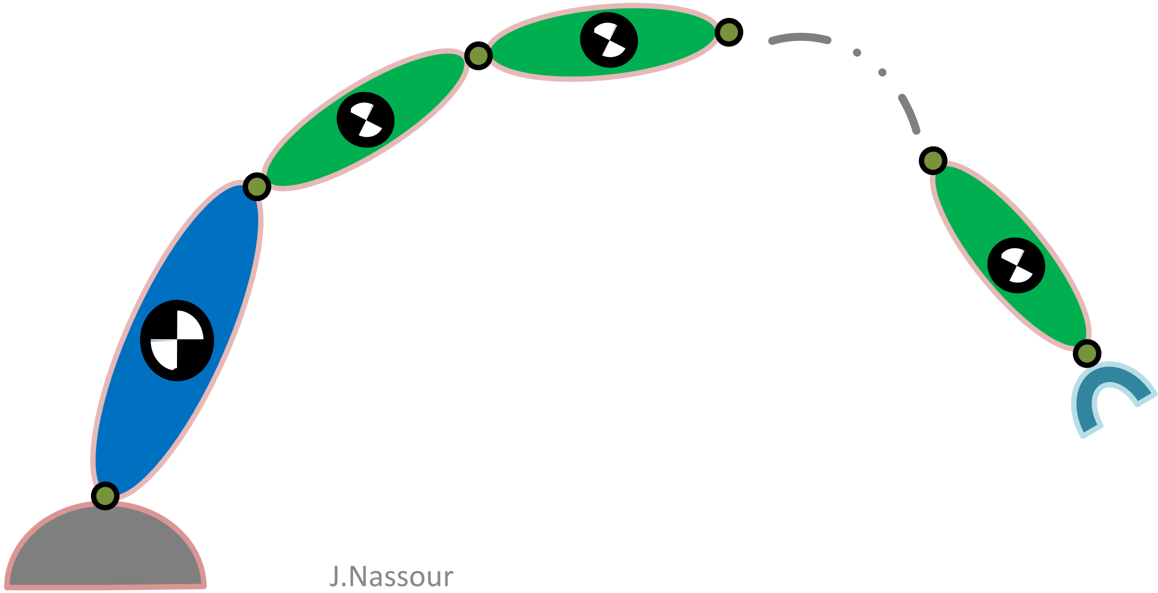
# Mass Matrix

$$M(q) = \begin{bmatrix} & & & \\ & m_{22} & \cdots & m_{2n} \\ & \vdots & \vdots & \vdots \\ & m_{n2} & \cdots & m_{nn} \end{bmatrix}$$



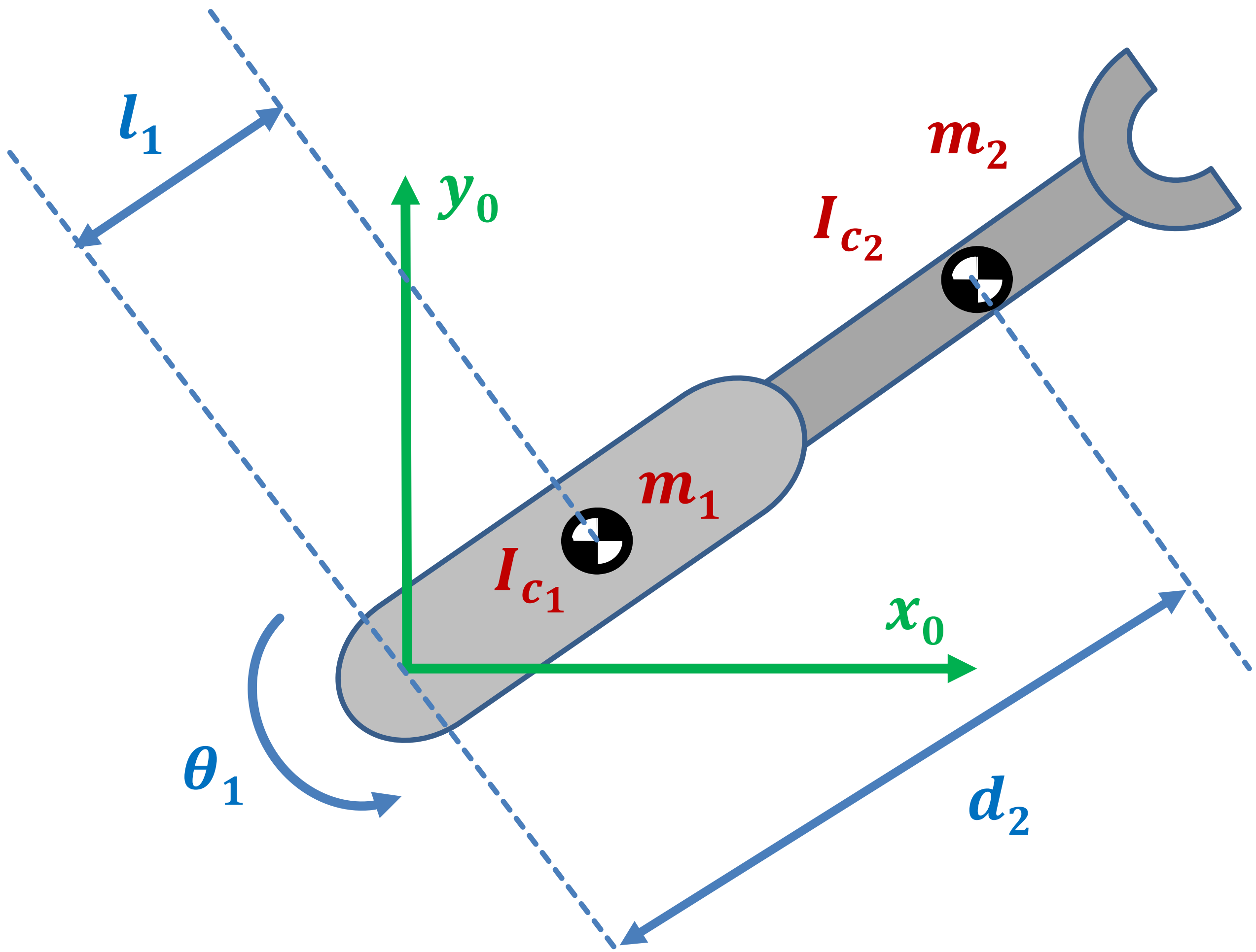
# Mass Matrix

$$M(q) = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix}$$



# Example RP

Work out the mass matrix $\boldsymbol{M}$.



# Example RP

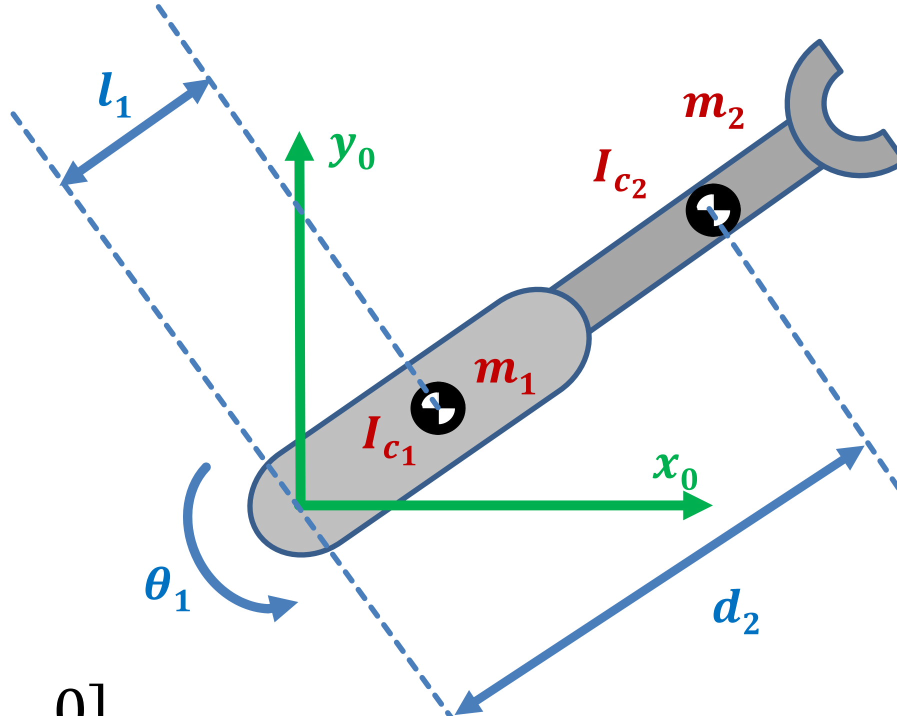
Work out the mass matrix $\boldsymbol{M}$.

$$M = \sum_{i=1}^{n} \left( m_i \mathcal{J}_{v_i}{}^{T} \mathcal{J}_{v_i} + \mathcal{J}_{\omega_i}{}^{T} I_{C_i} \mathcal{J}_{\omega_i} \right)$$

$$\mathcal{J}_{v_i} = \left[ \frac{\partial p_{C_i}}{\partial q_1} \quad \frac{\partial p_{C_i}}{\partial q_2} \quad \ldots \frac{\partial p_{C_i}}{\partial qi} \quad 0 \; \ldots 0 \right]$$

$$\mathcal{J}_{\omega_i} = [\rho_1 z_0 \quad \rho_2 z_1 \quad \ldots \quad \rho_i z_{i\,-\,1} \quad 0 \; \ldots 0]$$

$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$




Do it!

# Example RP

Work out the mass matrix $\boldsymbol{M}$.

$$M = \sum_{i=1}^{n} (m_i \mathcal{J}_{v_i}{}^T \mathcal{J}_{v_i} + \mathcal{J}_{\omega_i}{}^T I_{C_i} \mathcal{J}_{\omega_i})$$

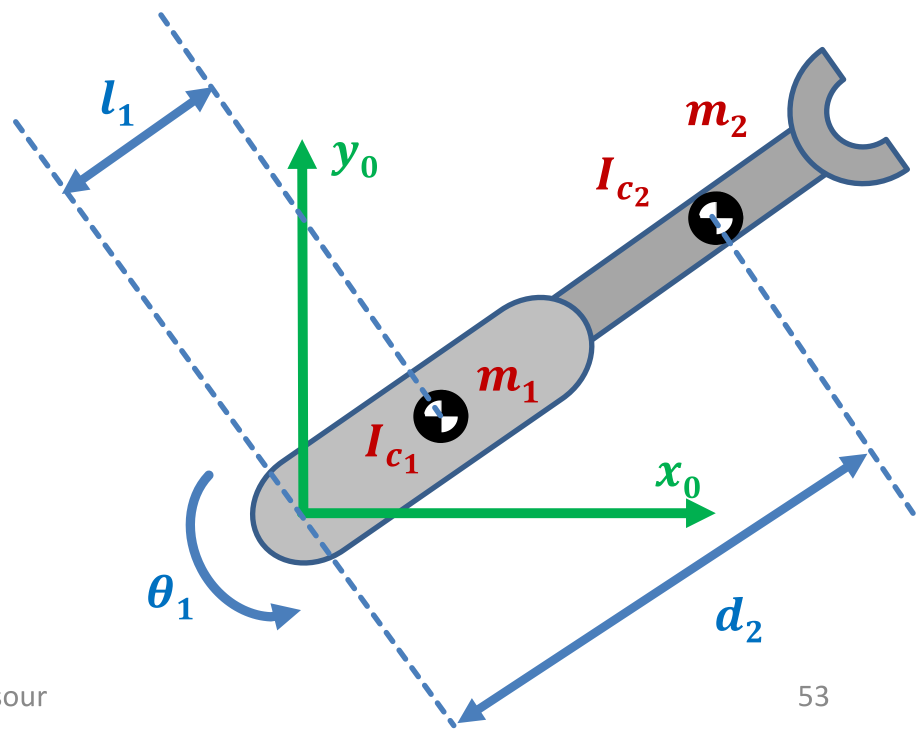
$$M = m_1 \mathcal{J}_{v_1}{}^T \mathcal{J}_{v_1} + \mathcal{J}_{\omega_1}{}^T I_{C_1} \mathcal{J}_{\omega_1} + m_2 \mathcal{J}_{v_2}{}^T \mathcal{J}_{v_2} + \mathcal{J}_{\omega_2}{}^T I_{C_2} \mathcal{J}_{\omega_2}$$

In frame ($o_0$,$x_0$,$y_0$,$z_0$)

$$p_{C_1} = \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{bmatrix}, p_{C_2} = \begin{bmatrix} d_2 c_1 \\ d_2 s_1 \\ 0 \end{bmatrix}$$

$$\mathcal{J}_{v_1} = \begin{bmatrix} -l_1 s_1 & 0 \\ l_1 c_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathcal{J}_{v_2} = \begin{bmatrix} -d_2 s_1 & c_1 \\ d_2 c_1 & s_1 \\ 0 & 0 \end{bmatrix}$$



# Example RP

Work out the mass matrix $\boldsymbol{M}$.

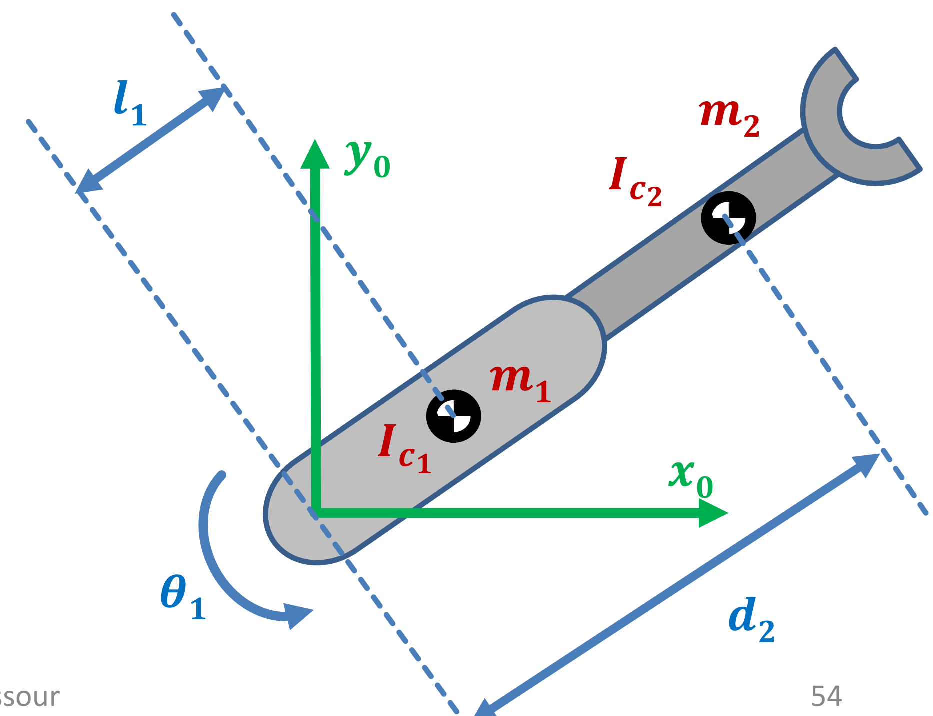
$$M = m_1 \mathcal{J}_{v_1}{}^T \mathcal{J}_{v_1} + \mathcal{J}_{\omega_1}{}^T I_{C_1} \mathcal{J}_{\omega_1} + m_2 \mathcal{J}_{v_2}{}^T \mathcal{J}_{v_2} + \mathcal{J}_{\omega_2}{}^T I_{C_2} \mathcal{J}_{\omega_2}$$

$$\mathcal{J}_{v_1} = \begin{bmatrix} -l_1 s_1 & 0 \\ l_1 c_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathcal{J}_{v_2} = \begin{bmatrix} -d_2 s_1 & c_1 \\ d_2 c_1 & s_1 \\ 0 & 0 \end{bmatrix}$$

$$m_1 \mathcal{J}_{v_1}{}^T \mathcal{J}_{v_1} = \begin{bmatrix} m_1 l_1^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$m_2 \mathcal{J}_{v_2}{}^T \mathcal{J}_{v_2} = \begin{bmatrix} m_2 d_2^2 & 0 \\ 0 & m_2 \end{bmatrix}$$



# Example RP

Work out the mass matrix $\boldsymbol{M}$.

$$M = m_1 \mathcal{J}_{v_1}{}^T \mathcal{J}_{v_1} + \mathcal{J}_{\omega_1}{}^T I_{C_1} \mathcal{J}_{\omega_1} + m_2 \mathcal{J}_{v_2}{}^T \mathcal{J}_{v_2} + \mathcal{J}_{\omega_2}{}^T I_{C_2} \mathcal{J}_{\omega_2}$$

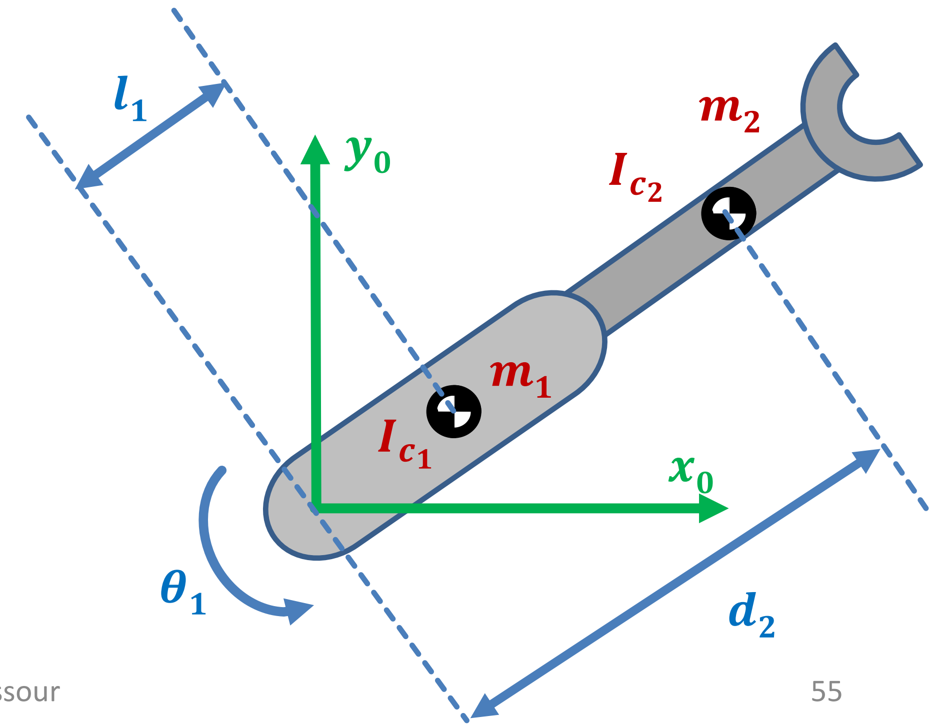
$$\mathcal{J}_{\omega_i} = [\rho_1 z_0 \quad \rho_2 z_1 \quad \cdots \quad \rho_i z_{i\,-\,1} \quad 0 \;\ldots 0]$$

$$\mathcal{J}_{\omega_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\mathcal{J}_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\mathcal{J}_{\omega_1}{}^T I_{C_1} \mathcal{J}_{\omega_1} = \begin{bmatrix} I_{zz1} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathcal{J}_{\omega_2}{}^T I_{C_2} \mathcal{J}_{\omega_2} = \begin{bmatrix} I_{zz2} & 0 \\ 0 & 0 \end{bmatrix}$$



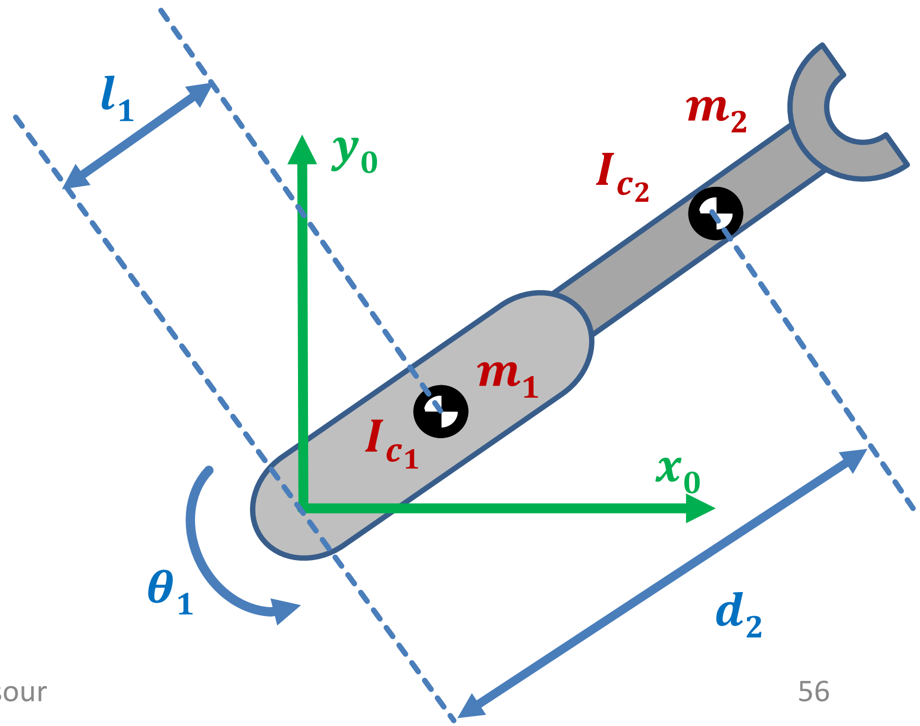
# Example RP

Work out the mass matrix $\boldsymbol{M}$.

$$M = m_1 \mathcal{J}_{v_1}{}^T \mathcal{J}_{v_1} + \mathcal{J}_{\omega_1}{}^T I_{C_1} \mathcal{J}_{\omega_1} + m_2 \mathcal{J}_{v_2}{}^T \mathcal{J}_{v_2} + \mathcal{J}_{\omega_2}{}^T I_{C_2} \mathcal{J}_{\omega_2}$$

$$M = \begin{bmatrix} m_1 l_1^2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} m_2 d_2^2 & 0 \\ 0 & m_2 \end{bmatrix} + \begin{bmatrix} I_{zz1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} I_{zz2} & 0 \\ 0 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} m_1 l_1^2 + I_{zz1} + m_2 d_2^2 + I_{zz2} & 0 \\ 0 & m_2 \end{bmatrix}$$



# Dynamic Equations

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q} = M\ddot{q} + \dot{M}\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \frac{\partial M}{\partial q_1} \dot{q} \\ \vdots \\ \dot{q}^T \frac{\partial M}{\partial qn} \dot{q} \end{bmatrix} = M\ddot{q} + V(q, \dot{q})$$

$$M(q)\ddot{q} + V(q, \dot{q}) + G(q) = \tau$$

$$K = \frac{1}{2}\dot{q}^T M(q)\dot{q} \Longrightarrow M(q)$$

$$M(q) \Longrightarrow V(q, \dot{q})$$

# Centrifugal and Coriolis Forces

$$\boldsymbol{V(q,\dot{q}) = \dot{M}\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \dfrac{\partial M}{\partial q_1}\dot{q} \\ \vdots \\ \dot{q}^T \dfrac{\partial M}{\partial qn}\dot{q} \end{bmatrix}}$$

$$M = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{bmatrix}$$

$$\boldsymbol{M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \tau}$$

# Centrifugal and Coriolis Forces

$$V(q,\dot{q}) = \dot{M}\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \dfrac{\partial M}{\partial q_1}\dot{q} \\ \vdots \\ \dot{q}^T \dfrac{\partial M}{\partial qn}\dot{q} \end{bmatrix}$$

For robot with 2 DOF…

$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \tau$$

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

# Centrifugal and Coriolis Forces

$$\boldsymbol{V(q,\dot{q}) = \dot{M}\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \frac{\partial M}{\partial q_1}\dot{q} \\ \vdots \\ \dot{q}^T \frac{\partial M}{\partial qn}\dot{q} \end{bmatrix}}$$

$$\boldsymbol{V(q,\dot{q}) = \begin{bmatrix} \dot{m}_{11} & \dot{m}_{12} \\ \dot{m}_{12} & \dot{m}_{22} \end{bmatrix}\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \begin{bmatrix} m_{111} & m_{121} \\ m_{121} & m_{221} \end{bmatrix}\dot{q} \\ \dot{q}^T \begin{bmatrix} m_{112} & m_{122} \\ m_{122} & m_{222} \end{bmatrix}\dot{q} \end{bmatrix}}$$

$$m_{ijk} = \frac{\partial m_{ij}}{\partial qk}$$

$$\dot{m}_{ij} = m_{ij1}\dot{q}_1 + m_{ij2}\dot{q}_2 + \ \ldots + m_{ijk}\dot{q}_k$$

# Centrifugal and Coriolis Forces

$$V(q,\dot{q}) = \begin{bmatrix} \dot{m}_{11} & \dot{m}_{12} \\ \dot{m}_{12} & \dot{m}_{22} \end{bmatrix} \dot{q} - \frac{1}{2} \begin{bmatrix} \dot{q}^T \begin{bmatrix} m_{111} & m_{121} \\ m_{121} & m_{221} \end{bmatrix} \dot{q} \\ \dot{q}^T \begin{bmatrix} m_{112} & m_{122} \\ m_{122} & m_{222} \end{bmatrix} \dot{q} \end{bmatrix}$$

$$V(q,\dot{q}) = \begin{bmatrix} \frac{1}{2}(m_{111} + m_{111} - m_{111}) & \frac{1}{2}(m_{122} + m_{122} - m_{221}) \\ \frac{1}{2}(m_{211} + m_{211} - m_{112}) & \frac{1}{2}(m_{222} + m_{222} - m_{222}) \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}$$

$$+ \begin{bmatrix} m_{112} + m_{121} - m_{121} \\ m_{212} + m_{221} - m_{122} \end{bmatrix} [\dot{q}_1 \dot{q}_2]$$

# Centrifugal and Coriolis Forces

$$V(q,\dot{q}) = \begin{bmatrix} \frac{1}{2}(m_{111} + m_{111} - m_{111}) & \frac{1}{2}(m_{122} + m_{122} - m_{221}) \\ \frac{1}{2}(m_{211} + m_{211} - m_{112}) & \frac{1}{2}(m_{222} + m_{222} - m_{222}) \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}$$

$$+ \begin{bmatrix} m_{112} + m_{121} - m_{121} \\ m_{212} + m_{221} - m_{122} \end{bmatrix} [\dot{q}_1 \dot{q}_2]$$

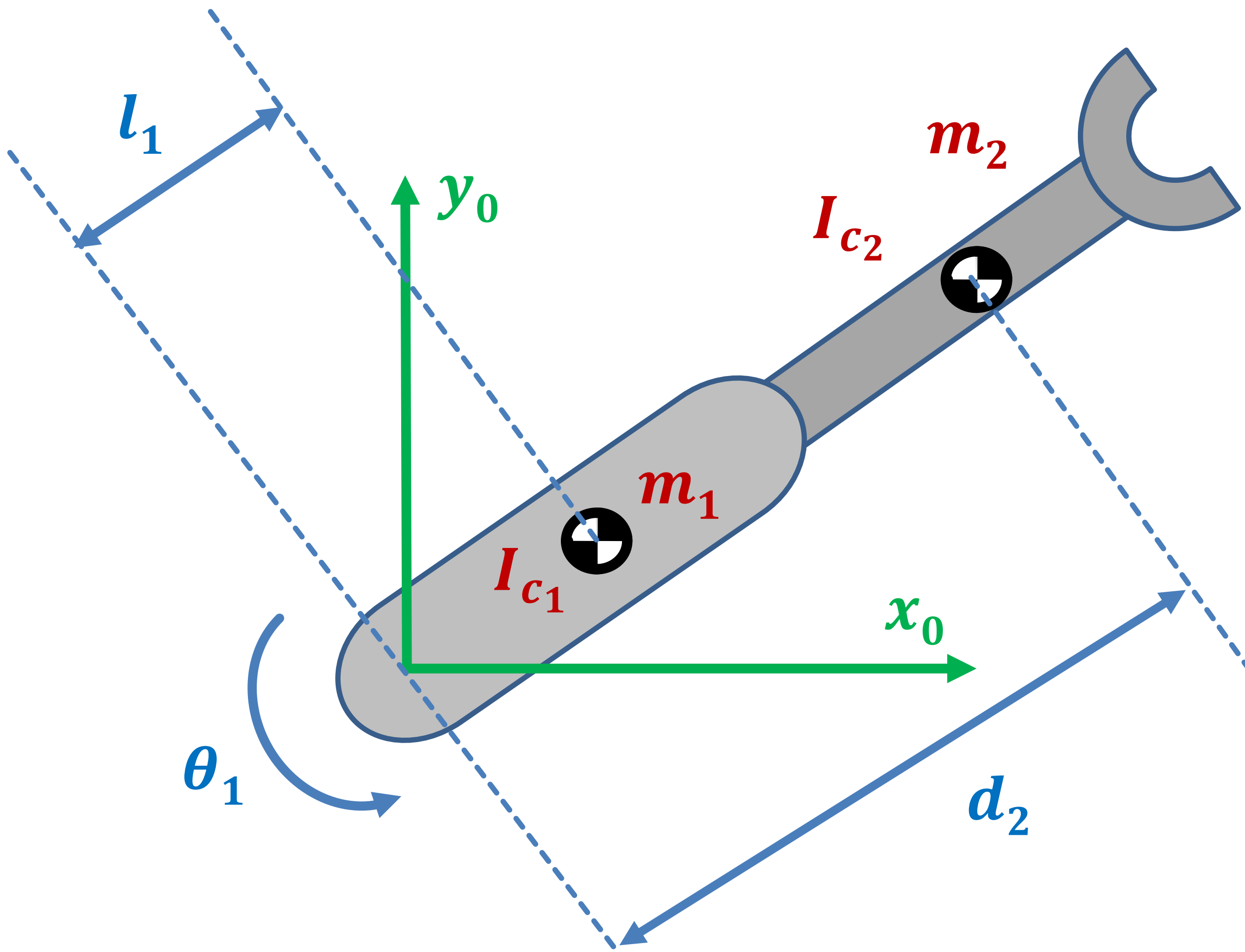
Christoffel Symbols: $b_{ijk} = \frac{1}{2}(m_{ijk} + m_{ikj} - m_{jki})$

$$\boldsymbol{V(q,\dot{q})} = \underbrace{\begin{bmatrix} \boldsymbol{b_{111}} & \boldsymbol{b_{122}} \\ \boldsymbol{b_{211}} & \boldsymbol{b_{222}} \end{bmatrix}}_{\boldsymbol{C(q)}} \begin{bmatrix} \boldsymbol{\dot{q}_1^2} \\ \boldsymbol{\dot{q}_2^2} \end{bmatrix} + \underbrace{\begin{bmatrix} \boldsymbol{2b_{112}} \\ \boldsymbol{2b_{212}} \end{bmatrix}}_{\boldsymbol{B(q)}} [\boldsymbol{\dot{q}_1 \dot{q}_2}]$$

Terms of the type $\boldsymbol{\dot{q}_i^2}$ are called centrifugal.
Terms of the type $\boldsymbol{\dot{q}_i \dot{q}_k}$ are called coriolis.

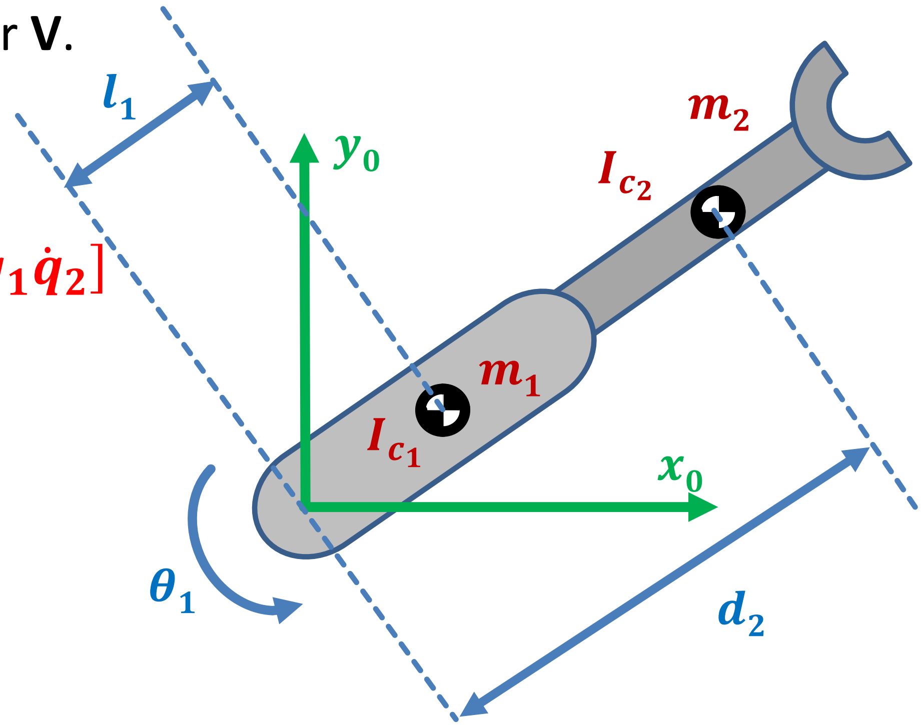
# Example RP

Work out the Centrifugal and Coriolis Vector **V**.



# Example RP

Work out the Centrifugal and Coriolis Vector **V**.




$$V(q,\dot{q}) = \overbrace{\begin{bmatrix} b_{111} & b_{122} \\ b_{211} & b_{222} \end{bmatrix}}^{C(q)} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} + \overbrace{\begin{bmatrix} 2b_{112} \\ 2b_{212} \end{bmatrix}}^{B(q)} [\dot{q}_1 \dot{q}_2]$$

$$b_{ijk} = \frac{1}{2}(m_{ijk} + m_{ikj} - m_{jki})$$

$$m_{ijk} = \frac{\partial m_{ij}}{\partial q_k}$$

$$B = \begin{bmatrix} 2b_{112} \\ 0 \end{bmatrix} = \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & b_{122} \\ b_{211} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} m_1 l_1^2 + I_{zz1} + m_2 d_2^2 + I_{zz2} & 0 \\ 0 & m_2 \end{bmatrix}$$

$$V = \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix} [\dot{\theta}_1 \dot{d}_2] + \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{d}_2^2 \end{bmatrix}$$

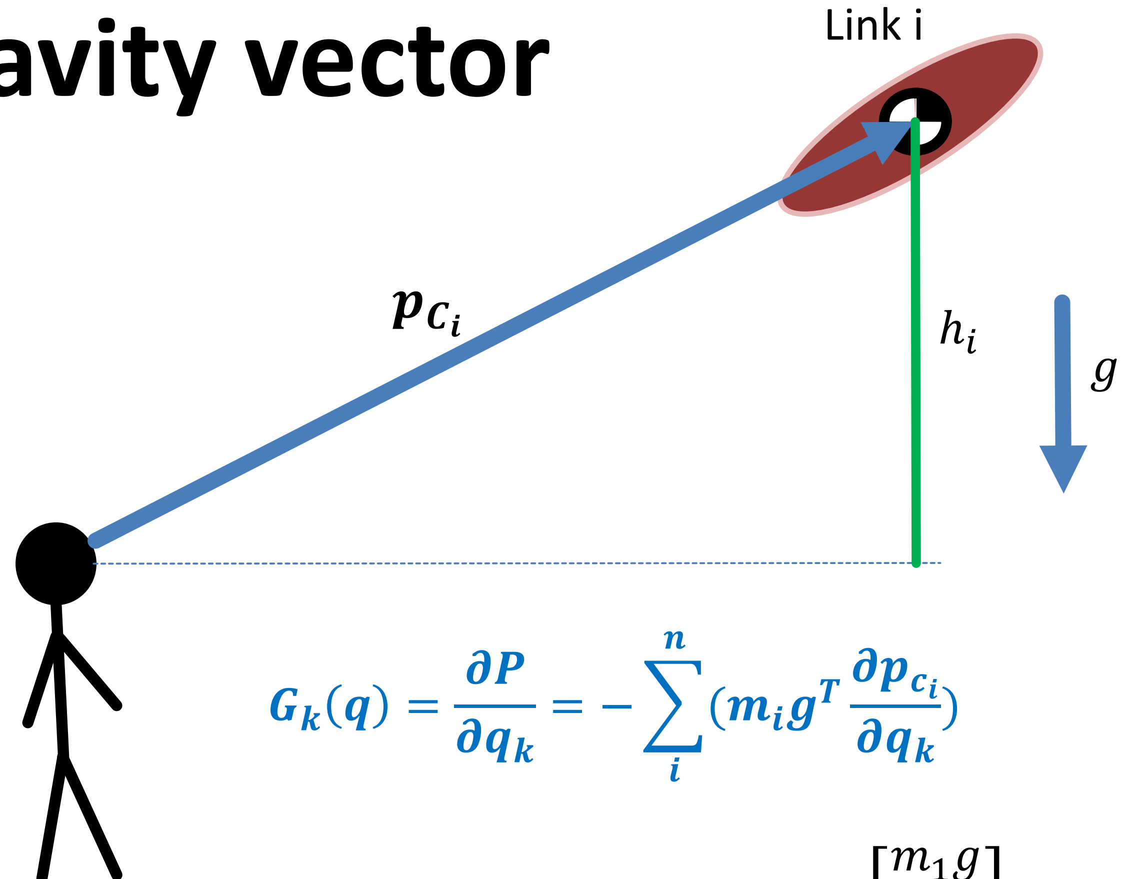
# Dynamic Equations

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{V}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{\tau}$$

✓ ✓ ?

**Gravity vector?**

# Gravity vector




$$\boldsymbol{G(q) = \frac{\partial P}{\partial q}}$$

Potential energy:

$$P_i = m_i g_0 h_i + P_0$$

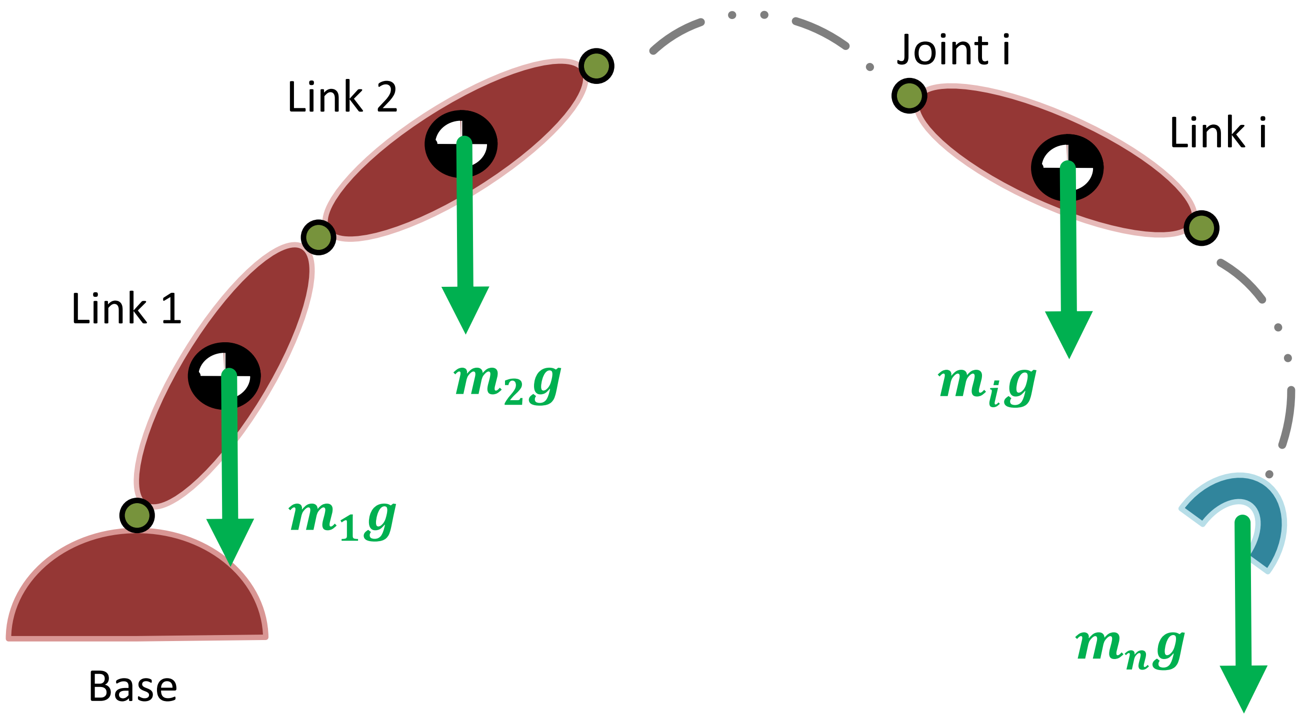
$$P_i = m_i\left(-g^T . p_{c_i}\right)$$

$$P = \sum_i P_i$$

$$\boldsymbol{G_k(q) = \frac{\partial P}{\partial q_k} = -\sum_i^n \left(m_i g^T \frac{\partial p_{c_i}}{\partial q_k}\right)}$$

$$G = -\begin{bmatrix} \mathcal{J}_{v_1}{}^T & \mathcal{J}_{v_2}{}^T & \cdots & \mathcal{J}_{v_n}{}^T \end{bmatrix} \begin{bmatrix} m_1 g \\ m_2 g \\ \vdots \\ m_n g \end{bmatrix}$$

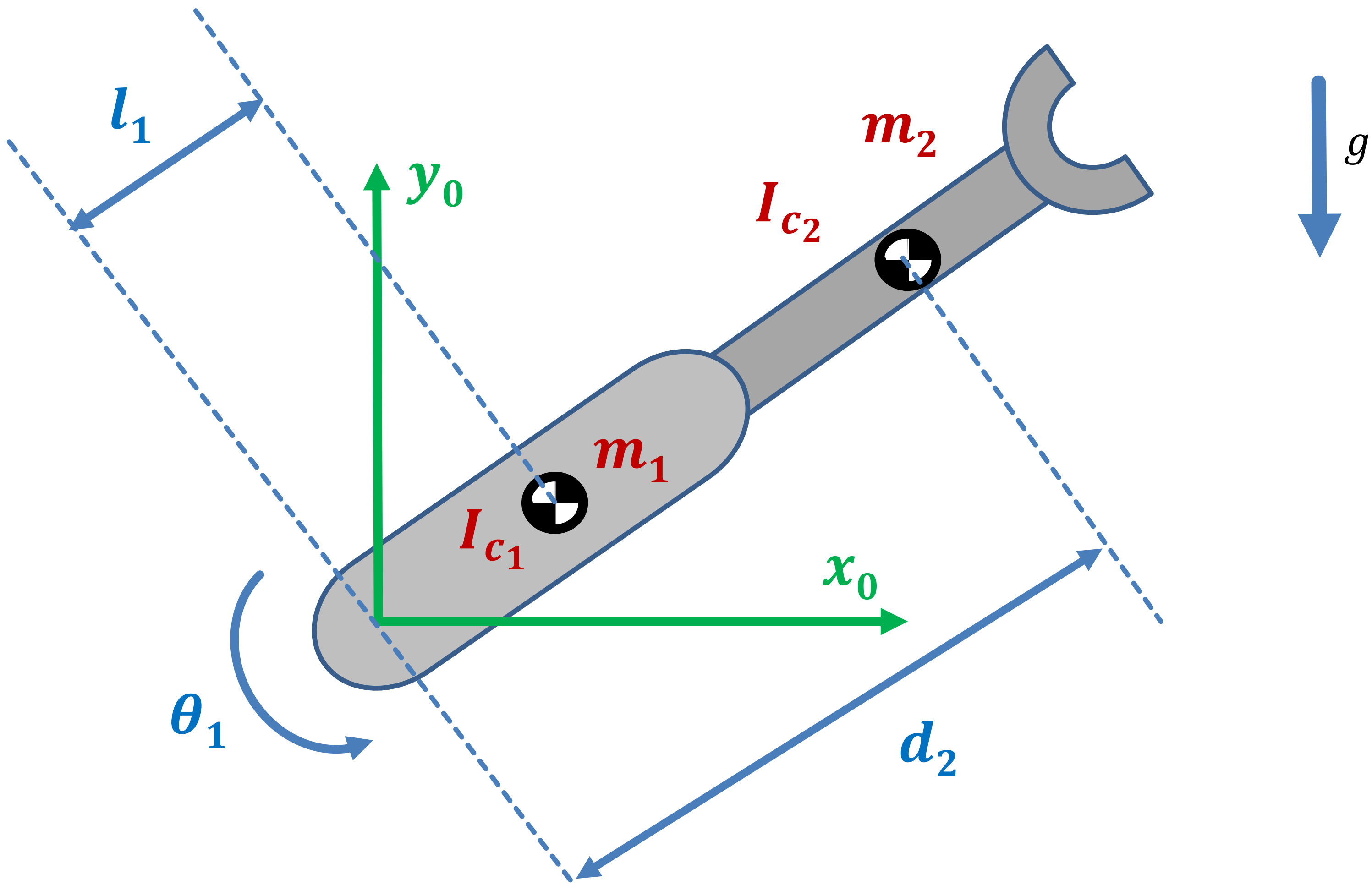
# Gravity vector




$$G = -\begin{bmatrix} \mathcal{J}_{v_1}{}^T & \mathcal{J}_{v_2}{}^T & \cdots & \mathcal{J}_{v_n}{}^T \end{bmatrix} \begin{bmatrix} m_1 g \\ m_2 g \\ \vdots \\ m_n g \end{bmatrix}$$

# Example RP

Work out the gravity vector **G**.

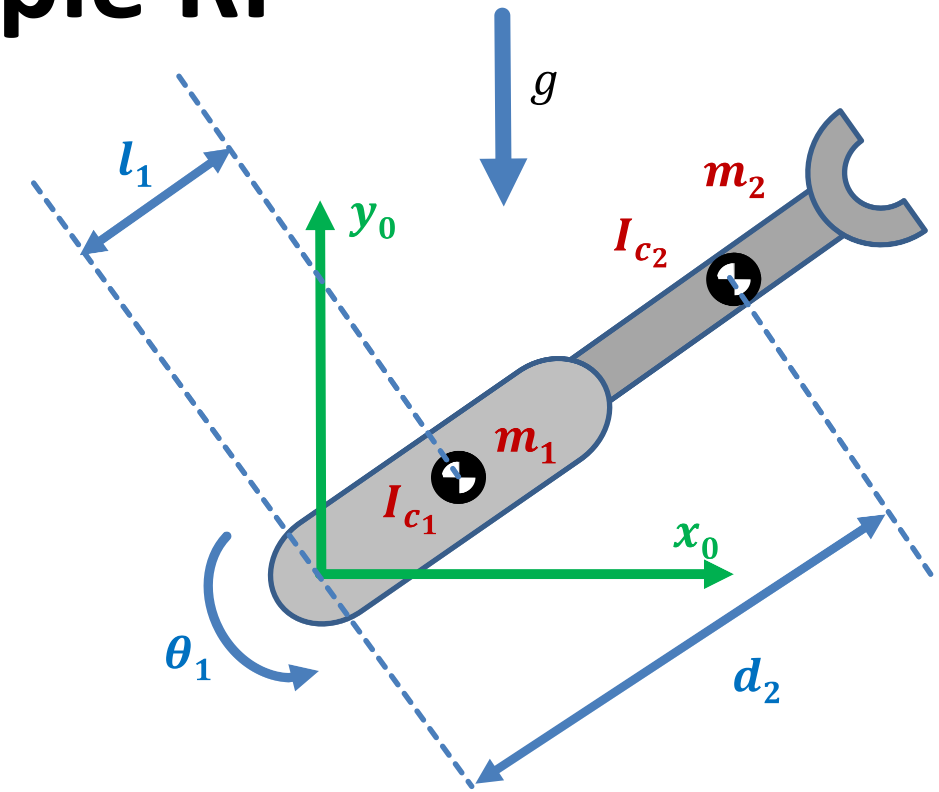


# Example RP

Work out the gravity vector **G**.

$$G = -\begin{bmatrix} \mathcal{J}_{v_1}{}^T & \mathcal{J}_{v_2}{}^T & \cdots & \mathcal{J}_{v_n}{}^T \end{bmatrix} \begin{bmatrix} m_1\boldsymbol{g} \\ m_2\boldsymbol{g} \\ \vdots \\ m_n\boldsymbol{g} \end{bmatrix}$$

$$G = -\mathcal{J}_{v_1}{}^T m_1\boldsymbol{g} - \mathcal{J}_{v_2}{}^T m_2\boldsymbol{g}$$




In frame $(o_0,x_0,y_0,z_0)$:

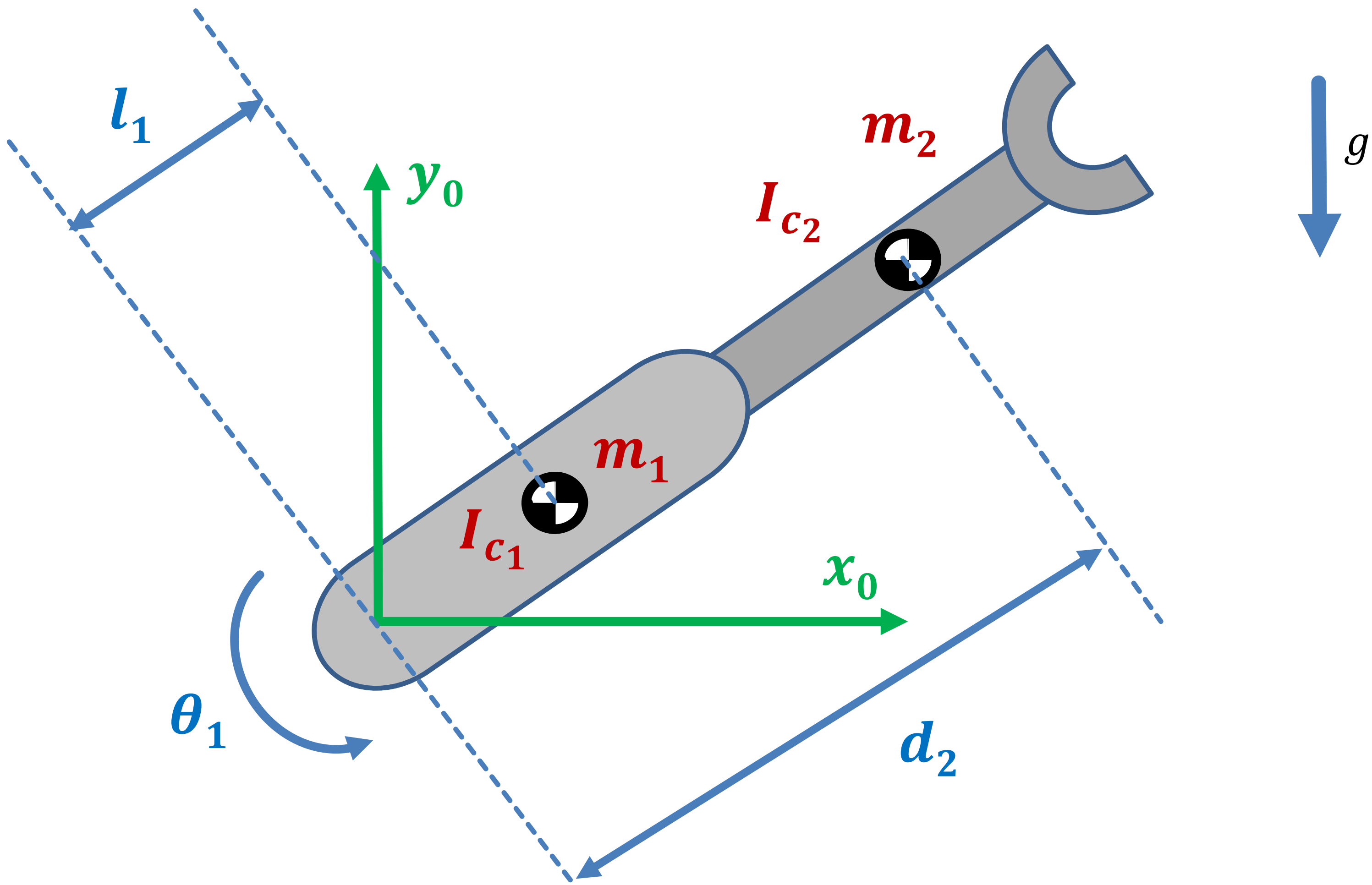
$$\boldsymbol{g} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}$$

The gravity vector is:

$$G = -\begin{bmatrix} -l_1 s_1 & 0 \\ l_1 c_1 & 0 \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ -m_1 g \\ 0 \end{bmatrix} - \begin{bmatrix} -d_2 s_1 & c_1 \\ d_2 c_1 & s_1 \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ -m_2 g \\ 0 \end{bmatrix} = \begin{bmatrix} (m_1 l_1 + m_2 d_2) g c_1 \\ m_2 g s_1 \end{bmatrix}$$

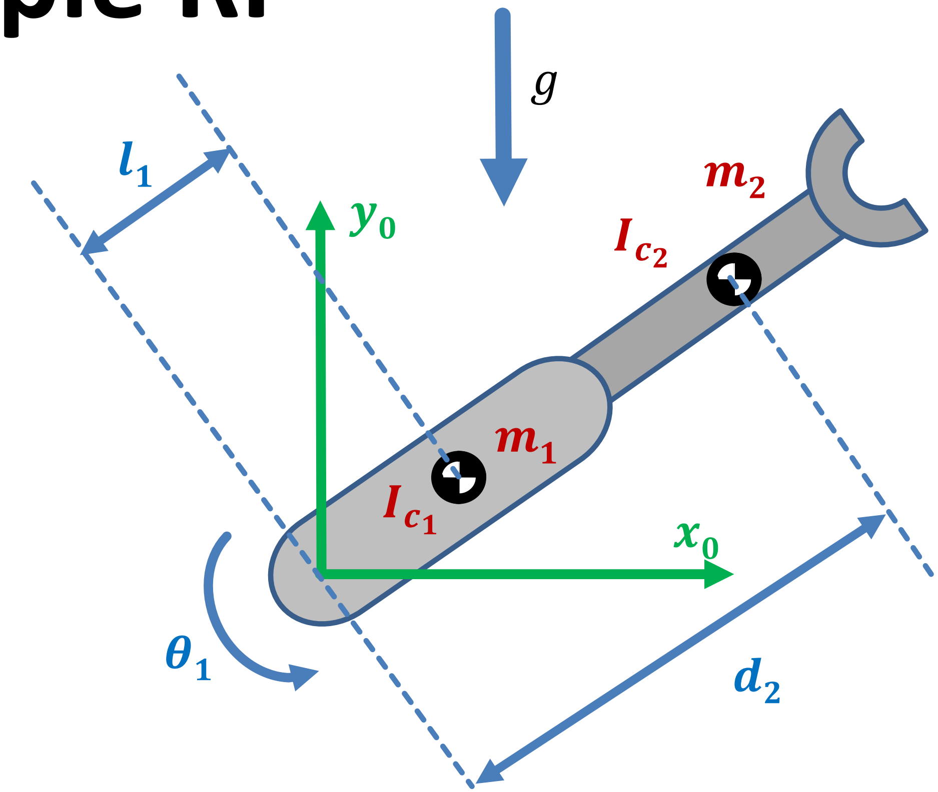
# Example RP

Work out the equation of motion.



# Example RP

Work out the equation of motion.




$$\begin{bmatrix} m_1 l_1^2 + I_{zz1} + m_2 d_2^2 + I_{zz2} & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{d}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \dot{d}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{d}_2^2 \end{bmatrix}$$

$$+ \begin{bmatrix} (m_1 l_1 + m_2 d_2) g c_1 \\ m_2 g s_1 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ f_2 \end{bmatrix}$$

$$\boldsymbol{(m_1 l_1^2 + I_{zz1} + m_2 d_2^2 + I_{zz2})\ddot{\theta}_1 + (2m_2 d_2)\dot{\theta}_1 \dot{d}_2 + (m_1 l_1 + m_2 d_2) g c_1 = \tau_1}$$

$$\boldsymbol{m_2 \ddot{d}_2 - m_2 d_2 \dot{\theta}_1^2 + m_2 g s_1 = f_2}$$

# Example PR

Work out the mass matrix $\boldsymbol{M}$.

$$M = \sum_{i=1}^{n} (m_i \mathcal{J}_{v_i}{}^T \mathcal{J}_{v_i} + \mathcal{J}_{\omega_i}{}^T I_{C_i} \mathcal{J}_{\omega_i})$$

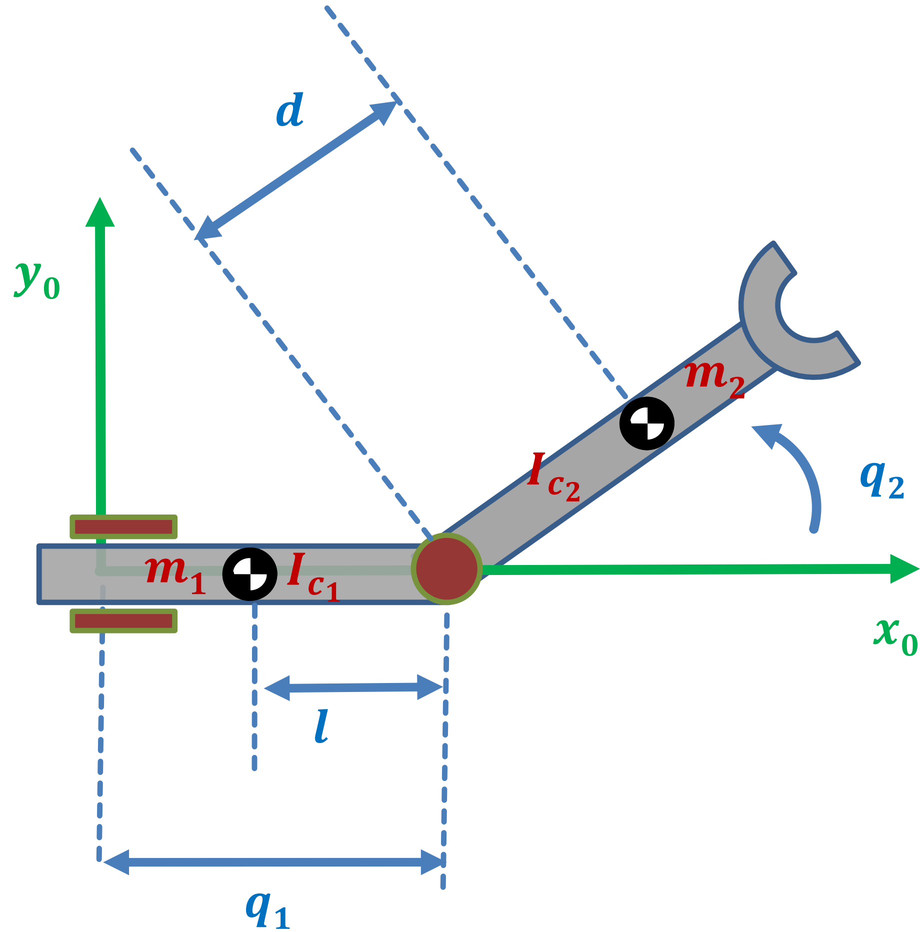
$$M = m_1 \mathcal{J}_{v_1}{}^T \mathcal{J}_{v_1} + \mathcal{J}_{\omega_1}{}^T I_{C_1} \mathcal{J}_{\omega_1} + m_2 \mathcal{J}_{v_2}{}^T \mathcal{J}_{v_2} + \mathcal{J}_{\omega_2}{}^T I_{C_2} \mathcal{J}_{\omega_2}$$

In frame ($o_0$,$x_0$,$y_0$,$z_0$)

$$p_{C_1} = \begin{bmatrix} q_1 - l \\ 0 \\ 0 \end{bmatrix}, p_{C_2} = \begin{bmatrix} q_1 + dc_2 \\ ds_2 \\ 0 \end{bmatrix}$$

$$\mathcal{J}_{v_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathcal{J}_{v_2} = \begin{bmatrix} 1 & -ds_2 \\ 0 & dc_2 \\ 0 & 0 \end{bmatrix}$$



# Example PR

Work out the mass matrix $\boldsymbol{M}$.

$$M = \sum_{i=1}^{n} (m_i \mathcal{J}_{v_i}{}^T \mathcal{J}_{v_i} + \mathcal{J}_{\omega_i}{}^T I_{C_i} \mathcal{J}_{\omega_i})$$

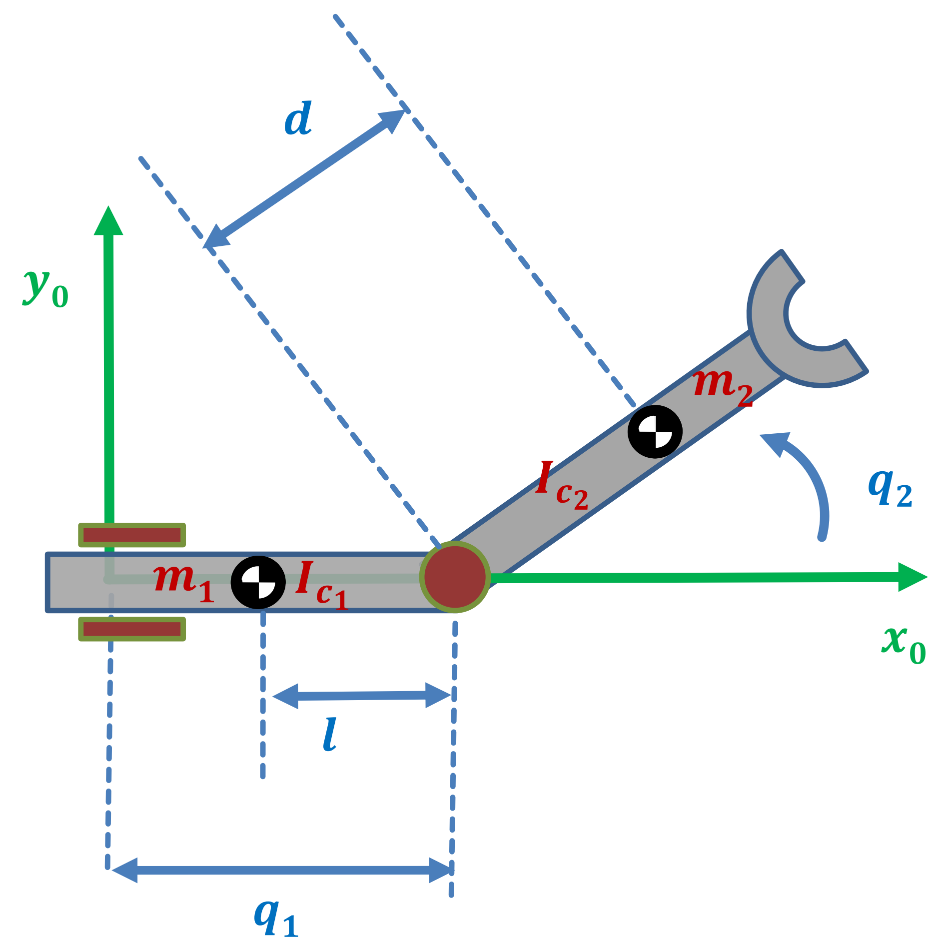
$$M = m_1 \mathcal{J}_{v_1}{}^T \mathcal{J}_{v_1} + \mathcal{J}_{\omega_1}{}^T I_{C_1} \mathcal{J}_{\omega_1} + m_2 \mathcal{J}_{v_2}{}^T \mathcal{J}_{v_2} + \mathcal{J}_{\omega_2}{}^T I_{C_2} \mathcal{J}_{\omega_2}$$

In frame $(o_0, x_0, y_0, z_0)$

$$p_{C_1} = \begin{bmatrix} q_1 - l \\ 0 \\ 0 \end{bmatrix}, p_{C_2} = \begin{bmatrix} q_1 + dc_2 \\ ds_2 \\ 0 \end{bmatrix}$$

$$\mathcal{J}_{\omega_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathcal{J}_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

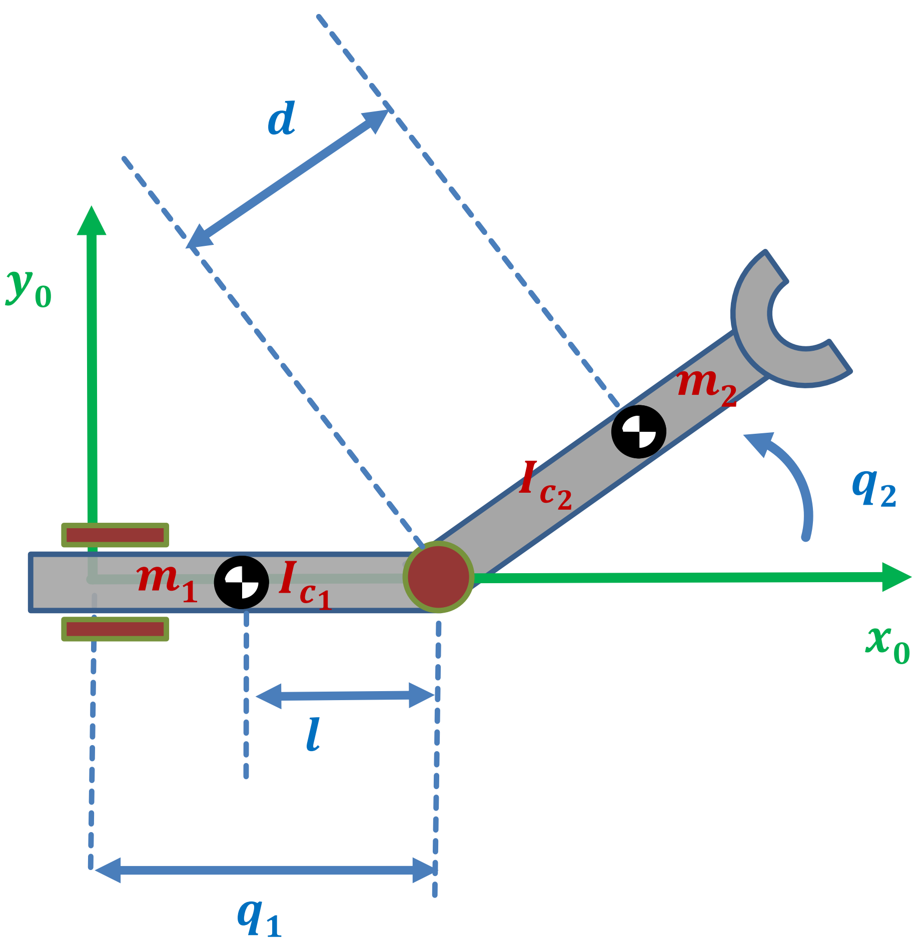


# Example PR

Work out the mass matrix $\boldsymbol{M}$.

$$M = m_1 \mathcal{J}_{v_1}{}^T \mathcal{J}_{v_1} + \mathcal{J}_{\omega_1}{}^T I_{C_1} \mathcal{J}_{\omega_1} + m_2 \mathcal{J}_{v_2}{}^T \mathcal{J}_{v_2} + \mathcal{J}_{\omega_2}{}^T I_{C_2} \mathcal{J}_{\omega_2}$$

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{m_1 + m_2} & \boldsymbol{-m_2 d s_2} \\ \boldsymbol{-m_2 d s_2} & \boldsymbol{I_{zz2} + m_2 d^2} \end{bmatrix}$$

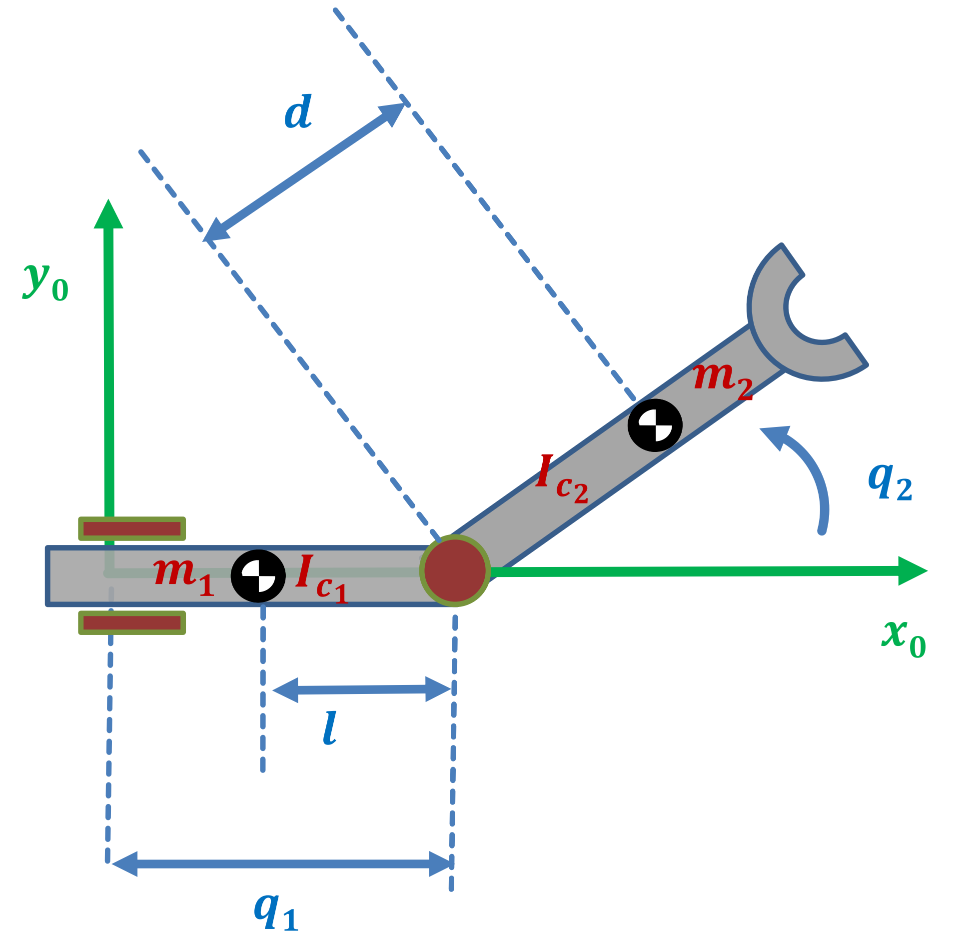


# Example PR

Work out the gravity vector **G**.

$$G = -\begin{bmatrix} \mathcal{J}_{v_1}{}^T & \mathcal{J}_{v_2}{}^T & \cdots & \mathcal{J}_{v_n}{}^T \end{bmatrix} \begin{bmatrix} m_1\boldsymbol{g} \\ m_2\boldsymbol{g} \\ \vdots \\ m_n\boldsymbol{g} \end{bmatrix}$$

$$G = -\mathcal{J}_{v_1}{}^T m_1\boldsymbol{g} - \mathcal{J}_{v_2}{}^T m_2\boldsymbol{g}$$




In frame $(o_0,x_0,y_0,z_0)$: $\boldsymbol{g} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}$

The gravity vector is:

$$G = -\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ -m_1 g \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & -ds_2 \\ 0 & dc_2 \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ -m_2 g \\ 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{dc_2 m_2 g} \end{bmatrix}$$

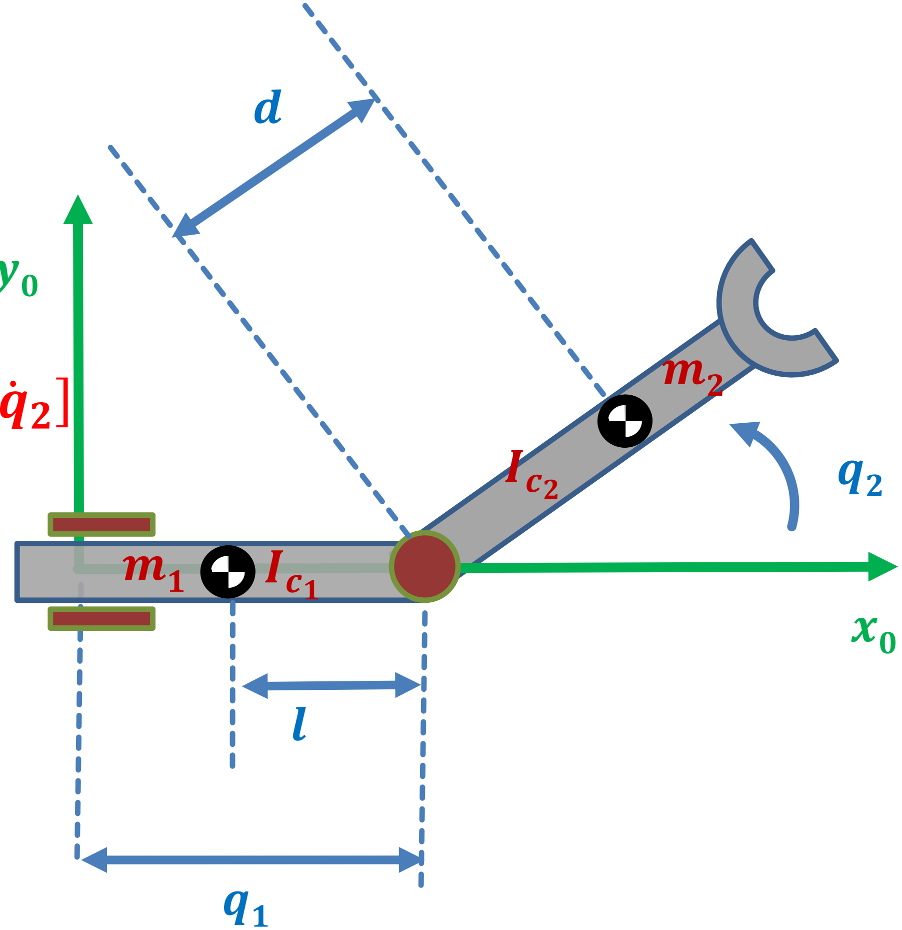
# Example PR

Work out the Centrifugal and Coriolis Vector **V**.

$$V(q,\dot{q}) = \begin{bmatrix} b_{111} & b_{122} \\ b_{211} & b_{222} \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} 2b_{112} \\ 2b_{212} \end{bmatrix} [\dot{q}_1 \dot{q}_2]$$

$$b_{ijk} = \frac{1}{2}(m_{ijk} + m_{ikj} - m_{jki})$$

$$m_{ijk} = \frac{\partial m_{ij}}{\partial q_k}$$




$$B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

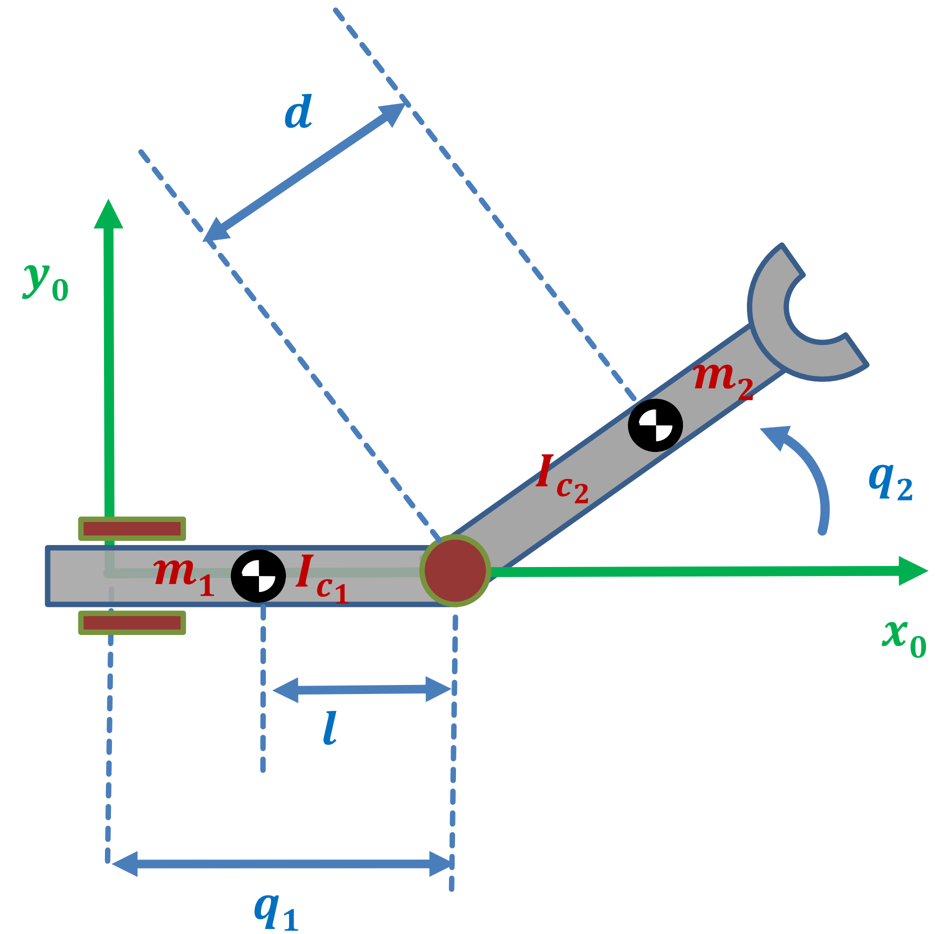
$$C = \begin{bmatrix} 0 & -m_2 d c_2 \\ 0 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} m_1 + m_2 & -m_2 d s_2 \\ -m_2 d s_2 & I_{zz2} + m_2 d^2 \end{bmatrix}$$

$$V = \begin{bmatrix} 0 \\ 0 \end{bmatrix} [\dot{q}_1 \dot{q}_2] + \begin{bmatrix} 0 & -m_2 d c_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}$$

# Example PR

Work out the equation of motion.




$$\begin{bmatrix} m_1 + m_2 & -m_2 d s_2 \\ -m_2 d s_2 & I_{zz2} + m_2 d^2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \end{bmatrix} [\dot{q}_1 \dot{q}_2] + \begin{bmatrix} 0 & -m_2 d c_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ d c_2 m_2 g \end{bmatrix} = \begin{bmatrix} f_1 \\ \tau_2 \end{bmatrix}$$

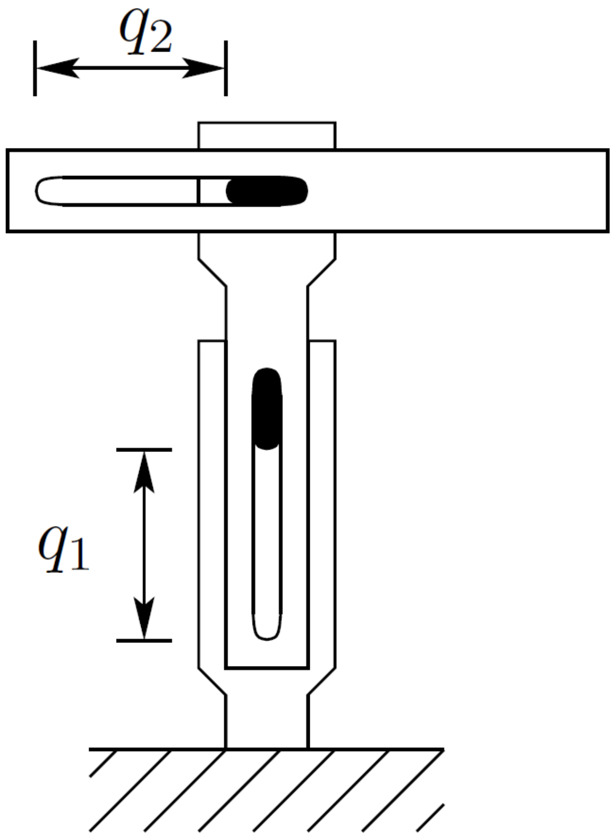
$$(\boldsymbol{m_1 + m_2})\ddot{q}_1 - \boldsymbol{m_2 d s_2}\ddot{q}_2 - \boldsymbol{m_2 d c_2}\dot{q}_2^2 = \boldsymbol{f_1}$$

$$-\boldsymbol{m_2 d s_2}\ddot{q}_1 + \left(\boldsymbol{I_{zz2} + m_2 d^2}\right)\ddot{q}_2 + \boldsymbol{d c_2 m_2 g} = \boldsymbol{\tau_2}$$

# Example PP

Work out the equation of motion.

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{m_1 + m_2} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{m_2} \end{bmatrix}$$

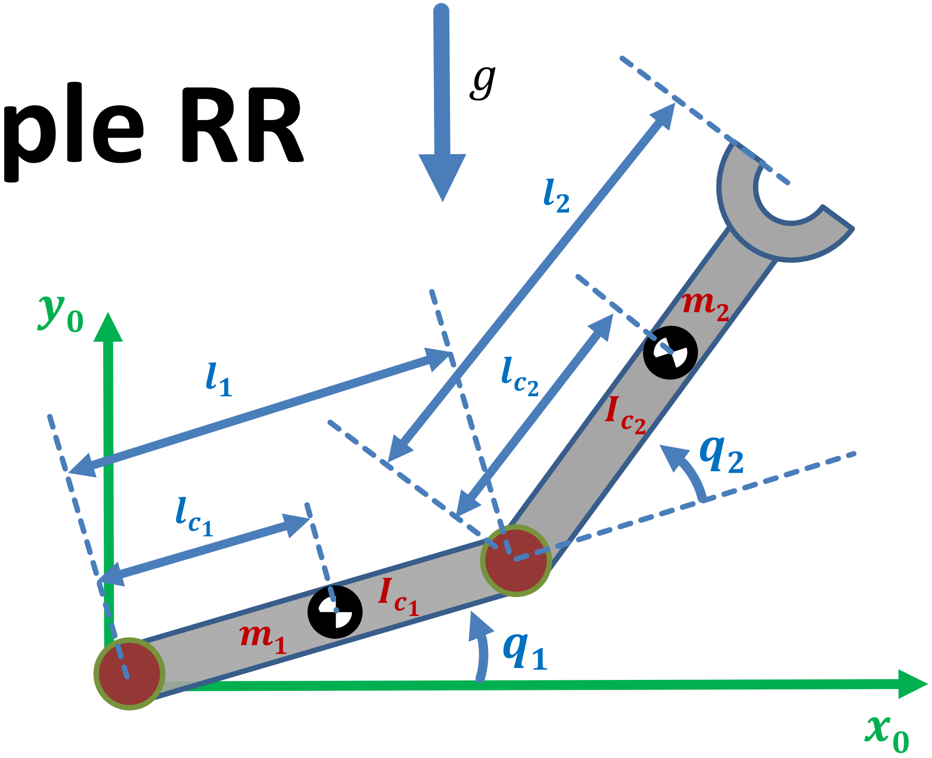



$$(\boldsymbol{m_1} + \boldsymbol{m_2})\ddot{\boldsymbol{q}}_1 + \boldsymbol{g}(\boldsymbol{m_1} + \boldsymbol{m_2}) = \boldsymbol{f_1}$$

$$\boldsymbol{m_2}\ddot{\boldsymbol{q}}_2 = \boldsymbol{f_2}$$

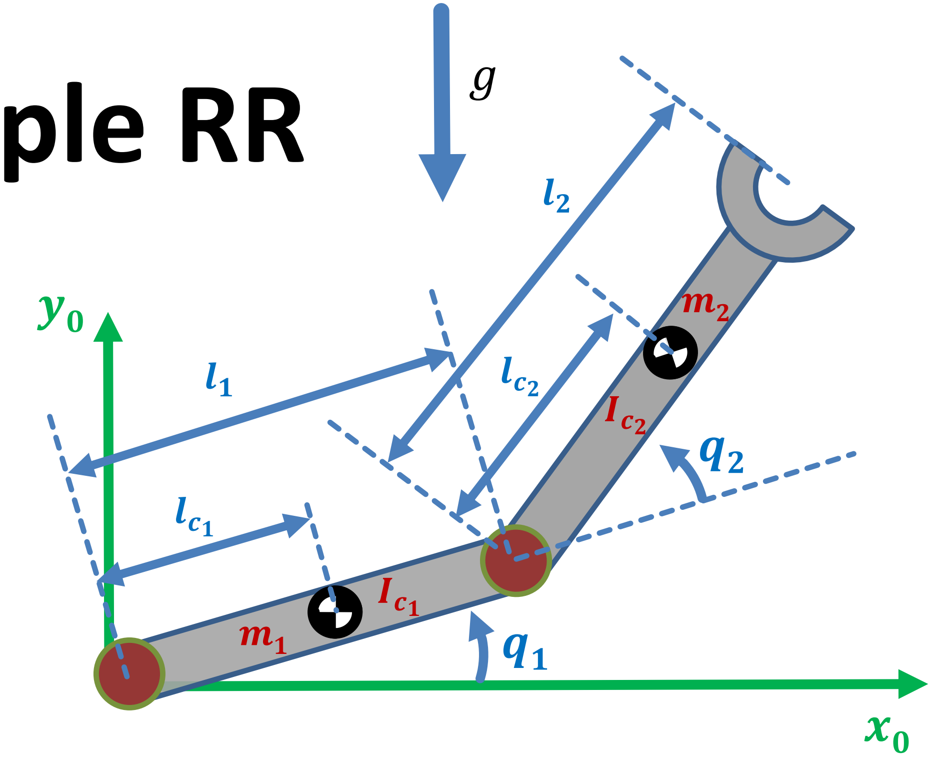
# Example RR

Work out the equation of motion.




$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \tau$$

# Example RR




Work out the mass matrix $\boldsymbol{M}$.

$$M = \sum_{i=1}^{n} (m_i \mathcal{J}_{v_i}{}^T \mathcal{J}_{v_i} + \mathcal{J}_{\omega_i}{}^T I_{C_i} \mathcal{J}_{\omega_i})$$

$$M = m_1 \mathcal{J}_{v_1}{}^T \mathcal{J}_{v_1} + \mathcal{J}_{\omega_1}{}^T I_{C_1} \mathcal{J}_{\omega_1} + m_2 \mathcal{J}_{v_2}{}^T \mathcal{J}_{v_2} + \mathcal{J}_{\omega_2}{}^T I_{C_2} \mathcal{J}_{\omega_2}$$

In frame ($o_0$,$x_0$,$y_0$,$z_0$)

$$p_{C_1} = \begin{bmatrix} l_{c_1} c_1 \\ l_{c_1} s_1 \\ 0 \end{bmatrix}, p_{C_2} = \begin{bmatrix} l_{c_2} c_{12} + l_1 c_1 \\ l_{c_2} s_{12} + l_1 s_1 \\ 0 \end{bmatrix}$$

$$\mathcal{J}_{v_1} = \begin{bmatrix} -l_{c_1} s_1 & 0 \\ l_{c_1} c_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathcal{J}_{v_2} = \begin{bmatrix} -l_{c_2} s_{12} - l_1 s_1 & -l_{c_2} s_{12} \\ l_{c_2} c_{12} + l_1 c_1 & l_{c_2} c_{12} \\ 0 & 0 \end{bmatrix}$$

# Example RR

Work out the mass matrix $\boldsymbol{M}$.

$$M = \sum_{i=1}^{n} (m_i {\mathcal{J}_{v_i}}^T \mathcal{J}_{v_i} + {\mathcal{J}_{\omega_i}}^T I_{C_i} \mathcal{J}_{\omega_i})$$

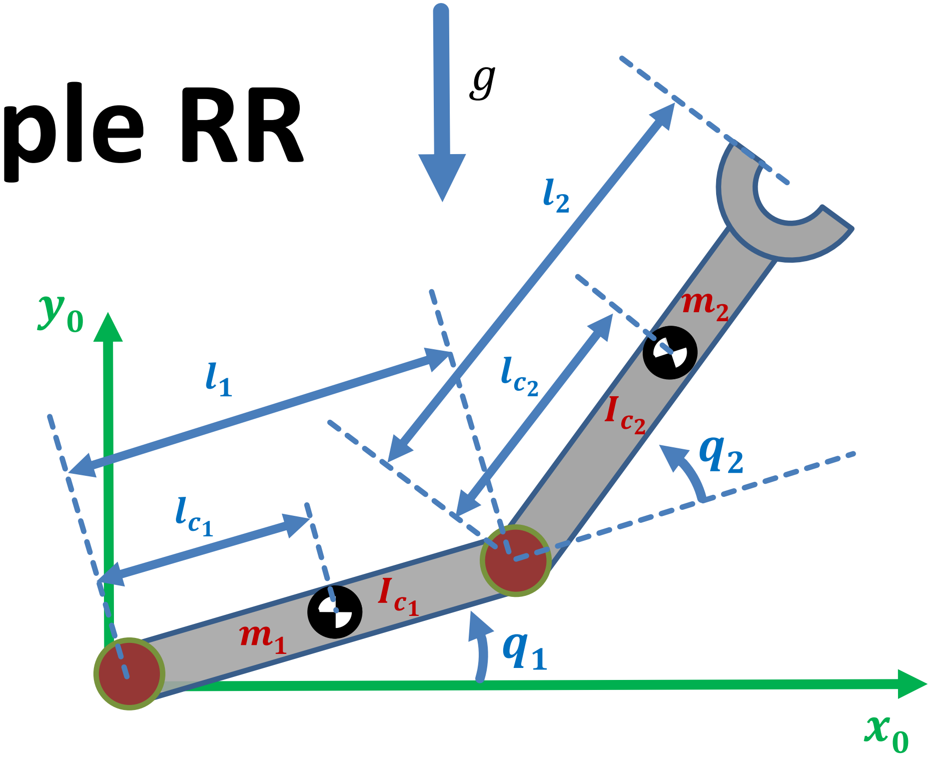
$$M = m_1 {\mathcal{J}_{v_1}}^T \mathcal{J}_{v_1} + {\mathcal{J}_{\omega_1}}^T I_{C_1} \mathcal{J}_{\omega_1} + m_2 {\mathcal{J}_{v_2}}^T \mathcal{J}_{v_2} + {\mathcal{J}_{\omega_2}}^T I_{C_2} \mathcal{J}_{\omega_2}$$

In frame $(o_0, x_0, y_0, z_0)$

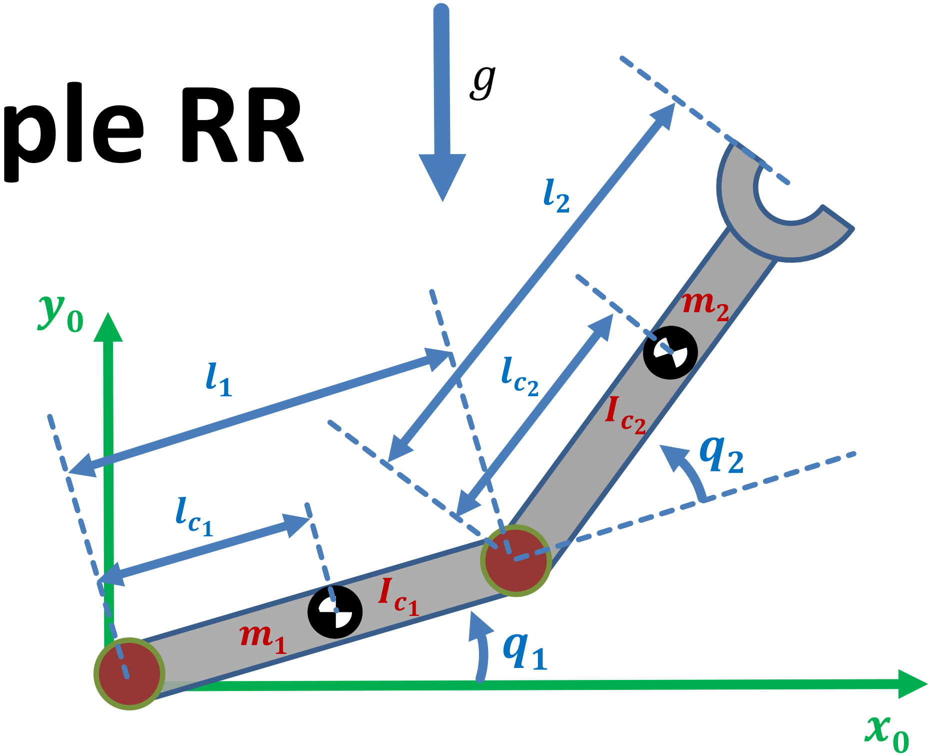
$$p_{C_1} = \begin{bmatrix} l_{c_1} c_1 \\ l_{c_1} s_1 \\ 0 \end{bmatrix}, p_{C_2} = \begin{bmatrix} l_{c_2} c_{12} + l_1 c_1 \\ l_{c_2} s_{12} + l_1 s_1 \\ 0 \end{bmatrix}$$

$$\mathcal{J}_{\omega_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\mathcal{J}_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$



# Example RR




Work out the mass matrix $\boldsymbol{M}$.

$$M = \sum_{i=1}^{n} (m_i \mathcal{J}_{v_i}{}^T \mathcal{J}_{v_i} + \mathcal{J}_{\omega_i}{}^T I_{C_i} \mathcal{J}_{\omega_i})$$

$$M = m_1 \mathcal{J}_{v_1}{}^T \mathcal{J}_{v_1} + \mathcal{J}_{\omega_1}{}^T I_{C_1} \mathcal{J}_{\omega_1} + m_2 \mathcal{J}_{v_2}{}^T \mathcal{J}_{v_2} + \mathcal{J}_{\omega_2}{}^T I_{C_2} \mathcal{J}_{\omega_2}$$

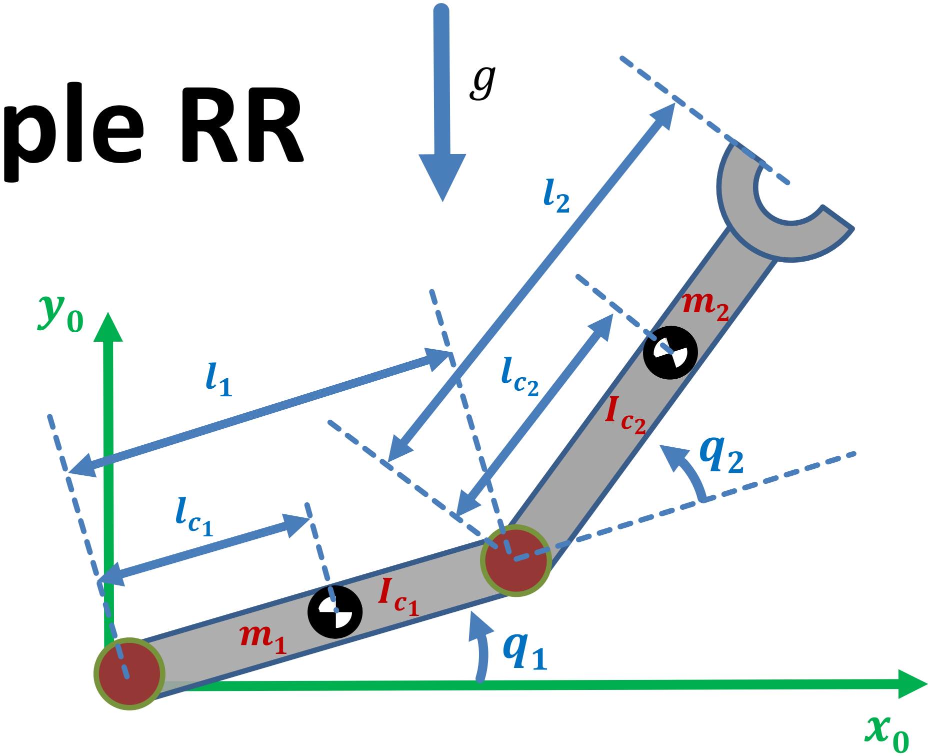
$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{m_2 l_1{}^2 + 2 m_2 l_1 l_{c_2} c_2 + m_1 l_{c_1}{}^2 + m_2 l_{c_2}{}^2 + I_{zz1}} & \boldsymbol{m_2 I_{c_2}{}^2 + l_1 m_2 l_{c_2} c_2 + I_{zz2}} \\ \boldsymbol{m_2 I_{c_2}{}^2 + l_1 m_2 I_{c_2} c_2 + I_{zz2}} & \boldsymbol{m_2 I_{c_2}{}^2 + I_{zz2}} \end{bmatrix}$$

# Example RR

Work out the gravity vector **G**.

$$G = -\begin{bmatrix} \mathcal{J}_{v_1}{}^T & \mathcal{J}_{v_2}{}^T & \cdots & \mathcal{J}_{v_n}{}^T \end{bmatrix} \begin{bmatrix} m_1\boldsymbol{g} \\ m_2\boldsymbol{g} \\ \vdots \\ m_n\boldsymbol{g} \end{bmatrix}$$

$$G = -\mathcal{J}_{v_1}{}^T m_1\boldsymbol{g} - \mathcal{J}_{v_2}{}^T m_2\boldsymbol{g}$$

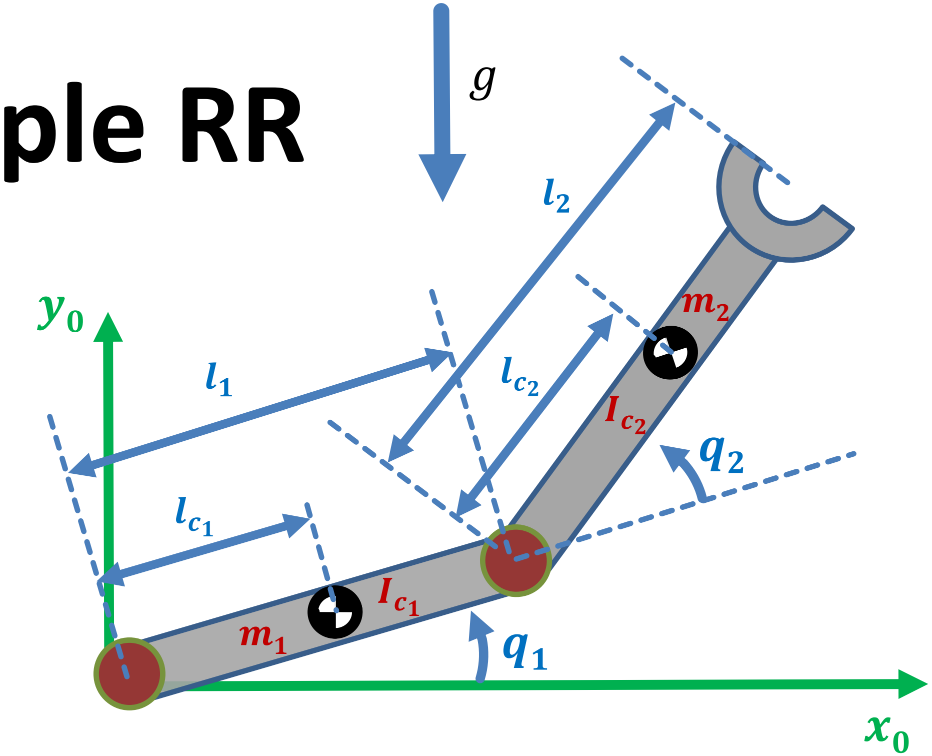



In frame ($o_0$,$x_0$,$y_0$,$z_0$): $\boldsymbol{g} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}$

The gravity vector is:

$$G = -\begin{bmatrix} -l_{c_1}s_1 & 0 \\ l_{c_1}c_1 & 0 \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ -m_1 g \\ 0 \end{bmatrix} - \begin{bmatrix} -l_{c_2}s_{12} - l_1 s_1 & -l_{c_2}s_{12} \\ l_{c_2}c_{12} + l_1 c_1 & l_{c_2}c_{12} \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ -m_2 g \\ 0 \end{bmatrix}$$

# Example RR




Work out the gravity vector **G**.

$$G = -\begin{bmatrix} \mathcal{J}_{v_1}{}^T & \mathcal{J}_{v_2}{}^T & \cdots & \mathcal{J}_{v_n}{}^T \end{bmatrix} \begin{bmatrix} m_1\boldsymbol{g} \\ m_2\boldsymbol{g} \\ \vdots \\ m_n\boldsymbol{g} \end{bmatrix}$$

$$G = -\mathcal{J}_{v_1}{}^T m_1\boldsymbol{g} - \mathcal{J}_{v_2}{}^T m_2\boldsymbol{g}$$

In frame ($o_0$,$x_0$,$y_0$,$z_0$): $\boldsymbol{g} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}$

The gravity vector is:

$$G = \begin{bmatrix} \boldsymbol{g m_2 (l_{c_2} c_{12}) + l_1 c_1 + g m_1 l_{c_1} c_1} \\ \boldsymbol{g m_2 l_{c_2} c_{12}} \end{bmatrix}$$

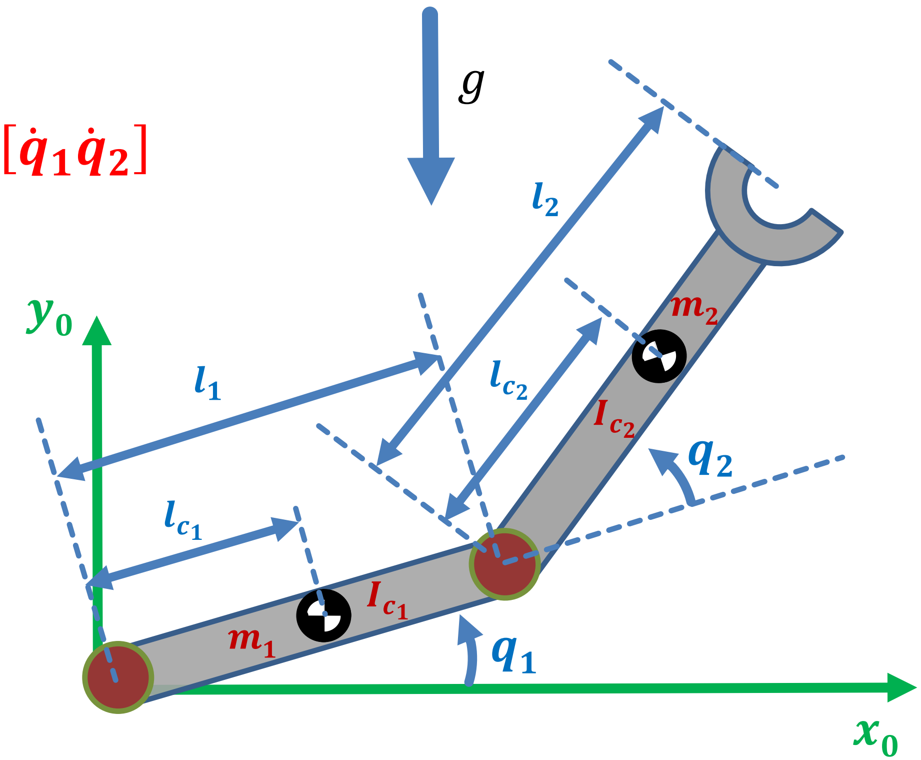
# Example RR

Work out the Centrifugal and Coriolis Vector **V**.

$$V(q,\dot{q}) = \begin{bmatrix} b_{111} & b_{122} \\ b_{211} & b_{222} \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} 2b_{112} \\ 2b_{212} \end{bmatrix} [\dot{q}_1 \dot{q}_2]$$

$$b_{ijk} = \frac{1}{2}(m_{ijk} + m_{ikj} - m_{jki})$$

$$m_{ijk} = \frac{\partial m_{ij}}{\partial q_k}$$




$$B = \begin{bmatrix} -2l_1 l_{c_2} m_2 s_2 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & -l_1 l_{c_2} m_2 s_2 \\ l_1 l_{c_2} m_2 s_2 & 0 \end{bmatrix}$$

$$V(q,\dot{q}) = \begin{bmatrix} -2l_1 l_{c_2} m_2 s_2 \\ 0 \end{bmatrix} [\dot{q}_1 \dot{q}_2] + \begin{bmatrix} 0 & -l_1 l_{c_2} m_2 s_2 \\ l_1 l_{c_2} m_2 s_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}$$

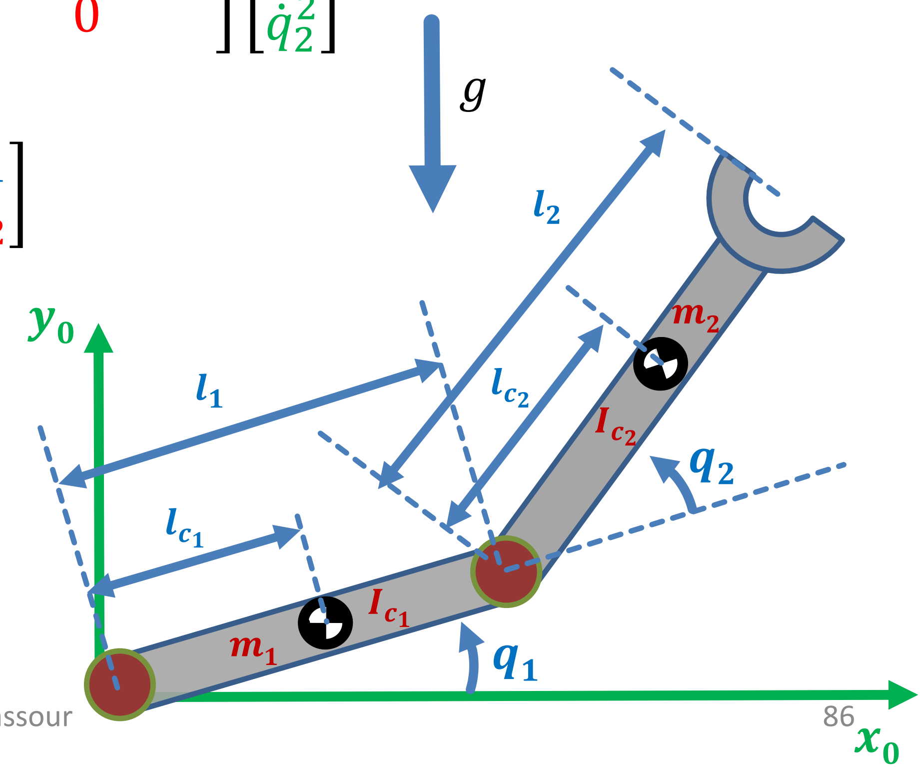
# Example RR

Work out the equation of motion.

$$\begin{bmatrix} m_2 l_1{}^2 + 2m_2 l_1 l_{c_2} c_2 + m_1 l_{c_1}{}^2 + m_2 l_{c_2}{}^2 + I_{zz1} & m_2 I_{c_2}{}^2 + l_1 m_2 l_{c_2} c_2 + I_{zz2} \\ m_2 I_{c_2}{}^2 + l_1 m_2 I_{c_2} c_2 + I_{zz2} & m_2 I_{c_2}{}^2 + I_{zz2} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} -2 l_1 l_{c_2} m_2 s_2 \\ 0 \end{bmatrix} [\dot{q}_1 \dot{q}_2] + \begin{bmatrix} 0 & -l_1 l_{c_2} m_2 s_2 \\ l_1 l_{c_2} m_2 s_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}$$

$$+ \begin{bmatrix} g m_2 (l_{c_2} c_{12}) + l_1 c_1 + g m_1 l_{c_1} c_1 \\ g m_2 l_{c_2} c_{12} \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$



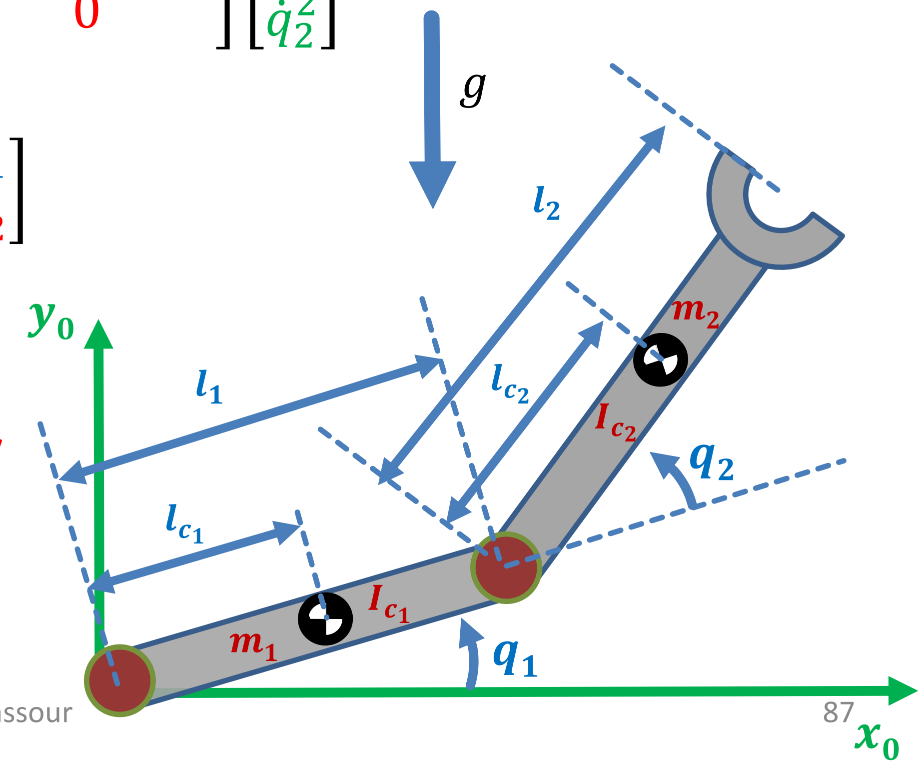
# Example RR

Work out the equation of motion.

$$\begin{bmatrix} m_2 {l_1}^2 + 2m_2 l_1 l_{c_2} c_2 + m_1 {l_{c_1}}^2 + m_2 {l_{c_2}}^2 + I_{zz1} & m_2 {I_{c_2}}^2 + l_1 m_2 l_{c_2} c_2 + I_{zz2} \\ m_2 {I_{c_2}}^2 + l_1 m_2 I_{c_2} c_2 + I_{zz2} & m_2 {I_{c_2}}^2 + I_{zz2} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} -2 l_1 l_{c_2} m_2 s_2 \\ 0 \end{bmatrix} [\dot{q}_1 \dot{q}_2] + \begin{bmatrix} 0 & -l_1 l_{c_2} m_2 s_2 \\ l_1 l_{c_2} m_2 s_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}$$

$$+ \begin{bmatrix} g m_2 (l_{c_2} c_{12}) + l_1 c_1 + g m_1 l_{c_1} c_1 \\ g m_2 l_{c_2} c_{12} \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$




**The centrifugal terms on one joint are proportional to the square of the velocity of the other joint, and are at a maximum when the two links are perpendicular.**

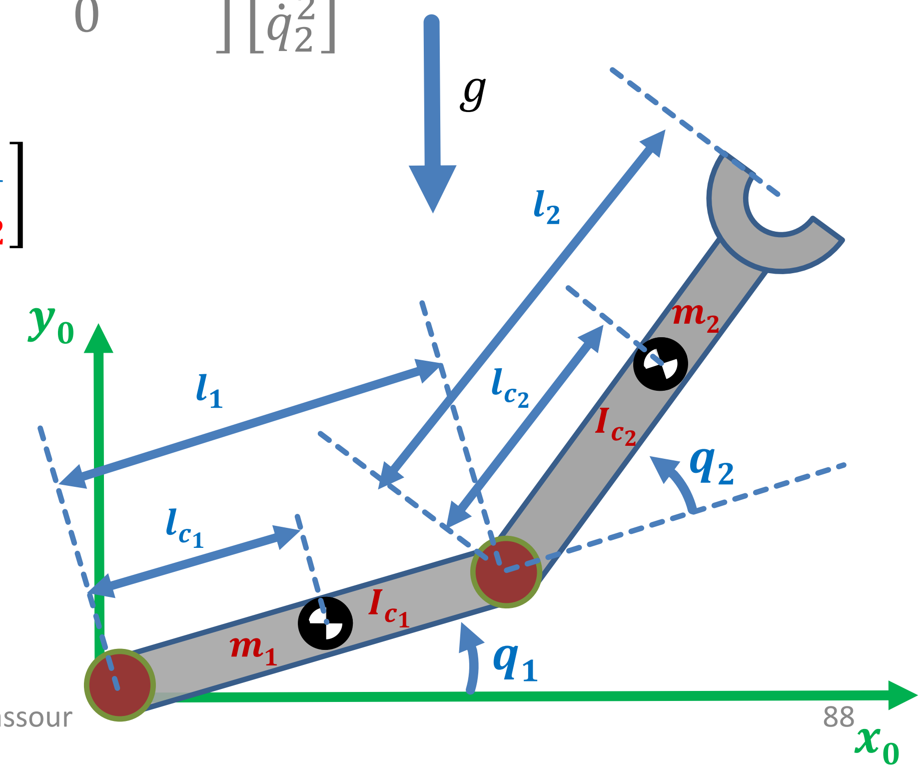
# Example RR

Work out the equation of motion.

$$\begin{bmatrix} m_2 {l_1}^2 + 2m_2 l_1 l_{c_2} c_2 + m_1 {l_{c_1}}^2 + m_2 {l_{c_2}}^2 + I_{zz1} & m_2 {I_{c_2}}^2 + l_1 m_2 l_{c_2} c_2 + I_{zz2} \\ m_2 {I_{c_2}}^2 + l_1 m_2 I_{c_2} c_2 + I_{zz2} & m_2 {I_{c_2}}^2 + I_{zz2} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} -2 l_1 l_{c_2} m_2 s_2 \\ 0 \end{bmatrix} [\dot{q}_1 \dot{q}_2] + \begin{bmatrix} 0 & -l_1 l_{c_2} m_2 s_2 \\ l_1 l_{c_2} m_2 s_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}$$

$$+ \begin{bmatrix} g m_2 \left( l_{c_2} c_{12} \right) + l_1 c_1 + g m_1 l_{c_1} c_1 \\ g m_2 l_{c_2} c_{12} \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$




**The Coriolis term is proportional to the product of the two joint velocities, and is also at a maximum when the two links are perpendicular.**