Interpolatory Wavelets on the Sphere

Daniel Potts and Manfred Tasche

Dedicated to Prof. E. W. Cheney on the occasion
of his 65th birthday

Abstract. In this paper, we construct an interpolatory wavelet basis on the unit sphere $S \subset \mathbb{R}^3$. Using spherical coordinates, we apply the tensor product of interpolatory trigonometric and algebraic polynomial wavelets. The described decomposition and reconstruction algorithms work in the frequency domain.

§0. Introduction

The wavelet theory is closely related to shift–invariant subspaces of $L^2(\mathbb{R}^d)$. Therefore, wavelets are naturally adapted to problems on the whole space $\mathbb{R}^d$. Very often, one has to deal with functions defined on a bounded domain such that the construction of wavelets on a domain is desirable. This problem is solved for the torus $\mathbf{T} := \mathbb{R}/2\pi\mathbf{Z}$ [5], and for compact intervals [1, 4]. In the sequel, we will apply 2π–periodic trigonometric polynomial wavelets [6] and algebraic polynomial wavelets on $I := [-1, 1]$ [3, 9] in order to construct interpolatory wavelets on the unit sphere $S \subset \mathbb{R}^3$. Other approachs to wavelets on the sphere can be found in [2, 7].

Using modified spherical coordinates $x := (x_1, x_2) \in H := \mathbf{T} \times I$, we can identify the sphere S with H by the mapping $\chi : H \to S$ with $(z_1, z_2, z_3) = \chi(x) \in S$,

$$z_1 := \cos x_1 \, \cos \frac{\pi x_2}{2}, \quad z_2 := \sin x_1 \, \cos \frac{\pi x_2}{2}, \quad z_3 := \sin \frac{\pi x_2}{2},$$

where $\mathbf{T} \times \{\pm 1\}$ corresponds to the north/south pole of S. Let $F : S \to \mathbb{R}$ be given and let $f := F \circ \chi : H \to \mathbb{R}$ be its coordinate representation.

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Then $F \in C^1(S)$ if and only if $f \in C^1(H)$ fulfils the following conditions (see [8])

$$f(x_1, \pm 1) = c_\pm, \tag{0.1}$$

$$(\partial_{x_2} f)(x_1, \pm 1) = a_\pm \cos x_1 + b_\pm \sin x_1, \tag{0.2}$$

$$(\partial_{x_1} \partial_{x_2} f)(x_1, \pm 1) = -a_\pm \sin x_1 + b_\pm \cos x_1, \tag{0.3}$$

where the mixed derivative exists for all $x_1 \in \mathbf{T}$. Here $a_\pm$, $b_\pm$, $c_\pm$ denote some constants. Note that (0.1) corresponds to the continuity of F at both poles of S. The conditions (0.2) – (0.3) assure that the tangent plane of F varies continuously at the poles of S.

The aim of this paper is to construct an almost smooth wavelet decomposition of functions defined on the unit sphere S. Roughly spoken, a function F is almost smooth on S, if $f := F \circ \chi \in C^1(H)$ with (0.1) satisfies the conditions (0.2) – (0.3) approximately.

This paper is organized as follows. In Sections 1 and 2, we briefly recall the construction of trigonometric polynomial wavelets on $\mathbf{T}$ and of algebraic polynomial wavelets on I. Using tensor product approach, we introduce a multiresolution of the weighted Hilbert space $L^2_w(H)$ in Section 3. Finally, we describe decomposition and reconstruction algorithms.

§1. Trigonometric polynomial wavelets on T

In the following, we sketch the construction of interpolatory trigonometric polynomial wavelets on $\mathbf{T}$. For $f \in L^2(\mathbf{T})$, let

$$c_u^1(f) := \frac{1}{2\pi} \int_0^{2\pi} f(t)\, e^{-\mathrm{i}ut}\, \mathrm{d}t \quad (u \in \mathbf{Z}).$$

With the Dirichlet kernel

$$D_l^1 := \frac{1}{2} + \sum_{k=1}^{l} \cos(k\cdot) \quad (l \in \mathbb{N})$$

and $M_j := 2^{j+3}$ ($j \in \mathbb{N}_0$), we consider the de la Vallée Poussin kernel as scaling function of level j

$$M_j\, \varphi_j^1 := \frac{1}{2^j} \sum_{l=3\cdot 2^j}^{5\cdot 2^j - 1} D_l^1.$$

Hence we have $c_u^1(\varphi_j^1) = c_{-u}^1(\varphi_j^1)$ ($u \in \mathbf{Z}$) and for $u \in \mathbb{N}_0$

$$M_j\, c_u^1(\varphi_j^1) = \begin{cases} 1 & u = 0, \ldots, 3\cdot 2^j, \\ 2^{-1}(5 - 2^{-j}u) & u = 3\cdot 2^j + 1, \ldots, 5\cdot 2^j - 1, \\ 0 & u = 5\cdot 2^j, 5\cdot 2^j + 1, \ldots . \end{cases}$$

Note that the transformed two–scale relation of φ_j^1 reads as follows

$$c_u^1(\varphi_j^1) = A_{j+1,u}^1\, c_u^1(\varphi_{j+1}^1) \quad (u \in \mathbf{Z}) \tag{1.1}$$

with $A_{j+1,u}^1 = A_{j+1,-u}^1 = A_{j+1,u+M_{j+1}}$ $(u \in \mathbf{Z})$ and

$$A_{j+1,u}^1 := \begin{cases} 2 & u = 0, \dots, 3 \cdot 2^j, \\ 5 - 2^{-j}u & u = 3 \cdot 2^j + 1, \dots, 5 \cdot 2^j - 1, \\ 0 & u = 5 \cdot 2^j, \dots, M_j. \end{cases}$$

With $h_j^1 := 2\pi/M_j$ we consider the shift operators $\sigma_{j,k}^1 : L^2(\mathbf{T}) \to L^2(\mathbf{T})$ defined by $\sigma_{j,k}^1 f := f(\cdot - kh_j^1)$ $(k \in I_j^1)$, where I_j^1 denotes the index set $\{0, \dots, M_j - 1\}$. Introduce the shifted scaling functions $\varphi_{j,k}^1 := \sigma_{j,k}^1\, \varphi_j^1$. Then $\varphi_{j,k}^1$ are Lagrange fundamental functions with respect to the equidistant grid $G_j^1 := h_j^1\, I_j^1$, since we have $\varphi_{j,k}^1(lh_j^1) = \delta_{k,l}$ $(k, l \in I_j^1)$.

As sample space of level j we use $V_j^1 := \operatorname{span}\{\varphi_{j,k}^1 : k \in I_j^1\}$. Obviously, the operator $L_j^1 : C(\mathbf{T}) \to V_j^1$ defined by

$$L_j^1 f := \sum_{k \in I_j^1} f(kh_j^1)\, \varphi_{j,k}^1 \quad (f \in C(\mathbf{T}))$$

is the interpolation projector onto V_j^1 with respect to the grid G_j^1. Then $\{V_j^1\}_{j=0}^{\infty}$ forms a multiresolution of $L^2(\mathbf{T})$ (see [5, 6]).

Setting $\psi_j^1 := (2\varphi_{j+1}^1 - \varphi_j^1)(\cdot - h_{j+1}^1)$, then we obtain $c_u^1(\psi_j^1) = \overline{c_{-u}^1(\psi_j^1)}$ $(u \in \mathbf{Z})$ and for $u \in \mathbb{N}_0$

$$M_{j+1}\, c_u^1(\psi_j^1) = \begin{cases} (2^{-j}u - 3)\, \omega_{j+1}^u & u = 3 \cdot 2^j + 1, \dots, 5 \cdot 2^j - 1, \\ 2\, \omega_{j+1}^u & u = 5 \cdot 2^j, \dots, 3 \cdot 2^{j+1}, \\ (5 - 2^{-j-1}u)\, \omega_{j+1}^u & u = 3 \cdot 2^{j+1} + 1, \dots, 5 \cdot 2^{j+1} - 1, \\ 0 & \text{otherwise} \end{cases}$$

with $\omega_j := \exp(-2\pi \mathrm{i}/M_j)$. Hence the transformed two-scale relation of ψ_j^1 reads as follows

$$c_u^1(\psi_j^1) = B_{j+1,u}^1\, c_u^1(\varphi_{j+1}^1) \quad (u \in \mathbf{Z}) \tag{1.2}$$

with $B_{j+1,u}^1 = \overline{B_{j+1,-u}^1} = B_{j+1,u+M_{j+1}}$ and

$$B_{j+1,u}^1 := \begin{cases} 0 & u = 0, \dots, 3 \cdot 2^j, \\ (2^{-j}u - 3)\, \omega_{j+1}^u & u = 3 \cdot 2^j + 1, \dots, 5 \cdot 2^j - 1, \\ 2\, \omega_{j+1}^u & u = 5 \cdot 2^j, \dots, M_j. \end{cases}$$

Let $W_j^1 := V_{j+1}^1 \ominus V_j^1$ be the wavelet space of level j. Then we obtain that $W_j^1 = \operatorname{span}\{\psi_{j,k}^1 := \psi_j^1(\cdot - kh_j^1) : k \in I_j^1\}$.

§2. Algebraic polynomial wavelets on I

Now we briefly describe interpolatory polynomial wavelets on the interval I. Let $w(x) := (1-x^2)^{-1/2}$ $(x \in (-1,\, 1))$. For $f \in L^2_w(I)$, let

$$c^2_n(f) := \frac{2}{\pi} \int_I w(x)\, f(x)\, T_n(x)\, \mathrm{d}x \quad (n \in \mathbb{N}_0),$$

where T_n denotes the n-th Chebyshev polynomial. With the Chebyshev–Dirichlet kernel

$$D^2_l := \frac{1}{2} + \sum_{k=1}^{l} T_k \quad (l \in \mathbb{N})$$

and $N_j := 2^{j+2}$ $(j \in \mathbb{N}_0)$, we consider the corresponding de la Vallée Poussin mean as scaling function φ^2_j of level j:

$$N_j\, \varphi^2_j := \frac{1}{2^j} \sum_{l=3\cdot 2^j}^{5\cdot 2^j-1} D^2_l .$$

Hence we have

$$N_j\, c^2_n(\varphi^2_j) = \begin{cases} 2 & n = 0, \dots, 3\cdot 2^j, \\ 5 - 2^{-j} n & n = 3\cdot 2^j + 1, \dots, 5\cdot 2^j - 1, \\ 0 & n = 5\cdot 2^j, 5\cdot 2^j + 1, \dots . \end{cases}$$

The transformed two–scale relation of φ^2_j reads as follows

$$c^2_n(\varphi^2_j) = A^2_{j+1,n}\, c^2_n(\varphi^2_{j+1}) \quad (n \in \mathbb{N}_0) \tag{2.1}$$

with $A^2_{j+1,n} = A^2_{j+1,n+N_{j+2}}$, $A^2_{j+1,N_{j+2}-l} = A^2_{j+1,l}$ $(l = 0, \dots, N_{j+1})$ and

$$A^2_{j+1,n} := \begin{cases} 2 & n = 0, \dots, 3\cdot 2^j, \\ 5 - 2^{-j} n & n = 3\cdot 2^j + 1, \dots, 5\cdot 2^j - 1, \\ 0 & n = 5\cdot 2^j, \dots, N_{j+1}. \end{cases}$$

Setting $h^2_{j,v} := \cos \frac{v\pi}{N_j}$ $(v \in \mathbf{Z})$, we introduce the Chebyshev–shift operators $\sigma^2_{j,v} : L^2_w(I) \to L^2_w(I)$ by $c^2_n(\sigma^2_{j,v} f) = h^2_{j,nv}\, c^2_n(f)$ $(f \in L^2_w(I))$. Put $\varphi^2_{j,v} := \sigma^2_{j,v}\, \varphi^2_j$ $(v \in \mathbf{Z})$. Then $\varphi^2_{j,n} \in \Pi_{5\cdot 2^j - 1}$ fulfils the interpolation properties $\varphi^2_{j,n}(h^2_{j,l}) = \varepsilon^{-1}_{j,l}\, \delta_{n,l}$ with $\varepsilon_{j,0} = \varepsilon_{j,N_j} := 1/2$, $\varepsilon_{j,l} := 1$ $(l = 1, \dots, N_j - 1)$. Hence $\varphi^2_{j,n}$ are modified Lagrange fundamental polynomials related to the Chebyshev grid $G^2_j := \{h^2_{j,k} : k \in I^2_j\}$ with $I^2_j := \{0, \dots, N_j\}$.

As sample space of level j ($j \in \mathbb{N}_0$) we use $V_j^2 := \operatorname{span}\{\varphi_{j,k}^2 : k \in I_j^2\}$. Obviously, the operator $L_j^2 : C(I) \to V_j^2$ defined by

$$L_j^2 f := \sum_{k \in I_j^2} \varepsilon_{j,k}\, f(h_{j,k}^2)\, \varphi_{j,k}^2 \quad (f \in C(I))$$

is the interpolation projector onto V_j^2 with respect to the grid G_j^2. Then $\{V_j^2\}_{j=0}^{\infty}$ forms a multiresolution of $L_w^2(I)$ (see [3, 4, 9]).

Introducing $\psi_{j+1}^2 \in V_{j+1}^2$ by

$$N_j\, c_n^2(\psi_{j+1}^2) := \begin{cases} 2^{-j}n - 3 & n = 3\cdot 2^j + 1, \ldots, 5\cdot 2^j - 1, \\ 2 & n = 5\cdot 2^j, \ldots, 3\cdot 2^{j+1}, \\ 5 - 2^{-j-1}n & n = 3\cdot 2^{j+1} + 1, \ldots, 5\cdot 2^{j+1} - 1, \\ 0 & \text{otherwise}, \end{cases}$$

then we obtain the transformed two–scale relation of ψ_{j+1}^2

$$c_n^2(\psi_{j+1}^2) = B_{j+1,n}^2\, c_n^2(\varphi_{j+1}^2) \quad (n \in \mathbb{N}_0) \tag{2.2}$$

with $B_{j+1,n}^2 = B_{j+1,n+N_{j+2}}^2$ ($n \in \mathbb{N}_0$), $B_{j+1,N_{j+2}-l}^2 = B_{j+1,l}^2$ ($l \in I_{j+1}^2$) and

$$B_{j+1,n}^2 := \begin{cases} 0 & n = 0, \ldots, 3\cdot 2^j, \\ 2^{-j}n - 3 & n = 3\cdot 2^j + 1, \ldots, 5\cdot 2^j - 1, \\ 2 & n = 5\cdot 2^j, \ldots, N_{j+1}. \end{cases}$$

Let $W_j^2 := V_{j+1}^2 \ominus V_j^2$ be the wavelet space of level j. Then we have $W_j^2 = \operatorname{span}\{\psi_{j+1,2k+1}^2 := \sigma_{j+1,2k+1}^2 \psi_{j+1}^2 : k \in K_j^2\}$ with $K_j^2 := \{0, \ldots, N_j - 1\}$.

§3. Multiresolution of $L_w^2(H)$

Let $L_w^2(H)$ be the Hilbert space of all functions $f : H \to \mathbb{R}$ with

$$2\pi^2\, \|f\|^2 := \int_D w(x_2)\, |f(x)|^2\, \mathrm{d}x < \infty$$

with $D := [0, 2\pi] \times I$ and $x := (x_1, x_2)$. Applying the tensor product method, we introduce a multiresolution of $L_w^2(H)$. For $f \in L_w^2(H)$, let

$$\pi^2\, c_k(f) := \int_D w(x_2)\, f(x)\, e^{-\mathrm{i}k_1 x_1}\, T_{k_2}(x_2)\, \mathrm{d}x$$

with $k := (k_1, k_2) \in \mathbf{Z} \times \mathbb{N}_0$. As *sample space of level* j we use $V_j := V_j^1 \otimes V_j^2$. Hence we have $V_j = \operatorname{span}\{\varphi_{j,k} : k \in I_j\}$ with $I_j := I_j^1 \times I_j^2$

and $\varphi_{j,k} := \varphi^1_{j,k_1} \otimes \varphi^2_{j,k_2}$. By (1.1) and (2.1), the transformed two–scale relation of the j-th scaling function $\varphi_j := \varphi^1_j \otimes \varphi^2_j$ reads as follows

$$c_k(\varphi_j) = A_{j+1,k}\, c_k(\varphi_{j+1}) \quad (k \in \mathbf{Z} \times \mathbb{N}_0) \tag{3.1}$$

with $A_{j+1,k} := A^1_{j+1,k_1} A^2_{j+1,k_2}$. Since we have $\epsilon_{j,l_2}\, \varphi_{j,k}(h_{j,l}) = \delta_{k_1,l_1}\, \delta_{k_2,l_2}$ $(k, l \in I_j)$ for $h_{j,l} := (l_1 h^1_j, h^2_{j,l_2})$, the shifted scaling functions $\varphi_{j,k}$ are modified Lagrange functions with respect to the grid $G_j := G^1_j \times G^2_j$. Therefore, the operator $L_j := L^1_j \otimes L^2_j : C(H) \to V_j$ given by

$$L_j f := \sum_{k \in I_j} \varepsilon_{j,k_2}\, f(h_{j,k})\, \varphi_{j,k} \quad (f \in C(H))$$

is the interpolation projector onto V_j with respect to the grid G_j. The points of $\chi(G_j)$ are lying more dense at the poles of S. If $f \in C^1(H)$ with (0.1) – (0.3) is given, then $L_j f$ fulfils (0.1) exactly and (0.2) – (0.3) approximately in the following sense that the partial derivatives in (0.2) – (0.3) must be replaced by corresponding difference quotients on G_j. From $L_j f|G_j = f|G_j$ it follows that $L_j f$ is almost smooth. Then $\{V_j\}_{j=0}^{\infty}$ forms a multiresolution of $L^2_w(H)$ in the following sense:

(M 1) $\quad V_j \subset V_{j+1}, \quad \mathrm{clos}(\bigcup_{j=0}^{\infty} V_j) = L^2_w(H)$.

(M 2) For all $j \in \mathbb{N}_0$ and for any $\alpha_{j,k} \in \mathbb{R}$ $(k \in I_j)$, we have

$$\frac{1}{4} \sum_{k \in I_j} \varepsilon_{j,k_2}\, |\alpha_{j,k}|^2 \le M_j N_j \, \| \sum_{k \in I_j} \varepsilon_{j,k_2}\, \alpha_{j,k}\, \varphi_{j,k} \|^2 \le \sum_{k \in I_j} \varepsilon_{j,k_2}\, |\alpha_{j,k}|^2 .$$

As known, the orthogonal complement W_j of V_j in V_{j+1} is generated by V^1_j, V^2_j, W^1_j and W^2_j. We obtain that $W_j = W^{\mathrm{I}}_j \oplus W^{\mathrm{II}}_j \oplus W^{\mathrm{III}}_j$ with the *wavelet spaces* $W^{\mathrm{I}}_j := V^1_j \otimes W^2_j$, $W^{\mathrm{II}}_j := W^1_j \otimes V^2_j$ and $W^{\mathrm{III}}_j := W^1_j \otimes W^2_j$. Then we have

$$W^{\mathrm{I,\,III}}_j = \mathrm{span}\, \{\psi^{\mathrm{I,\,III}}_{j,k} : k \in K_j\}, \quad W^{\mathrm{II}}_j = \mathrm{span}\, \{\psi^{\mathrm{II}}_{j,k} : k \in I_j\}$$

with $K_j := I^1_j \times K^2_j$ and with $\psi^{\mathrm{I}}_{j,k} := \varphi^1_{j,k_1} \otimes \psi^2_{j+1,2k_2+1}$, $\psi^{\mathrm{II}}_{j,k} := \psi^1_{j,k_1} \otimes \varphi^2_{j,k_2}$ and $\psi^{\mathrm{III}}_{j,k} := \psi^1_{j,k_1} \otimes \psi^2_{j+1,2k_2+1}$. By (1.1) – (1.2) and (2.1) – (2.2), the transformed two–scale relations of the wavelets $\psi^{\mathrm{I}}_j := \varphi^1_j \otimes \psi^2_{j+1}$, $\psi^{\mathrm{II}}_j := \psi^1_j \otimes \varphi^2_j$ and $\psi^{\mathrm{III}}_j := \psi^1_j \otimes \psi^2_{j+1}$ read as follows

$$c_l(\psi^{\mathrm{I-III}}_j) = B^{\mathrm{I-III}}_{j+1,l}\, c_l(\varphi_{j+1}) \quad (l = (l_1, l_2) \in \mathbf{Z} \times \mathbb{N}_0) \tag{3.2}$$

with $B^{\mathrm{I}}_{j+1,l} := A^1_{j+1,l_1}\, B^2_{j+1,l_2}$, $B^{\mathrm{II}}_{j+1,l} := B^1_{j+1,l_1}\, A^2_{j+1,l_2}$ and $B^{\mathrm{III}}_{j+1,l} := B^1_{j+1,l_1}\, B^2_{j+1,l_2}$.

§4. Decomposition and reconstruction algorithms

Now we derive efficient decomposition and reconstruction algorithms. In order to decompose a given function $f_{j+1} \in V_{j+1}$ of the form

$$f_{j+1} = \sum_{k \in I_{j+1}} \varepsilon_{j+1,k_2}\, \alpha_{j+1,k}\, \varphi_{j+1,k}\,, \tag{4.1}$$

uniquely determined functions $f_j \in V_j$ and $g_j^{\mathrm{I-III}} \in W_j^{\mathrm{I-III}}$ have to be found such that $f_{j+1} = f_j + g_j^{\mathrm{I}} + g_j^{\mathrm{II}} + g_j^{\mathrm{III}}$. Assume that the coefficients $\alpha_{j+1,k} \in \mathbb{R}$ of f_{j+1} or their transformed data

$$\hat{\alpha}_{j+1,l} := \sum_{k \in I_{j+1}} \varepsilon_{j+1,k_2}\, \alpha_{j+1,k}\, \omega_{j+1}^{k_1 l_1} \cos \frac{k_2 l_2 \pi}{N_{j+1}} \tag{4.2}$$

are known. The wanted functions $f_j \in V_j$ and $g_j^{\mathrm{I-III}} \in W_j^{\mathrm{I-III}}$ can be uniquely represented in the form (4.1) and

$$g_j^{\mathrm{I,III}} = \sum_{k \in K_j} \beta_{j,k}^{\mathrm{I,III}}\, \psi_{j+1,k}^{\mathrm{I,III}}\,, \quad g_j^{\mathrm{II}} = \sum_{k \in I_j} \varepsilon_{j,k_2}\, \beta_{j,k}^{\mathrm{II}}\, \psi_{j+1,k}^{\mathrm{II}}$$

with unknown coefficients $\alpha_{j,k}$, $\beta_{j,k}^{\mathrm{I-III}} \in \mathbb{R}$. Let $\hat{\alpha}_{j,l}$, $\hat{\beta}_{j,l}^{\mathrm{I-III}} \in \mathbb{C}$ be transformed data given in the form (4.2) or by

$$\hat{\beta}_{j,l}^{\mathrm{I,III}} := \sum_{k \in I_j} \beta_{j,k}^{\mathrm{I,III}}\, \omega_j^{k_1 l_1} \cos \frac{(2k_2+1) l_2 \pi}{N_{j+1}}, \tag{4.3}$$

$$\hat{\beta}_{j,l}^{\mathrm{II}} := \sum_{k \in K_j} \varepsilon_{j,k_2}\, \beta_{j,k}^{\mathrm{II}}\, \omega_j^{k_1 l_1} \cos \frac{k_2 l_2 \pi}{N_j}. \tag{4.4}$$

Note that the discrete transforms (4.2) – (4.4) and their inverses can be realized by the row–column method using fast Fourier transform and fast algorithms of discrete cosine transforms.

In order to reconstruct $f_{j+1} \in V_{j+1}$ ($j \in \mathbb{N}_0$), we have to compute the sum $f_{j+1} = f_j + g_j^{\mathrm{I}} + g_j^{\mathrm{II}} + g_j^{\mathrm{III}}$ with given functions $f_j \in V_j$ and $g_j^{\mathrm{I-III}} \in W_j^{\mathrm{I-III}}$. Assume that $\alpha_{j,k}$, $\beta_{j,k}^{\mathrm{I-III}} \in \mathbb{R}$ or their transformed data are known. Then $f_{j+1} \in V_{j+1}$ can be uniquely represented in the form (4.1).

The decomposition and reconstruction algorithms are based on the following connection between $\hat{\alpha}_{j+1,l}$ and $\hat{\alpha}_{j,l}$, $\hat{\beta}_{j,l}^{\mathrm{I-III}}$. Introducing for $l = (l_1, l_2) \in I_j$ the two-scale symbol matrices

$$S_{j+1,l_1}^1 := \begin{pmatrix} A_{j+1,l_1}^1 & B_{j+1,l_1}^1 \\ A_{j+1,M_j+l_1}^1 & B_{j+1,M_j+l_1}^1 \end{pmatrix},$$

$$S_{j+1,l_2}^2 := \begin{pmatrix} A_{j+1,l_2}^2 & B_{j+1,l_2}^2 \\ A_{j+1,N_{j+1}-l_2}^2 & -B_{j+1,N_{j+1}-l_2}^2 \end{pmatrix}$$

and the Kronecker product $S_{j+1,l} := S^1_{j+1,l_1} \otimes S^2_{j+1,l_2}$, then we obtain by (3.1) – (3.2) that for $l \in K_j$

$$\begin{pmatrix} \hat{\alpha}_{j+1,l_1,l_2} \\ \hat{\alpha}_{j+1,l_1,N_{j+1}-l_2} \\ \hat{\alpha}_{j+1,M_j+l_1,l_2} \\ \hat{\alpha}_{j+1,M_j+l_1,N_{j+1}-l_2} \end{pmatrix} = S_{j+1,l} \begin{pmatrix} \hat{\alpha}_{j,l} \\ \hat{\beta}^{\mathrm{I}}_{j,l} \\ \hat{\beta}^{\mathrm{II}}_{j,l} \\ \hat{\beta}^{\mathrm{III}}_{j,l} \end{pmatrix}$$

and for $l_1 \in I^1_j$

$$\begin{pmatrix} \hat{\alpha}_{j+1,l_1,N_j} \\ \hat{\alpha}_{j+1,M_j+l_1,N_j} \end{pmatrix} = S^1_{j+1,l_1} \begin{pmatrix} \hat{\alpha}_{j,l_1,N_j} \\ \hat{\beta}^{\mathrm{II}}_{j,l_1,N_j} \end{pmatrix}.$$

Note that $S_{j+1,l}$ ($l \in K_j$) and S^1_{j+1,l_1} ($l_1 \in I^1_j$) are regular matrices.

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