Similarity Transformations

The matrix representation of a general linear transformation is transformed from one frame to another using a so-called similarity transformation.

For example, if $A$ is the matrix representation of a given linear transformation in $o_0 \ x_0 \ y_0 \ z_0$ and $B$ is the representation of the same linear transformation in $o_1 \ x_1 \ y_1 \ z_1$ then $A$ and $B$ are related as:

$$B = (R^0_1)^{-1} AR^0_1$$

where $R^0_1$ is the coordinate transformation between frames $o_1 \ x_1 \ y_1 \ z_1$ and $o_0 \ x_0 \ y_0 \ z_0$. In particular, if $A$ itself is a rotation, then so is $B$, and thus the use of similarity transformations allows us to express the same rotation easily with respect to different frames.
Example

Suppose frames \( o_0 \ x_0 \ y_0 \ z_0 \) and \( o_1 \ x_1 \ y_1 \ z_1 \) are related by the rotation

\[
R_1^0 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
-1 & 0 & 0
\end{bmatrix}
\]

If \( A = R_z \) relative to the frame \( o_0 \ x_0 \ y_0 \ z_0 \), then, relative to frame \( o_1 \ x_1 \ y_1 \ z_1 \) we have

\[
B = (R_1^0)^{-1} A R_1^0
\]

\[
B = (R_1^0)^{-1} A R_1^0 = \begin{bmatrix}
1 & 0 & 0 \\
0 & c_\theta & s_\theta \\
0 & -s_\theta & c_\theta
\end{bmatrix}
\]

\( B \) is a rotation about the \( zo \) - axis but expressed relative to the frame \( o_1 \ x_1 \ y_1 \ z_1 \).
Rotation With Respect To The Current Frame

The matrix \( \mathbf{R}_1^0 \) represents a rotational transformation between the frames \( o_0 \, x_0 \, y_0 \, z_0 \) and \( o_1 \, x_1 \, y_1 \, z_1 \).

Suppose we now add a third coordinate frame \( o_2 \, x_2 \, y_2 \, z_2 \) related to the frames \( o_0 \, x_0 \, y_0 \, z_0 \) and \( o_1 \, x_1 \, y_1 \, z_1 \) by rotational transformations. A given point \( \mathbf{p} \) can then be represented by coordinates specified with respect to any of these three frames: \( \mathbf{p}^0 \), \( \mathbf{p}^1 \) and \( \mathbf{p}^2 \).

The relationship among these representations of \( \mathbf{p} \) is:

\[
\begin{align*}
\mathbf{p}^0 &= \mathbf{R}_1^0 \mathbf{p}^1 \\
\mathbf{p}^1 &= \mathbf{R}_2^1 \mathbf{p}^2 \\
\mathbf{p}^0 &= \mathbf{R}_2^0 \mathbf{p}^2 \\
\end{align*}
\]

where each \( \mathbf{R}_j^i \) is a rotation matrix.
Composition Law for Rotational Transformations

In order to transform the coordinates of a point $p$ from its representation $p^2$ in the frame $o_2 x_2 y_2 z_2$ to its representation $p^0$ in the frame $o_0 x_0 y_0 z_0$, we may first transform to its coordinates $p^1$ in the frame $o_1 x_1 y_1 z_1$ using $R^1_2$ and then transform $p^1$ to $p^0$ using $R^0_1$.

\[
\begin{align*}
p^0 & = R^0_1 p^1 \\
p^1 & = R^1_2 p^2 \\
p^0 & = R^0_2 p^2
\end{align*}
\]

\[
p^0 = R^0_1 R^1_2 p^2
\]

\[
R^0_2 = R^0_1 R^1_2
\]
Composition Law for Rotational Transformations

\[ R^0_2 = R^0_1 R^1_2 \]

Suppose initially that all three of the coordinate frames are coincide.

We first rotate the frame \( o_2 x_2 y_2 z_2 \) relative to \( o_0 x_0 y_0 z_0 \) according to the transformation \( R^0_1 \).

Then, with the frames \( o_1 x_1 y_1 z_1 \) and \( o_2 x_2 y_2 z_2 \) coincident, we rotate \( o_2 x_2 y_2 z_2 \) relative to \( o_1 x_1 y_1 z_1 \) according to the transformation \( R^1_2 \).

In each case we call the frame relative to which the rotation occurs the current frame.

**Coincident**: lie exactly on top of each other
Example

Suppose a rotation matrix \( R \) represents
- a rotation of angle \( \phi \) about the current \( y \) – axis followed by
- a rotation of angle \( \theta \) about the current \( z \) – axis.

\[
R = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{pmatrix}
\]

\[
R_{x,\theta} = \begin{pmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{pmatrix}
\]

\[
R_{z,\theta} = \begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

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Example

Suppose a rotation matrix $R$ represents

- a rotation of angle $\phi$ about the current $y - axis$ followed by
- a rotation of angle $\theta$ about the current $z - axis$.

$$R = R_{y,\phi} R_{z,\theta}$$

$$R_{x,\theta} = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{bmatrix}$$

$$R_{z,\theta} = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}$$
Example

Suppose a rotation matrix $R$ represents

- a rotation of angle $\phi$ about the current $y$-axis followed by
- a rotation of angle $\theta$ about the current $z$-axis.

\[
R = R_{y,\phi} R_{z,\theta} = \begin{bmatrix}
    c\phi & 0 & s\phi \\
    0 & 1 & 0 \\
    -s\phi & 0 & c\phi
\end{bmatrix}
\begin{bmatrix}
    c\theta & -s\theta & 0 \\
    s\theta & c\theta & 0 \\
    0 & 0 & 1
\end{bmatrix}
\]

\[
R_{x,\theta} = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix}
\]

\[
R_{y,\theta} = \begin{bmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{bmatrix}
\]

\[
R_{z,\theta} = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
Example

Suppose a rotation matrix $R$ represents
- a rotation of angle $\phi$ about the current $y -$ axis followed by
- a rotation of angle $\theta$ about the current $z -$ axis.

\[
R = R_{y, \phi} R_{z, \theta} = \begin{bmatrix}
  c_{\phi} & 0 & s_{\phi} \\
  0 & 1 & 0 \\
  -s_{\phi} & 0 & c_{\phi}
\end{bmatrix} \begin{bmatrix}
  c_{\theta} & -s_{\theta} & 0 \\
  s_{\theta} & c_{\theta} & 0 \\
  0 & 0 & 1
\end{bmatrix}
\]

\[
R_{x, \theta} = \begin{bmatrix}
  1 & 0 & 0 \\
  0 & \cos \theta & -\sin \theta \\
  0 & \sin \theta & \cos \theta
\end{bmatrix}
\]

\[
R_{y, \theta} = \begin{bmatrix}
  \cos \theta & 0 & \sin \theta \\
  0 & 1 & 0 \\
  -\sin \theta & 0 & \cos \theta
\end{bmatrix}
\]

\[
R_{z, \theta} = \begin{bmatrix}
  \cos \theta & -\sin \theta & 0 \\
  \sin \theta & \cos \theta & 0 \\
  0 & 0 & 1
\end{bmatrix}
\]
Example

Suppose a rotation matrix $R$ represents

- a rotation of angle $\theta$ about the current $z$–axis followed by
- a rotation of angle $\phi$ about the current $y$–axis

\[
R' = R_x,\theta \cdot R_y,\theta \cdot R_z,\theta
\]

\[
R_x,\theta = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix}
\]

\[
R_y,\theta = \begin{bmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{bmatrix}
\]

\[
R_z,\theta = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
Example

Suppose a rotation matrix $R$ represents

- a rotation of angle $\theta$ about the current $z - axis$ followed by
- a rotation of angle $\phi$ about the current $y - axis$

$$R' = R_{z,\theta} R_{y,\phi}$$

$$= \begin{bmatrix} c_{\theta} & -s_{\theta} & 0 \\ s_{\theta} & c_{\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_{\phi} & 0 & s_{\phi} \\ 0 & 1 & 0 \\ -s_{\phi} & 0 & c_{\phi} \end{bmatrix}$$


$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

$$R_{z,\theta} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
Example

Suppose a rotation matrix $R$ represents

- a rotation of angle $\theta$ about the current $z-\text{axis}$ followed by
- a rotation of angle $\phi$ about the current $y-\text{axis}$

$$
R' = R_{z,\theta} R_{y,\phi} =
\begin{bmatrix}
c_{\theta} & -s_{\phi} & 0 \\
s_{\theta} & c_{\theta} & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
c_{\phi} & 0 & s_{\phi} \\
0 & 1 & 0 \\
-s_{\phi} & 0 & c_{\phi}
\end{bmatrix}
= 
\begin{bmatrix}
c_{\theta}c_{\phi} & -s_{\theta} & c_{\theta}s_{\phi} \\
s_{\theta}c_{\phi} & c_{\theta} & s_{\theta}s_{\phi} \\
-s_{\phi} & 0 & c_{\phi}
\end{bmatrix}
$$

Rotational transformations do not commute

$R \neq R'$
Rotation With Respect To The Fixed Frame

Performing a sequence of rotations, each about a given fixed coordinate frame, rather than about successive current frames.

For example we may wish to perform a rotation about \( x_0 \) followed by a rotation about \( y_0 \) (and not \( y_1 \)). We will refer to \( o_0 x_0 y_0 z_0 \) as the fixed frame. In this case the composition law given before is not valid.

The composition law that was obtained by multiplying the successive rotation matrices in the reverse order from that given by is not valid.
Rotation with Respect to the Fixed Frame

Suppose we have two frames $o_0 x_0 y_0 z_0$ and $o_1 x_1 y_1 z_1$ related by the rotational transformation $R_1^0$.

If $R$ represents a rotation relative to $o_0 x_0 y_0 z_0$, the representation for $R$ in the current frame $o_1 x_1 y_1 z_1$ is given by:

$$(R_1^0)^{-1} RR_1^0.$$

With applying the composition law for rotations about the current axis:

$$R_2^0 = R_1^0 [(R_1^0)^{-1} RR_1^0] = RR_1^0$$

Reminder:

Similarity Transformations

$$B = (R_1^0)^{-1} AR_1^0$$

composition law for rotations about the current axis

$$R_2^0 = R_1^0 R_2^1$$
Example

Suppose a rotation matrix $R$ represents
- a rotation of angle $\phi$ about $y_0$—axis followed by
- a rotation of angle $\theta$ about the fixed $z_0$—axis

The second rotation about the fixed axis is given by

$$ R_{y,-\phi} R_{z,\theta} R_{y,\phi} $$

which is the basic rotation about the z-axis expressed relative to the frame $o_1 x_1 y_1 z_1$ using a similarity transformation.

Reminder:

Similarity Transformations

$$ B = (R_1^0)^{-1} AR_1^0 $$

composition law for rotations about the current axis

$$ R_2^0 = R_1^0 R_2^1 $$

composition law for rotations about the fixed axis

$$ R_2^0 = R_1^0 [(R_1^0)^{-1} R R_1^0] $$

$$ = R R_1^0 $$
Example

Suppose a rotation matrix $R$ represents:
- a rotation of angle $\phi$ about $y_0$—axis followed by
- a rotation of angle $\theta$ about the fixed $z_0$—axis.

Therefore, the composition rule for rotational transformations:

\[
p_0^0 = p_1^1 = p_2^2
\]
Example

Suppose a rotation matrix $R$ represents

- a rotation of angle $\phi$ about $y_0$ — *axis* followed by
- a rotation of angle $\theta$ about the fixed $z_0$ — *axis*

$$
\begin{align*}
p^0 &= R_{y, \phi} p^1 \\
      &= R_{y, \phi} \left[ R_{y, -\phi} R_{z, \theta} R_{y, \phi} \right] p^2 \\
      &= R_{z, \theta} R_{y, \phi} p^2
\end{align*}
$$
Example

Suppose a rotation matrix $R$ represents

• a rotation of angle $\phi$ about $y_0$ — *axis* followed by

• a rotation of angle $\theta$ about the *fixed* $z_0$—*axis*

\[
\begin{align*}
p^0 & = R_{y,\phi} p^1 \\
& = R_{y,\phi} \left[ R_{y,-\phi} R_{z,\theta} R_{y,\phi} \right] p^2 \\
& = R_{z,\theta} R_{y,\phi} p^2
\end{align*}
\]

Suppose a rotation matrix $R$ represents

• a rotation of angle $\phi$ about the *current* $y$ — *axis* followed by

• a rotation of angle $\theta$ about the *current* $z$ — *axis*.

\[
R = R_{y,\phi} R_{z,\theta}
\]
Summary

To note that we obtain the same basic rotation matrices, but in the reverse order.

\[
R_2^0 = R_1^0 R_2^1
\]

Rotation with Respect to the Current Frame

\[
R_2^0 = R_2^0 R_1^0
\]

Rotation with Respect to the Fixed Frame
Rules for Composition of Rotational Transformations

We can summarize the rule of composition of rotational transformations by:

Given a fixed frame \( o_0 x_0 y_0 z_0 \) a current frame \( o_1 x_1 y_1 z_1 \), together with rotation matrix \( R_1^0 \) relating them, if a third frame \( o_2 x_2 y_2 z_2 \) is obtained by a rotation \( R \) performed relative to the current frame then post-multiply \( R_1^0 \) by \( R = R_2^1 \) to obtain

\[
R_2^0 = R_1^0 R_2^1
\]

If the second rotation is to be performed relative to the fixed frame then it is both confusing and inappropriate to use the notation \( R_2^1 \) to represent this rotation. Therefore, if we represent the rotation by \( R \), we pre-multiply \( R_1^0 \) by \( R \) to obtain

\[
R_2^0 = R R_1^0
\]

In each case \( R_2^0 \) represents the transformation between the frames \( o_0 x_0 y_0 z_0 \) and \( o_2 x_2 y_2 z_2 \).
Example

Find \( \mathbf{R} \) for the following sequence of basic rotations:

1. A rotation of \( \Theta \) about the current x-axis
2. A rotation of \( \phi \) about the current z-axis
3. A rotation of \( \alpha \) about the fixed z-axis
4. A rotation of \( \beta \) about the current y-axis
5. A rotation of \( \delta \) about the fixed x-axis

\[
\mathbf{R} =
\]

Reminder:

Rotation with Respect to the Current Frame

\[
\mathbf{R}_2^0 = \mathbf{R}_1^0 \mathbf{R}_2^1
\]

Rotation with Respect to the Fixed Frame

\[
\mathbf{R}_2^0 = \mathbf{R} \mathbf{R}_1^0
\]
Example

Find \( R \) for the following sequence of basic rotations:

1. A rotation of \( \Theta \) about the current \( x \)-axis
2. A rotation of \( \phi \) about the current \( z \)-axis
3. A rotation of \( \alpha \) about the fixed \( z \)-axis
4. A rotation of \( \beta \) about the current \( y \)-axis
5. A rotation of \( \delta \) about the fixed \( x \)-axis

\[
R = R_{x, \theta}
\]

Reminder:

Rotation with Respect to the Current Frame

\[
R_2^0 = R_1^0 R_2^1
\]

Rotation with Respect to the Fixed Frame

\[
R_2^0 = R R_1^0
\]
Example

Find $R$ for the following sequence of basic rotations:

1. A rotation of $\Theta$ about the current x-axis
2. A rotation of $\phi$ about the current z-axis
3. A rotation of $\alpha$ about the fixed z-axis
4. A rotation of $\beta$ about the current y-axis
5. A rotation of $\delta$ about the fixed x-axis

$$R = R_{x,\theta} R_{z,\phi}$$

Reminder:

Rotation with Respect to the Current Frame

$$R_2^0 = R_1^0 R_2^1$$

Rotation with Respect to the Fixed Frame

$$R_2^0 = R R_1^0$$
Example

Find $\mathbf{R}$ for the following sequence of basic rotations:

1. A rotation of $\Theta$ about the current $x$-axis
2. A rotation of $\phi$ about the current $z$-axis
3. A rotation of $\alpha$ about the fixed $z$-axis
4. A rotation of $\beta$ about the current $y$-axis
5. A rotation of $\delta$ about the fixed $x$-axis

$\mathbf{R} = R_{z,\alpha} R_{x,\theta} R_{z,\phi}$

Reminder:

Rotation with Respect to the Current Frame

$R_{2}^{0} = R_{1}^{0} R_{2}^{1}$

Rotation with Respect to the Fixed Frame

$R_{2}^{0} = \mathbf{R} R_{1}^{0}$
Example

Find $R$ for the following sequence of basic rotations:

1. A rotation of $\Theta$ about the current $x$-axis
2. A rotation of $\phi$ about the current $z$-axis
3. A rotation of $\alpha$ about the fixed $z$-axis
4. A rotation of $\beta$ about the current $y$-axis
5. A rotation of $\delta$ about the fixed $x$-axis

$$R = R_{z, \alpha} R_{x, \theta} R_{z, \phi} R_{y, \beta}$$

Reminder:

Rotation with Respect to the Current Frame

$$R_2^0 = R_1^0 R_2^1$$

Rotation with Respect to the Fixed Frame

$$R_2^0 = R R_1^0$$
Example

Find $R$ for the following sequence of basic rotations:

1. A rotation of $\Theta$ about the current $x$-axis
2. A rotation of $\phi$ about the current $z$-axis
3. A rotation of $\alpha$ about the fixed $z$-axis
4. A rotation of $\beta$ about the current $y$-axis
5. A rotation of $\delta$ about the fixed $x$-axis

$$R = R_{x,\delta} R_{z,\alpha} R_{x,\theta} R_{z,\phi} R_{y,\beta}$$

**Reminder:**

Rotation with Respect to the Current Frame

$$R_2^0 = R_1^0 R_2^1$$

Rotation with Respect to the Fixed Frame

$$R_2^0 = R R_1^0$$
Example

Find $\mathbf{R}$ for the following sequence of basic rotations:

1. A rotation of $\delta$ about the fixed $x$-axis
2. A rotation of $\beta$ about the current $y$-axis
3. A rotation of $\alpha$ about the fixed $z$-axis
4. A rotation of $\phi$ about the current $z$-axis
5. A rotation of $\Theta$ about the current $x$-axis

Reminder:

Rotation with Respect to the Current Frame

$$R^0_2 = R^0_1 R^1_2$$

Rotation with Respect to the Fixed Frame

$$R^0_2 = \mathbf{R} R^0_1$$
Rotations in Three Dimensions

Each axis of the frame \( o_1 x_1 y_1 z_1 \) is projected onto the coordinate frame \( o_0 x_0 y_0 z_0 \).

The resulting rotation matrix is given by

\[
R_1^0 = \begin{bmatrix}
x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\
x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\
x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0
\end{bmatrix}
\]

The nine elements \( r_{ij} \) in a general rotational transformation \( \mathbf{R} \) are not independent quantities.

\( \mathbf{R} \in \text{SO}(3) \)

Where \( \text{SO}(n) \) denotes the Special Orthogonal group of order \( n \).
Rotations In Three Dimensions

For any $R \in SO(n)$ The following properties hold

- $R^T = R^{-1} \in SO(n)$
- The columns and the rows of $R$ are mutually orthogonal
- Each column and each row of $R$ is a unit vector
- $\det R = 1$ (the determinant)

Where $SO(n)$ denotes the Special Orthogonal group of order $n$.

Example for $R^T = R^{-1} \in SO(2)$:

$$
\begin{bmatrix}
\cos(-\theta) & -\sin(-\theta) \\
\sin(-\theta) & \cos(-\theta)
\end{bmatrix}
= 
\begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}
= 
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}^T
$$
Parameterizations Of Rotations

The nine elements $r_{ij}$ in a general rotational transformation $R$ are not independent quantities.

$R \in SO(3)$

Where $SO(n)$ denotes the Special Orthogonal group of order $n$.

As each column of $R$ is a unit vector, then we can write:

$$\sum_i r_{ij}^2 = 1, \quad j \in \{1, 2, 3\}$$

As the columns of $R$ are mutually orthogonal, then we can write:

$$r_{1i}r_{1j} + r_{2i}r_{2j} + r_{3i}r_{3j} = 0, \quad i \neq j$$

Together, these constraints define six independent equations with nine unknowns, which implies that there are three free variables.
Parameterizations of Rotations

We present three ways in which an arbitrary rotation can be represented using only three independent quantities:

- Euler Angles representation
- Roll-Pitch-Yaw representation
- Axis/Angle representation
Euler Angles Representation

We can specify the orientation of the frame $o_1 x_1 y_1 z_1$ relative to the frame $o_0 x_0 y_0 z_0$ by three angles $(\phi, \theta, \psi)$, known as Euler Angles, and obtained by three successive rotations as follows:

1. rotation about the $z$-axis by the angle $\phi$
2. rotation about the current $y$-axis by the angle $\theta$
3. rotation about the current $z$-axis by the angle $\psi$
Euler Angles Representation

\[ R_{ZYX} = R_{z,\phi} R_{y,\theta} R_{z,\psi} \]

\[ = \begin{bmatrix}
    c_\phi & -s_\phi & 0 \\
    s_\phi & c_\phi & 0 \\
    0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
    c_\theta & 0 & s_\theta \\
    0 & 1 & 0 \\
    -s_\theta & 0 & c_\theta
\end{bmatrix}
= \begin{bmatrix}
    c_\psi & -s_\psi & 0 \\
    s_\psi & c_\psi & 0 \\
    0 & 0 & 1
\end{bmatrix} \]

\[ = \begin{bmatrix}
    c_\phi c_\psi c_\theta - s_\phi s_\psi
    -c_\phi c_\psi s_\theta - s_\phi c_\psi
    c_\phi s_\theta \\
    s_\phi c_\psi c_\theta + c_\phi s_\theta
    -s_\phi c_\psi s_\theta + c_\phi c_\psi
    s_\phi s_\theta \\
    -s_\theta c_\psi
    s_\theta s_\psi
    c_\theta
\end{bmatrix} \]
\[
\begin{bmatrix}
\cos(p) & -\sin(p) & 0 \\
\sin(p) & \cos(p) & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\cdot
\begin{bmatrix}
\cos(t) & 0 & \sin(t) \\
0 & 1 & 0 \\
-\sin(t) & 0 & \cos(t) \\
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 \\
0 & \cos(\alpha) & -\sin(\alpha) \\
0 & \sin(\alpha) & \cos(\alpha) \\
\end{bmatrix}
\]

**Answer**

\[
\begin{bmatrix}
\cos(p)\cos(t) & -\sin(p)\cos(\alpha) + \cos(p)\sin(t)\sin(\alpha) & \sin(p)\sin(\alpha) + \cos(p)\sin(t)\cos(\alpha) \\
\sin(p)\cos(t) & \cos(p)\cos(\alpha) + \sin(p)\sin(t)\sin(\alpha) & -\cos(p)\sin(\alpha) + \sin(p)\sin(t)\cos(\alpha) \\
-\sin(t) & \cos(t)\sin(\alpha) & \cos(t)\cos(\alpha) \\
\end{bmatrix}
\]
Reminder:

Trigonometry (Atan vs. Atan2)
Trigonometry (Atan vs. Atan2)

\[
\begin{align*}
\text{atan2}(y, x) &= \begin{cases} 
\arctan \frac{y}{x} & x > 0 \\
\arctan \frac{y}{x} + \pi & y \geq 0, x < 0 \\
\arctan \frac{y}{x} - \pi & y < 0, x < 0 \\
+\frac{\pi}{2} & y > 0, x = 0 \\
-\frac{\pi}{2} & y < 0, x = 0 \\
\text{undefined} & y = 0, x = 0
\end{cases}
\end{align*}
\]

\[\text{atan}(y/x)\quad \text{atan2}(y, x)\]
Reminder:

**Trigonometry (Atan vs. Atan2)**

tan(angle) = opposite/adjacent
atan(opposite/adjacent) = angle
Euler Angles Representation

Given a matrix \( R \in SO(3) \)

Determine a set of Euler angles \( \phi, \theta, \) and \( \psi \) so that \( R = R_{ZYX} \)

\[
R_{ZYX} = \begin{bmatrix}
    c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\
    s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\
    -s_\theta c_\psi & s_\theta s_\psi & c_\theta
\end{bmatrix}
\]

If \( r_{13} \neq 0 \) and \( r_{23} \neq 0 \),
it follows that: \( \phi = \text{Atan2}(r_{23}, r_{13}) \)
where the function \( \text{Atan2}(y, x) \) computes the arctangent of the ration \( y/x \).
Then squaring the summing of the elements (1,3) and (2,3) and using the element (3,3) yields:

\[
\theta = \text{Atan2}(+\sqrt{r_{13}^2 + r_{23}^2}, r_{33}) \quad \text{or} \quad \theta = \text{Atan2}(-\sqrt{r_{13}^2 + r_{23}^2}, r_{33})
\]

If we consider the first choice then \( \sin(\theta) > 0 \) then: \( \psi = \text{Atan2}(r_{32}, -r_{31}) \)

If we consider the second choice then \( \sin(\theta) < 0 \) then: \( \psi = \text{Atan2}(-r_{32}, r_{31}) \)

and \( \phi = \text{Atan2}(-r_{23}, -r_{13}) \)
Euler Angles Representation

Given a matrix \( R \in SO(3) \)

Determine a set of Euler angles \( \phi, \theta, \) and \( \psi \) so that \( R = R_{ZYX} \)

\[
R_{ZYX} = \begin{bmatrix}
    c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\
    s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\
    -s_\theta c_\psi & s_\theta s_\psi & c_\theta
\end{bmatrix}
\]

If \( r_{13} = r_{23} = 0 \), then the fact that \( R \) is orthogonal implies that \( r_{33} = \pm 1 \) and that \( r_{31} = r_{32} = 0 \) thus \( R \) has the form:

\[
R = \begin{bmatrix}
    r_{11} & r_{12} & 0 \\
    r_{21} & r_{22} & 0 \\
    0 & 0 & \pm 1
\end{bmatrix}
\]

If \( r_{33} = +1 \) then \( c_\theta = 1 \) and \( s_\theta = 0 \), so that \( \theta = 0 \).

\[
\begin{bmatrix}
    c_\phi c_\psi - s_\phi s_\psi & -c_\phi s_\psi - s_\phi c_\psi & 0 \\
    s_\phi c_\psi + c_\phi s_\psi & -s_\phi c_\psi + c_\phi c_\psi & 0 \\
    0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
    c_{\phi+\psi} & -s_{\phi+\psi} & 0 \\
    s_{\phi+\psi} & c_{\phi+\psi} & 0 \\
    0 & 0 & 1
\end{bmatrix}
\]

Thus, the sum \( \phi + \psi \) can be determined as \( \phi + \psi = \text{Atan2}(r_{21}, r_{11}) = \text{Atan2}(- r_{12}, r_{22}) \)

There is infinity of solutions.
Euler Angles Representation

Given a matrix $R \in SO(3)$

**Determine** a set of Euler angles $\phi$, $\theta$, and $\psi$ so that $R = R_{ZYZ}$

$$
R_{ZYZ} = \begin{bmatrix}
C_\phi C_\theta C_\psi - S_\phi S_\psi & -C_\phi C_\theta S_\psi - S_\phi C_\psi & C_\phi S_\theta \\
S_\phi C_\theta C_\psi + C_\phi S_\psi & -S_\phi C_\theta S_\psi + C_\phi C_\psi & S_\phi S_\theta \\
-S_\theta C_\psi & S_\theta S_\psi & C_\theta
\end{bmatrix}
$$

If $r_{13} = r_{23} = 0$, then the fact that $R$ is orthogonal implies that $r_{33} = \pm 1$ and that $r_{31} = r_{32} = 0$ thus $R$ has the form:

$$
R = \begin{bmatrix}
r_{11} & r_{12} & 0 \\
r_{21} & r_{22} & 0 \\
0 & 0 & \pm 1
\end{bmatrix}
$$

If $r_{33} = -1$ then $c_\theta = -1$ and $s_\theta = 0$, so that $\theta = \pi$.

Thus, the $\phi - \psi$ can be determined as $\phi - \psi = \text{Atan2}(-r_{12}, -r_{11}) = \text{Atan2}(r_{21}, r_{22})$

As before there is infinity of solutions.
Yaw-Pitch-Roll Representation

A rotation matrix $R$ can also be described as a product of successive rotations about the principal coordinate axes $o_0 x_0 y_0 z_0$ taken in a specific order. These rotations define the roll, pitch, and yaw angles, which we shall also denote $(\phi, \theta, \psi)$.

We specify the order in three successive rotations as follows:
1. Yaw rotation about $x_0$ — axis by the angle $\psi$
2. Pitch rotation about $y_0$ — axis by the angle $\theta$
3. Roll rotation about $z_0$ — axis by the angle $\phi$

Since the successive rotations are relative to the fixed frame, the resulting transformation matrix is given by:

$$R_{XYZ} = \ldots$$
Yaw-Pitch-Roll Representation

A rotation matrix $\mathbf{R}$ can also be described as a product of successive rotations about the principal coordinate axes $\mathbf{o}_0\ x_0\ y_0\ z_0$ taken in a specific order. These rotations define the roll, pitch, and yaw angles, which we shall also denote $(\phi, \theta, \psi)$.

We specify the order in three successive rotations as follows:

1. Yaw rotation about $x_0$ − axis by the angle $\psi$
2. Pitch rotation about $y_0$ − axis by the angle $\theta$
3. Roll rotation about $z_0$ − axis by the angle $\phi$

Since the successive rotations are relative to the fixed frame, the resulting transformation matrix is given by:

$$
\mathbf{R}_{XYZ} = \mathbf{R}_{z,\phi}\mathbf{R}_{y,\theta}\mathbf{R}_{x,\psi}
$$
Yaw-Pitch-Roll Representation

A rotation matrix $R$ can also be described as a product of successive rotations about the principal coordinate axes $o_0 x_0 y_0 z_0$ taken in a specific order. These rotations define the roll, pitch, and yaw angles, which we shall also denote $(\phi, \theta, \psi)$

We specify the order in three successive rotations as follows:
1. Yaw rotation about $x_0$ – axis by the angle $\psi$
2. Pitch rotation about $y_0$ – axis by the angle $\theta$
3. Roll rotation about $z_0$ – axis by the angle $\phi$

Since the successive rotations are relative to the fixed frame, the resulting transformation matrix is given by:

$$R_{XYZ} = R_z,\phi R_y,\theta R_x,\psi$$

$$= \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\psi & -s\psi \\ 0 & s\psi & c\psi \end{bmatrix}$$
Yaw-Pitch-Roll Representation

A rotation matrix $R$ can also be described as a product of successive rotations about the principal coordinate axes $o_0 x_0 y_0 z_0$ taken in a specific order. These rotations define the roll, pitch, and yaw angles, which we shall also denote $(\phi, \theta, \psi)$.

We specify the order in three successive rotations as follows:
1. Yaw rotation about $x_0$—axis by the angle $\psi$
2. Pitch rotation about $y_0$—axis by the angle $\theta$
3. Roll rotation about $z_0$—axis by the angle $\phi$

Since the successive rotations are relative to the fixed frame, the resulting transformation matrix is given by:

$$R_{XYZ} = R_{z,\phi}R_{y,\theta}R_{x,\psi}$$

$$= \begin{bmatrix}
    c_\phi & -s_\phi & 0 \\
    s_\phi & c_\phi & 0 \\
    0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    c_\theta & 0 & s_\theta \\
    0 & 1 & 0 \\
    -s_\theta & 0 & c_\theta
\end{bmatrix}
\begin{bmatrix}
    1 & 0 & 0 \\
    0 & c_\psi & -s_\psi \\
    0 & s_\psi & c_\psi
\end{bmatrix}$$

$$= \begin{bmatrix}
    c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\
    s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\
    -s_\theta & c_\phi s_\psi & c_\theta c_\psi
\end{bmatrix}$$
Yaw-Pitch-Roll Representation

A rotation matrix $R$ can also be described as a product of successive rotations about the principal coordinate axes $o_0 x_0 y_0 z_0$ taken in a specific order. These rotations define the roll, pitch, and yaw angles, which we shall also denote ($\phi, \theta, \psi$).

We specify the order in three successive rotations as follows:
1. Yaw rotation about $x_0$—axis by the angle $\psi$
2. Pitch rotation about $y_0$—axis by the angle $\theta$
3. Roll rotation about $z_0$—axis by the angle $\phi$

Since the successive rotations are relative to the fixed frame, the resulting transformation matrix is given by:

$$R_{XYZ} = R_z,\phi R_y,\theta R_x,\psi$$
Yaw-Pitch-Roll Representation

A rotation matrix $R$ can also be described as a product of successive rotations about the principal coordinate axes $o_0 x_0 y_0 z_0$ taken in a specific order. These rotations define the roll, pitch, and yaw angles, which we shall also denote $(\phi, \theta, \psi)$.

We specify the order in three successive rotations as follows:
1. Yaw rotation about $x_0$ axis by the angle $\psi$
2. Pitch rotation about $y_0$ axis by the angle $\theta$
3. Roll rotation about $z_0$ axis by the angle $\phi$

Since the successive rotations are relative to the fixed frame, the resulting transformation matrix is given by:

$$R_{XYZ} = R_{z, \phi} R_{y, \theta} R_{x, \psi}$$

Instead of yaw-pitch-roll relative to the fixed frames we could also interpret the above transformation as roll-pitch-yaw, in that order, each taken with respect to the current frame. The end result is the same matrix.
Yaw-Pitch-Roll Representation

Find the inverse solution to a given rotation matrix $R$.

$$R_{XYZ} = \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}$$

Determine a set of Roll-Pitch-Yaw angles $\phi, \theta, \text{and } \psi$ so that $R = R_{XYZ}$
Axis/Angle Representation

Rotations are not always performed about the principal coordinate axes. We are often interested in a rotation about an arbitrary axis in space. This provides both a convenient way to describe rotations, and an alternative parameterization for rotation matrices.

Let \( k = [k_x, k_y, k_z]^T \), expressed in the frame \( o_0 x_0 y_0 z_0 \), be a unit vector defining an axis. We wish to derive the rotation matrix \( R_{k, \theta} \) representing a rotation of \( \theta \) about this axis.

A possible solution is to rotate first \( k \) by the angles necessary to align it with \( z \), then to rotate by \( \theta \) about \( z \), and finally to rotate by the angles necessary to align the unit vector with the initial direction.
Axis/Angle Representation

Rotations are not always performed about the principal coordinate axes. We are often interested in a rotation about an arbitrary axis in space. This provides both a convenient way to describe rotations, and an alternative parameterization for rotation matrices.

Let \( \mathbf{k} = [k_x, k_y, k_z]^T \), expressed in the frame \( o_0 x_0 y_0 z_0 \), be a unit vector defining an axis. We wish to derive the rotation matrix \( R_{k,\theta} \) representing a rotation of \( \theta \) about this axis.

The sequence of rotations to be made with respect to axes of fixed frame is the following:

- Align \( \mathbf{k} \) with \( \mathbf{z} \) (which is obtained as the sequence of a rotation by \( -\alpha \) about \( \mathbf{z} \) and a rotation of \( -\beta \) about \( \mathbf{y} \)).
- Rotate by \( \theta \) about \( \mathbf{z} \).
- Realign with the initial direction of \( \mathbf{k} \), which is obtained as the sequence of a rotation by \( \beta \) about \( \mathbf{y} \) and a rotation by \( \alpha \) about \( \mathbf{z} \).

\[
R_{k,\theta} = R_{z,\alpha} R_{y,\beta} R_{z,\theta} R_{y,-\beta} R_{z,-\alpha}
\]
Axis/Angle Representation

\[ R_{k,\theta} = R_{z,\alpha} R_{y,\beta} R_{z,\theta} R_{y,-\beta} R_{z,-\alpha} \]

\[ \sin \alpha = \frac{k_y}{\sqrt{k_x^2 + k_y^2}} \]
\[ \cos \alpha = \frac{k_x}{\sqrt{k_x^2 + k_y^2}} \]
\[ \sin \beta = \sqrt{k_x^2 + k_y^2} \]
\[ \cos \beta = k_z \]
Axis/Angle Representation

Rotations are not always performed about the principal coordinate axes. We are often interested in a rotation about an arbitrary axis in space. This provides both a convenient way to describe rotations, and an alternative parameterization for rotation matrices.

\[ R_{k,\theta} = R_{z,\alpha} R_{y,\beta} R_{z,\theta} R_{y,-\beta} R_{z,-\alpha} \]

\[
R_{k,\theta} = \begin{bmatrix}
  k_x^2 c_\theta + 1 & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\
  k_x k_y v_\theta + k_z s_\theta & k_y^2 c_\theta + 1 & k_y k_z v_\theta - k_x s_\theta \\
  k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 c_\theta + 1
\end{bmatrix}
\]

\[ v_\theta = \text{vers} \, \theta = 1 - c_\theta. \]
**Axis/Angle Representation**

Any rotation matrix \( R \in SO(3) \) can be represented by a single rotation about a suitable axis in space by a suitable angle.

\[
R = R_{k, \theta}
\]

where \( k \) is a unit vector defining the axis of rotation, and \( \theta \) is the angle of rotation about \( k \).

The matrix \( R_{k, \theta} \) is called the axis/angle representation of \( R \).

**Given \( R \) find \( \theta \) and \( k \):**

\[
\theta = \cos^{-1} \left( \frac{Tr(R) - 1}{2} \right)
\]

\[
= \cos^{-1} \left( \frac{r_{11} + r_{22} + r_{33} - 1}{2} \right)
\]

\[
k = \frac{1}{2 \sin \theta} \begin{bmatrix}
    r_{32} - r_{23} \\
    r_{13} - r_{31} \\
    r_{21} - r_{12}
\end{bmatrix}
\]

**Reminder:**

\[
\text{trace}(A) = tr(A) = \sum_{i=1}^{n} a_{ii}
\]
The axis/angle representation is not unique since a rotation of \(-\theta\) about \(-k\) is the same as a rotation of \(\theta\) about \(k\).

\[ R_{k,\theta} = R_{-k,-\theta} \]

If \(\theta = 0\) then \(R\) is the identity matrix and the axis of rotation is undefined.
Example

Suppose $\mathbf{R}$ is generated by a rotation of $90^\circ$ about $z_0$ followed by a rotation of $30^\circ$ about $y_0$ followed by a rotation of $60^\circ$ about $x_0$. Find the axis/angle representation of $\mathbf{R}$

$$\mathbf{R} = R_{x,60} R_{y,30} R_{z,90}$$

Reminder:
The axis/angle representation of $\mathbf{R}$

$$\theta = \cos^{-1} \left( \frac{Tr(\mathbf{R}) - 1}{2} \right)$$

$$= \cos^{-1} \left( \frac{r_{11} + r_{22} + r_{33} - 1}{2} \right)$$

$$k = \frac{1}{2 \sin \theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$
Example

Suppose \( R \) is generated by a rotation of 90° about \( z_0 \) followed by a rotation of 30° about \( y_0 \) followed by a rotation of 60° about \( y_0 \). Find the axis/angle representation of \( R \)

\[
R = R_{x,60} R_{y,30} R_{z,90}
\]

\[
= \begin{bmatrix}
0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\
\frac{\sqrt{3}}{2} & \frac{1}{4} & \frac{\sqrt{3}}{4} \\
\end{bmatrix}
\]

\[
Tr(R) =
\]

\[
\theta = \cos^{-1} \left( \frac{Tr(R) - 1}{2} \right)
\]

\[
= \cos^{-1} \left( \frac{r_{11} + r_{22} + r_{33} - 1}{2} \right)
\]

\[
k = \frac{1}{2 \sin \theta} \begin{bmatrix}
r_{32} - r_{23} \\
r_{13} - r_{31} \\
r_{21} - r_{12} \\
\end{bmatrix}
\]

Reminder:
The axis/angle representation of \( R \)
Example

Suppose $R$ is generated by a rotation of $90^\circ$ about $z_0$ followed by a rotation of $30^\circ$ about $y_0$ followed by a rotation of $60^\circ$ about $y_0$. Find the axis/angle representation of $R$

$$R = R_{x,60} R_{y,30} R_{z,90}$$

$$= \begin{bmatrix}
0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\
\frac{\sqrt{3}}{2} & \frac{1}{4} & \frac{\sqrt{3}}{4}
\end{bmatrix}$$

$Tr(R) = 0$

$$\theta = \cos^{-1}\left(\frac{Tr(R) - 1}{2}\right)$$

$$= \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$$k = \frac{1}{2 \sin \theta} \begin{bmatrix}
r_{32} - r_{23} \\
r_{13} - r_{31} \\
r_{21} - r_{12}
\end{bmatrix}$$
Example

Suppose $R$ is generated by a rotation of $90^\circ$ about $z_0$ followed by a rotation of $30^\circ$ about $y_0$ followed by a rotation of $60^\circ$ about $y_0$. Find the axis/angle representation of $R$

\[
R = R_{x,60} R_{y,30} R_{z,90}
\]
\[
= \begin{bmatrix}
0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\
\frac{\sqrt{3}}{2} & \frac{1}{4} & \frac{\sqrt{3}}{4}
\end{bmatrix}
\]

$Tr(R) = 0$

$\theta = \cos^{-1} \left( \frac{Tr(R) - 1}{2} \right)$

$\theta = \cos^{-1} \left( \frac{r_{11} + r_{22} + r_{33} - 1}{2} \right)$

$k = \frac{1}{2 \sin \theta} \begin{bmatrix}
r_{32} - r_{23} \\
r_{13} - r_{31} \\
r_{21} - r_{12}
\end{bmatrix}$
Example

Suppose \( \mathbf{R} \) is generated by a rotation of 90° about \( z_0 \) followed by a rotation of 30° about \( y_0 \) followed by a rotation of 60° about \( x_0 \). Find the axis/angle representation of \( \mathbf{R} \).

\[
\mathbf{R} = \mathbf{R}_{x,60} \mathbf{R}_{y,30} \mathbf{R}_{z,90}
\]
\[
= \begin{bmatrix}
0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\
\frac{\sqrt{3}}{2} & \frac{1}{4} & \frac{\sqrt{3}}{4}
\end{bmatrix}
\]

\( Tr(\mathbf{R}) = 0 \)

\[
\theta = \cos^{-1} \left( \frac{Tr(\mathbf{R}) - 1}{2} \right) = \cos^{-1} \left( \frac{r_{11} + r_{22} + r_{33} - 1}{2} \right)
\]

\[
k = \frac{1}{2 \sin \theta} \begin{bmatrix}
r_{32} - r_{23} \\
r_{13} - r_{31} \\
r_{21} - r_{12}
\end{bmatrix}
\]
Axis/Angle Representation

The above axis/angle representation characterizes a given rotation by four quantities, namely the three components of the equivalent axis $\mathbf{k}$ and the equivalent angle $\theta$. However, since the equivalent axis $\mathbf{k}$ is given as a unit vector only two of its components are independent. The third is constrained by the condition that $\mathbf{k}$ is of unit length. Therefore, only **three independent quantities** are required in this representation of a rotation $\mathbf{R}$. We can represent the equivalent axis/angle by a single vector $\mathbf{r}$ as:

$$
\mathbf{r} = (r_x, r_y, r_z)^T = (\theta k_x, \theta k_y, \theta k_z)^T
$$

since $\mathbf{k}$ is a unit vector, the length of the vector $\mathbf{r}$ is the equivalent angle $\theta$ and the direction of $\mathbf{r}$ is the equivalent axis $\mathbf{k}$. 
Rigid Motions

A rigid motion is a pure translation together with a pure rotation.

A rigid motion is an ordered pair $(d, R)$ where $d \in \mathbb{R}^3$ and $R \in SO(3)$. The group of all rigid motions is known as the **Special Euclidean Group** and is denoted by $SE(3)$. We see then that $SE(3) = \mathbb{R}^3 \times SO(3)$.
One Rigid Motion

If frame $o_1 \ x_1 \ y_1 \ z_1$ is obtained from frame $o_0 \ x_0 \ y_0 \ z_0$ by first applying a rotation specified by $R_1^0$.
One Rigid Motion

If frame $o_1x_1y_1z_1$ is obtained from frame $o_0x_0y_0z_0$ by first applying a rotation specified by $R^0_1$ followed by a translation given (with respect to $o_0x_0y_0z_0$) by $d^0_1$.
If frame $o_1 x_1 y_1 z_1$ is obtained from frame $o_0 x_0 y_0 z_0$ by first applying a rotation specified by $R^0_1$ followed by a translation given (with respect to $o_0 x_0 y_0 z_0$) by $d^0_1$. 
One Rigid Motion

If frame $o_1 x_1 y_1 z_1$ is obtained from frame $o_0 x_0 y_0 z_0$ by first applying a rotation specified by $R_1^0$ followed by a translation given (with respect to $o_0 x_0 y_0 z_0$) by $d_1^0$, then the coordinates $p^0$ are given by:
One Rigid Motion

If frame $o_1 x_1 y_1 z_1$ is obtained from frame $o_0 x_0 y_0 z_0$ by first applying a rotation specified by $R_1^0$ followed by a translation given (with respect to $o_0 x_0 y_0 z_0$) by $d_1^0$, then the coordinates $p^0$ are given by:

$$p^0 = R_1^0 \ p^1 + d_1^0$$
One Rigid Motion

If frame \( o_1 x_1 y_1 z_1 \) is obtained from frame \( o_0 x_0 y_0 z_0 \) by first applying a rotation specified by \( R_1^0 \) followed by a translation given (with respect to \( o_0 x_0 y_0 z_0 \)) by \( d_1^0 \), then the coordinates \( p^0 \) are given by:

\[
p^0 = R_1^0 \; p^1 + d_1^0
\]
Two Rigid Motions

If frame $o_2 x_2 y_2 z_2$ is obtained from frame $o_1 x_1 y_1 z_1$ by first applying a rotation specified by $R^1_2$ followed by a translation given (with respect to $o_1 x_1 y_1 z_1$) by $d^1_2$. If frame $o_1 x_1 y_1 z_1$ is obtained from frame $o_0 x_0 y_0 z_0$ by first applying a rotation specified by $R^0_1$ followed by a translation given (with respect to $o_0 x_0 y_0 z_0$) by $d^0_1$, find the coordinates $p^0$.

For the first rigid motion:

$$p^0 = R^0_1 p^1 + d^0_1$$

For the second rigid motion:

$$p^1 = R^1_2 p^2 + d^1_2$$

Both rigid motions can be described as one rigid motion:

$$p^0 = R^0_2 R^1_2 p^2 + R^0_1 d^1_2 + d^0_1$$
Two Rigid Motions

The orientation transformations can simply be multiplied together.

The translation transformation is the sum of:
- \( d^0_1 \) the vector from the origin \( o_0 \) to the origin \( o_1 \) expressed with respect to \( o_0 x_0 y_0 z_0 \).
- \( R^0_1 d^1_2 \) the vector from \( o_1 \) to \( o_2 \) expressed in the orientation of the coordinate system \( o_0 x_0 y_0 z_0 \).

\[
p^0 = R^0_1 R^1_2 p^2 + R^0_1 d^1_2 + d^0_1
\]
Three Rigid Motions

\[ p^0 = ? \]
Homogeneous Transformations

A long sequence of rigid motions, find $p^0$.

$$p^0 = R_n^0 p^n + d_n^0$$

$\textbf{p}^0 = ?$
Homogeneous Transformations

A long sequence of rigid motions, find \( p^0 \).

\[
p^0 = R_n^0 \ p^n + d_n^0
\]

Represent rigid motions in **matrix** so that composition of rigid motions can be reduced to matrix multiplication as was the case for composition of rotations.

\[
p^0 = ?
\]
Homogeneous Transformations

A long sequence of rigid motions, find $p^0$.

\[ p^0 = R_n^0 \ p^n + d_n^0 \]

Represent rigid motions in matrix so that composition of rigid motions can be reduced to matrix multiplication as was the case for composition of rotations.

\[ H = \begin{bmatrix} R & d \end{bmatrix} ; \quad d \in \mathbb{R}^3, \ R \in SO(3) \]
Homogeneous Transformations

A long sequence of rigid motions, find $p^0$. 

$$p^0 = R_n^0 p^n + d_n^0$$

Represent rigid motions in matrix so that composition of rigid motions can be reduced to matrix multiplication as was the case for composition of rotations.

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} ; \quad d \in \mathbb{R}^3, \quad R \in SO(3)$$

Transformation matrices of the form $H$ are called homogeneous transformations.

A homogeneous transformation is therefore a matrix representation of a rigid motion.
Homogeneous Transformations

A long sequence of rigid motions, find $p^0$.

$$p^0 = R_n^0 p^n + d_n^0$$

Represent rigid motions in \textbf{matrix} so that composition of rigid motions can be reduced to matrix multiplication as was the case for composition of rotations.

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}; \quad d \in \mathbb{R}^3, \quad R \in SO(3)$$

The inverse transformation $H^{-1}$ is given by

$$H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}$$
Ex. : Two Rigid Motions

\[ R^0_2 \] The orientation transformations can simply be multiplied together.

\[ d^0_2 \] The translation transformation is the sum of:

- \[ d^0_1 \] the vector from the origin \( o_0 \) to the origin \( o_1 \) expressed with respect to \( o_0 \ x_0 \ y_0 \ z_0 \).
- \( R^1_1 \ d^1_2 \) the vector from \( o_1 \) to \( o_2 \) expressed in the orientation of the coordinate system \( o_0 \ x_0 \ y_0 \ z_0 \).

\[ p^0 = R^0_1 \ R^1_2 \ p^2 + R^0_1 \ d^1_2 + d^0_1 \]

\[ H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} ; \ d \in \mathbb{R}^3, \ R \in SO(3) \]
Ex. : Two Rigid Motions

\( R^0_2 \) The orientation transformations can simply be multiplied together.

\( d^0_2 \) The translation transformation is the sum of:
- \( d^0_1 \) the vector from the origin \( o_0 \) to the origin \( o_1 \) expressed with respect to \( o_0 x_0 y_0 z_0 \).
- \( R^0_1 \ d^1_2 \) the vector from \( o_1 \) to \( o_2 \) expressed in the orientation of the coordinate system \( o_0 x_0 y_0 z_0 \).

\[
p^0 = R^0_1 \ R^1_2 \ p^2 + R^0_1 \ d^1_2 + d^0_1
\]

\[
H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}; \ d \in \mathbb{R}^3, R \in SO(3)
\]

\[
\begin{bmatrix} R^0_1 & d^0_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R^1_2 & d^1_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R^0_1 & R^1_2 \\ 0 & 1 \end{bmatrix}
\]
Ex. : Two Rigid Motions

$R_2^0$ The orientation transformations can simply be multiplied together.

$d_2^0$ The translation transformation is the sum of:

- $d_1^0$ the vector from the origin $o_0$ to the origin $o_1$ expressed with respect to $o_0 x_0 y_0 z_0$.
- $R_1^0 d_2^1$ the vector from $o_1$ to $o_2$ expressed in the orientation of the coordinate system $o_0 x_0 y_0 z_0$.

$$p^0 = R_1^0 R_2^1 p^2 + R_1^0 d_2^1 + d_1^0$$

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}; \quad d \in \mathbb{R}^3, R \in SO(3)$$

$$\begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1^0 & R_2^1 & R_1^0 d_2^1 + d_1^0 \\ 0 & 1 \end{bmatrix}$$
**Ex. : Two Rigid Motions**

\( R^0_2 \) The orientation transformations can simply be multiplied together.

\( d^0_2 \) The translation transformation is the sum of:

- \( d^0_1 \) the vector from the origin \( o_0 \) to the origin \( o_1 \) expressed with respect to \( o_0 \) \textbf{x} \textbf{y} \textbf{z}.
- \( R^0_1 \ d^1_2 \) the vector from \( o_1 \) to \( o_2 \) expressed in the orientation of the coordinate system \( o_0 \) \textbf{x} \textbf{y} \textbf{z}.

\[
p^0 = R^0_1 \ R^1_2 \ p^2 + R^0_1 \ d^1_2 + d^0_1
\]

\[
H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}; \ d \in \mathbb{R}^3, R \in SO(3)
\]

\[
\begin{bmatrix} R^0_1 & d^0_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R^1_2 & d^1_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R^0_1 & R^1_2 & R^0_1 \ d^1_2 + d^0_1 \\ 0 & 1 \end{bmatrix}
\]

We must augment the vectors \( p^0, p^1 \) and \( p^2 \) by the addition of a fourth component of 1:

\[
p^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix}, \ p^1 = \begin{bmatrix} p^1 \\ 1 \end{bmatrix}, \ p^2 = \begin{bmatrix} p^2 \\ 1 \end{bmatrix}
\]
Homogeneous Transformations

\[ P^0 = H^0_1 P^1 \]
\[ P^0 = H^0_2 P^2 \]
\[ \ldots \]
\[ P^0 = H^0_n P^n \]

\[ P^0 = \begin{bmatrix} p^0 \end{bmatrix} \]
Basic Homogeneous Transformations

\[
\text{Trans}_{x,a} = \begin{bmatrix}
1 & 0 & 0 & a \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\text{Rot}_{x,\alpha} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & c_\alpha & -s_\alpha & 0 \\
0 & s_\alpha & c_\alpha & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\text{Trans}_{y,b} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & b \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\text{Rot}_{y,\beta} = \begin{bmatrix}
c_\beta & 0 & s_\beta & 0 \\
0 & 1 & 0 & 0 \\
-s_\beta & 0 & c_\beta & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\text{Trans}_{z,c} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & c \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\text{Rot}_{z,\gamma} = \begin{bmatrix}
c_\gamma & -s_\gamma & 0 & 0 \\
s_\gamma & c_\gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
Homogeneous Transformations

\[ H_1^0 = \begin{bmatrix} n_x & s_x & a_x & d_x \\ n_y & s_y & a_y & d_y \\ n_z & s_z & a_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n & s & a & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\( n \) is a vector representing the direction of \( x_1 \) in the \( o_0 \, x_0 \, y_0 \, z_0 \) system

\( s \) is a vector representing the direction of \( y_1 \) in the \( o_0 \, x_0 \, y_0 \, z_0 \) system

\( a \) is a vector representing the direction of \( z_1 \) in the \( o_0 \, x_0 \, y_0 \, z_0 \) system
Composition Rule For Homogeneous Transformations

Given a homogeneous transformation $H_1^0$ relating two frames, if a second rigid motion, represented by $H$ is performed relative to the current frame, then:

$$H_2^0 = H_1^0 H$$

whereas if the second rigid motion is performed relative to the fixed frame, then:

$$H_2^0 = H H_1^0$$
Example

Find $H$ for the following sequence of

1. a rotation by $\alpha$ about the current $x-axis$, followed by
2. a translation of $b$ units along the current $x-axis$, followed by
3. a translation of $d$ units along the current $z-axis$, followed by
4. a rotation by angle $\Theta$ about the current $z-axis$

$$H =$$

Reminder:

Transformation with respect to the current frame
$$H_2^0 = H_1^0 H$$

Transformation with respect to the fixed frame
$$H_2^0 = HH_1^0$$
Example

Find $H$ for the following sequence of

1. a rotation by $\alpha$ about the current $x$ - axis, followed by
2. a translation of $b$ units along the current $x$ - axis, followed by
3. a translation of $d$ units along the current $z$ - axis, followed by
4. a rotation by angle $\Theta$ about the current $z$ - axis

$$H = Rot_{x,\alpha}$$

Reminder:

Transformation with respect to the current frame
$$H_{2}^{0} = H_{1}^{0} H$$

Transformation with respect to the fixed frame
$$H_{2}^{0} = HH_{1}^{0}$$
Example

Find $H$ for the following sequence of

1. a rotation by $\alpha$ about the current $x$—axis, followed by
2. a translation of $b$ units along the current $x$—axis, followed by
3. a translation of $d$ units along the current $z$—axis, followed by
4. a rotation by angle $\Theta$ about the current $z$—axis

$$H = \text{Rot}_{x, \alpha} \text{Trans}_{x, b}$$

Reminder:

Transformation with respect to the current frame

$$H_2^0 = H_1^0 H$$

Transformation with respect to the fixed frame

$$H_2^0 = H H_1^0$$
Example

Find $H$ for the following sequence of

1. a rotation by $\alpha$ about the current $x - axis$, followed by
2. a translation of $b$ units along the current $x - axis$, followed by
3. a translation of $d$ units along the current $z - axis$, followed by
4. a rotation by angle $\Theta$ about the current $z - axis$

$$H = Rot_{x, \alpha} \ Trans_{x, b} \ Trans_{z, d}$$

Reminder:

Transformation with respect to the current frame

$$H_2^0 = H_1^0 \ H$$

Transformation with respect to the fixed frame

$$H_2^0 = H \ H_1^0$$
Example

Find $H$ for the following sequence of

1. a rotation by $\alpha$ about the current $x$–axis, followed by
2. a translation of $b$ units along the current $x$–axis, followed by
3. a translation of $d$ units along the current $z$–axis, followed by
4. a rotation by angle $\Theta$ about the current $z$–axis

$$H = \text{Rot}_{x, \alpha} \text{Trans}_{x, b} \text{Trans}_{z, d} \text{Rot}_{z, \Theta}$$

Reminder:

Transformation with respect to the current frame

$$H_2^0 = H_1^0 H$$

Transformation with respect to the fixed frame

$$H_2^0 = H H_1^0$$
Example

Find $H$ for the following sequence of

1. a rotation by $\alpha$ about the current $x$ – axis, followed by
2. a translation of $b$ units along the current $x$ – axis, followed by
3. a translation of $d$ units along the current $z$ – axis, followed by
4. a rotation by angle $\Theta$ about the current $z$ – axis

$H = \text{Rot}_{x,\alpha} \text{Trans}_{x,b} \text{Trans}_{z,d} \text{Rot}_{z,\theta}$

$= \begin{bmatrix}
c_\theta & -s_\theta & 0 & b \\
c_\alpha s_\theta & c_\alpha c_\theta & -s_\alpha & -d s_\alpha \\
s_\alpha s_\theta & s_\alpha c_\theta & c_\alpha & d c_\alpha \\
0 & 0 & c_\alpha & 1
\end{bmatrix}$

Reminder:

Transformation with respect to the current frame

$H^0_2 = H^0_1 H$

Transformation with respect to the fixed frame

$H^0_2 = H H^0_1$
Example

Find the homogeneous transformations $H_1^0, H_2^0, H_2^1$ representing the transformations among the three frames shown. Show that $H_2^0 = H_1^0 H_2^1$.