Statistical Assessment of Eigenvector-Based Target Decomposition Theorems in Radar Polarimetry

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Abstract—This paper concerns the analytical study of the eigen decomposition of hermitian, positive semidefinite matrices applied to PolSAR data analysis. Based on the Gaussian scattering assumption for multidimensional SAR data, the joint distribution of the sample eigenvalues of the coherency, or covariance, matrices is derived for a general case. The distribution is particularized for PolSAR data, and the moments of the sample eigenvalues, the entropy (H), and the anisotropy (A) are analyzed.

SAR Polarimetry, target decomposition theorems, speckle noise, statistical characterization.

I. INTRODUCTION

Nowadays, radar polarimetry has become a valuable technique in quantitative remote sensing applications. This importance lies in the possibility to link the microwave being scattered by a given target with its physical properties. The step to pass from recorded data to physical information is performed by the so-called Target Decomposition (TD) theorems [1]. In the study and characterization of natural targets from polarimetric data, TD theorems based on the eigen decomposition of the coherency matrix T have been revealed as the most suitable tool to perform data interpretation [1-2]. This type of TD theorems have been successfully employed in a broad set of applications, among which, it can be mentioned: contrast optimization, unsupervised image classification, quantitative surface scattering, landslide characterization, etc...

Radar polarimetry is usually considered in the frame of Synthetic Aperture Radar (SAR) technology as a consequence of the high spatial resolution this type of systems can achieve and, also, due to its independence from weather conditions and from the day/night cycle [3]. Nevertheless, as a consequence of the SAR’s coherent nature, recorded data are affected by speckle noise [3-5]. The presence of this noise component imposes, therefore, an estimation step, in which, the coherency matrix T, or the equivalent covariance matrix C, need to be estimated from data. Hence, one can conclude that speckle noise shall also affect the eigen decomposition of T, and the information extracted from it.

On the one hand, speckle noise in one-dimensional SAR systems presents a multiplicative noise model [5]. On the other hand, for multidimensional SAR systems, speckle is characterized by a multiplicative/additive noise model [6]. The objective of this paper is to go a step beyond presenting, by the first time, an analytical study the effects of the speckle noise over the information extracted from T, or C, through its eigen decomposition. In the following sections, the joint probability density function (pdf) of the sample eigenvalues is obtained. Then, the pdf is employed to characterize, statistically, the sample eigenvalues, as well as, the secondary parameters: entropy (H) and anisotropy (A).

II. POLARIMETRIC SAR DATA DESCRIPTION

A polarimetric SAR (PolSAR) can be considered as a particularization of a more general multidimensional SAR system. The following development is obtained in its most general form, being particularized to the PolSAR case when necessary.

A m-dimensional SAR system acquires a set of complex SAR images, S_i for i=1..m, which can be represented by the complex target vector

\[ \mathbf{k} = [S_1, S_2, ..., S_m]^T \]  

where T indicates transpose. For distributed targets, under the Gaussian scattering assumption, \( \mathbf{k} \) is described by the pdf [7]

\[ p_k (\mathbf{k}) = \frac{1}{\pi^m} |C| \exp \left( -\mathbf{k}^H C^{-1} \mathbf{k} \right) \]  

(2)

where \( ^H \) denotes transpose complex conjugation. Consequently, \( E\{|\mathbf{k}|=0 \} \), and all the statistical behavior of \( \mathbf{k} \) is embedded within the \( m \times m \), positive semidefinite, hermitian covariance matrix C

\[ C = E\{\mathbf{kk}^H\} \]  

(3)

In order to describe PolSAR data, an alternative vectorization of the SAR images set, based on the Pauli basis, could be considered [1]. The alternative target vector \( \mathbf{k}_p \) gives rise to the coherency matrix T, in the same way \( \mathbf{C} \) is derived from \( \mathbf{k} \). \( \mathbf{C} \) and T share several properties, among them, it has to be emphasized that both present the same eigenvalues. In what it follows, a formulation based on \( \mathbf{C} \) is considered, as it simplifies the general statistical analysis of the sample eigenvalues.

Due to data variability (2), i.e., speckle noise, \( \mathbf{C} \) must be estimated from data. By assuming local homogeneity and ergodicity in mean, the expectation operator in (3) can be substituted by special averaging. Hence, the estimation of \( \mathbf{C} \) is

\[ \mathbf{Z}_s = \frac{1}{\mu} \sum_{n=1}^{\mu} \mathbf{k}_n \mathbf{k}_n^H \]  

(4)
where $n$ is the number of averaged samples of looks, $k_i$ denotes the target vector of the $i^{th}$ sample, and $Z_m$ receives the name of sample covariance matrix. $Z_m$, which consists of the Maximum Likelihood Estimator (MLE) of $C$, is a multidimensional random variable characterized by the Wishart pdf [8], for $n \geq 2m$,

$$p_{Z_m}(Z_m) = \frac{n^{m/2} |Z_m|^{-n/2}}{|\Gamma(n)|^{1/2}} \exp \left( -\frac{1}{2} tr(\mathcal{C}^{-1}Z_m) \right)$$  \hspace{1cm} (5)$$

where $tr(X)$ is the exponential of the trace of $X$, and the multivariate gamma function is defined as

$$\Gamma_m(n) = \frac{n^{m/2} \prod_{i=1}^{m} (n-i+1)}{(n-1)!}.$$  \hspace{1cm} (6)

### A. Eigenvalue/Eigenvector Decomposition

$Z_m$ can be diagonalized by means of the eigen decomposition of hermitian matrices

$$Z_m = Q\Xi Q^H$$  \hspace{1cm} (7)

where $Q$ is a unitary matrix which orthonormal columns, $q_i$, $i=1,...,m$, are the eigenvectors of $Z_m$, and

$$\Xi = \begin{bmatrix} 
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_m 
\end{bmatrix}$$  \hspace{1cm} (8)

where $\{\lambda_1,\lambda_2,\ldots,\lambda_m\}$ consists of the positive, sample eigenvalues of $Z_m$. $\{\lambda_1,\lambda_2,\ldots,\lambda_m\}$ is a $m$-dimensional random variable, which statistical behavior is completely characterized in Section III. Similarly, the diagonalization of $C$ can be defined as in (7). However, the true eigenvalues are considered now

$$C = \begin{bmatrix} 
l_1 & 0 & \cdots & 0 \\
0 & l_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & l_m 
\end{bmatrix}$$  \hspace{1cm} (9)

The true eigenvalues $\{l_1,l_2,\ldots,l_m\}$ are estimated by means of the sample eigenvalues $\{\lambda_1,\lambda_2,\ldots,\lambda_m\}$. Hence, it should be expected that $E\{\Xi\} = \Sigma$.

### III. SAMPLE EIGENVALUES JOINT DISTRIBUTION

The sample eigenvalues joint pdf, i.e., $p_{\Xi}(\Xi)$, is derived from (5), first, by obtaining the Jacobian governing (7), and, second, integrating out the dependence on $Q$. Since the eigenvector decomposition is unique up to $m$ phase components, this Jacobian is square. Consequently, the $m^2$ independent parameters of $Z_m$ are divided into the $m$ independent parameters of $\Xi$, and the corresponding $m(m-1)$ plus the $m$ phase components of $Q$.

In multidimensional statistics, the direct calculation of the Jacobian's determinant of a transformation of variables can be extremely complicated and tedious [9]. This drawback is overcome by considering the exterior product of differential forms introduced by Elie Cartan [10], who extended the concept of skew-symmetric or exterior product, first developed by Herman G. Grassmann, to the study of exterior differential forms. This theory is based on the observation that the exterior product of independent differential forms behaves as the Jacobian’s determinant. Let’s consider a transformation

$$\phi: \mathbb{R}^m \rightarrow \mathbb{R}^m$$

from the independent $m$ coordinates $\{x_1,x_2,\ldots,x_m\}$, which can be though as the parameters of $X$, to the independent ones $\{y_1,y_2,\ldots,y_m\}$, i.e., the parameters of $Y$. Hence,

$$dy = \partial_y dy = \partial_x d\phi x$$

where the symbol $\lambda$ represents the exterior product, and $d\phi x$ denotes the exterior product of the independent differential forms of $X$. In (10), $|\partial_y dy|$ can be identified, hence, as the determinant of the Jacobian of the transformation $\phi$. In the case of the eigen decomposition (7) for hermitian matrices, considering the $m$ independent parameters of $\Xi$ and the $(m-1)$ of $Q$

$$d\Xi = \prod_{i=1}^{m} (\lambda_i - \lambda_j)^2 (Q^H dQ)(d\Xi)$$  \hspace{1cm} (11)

in which

$$\infty > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0.$$  \hspace{1cm} (12)

Since $p_{X}(X) = p_{Y}(Y) dV$, introducing (11) in (5)

$$p_{\Xi\lambda}(Q\Xi\lambda^H) = \frac{n^{m/2} \prod_{i=1}^{m} (\lambda_i - \lambda_j)^2}{\prod_{i=1}^{m} (\lambda_i - \lambda_j)^2} \exp \left( -\frac{1}{2} tr(\mathcal{C}^{-1}Q\Xi\lambda^H) \right)$$  \hspace{1cm} (13)

is obtained. From (13), the marginal distribution $p_{\Xi}(\Xi)$ is derived by integrating the dependence on the $m^2$ parameters of $Q$ over the space of $m \times m$ unitary matrices, also called Unitary Group, i.e., $U(m)$.

#### A. Integral Expression

In (13), it is necessary to identify what is the term $\mathcal{C}^{H} dQ = \lambda_1^{m_1} \lambda_2^{m_2} \cdots d\lambda_i$ in order to integrate the dependence on $Q$. $\mathcal{C}^{H} dQ$ consists of an invariant differential form, i.e., a differential volume, which defines a unique measure on $U(m)$, called Haar measure. Hence, this measure allows to integrate over $U(m)$. Nevertheless, it is more convenient to define a normalized differential form $d\phi$, respect to which, the integral over all the space is equal to 1, establishing, consequently, a probability space. Hence,

$$d\phi = \frac{Q^H dQ}{Vol[U(m)]} = \frac{2^n \pi^{mn}}{\pi^{(m-1)/2} \prod_{i=1}^{m} \Gamma(n-i+1)}$$  \hspace{1cm} (14)

where $Vol[U(m)]$ defines the volume of $U(m)$. Therefore,

$$\int_{U(m)} Q^H dQ = Vol[U(m)] \int_{U(m)} d\phi = 1.$$  \hspace{1cm} (15)

Finally, introducing (14) in (13), and considering (6), the integral expression of the sample eigenvalues pdf is obtained as

$$p_{\Xi}(\Xi) = \frac{n^{m-n} \pi^{m-n} \prod_{i=1}^{m} (\lambda_i - \lambda_j) \int \exp \left( -\frac{1}{2} tr(\mathcal{C}^{-1}Q\Xi\phi) \right) }{\prod_{i=1}^{m} \Gamma(n) \prod_{i=1}^{m} \Gamma(n-i+1) \int \exp \left( -\frac{1}{2} tr(\mathcal{C}^{-1}Q\Xi\phi) \right) }.$$  \hspace{1cm} (16)

#### B. Infinite Series Expression

Equation (16) is not useful for practical purposes due to the presence of the multidimensional integral over $U(m)$. It must be taken into consideration that the integral term in (16) is not with respect to the real and imaginary parts of the entries of $Q$, but with respect to its independent parameters. Consequently, it
can not be solved by a simple term-by-term integration. Fortunately, the integral in (16) can be obtained in the frame of the Group Representation [11] and the Complex Zonal Polynomials [12] theories. Formally, a representation of a group $G$ is a group action of $G$ on a vector space $V$ by invertible linear maps. On the one hand, $G$ is considered, here, as the general linear group of $m \times m$ matrices with complex entries, i.e., $GL(m, \mathbb{C})$. On the other hand, $V$ is considered as the space $V_i$ of polynomials $p(S)$ of degree $k$, which are homogeneous in the entries of a $m \times m$, hermitian, positive definite matrix $S$. The interesting result behind the representation of $GL(m, \mathbb{C})$ on $V_i$ is that it induces a decomposition of $V_i$ into a set of irreducible invariant subspaces $V$

$$V_i = \bigoplus V$$

where $(k, k, ..., k)$ runs through all the partitions of $k$ into at most $m$ parts. According to a theorem of Cartan, each $V$ will contain a 1-dimensional subspace of polynomials invariant under the unitary group $U(m)$. A polynomial $\tilde{C} (S)$, which generates this subspace, is called the complex zonal polynomial of the representation [12]. $\tilde{C} (S)$ is a symmetric, homogeneous polynomial of degree $k$ in the eigenvalues of $S$; equivalently, $\tilde{C} (QSO^k) = \tilde{C} (S)$ for $Q \in U(m)$. Then, the polynomial $(tr S)^k \in V_i$ has a unique decomposition

$$\tilde{C} (S) \in V_i$$

into the polynomials $\tilde{C} (S) \in V_i$. Since any real analytic function $f(S)$, invariant under the congruence transformation of $GL(m, \mathbb{C})$, has a power series expansion, i.e., a Taylor decomposition, then, $f(S)$ can be decomposed into the different irreducible invariant spaces $V$. In other words, $f(S)$ can be decomposed in a power series expansion determined by zonal polynomials

$$f(S) = \sum_{i=0}^{\infty} \sum_{\lambda} c_i \tilde{C}_i (S)$$

(20)

where $c_i$, which does not depend on $S$, is the coefficient of the complex zonal polynomial $\tilde{C} (S)$.

The integral term in (16) can be, then, solved in the frame of the complex zonal polynomials by considering, following Takemura [12], the splitting property

$$\int tr (-nC^{-1}Q \Xi Q^h) (\Xi) dQ = \sum_{i=0}^{\infty} \sum_{\lambda} \frac{1}{k!} \tilde{C}_i (-nC^{-1}) \tilde{C} (\Xi)$$

(22)

$$= \sum_{i=0}^{\infty} \sum_{\lambda} \frac{1}{k!} \tilde{C} (-n\Sigma^{-1}) \tilde{C} (\Xi) = \tilde{F} (-n\Sigma^{-1}, \Xi)$$

where $\tilde{F} (X, Y)$ is the complex hypergeometric function of double matrix argument. Considering the expansion (21) of $etr(-nC^{-1}Q \Xi Q^h)$ in (16), and applying the splitting property (22), $p_n (\Xi)$ can be written as

$$p_n (\Xi) = \sum_{i=0}^{m(n-1)} \frac{\prod_{i=1}^{\lambda_i} \lambda_i \prod_{i=2}^{\lambda_i} (-\lambda_i)}{\Gamma_m (n)} \tilde{F} (-n\Sigma^{-1}, \Xi).$$

(25)

Equation (25) is an analytical expression of the joint pdf of the sample eigenvalues, but, as given in (22), it is expressed as a function of an infinite series. The complex hypergeometric function of double matrix argument presents poor convergence properties, making necessary to evaluate a high number of its terms. This prevents, again, to use (25) for practical purposes.

C. Determinant Expression

The solution to (25) can be obtained from Soliton Theory and specifically in the associated concept of Tau Functions [13]. A tau function has to be considered as a sort of potential which give rise to certain hierarchies of differential equations. The special class of tau functions of hypergeometric type, expressed in the so-called Hirota-Miwa variables, allows to write

$$\tilde{F} (X, Y) = \sum_{i=0}^{\infty} \frac{\prod_{i=1}^{\lambda_i} \lambda_i \prod_{i=2}^{\lambda_i} (-\lambda_i)}{\Gamma_m (n)} \exp \left( \sum_{i=0}^{\infty} \frac{1}{k!} \tilde{C}_i (-nC^{-1}) \tilde{C} (\Xi) \right).$$

(26)

where $x = [x_1, x_2, ..., x_n]$ and $y = [y_1, y_2, ..., y_n]$ represent the eigenvalues of $X$ and $Y$, respectively, and $\Delta(x)$ and $\Delta(y)$ are their Vandermonde determinants, which definition is $\Delta(x) = \prod_{i<j} (x_i - x_j)$. Hence, (17) consists of a determinant expression of $\tilde{F} (X, Y)$. Equation, (17), can be introduced within (25), leading to a determinant expression for the joint sample eigenvalues pdf

$$p_n (\Xi) = \frac{\prod_{i=1}^{\lambda_i} \lambda_i \prod_{i=2}^{\lambda_i} (-\lambda_i)}{\Gamma_m (n)} \sum_{\pi} \prod_{i=1}^{\lambda_i} \exp \left( -n \lambda_i \frac{\pi}{\lambda_i} \right) \prod_{i,j} (\lambda_i - \lambda_j) \sum_{\pi} \exp \left( -n \lambda_i \frac{\pi}{\lambda_i} \right).$$

(23)

Equation (19) can be slightly simplified by considering the determinant as a sum over the symmetric group of degree $m, S_m$

$$p_n (\Xi) = \prod_{i=1}^{\lambda_i} \lambda_i \prod_{i=2}^{\lambda_i} (-\lambda_i) \sum_{\pi} \exp \left( -n \lambda_i \frac{\pi}{\lambda_i} \right) \prod_{i,j} (\lambda_i - \lambda_j) \sum_{\pi} \exp \left( -n \lambda_i \frac{\pi}{\lambda_i} \right).$$

(23)

where $\pi$ denotes a permutation of $\{1, 2, ..., m\}$, $\pi\Omega (\pi)$ is the parity of the permutation, and

$$K (m, n) = \prod_{i=1}^{\lambda_i} \lambda_i \prod_{i=2}^{\lambda_i} (-\lambda_i) \sum_{\pi} \exp \left( -n \lambda_i \frac{\pi}{\lambda_i} \right) \prod_{i,j} (\lambda_i - \lambda_j) \sum_{\pi} \exp \left( -n \lambda_i \frac{\pi}{\lambda_i} \right).$$

(24)

As one can conclude from (23), that the joint pdf of the sample eigenvalues $\{\lambda_1, \lambda_2, ..., \lambda_m\}$ depends on the true value of the eigenvalues $\{l_1, l_2, ..., l_n\}$, the number of averaged samples or looks $n$, and the number of data channels $m$. Besides, (23) can not be written as the product of $m$ separated pdfs, one for each sample eigenvalue. On the one hand, it demonstrates that the sample eigenvalues are not uncorrelated. On the other hand, in order to obtain the pdf of a particular eigenvalue, the rest need to be integrated out.

D. Sample Eigenvalues Distribution for PolSAR

Equation (23) in now particularized for PolSAR data for backscattering direction case. Hence, $m=3$ and $n=2$, as imposed by (5). In this situation, it is not possible to derive a general expression for $p_n (\lambda_i)$, for $i=1, 2, \lambda$, due to the complexity of the distribution presented in (23). Consequently, an analysis is only possible through numerical methods. Figure 1 presents the result of this integration process, showing, for each sample

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eigenvalues are asymptotic non-biased estimators of the true pdf is derived. This pdf allows to demonstrate that the sample eigenvalues are asymptotic non-biased estimators of the true eigenvalues of the group representation theory, the joint sample eigenvalues decomposition applied to PolSAR data analysis. On the basis of the eigenvalue, the evolution of its corresponding pdf as a function of the number of averaged samples $n$, Figure 2 shows the evolution of the average values of the sample eigenvalues as a function of $n$. Hence, one can conclude from Figure 2 that the sample eigenvalues are asymptotic non-biased estimators of the true eigenvalues of $C$, or $T$.

Since the sample eigenvalues consist of asymptotic non-biased estimators, these biases shall also produce the entropy (H) and anisotropy (A) to be biased with respect to the corresponding true values. An in-depth analysis of (23) reveals that H is always asymptotically overestimated, independently of $\{l_1,l_2,\ldots,l_m\}$. Whereas low anisotropies are asymptotically overestimated, high ones are underestimated. Despite it is not shown in the paper, the variances of the sample eigenvalues $\{\lambda_1,\lambda_2,\ldots,\lambda_m\}$ decrease by increasing the number of averaged samples. Consequently, the variances of $H$ and $A$ also decrease with the number of looks.

As a consequence of the asymptotic behavior of the sample eigenvalues, it can be concluded that a minimum number of averages samples is necessary in order to recover the true information. This minimum should be located in an averaging window of $7 \times 7$, or $9 \times 9$, pixels.

V. CONCLUSIONS

This paper analyses the statistical behavior of the eigen decomposition applied to PolSAR data analysis. On the basis of the group representation theory, the joint sample eigenvalues pdf is derived. This pdf allows to demonstrate that the sample eigenvalues are asymptotic non-biased estimators of the true eigenvalues. Finally, the selection of the minimum number of looks necessary to estimate the correct information is justified.

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