Forward Kinematics

Serial link manipulators

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Suggested literature

• Robot Modeling and Control
• Robotics: Modelling, Planning and Control
Reminder: Right Hand Rules

Cross product

\[ \mathbf{a} \times \mathbf{b} \]
Reminder: Right Hand Rules

A right-handed coordinate frame

The first three fingers of your right hand which indicate the relative directions of the $x$-, $y$- and $z$-axes respectively.
Rotation about a vector

Wrap your right hand around the vector with your thumb (your x-finger) in the direction of the arrow. The curl of your fingers indicates the direction of increasing angle.
Kinematics

The problem of kinematics is to describe the motion of the manipulator without consideration of the forces and torques causing that motion.

The kinematic description is therefore a geometric one.
Forward Kinematics

Determine the position and orientation of the end-effector given the values for the joint variables of the robot.

Robot Manipulators are composed of links connected by joints to form a kinematic chain.
Robot Manipulators

Revolute joint (**R**): allows a relative rotation about a single axis.

Prismatic joint (**P**): allows a linear motion along a single axis (extension or retraction).

Spherical wrist: A three degree of freedom rotational joint with all three axes of rotation crossing at a point is typically called a spherical wrist.
The Workspace Of A Robot

The total volume its end-effector could sweep as the robot executes all possible motions. It is constrained by the geometry of the manipulator as well as mechanical limits imposed on the joints.
Symbolic representation of robot joints

A three-link arm with three revolute joints was denoted by RRR.

Joint variables, denoted by $\theta$ for a revolute joint and $d$ for the prismatic joint, represent the relative displacement between adjacent links.
Articulated Manipulators (RRR)
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Also called: Anthropomorphic Manipulators

Three joints of the rotational type (RRR). It resembles the human arm. The second joint axis is perpendicular to the first one. The third joint axis is parallel to the second one. The workspace of the anthropomorphich robot arm, encompassing all the points that can be reached by the robot end point.
Elbow Manipulator (RRR)
Spherical Manipulator

The Stanford Arm
Spherical Manipulator RRP

Two rotation and one translation (RRP). The second joint axis is perpendicular to the first one and the third axis is perpendicular to the second one.

Structure

Workspace
Spherical Manipulator RRP

Two rotation and one translation (RRP). The second joint axis is perpendicular to the first one and the third axis is perpendicular to the second one. The workspace of the robot arm has a spherical shape as in the case of the anthropomorphic robot arm.
Spherical Manipulator RRR

Workspace?

Structure
SCARA Manipulator

Two joints are rotational and one is translational (RRP). The axes of all three joints are parallel.

Workspace
SCARA Manipulator

Two joints are rotational and one is translational (RRP). The axes of all three joints are parallel. The workspace of SCARA robot arm is of cylindrical shape.
One rotational and two translational (RPP).
The axis of the second joint is parallel to the first axis.
The third joint axis is perpendicular to the second one.
Cylindrical Manipulator

One rotational and two translational (RPP).
The axis of the second joint is parallel to the first axis.
The third joint axis is perpendicular to the second one.
The Cartesian Manipulators

Workspace

Three joints of the translational type (PPP). The joint axes are perpendicular one to another.
The Cartesian Manipulators

Workspace

Three joints of the translational type (PPP). The joint axes are perpendicular one to another.
Configuration Parameters

A set of *position* parameters that describes the full configuration of the system.
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9 parameters/link
Generalized Coordinates

A set of independent configuration parameters

\[ 6 \text{ parameters/link} \begin{cases} 
3 \text{ positions} \\
3 \text{ orientations}
\end{cases} \]
**Generalized Coordinates**

A set of independent configuration parameters

6 parameters/link \( \left\{ \begin{array}{l} 3 \text{ positions} \\ 3 \text{ orientations} \end{array} \right\} \)

6n parameters for n moving links

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Generalized Coordinates

A set of independent configuration parameters

6 parameters/link \{ 3 positions, 3 orientations \}

5 Constraint

6n parameters for n moving links
Generalized Coordinates

A set of independent configuration parameters

6 parameters/link \( \{ \begin{align*} &3 \text{ positions} \\ &3 \text{ orientations} \end{align*} \)
Generalized Coordinates

A set of independent configuration parameters

\[ \text{6 parameters/link} \begin{cases} 3 \text{ positions} \\ 3 \text{ orientations} \end{cases} \]

\[ \text{5 Constraint} \]

\[ \text{6n parameters for n moving links} \]
\[ \text{5n constraints for n joints} \]
\[ \text{D.O.F:} \ 6n - 5n = n \]
Generalized Coordinates

D.O.F: \( n \) joints + ?
Generalized Coordinates

The robot is free to move forward/backward, up/down, left/right (translation in three perpendicular axes) combined with rotation about three perpendicular axes, often termed pitch, yaw, and roll.
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D.O.F: n joints + 6
Operational Coordinates

End-effector configuration parameters are a set of $m$ parameters $(x_1, x_2, x_3, \ldots, x_m)$ that completely specify the end-effector position and orientation with respect to the frame $o_0 x_0 y_0 z_0$.

$o_{n+1}$ is the operational point.

A set $(x_1, x_2, x_3, \ldots, x_{m_0})$ of independent configuration Parameters $m_0$: number of degree of freedom of the end-effector.
Operational Coordinates

Is also called Operational Space
Joint Coordinates

Is also called Joint Space
Joint Space -> Operational Space

Determine the position and orientation of the end-effector given the values for the joint variables of the robot.
A robot is said to be redundant if $n > m_0$. 
Degree of redundancy: $n - m_0$

**how many solutions exist?**

$m_0 = 3$
$n = 4$
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Redundancy

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Degree of redundancy: $n - m_0$

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$m_0 = 3$
$n = 3$
Kinematic Arrangements

The objective of forward kinematic analysis is to determine the cumulative effect of the entire set of joint variables, that is, to determine the position and orientation of the end effector given the values of these joint variables.

We assume that each joint has one D.O.F

The action of each joint can be described by one real number:
- the angle of rotation in the case of a revolute joint or
- the displacement in the case of a prismatic joint.

When joint $i$ is actuated, link $i$ moves.

$q_i$ is the joint variable

$$q_i = \begin{cases} 
\theta_i & \text{if joint } i \text{ is revolute} \\
\delta_i & \text{if joint } i \text{ is prismatic}
\end{cases}$$
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Spherical wrist 3 D.O.F

spherical wrist:
RRR
Links’ lengths = 0
Kinematic Arrangements

To perform the kinematic analysis, we attach a coordinate frame rigidly to each link. In particular, we attach \( o_i x_i \ y_i \ z_i \) to \textit{link i}. This means that, whatever motion the robot executes, the coordinates of any point \( p \) on link \( i \) are constant when expressed in the \( i^{th} \) coordinate frame \( p_i = \text{constant} \). When \textit{joint i} is actuated, \textit{link i} and its attached frame, \( o_i x_i \ y_i \ z_i \), experience a resulting motion.

The frame \( o_0 x_0 \ y_0 \ z_0 \), which is attached to the robot base, is referred to as the \textit{reference frame}. 
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Joint And Link Labelling
Joint And Link Labelling

Link 0 (fixed)  \( o_0x_0, y_0, z_0 \)
Base frame
Joint And Link Labelling

Joint variable $\theta_1$

Link 1

Joint 1

Link 0 (fixed)

Base frame

Initial positions: $o_0x_0, y_0, z_0$
Joint And Link Labelling

Joint variable $\theta_1$

Joint variable $\theta_2$

Link 0 (fixed)

$o_0x_0 \ y_0 \ z_0$

Base frame

Joint 1

Link 1

Joint 2

$o_1x_1 \ y_1 \ z_1$
Joint And Link Labelling

Joint variable $\theta_1$

Joint variable $\theta_2$

Joint variable $\theta_3$

Link 0 (fixed)

Joint 1

Link 1

Joint 2

Link 2

Joint 3

Link 3

Base frame

"o_0 x_0 y_0 z_0"

"o_1 x_1 y_1 z_1"

"o_2 x_2 y_2 z_2"
Joint And Link Labelling

Do we need a specific way to orientate the axes?
Suppose $A_i$ is the homogeneous transformation matrix that describes the position and orientation of $o_i x_i y_i z_i$ with respect to $o_{i-1} x_{i-1} y_{i-1} z_{i-1}$. $A_i$ is derived from joint and link $i$. $A_i$ is a function of only a single joint variable.

$$A_i = A_i(q_i)$$

$$A_i(q_i) = \begin{bmatrix} R^{i-1}_i & o^{i-1}_i \\ 0 & 1 \end{bmatrix}$$
Transformation Matrix

The position and the orientation of the end effector (reference frame \( o_n x_n y_n z_n \)) with respect to the base (reference frame \( o_0 x_0 y_0 z_0 \)) can be expressed by the transformation matrix:

\[
H = T^n_0 = A_1(q_1) \cdots A_n(q_n) = \begin{bmatrix}
R^n_0 & o^n_0 \\
0 & 1
\end{bmatrix}
\]

The position and the orientation of a reference frame \( o_j x_j y_j z_j \) with respect to a reference frame \( o_i x_i y_i z_i \) can be expressed by the transformation matrix:

\[
T^i_j = \begin{cases}
A_{i+1}A_{i+2} \cdots A_{j-1} A_j & \text{if } i < j \\
I & \text{if } i = j \\
(T^i_j)^{-1} & \text{if } i > j
\end{cases}
\]
Transformation Matrix

\[ T^i_j = \begin{cases} 
A_{i+1}A_{i+2} \ldots A_{j-1}A_j & \text{if } i < j \\
I & \text{if } i = j \\
(T^i_j)^{-1} & \text{if } i > j 
\end{cases} \]

if \( i < j \) then

\[ T^i_j = A_{i+1}A_{i+2} \ldots A_{j-1}A_j = \begin{bmatrix} R^i_j & o^i_j \\
0 & 1 \end{bmatrix} \]

The orientation part: \( R^i_j = R_{i+1}^i \ldots R_{j-1}^j \)

The translation part: \( o^i_j = o_{j-1}^i + R_{j-1}^i o_{j}^{j-1} \)