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Programming with Nonequispaced FFT

Solution 1 **Basics and Matlab**

Exercise 1:
The function $g(x) = \sum_{r\in\mathbb{Z}} f(x+r)$ is one-periodic and integrable on $[-\frac{1}{2}, \frac{1}{2}]$. Moreover, the right hand side converges absolutely. The Fourier coefficients of g obey

$$\begin{aligned} c_k(g) &= \int_{-1/2}^{1/2} g(x)\,\mathrm{e}^{-2\pi\mathrm{i}kx}\mathrm{d}x \\ &= \int_{-1/2}^{1/2} \sum_{r\in\mathbb{Z}} f(x+r)\,\mathrm{e}^{-2\pi\mathrm{i}kx}\mathrm{e}^{-2\pi\mathrm{i}kr}\mathrm{d}x \\ &= \sum_{r\in\mathbb{Z}} \int_{r-1/2}^{r+1/2} f(x)\,\mathrm{e}^{-2\pi\mathrm{i}kx}\mathrm{d}x \\ &= \int_{\mathbb{R}} f(x)\mathrm{e}^{2\pi\mathrm{i}kx}\mathrm{d}x. \end{aligned}$$

Due to the decay of $\hat{f}$, these Fourier coefficients are summable and thus the Fourier series of g converges absolutely.

Exercise 2:

1. For $p = 1$, $k \in \mathbb{Z}$, and $f \in C^1(\mathbb{T})$ holds

$$\begin{aligned} c_k\left(f^{(1)}\right) &= \int_{-1/2}^{1/2} f^{(1)}(x)\,\mathrm{e}^{-2\pi\mathrm{i}kx}\mathrm{d}x \\ &= 0 - \left[-2\pi\mathrm{i}k \int_{-1/2}^{1/2} f(x)\,\mathrm{e}^{-2\pi\mathrm{i}kx}\mathrm{d}x\right] \\ &= (2\pi\mathrm{i}k)^1\, c_k(f). \end{aligned}$$

2. For real-valued $f \in L^2(\mathbb{T})$ and $k \in \mathbb{Z}$ the Fourier-coefficients satisfy

$$\begin{aligned}
\overline{c_{-k}(f)} &= \overline{\int_{-1/2}^{1/2} f(x)\, \mathrm{e}^{2\pi \mathrm{i} k x} \mathrm{d}x} \\
&= \int_{-1/2}^{1/2} \overline{f(x)} \mathrm{e}^{-2\pi \mathrm{i} k x} \mathrm{d}x \\
&= c_k(f).
\end{aligned}$$

3. For even $f \in L^2(\mathbb{T})$, i.e. $f(x) = f(-x)$, and $k \in \mathbb{Z}$ the Fourier-coefficients satisfy

$$\begin{aligned}
c_{-k}(f) &= \int_{-1/2}^{1/2} f(x)\, \mathrm{e}^{2\pi \mathrm{i} k x} \mathrm{d}x \\
&= -\int_{1/2}^{-1/2} f(-x)\, \mathrm{e}^{-2\pi \mathrm{i} k x} \mathrm{d}x \\
&= c_k(f).
\end{aligned}$$

4. For $j, k = -N/2, \ldots, N/2 - 1$ holds, see slides, page 15 and 19-20,

$$\begin{aligned}
\left(\tilde{\mathbf{F}}_N^{\mathrm{H}} \tilde{\mathbf{F}}_N\right)_{j,k} &= \frac{1}{N} \sum_{l=-N/2}^{N/2-1} \mathrm{e}^{2\pi \mathrm{i} l (j-k)/N} \\
&= \delta_{k,j}.
\end{aligned}$$

Exercise 3:
Convolution theorems.

1.

$$\begin{aligned}
c_k(f *_p g) &= c_k\left(\int_{\mathbb{T}} f(t)\, g(x-t)\, \mathrm{d}t \right) \\
&= \int_{\mathbb{T}} \int_{\mathbb{T}} f(t)\, g(x-t)\, \mathrm{e}^{-2\pi \mathrm{i} k x} \mathrm{d}t \mathrm{d}x \\
&= \int_{\mathbb{T}} f(t)\, \mathrm{e}^{-2\pi \mathrm{i} k t} \left[\int_{\mathbb{T}} g(x-t)\, \mathrm{e}^{-2\pi \mathrm{i} k (x-t)} \mathrm{d}x \right] \mathrm{d}t \\
&= c_k(f)\, c_k(g).
\end{aligned}$$

$$
\begin{aligned}
(\mathbf{c}(f) *_d \mathbf{c}(g))_k &= \sum_{l\in\mathbb{Z}} c_l(f)\, c_{(k-l)}(g) \\
&= \sum_{l\in\mathbb{Z}} \int_{\mathbb{T}} f(t)\,\mathrm{d}t \int_{\mathbb{T}} g(x)\,\mathrm{e}^{-2\pi\mathrm{i}(k-l)x}\mathrm{d}x \\
&= \int_{\mathbb{T}} f(t) \left[\sum_{l\in\mathbb{Z}} \left(\int_{\mathbb{T}} g(x)\,\mathrm{e}^{-2\pi\mathrm{i}(k-l)x}\mathrm{d}x\right) \mathrm{e}^{2\pi\mathrm{i}(k-l)t}\right] \mathrm{e}^{-2\pi\mathrm{i}kt}\mathrm{d}t \\
&= \int_{\mathbb{T}} f(t) \left[\sum_{l\in\mathbb{Z}} c_{k-l}(g)\,\mathrm{e}^{2\pi\mathrm{i}(k-l)t}\right] \mathrm{e}^{-2\pi\mathrm{i}kt}\mathrm{d}t \\
&= \int_{\mathbb{T}} f(t)\, g(t)\,\mathrm{e}^{-2\pi\mathrm{i}kt}\mathrm{d}t \\
&= c_k(fg)\,.
\end{aligned}
$$

$$
\begin{aligned}
\hat{f}_k \hat{g}_k &= \sum_{j=-n/2}^{n/2-1} f_j \mathrm{e}^{-2\pi\mathrm{i}kj/n} \sum_{l=-n/2}^{n/2-1} g_l \mathrm{e}^{-2\pi\mathrm{i}kl/n} \\
&= \sum_{j=-n/2}^{n/2-1} f_j \mathrm{e}^{-2\pi\mathrm{i}kj/n} \sum_{l=-n/2}^{n/2-1} g_{l-j} \mathrm{e}^{-2\pi\mathrm{i}k(l-j)/n} \\
&= \sum_{l=-n/2}^{n/2-1} \left[\sum_{j=-n/2}^{n/2-1} f_j g_{l-j}\right] \mathrm{e}^{-2\pi\mathrm{i}kl/n} \\
&= \left(\mathbf{F}_n (\mathbf{f} *_c \mathbf{g})\right)_k\,.
\end{aligned}
$$

Exercise 4:

3.
```
function f=ndft(x,f_hat)

M=length(x);
freq=(-N/2):(N/2-1);
f=zeros(M,1);
for k=1:N
  f=f+f_hat(k).*exp(-2*pi*i*x*freq(k));
end;
```

Exercise 5:
Owing to Exercise 3.3 and 2.4, we have

$$\mathbf{G} = \tilde{\mathbf{F}}_n^{\mathrm{H}} \mathrm{diag}(\tilde{\mathbf{F}}_n \mathbf{g}) \tilde{\mathbf{F}}_n$$

Fast matrix vector multiplication with a Toeplitz matrix can be realised by

```
function b=fast_toeplitz(c,r,x)

c1=[c;flipud(r(2:end))];
b=(size(c,1)+size(r,1)-1)*fft(ifft(c1).*ifft([x;zeros(size(c))]));
b=b(1:size(c,1));
```