Implementations in form of Vector Symbolic Architectures

1. Possible vector spaces
2. Implementation of the necessary operations
3. Practical Part with experiments

Smolenskys [1] Tensor Product as one of the first VSA

u

$|u| = 1$

$$\begin{pmatrix} 0.34 \\ 0.41 \\ 0.28 \\ 0.01 \\ 0.36 \\ . \\ . \\ . \\ . \\ 0.40 \\ 0.29 \\ 0.32 \\ 0.32 \\ 0.17 \end{pmatrix}$$

Bundling:

→ *element-wise addition and normalization*

Binding:

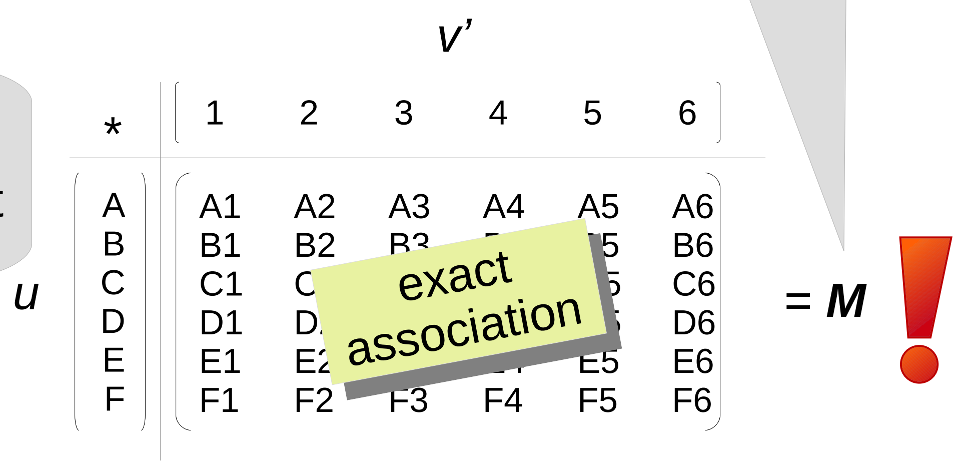
→ *outer product*

Need for an unbind operator - some VSAs are not self-inverse

Unbinding:

→ *normalized inner product*

We want an output space as the input → a **vector**



$$unbind(v, \boldsymbol{M}) = \frac{\boldsymbol{M} \cdot v}{|v| \cdot |u|} = \boldsymbol{M} \cdot v$$

exact recovering

[1] P. Smolensky. Tensor product variable binding and the representation of symbolic structures in connectionist systems. Artif. Intell., 46(1-2):159–216, 1990.

1. Different Vector Spaces

Bipolar	**Binary**			**Real**	**Complex**
[-1,1]	{-1,1}	{0, 1}	sparse	$\sim\mathcal{N}(0,1/D)$	$\measuredangle \in [-\pi,\pi]$
0.62	-1	0	1	-0.43	$e^{-2.65\,i}$
0.81	-1	1	0	1.13	$e^{-0.36\,i}$
-0.74	1	1	0	0.54	$e^{-2.47\,i}$
0.82	1	0	0	0.42	$e^{2.90\,i}$
0.26	-1	1	0	-0.02	$e^{-3.11\,i}$
-0.80	1	0	0	0.26	$e^{1.72\,i}$
-0.44	-1	0	0	-1.83	$e^{1.99\,i}$
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
0.94	-1	0	0	-1.23	$e^{1.88\,i}$
0.91	1	1	0	-0.46	$e^{-0.43\,i}$
-0.02	-1	0	0	1.27	$e^{2.58\,i}$
0.60	1	1	0	1.32	$e^{-1.99\,i}$
-0.71	-1	0	1	1.83	$e^{-1.48\,i}$
-0.15	1	1	0	-1.32	$e^{-2.22\,i}$

Bipolar [2], [3] | Binary | Real | Complex

[-1,1]	{-1,1}
0.62	-1
0.81	-1
-0.74	1
0.82	1
0.26	-1
-0.80	1
-0.44	-1
.	.
.	.
.	.
.	.
0.94	-1
0.91	1
[illegible]	-1
-0.15	1

approx. self-inverse

exact self-inverse

How can we implement a <u>bundling</u> operator?
The properties should be:

- output is similar to its inputs

→ *element-wise addition and transfer to vector space*

How can we implement a <u>binding</u> and <u>unbinding</u> operator?
The properties should be:

- output of binding is dissimilar to its inputs
- unbinding recovers the original vectors

→ *element-wise multiplication for both*

How can we implement a <u>similarity</u> metrics?

→ *cosine angle*

[2] Ross W. Gayler. Multiplicative binding, representation operators, and analogy. In Advances in analogy research: Integr. of theory and data from the cogn., comp., and neural sciences, Bulgaria, 1998
[3] Ross W. Gayler and Simon D. Levy. A distributed basis for analogical mapping. New Frontiers in Analogy Research, Proceedings of the Second International Conference on Analogy, ANALOGY-2009, pages 165–174, 2009.

Bipolar [2], [3] | Binary | Real | Complex

[-1,1]	{-1,1}
0.62	-1
0.81	-1
-0.74	1
0.82	1
0.26	-1
-0.80	1
-0.44	-1
.	.
.	.
.	.
.	.
0.94	-1
0.91	1
-0.0?	-1
-0.15	1

approx. self-inverse

exact self-inverse

Remember the Tensor-product:
element-wise multiplication
is the diagonal of the outer product

A1	A2	A3	A4	A5	A6
B1	**B2**	B3	B4	B5	B6
C1	C2	**C3**	C4	C5	C6
D1	D2	D3	**D4**	D5	D6
E1	E2	E3	E4	**E5**	E6
F1	F2	F3	F4	F5	**F6**

How can we implement a bundling operator?
The properties should be:
... inputs
... *nd transfer to vector space*
...w simple it is ...
...**inding operator?**
...similar to its inputs
unbinding re... the original vectors
→ *element-wise multiplication for both*

How can we implement a similarity metrics?

→ *cosine angle*

[2] Ross W. Gayler. Multiplicative binding, representation operators, and analogy. In Advances in analogy research: Integr. of theory and data from the cogn., comp., and neural sciences, Bulgaria, 1998
[3] Ross W. Gayler and Simon D. Levy. A distributed basis for analogical mapping. New Frontiers in Analogy Research, Proceedings of the Second International Conference on Analogy, ANALOGY-2009, pages 165–174, 2009.

Bipolar | **Binary** [4], [5] | Real | Complex

{0, 1}	sparse
0	1
1	0
1	0
0	0
1	0
0	0
0	0
.	.
.	.
.	.
.	.
0	0
1	0
0	0
1	0
0	1
1	0

Implementation of the bundling?

→ *element-wise addition and thresholding* | → *element-wise OR*

Implementation of the binding?

→ *element-wise XOR* | → *shifting (or CDT)*

Implementation of the unbinding?

→ *element-wise XOR* (exact self-inverse) | → *shifting* (exact inverse)

Implementation of the similarity metrics?

→ *Hamming distance (inverse)* | → *overlap*

[4] Pentti Kanerva. Fully distributed representation. In Proc. of Real World Computing Symposium, pages 358–365, Tokyo, Japan, 1997.
[5] D. A. Rachkovskij. Representation and processing of structures with binary sparse distributed codes. IEEE Transactions on Knowledge and Data Engineering, 13(2):261–276, 2001.

Bipolar | Binary | **Real** [6] | Complex

~(0,1/D)

-0.43
1.13

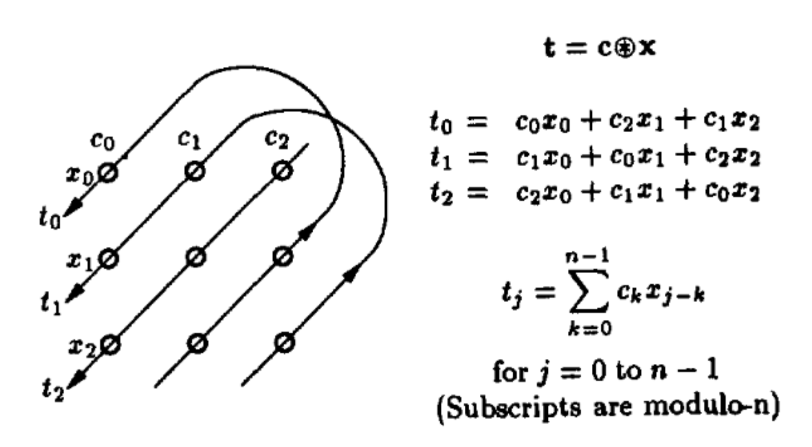
Implementation of the bundling?

→ *element-wise addition and normalize to length of one*

Implementation of the binding?

→ *circular convolution*

Visualization from [6]



$$t_j = \sum_{k=0}^{n-1} c_k x_{j-k}$$

Implementation of the unbinding?

→ *circular correlation*

Approx. inverse

Implementation of the similarity metrics?

→ *dot product*

1.32
1.83
-1.32

[6] Tony A. Plate. Holographic Reduced Representations. IEEE Transactions on Neural Networks, 6(3):623–641, 1995

Bipolar | Binary | Real | **Complex** [7]

$\measuredangle \in [-\pi, \pi]$

$$\begin{pmatrix} e^{-2.65 i} \\ e^{-0.36 i} \\ e^{-2.47 i} \\ e^{2.90 i} \\ e^{-3.11 i} \\ e^{1.72 i} \\ e^{1.99 i} \\ . \\ . \\ . \\ . \\ e^{1.88 i} \\ e^{-0.43 i} \\ e^{2.58 i} \\ e^{-1.99 i} \\ e^{-1.48 i} \\ e^{-2.22 i} \end{pmatrix}$$

Implementation of the bundling?

→ *element-wise addition*

Implementation of the binding?

→ *element-wise multiplication*

Implementation of the unbinding?

→ *element-wise multiplication with conjugated values*

exact inverse

Implementation of the similarity metrics?

→ sum of cosine of angle differences

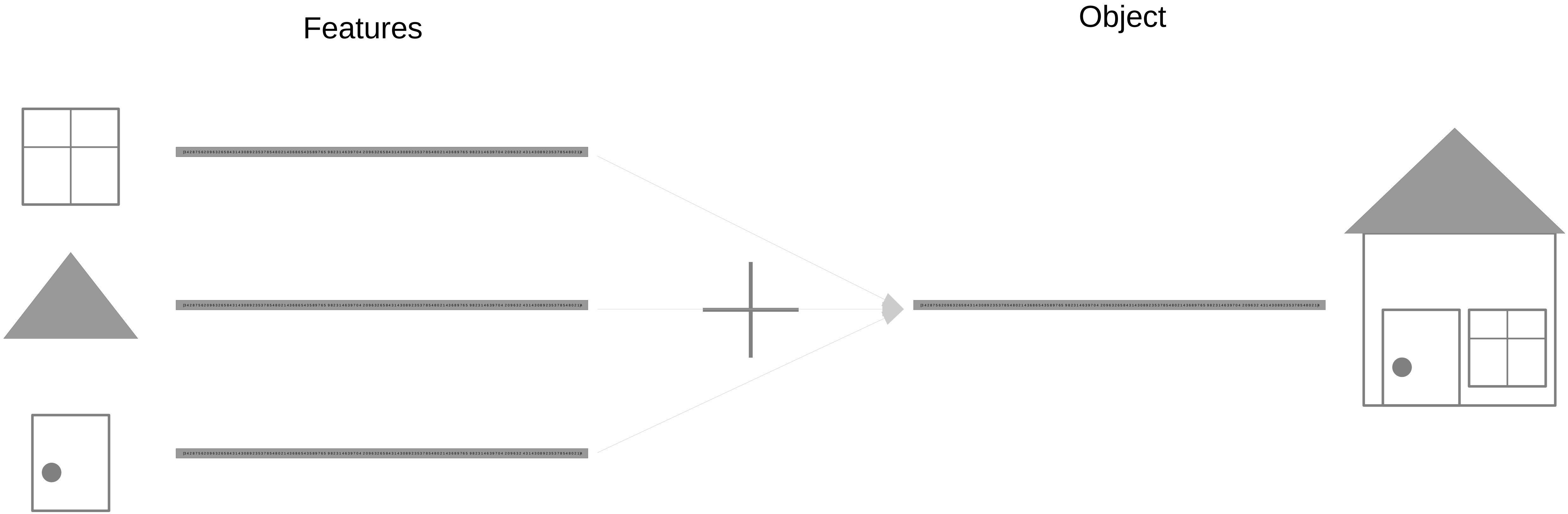
[7] Tony Alexander Plate. Distributed Representations and Nested Compositional Structure. PhD thesis, Toronto, Ont., Canada, 1994.

Questions to be answered:

a) ***How efficiently can the different VSAs store information into one representation?*** **(bundling)**

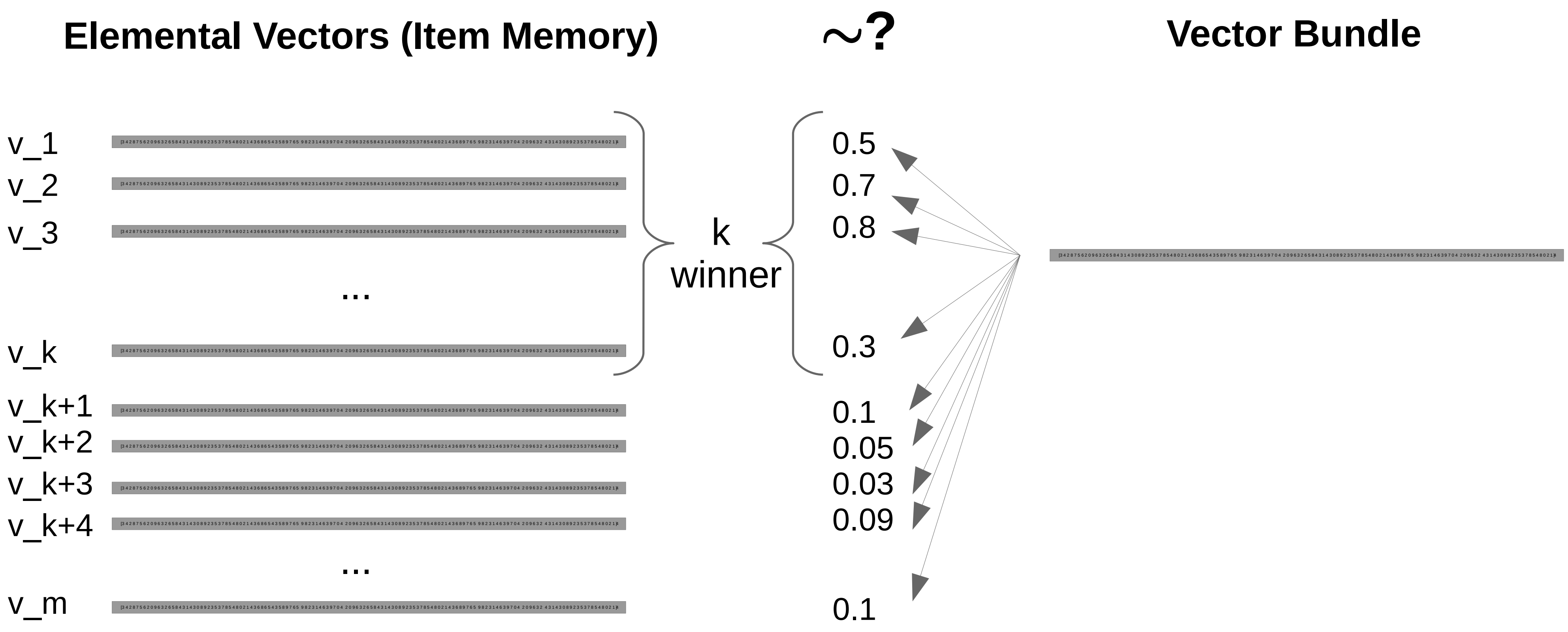
b) ***How much is the binding-unbinding disturbed by bundling (noise)?***

a) Overlaying given vectors

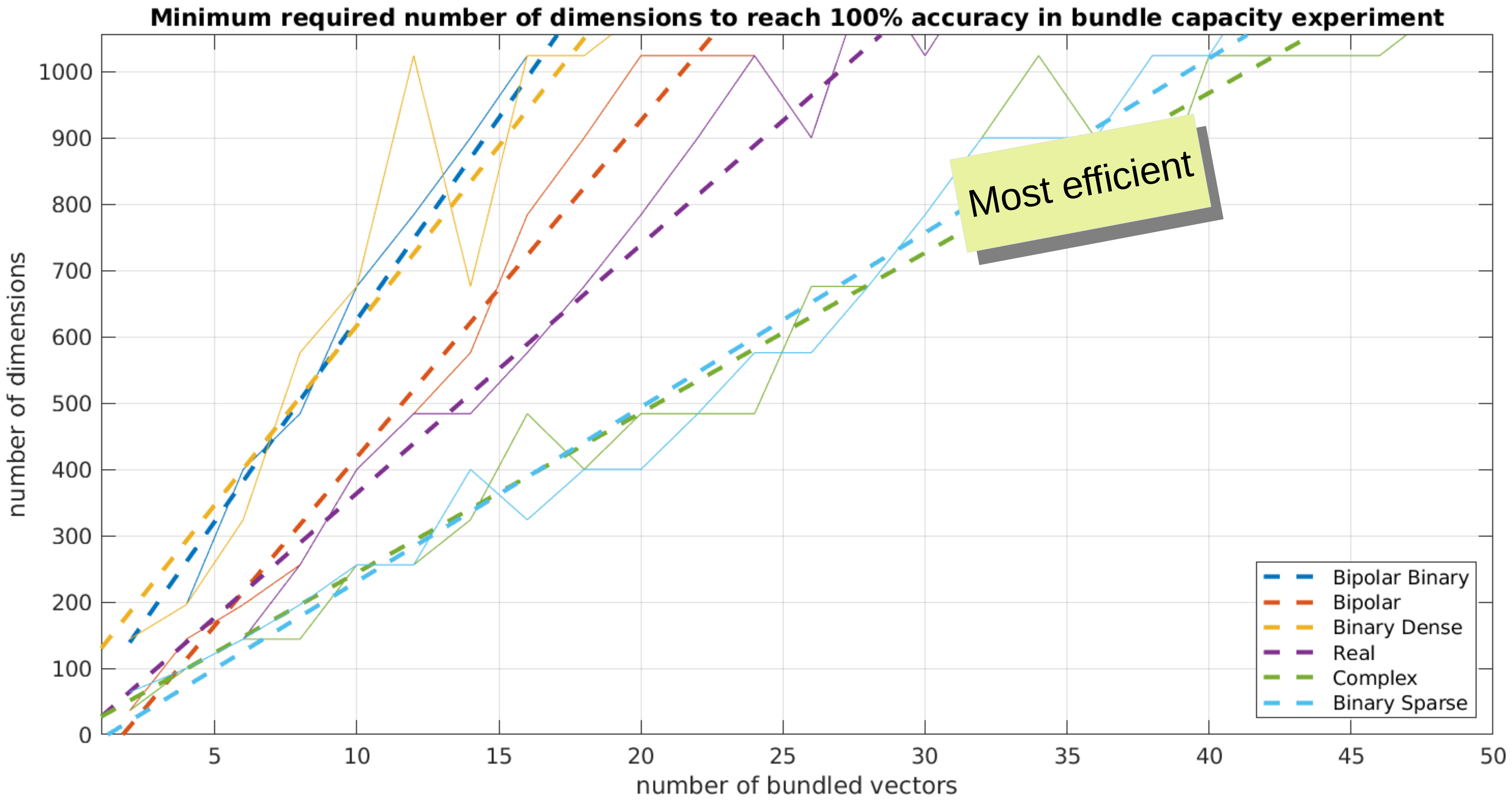


Question: ***How efficiently can the different VSAs store (bundle) information into one representation?***

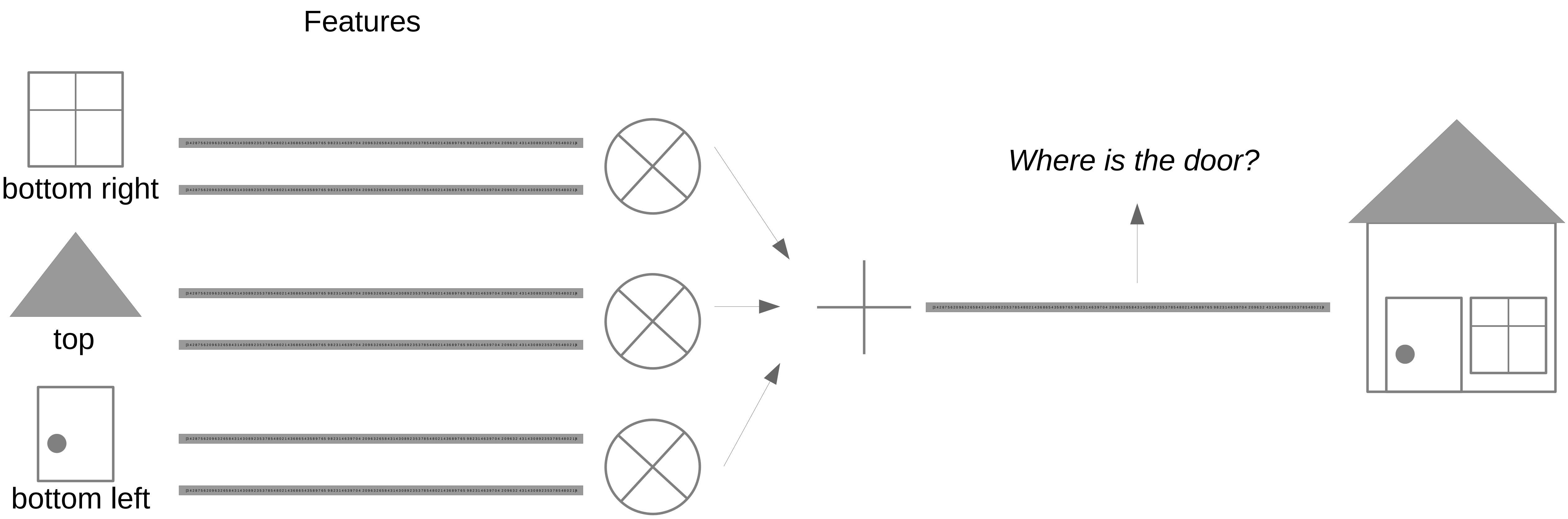
a) Overlaying given vectors – the experiment



a) Overlaying of given vectors – the results



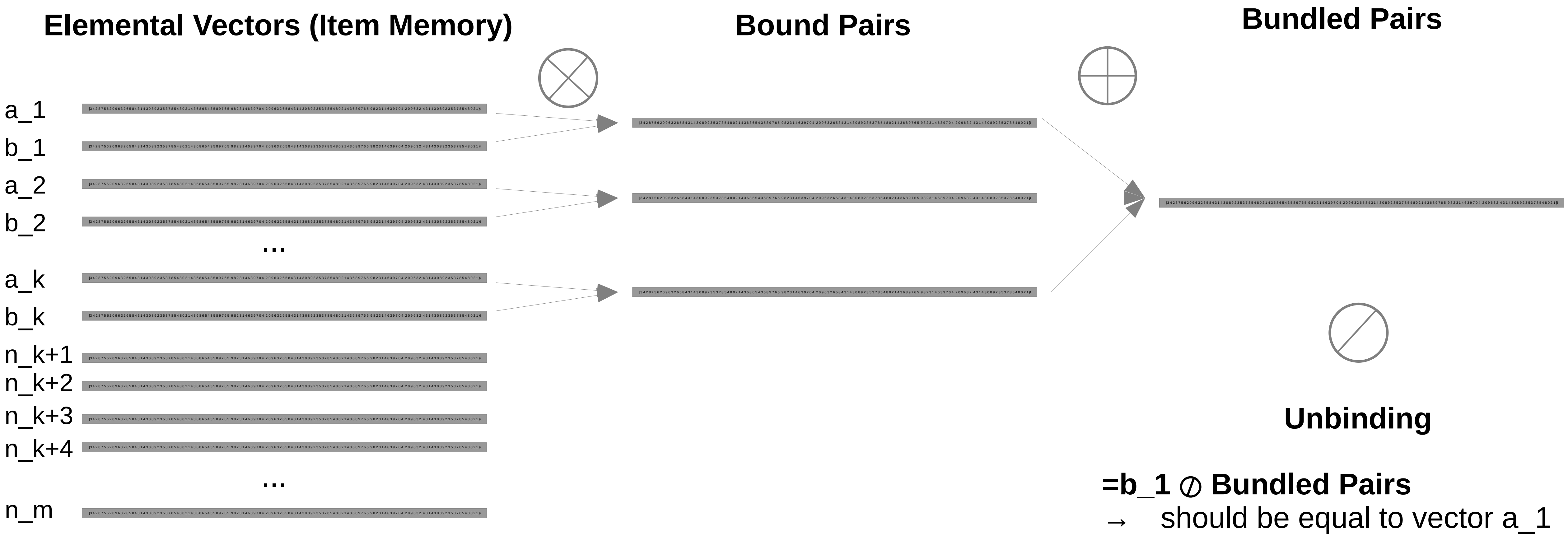
b) Binding and Bundling (as in [8])



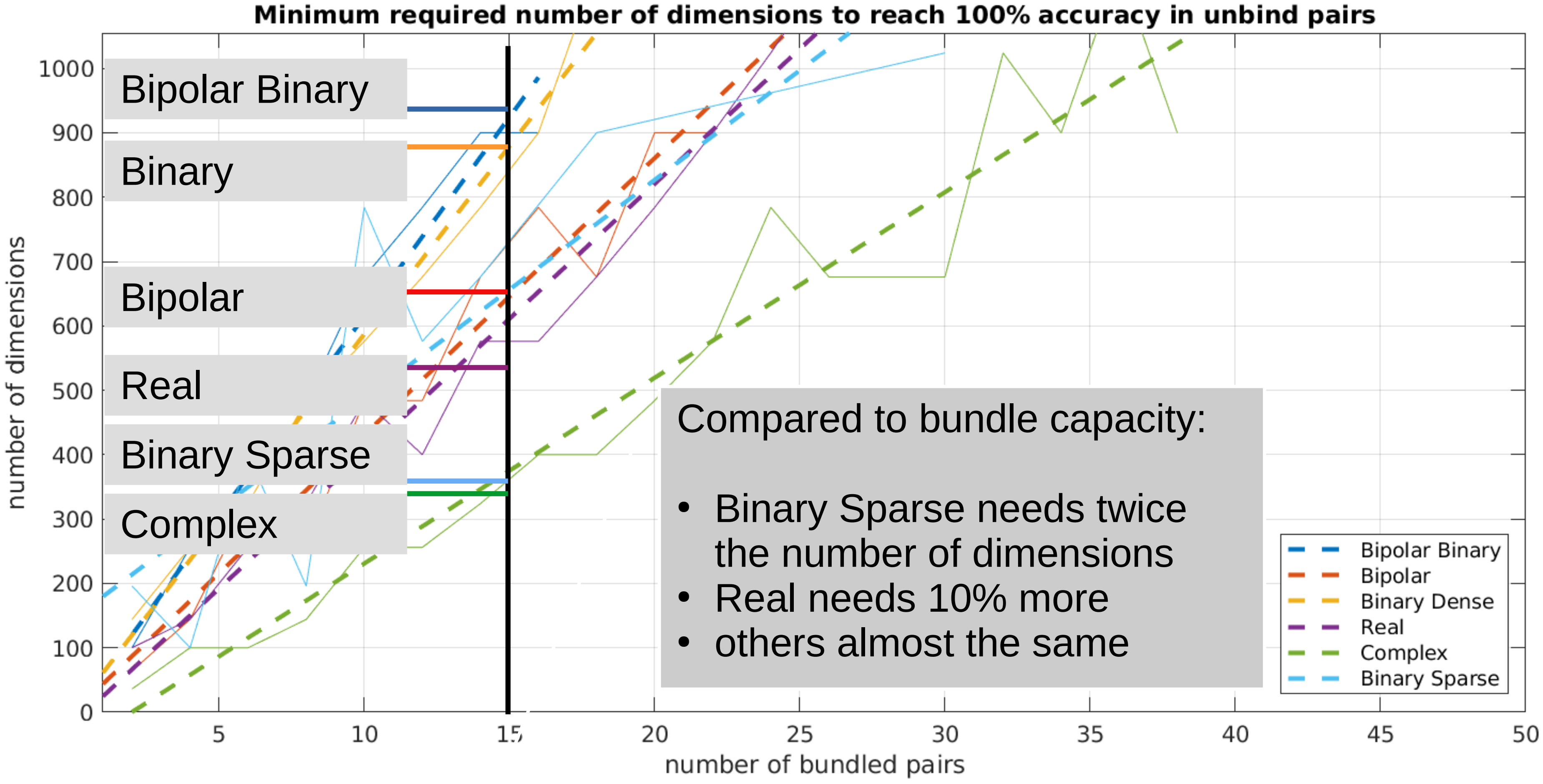
Question: ***How much is the binding-unbinding disturbed by bundling?***

[8] Chris Eliasmith. How to Build a Brain: A Neural Architecture for Biological Cognition. Oxford Univ. Press, 2013

b) Binding and Bundling – the experiment



b) Binding and bundling together – the results



Compared to bundle capacity:

- Binary Sparse needs twice the number of dimensions
- Real needs 10% more
- others almost the same

Summary

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0.82	1	0	0	0.42	$e^{2.90 i}$
0.26	-1	1			
-0.80	1	0			
-0.44	-1	0	0	-1.83	$e^{1.99 i}$
.	.	.		.	
.	.	.		.	
.	.	.	.	.	.
				.	
				-1.23	
				-0.46	
-0.02	-1	0	0	1.27	$e^{2.58 i}$
0.60				1.32	$e^{-1.99 i}$
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-0.15	1	1	0	-1.32	$e^{-2.22 i}$

- self-inverse (Bipolar, Binary)
- robust binding-unbinding (Bipolar, Binary)
- energy efficient ({-1,1}, {0, 1})
- non-commutative unbinding (sparse, Real, Complex)
- efficient bundling (sparse)
- less robust unbinding (sparse)
- Highly energy efficient (sparse)
- efficient bundling (Complex)
- robust binding-unbinding (Complex)

Summary

Bipolar		Binary		Real	Complex
[-1,1]	{-1,1}	{0, 1}	sparse	~(0,1/D)	$\sphericalangle \in [-\pi,\pi]$
0.62	1	0	1		
			0		
0.82	1	0	0		
0.26	-1	1			
-0.80	1	0			
-0.44	-1	0	0		
.	.	.			
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.	.	.	.	.	.
				.	
				-1.23	
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-0.02	-1	0	0	1.27	$e^{2.58 i}$
0.60				1.32	$e^{-1.99 i}$
-0.71				1.83	$e^{-1.48 i}$
-0.15	1	1	0	-1.32	$e^{-2.22 i}$

self-inverse

efficient bundling

bundling

robust binding-unbinding

less robust unbinding

robust binding-unbinding

energy efficient

Highly energy efficient

Remember the "Dollar of Mexico" Example:

Representation of *USA* and *Mexico* is ***F***

Query:

$Dol \otimes \mathbf{F} = \ldots\ Dol \otimes Dol \otimes Pes \ldots$