A unified approach to scattered data approximation on $\mathbb{S}^3$ and $\mathrm{SO}(3)$

Manuel Gräf*

In this paper we use the connection between the rotation group SO(3) and the three-dimensional Euclidean sphere $\mathbb{S}^3$ in order to carry over results on the sphere $\mathbb{S}^3$ directly to the rotation group SO(3) and vice versa. More precisely, these results connect properties of sampling sets and quadrature formulae on SO(3) and $\mathbb{S}^3$ respectively. Furthermore we relate Marcinkiewicz-Zygmund inequalities and conditions for the existence of positive quadrature formulae on the rotation group SO(3) to those on the sphere $\mathbb{S}^3$, respectively.

Keywords and Phrases : rotation group SO(3), sphere $\mathbb{S}^3$, quaternions, scattered data, sampling sets, quadrature formulae, Marcinkiewicz-Zygmund inequalities

AMS subject classification. : 65T99, 43A90, 65D32

1 Introduction

Scattered data approximation on various domains is a problem with many applications in science and engineering, cf. [17, 3, 15]. Typical ingredients to treat such multivariate approximation problems are studies of scattered sampling nodes and quadrature formulae. For example Schmid was able to present in [19] a trade-off result, based on a paper of Schaback [16] for approximation problems using positive definite basis functions on the rotation group SO(3). There was shown that it is impossible to come up with a positive definite basis function that enables one to keep the estimate on the approximation error and the condition number of the associated interpolation matrix arbitrarily small simultaneously. Another important result comes from Gröchenig. He investigated in [10] the problem of the reconstruction of band-limited functions from scattered sampling data and arrived on Marcinkiewicz-Zygmund type inequalities. Marcinkiewicz-Zygmund inequalities provide a norm equivalence between the discrete l_p norm of the sampling values

*Chemnitz University of Technology, Department of Mathematics, 09107 Chemnitz, Germany, m.graef@mathematik.tu-chemnitz.de

and the continuous L_p norm of the sampled function under certain conditions on the geometric properties of the scattered sampling nodes. Mhaskar, Narcowich and Ward proved in [12] such inequalities for the d-dimensional sphere $\mathbb{S}^d$ and they used them in order to derive assertions for (positive) quadrature formulae of scattered data. In later investigations such inequalities were improved and used for stability results of scattered data approximation methods as well, cf. [11, 6]. Recently, one was able to state such results of scattered data approximation on the rotation group SO(3) by using similar methods, we refer to [18, 8] and the dissertation of Schmid [20].

All the above results reveal the importance of the distribution of the sampling sets. In [9] a simple construction was proposed which leads to well distributed sampling sets on the rotation group SO(3). The same construction can also be used in order to generate quadrature formulae on the rotation group from quadratures on the sphere $\mathbb{S}^2$. In particular, one obtains immediately t-designs, which are equal weight quadratures, on the rotation group SO(3) from t-designs on the $\mathbb{S}^2$. In addition the authors presented a method, based on a fast algorithm for nonequispaced Fourier transforms on the rotation group SO(3), cf. [14], for the computation of nonnegative quadrature weights for scattered sampling sets on SO(3).

The aim of this paper is to consider the results on the sphere $\mathbb{S}^3$ and rotation group SO(3) from a common point of view. Therefore, we recapitulate the well-known connection between the rotation group SO(3) and the sphere $\mathbb{S}^3$ by quaternions, see for example [2]. Using this connection we show that the natural metrics, measures and polynomial spaces on these manifolds are essentially the same. From these facts we carry over assertions of sampling sets and quadrature formulae from $\mathbb{S}^3$ to SO(3) and vice versa. As an application of these connections we present new proofs for the existence of nonnegative quadrature weights and Marcinkiewicz-Zygmund inequalities on the rotation group SO(3) based on results on $\mathbb{S}^3$. Moreover we obtain immediately constructions of well distributed sampling sets and quadrature formulae on the sphere $\mathbb{S}^3$ by using the results from [9]. As a consequence we are able to construct t-designs on $\mathbb{S}^3$ from t-designs on the sphere $\mathbb{S}^2$ and can compute nonnegative quadrature weights for antipodal scattered sampling sets on $\mathbb{S}^3$ in a fast way.

The outline of this paper is as follows. For the paper to be self contained we introduce in Section 2 the necessary notations for deriving the well-known isomorphism between the rotation group SO(3) and the three-dimensional projective space $\mathbb{S}^3_*$, where antipodal points on the sphere $\mathbb{S}^3$ are identified. Using this connection we show that the even polynomials on $\mathbb{S}^3$ correspond to the polynomials on SO(3). With the mentioned results at hand we connect in Section 3 sampling sets and quadrature formulae on these manifolds. Afterwards this connection is applied to recent results on the rotation group SO(3).

2 Preliminary: relations between quaternions and rotations

Throughout this paper we use the notation

$$\mathbb{S}^d := \{\boldsymbol{x} = (x_1, \ldots, x_{d+1})^\top \in \mathbb{R}^{d+1} \ : \ \|\boldsymbol{x}\|_2 = 1\}$$

for the d-dimensional sphere. The rotation group of the n-dimensional Euclidean space is denoted by

$$\mathrm{SO}(n) := \left\{ \boldsymbol{G} \in \mathbb{R}^{n\times n} : \boldsymbol{G}^\top = \boldsymbol{G}^{-1}, \det \boldsymbol{G} = 1 \right\},$$

and the group operation is given by composition. The rotation group SO(3) can be naturally parameterized by

$$\boldsymbol{R}(\boldsymbol{r},\alpha) := (1-\cos(\alpha))\boldsymbol{r}\boldsymbol{r}^\top + \begin{pmatrix} \cos(\alpha) & -z\sin(\alpha) & y\sin(\alpha) \\ z\sin(\alpha) & \cos(\alpha) & -x\sin(\alpha) \\ -y\sin(\alpha) & x\sin(\alpha) & \cos(\alpha) \end{pmatrix}, \tag{2.1}$$

with rotation axis $\boldsymbol{r} = (x,y,z)^\top \in \mathbb{S}^2$ and rotation angle $\alpha \in [0,\pi]$. In the following we introduce the space of unit quaternions which establishes the connection of the sphere $\mathbb{S}^3$ to the rotation group SO(3). A unit quaternion is given by

$$\boldsymbol{q} := \begin{pmatrix} s \\ \boldsymbol{v} \end{pmatrix} \in \mathbb{S}^3,$$

which consists of a scalar part $s \in \mathbb{R}$ and a vector part $\boldsymbol{v} \in \mathbb{R}^3$. Via the multiplication formula

$$\boldsymbol{q}_1 \cdot \boldsymbol{q}_2 := \begin{pmatrix} s_1 s_2 - \boldsymbol{v}_1^\top \boldsymbol{v}_2 \\ s_1\boldsymbol{v}_2 + s_2\boldsymbol{v}_1 + \boldsymbol{v}_1 \times \boldsymbol{v}_2 \end{pmatrix}, \quad \boldsymbol{q}_1 = \begin{pmatrix} s_1 \\ \boldsymbol{v}_1 \end{pmatrix}, \boldsymbol{q}_2 = \begin{pmatrix} s_2 \\ \boldsymbol{v}_2 \end{pmatrix} \in \mathbb{S}^3$$

one easily checks that the space of the unit quaternions forms a group. The inverse element of $\boldsymbol{q} = (s, \boldsymbol{v}^\top)^\top$ is the conjugated quaternion

$$\overline{\boldsymbol{q}} := \begin{pmatrix} s \\ -\boldsymbol{v} \end{pmatrix}.$$

The connection between quaternions and rotations is given by the following operation. For a vector $\boldsymbol{p} \in \mathbb{S}^2$ we define the action of a unit quaternion $\boldsymbol{q} \in \mathbb{S}^3$ by

$$\boldsymbol{p}(\boldsymbol{q}) := \boldsymbol{q} \cdot \begin{pmatrix} 0 \\ \boldsymbol{p} \end{pmatrix} \cdot \overline{\boldsymbol{q}}. \tag{2.2}$$

If we parameterize a quaternion $\boldsymbol{q}$ by an axis $\boldsymbol{r} = (x,y,z)^\top \in \mathbb{S}^2$ and an angle $\alpha \in [0,\pi]$ due to

$$\boldsymbol{q}(\boldsymbol{r},\alpha) := \begin{pmatrix} \cos\frac{\alpha}{2} \\ \sin\frac{\alpha}{2}\boldsymbol{r} \end{pmatrix} \tag{2.3}$$

we obtain the identity

$$\boldsymbol{p}(\boldsymbol{q}(\boldsymbol{r},\alpha)) = \begin{pmatrix} 0 \\ \boldsymbol{R}(\boldsymbol{r},\alpha)\boldsymbol{p} \end{pmatrix}, \qquad \boldsymbol{p} \in \mathbb{S}^2, \tag{2.4}$$

which shows that the actions of rotations and quaternions on vectors $\boldsymbol{p} \in \mathbb{S}^2$ are identical. Form this observation we call $\boldsymbol{G} := \boldsymbol{R}(\boldsymbol{r},\alpha)$ the corresponding rotation of the quaternion

$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{r}, \alpha)$. Since the quaternion multiplication is associative it follows immediately that the quaternion multiplication of two unit quaternions $\boldsymbol{q}_1, \boldsymbol{q}_2 \in \mathbb{S}^3$ is consistent with the composition of the corresponding rotations $\boldsymbol{G}_1, \boldsymbol{G}_2 \in \mathrm{SO}(3)$, i.e.,

$$\begin{pmatrix} 0 \\ \boldsymbol{G}_2\boldsymbol{G}_1\boldsymbol{p} \end{pmatrix} = \boldsymbol{p}(\boldsymbol{q}_2 \cdot \boldsymbol{q}_1), \qquad \boldsymbol{p} \in \mathbb{S}^2.$$

By definition (2.3) the quaternion $\boldsymbol{q}(\boldsymbol{r}, \alpha) = (s, \boldsymbol{v}^\top)^\top$ is in the upper hemisphere of $\mathbb{S}^3$, i.e., $s \geq 0$. Since $\boldsymbol{q}$ and $-\boldsymbol{q}$ corresponds by (2.2) and (2.4) to the same rotation one obtains that $\mathbb{S}^3$ is a double cover of SO(3), cf. [2, Chap. III, Sect. 10]. So it is convenient to pass to the quotient space $\mathbb{S}^3_* := \mathbb{S}^3/\{1, -1\}$, where we identify antipodal quaternions. This space is also known as the three-dimensional projective space. With it we obtain an isomorphism

$$\boldsymbol{q}_* : \mathrm{SO}(3) \to \mathbb{S}^3_*, \qquad \boldsymbol{q}_*(\boldsymbol{R}(\boldsymbol{r}, \alpha)) := \{\pm \boldsymbol{q}(\boldsymbol{r}, \alpha)\} = \{-\boldsymbol{q}(\boldsymbol{r}, \alpha), \boldsymbol{q}(\boldsymbol{r}, \alpha)\}, \tag{2.5}$$

between SO(3) and the projective space $\mathbb{S}^3_*$ with the property

$$\boldsymbol{q}_*(\boldsymbol{G}_2\boldsymbol{G}_1) = \boldsymbol{q}_*(\boldsymbol{G}_2) \cdot \boldsymbol{q}_*(\boldsymbol{G}_1) := \{\pm \boldsymbol{q}_2 \cdot \boldsymbol{q}_1\}. \tag{2.6}$$

For simplicity we use a representative $\boldsymbol{x} \in \mathbb{S}^3$ to identify the corresponding element of the projective space $\mathbb{S}^3_*$ by $\boldsymbol{x}_* := \{\pm \boldsymbol{x}\}$.

We introduce on the sphere $\mathbb{S}^3$the natural metric

$$\mathrm{d}_{\mathbb{S}^3}(\boldsymbol{x}_1, \boldsymbol{x}_2) := \arccos(\boldsymbol{x}_1^\top \boldsymbol{x}_2), \qquad \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{S}^3, \tag{2.7}$$

and the natural rotation invariant surface measure $\mu_{\mathbb{S}^3}$, with normalization $\int_{\mathbb{S}^3} \mathrm{d}\mu_{\mathbb{S}^3}(\boldsymbol{x}) = 1$. These definitions induce a metric and a measure on $\mathbb{S}^3_*$ respectively. More precisely we define the metric by

$$\mathrm{d}_{\mathbb{S}^3_*}(\boldsymbol{x}_{1*}, \boldsymbol{x}_{2*}) := \min_{\substack{\boldsymbol{y}_1 = \pm \boldsymbol{x}_1, \\ \boldsymbol{y}_2 = \pm \boldsymbol{x}_2}} \mathrm{d}_{\mathbb{S}^3}(\boldsymbol{y}_1, \boldsymbol{y}_2) = \arccos |\boldsymbol{x}_1 \boldsymbol{x}_2^\top|, \quad \boldsymbol{x}_{1*} = \{\pm \boldsymbol{x}_1\},\ \boldsymbol{x}_{2*} = \{\pm \boldsymbol{x}_2\} \in \mathbb{S}^3_* \tag{2.8}$$

and the measure by

$$\mu_{\mathbb{S}^3_*}(\Omega_*) = \mu_{\mathbb{S}^3}(\Omega) + \mu_{\mathbb{S}^3}(-\Omega), \qquad \Omega_* = \{\boldsymbol{x}_* \ : \ \boldsymbol{x} \in \Omega\} \subset \mathbb{S}^3_*,\ \Omega \subset \mathbb{S}^3, \tag{2.9}$$

where we assume $\mu_{\mathbb{S}^3}(\Omega \cap -\Omega) = 0$. Furthermore, the distance between two rotations $\boldsymbol{G}_1, \boldsymbol{G}_2 \in \mathrm{SO}(3)$ is naturally given by

$$\mathrm{d}_{\mathrm{SO}(3)}(\boldsymbol{G}_1, \boldsymbol{G}_2) := \alpha(\boldsymbol{G}_1^\top \boldsymbol{G}_2) := \arccos\left(\frac{1}{2}(\mathrm{trace}\, \boldsymbol{G}_1^\top \boldsymbol{G}_2 - 1)\right), \tag{2.10}$$

where $\alpha(\boldsymbol{G})$ is the rotation angle of the rotation $\boldsymbol{G} \in \mathrm{SO}(3)$. Moreover, we use the normalized (translation invariant) Haar measure $\mu_{\mathrm{SO}(3)}$, i.e.,

$$\mu_{\mathrm{SO}(3)}(\Omega) = \mu_{\mathrm{SO}(3)}(\boldsymbol{H} \cdot \Omega), \qquad \boldsymbol{H} \in \mathrm{SO}(3),\ \Omega \subset \mathrm{SO}(3),$$

with $\int_{\mathrm{SO}(3)} \mathrm{d}\mu_{\mathrm{SO}(3)}(\boldsymbol{G}) = 1$, where $\boldsymbol{H} \cdot \Omega := \{\boldsymbol{H}\boldsymbol{G} \;:\; \boldsymbol{G} \in \Omega\}$ is the translated set.

The following Lemma 2.1 seems to be well-known, but since the author was not able to find it in the literature we give a proof. It states by virtue of the isomorphism $\boldsymbol{q}_*$, cf. (2.5), that the induced metric (2.8) on the projective space $\mathbb{S}^3_*$ coincides with the metric on the rotation group SO(3), as well as the induced measure (2.9) coincides with the normalized Haar measure.

Lemma 2.1. *The isomorphism* $\boldsymbol{q}_* : \mathrm{SO}(3) \to \mathbb{S}^3_*$, *cf.* (2.5) *yields the identities*

$$\begin{aligned} \mathrm{d}_{\mathrm{SO}(3)}(\boldsymbol{G}_1, \boldsymbol{G}_2) &= 2\mathrm{d}_{\mathbb{S}^3_*}(\boldsymbol{q}_*(\boldsymbol{G}_1), \boldsymbol{q}_*(\boldsymbol{G}_2)), \quad && \boldsymbol{G}_1, \boldsymbol{G}_2 \in \mathrm{SO}(3), \\ \mu_{\mathrm{SO}(3)}(\Omega) &= \mu_{\mathbb{S}^3_*}(\boldsymbol{q}_*(\Omega)), && \Omega \subset \mathrm{SO}(3), \end{aligned} \tag{2.11}$$

where $\boldsymbol{q}_*(\Omega) := \{\boldsymbol{q}_*(\boldsymbol{G}) \;:\; \boldsymbol{G} \in \Omega\}$.

Proof. We obtain the first identity by using for $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ the parameterization with rotation axes $\boldsymbol{r}_1, \boldsymbol{r}_2 \in \mathbb{S}^2$ and rotation angles $\alpha_1, \alpha_2 \in [0, \pi]$, cf. (2.1). This yields with (2.8) the desired result

$$\begin{aligned} \mathrm{d}_{\mathrm{SO}(3)}(\boldsymbol{R}(\boldsymbol{r}_1, \alpha_1), \boldsymbol{R}(\boldsymbol{r}_2, \alpha_2)) &= 2 \arccos \left| \boldsymbol{r}_1^\top \boldsymbol{r}_2 \sin\frac{\alpha_1}{2} \sin\frac{\alpha_2}{2} + \cos\frac{\alpha_1}{2} \cos\frac{\alpha_2}{2} \right| \\ &= 2 \arccos \left| \boldsymbol{q}(\boldsymbol{r}_1, \alpha_1)^\top \boldsymbol{q}(\boldsymbol{r}_2, \alpha_2) \right| \\ &= 2\,\mathrm{d}_{\mathbb{S}^3_*}(\boldsymbol{q}_*(\boldsymbol{R}(\boldsymbol{r}_1, \alpha_1)), \boldsymbol{q}_*(\boldsymbol{R}(\boldsymbol{r}_2, \alpha_2))). \end{aligned}$$

In order to prove the second identity we make use of the uniqueness of the Haar measure, cf. [7, Section 2.2]. Therefore let $\boldsymbol{H} \in \mathrm{SO}(3)$ and $\Omega \subset \mathrm{SO}(3)$ be given. From equation (2.6) we infer that

$$\boldsymbol{q}_*(\boldsymbol{H} \cdot \Omega) = \{\boldsymbol{q}_*(\boldsymbol{H}) \cdot \boldsymbol{q}_*(\boldsymbol{G}) \;:\; \boldsymbol{G} \in \Omega\}.$$

Since the quaternionic multiplication $\boldsymbol{q} \cdot \boldsymbol{x}$ of a vector $\boldsymbol{x} \in \mathbb{R}^4$ with an unit quaternion $\boldsymbol{q} \in \mathbb{S}^3$ can be considered as a linear transformation $\boldsymbol{T}\boldsymbol{x}$ with determinant $|\det \boldsymbol{T}| = 1$, we conclude that the antipodal quaternions $\boldsymbol{q}_*(\boldsymbol{H})$ represent some rotation $\boldsymbol{T} \in \mathrm{SO}(4)$ or $-\boldsymbol{T} \in \mathrm{SO}(4)$. Let without loss of generality $\boldsymbol{T} \in \mathrm{SO}(4)$, then we arrive at

$$\boldsymbol{q}_*(\boldsymbol{H} \cdot \Omega) = \{\{\pm \boldsymbol{T}\boldsymbol{x}\} \;:\; \boldsymbol{x}_* = \boldsymbol{q}_*(\boldsymbol{G}),\, \boldsymbol{G} \in \Omega\} =: \boldsymbol{T} \cdot \boldsymbol{q}_*(\Omega).$$

Together with the rotation invariance of the measure $\mu_{\mathbb{S}^3}$ on the sphere $\mathbb{S}^3$ and the definition (2.9) we obtain the translation invariance of the induced measure

$$\mu_{\mathbb{S}^3_*}(\boldsymbol{q}_*(\boldsymbol{H} \cdot \Omega)) = \mu_{\mathbb{S}^3_*}(\boldsymbol{T} \cdot \boldsymbol{q}_*(\Omega)) = \mu_{\mathbb{S}^3_*}(\boldsymbol{q}_*(\Omega)).$$

With the normalization $1 = \mu_{\mathrm{SO}(3)}(\mathrm{SO}(3)) = \mu_{\mathbb{S}^3_*}(\mathbb{S}^3_*)$ and the uniqueness of the Haar measure we conclude that these two measures must coincide. ∎

For our considerations we introduce the harmonic spaces on $\mathbb{S}^3$ and SO(3). The spaces of square integrable functions are denoted by

$$\begin{aligned} L^2(\mathbb{S}^3) &:= \left\{ f : \mathbb{S}^3 \to \mathbb{C} \;:\; \int_{\mathbb{S}^3} |f(\boldsymbol{x})|^2 \mathrm{d}\mu_{\mathbb{S}^3}(\boldsymbol{x}) \right\}, \\ L^2(\mathrm{SO}(3)) &:= \left\{ f : \mathrm{SO}(3) \to \mathbb{C} \;:\; \int_{\mathrm{SO}(3)} |f(\boldsymbol{G})|^2 \mathrm{d}\mu_{\mathrm{SO}(3)}(\boldsymbol{G}) \right\}, \end{aligned}$$

where the standard orthonormal bases are given by spherical harmonics Y_l^k, $l \in \mathbb{N}_0$, $k = 0, \dots, (l+1)^2$, and by the Wigner D-functions $D_l^{m,m'}$, $l \in \mathbb{N}_0$, $m, m' = -l, \dots, l$, respectively, cf. [13, 21]. Furthermore we define for degree $l \in \mathbb{N}_0$ the harmonic spaces

$$\Gamma_l(\mathbb{S}^3) := \operatorname{span}\left\{ Y_l^k \ : \ k = 0, \dots, (l+1)^2 \right\},$$
$$\Gamma_l(\mathrm{SO}(3)) := \operatorname{span}\left\{ D_l^{m,m'} \ : \ m, m' = -l, \dots, l \right\},$$

and

$$\Pi_N(\mathbb{S}^3) := \bigoplus_{l=0}^{N} \Gamma_l(\mathbb{S}^3), \quad \Pi_N(\mathrm{SO}(3)) := \bigoplus_{l=0}^{N} \Gamma_l(\mathrm{SO}(3)),$$

where the firsts consist of all polynomials of degree l and the latter of all polynomials of degree at most N.

3 Sampling sets, quadratures and Marcinkiewicz-Zygmund inequalities

In the following we present three theorems which connect sampling sets, quadrature formulae and Marcinkiewicz-Zygmund inequalities on the sphere $\mathbb{S}^3$ to those on the rotation group SO(3) respectively.

First of all we consider finite subsets $\mathcal{X}(\mathcal{M})$ of a metric space $(\mathcal{M}, \mathrm{d}_\mathcal{M})$ with metric $\mathrm{d}_\mathcal{M}$. In this paper the space $\mathcal{M}$ stands for the sphere $\mathbb{S}^3$ and the rotation group SO(3). In order to describe the quality of such a sampling set $\mathcal{X}(\mathcal{M})$ we introduce the following two parameters. The first one is the separation distance

$$q(\mathcal{X}(\mathcal{M})) := \min_{\boldsymbol{y} \neq \boldsymbol{x} \in \mathcal{X}(\mathcal{M})} \mathrm{d}_\mathcal{M}(\boldsymbol{x}, \boldsymbol{y}), \tag{3.1}$$

which is the minimal distance of two distinct nodes of the sampling set $\mathcal{X}(\mathcal{M})$. On the other hand the mesh norm

$$\delta(\mathcal{X}(\mathcal{M})) := 2 \max_{\boldsymbol{y} \in \mathcal{M}} \min_{\boldsymbol{x} \in \mathcal{X}(\mathcal{M})} \mathrm{d}_\mathcal{M}(\boldsymbol{x}, \boldsymbol{y}) \tag{3.2}$$

describes the “density” of $\mathcal{X}(\mathcal{M})$ in $\mathcal{M}$.

Furthermore, we say a quadrature rule

$$Q(\mathcal{M}) := \{(\boldsymbol{x}_i, w_i) \mid i = 0, \dots, M-1\}$$

with sampling nodes $\boldsymbol{x}_i \in \mathcal{M}$ and quadrature weights $w_i \in \mathbb{C}$ has degree of exactness N, if for all polynomials $f \in \Pi_N(\mathcal{M})$ the relation

$$\sum_{i=0}^{M-1} w_i f(\boldsymbol{x}_i) = \int_\mathcal{M} f(\boldsymbol{x}) \mathrm{d}\mu_\mathcal{M}(\boldsymbol{x})$$

is valid.

The following Theorem 3.1 states that antipodal sampling sets on the sphere $\mathbb{S}^3$ can be considered as sampling sets on the rotation group SO(3) and that they share the same metric properties.

Theorem 3.1. *For a sampling set* $\mathcal{X}(\mathbb{S}^3) := \{\pm\boldsymbol{x}_0, \dots, \pm\boldsymbol{x}_{M-1}\}$ *with* $2M \geq 4$ *antipodal nodes on the sphere* $\mathbb{S}^3$ *and the corresponding sampling set* $\mathcal{X}(\mathrm{SO}(3)) := \{\boldsymbol{q}_*^{-1}(\boldsymbol{x}_{0*}), \dots, \boldsymbol{q}_*^{-1}(\boldsymbol{x}_{M-1*})\}$ *with* M *nodes on the rotation group* $\mathrm{SO}(3)$ *the separation distances and mesh norms obey*

$$q(\mathcal{X}(\mathbb{S}^3)) = \frac{1}{2} q(\mathcal{X}(\mathrm{SO}(3))) \quad \text{and} \quad \delta(\mathcal{X}(\mathbb{S}^3)) = \frac{1}{2}\delta(\mathcal{X}(\mathrm{SO}(3))),$$

respectively.

Proof. At first, we observe that the extremal nodes of the antipodal sampling set $\mathcal{X}(\mathbb{S}^3)$ in the definitions of the separation distance (3.1) and the mesh norm (3.2) occur in antipodal pairs. Hence, by the definition of the distance (2.8) we conclude for the sampling set $\mathcal{X}(\mathbb{S}^3_*) := \{\boldsymbol{x}_{0*}, \dots, \boldsymbol{x}_{M-1*}\}$ the relations $q(\mathcal{X}(\mathbb{S}^3)) = q(\mathcal{X}(\mathbb{S}^3_*))$ and $\delta(\mathcal{X}(\mathbb{S}^3)) = \delta(\mathcal{X}(\mathbb{S}^3_*))$. So the assumption follows from Lemma 2.1. ■

A similar connection is valid for the harmonic spaces on $\mathbb{S}^3$ and $\mathrm{SO}(3)$, where we identify even functions on the sphere $\mathbb{S}^3$ with functions on the rotation group $\mathrm{SO}(3)$. For an even function $f : \mathbb{S}^3 \to \mathbb{C}$ we define the function $\tilde{f} : \mathbb{S}^3_* \to \mathbb{C}$ by

$$\tilde{f}(\boldsymbol{q}_*) := f(\pm\boldsymbol{q}), \tag{3.3}$$

where the isomorphism (2.5) yields a function $\bar{f} : \mathrm{SO}(3) \to \mathbb{C}$ by $\bar{f}(\boldsymbol{G}) = \tilde{f}(\boldsymbol{q}_*(\boldsymbol{G}))$, $\boldsymbol{G} \in \mathrm{SO}(3)$. Conversely, an arbitrarily function $\bar{f} : \mathrm{SO}(3) \to \mathbb{C}$ can be continued uniquely to an even function $f : \mathbb{S}^3 \to \mathbb{C}$.

Lemma 3.2. *The isomorphism* $\boldsymbol{q}_* : \mathrm{SO}(3) \to \mathbb{S}^3_*$, *cf.* (2.5), *and the definition* (3.3) *yield the following equivalence*

$$f(\cdot) \in \Gamma_{2N}(\mathbb{S}^3) \qquad \Leftrightarrow \qquad \tilde{f}(\boldsymbol{q}_*(\cdot)) \in \Gamma_N(\mathrm{SO}(3)). \tag{3.4}$$

Proof. The addition theorems, cf. [13, Theorem 2] and [21, Section 4.7 and 4.14],

$$\sum_{k=0}^{(l+1)^2} Y_l^k(\boldsymbol{x})\overline{Y_l^k(\boldsymbol{y})} = (l+1)U_l(\cos \mathrm{d}_{\mathbb{S}^3}(\boldsymbol{x},\boldsymbol{y})), \qquad \boldsymbol{x},\boldsymbol{y} \in \mathbb{S}^3,\ l \in \mathbb{N}_0,$$

$$\sum_{m,m'=-l}^{l} D_l^{m,m'}(\boldsymbol{G})\overline{D_l^{m,m'}(\boldsymbol{H})} = (2l+1)U_{2l}\left(\cos\frac{\mathrm{d}_{\mathrm{SO}(3)}(\boldsymbol{G},\boldsymbol{H})}{2}\right), \boldsymbol{G},\boldsymbol{H} \in \mathrm{SO}(3),\ l \in \mathbb{N}_0,$$

where U_n denotes the n-th Chebyshev polynomial of second kind, lead to reproducing kernels of the spaces $\Gamma_l(\mathbb{S}^3)$ and $\Gamma_l(\mathrm{SO}(3))$ respectively. Similarly as stated in [13, Theorem 3] we can express the polynomials on $\mathbb{S}^3$ and $\mathrm{SO}(3)$ by finite linear combinations of translated kernels, that is

$$\begin{aligned}\Gamma_l(\mathbb{S}^3) &= \left\{\sum_{j=0}^{M-1} a_j U_l(\cos \mathrm{d}_{\mathbb{S}^3}(\boldsymbol{x}_j,\cdot)) \ :\ M \in \mathbb{N},\ a_j \in \mathbb{C},\ \boldsymbol{x}_j \in \mathbb{S}^3\right\},\\ \Gamma_l(\mathrm{SO}(3)) &= \left\{\sum_{j=0}^{M-1} a_j U_{2l}\left(\cos\frac{\mathrm{d}_{\mathrm{SO}(3)}(\boldsymbol{G}_j,\cdot)}{2}\right) \ :\ M \in \mathbb{N},\ a_j \in \mathbb{C},\ \boldsymbol{G}_j \in \mathrm{SO}(3)\right\}.\end{aligned} \tag{3.5}$$

Now, let $f \in \Gamma_{2N}(\mathbb{S}^3)$ be given. Then there exists $M \in \mathbb{N}$, $\boldsymbol{x}_j \in \mathbb{S}^3$ and $a_j \in \mathbb{C}$, $j = 0, \ldots, M-1$, with

$$f(\boldsymbol{x}) = \sum_{j=0}^{M-1} a_j U_{2N}(\cos \mathrm{d}_{\mathbb{S}^3}(\boldsymbol{x}_j, \boldsymbol{x})), \qquad \boldsymbol{x} \in \mathbb{S}^3,$$

and since f is even we infer from Lemma 2.1 the relation

$$\begin{aligned} f(\boldsymbol{x}) &= \sum_{j=0}^{M-1} a_j U_{2N}(\cos \mathrm{d}_{\mathbb{S}^3}(\boldsymbol{x}_j, \boldsymbol{x})) = \sum_{j=0}^{M-1} a_j U_{2N}(\cos \mathrm{d}_{\mathbb{S}^3}(\pm\boldsymbol{x}_j, \pm\boldsymbol{x})) \\ &= \sum_{j=0}^{M-1} a_j U_{2N}\left(\cos \frac{\mathrm{d}_{\mathrm{SO}(3)}\big(\boldsymbol{q}_*^{-1}(\boldsymbol{x}_{j*}), \boldsymbol{q}_*^{-1}(\boldsymbol{x}_*)\big)}{2}\right). \end{aligned}$$

Hence, it is $\tilde{f}(\boldsymbol{q}_*(\cdot)) \in \Gamma_N(\mathrm{SO}(3))$ by equation (3.5). The other direction follows similarly. ■

The above Theorem yields the following Corollary 3.3 which states an equivalence between antipodal quadrature formulae on the sphere $\mathbb{S}^3$ and quadrature formulae on the rotation group SO(3).

Corollary 3.3. A quadrature $Q(\mathbb{S}^3)$ with antipodal nodes $\mathcal{X}(\mathbb{S}^3) := \{\pm\boldsymbol{x}_0, \ldots, \pm\boldsymbol{x}_{M-1})\}$ and corresponding weights w_i, $i = 0, \ldots, M-1$, on $\mathbb{S}^3$, integrates exactly all polynomials up to degree $2N+1$, i.e.,

$$\int_{\mathbb{S}^3} f(\boldsymbol{x}) \mathrm{d}\mu_{\mathbb{S}^3}(\boldsymbol{x}) = \sum_{i=0}^{M-1} w_i(f(-\boldsymbol{x}_i) + f(\boldsymbol{x}_i)), \qquad f \in \Pi_{2N+1}(\mathbb{S}^3),$$

if and only if the quadrature $Q(\mathrm{SO}(3))$ with weights $\tilde{w}_i = 2w_i$ and nodes $\mathcal{X}(\mathrm{SO}(3)) := \{\boldsymbol{q}_*^{-1}(\boldsymbol{x}_{0*}), \ldots, \boldsymbol{q}_*^{-1}(\boldsymbol{x}_{M-1*})\}$ integrates exactly all polynomials upto degree N, i.e.,

$$\int_{\mathrm{SO}(3)} g(\boldsymbol{G}) \mathrm{d}\mu_{\mathrm{SO}(3)}(\boldsymbol{G}) = \sum_{i=0}^{M-1} \tilde{w}_i g(\boldsymbol{q}_*^{-1}(\boldsymbol{x}_{i*})), \qquad g \in \Pi_N(\mathrm{SO}(3)).$$

Proof. For the constant functions $f \equiv 1$ and $g \equiv 1$ the assertion is trivial, since the measures are normalized

$$\int_{\mathrm{SO}(3)} 1 \mathrm{d}\mu_{\mathrm{SO}(3)}(\boldsymbol{G}) = 1 = \int_{\mathbb{S}^3} 1 \mathrm{d}\mu_{\mathbb{S}^3}(\boldsymbol{x}).$$

Furthermore, arbitrary odd functions f on $\mathbb{S}^3$ are integrated exactly by

$$\int_{\mathbb{S}^3} f(\boldsymbol{x}) \mathrm{d}\mu_{\mathbb{S}^3}(\boldsymbol{x}) = 0 = \sum_{i=0}^{M-1} w_i(-f(\boldsymbol{x}_i) + f(\boldsymbol{x}_i)).$$

Hence, the assertion follows by Lemma 3.2. ■

Remark 3.4. For the quadrature $Q(\mathbb{S}^3)$ with antipodal nodes we can assume without loss of generality that the weights for antipodal nodes are equal. Since, the quadrature keeps the same degree of exactness by setting $w_i = \frac{1}{2}(w_i^+ + w_i^-)$, if w_i^+, w_i^- are the weights for $\boldsymbol{x}_i$, $-\boldsymbol{x}_i$, respectively. □

For a construction of well distributed sampling sets and quadrature formulae on the rotation group SO(3) we refer to [9]. The proposed construction is based on a tensor like product of spheres $\mathbb{S}^2$ and $\mathbb{S}^1$. Therefore, "nice" sampling sets and quadrature formulae on these spheres yield "nice" sampling sets and quadrature formulae on the rotation group SO(3). Hence, by Corollary 3.3 and Theorem 3.1 one obtains easily quadrature formulae and well distributed sampling sets on the sphere $\mathbb{S}^3$ via the isomorphism $\boldsymbol{q}_* : \mathrm{SO}(3) \to \mathbb{S}^3_*$, as well. It was also shown in [9] that the finite three-dimensional rotation groups $\mathcal{X}_\mathrm{T}, \mathcal{X}_\mathrm{O}, \mathcal{X}_\mathrm{I}$ of the tetrahedron, octahedron (or hexahedron) and icosahedron (or dodecahedron), respectively, are t-designs on the rotation group SO(3). More precisely, they are sampling sets of quadrature formulae with equal weights for degree $N = 2, 3$ and 5, respectively. The antipodal vertices of the four-dimensional polyhedra 24-cell and 600-cell can be identified with the tetrahedral group and the icosahedral group respectively. Hence, the former statement is equivalent to the assertion that these form a 5-design and a 11-design on the three-dimensional sphere, which are well known facts, cf. [1, Example 2.8].

Moreover, the above results lead to a new proof of a necessary condition for the existence of nonnegative quadrature weights on the rotation group SO(3), cf. [9, Theorem 3.3]. We remark that the condition there is exactly the same as stated in the following Theorem 3.5.

Theorem 3.5. *Let the sampling set* $\mathcal{X}_N(\mathrm{SO}(3)) = \{\boldsymbol{G}_0, \ldots, \boldsymbol{G}_{M-1}\} \subset \mathrm{SO}(3)$ *support nonnegative quadrature weights* w_k, $k = 0, \ldots, M-1$, *integrating exactly all polynomials in* $\Pi_N(\mathrm{SO}(3))$, $N \in \mathbb{N}$, *then the mesh norm satisfies*

$$\delta(\mathcal{X}_N(\mathrm{SO}(3))) \leq \frac{4\pi}{N+2}. \tag{3.6}$$

Proof. Let $\mathcal{X}_{2N+1}(\mathbb{S}^3) := \{\pm\boldsymbol{x}_0, \ldots, \pm\boldsymbol{x}_{M-1}\}$ be the corresponding antipodal sampling set on $\mathbb{S}^3$, which supports by assumption and Corollary 3.3 nonnegative quadrature weights. Then from [15, Theorem 6.21] we have the bound

$$\delta(\mathcal{X}_{2N+1}(\mathbb{S}^3)) \leq 2 \arccos z_{N+1} = \frac{2\pi}{N+2},$$

where $z_{N+1} = \cos\frac{\pi}{N+2}$ is the greatest zero of the Chebyshev polynomial of second kind U_{N+1}. Actually the bound stated by Reimer is given for quadratures with degree of exactness $2N+2$, but if we follow the proof of [15, Theorem 6.21] line by line we see that this is also true for degree of exactness $2N+1$. By Theorem 3.1 the assertion follows from

$$\delta(\mathcal{X}_N(\mathrm{SO}(3))) = 2\delta(\mathcal{X}_{2N+1}(\mathbb{S}^3)) \leq \frac{4\pi}{N+2}.$$

■

Finally we consider Marcinkiewicz-Zygmund inequalities on the manifolds $\mathcal{M} \in \{\mathbb{S}^3, \mathrm{SO}(3)\}$. Therefor we introduce for a given sampling set $\mathcal{X}(\mathcal{M}) := \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{M-1}\}$ its associated partition $\mathcal{R}(\mathcal{M}) := \{\Omega_0, \ldots, \Omega_{M-1}\}$ of closed regions $\Omega_i \subset \mathcal{M}$, $i = 0, \ldots, M-1$, where we require that $\boldsymbol{x}_j$ is an interior point of Ω_j, i.e., $\boldsymbol{x}_j \in \mathring{\Omega}_j$. Moreover, the regions cover the whole space $\mathcal{M} = \bigcup_{i=0}^{M-1} \Omega_i$ and share no common interior point, i.e., $\mathring{\Omega}_j \cap \mathring{\Omega}_i = \emptyset$ for $i \neq j$. For such partitions $\mathcal{R}(\mathcal{M})$ we define the partition norm

$$\|\mathcal{R}(\mathcal{M})\| := \max_{i=0,\ldots,M-1} \max_{\boldsymbol{x},\boldsymbol{y} \in \Omega_i} \mathrm{d}_{\mathcal{M}}(\boldsymbol{x}, \boldsymbol{y}) \tag{3.7}$$

and obtain a weighted l^p norm over the sampling values of a continuous function $f : \mathcal{M} \to \mathbb{C}$ as follows

$$\|f\|_{\mathcal{X}(\mathcal{M}),p} := \begin{cases} \left(\sum_{i=0}^{M-1} |f(\boldsymbol{x}_i)|^p \mathrm{d}\mu_{\mathcal{M}}(\Omega_i) \right)^{\frac{1}{p}} & , 1 \le p < \infty, \\ \max_{\boldsymbol{x} \in \mathcal{X}(\mathcal{M})} |f(\boldsymbol{x})| & , p = \infty. \end{cases}$$

Additionally, for $1 \le p \le \infty$ the L^p norms of measurable functions $f : \mathcal{M} \to \mathbb{C}$ are defined as usual by

$$\|f\|_p := \begin{cases} \left(\int_{\mathcal{M}} |f(\boldsymbol{x})|^p \mathrm{d}\mu_{\mathcal{M}}(\boldsymbol{x}) \right)^{\frac{1}{p}} & , 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{\boldsymbol{x} \in \mathcal{M}} |f(\boldsymbol{x})| & , p = \infty. \end{cases}$$

Marcinkiewicz-Zygmund inequalities state the equivalence between L_p norms of polynomials and l_p norms of their sampling values under certain conditions on the sampling set. These inequalities are an important tool in approximation theory, see the references [10, 12, 11, 6] to name but a few. For example, in the case $p = 1$ these inequalities yield, in contrast to Theorem 3.5, sufficient conditions for the existence of nonnegative quadrature weights as presented in [12]. Generalizations to Riemannian manifolds are given in [5, 4].

Marcinkiewicz-Zygmund inequalities on the rotation group SO(3) were already established in [18] by using reproducing kernel techniques on the rotation group directly. Here we derive such inequalities from Marcinkiewicz-Zygmund inequalities on the sphere $\mathbb{S}^3$, cf. [12, 6]. Theorem 3.6 shows that the involved constants on the rotation group SO(3) do not exceed the corresponding constants on the sphere $\mathbb{S}^3$.

Theorem 3.6. *Let $1 \le p \le \infty$ and a constant $C_p > 0$ be given.*

If for arbitrary sampling sets $\mathcal{X}(\mathbb{S}^3) = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{M-1}\}$ with associated partitions $\mathcal{R}(\mathbb{S}^3) = \{\Omega_0, \ldots, \Omega_{M-1}\}$ the Marcinkiewicz-Zygmund inequalities

$$\begin{aligned} (1-\eta)\, \|f\|_p^p \le \|f\|_{\mathcal{X}(\mathbb{S}^3),p}^p \le (1+\eta)\, \|f\|_p^p \qquad & (1 \le p < \infty), \\ (1-\eta)\, \|f\|_\infty \le \|f\|_{\mathcal{X}(\mathbb{S}^3),\infty} \le (1+\eta)\, \|f\|_\infty \qquad & (p = \infty) \end{aligned} \tag{3.8}$$

are valid for all polynomials $f \in \Pi_N(\mathbb{S}^3)$ with degree $N \le C_p \eta / \|\mathcal{R}(\mathbb{S}^3)\|$, $\eta \in (0, 1)$, *then for arbitrary sampling sets $\mathcal{X}(\mathrm{SO}(3)) = \{\boldsymbol{G}_0, \ldots, \boldsymbol{G}_{M-1}\}$ with associated partitions $\mathcal{R}(\mathrm{SO}(3)) = \{\bar{\Omega}_0, \ldots, \bar{\Omega}_{M-1}\}$ the Marcinkiewicz-Zygmund inequalities*

$$\begin{aligned} (1-\eta)\,\|g\|_p^p \le \|g\|_{\mathcal{X}(\mathrm{SO}(3)),p}^p \le (1+\eta)\,\|g\|_p^p \qquad & (1 \le p < \infty), \\ (1-\eta)\,\|g\|_\infty \le \|g\|_{\mathcal{X}(\mathrm{SO}(3)),\infty} \le (1+\eta)\,\|g\|_\infty \qquad & (p = \infty) \end{aligned} \tag{3.9}$$

are valid for all polynomials $g \in \Pi_N(\mathrm{SO}(3))$ with degree $N \le C_p\eta/\|\mathcal{R}(\mathrm{SO}(3))\|$, $\eta \in (0,1)$.

Proof. Let $g \in \Pi_N(\mathrm{SO}(3))$ be given. From Lemma 3.2 we know, there exists a unique even extension $f \in \Pi_{2N}(\mathbb{S}^3)$, i.e.,

$$g(\boldsymbol{G}) = \tilde{f}(\boldsymbol{q}_*(\boldsymbol{G})), \quad \boldsymbol{G} \in \mathrm{SO}(3). \tag{3.10}$$

Furthermore, we consider to the sampling set $\mathcal{X}(\mathrm{SO}(3))$ and its associated partition $\mathcal{R}(\mathrm{SO}(3))$ the corresponding antipodal sampling set $\mathcal{X}(\mathbb{S}^3) := \{\pm\boldsymbol{x}_0, \ldots, \pm\boldsymbol{x}_{M-1}\}$ and its associated antipodal partition $\mathcal{R}(\mathbb{S}^3) := \{\pm\Omega_0, \ldots, \pm\Omega_{M-1}\}$ on $\mathbb{S}^3$, respectively, i.e.,

$$\boldsymbol{q}_*(\boldsymbol{G}_i) = \{\pm\boldsymbol{x}_i\}, \quad \boldsymbol{q}_*(\bar{\Omega}_i) = \{\pm\Omega_i\}, \qquad i = 0, \ldots, M-1.$$

Since the partition norm of $\mathcal{R}(\mathbb{S}^3)$ obeys the relation $\|\mathcal{R}(\mathbb{S}^3)\| = \frac{1}{2}\|\mathcal{R}(\mathrm{SO}(3))\|$, cf. Lemma 2.1 and Theorem 3.1, we obtain from the condition $N \le C_p\eta/\|\mathcal{R}(\mathrm{SO}(3))\|$ that

$$2N \le 2C_p\eta/\|\mathcal{R}(\mathrm{SO}(3))\| = C_p\eta/\|\mathcal{R}(\mathbb{S}^3)\|.$$

Hence, by the Marcinkiewicz-Zygmund inequalities (3.8) on the sphere $\mathbb{S}^3$ we have

$$(1-\eta)\,\|f\|_p^p \le \|f\|_{\mathcal{X}(\mathbb{S}^3),p}^p \le (1+\eta)\,\|f\|_p^p.$$

In the case $p = \infty$ the assertion (3.9) follows immediately from (3.10). In the cases $1 \le p < \infty$ we infer from Lemma 2.1 that

$$\|f\|_p^p = \int |f(\boldsymbol{x})|^p \mathrm{d}\mu_{\mathbb{S}^3}(\boldsymbol{x}) = \int |\tilde{f}(\boldsymbol{x}_*)|^p \mathrm{d}\mu_{\mathbb{S}^3_*}(\boldsymbol{x}_*) = \int |g(\boldsymbol{G})|^p \mathrm{d}\mu_{\mathrm{SO}(3)}(\boldsymbol{G}) = \|g\|_p^p,$$

and derive from definition (2.8) the relation $\mu_{\mathbb{S}^3}(\Omega_i) = \mu_{\mathbb{S}^3}(-\Omega_i) = \frac{1}{2}\mu_{\mathbb{S}^3_*}(\Omega_{i*}) = \frac{1}{2}\mu_{\mathrm{SO}(3)}(\bar{\Omega}_i)$. Together with $f(\pm\boldsymbol{x}_i) = g(\boldsymbol{q}_*^{-1}(\boldsymbol{x}_{i*})) = g(\boldsymbol{G}_i)$ we obtain the assertion (3.9). ■

Remark 3.7. The converse statement of Theorem 3.6 need not to be true, due to the missing counterpart on the rotation group SO(3) for odd functions on the sphere $\mathbb{S}^3$. Hence, this approach might not lead to sharp conditions and one has to operate on the rotation group directly as proposed by Schmid in [18, 20]. □

One should not that by no means it is obvious that the constants C_p in Theorem 3.6 exists for every $1 < p < \infty$, but such results were recently proved in a broad context in [4]. In the case $p \in \{1, \infty\}$ we can apply Theorem 3.6 to [6, Theorem 4.2] and obtain explicit constants for Marcinkiewicz-Zygmund inequalities on the rotation group SO(3). Unfortunately the Marcinkiewicz-Zygmund inequalities in [6, Theorem 4.2] and [20, Theorem 4.22] are stated in slightly different form such that a direct comparison of the involved constants is pointless.

4 Conclusion

In this paper we showed that in the setting of scattered data approximation the rotation group SO(3) and the sphere $\mathbb{S}^3$ can be treated almost equally. That is for scattered sampling nodes one obtains in the natural metrics the same values for the mesh norm and the separation distance up to a proportional factor, if we identify antipodal nodes on $\mathbb{S}^3$ with a rotation in SO(3). For the polynomial spaces we can identify even functions on $\mathbb{S}^3$ with functions on SO(3). However this leads to a discrepancy in the approximation behavior for scattered data on those manifolds, since the odd functions on $\mathbb{S}^3$ lack a counter part on SO(3).

Acknowledgment

The author gratefully acknowledge support by German Research Foundation within the project PO 711/9-2 and thanks the referees for their valuable suggestions.

References

[1] E. Bannai and E. Bannai. A survey on spherical designs and algebraic combinatorics on spheres. *European J. Combin.*, 30:1392 – 1425, 2009.

[2] G. E. Bredon. *Topology and Geometry*, volume 139 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1993.

[3] M. D. Buhmann. *Radial Basis Functions*. Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, Cambridge, 2003.

[4] F. Filbir and H. Mhaskar. Marcinkiewicz-Zygmund measures on manifolds. *arXiv:1006.5123v2 [math.CA]*, 2010.

[5] F. Filbir and H. Mhaskar. A quadrature formula for diffusion polynomials corresponding to a generalized heat kernel. *J. Fourier Anal. Appl.*, 16:629 – 657, 2010.

[6] F. Filbir and W. Themistoclakis. Polynomial approximation on the sphere using scattered data. *Math. Nachr.*, 281:650 – 668, 2008.

[7] G. B. Folland. *A course in abstract harmonic analysis*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[8] M. Gräf and S. Kunis. Stability results for scattered data interpolation on the rotation group. *Electron. Trans. Numer. Anal.*, 31:30 – 39, 2008.

[9] M. Gräf and D. Potts. Sampling sets and quadrature formulae on the rotation group. *Numer. Funct. Anal. Optim.*, 30:665 – 688, 2009.

[10] K. Gröchenig. Reconstruction algorithms in irregular sampling. *Math. Comput.*, 59:181 – 194, 1992.

[11] J. Keiner, S. Kunis, and D. Potts. Efficient reconstruction of functions on the sphere from scattered data. *J. Fourier Anal. Appl.*, 13:435 – 458, 2007.

[12] H. N. Mhaskar, F. J. Narcowich, and J. D. Ward. Spherical Marcinkiewicz-Zygmund inequalities and positive quadrature. *Math. Comput.*, 70:1113 – 1130, 2001. Corrigendum on the positivity of the quadrature weights in 71:453 – 454, 2002.

[13] C. Müller. *Spherical Harmonics*. Springer, Aachen, 1966.

[14] D. Potts, J. Prestin, and A. Vollrath. A fast algorithm for nonequispaced Fourier transforms on the rotation group. *Numer. Algorithms*, 52:355 – 384, 2009.

[15] M. Reimer. *Multivariate Polynomial Approximation*. Birkhäuser Verlag, Basel, 2003.

[16] R. Schaback. Error estimates and condition numbers for radial basis function interpolation. *Adv. Comput. Math.*, 3:251 – 264, 1995.

[17] R. Schaback. Reconstruction of multivariate functions from scattered data. `http://www.num.math.uni-goettingen.de/schaback/`, 1997. manuscript.

[18] D. Schmid. Marcinkiewicz-Zygmund inequalities and polynomial approximation from scattered data on $SO(3)$. *Numer. Funct. Anal. Optim.*, 29:855 – 882, 2008.

[19] D. Schmid. A trade-off principle in connection with the approximation by positive definite kernels. In C. K. Chui, M. Neamtu, and L. L. Schumaker, editors, *Approximation Theory XII: San Antonio*, pages 348 – 359, Brentwood, 2008. Nashboro Press.

[20] D. Schmid. Scattered data approximation on the rotation group and generalizations. Dissertation, Fakultät für Mathematik, Technische Universität München, 2009.

[21] D. Varshalovich, A. Moskalev, and V. Khersonskii. *Quantum Theory of Angular Momentum*. World Scientific Publishing, Singapore, 1988.