Numerical Fourier Analysis: Theory and Applications

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based on the book "Numerical Fourier Analysis" by

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Outline

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Fourier coefficients and Fourier series

A complex-valued function $f : \mathbb{R} \to \mathbb{C}$ is 2π-*periodic* or *periodic with period* 2π, if $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$.
In the following, we identify any 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$ with the corresponding function $f : \mathbb{T} \to \mathbb{C}$ defined on the *torus* $\mathbb{T}$ of length 2π.
The torus $\mathbb{T}$ can be considered as quotient space $\mathbb{R}/(2\pi\mathbb{Z})$ or its representatives, e.g. the interval $[0, 2\pi]$ with identified endpoints 0 and 2π. For short, one can also geometrically think of the unit circle with circumference 2π.
Typical examples of 2π-periodic functions are 1, $\cos(n\cdot)$, $\sin(n\cdot)$ for each angular frequency $n \in \mathbb{N}$ and the complex exponentials $\mathrm{e}^{\mathrm{i}k\cdot}$ for each $k \in \mathbb{Z}$.

By $C(\mathbb{T})$ we denote the Banach space of all continuous functions $f : \mathbb{T} \to \mathbb{C}$ with the norm

$$\|f\|_{C(\mathbb{T})} := \max_{x \in \mathbb{T}} |f(x)|$$

and by $C^r(\mathbb{T})$, $r \in \mathbb{N}$ the Banach space of r-times continuously differentiable functions $f : \mathbb{T} \to \mathbb{C}$ with the norm

$$\|f\|_{C^r(\mathbb{T})} := \|f\|_{C(\mathbb{T})} + \|f^{(r)}\|_{C(\mathbb{T})} .$$

Clearly, we have $C^r(\mathbb{T}) \subset C^s(\mathbb{T})$ for $r > s$.

Let $L_p(\mathbb{T})$, $1 \le p \le \infty$ be the Banach space of (equivalence classes of) measurable functions $f : \mathbb{T} \to \mathbb{C}$ with finite norm

$$\|f\|_{L_p(\mathbb{T})} := \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^p \, \mathrm{d}x \right)^{1/p}, \quad 1 \le p < \infty\,,$$
$$\|f\|_{L_\infty(\mathbb{T})} := \operatorname{ess\,sup}\{|f(x)| : x \in \mathbb{T}\}\,.$$

If a 2π-periodic function f is integrable on $[-\pi, \pi]$, then we have

$$\int_{-\pi}^{\pi} f(x) \, \mathrm{d}x = \int_{-\pi+a}^{\pi+a} f(x) \, \mathrm{d}x$$

for all $a \in \mathbb{R}$ so that we can integrate over any interval of length 2π.

Using Hölder's inequality it can be shown that the spaces $L_p(\mathbb{T})$ for $1 \le p \le \infty$ are continuously embedded as

$$L_1(\mathbb{T}) \supset L_2(\mathbb{T}) \supset \ldots \supset L_\infty(\mathbb{T}).$$

In the following we are mainly interested in the Hilbert space $L_2(\mathbb{T})$ consisting of all absolutely square-integrable functions $f : \mathbb{T} \to \mathbb{C}$ with inner product and norm

$$\langle f, g \rangle_{L_2(\mathbb{T})} := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, \overline{g(x)}\, \mathrm{d}x\,,$$

$$\|f\|_{L_2(\mathbb{T})} := \Big(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, \mathrm{d}x \Big)^{1/2}.$$

If it is clear from the context which inner product or norm is addressed, we abbreviate $\langle f, g \rangle := \langle f, g \rangle_{L_2(\mathbb{T})}$ and $\|f\| := \|f\|_{L_2(\mathbb{T})}$. For all $f, g \in L_2(\mathbb{T})$ it holds the Cauchy–Schwarz inequality

$$|\langle f, g \rangle_{L_2(\mathbb{T})}| \le \|f\|_{L_2(\mathbb{T})}\, \|g\|_{L_2(\mathbb{T})}\,.$$

Theorem 1

The set of complex exponentials

$$\left\{\mathrm{e}^{\mathrm{i}k\cdot} = \cos(k\cdot) + \mathrm{i}\,\sin(k\cdot) :\; k \in \mathbb{Z}\right\} \tag{1}$$

forms an orthonormal basis of $L_2(\mathbb{T})$.

Proof: By definition, an orthonormal basis is a complete orthonormal system. First we show the orthonormality of the complex exponentials in (1). We have

$$\langle \mathrm{e}^{\mathrm{i}k\cdot}, \mathrm{e}^{\mathrm{i}j\cdot} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(k-j)x}\, \mathrm{d}x\,,$$

which implies for integers $k = j$

$$\langle \mathrm{e}^{\mathrm{i}k\cdot},\, \mathrm{e}^{\mathrm{i}k\cdot} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1\, \mathrm{d}x = 1.$$

On the other hand we obtain for distinct integers j, k

$$\begin{aligned} \langle \mathrm{e}^{\mathrm{i}k\cdot},\, \mathrm{e}^{\mathrm{i}j\cdot} \rangle &= \frac{1}{2\pi\mathrm{i}(k-j)} \Big(\mathrm{e}^{\pi\mathrm{i}(k-j)} - \mathrm{e}^{-\pi\mathrm{i}(k-j)} \Big) \\ &= \frac{2\mathrm{i}\, \sin \pi(k-j)}{2\pi\mathrm{i}(k-j)} = 0\,. \end{aligned}$$

Now we prove the completeness of the set (1). We have to show that $\left\langle f, \mathrm{e}^{\mathrm{i}k\cdot} \right\rangle = 0$ for all $k \in \mathbb{Z}$ implies $f = 0$.
First we consider a continuous function $f \in C(\mathbb{T})$ having $\langle f, \mathrm{e}^{\mathrm{i}k\cdot} \rangle = 0$ for all $k \in \mathbb{Z}$.
Let us denote by

$$\mathcal{T}_n := \Big\{ \sum_{k=-n}^{n} c_k \mathrm{e}^{\mathrm{i}k\cdot} : \, c_k \in \mathbb{C} \Big\} \tag{2}$$

the space of all trigonometric polynomials of degree $\leq n$. By the approximation theorem of Weierstrass there exists for any function $f \in C(\mathbb{T})$ a sequence $\big\{ p_n \big\}_{n=1}^{\infty}$ of trigonometric polynomials $p_n \in \mathcal{T}_n$, which converges uniformly to f, i.e.

$$\max_{x \in \mathbb{T}} \big| f(x) - p_n(x) \big| \to 0 \quad \text{for } n \to \infty.$$

By assumption we have

$$\langle f, p_n \rangle = \left\langle f, \sum_{k=-n}^{n} c_k \, \mathrm{e}^{\mathrm{i}k\cdot} \right\rangle = \sum_{k=-n}^{n} \overline{c}_k \left\langle f, \mathrm{e}^{\mathrm{i}k\cdot} \right\rangle = 0.$$

Hence we conclude

$$\|f\|^2 = \langle f, f \rangle - \langle f, p_n \rangle = \langle f, f - p_n \rangle \to 0 \tag{3}$$

as $n \to \infty$, so that $f = 0$.
Now let $f \in L_2(\mathbb{T})$ with $\langle f, \mathrm{e}^{\mathrm{i}k\cdot} \rangle = 0$ for all $k \in \mathbb{Z}$ be given. Then

$$h(x) := \int_0^x f(t) \, \mathrm{d}t, \quad x \in [0, 2\pi),$$

is an absolutely continuous function satisfying $h'(x) = f(x)$ almost everywhere.

We have further $h(0) = h(2\pi) = 0$. For $k \in \mathbb{Z}\setminus\{0\}$ we obtain

$$\begin{aligned}
\langle h, \mathrm{e}^{\mathrm{i}k\cdot}\rangle &= \frac{1}{2\pi} \int_0^{2\pi} h(x)\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x \\
&= -\frac{1}{2\pi\mathrm{i}k}\, h(x)\, \mathrm{e}^{-\mathrm{i}kx}\Big|_0^{2\pi} + \frac{1}{2\pi\mathrm{i}k} \int_0^{2\pi} \underbrace{h'(x)}_{=f(x)}\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x \\
&= \frac{1}{2\pi\mathrm{i}k}\, \langle f, \mathrm{e}^{\mathrm{i}k\cdot}\rangle = 0\,.
\end{aligned}$$

Hence the 2π-periodically continued continuous function $h - \langle h, 1\rangle$ fulfills $\big\langle h - \langle h, 1\rangle, \mathrm{e}^{\mathrm{i}k\cdot}\big\rangle = 0$ for all $k \in \mathbb{Z}$. Using the first part of this proof, we obtain $h = \langle h, 1\rangle = \text{const}$. Since $f(x) = h'(x) = 0$ almost everywhere, this yields the assertion. ■

Once we have an orthonormal basis of a Hilbert space, we can represent its elements with respect to this basis. Let us consider the finite sum

$$S_n f := \sum_{k=-n}^{n} c_k(f)\, \mathrm{e}^{\mathrm{i}k\cdot} \in \mathcal{T}_n\,,$$

$$c_k(f) := \left\langle f, \mathrm{e}^{\mathrm{i}k\cdot} \right\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x\,,$$

called *nth Fourier partial sum* of f with the *Fourier coefficients* $c_k(f)$. By definition $S_n : L_2(\mathbb{T}) \to L_2(\mathbb{T})$ is a linear operator which possesses the following important approximation property.

Lemma 2

The Fourier partial sum operator $S_n : L_2(\mathbb{T}) \to L_2(\mathbb{T})$ is an orthogonal projector onto $\mathcal{T}_n$, i.e.

$$\|f - S_n f\| = \min\{\|f - p\| : p \in \mathcal{T}_n\}$$

for arbitrary $f \in L_2(\mathbb{T})$. In particular, it holds

$$\|f - S_n f\|^2 = \|f\|^2 - \sum_{k=-n}^{n} |c_k(f)|^2. \tag{4}$$

Proof: For each trigonometric polynomial

$$p = \sum_{k=-n}^{n} c_k \,\mathrm{e}^{\mathrm{i}k\cdot} \tag{5}$$

with arbitrary $c_k \in \mathbb{C}$ and all $f \in L_2(\mathbb{T})$ we have

$$\begin{aligned}
\|f - p\|^2 &= \|f\|^2 - \langle f,\, p\rangle - \langle p,\, f\rangle + \|p\|^2 \\
&= \|f\|^2 + \sum_{k=-n}^{n} \big(- \overline{c_k}\, c_k(f) - c_k\, \overline{c_k(f)} + |c_k|^2 \big) \\
&= \|f\|^2 - \sum_{k=-n}^{n} |c_k(f)|^2 + \sum_{k=-n}^{n} |c_k - c_k(f)|^2 .
\end{aligned}$$

Thus,

$$\|f - p\|^2 \geq \|f\|^2 - \sum_{k=-n}^{n} |c_k(f)|^2,$$

where equality holds only in the case $c_k = c_k(f)$, $k = -n, \ldots, n$, i.e. if and only if $p = S_n f$.

For $p \in \mathcal{T}_n$ of the form (5), the corresponding Fourier coefficients are $c_k(p) = c_k$ for $k = -n, \ldots, n$ and $c_k(p) = 0$ for all $|k| > n$.

Thus we have $S_n p = p$ and $S_n(S_n f) = S_n f$ for arbitrary $f \in L_2(\mathbb{T})$. Hence $S_n : L_2(\mathbb{T}) \to L_2(\mathbb{T})$ is a projection onto $\mathcal{T}_n$. By

$$\langle S_n f, g \rangle = \sum_{k=-n}^{n} c_k(f)\, \overline{c_k(g)} = \langle f, S_n g \rangle$$

for all $f, g \in L_2(\mathbb{T})$, the Fourier partial sum operator S_n is selfadjoint, i.e., S_n is an orthogonal projection. Moreover, S_n has the operator norm $\|S_n\|_{L_2(\mathbb{T}) \to L_2(\mathbb{T})} = 1$. ■

As an immediate consequence of Lemma 2 we obtain the following

Theorem 3

Every function $f \in L_2(\mathbb{T})$ has a unique representation of the form

$$f = \sum_{k\in\mathbb{Z}} c_k(f)\,\mathrm{e}^{\mathrm{i}k\cdot}, \quad c_k(f) := \langle f, \mathrm{e}^{\mathrm{i}k\cdot}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x\,, \tag{6}$$

where the series $(S_n f)_{n=0}^{\infty}$ converges in $L_2(\mathbb{T})$ to f, i.e.

$$\lim_{n\to\infty} \|S_n f - f\| = 0\,.$$

Further the Parseval equation is fulfilled

$$\|f\|^2 = \sum_{k\in\mathbb{Z}} \left|\langle f, \mathrm{e}^{\mathrm{i}k\cdot}\rangle\right|^2 = \sum_{k\in\mathbb{Z}} |c_k(f)|^2 < \infty\,. \tag{7}$$

Proof: By Lemma 2, we know that for each $n \in \mathbb{N}_0$

$$\|S_n f\|^2 = \sum_{k=-n}^{n} |c_k(f)|^2 \leq \|f\|^2 < \infty\,.$$

For $n \to \infty$, we obtain *Bessel's inequality*

$$\sum_{k=-\infty}^{\infty} |c_k(f)|^2 \leq \|f\|^2\,.$$

Consequently, for arbitrary $\varepsilon > 0$, there exists an index $N(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{|k|>N(\varepsilon)} |c_k(f)|^2 < \varepsilon\,.$$

For $m > n \geq N(\varepsilon)$ we obtain

$$\|S_m f - S_n f\|^2 = \left(\sum_{k=-m}^{-n-1} + \sum_{k=n+1}^{m}\right) |c_k(f)|^2 \leq \sum_{|k|>N(\varepsilon)} |c_k(f)|^2 < \varepsilon\,.$$

Hence $(S_n f)_{n=0}^{\infty}$ is a Cauchy sequence. In the Hilbert space $L_2(\mathbb{T})$, each Cauchy sequence is convergent. Assume that $\lim_{n\to\infty} S_n f = g$ with $g \in L_2(\mathbb{T})$. Since

$$\langle g,\, \mathrm{e}^{\mathrm{i}k\cdot}\rangle = \lim_{n\to\infty}\langle S_n f,\, \mathrm{e}^{\mathrm{i}k\cdot}\rangle = \lim_{n\to\infty}\langle f,\, S_n \mathrm{e}^{\mathrm{i}k\cdot}\rangle = \langle f,\, \mathrm{e}^{\mathrm{i}k\cdot}\rangle$$

for all $k \in \mathbb{Z}$, we conclude by Theorem 1 that $f = g$. Letting $n \to \infty$ in (4) we obtain the Parseval equation (7). ■

The representation (6) is the so-called *Fourier series* of f. Figure 1 shows 2π-periodic functions as superposition of two 2π-periodic functions.

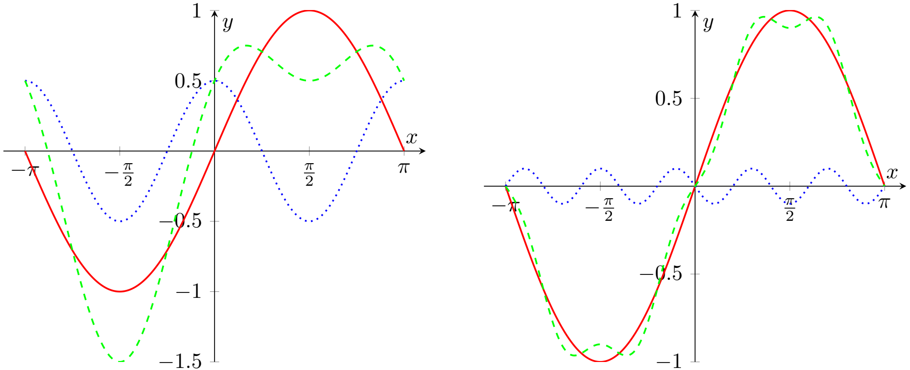


Figure 1: Two 2π-periodic functions $\sin x + \frac{1}{2}\cos(2x)$ (left) and $\sin x - \frac{1}{10}\sin(4x)$ as superpositions of sine and cosine functions.

Clearly, the partial sums of the Fourier series are the Fourier partial sums. The constant term $c_0(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$ in the Fourier series of f is the mean value of f.

Remark 4

For fixed $L > 0$, a function $f : \mathbb{R} \to \mathbb{C}$ is called L-periodic, if $f(x+L) = f(x)$ for all $x \in \mathbb{R}$. By substitution we see that the Fourier series of an L-periodic function f reads as follows

$$f = \sum_{k\in\mathbb{Z}} c_k^{(L)}(f)\,\mathrm{e}^{2\pi\mathrm{i}k\cdot/L}\,, \quad c_k^{(L)}(f) := \frac{1}{L}\int_{-L/2}^{L/2} f(x)\,\mathrm{e}^{-2\pi\mathrm{i}kx/L}\,\mathrm{d}x\,. \tag{8}$$

In polar coordinates we can represent the Fourier coefficients in the form

$$c_k(f) = \frac{1}{2}\, r_k\, \mathrm{e}^{\mathrm{i}\varphi_k}, \quad r_k := 2|c_k(f)|,\ \varphi_k := \mathrm{atan2}\big(\,\mathrm{Im}\ c_k(f),\, \mathrm{Re}\ c_k(f)\big), \tag{9}$$

where

$$\mathrm{atan2}(y, x) := \begin{cases} \arctan \frac{y}{x} & \text{if } x > 0\,, \\ \arctan \frac{y}{x} + \pi & \text{if } x < 0,\ y \ge 0\,, \\ \arctan \frac{y}{x} - \pi & \text{if } x < 0,\ y < 0\,, \\ \frac{\pi}{2} & \text{if } x = 0,\ y > 0\,, \\ -\frac{\pi}{2} & \text{if } x = 0,\ y < 0\,, \\ 0 & \text{if } x = y = 0\,. \end{cases}$$

Then $\big(|c_k(f)|\big)_{k\in\mathbb{Z}} = \frac{1}{2}(r_k)_{k\in\mathbb{Z}}$ is called the *spectrum* or *modulus* of f and $\big(\varphi_k\big)_{k\in\mathbb{Z}}$ the *phase* of f.

For fixed $a \in \mathbb{R}$, the 2π-*periodic extension* of a function $f : [-\pi + a, \pi + a) \to \mathbb{C}$ to the whole line $\mathbb{R}$ is given by $f(x + 2\pi n) := f(x)$ for all $x \in [-\pi + a, \pi + a)$ and all $n \in \mathbb{Z}$. Often we have $a = 0$ or $a = \pi$.

Example 5

Consider the 2π-periodic extension of the real-valued function $f(x) = \mathrm{e}^{-x}$, $x \in (-\pi, \pi)$ with $f(\pm\pi) = \cosh \pi = \frac{1}{2}(\mathrm{e}^{-\pi} + \mathrm{e}^{\pi})$. Then the Fourier coefficients $c_k(f)$ are given by

$$\begin{aligned} c_k(f) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-(1+\mathrm{i}k)x} \, \mathrm{d}x \\ &= -\frac{1}{2\pi\,(1+\mathrm{i}k)} \left(\mathrm{e}^{-(1+\mathrm{i}k)\pi} - \mathrm{e}^{(1+\mathrm{i}k)\pi} \right) = \frac{(-1)^k \sinh \pi}{(1+\mathrm{i}\,k)\,\pi} \,. \end{aligned}$$

Figure 2 shows both the 8-th and 16-th Fourier partial sum $S_8 f$ and $S_{16} f$. $\square$

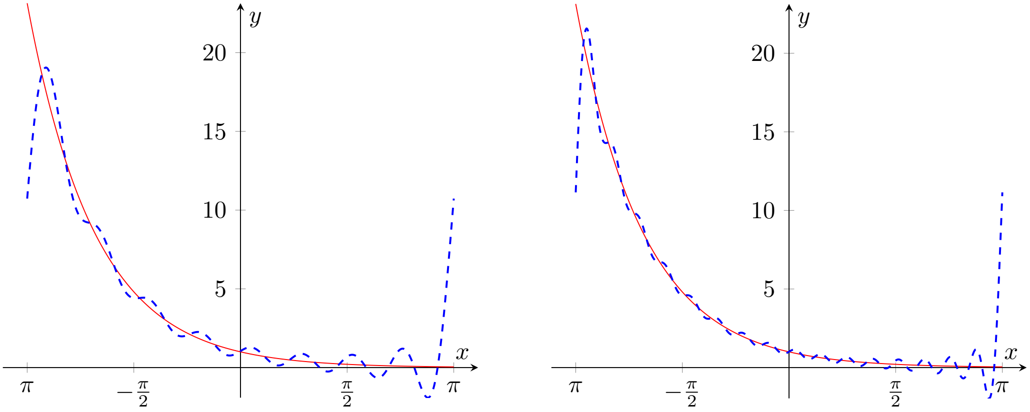


Figure 2: The 2π-periodic function f given by $f(x) := \mathrm{e}^{-x}$, $x \in (-\pi, \pi)$, with $f(\pm\pi) = \cosh(\pi)$ and its Fourier partial sums $S_8 f$ (left) and $S_{16} f$ (right).

For $f \in L_2(\mathbb{T})$ we have (7). Thus the Fourier coefficients $c_k(f)$ converge to zero as $|k| \to \infty$. Since

$$|c_k(f)| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)| \, \mathrm{d}x = \|f\|_{L_1(\mathbb{T})} \,,$$

the integrals

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, \mathrm{e}^{-\mathrm{i}kx} \, \mathrm{d}x \,, \quad k \in \mathbb{Z}$$

also exist for all functions $f \in L_1(\mathbb{T})$, i.e., the Fourier coefficients are well-defined for any function of $L_1(\mathbb{T})$. The next lemma contains simple properties of Fourier coefficients.

Lemma 6

The Fourier coefficients of $f, g \in L_1(\mathbb{T})$ *have the following properties for all* $k \in \mathbb{Z}$*:*

1. *Linearity: For all* $\alpha, \beta \in \mathbb{C}$*,*

$$c_k(\alpha f + \beta g) = \alpha\, c_k(f) + \beta\, c_k(g)\,.$$

2. *Translation – Modulation: For all* $x_0 \in [0,\, 2\pi)$ *and* $k_0 \in \mathbb{Z}$*,*

$$c_k(f(\cdot - x_0)) = \mathrm{e}^{-\mathrm{i}kx_0}\, c_k(f)\,,$$
$$c_k(\mathrm{e}^{-\mathrm{i}k_0 \cdot}\, f) = c_{k+k_0}(f)\,.$$

   In particular $|c_k(f(\cdot - x_0))| = |c_k(f)|$*, i.e., translation does not change the spectrum of* f*.*

3. *Differentiation – Multiplication: For absolute continuous functions* $f \in L_1(\mathbb{T})$ *with* $f' \in L_1(\mathbb{T})$ *we have*

$$c_k(f') = \mathrm{i}\, k\, c_k(f)\,.$$

Proof: The first property follows directly from the linearity of the integral.
The translation–modulation property can be seen as

$$\begin{aligned} c_k(f(\cdot - x_0)) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x - x_0)\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, \mathrm{e}^{-\mathrm{i}k(y+x_0)}\, \mathrm{d}y = \mathrm{e}^{-\mathrm{i}kx_0}\, c_k(f), \end{aligned}$$

and similarly for the modulation–translation property.
For the differentiation property recall that an absolute continuous function has a derivative almost everywhere. Then we obtain by integration by parts

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{i}k\, f(x)\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x)\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x = c_k(f'). \qquad \blacksquare$$

The complex Fourier series

$$f = \sum_{k\in\mathbb{Z}} c_k(f)\,\mathrm{e}^{\mathrm{i}k\cdot}$$

can be rewritten using Euler's formula $\mathrm{e}^{\mathrm{i}k\cdot} = \cos(k\cdot) + \mathrm{i}\,\sin(k\cdot)$ as

$$f = \frac{1}{2}\,a_0(f) + \sum_{k=1}^{\infty} \big(a_k(f)\,\cos(k\cdot) + b_k(f)\,\sin(k\cdot)\big)\,, \tag{10}$$

where

$$\begin{aligned} a_k(f) &= c_k(f) + c_{-k}(f) = 2\,\langle f, \cos(k\cdot)\rangle\,, \quad k \in \mathbb{N}_0\,,\\ b_k(f) &= \mathrm{i}\,\big(c_k(f) - c_{-k}(f)\big) = 2\,\langle f, \sin(k\cdot)\rangle\,, \quad k \in \mathbb{N}\,. \end{aligned}$$

Consequently $\big\{1, \sqrt{2}\,\cos(k\cdot)\colon\, k\in\mathbb{N}\big\} \cup \big\{\sqrt{2}\,\sin(k\cdot)\colon\, k\in\mathbb{N}\big\}$ form also an orthonormal basis of $L_2(\mathbb{T})$. If $f:\mathbb{T}\to\mathbb{R}$ is a real-valued function, then $c_k(f) = \overline{c_{-k}(f)}$ and (10) is the *real Fourier series* of f.

Using polar coordinates (9), the Fourier series of a real-valued function $f \in L_2(\mathbb{T})$ can be written in the form

$$f = \frac{1}{2}\, a_0(f) + \sum_{k=1}^{\infty} r_k\, \sin(k \cdot + \varphi_k).$$

with sine oscillations of amplitudes $r_k = 2|c_k|$, angular frequencies k and phase shifts φ_k. For even and odd functions the Fourier series simplify to pure cosine resp. sine series.

Lemma 7

If $f \in L_2(\mathbb{T})$ is even, i.e. $f(x) = f(-x)$ for all $x \in \mathbb{T}$, then $c_k(f) = c_{-k}(f)$ for all $k \in \mathbb{Z}$ and f can be represented as a Fourier cosine series

$$f = c_0(f) + 2\sum_{k=1}^{\infty} c_k(f)\cos(k\cdot) = \frac{1}{2}a_0(f) + \sum_{k=1}^{\infty} a_k(f)\cos(k\cdot)\,.$$

If $f \in L_2(\mathbb{T})$ is odd, i.e. $f(x) = -f(-x)$ for all $x \in \mathbb{T}$, then $c_k(f) = -c_{-k}(f)$ for all $k \in \mathbb{Z}$ and f can be represented as a Fourier sine series

$$f = 2\mathrm{i}\sum_{k=1}^{\infty} c_k(f)\sin(k\cdot) = \sum_{k=1}^{\infty} b_k(f)\sin(k\cdot).$$

The simple proof of Lemma 7 is left as an exercise.

Example 8

The 2π-periodic extension of the function $f(x) = x^2$, $x \in [-\pi, \pi)$ is even and has the Fourier cosine series

$$\frac{\pi^2}{3} + 4 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(k\cdot) \,. \qquad \square$$

Example 9

The 2π-periodic extension of the function $s(x) = \frac{\pi - x}{2\pi}$, $x \in (0, 2\pi)$, with $s(0) = 0$ is odd and has jump discontinuities at $2\pi k$, $k \in \mathbb{Z}$, of unit height. This so-called *sawtooth function* has the Fourier sine series

$$\sum_{k=1}^{\infty} \frac{1}{\pi\, k} \sin(k\cdot)\,.$$

Applying the Parseval equation (7) we obtain

$$\sum_{k=1}^{\infty} \frac{1}{2\pi^2 k^2} = \|s\|^2 = \frac{1}{12}.$$

This implies $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. The last equation can be also obtained from the Fourier series in Example 8 by setting $x := \pi$ and assuming that the series converges in this point. $\square$

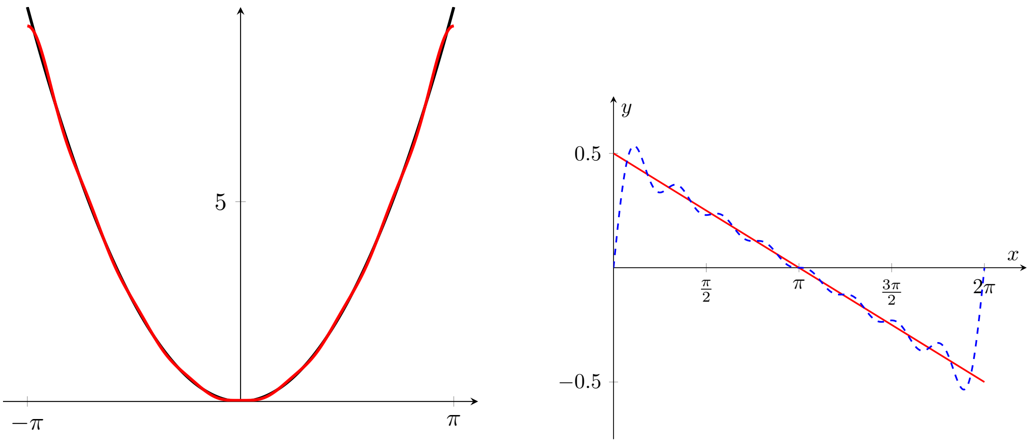


Figure 3: The Fourier partial sums $S_8 f$ of the even 2π-periodic function f given by $f(x) := x^2$, $x \in [-\pi, \pi)$ (left) and of the odd 2π-periodic function f given by $f(x) = \frac{1}{2} - \frac{x}{2\pi}$, $x \in (0, 2\pi)$, with $f(0) = f(2\pi) = 0$ (right).

Example 10

We consider the 2π-periodic extension of the *rectangular pulse function* $f : [-\pi, \pi) \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \in (-\pi, 0), \\ 1 & \text{if } x \in (0, \pi) \end{cases}$$

and $f(-\pi) = f(0) = \frac{1}{2}$. The function $f - \frac{1}{2}$ is odd and the Fourier series of f reads

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi} \sin\big((2n-1)\cdot\big) . \qquad \square$$

Convolution of periodic functions

The *convolution* of two 2π-periodic functions $f, g \in L_1(\mathbb{T})$ is the function $h = f * g$ given by

$$h(x) := (f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, g(x-y)\, \mathrm{d}y\,.$$

Using the substitution $y = x - t$, we see

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)\, g(t)\, \mathrm{d}t = (g * f)(x)$$

so that the convolution is commutative. It is easy to check that it is also associative and distributive. Furthermore, the convolution is translation invariant

$$(f(\cdot - t) * g)(x) = (f * g)(x-t)\,.$$

If g is an even function, i.e. $g(x) = g(-x)$ for all $x \in \mathbb{R}$, then

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, g(y-x)\, \mathrm{d}y.$$

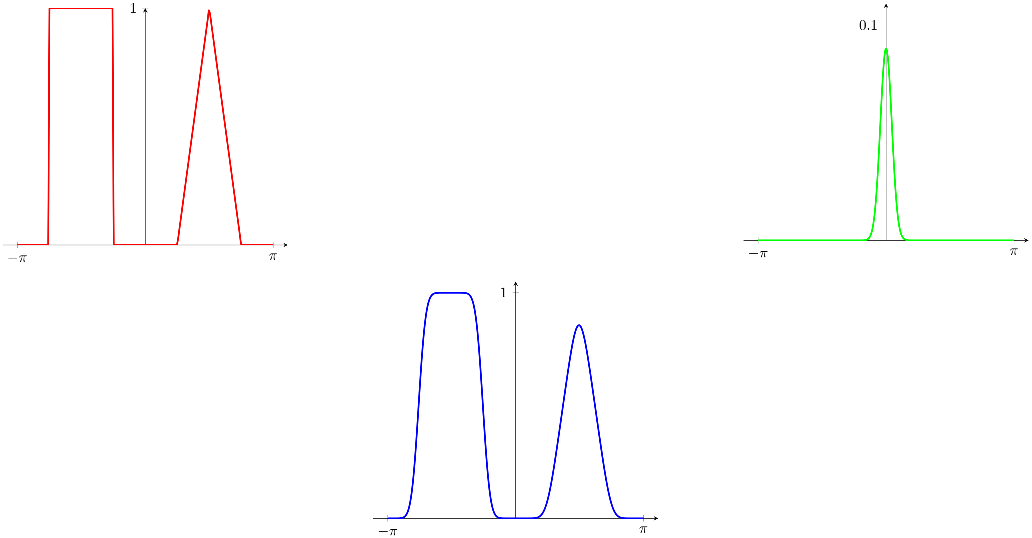


Figure 4: Top: Two 2π-periodic functions f (red) and g (green). Down: The corresponding convolution $f * g$.

Theorem 11

1. *Let* $f \in L_p(\mathbb{T})$, $1 \le p \le \infty$ *and* $g \in L_1(\mathbb{T})$ *be given. Then* $f * g$ *exists almost everywhere and* $f * g \in L_p(\mathbb{T})$. *Further we have the Young inequality*

$$\|f * g\|_{L_p(\mathbb{T})} \le \|f\|_{L_p(\mathbb{T})} \|g\|_{L_1(\mathbb{T})}.$$

2. *Let* $f \in L_p(\mathbb{T})$ *and* $g \in L_q(\mathbb{T})$, *where* $1 \le p, q \le \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then* $f * g$ *exists for every* $x \in \mathbb{T}$ *and* $f * g \in C(\mathbb{T})$. *It holds*

$$\|f * g\|_{C(\mathbb{T})} \le \|f\|_{L_p(\mathbb{T})} \|g\|_{L_q(\mathbb{T})}.$$

3. *Let* $f \in L_p(\mathbb{T})$ *and* $g \in L_q(\mathbb{T})$, *where* $1 \le p, q, r \le \infty$ *and* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. *Then* $f * g \in L_r(\mathbb{T})$ *and we have the generalized Young inequality*

$$\|f * g\|_{L_r(\mathbb{T})} \le \|f\|_{L_p(\mathbb{T})} \|g\|_{L_q(\mathbb{T})}.$$

Proof: 1. For $f \in L_p(\mathbb{T})$ and $g \in L_1(\mathbb{T})$ we obtain by generalized Minkowski's inequality

$$\begin{aligned}
\|f * g\|_{L_p(\mathbb{T})} &= \frac{1}{2\pi} \Big\| \int_{-\pi}^{\pi} g(y)\, f(\cdot - y)\, \mathrm{d}y \Big\|_{L_p(\mathbb{T})} \\
&\le \frac{1}{2\pi} \int_{-\pi}^{\pi} \|g(y)\, f(\cdot - y)\|_{L_p(\mathbb{T})}\, \mathrm{d}y \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(y)|\, \|f(\cdot - y)\|_{L_p(\mathbb{T})}\, \mathrm{d}y \\
&= \|g\|_{L_1(\mathbb{T})} \|f\|_{L_p(\mathbb{T})}.
\end{aligned}$$

2. Let $f \in L_p(\mathbb{T})$ and $g \in L_q(\mathbb{T})$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$ be given. By Hölder's inequality it follows

$$|(f * g)(x)| \leq \Big(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x-y)|^p \, \mathrm{d}y\Big)^{1/p} \Big(\frac{1}{2\pi} \int_{-\pi}^{\pi} |g(y)|^q \, \mathrm{d}y\Big)^{1/q}$$
$$\leq \|f\|_{L_p(\mathbb{T})} \, \|g\|_{L_q(\mathbb{T})}$$

and consequently

$$|(f * g)(x+t) - (f * g)(x)| \leq \|f(\cdot + t) - f\|_{L_p(\mathbb{T})} \|g\|_{L_q(\mathbb{T})}.$$

Now the second assertion follows, since the translation is continuous in the $L_p(\mathbb{T})$ norm (see [5, Proposition 8.5]), i.e. $\|f(\cdot + t) - f\|_{L_p(\mathbb{T})} \to 0$ as $t \to 0$.
The case $p = 1$ is straightforward.

3. Finally, let $f \in L_p(\mathbb{T})$ and $g \in L_q(\mathbb{T})$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$ for $1 \le p, q, r \le \infty$ be given. The case $r = \infty$ is described in Part 2 so that it remains to consider $1 \le r < \infty$. Then $p \le r$ and $q \le r$, since otherwise we would get the contradiction $q < 1$ resp. $p < 1$. Set $s := p\left(1 - \frac{1}{q}\right) = 1 - \frac{p}{r} \in [0, 1)$ and $t := \frac{r}{q} \in [1, \infty)$. Define q' by $\frac{1}{q} + \frac{1}{q'} = 1$. Then we obtain by Hölder's inequality

$$\begin{aligned} h(x) &:= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x-y)g(y)| \,\mathrm{d}y = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x-y)|^{1-s} \, |g(y)| \, |f(x-y)|^{s} \,\mathrm{d}y \\ &\le \Big(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x-y)|^{(1-s)q} \, |g(y)|^{q} \,\mathrm{d}y \Big)^{1/q} \Big(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x-y)|^{sq'} \,\mathrm{d}y \Big)^{1/q'} . \end{aligned}$$

Using that by definition $sq' = p$ and $q/q' = (sq)/p$, this implies

$$\begin{aligned} h^q(x) &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-y)|^{(1-s)q}\,|g(y)|^q\,\mathrm{d}y \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x-y)|^p\,\mathrm{d}y\Big)^{q/q'} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-y)|^{(1-s)q}\,|g(y)|^q\,\mathrm{d}y \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x-y)|^p\,\mathrm{d}y\Big)^{(sq)/p} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-y)|^{(1-s)q}\,|g(y)|^q\,\mathrm{d}y\,\|f\|_{L_p(\mathbb{T})}^{sq} \end{aligned}$$

such that

$$\begin{aligned} \|h\|_{L_r(\mathbb{T})}^q &= \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}|h(x)|^{qt}\,\mathrm{d}x\Big)^{q/(qt)} = \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}|h^q(x)|^{t}\,\mathrm{d}x\Big)^{1/t} = \|h^q\|_{L_t(\mathbb{T})} \\ &\le \|f\|_{L_p(\mathbb{T})}^{sq}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x-y)|^{(1-s)q}\,|g(y)|^q\,\mathrm{d}y\Big)^t\,\mathrm{d}x\Big)^{1/t} \end{aligned}$$

and further by $(1-s)qt = p$ and generalized Minkowski's inequality

$$\begin{aligned} \|h\|_{L_r(\mathbb{T})}^q &\le \|f\|_{L_p(\mathbb{T})}^{sq}\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x-y)|^{(1-s)qt}\,|g(y)|^{qt}\,\mathrm{d}x\Big)^{1/t}\,\mathrm{d}y \\ &= \|f\|_{L_p(\mathbb{T})}^{sq}\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(y)|^q\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x-y)|^{(1-s)qt}\,\mathrm{d}x\Big)^{1/t}\,\mathrm{d}y \\ &= \|f\|_{L_p(\mathbb{T})}^{sq}\,\|f\|_{L_{(1-s)qt}}^{(1-s)q}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(y)|^q\,\mathrm{d}y = \|f\|_{L_p(\mathbb{T})}^{q}\,\|g\|_{L_q(\mathbb{T})}^{q}. \end{aligned}$$

Taking the q-th root finishes the proof. Alternatively Part 3 can be proved using the

The convolution of an $L_1(\mathbb{T})$ function and an $L_p(\mathbb{T})$ function with $1 \leq p < \infty$ is in general not defined pointwise as the following example shows.

Example 12

We consider the 2π-periodic extension of $f : [-\pi, \pi) \to \mathbb{R}$ given by

$$f(y) := \begin{cases} |y|^{-3/4} & y \in [-\pi, \pi) \setminus \{0\}\,, \\ 0 & y = 0\,. \end{cases} \tag{11}$$

The extension denoted by f is even and belongs to $L_1(\mathbb{T})$. The convolution $(f * f)(x)$ is finite for all $x \in [-\pi, \pi) \setminus \{0\}$. However, for $x = 0$, this does not hold true, since

$$\int_{-\pi}^{\pi} f(y)\, f(-y)\, \mathrm{d}y = \int_{-\pi}^{\pi} |y|^{-3/2}\, \mathrm{d}y = \infty. \qquad \square$$

The following lemma describes the *convolution property of Fourier series*.

Lemma 13

For $f, g \in L_1(\mathbb{T})$ it holds

$$c_k(f * g) = c_k(f)\, c_k(g)\,, \quad k \in \mathbb{Z}.$$

Proof.

Using the 2π-periodicity of g we obtain

$$\begin{aligned} c_k(f * g) &= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \Big(\int_{-\pi}^{\pi} f(y)\, g(x-y)\, \mathrm{d}y \Big) \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x \\ &= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(y)\, \mathrm{e}^{-\mathrm{i}ky}\, g(x-y)\, \mathrm{e}^{-\mathrm{i}k(x-y)}\, \mathrm{d}y\, \mathrm{d}x \\ &= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} f(y)\, \mathrm{e}^{-\mathrm{i}ky}\, \mathrm{d}y \int_{-\pi}^{\pi} g(t)\, \mathrm{e}^{-\mathrm{i}kt}\, \mathrm{d}t = c_k(f)\, c_k(g)\,. \end{aligned}$$

■

The convolution of functions with certain functions, so-called *kernels*, is of particular interest.

Example 14

The nth *Dirichlet kernel* for $n \in \mathbb{N}_0$ is defined by

$$D_n(x) := \sum_{k=-n}^{n} \mathrm{e}^{\mathrm{i}kx}\,, \quad x \in \mathbb{R}\,. \tag{12}$$

By Euler's formula it follows

$$D_n(x) = 1 + 2\sum_{k=1}^{n} \cos(kx)\,.$$

Obviously, $D_n \in \mathcal{T}_n$ is real-valued and even. For $x \in (0, \pi]$ and $n \in \mathbb{N}$, we can express $\sin \frac{x}{2}\, D_n(x)$ as telescope sum

$$\begin{aligned}
\left(\sin \frac{x}{2}\right) D_n(x) &= \sin \frac{x}{2} + \sum_{k=1}^{n} 2 \cos(kx) \sin \frac{x}{2} \\
&= \sin \frac{x}{2} + \sum_{k=1}^{n} \left(\sin \frac{(2k+1)x}{2} - \sin \frac{(2k-1)x}{2} \right) \\
&= \sin \frac{(2n+1)x}{2} \,.
\end{aligned}$$

Thus, the nth Dirichlet kernel can be represented as a fraction

$$D_n(x) = \frac{\sin \frac{(2n+1)x}{2}}{\sin \frac{x}{2}}\,, \quad x \in [-\pi, \pi) \setminus \{0\} \tag{13}$$

with $D_n(0) = 2n + 1$.

Figure 5 depicts the Dirichlet kernel D_8. The Fourier coefficients of D_n are

$$c_k(D_n) = \begin{cases} 1 & k = -n, \dots, n, \\ 0 & |k| > n\,. \end{cases}$$

For $f \in L_1(\mathbb{T})$ with Fourier coefficients $c_k(f)$, $k \in \mathbb{Z}$, we obtain by Lemma 13 that

$$f * D_n = \sum_{k=-n}^{n} c_k(f)\, \mathrm{e}^{\mathrm{i}k\cdot} = S_n f\,,$$

which is just the nth Fourier partial sum of f and hence its orthogonal projection onto the space of trigonometric polynomials $\mathcal{T}_n$.

By the following calculations, the Dirichlet kernel fulfills

$$\|D_n\|_{L_1(\mathbb{T})} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(x)| \, \mathrm{d}x \geq \frac{4}{\pi^2} \ln n \,. \tag{14}$$

Note that $\|D_n\|_{L_1(\mathbb{T})}$ are called *Lebesgue constants*. Since $\sin x \leq x$ for $x \in [0, \frac{\pi}{2})$ we get by (13) that

$$\|D_n\|_{L_1(\mathbb{T})} = \frac{1}{\pi} \int_0^{\pi} \frac{|\sin((2n+1)x/2)|}{\sin(x/2)} \mathrm{d}x \geq \frac{2}{\pi} \int_0^{\pi} \frac{|\sin((2n+1)x/2)|}{x} \mathrm{d}x.$$

Substituting $y = \frac{2n+1}{2} x$ results in

$$\begin{aligned}
\|D_n\|_{L_1(\mathbb{T})} &\geq \frac{2}{\pi} \int_0^{(n+\frac{1}{2})\pi} \frac{|\sin y|}{y} \, \mathrm{d}y \\
&\geq \frac{2}{\pi} \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin y|}{y} \, \mathrm{d}y \geq \frac{2}{\pi} \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin y|}{k\pi} \, \mathrm{d}y \\
&= \frac{4}{\pi^2} \sum_{k=1}^{n} \frac{1}{k} \geq \frac{4}{\pi^2} \int_1^{n+1} \frac{1}{x} \, \mathrm{d}x \geq \frac{4}{\pi^2} \ln n.
\end{aligned}$$

The Lebesgue constants fulfill

$$\|D_n\|_{L_1(\mathbb{T})} = \frac{4}{\pi^2} \ln n + \mathcal{O}(1) \,, \quad n \to \infty \,. \qquad \square$$

Example 15

The *n*th *Fejér kernel* for $n \in \mathbb{N}_0$ is defined by

$$F_n := \frac{1}{n+1} \sum_{j=0}^{n} D_j \in \mathcal{T}_n \,. \tag{15}$$

By (13) and (15) we obtain $F_n(0) = n + 1$ and for $x \in [-\pi, \pi) \setminus \{0\}$

$$F_n(x) = \frac{1}{n+1} \sum_{j=0}^{n} \frac{\sin\big((j + \frac{1}{2})x\big)}{\sin \frac{x}{2}} \,.$$

Multiplying the numerator and denominator of each right-hand fraction by $2\sin\frac{x}{2}$ and replacing the product of sines in the numerator by the differences $\cos(jx) - \cos\big((j+1)x\big)$, we find by cascade summation that F_n can be represented in the form

$$F_n(x) = \frac{1}{2(n+1)} \frac{1-\cos\big((n+1)x\big)}{\big(\sin\frac{x}{2}\big)^2} = \frac{1}{n+1}\left(\frac{\sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}}\right)^2. \tag{16}$$

In contrast to the Dirichlet kernel the Fejér kernel is non-negative. Figure 6 shows the Fejér kernel F_8. The Fourier coefficients of F_n are

$$c_k(F_n) = \begin{cases} 1 - \frac{|k|}{n+1} & k = -n, \dots, n\,, \\ 0 & |k| > n\,. \end{cases}$$

Using the convolution property, the convolution $f * F_n$ for arbitrary $f \in L_1(\mathbb{T})$ is given by

$$\sigma_n f := f * F_n = \sum_{k=-n}^{n} \Big(1 - \frac{|k|}{n+1}\Big)\, c_k(f)\, \mathrm{e}^{\mathrm{i}k\cdot}\,. \tag{17}$$

Then $\sigma_n f$ is called the *nth Fejér sum* or *nth Cesàro sum* of f. Further, we have

$$\|F_n\|_{L_1(\mathbb{T})} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(x)\,\mathrm{d}x = 1. \qquad \square$$

Example 16

The *n*th *de la Vallée Poussin kernel* V_{2n} for $n \in \mathbb{N}$ is defined by

$$V_{2n} = \frac{1}{n} \sum_{j=n}^{2n-1} D_j = 2\, F_{2n-1} - F_{n-1} = \sum_{k=-2n}^{2n} c_k(V_{2n})\, \mathrm{e}^{\mathrm{i}k\cdot}$$

with the Fourier coefficients

$$c_k(V_{2n}) = \begin{cases} 2 - \frac{|k|}{n} & k = -2n, \ldots, -(n+1), n+1, \ldots, 2n\,, \\ 1 & k = -n, \ldots, n\,, \\ 0 & |k| > 2n\,. \end{cases} \qquad \square$$

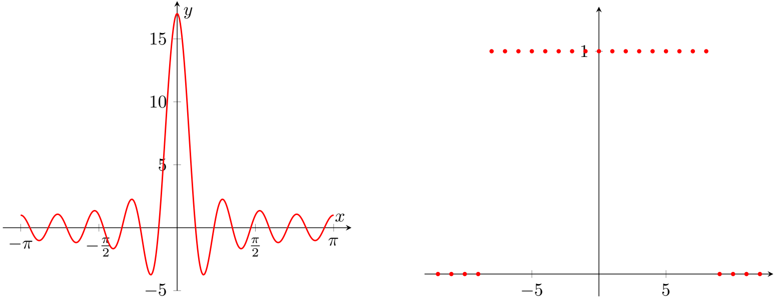
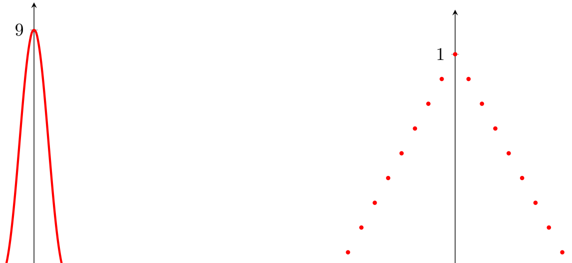


Figure 5: The Dirichlet kernel D_8 (left) and its Fourier coefficients $c_k(D_8)$ (right).



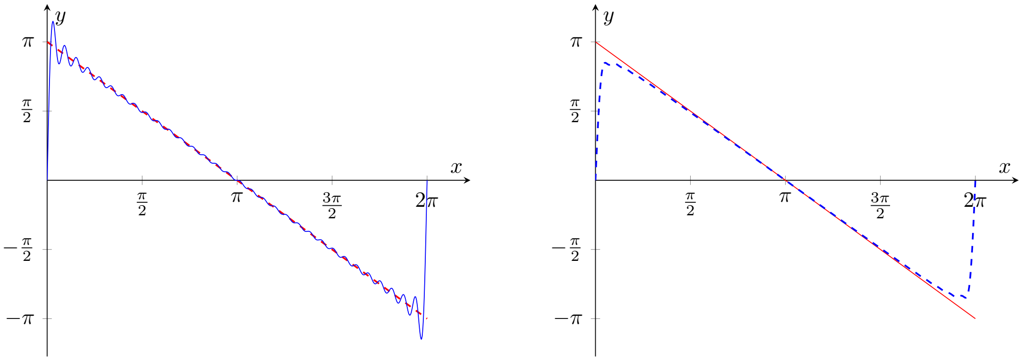


Figure 7: The convolution $f * D_{32}$ of the 2π-periodic sawtooth function f and the Dirichlet kernel D_{32} approximates f quite good except at the jump discontinuities (left). The convolution $f * F_{32}$ of f and the Fejér kernel F_{32} approximates f not as good as $f * D_{32}$, but it does not oscillates near the jump discontinuities (right).

By Theorem 11 the convolution of two $L_1(\mathbb{T})$ functions is again a function in $L_1(\mathbb{T})$. The space $L_1(\mathbb{T})$ forms together with the addition and the convolution a so-called Banach algebra. Unfortunately, there does not exist an identity element with respect to $*$, i.e., there is no function $g \in L_1(\mathbb{T})$ such that $f * g = f$ for all $f \in L_1(\mathbb{T})$. As a remedy we can define approximate identities.

A sequence $(K_n)_{n\in\mathbb{N}}$ of functions $K_n \in L_1(\mathbb{T})$ is called an *approximate identity* or a *summation kernel* , if it satisfies the following properties:

1. $\frac{1}{2\pi} \int\limits_{-\pi}^{\pi} K_n(x)\,\mathrm{d}x = 1$ for all $n \in \mathbb{N}$,
2. $\|K_n\|_{L_1(\mathbb{T})} = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} |K_n(x)|\,\mathrm{d}x \leq C < \infty$ for all $n \in \mathbb{N}$,
3. $\lim\limits_{n\to\infty} \left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right) |K_n(x)|\,\mathrm{d}x = 0$ for each $0 < \delta < \pi$.

Theorem 17

For an approximate identity $(K_n)_{n\in\mathbb{N}}$ *it holds*

$$\lim_{n\to\infty} \|K_n * f - f\|_{C(\mathbb{T})} = 0$$

for all $f \in C(\mathbb{T})$.

Proof: Since a continuous function is uniformly continuous on a compact interval, for all $\varepsilon > 0$ there exists a number $\delta > 0$ so that for all $|u| < \delta$

$$\|f(\cdot - u) - f\|_{C(\mathbb{T})} < \varepsilon \,. \tag{18}$$

Using the first property of an approximate identity, we obtain

$$\begin{aligned}
&\|K_n * f - f\|_{C(\mathbb{T})} \\
&= \sup_{x\in\mathbb{T}} \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\,K_n(u)\,\mathrm{d}u - f(x)\Big| \\
&= \sup_{x\in\mathbb{T}} \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi} \big(f(x-u) - f(x)\big)\,K_n(u)\,\mathrm{d}u\Big| \\
&\le \frac{1}{2\pi}\sup_{x\in\mathbb{T}} \int_{-\pi}^{\pi} |f(x-u) - f(x)|\,|K_n(u)|\,\mathrm{d}u \\
&= \frac{1}{2\pi}\sup_{x\in\mathbb{T}} \Big(\int_{-\pi}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\pi}\Big)|f(x-u) - f(x)|\,|K_n(u)|\,\mathrm{d}u \,.
\end{aligned}$$

By (18) the right-hand side can be estimated as

$$\frac{\varepsilon}{2\pi} \int_{-\delta}^{\delta} |K_n(u)| \,\mathrm{d}u + \frac{1}{2\pi} \sup_{x\in\mathbb{T}} \Big(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} \Big) |f(x-u) - f(x)| \, |K_n(u)| \,\mathrm{d}u \,.$$

By the Properties 2. and 3. of the reproducing kernel K_n, we obtain for sufficiently large $n \in \mathbb{N}$ that

$$\|K_n * f - f\|_{C(\mathbb{T})} \le \varepsilon \, C + \frac{1}{\pi} \, \|f\|_{C(\mathbb{T})} \, \varepsilon.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, this yields the assertion. ■

Example 18

The sequence $(D_n)_{n\in\mathbb{N}}$ of Dirichlet kernels defined in Example 14 is not an approximate identity, since $\|D_n\|_{L_1(\mathbb{T})}$ is not uniformly bounded for all $n \in \mathbb{N}$ by (14). Indeed we will see in the next section that $S_n f = D_n * f$ does in general not converge uniformly to $f \in C(\mathbb{T})$ for $n \to \infty$. A general remedy in such cases consists in considering the Cesàro mean as shown in the next example. $\square$

Example 19

The sequence $(F_n)_{n\in\mathbb{N}}$ of Fejér kernels defined in Example 15 possesses by definition the first two properties of an approximate identity and also fulfills the third one by (16) and

$$\begin{aligned}\Big(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\Big) F_n(x)\,\mathrm{d}x &= 2\int_{\delta}^{\pi} F_n(x)\,\mathrm{d}x \\ &= \frac{2}{n+1}\int_{\delta}^{\pi}\Big(\frac{\sin((n+1)x/2)}{\sin(x/2)}\Big)^2\,\mathrm{d}x \\ &\le \frac{2}{n+1}\int_{\delta}^{\pi}\frac{\pi^2}{x^2}\,\mathrm{d}x = \frac{2\pi}{n+1}\Big(\frac{\pi}{\delta}-1\Big)\,.\end{aligned}$$

The right-hand side tends to zero as $n\to\infty$ so that $(F_n)_{n\in\mathbb{N}}$ is an approximate identity. It is not hard to verify that the sequence $(V_{2n})_{n\in\mathbb{N}}$ of de la Vallée Poussin kernels defined in Example 16 is also an approximate identity. $\square$

From Theorem 17 and Example 19 it follows immediately

Theorem 20 (Approximation Theorem of Fejér)

If $f \in C(\mathbb{T})$, then the Fejér sums $\sigma_n f$ converge uniformly to f as $n \to \infty$. If $m \leq f(x) \leq M$ for all $x \in \mathbb{T}$ with $m, M \in \mathbb{R}$, then $m \leq (\sigma_n f)(x) \leq M$ for all $n \in \mathbb{N}$.

Proof: Since $(F_n)_{n\in\mathbb{N}}$ is an approximate identity, the Fejér sums $\sigma_n f$ converge uniformly to f as $n \to \infty$. If a real-valued function $f : \mathbb{T} \to \mathbb{R}$ fulfills the estimate $m \leq f(x) \leq M$ for all $x \in \mathbb{T}$ with certain constants m, $M \in \mathbb{R}$, then

$$(\sigma_n f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(y)\, f(x-y)\,\mathrm{d}y$$

fulfills also $m \leq (\sigma_n f)(x) \leq M$ for all $x \in \mathbb{T}$, since $F_n(y) \geq 0$ and $\frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(y)\,\mathrm{d}y = c_0(F_n) = 1$. ■

The Theorem 20 of Fejér has many important consequences such as

Theorem 21 (Approximation Theorem of Weierstrass)

If $f \in C(\mathbb{T})$, then for each $\varepsilon > 0$ there exists a trigonometric polynomial $p = \sigma_n f \in \mathcal{T}_n$ of sufficiently large degree n such that $\|f - p\|_{C(\mathbb{T})} < \varepsilon$. Further this trigonometric polynomial p is a weighted Fourier partial sum given by (17).

Pointwise and uniform convergence of Fourier series

It was shown that a Fourier series of an arbitrary function $f \in L_2(\mathbb{T})$ converges in the norm of $L_2(\mathbb{T})$, i.e.

$$\lim_{n\to\infty} \|S_n f - f\|_{L_2(\mathbb{T})} = \lim_{n\to\infty} \|f * D_n - f\|_{L_2(\mathbb{T})} = 0\,.$$

In general convergence of a sequence a.e. does not result in L_p, $p \in [1,\infty]$ convergence. Conversely, L_p convergence only implies convergence a.e. of a subsequence. In 1966, L. Carleson proved the fundamental result that the Fourier series of an arbitrary function $f \in L_p(\mathbb{T})$, $1 < p < \infty$ converges almost everywhere. Kolmogorov (1923) showed that the analogue of Carlson's result for $L_1(\mathbb{T})$ is false. A natural question is whether the Fourier series of every function $f \in C(\mathbb{T})$ converges *uniformly* or at least *pointwise* to f.

In fact, many mathematicians like Riemann, Weierstrass and Dedekind conjectured over long time that the Fourier series of a function $f \in C(\mathbb{T})$ converges pointwise to f. Unfortunately, we have in general neither pointwise nor uniform convergence of the Fourier series of a function $f \in C(\mathbb{T})$. A concrete counterexample was constructed by Du Bois–Reymond in 1876 and was a quite remarkable surprise. It was shown that there exists a (real-valued) function $f \in C(\mathbb{T})$ such that

$$\lim_{n\to\infty} \sup |S_n f(0)| = \infty.$$

To see that pointwise convergence fails in general we need the following principle of uniform boundedness of sequences of linear operators, see e.g. [20, Kor.2.4].

Theorem 22 (Theorem of Banach–Steinhaus)

Let X be a Banach space with a dense subset $D \subset X$ and Y a normed space. Further let $T_n : X \to Y$ for $n \in \mathbb{N}$, and $T : X \to Y$ be linear bounded operators. Then it holds

$$Tf := \lim_{n\to\infty} T_n f \tag{19}$$

for all $f \in X$ if and only if

1. $\|T_n\|_{X\to Y} \leq \mathrm{const} < \infty$ *for all* $n \in \mathbb{N}$, *and*
2. $\lim_{n\to\infty} T_n p = Tp$ *for all* $p \in D$.

Theorem 23

There exists a function $f \in C(\mathbb{T})$ whose Fourier series does not converge pointwise.

Proof: Applying Theorem 22 of Banach–Steinhaus, we choose $X = C(\mathbb{T})$, $Y = \mathbb{C}$ and $D = \bigcup_{n=0}^{\infty} \mathcal{T}_n$. By the Approximation Theorem 21 of Weierstrass, the set D of all trigonometric polynomials is dense in $C(\mathbb{T})$. Then we consider the linear bounded functionals $T_n f := (S_n f)(0)$ for $n \in \mathbb{N}$ and $Tf := f(0)$ for $f \in C(\mathbb{T})$. Note that instead of 0 we can choose any fixed $x_0 \in \mathbb{T}$.
By $S_n p = p$ for each $p \in D$ and sufficiently large n, in particular we have $\lim_{n\to\infty} S_n p(0) = p(0)$. But the norms $\|T_n\|_{C(\mathbb{T})\to\mathbb{C}}$ are not uniformly bounded with respect to n, because $\|T_n\|_{C(\mathbb{T})\to\mathbb{C}} = \|D_n\|_{L_1(\mathbb{T})}$ are not uniformly bounded by (14). Thus by the Banach–Steinhaus Theorem 22 there exists a function $f \in C(\mathbb{T})$ whose Fourier series does not converge in the point 0.

Finally we determine the norm $\|T_n\|_{C(\mathbb{T})\to\mathbb{C}}$. From

$$
\begin{aligned}
|T_n f| &= |S_n f(0)| = |(D_n * f)(0)| \\
&= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x)\, f(x)\, \mathrm{d}x\right| \\
&\leq \|f\|_{C(\mathbb{T})}\|D_n\|_{L_1(\mathbb{T})}
\end{aligned}
$$

for arbitrary $f \in C(\mathbb{T})$ it follows $\|T_n\|_{C(\mathbb{T})\to\mathbb{C}} \leq \|D_n\|_{L_1(\mathbb{T})}$. To verify the opposite direction consider for an arbitrary $\varepsilon > 0$ the function

$$
f_\varepsilon := \frac{D_n}{|D_n| + \varepsilon} \in C(\mathbb{T}),
$$

which has $C(\mathbb{T})$ norm smaller than 1.

Then

$$
\begin{aligned}
|T_n f_\varepsilon| = (D_n * f_\varepsilon)(0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|D_n(x)|^2}{|D_n(x)| + \varepsilon}\,\mathrm{d}x \\
&\geq \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|D_n(x)|^2 - \varepsilon^2}{|D_n(x)| + \varepsilon}\,\mathrm{d}x \\
&\geq \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(x)|\,\mathrm{d}x - \varepsilon\Big)\,\|f_\varepsilon\|_{C(\mathbb{T})}
\end{aligned}
$$

implies $\|T_n\|_{C(\mathbb{T})\to\mathbb{C}} \geq \|D_n\|_{L_1(\mathbb{T})} - \varepsilon$. For $\varepsilon \to 0$ we obtain the assertion. ■

In the following we will see that for frequently appearing classes of functions stronger convergence results can be proved. A function $f\colon \mathbb{T} \to \mathbb{C}$ is called *piecewise continuously differentiable*, if there exist finitely many points $0 \le x_0 < x_1 < \ldots < x_{n-1} < 2\pi$ such that f is continuously differentiable on each subinterval (x_j, x_{j+1}), $j = 0, \ldots, n-1$ with $x_n = x_0 + 2\pi$, and the left and right limits $f(x_j \pm 0)$, $f'(x_j \pm 0)$ for $j = 0, \ldots, n$ exist and are finite. In the case $f(x_j - 0) \neq f(x_j + 0)$, the piecewise continuously differentiable function $f : \mathbb{T} \to \mathbb{C}$ has a *jump discontinuity* at x_j with jump height $|f(x_j + 0) - f(x_j - 0)|$. Simple examples of piecewise continuously differentiable functions $f : \mathbb{T} \to \mathbb{C}$ are the the sawtooth function and the rectangular pulse function (see Examples 9 and 10). This definition is illustrated in Figure 8.

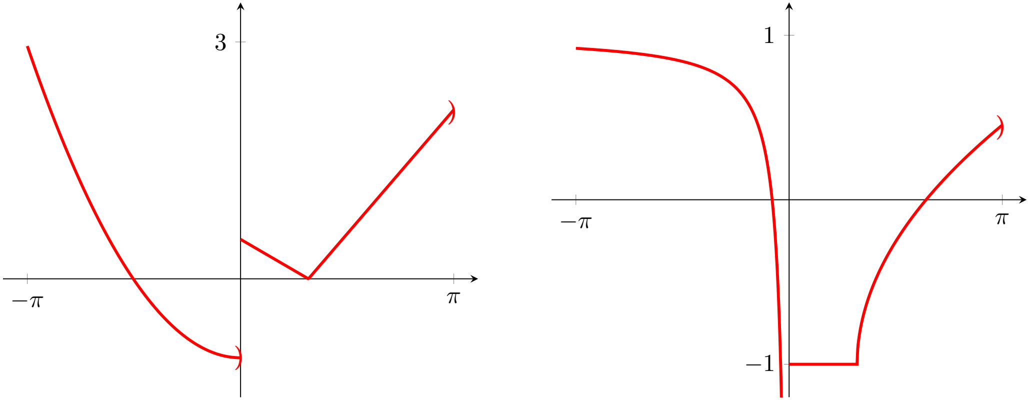


Figure 8: A piecewise continuously differentiable function (left) and a function that is not piecewise continuously differentiable (right).

The next convergence statements will use the following result of Riemann–Lebesgue.

Lemma 24 (Lemma of Riemann–Lebesgue)

Let $f \in L_1\big(\overline{(a,b)}\big)$ with $-\infty \le a < b \le \infty$ be given. Then the following relations hold

$$\lim_{|v|\to\infty} \int_a^b f(x)\,\mathrm{e}^{-\mathrm{i}xv}\,\mathrm{d}x = 0\,,$$

$$\lim_{|v|\to\infty} \int_a^b f(x)\,\sin(xv)\,\mathrm{d}x = 0\,, \qquad \lim_{|v|\to\infty} \int_a^b f(x)\,\cos(xv)\,\mathrm{d}x = 0\,.$$

Especially, for $f \in L_1(\mathbb{T})$ we have

$$\lim_{|k|\to\infty} c_k(f) = \frac{1}{2\pi} \lim_{|k|\to\infty} \int_{-\pi}^{\pi} f(x)\,\mathrm{e}^{-\mathrm{i}xk}\,\mathrm{d}x = 0\,.$$

Proof: We prove only

$$\lim_{|v|\to\infty} \int_a^b f(x)\, p(vx)\, \mathrm{d}x = 0 \tag{20}$$

for $p(t) = \mathrm{e}^{-\mathrm{i}t}$. The other cases $p(t) = \sin t$ and $p(t) = \cos t$ can be shown analogously.

For the characteristic function $\chi_{[\alpha,\beta]}$ of a finite interval $[\alpha, \beta] \subseteq [a, b]$ it follows for $v \neq 0$ that

$$\Big| \int_a^b \chi_{[\alpha,\beta]}(x)\, \mathrm{e}^{-\mathrm{i}xv}\, \mathrm{d}x \Big| = \Big| -\frac{1}{\mathrm{i}v} \left(\mathrm{e}^{-\mathrm{i}v\beta} - \mathrm{e}^{-\mathrm{i}v\alpha}\right) \Big| \leq \frac{2}{|v|}.$$

This becomes arbitrarily small as $|v| \to \infty$ so that characteristic functions and also all linear combinations of characteristic functions (i.e. step functions) fulfill the assertion.

The set of all step functions is dense in $L_1([a,b])$, i.e. for any $\varepsilon > 0$ and $f \in L_1([a,b])$ there exists a step function φ such that

$$\|f - \varphi\|_{L_1([a,b])} = \int_a^b |f(x) - \varphi(x)| \, \mathrm{d}x < \varepsilon.$$

By

$$\begin{aligned} \Big| \int_a^b f(x) \, \mathrm{e}^{-\mathrm{i}xv} \, \mathrm{d}x \Big| &\leq \Big| \int_a^b (f(x) - \varphi(x)) \, \mathrm{e}^{-\mathrm{i}xv} \, \mathrm{d}x \Big| + \Big| \int_a^b \varphi(x) \, \mathrm{e}^{-\mathrm{i}xv} \, \mathrm{d}x \Big| \\ &\leq \varepsilon + \Big| \int_a^b \varphi(x) \, \mathrm{e}^{-\mathrm{i}xv} \, \mathrm{d}x \Big| \end{aligned}$$

we obtain the assertion. ■

Next we formulate a localization principle, which states that the convergence behavior of a Fourier series of a function $f \in L_1(\mathbb{T})$ at a point x_0 depends merely on the values of f in some arbitrarily small neighborhood – despite the fact that the Fourier coefficients are determined by all function values on $\mathbb{T}$.

Theorem 25 (Riemann's Localization Principle)

Let $f \in L_1(\mathbb{T})$ and $x_0 \in \mathbb{R}$ be given. Then we have

$$\lim_{n\to\infty} S_n f(x_0) = c$$

for some $c \in \mathbb{R}$ if and only if for some $\delta \in (0, \pi]$

$$\lim_{n\to\infty} \int_0^\delta \big(f(x_0 - t) + f(x_0 + t) - 2\,c\big)\, D_n(t)\,\mathrm{d}t = 0\,.$$

Proof: Since $D_n \in C(\mathbb{T})$ is even, we get

$$\begin{aligned} S_n f(x_0) &= \frac{1}{2\pi} \Big(\int_{-\pi}^{0} + \int_{0}^{\pi}\Big) f(x_0 - t)\, D_n(t)\, \mathrm{d}t \\ &= \frac{1}{2\pi} \int_0^\pi \big(f(x_0 - t) + f(x_0 + t)\big)\, D_n(t)\, \mathrm{d}t\,. \end{aligned}$$

Using $\pi = \int_0^\pi D_n(t)\, \mathrm{d}t$, we conclude further

$$S_n f(x_0) - c = \frac{1}{2\pi} \int_0^\pi \big(f(x_0 - t) + f(x_0 + t) - 2\,c\big)\, D_n(t)\, \mathrm{d}t.$$

By Example 14, we have $D_n(t) = \sin\big((n + \frac{1}{2})t\big)/\sin\frac{t}{2}$ for $t \in (0, \pi]$. By the Lemma 24 of Riemann–Lebesgue we obtain

$$\lim_{n\to\infty} \int_\delta^\pi \frac{f(x_0 - t) + f(x_0 + t) - 2\,c}{\sin\frac{t}{2}}\, \sin\big((n + \frac{1}{2})t\big)\, \mathrm{d}t = 0$$

and hence

$$\lim_{n\to\infty} S_n f(x_0) - c = \lim_{n\to\infty} \frac{1}{2\pi} \int_0^\delta \big(f(x_0 - t) + f(x_0 + t) - 2\,c\big)\, D_n(t)\, \mathrm{d}t\,,$$

if one of the limits exists. ■

For a complete proof of the main result on the convergence of Fourier series, we need some additional preliminaries. Here we follow mainly the ideas of [11, p. 137 and pp. 144-148].

Let a compact interval $[a, b] \subset \mathbb{R}$ with $-\infty < a < b < \infty$ be given. Then a function $\varphi : [a, b] \to \mathbb{C}$ is called a *function of bounded variation*, if

$$V_a^b(\varphi) := \sup \sum_{j=1}^{n} |\varphi(x_j) - \varphi(x_{j-1})| < \infty\,,$$

where the supremum is taken over all partitions $a = x_0 < x_1 < \ldots < x_n = b$ of $[a, b]$. The nonnegative number $V_a^b(\varphi)$ is the *total variation* of φ on $[a, b]$. We set $V_a^a(\varphi) := 0$. For instance, each monotone function $\varphi : [a, b] \to \mathbb{R}$ is a function of bounded variation with $V_a^b(\varphi) = |\varphi(b) - \varphi(a)|$. Because

$$|\varphi(x)| \le |\varphi(a)| + |\varphi(x) - \varphi(a)| \le |\varphi(a)| + V_a^x(\varphi) \le |\varphi(a)| + V_a^b(\varphi) < \infty$$

for all $x \in [a, b]$, each function of bounded variation is bounded on $[a, b]$.

Lemma 26

Let $\varphi : [a, b] \to \mathbb{C}$ *and* $\psi : [a, b] \to \mathbb{C}$ *be functions of bounded variation. Then for arbitrary* $\alpha \in \mathbb{C}$ *and* $c \in [a, b]$ *it holds*

$$\begin{aligned} V_a^b(\alpha\,\varphi) &= |\alpha|\, V_a^b(\varphi)\,, \\ V_a^b(\varphi + \psi) &\le V_a^b(\varphi) + V_a^b(\psi)\,, \\ V_a^b(\varphi) &= V_a^c(\varphi) + V_c^b(\varphi)\,, && (21) \\ \max\{V_a^b(\operatorname{Re}\varphi),\ V_a^b(\operatorname{Im}\varphi)\} &\le V_a^b(\varphi) \le V_a^b(\operatorname{Re}\varphi) + V_a^b(\operatorname{Im}\varphi)\,. && (22) \end{aligned}$$

The simple proof is omitted here. For details see e.g. [18, pp. 159–162].

Theorem 27 (Jordan Decomposition Theorem)

Let $\varphi : [a, b] \to \mathbb{C}$ *be a given function of bounded variation. Then there exist four nondecreasing functions* $\varphi_j : [a, b] \to \mathbb{R}$, $j = 1, \ldots, 4$, *such that* φ *possesses the Jordan decomposition*

$$\varphi = (\varphi_1 - \varphi_2) + \mathrm{i}\,(\varphi_3 - \varphi_4)\,,$$

where $\operatorname{Re}\varphi = \varphi_1 - \varphi_2$ *and* $\operatorname{Im}\varphi = \varphi_3 - \varphi_4$ *are functions of bounded variation. If* φ *is continuous, then* φ_j, $j = 1, \ldots, 4$, *are continuous too.*

Proof: From (22) it follows that $\operatorname{Re}\varphi$ and $\operatorname{Im}\varphi$ are functions of bounded variation. We decompose $\operatorname{Re}\varphi$. Obviously,

$$\varphi_1(x) := V_a^x(\operatorname{Re}\varphi)\,, \quad x \in [a,\, b]\,,$$

is nondecreasing by (21). Then

$$\varphi_2(x) := \varphi_1 - \operatorname{Re}\varphi(x)\,, \quad x \in [a,\, b]\,,$$

is nondecreasing too, since for $a \le x < y \le b$ it holds

$$\big|\operatorname{Re}\varphi(y) - \operatorname{Re}\varphi(x)\big| \le V_x^y(\operatorname{Re}\varphi) = \varphi_1(y) - \varphi_1(x)$$

and hence

$$\varphi_2(y) - \varphi_2(x) = \big(\varphi_1(y) - \varphi_1(x)\big) - \big(\operatorname{Re}\varphi(y) - \operatorname{Re}\varphi(x)\big) \ge 0\,.$$

Thus we obtain $\operatorname{Re}\varphi = \varphi_1 - \varphi_2$. Analogously, we can decompose $\operatorname{Im}\varphi = \varphi_3 - \varphi_4$. Using $\varphi = \operatorname{Re}\varphi + \mathrm{i}\,\operatorname{Im}\varphi$, we receive the above Jordan decomposition of φ. If φ is continuous at $x \in [a,\, b]$, then, by definition, each φ_j is continuous at x. ■

A 2π-periodic function $f : \mathbb{T} \to \mathbb{C}$ with $V_0^{2\pi}(f) < \infty$ is called a 2π-*periodic function of bounded variation*. By (21) a 2π-periodic function of bounded variation has the property $V_a^b(f) < \infty$ for each compact interval $[a, b] \subset \mathbb{R}$.

Example 28

Let $f : \mathbb{T} \to \mathbb{C}$ be a piecewise continuously differentiable function with jump discontinuities at distinct points $x_j \in (0, 2\pi)$, $j = 1, \ldots, n-1$. Assume that it holds $f(x) = \frac{1}{2}\big(f(x+0) + f(x-0)\big)$ for all $x \in [0, 2\pi)$. Then f is a 2π-periodic function of bounded variation, since

$$\begin{aligned} V_0^{2\pi}(f) &= \big|f(0+0) - f(0-0)\big| + \sum_{j=1}^{n-1} |f(x_j+0) - f(x_j-0)| \\ &+ \int_0^{2\pi} |f'(t)|\,\mathrm{d}t < \infty\,. \end{aligned}$$

The functions given in Examples 5, 8, 9, and 10 are 2π-periodic functions of bounded variation. $\square$

Lemma 29

There exists a constant $c_0 > 0$ such that for all α, $\beta \in [0, \pi]$ and all $n \in \mathbb{N}$ it holds

$$\Big| \int_\alpha^\beta D_n(t)\,\mathrm{d}t \Big| \le c_0 \,. \tag{23}$$

Proof: We introduce the function $h \in C[0, \pi]$ by

$$h(t) := \frac{1}{\sin \frac{t}{2}} - \frac{2}{t} \,, \quad t \in (0, \pi] \,,$$

and $h(0) := 0$. This continuous function h is increasing, i.e., we have $0 \le h(t) \le h(\pi) < \frac{1}{2}$ for all $t \in [0, \pi]$. Using (13), for arbitrary α, $\beta \in [0, \pi]$ we estimate

$$\begin{aligned} \Big| \int_\alpha^\beta D_n(t)\,\mathrm{d}t \Big| &\le \Big| \int_\alpha^\beta h(t) \sin\Big(n + \frac{1}{2}\Big) t \,\mathrm{d}t \Big| + 2 \Big| \int_\alpha^\beta \frac{\sin\big(n + \frac{1}{2}\big) t}{t} \,\mathrm{d}t \Big| \\ &\le \frac{\pi}{2} + 2 \Big| \int_\alpha^\beta \frac{\sin\big(n + \frac{1}{2}\big) t}{t} \,\mathrm{d}t \Big| \,. \end{aligned}$$

By the sine integral

$$\mathrm{Si}(x) := \int_0^x \frac{\sin t}{t}\,\mathrm{d}t\,, \quad x \in \mathbb{R}\,,$$

it holds for all $\gamma \ge 0$ (see Lemma 37)

$$\Big|\int_0^\gamma \frac{\sin x}{x}\,\mathrm{d}x\Big| \le \mathrm{Si}(\pi) < 2\,.$$

From

$$\int_\alpha^\beta \frac{\sin\big(n+\frac{1}{2}\big)t}{t}\,\mathrm{d}t = \int_0^{(n+\frac{1}{2})\,\beta} \frac{\sin x}{x}\,\mathrm{d}x - \int_0^{(n+\frac{1}{2})\,\alpha} \frac{\sin x}{x}\,\mathrm{d}x$$

it follows that

$$\Big|\int_\alpha^\beta \frac{\sin\big(n+\frac{1}{2}\big)t}{t}\,\mathrm{d}t\Big| \le 4\,,$$

i.e., (23) is fulfilled for the constant $c_0 = \frac{\pi}{2} + 8$. ■

Lemma 30

Assume that $0 < a < b < 2\pi$, $\delta > 0$ *and* $b - a + 2\delta < 2\pi$ *be given. Let* $\varphi : [a - \delta - \pi,\, b + \delta + \pi] \to \mathbb{R}$ *be nondecreasing, piecewise continuous function which is continuous on* $[a - \delta,\, b + \delta]$.

Then for each $\varepsilon > 0$ *there exists an index* $n_0(\varepsilon)$ *such that for all* $n > n_0(\varepsilon)$ *and all* $x \in [a,\, b]$

$$\Big| \int_0^{\pi} \big(\varphi(x + t) + \varphi(x - t) - 2\,\varphi(x)\big)\, D_n(t)\,\mathrm{d}t \Big| < \varepsilon\,.$$

Proof: 1. For $(x, t) \in [a-\delta,\, b+\delta] \times [0,\, \pi]$ we introduce the functions

$$
\begin{aligned}
g(x,t) &:= \varphi(x+t) + \varphi(x-t) - 2\,\varphi(x)\,, \\
h_1(x,t) &:= \varphi(x+t) - \varphi(x) \geq 0\,, \\
h_2(x,t) &:= \varphi(x) - \varphi(x-t) \geq 0
\end{aligned}
$$

such that $g = h_1 - h_2$. For fixed $x \in [a,\, b]$, both functions $h_j(x, \cdot)$, $j = 1,\, 2$, are nondecreasing on $[0,\, \pi]$. Since $h_j(\cdot, \pi)$, $j = 1,\, 2$, are piecewise continuous on $[a,\, b]$, there exists a constant $c_1 > 0$ such that for all $(x, t) \in [a,\, b] \times [0,\, \pi]$

$$
\big|h_j(x,t)\big| \leq c_1\,. \tag{24}
$$

Since φ is continuous on the compact interval $[a-\delta,\, b+\delta]$, the function φ is uniformly continuous on $[a-\delta,\, b+\delta]$, i.e., for each $\varepsilon > 0$ there exists $\beta \in (0,\, \delta)$ such that for all $y,\, z \in [a-\delta,\, b+\delta]$ with $|y-z| \leq \beta$ we have

$$
\big|\varphi(y) - \varphi(z)\big| < \frac{\varepsilon}{4\,c_0}\,.
$$

By the proof of Lemma 29 we can choose $c_0 = \frac{\pi}{2} + 8$. Hence we obtain for all $(x, t) \in [a,\, b] \times [0,\, \beta]$ and $j = 1,\, 2$

$$0 \le h_j(x,\ t) < \frac{\varepsilon}{4\, c_0}\,. \tag{25}$$

2. Now we split the integral

$$\int_0^{\pi} g(x, t)\, D_n(t)\, \mathrm{d}t = \int_0^{\beta} g(x, t)\, D_n(t)\, \mathrm{d}t + \int_{\beta}^{\pi} g(x, t)\, D_n(t)\, \mathrm{d}t \tag{26}$$

into a sum of two integrals, where the first integral can be written in the form

$$\int_0^{\beta} g(x, t)\, D_n(t)\, \mathrm{d}t = \int_0^{\beta} h_1(x, t)\, D_n(t)\, \mathrm{d}t - \int_0^{\beta} h_2(x, t)\, D_n(t)\, \mathrm{d}t\,. \tag{27}$$

Observing that $h_j(x,\cdot)$, $j=1,\,2$, are nondecreasing for fixed $x \in [a,\,b]$, we obtain by the second mean value theorem for integrals, see e.g. [18, pp. 328–329], that for certain $\alpha_j(x) \in [0,\,\beta]$

$$\begin{aligned}\int_0^\beta h_j(x,t)\,D_n(t)\,\mathrm{d}t &= h_j(x,0)\int_0^{\alpha_j(x)} D_n(t)\,\mathrm{d}t \\ &+ h_j(x,\beta)\int_{\alpha_j(x)}^\beta D_n(t)\,\mathrm{d}t \\ &= 0 + h_j(x,\beta)\int_{\alpha_j(x)}^\beta D_n(t)\,\mathrm{d}t\,, \quad j=1,\,2\,.\end{aligned}$$

By (23) and (25) this integral can be estimated for all $x \in [a,\,b]$ by

$$\Big|\int_0^\beta h_j(x,t)\,D_n(t)\,\mathrm{d}t\Big| \le \frac{\varepsilon}{4\,c_0}\,c_0 = \frac{\varepsilon}{4}$$

such that by (27) for all $x \in [a,\,b]$

$$\Big|\int_0^\beta g(x,t)\,D_n(t)\,\mathrm{d}t\Big| \le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}\,. \tag{28}$$

3. Next we consider the second integral in (26) which can be written as

$$\int_\beta^\pi g(x,t)D_n(t)\,\mathrm{d}t = \int_\beta^\pi h_1(x,t)D_n(t)\,\mathrm{d}t - \int_\beta^\pi h_2(x,t)\,D_n(t)\,\mathrm{d}t. \tag{29}$$

Since $h_j(x,\cdot)$, $j = 1, 2$, are nondecreasing for fixed $x \in [a,\ b]$, the second mean value theorem for integrals provides the existence of certain $\gamma_j(x) \in [\beta,\ \pi]$ such that

$$\int_\beta^\pi h_j(x,t)\,D_n(t)\mathrm{d}t = h_j(x,\beta)\int_\beta^{\gamma_j(x)} D_n(t)\mathrm{d}t + h_j(x,\pi)\int_{\gamma_j(x)}^\pi D_n(t)\,\mathrm{d}t. \tag{30}$$

From (13) it follows

$$\int_\beta^{\gamma_j(x)} D_n(t)\,\mathrm{d}t = \int_\beta^{\gamma_j(x)} \frac{1}{\sin\frac{t}{2}}\,\sin\big(n+\frac{1}{2}\big)t\,\mathrm{d}t\,.$$

Since $\left(\sin\frac{t}{2}\right)^{-1}$ is monotone on $[\beta,\, \gamma_j(x)]$, again by the second mean value theorem for integrals there exist $\eta_j(x) \in [\beta,\, \gamma_j(x)]$ with

$$\int_\beta^{\gamma_j(x)} D_n(t)\,\mathrm{d}t \;=\; \frac{1}{\sin\frac{\beta}{2}}\int_\beta^{\eta_j(x)} \sin\left(n+\frac{1}{2}\right)t\,\mathrm{d}t + \frac{1}{\sin\frac{\gamma_j(x)}{2}}\int_{\eta_j(x)}^{\gamma_j(x)} \sin\left(n+\frac{1}{2}\right)t\,\mathrm{d}t\,. \tag{31}$$

Now we estimate both integral in (31) such that

$$\Big|\int_\beta^{\eta_j(x)} \sin\left(n+\frac{1}{2}\right)t\,\mathrm{d}t\Big| \le \frac{4}{2n+1}\,,$$

$$\Big|\int_{\eta_j(x)}^{\gamma_j(x)} \sin\left(n+\frac{1}{2}\right)t\,\mathrm{d}t\Big| \le \frac{4}{2n+1}\,.$$

Applying the above inequalities, we see by (31) for all $x \in [a,\, b]$ and $j = 1,\, 2$ that

$$\Big|\int_\beta^{\gamma_j(x)} D_n(t)\,\mathrm{d}t\Big| \le \frac{8}{(2n+1)\,\sin\frac{\beta}{2}}\,. \tag{32}$$

Analogously, one can show for all $x \in [a,\, b]$ and $j = 1,\, 2$ that

$$\Big|\int_{\gamma_j(x)}^{\pi} D_n(t)\,\mathrm{d}t\Big| \le \frac{8}{(2n+1)\,\sin\frac{\beta}{2}}\,. \tag{33}$$

Using (24) and (30), the inequalities (32) and (33) yield for all $x \in [a,\, b]$ and $j = 1,\, 2$,

$$\Big|\int_{\beta}^{\pi} h_j(x,t)\, D_n(t)\,\mathrm{d}t\Big| \le \frac{16\, c_1}{(2n+1)\,\sin\frac{\beta}{2}}$$

and hence by (29)

$$\Big|\int_{\beta}^{\pi} g(x,t)\, D_n(t)\,\mathrm{d}t\Big| \le \frac{32\, c_1}{(2n+1)\,\sin\frac{\beta}{2}}.$$

Therefore for the chosen $\varepsilon > 0$ there exists an index $n_0(\varepsilon) \in \mathbb{N}$ such that for all $n > n_0(\varepsilon)$ and all $x \in [a,\, b]$,

$$\Big|\int_{\beta}^{\pi} g(x,t)\, D_n(t)\,\mathrm{d}t\Big| < \frac{\varepsilon}{2}\,. \tag{34}$$

Together with (26), (28), and (34) it follows for all $n > n_0(\varepsilon)$ and all $x \in [a, b]$ that

$$\Big| \int_0^{\pi} g(x,t)\, D_n(t)\, \mathrm{d}t \Big| < \varepsilon\,.$$

This completes the proof. ■

Based on Riemann's Localization Principle and these preliminaries, we can prove the following important theorem concerning pointwise convergence of the Fourier series of a piecewise continuously differentiable function f.

Theorem 31 (Convergence Theorem of Dirichlet–Jordan)

Let $f : \mathbb{T} \to \mathbb{C}$ be a piecewise continuously differentiable function. Then at every point $x_0 \in \mathbb{R}$, the Fourier series of f converges as

$$\lim_{n\to\infty} S_n f(x_0) = \frac{1}{2}\left(f(x_0+0) + f(x_0-0)\right).$$

In particular, if f is continuous at x_0, then

$$\lim_{n\to\infty} S_n f(x_0) = f(x_0).$$

Further the Fourier series of f converges uniformly on any closed interval $[a, b] \subset (0, 2\pi)$, if f is continuous on $[a-\delta, b+\delta]$ with certain $\delta > 0$. Especially, if $f \in C(\mathbb{T})$ is piecewise continuously differentiable, then the Fourier series of f converges uniformly to f on $\mathbb{R}$.

Proof: 1. By assumption there exists $\delta \in (0, \pi)$, such that f is continuously differentiable in $[x_0 - \delta, x_0 + \delta] \backslash \{x_0\}$. Let

$$M := \max_{t \in [-\pi, \pi]} \{|f'(t+0)|, |f'(t-0)|\}.$$

By the mean value theorem we conclude

$$|f(x_0 + t) - f(x_0 + 0)| \le t\, M\,, \quad |f(x_0 - t) - f(x_0 - 0)| \le t\, M$$

for all $t \in (0, \delta]$. This implies

$$\int_0^\delta \frac{|f(x_0 - t) + f(x_0 + t) - f(x_0 + 0) - f(x_0 - 0)|}{t}\, \mathrm{d}t \le 2\, M\delta < \infty\,.$$

By $\frac{t}{\pi} \le \sin \frac{t}{2}$ for $t \in [0,\, \pi]$ the function

$$h(t) := \frac{f(x_0 - t) + f(x_0 + t) - f(x_0 + 0) - f(x_0 - 0)}{t} \frac{t}{\sin \frac{t}{2}}, \quad t \in (0,\, \delta],$$

is absolutely integrable on $[0, \delta]$. By Lemma 24 of Riemann–Lebesgue we get

$$\lim_{n \to \infty} \int_0^\delta h(t)\, \sin((n + \frac{1}{2})t)\, \mathrm{d}t = 0\,.$$

Using Riemann's Localization Principle, Theorem 25, we obtain the assertion with $2\,c = f(x_0 + 0) + f(x_0 - 0)$.

2. By assumption and Example 28, the given function f is a 2π-periodic function of bounded variation. Then it follows that $V_{a-\delta-\pi}^{b+\delta+\pi}(f) < \infty$. By the Jordan Decomposition Theorem 27 the function f restricted on $[a - \delta - \pi,\, b + \delta + \pi]$ can be represented in the form

$$f = (\varphi_1 - \varphi_2) + \mathrm{i}\,(\varphi_3 - \varphi_4)\,,$$

where $\varphi_j :\, [a - \delta - \pi,\, b + \delta + \pi] \to \mathbb{R}$, $j = 1, \ldots, 4$, are nondecreasing and piecewise continuous. Since f is continuous on $[a,\, b]$, each φ_j, $j = 1, \ldots, 4$, is continuous on $[a,\, b]$ too. Applying Lemma 30, we obtain that for each $\varepsilon > 0$ there exists an index $N(\varepsilon) \in \mathbb{N}$ such that for $n > N(\varepsilon)$ and all $x \in [a,\, b]$,

$$|S_n f(x) - f(x)| = \frac{1}{2\pi}\,\Big|\int_0^{\pi} \big(f(x+t) + f(x-t) - 2\,f(x)\big)\,D_n(t)\,\mathrm{d}t\Big| < \varepsilon\,.$$

This completes the proof. ■

Example 32

The functions $f : \mathbb{T} \to \mathbb{C}$ given in Examples 5, 8, 9, and 10 are piecewise continuously differentiable. If $x_0 \in \mathbb{R}$ is a jump discontinuity of f, then the value $f(x_0)$ is equal to the mean $\frac{1}{2}\big(f(x_0 + 0) + f(x_0 - 0)\big)$ of right and left limits. By the Convergence Theorem 31 of Dirichlet–Jordan, the Fourier series of f converges to f in each point of $\mathbb{R}$. On each closed interval, which does not contain any discontinuity of f, the Fourier series converges uniformly. Since the piecewise continuously differentiable function of Example 8 is contained in $C(\mathbb{T})$, its Fourier series converges uniformly on $\mathbb{R}$. $\square$

Remark 33

The Convergence Theorem 31 of Dirichlet–Jordan is also valid for each 2π-periodic function $f : \mathbb{T} \to \mathbb{C}$ of bounded variation (see e.g. [18, pp. 546–547]). $\square$

A useful criterion for uniform convergence of the Fourier series of a function $f \in C(\mathbb{T})$ is given in the following theorem.

Theorem 34

If $f \in C(\mathbb{T})$ fulfills the condition

$$\sum_{k\in\mathbb{Z}} |c_k(f)| < \infty\,, \tag{35}$$

then the Fourier series of f converges uniformly to f. Each function $f \in C^1(\mathbb{T})$ has the property (35).

Proof: By the assumption (35) and

$$|c_k(f)\,\mathrm{e}^{\mathrm{i}k\cdot}| = |c_k(f)|\,,$$

the assertion follows from the Weierstrass criterion of uniform convergence.
Assume that $f \in C^1(\mathbb{T})$. By the Convergence Theorem 31 of Dirichlet–Jordan we know already that the Fourier series of f converges uniformly to f. This could be also seen as follows: By the differentiation property of the Fourier coefficients in Lemma 6, we have $c_k(f) = (\mathrm{i}k)^{-1}\,c_k(f')$ for all $k \neq 0$ and $c_0(f') = 0$. By Parseval's identity of $f' \in L_2(\mathbb{T})$ it follows

$$\|f'\|^2 = \sum_{k\in\mathbb{Z}} |c_k(f')|^2 < \infty\,.$$

Using Cauchy–Schwarz inequality, we get finally

$$\begin{aligned}\sum_{k\in\mathbb{Z}} |c_k(f)| &= |c_0(f)| + \sum_{k\neq 0} \frac{1}{k}\,|c_k(f')| \\ &\leq |c_0(f)| + \Big(\sum_{k\neq 0} \frac{1}{k^2}\Big)^{1/2} \Big(\sum_{k\neq 0} |c_k(f')|^2\Big)^{1/2} < \infty.\end{aligned}$$

This completes the proof. ■

Remark 35

If $f \in C^1(\mathbb{T})$*, then by the mean value theorem it follows that*

$$|f(x+h) - f(x)| \le |h| \max_{t \in \mathbb{T}} |f'(t)|$$

for all $x,\, x+h \in \mathbb{T}$*, that means* f *is Lipschitz continuous on* $\mathbb{T}$*. More generally, a function* $f \in C(\mathbb{T})$ *is called Hölder continuous of order* $\alpha \in (0,\, 1]$ *on* $\mathbb{T}$*, if*

$$|f(x+h) - f(x)| \le c\, |h|^{\alpha}$$

for all $x,\, x+h \in \mathbb{T}$ *with certain constant* $c \ge 0$ *which depends on* f*. One can show that the Fourier series of a function* $f \in C(\mathbb{T})$ *which is Hölder continuous of order* $\alpha \in (0,\, 1]$ *converges uniformly to* f *and it holds*

$$\|S_n f - f\|_{C(\mathbb{T})} = \mathcal{O}(n^{-\alpha} \ln n)\,, \quad n \to \infty$$

(see [22, Vol. I, p. 64]). □

In practice, the following convergence result of Fourier series for a sufficiently smooth, 2π–periodic function is very useful.

Theorem 36 (Theorem of Bernstein)

Let $f \in C^r(\mathbb{T})$ with fixed $r \in \mathbb{N}$ be given. Then the Fourier coefficients $c_k(f)$ have the form

$$c_k(f) = \frac{1}{(\mathrm{i}k)^r}\, c_k(f^{(r)}), \quad k \in \mathbb{Z} \setminus \{0\}. \tag{36}$$

Further the approximation error $f - S_n f$ can be estimated for all $n \in \mathbb{N} \setminus \{1\}$ by

$$\|f - S_n f\|_\infty \le c\, \|f^{(r)}\|_\infty \, \frac{\ln n}{n^r}, \tag{37}$$

where the constant $c > 0$ is independent of f and n.

Proof: 1. Repeated integration by parts provides (36). By Lemma 24 of Riemann–Lebesgue we known

$$\lim_{|k|\to\infty} c_k(f^{(r)}) = 0$$

such that

$$\lim_{|k|\to\infty} k^r\, c_k(f) = 0\,.$$

2. The nth partial sum of the Fourier series of $f^{(r)} \in C(\mathbb{T})$ can be written in the form

$$\big(S_n f^{(r)}\big)(x) = \frac{1}{\pi} \int_0^{\pi} \big(f^{(r)}(x+y) + f^{(r)}(x-y)\big)\, \frac{\sin(n+\frac{1}{2})y}{2\,\sin\frac{y}{2}}\, \mathrm{d}y\,.$$

Then we estimate

$$\begin{aligned}
\big|\big(S_n f^{(r)}\big)(x)\big| &\le \frac{2}{\pi}\,\|f^{(r)}\|_\infty \int_0^\pi \frac{|\sin(n+\frac{1}{2})y|}{2\,\sin\frac{y}{2}}\,\mathrm{d}y \\
&< \|f^{(r)}\|_\infty \int_0^\pi \frac{|\sin(n+\frac{1}{2})y|}{y}\,\mathrm{d}y \\
&= \|f^{(r)}\|_\infty \int_0^{(n+\frac{1}{2})\pi} \frac{|\sin u|}{u}\,\mathrm{d}u \\
&< \|f^{(r)}\|_\infty \Big(1 + \int_1^{(n+\frac{1}{2})\pi} \frac{1}{u}\,\mathrm{d}u\Big) \\
&= \|f^{(r)}\|_\infty \Big(1 + \ln\big(n + \frac{1}{2}\big)\pi\Big)\,.
\end{aligned}$$

For a convenient constant $c > 0$, we obtain for all $n \in \mathbb{N} \setminus \{1\}$ that

$$\|S_n f^{(r)}\|_\infty \le c\,\|f^{(r)}\|_\infty\,\ln n\,. \tag{38}$$

By Theorem 34 the Fourier series of f converges uniformly to f such that by (36)

$$
\begin{aligned}
f - S_n f &= \sum_{k=n+1}^{\infty} \big(c_k(f)\, \mathrm{e}^{\mathrm{i}k\cdot} + c_{-k}(f)\, \mathrm{e}^{-\mathrm{i}k\cdot} \big) \\
&= \sum_{k=n+1}^{\infty} \frac{1}{(\mathrm{i}k)^r} \big(c_k(f^{(r)}) \mathrm{e}^{\mathrm{i}k\cdot} + (-1)^r c_{-k}(f^{(r)})\, \mathrm{e}^{-\mathrm{i}k\cdot} \big). \qquad (39)
\end{aligned}
$$

3. For even smoothness $r = 2s$, $s \in \mathbb{N}$, we obtain by (39) that

$$
\begin{aligned}
f - S_n f &= (-1)^s \sum_{k=n+1}^{\infty} \frac{1}{k^r} \big(c_k(f^{(r)})\, \mathrm{e}^{\mathrm{i}k\cdot} + c_{-k}(f^{(r)})\, \mathrm{e}^{-\mathrm{i}k\cdot} \big) \\
&= (-1)^s \sum_{k=n+1}^{\infty} \frac{1}{k^r} \big(S_k f^{(r)} - S_{k-1} f^{(r)} \big).
\end{aligned}
$$

Obviously, for $N > n$ it holds the identity

$$\sum_{k=n+1}^{N} a_k\,(b_k - b_{k-1}) = a_N\,b_N - a_{n+1}\,b_n + \sum_{k=n+1}^{N-1} (a_k - a_{k+1})\,b_k \tag{40}$$

for arbitrary complex numbers a_k and b_k. We apply (40) to $a_k = k^{-r}$ and $b_k = S_k f^{(r)}$. Then for $N \to \infty$ we receive

$$f - S_n f = (-1)^{s+1}\,\frac{1}{(n+1)^r}\,S_n f^{(r)} + (-1)^s \sum_{k=n+1}^{\infty} \big(\frac{1}{k^r} - \frac{1}{(k+1)^r}\big)\,S_k f^{(r)}\,, \tag{41}$$

since by (38)

$$\frac{1}{N^r}\,\|S_N f^{(r)}\|_\infty \le c\,\|f^{(r)}\|_\infty\,\frac{\ln N}{N^r} \to 0 \quad \text{as } N \to \infty\,.$$

Thus we can estimate the approximation error (41) by

$$\|f - S_n f\|_\infty \le c\,\|f^{(r)}\|_\infty \Big(\frac{\ln n}{(n+1)^r} + \sum_{k=n+1}^{\infty} \big(\frac{1}{k^r} - \frac{1}{(k+1)^r}\big)\,\ln k\Big)\,.$$

Using the identity (40) for $a_k = \ln k$ and $b_k = -(k+1)^{-r}$, we see that

$$\sum_{k=n+1}^{\infty} \Big(\frac{1}{k^r} - \frac{1}{(k+1)^r}\Big) \ln k = \frac{\ln(n+1)}{(n+1)^r} + \sum_{k=n+1}^{\infty} \frac{1}{(k+1)^r} \ln\Big(1 + \frac{1}{k}\Big),$$

since $(N+1)^{-k} \ln N \to 0$ as $N \to \infty$. From $\ln(1+\frac{1}{k}) < \frac{1}{k}$ it follows that

$$\begin{aligned} \sum_{k=n+1}^{\infty} \frac{1}{(k+1)^r} \ln\Big(1+\frac{1}{k}\Big) &< \sum_{k=n+1}^{\infty} \frac{1}{k\,(k+1)^r} < \sum_{k=n+1}^{\infty} \frac{1}{k^{r+1}} \\ &< \int_n^{\infty} \frac{1}{x^{r+1}}\,\mathrm{d}x = \frac{1}{r\,n^r}\,. \end{aligned}$$

Hence for convenient constant $c_1 > 0$ we have

$$\|f - S_n f\|_\infty \le c_1\, \|f^{(r)}\|_\infty\, \frac{1}{n^r}\,(1 + \ln n)\,.$$

This inequality implies (37) for even r.

4. The case of odd smoothness $r = 2s + 1$, $s \in \mathbb{N}_0$, can be handled similarly as the case of even r. By (39) we obtain

$$\begin{aligned} f - S_n f &= (-1)^s \, \mathrm{i} \sum_{k=n+1}^{\infty} \frac{1}{k^r} \big(c_{-k}(f^{(r)}) \, \mathrm{e}^{-\mathrm{i}k\cdot} - c_k(f^{(r)}) \, \mathrm{e}^{\mathrm{i}k\cdot} \big) \\ &= (-1)^s \sum_{k=n+1}^{\infty} \frac{1}{k^r} \big(\tilde{S}_k f^{(r)} - \tilde{S}_{k-1} f^{(r)} \big) \end{aligned} \tag{42}$$

with the *nth partial sum of the conjugate Fourier series* of $f^{(r)}$

$$\tilde{S}_n f^{(r)} := \mathrm{i} \sum_{j=1}^{n} \big(c_{-j}(f^{(r)}) \, \mathrm{e}^{-\mathrm{i}j\cdot} - c_j(f^{(r)}) \, \mathrm{e}^{\mathrm{i}j\cdot} \big) \, .$$

From

$$
\begin{aligned}
& \mathrm{i}\left(c_{-j}(f^{(r)})\,\mathrm{e}^{-\mathrm{i}jx} - c_j(f^{(r)})\,\mathrm{e}^{\mathrm{i}jx}\right) \\
= \;& -\frac{1}{\pi}\int_{-\pi}^{\pi} f^{(r)}(y)\,\sin j(y-x)\mathrm{d}y \\
= \;& -\frac{1}{\pi}\int_{-\pi}^{\pi} f^{(r)}(x+y)\,\sin(jy)\mathrm{d}y \\
= \;& -\frac{1}{\pi}\int_{0}^{\pi} \left(f^{(r)}(x+y) - f^{(r)}(x-y)\right)\,\sin(jy)\mathrm{d}y
\end{aligned}
$$

and

$$
\sum_{j=1}^{n}\sin(jy) = \frac{\cos\frac{y}{2} - \cos(n+\frac{1}{2})y}{2\,\sin\frac{y}{2}}\,, \quad y \in \mathbb{R}\setminus 2\pi\,\mathbb{Z}\,,
$$

it follows that

$$
(\tilde{S}_n f^{(r)})(x) = -\frac{1}{\pi}\int_{0}^{\pi}\left(f^{(r)}(x+y) - f^{(r)}(x-y)\right)\frac{\cos\frac{y}{2} - \cos(n+\frac{1}{2})y}{2\,\sin\frac{y}{2}}\,\mathrm{d}y
$$

and hence

$$\begin{aligned}
|(\tilde{S}_n f^{(r)})(x)| &\le \frac{2}{\pi}\,\|f^{(r)}\|_\infty \int_0^\pi \frac{|\cos\frac{y}{2} - \cos(n+\frac{1}{2})y|}{2\,\sin\frac{y}{2}}\,\mathrm{d}y \\
&= \frac{4}{\pi}\,\|f^{(r)}\|_\infty \int_0^\pi \frac{|\sin\frac{ny}{2}\ \ \sin\frac{(n+1)y}{2}|}{2\,\sin\frac{y}{2}}\,\mathrm{d}y \\
&< \frac{4}{\pi}\,\|f^{(r)}\|_\infty \int_0^\pi \frac{|\sin\frac{(n+1)y}{2}|}{2\,\sin\frac{y}{2}}\,\mathrm{d}y\,.
\end{aligned}$$

Similarly as in step 2, we obtain for any $n \in \mathbb{N} \setminus \{1\}$

$$\|\tilde{S}_n f^{(r)}\|_\infty \le c\,\|f^{(r)}\|_\infty \ln n$$

with some constant $c > 0$.
Now we apply the identity (40) to $a_k = k^{-r}$ and $b_k = \tilde{S}_k f^{(r)}$.

For $N \to \infty$ it follows from (42) that

$$f - S_n f = (-1)^{s+1} \frac{1}{(n+1)^r} \tilde{S}_n f^{(r)} + (-1)^s \sum_{k=n+1}^{\infty} \Big(\frac{1}{k^r} - \frac{1}{(k+1)^r}\Big) \tilde{S}_k f^{(r)} .$$

Thus we obtain the estimate

$$\|f - S_n f\|_\infty \le c\, \|f^{(r)}\|_\infty \Big(\frac{\ln n}{(n+1)^r} + \sum_{k=n+1}^{\infty} \Big(\frac{1}{k^r} - \frac{1}{(k+1)^r}\Big) \ln k \Big) .$$

We proceed as in step 3 and show the estimate (37) for odd r. ■

Roughly speaking we can say by Theorem 36 of Bernstein:
The smoother a function $f : \mathbb{T} \to \mathbb{C}$ is, the faster its Fourier coefficients $c_k(f)$ tend to zero as $|k| \to \infty$ and the faster its Fourier series converges uniformly to f.

Gibbs phenomenon

Let $f : \mathbb{T} \to \mathbb{C}$ be a piecewise continuously differentiable function with a jump discontinuity at $x_0 \in \mathbb{R}$. Then Theorem 31 of Dirichlet–Jordan implies

$$\lim_{n\to\infty} (S_n f)(x_0) = \frac{f(x_0 - 0) + f(x_0 + 0)}{2}.$$

Clearly, the Fourier series of f cannot converge uniformly in any small neighborhood of x_0, because the uniform limit of the continuous functions $S_n f$ would be continuous. The *Gibbs phenomenon* describes the bad convergence behavior of the Fourier sums $S_n f$ in a small neighborhood of x_0. If $n \to \infty$, then $S_n f$ overshoot and undershoot f near the jump discontinuity at x_0, see the right Figure 3.

First we analyze the convergence of the Fourier partial sums $S_n s$ of the sawtooth function s from Example 9 which is piecewise linear with $s(0) = 0$ and therefore piecewise continuously differentiable. The nth Fourier partial sum $S_n s$ reads as

$$(S_n s)(x) = \sum_{k=1}^{n} \frac{1}{\pi k} \sin(kx).$$

By the Theorem 31 of Dirichlet–Jordan, $(S_n s)(x)$ converges to $s(x)$ as $n \to \infty$ at each point $x \in \mathbb{R} \setminus \{2k\pi : \ k \in \mathbb{Z}\}$ such that

$$s(x) = \sum_{k=1}^{\infty} \frac{1}{\pi k} \sin(kx)\,.$$

Now we compute $S_n s$ in a neighborhood of the jump discontinuity at $x_0 = 0$. By Example 14 we have

$$\frac{1}{2} + \sum_{k=1}^{n} \cos(kx) = \frac{1}{2} D_n(t), \quad t \in \mathbb{R},$$

and hence by integration

$$\begin{aligned}\frac{x}{2\pi} + (S_n s)(x) &= \frac{1}{2\pi} \int_0^x D_n(t)\,\mathrm{d}t = \frac{1}{\pi} \int_0^{x/2} \frac{\sin((2n+1)t)}{t}\,\mathrm{d}t \\ &\quad + \frac{1}{\pi} \int_0^{x/2} h(t) \sin((2n+1)t)\,\mathrm{d}t, \end{aligned} \tag{43}$$

where the function

$$h(t) := \begin{cases} (\sin t)^{-1} - t^{-1} & t \in [-\pi, \pi] \backslash \{0\}, \\ 0 & t = 0 \end{cases}$$

is continuously differentiable in $(-\pi, \pi)$.

Integration by parts yields

$$\frac{1}{\pi} \int_0^{x/2} h(t) \sin((2n+1)t) \, \mathrm{d}t = \mathcal{O}(n^{-1}) \,, \quad n \to \infty \,.$$

Using the *sine integral*

$$\mathrm{Si}(y) := \int_0^y \frac{\sin t}{t} \mathrm{d}t \,, \quad y \in \mathbb{R} \,,$$

we obtain

$$(S_n s)(x) = \frac{1}{\pi} \mathrm{Si}\big((n+\frac{1}{2})x\big) - \frac{x}{2\pi} + \mathcal{O}(n^{-1}) \,, \quad n \to \infty \,. \tag{44}$$

Lemma 37

The sine integral has the property

$$\lim_{y\to\infty} \mathrm{Si}(y) = \int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t = \frac{\pi}{2}\,.$$

Further $\mathrm{Si}(\pi)$ *is the maximum value of the sine integral.*

Proof: Introducing

$$a_k := \int_{k\pi}^{(k+1)\pi} \frac{\sin t}{t}\,\mathrm{d}t\,, \quad k \in \mathbb{N}_0\,,$$

we see that $\operatorname{sgn} a_k = (-1)^k$, $|a_k| > |a_{k+1}|$ and $\lim_{k\to\infty}|a_k| = 0$. By the Leibniz criterion for alternating series we obtain that

$$\int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t = \sum_{k=0}^{\infty} a_k < \infty\,,$$

i.e., $\lim_{y\to\infty} \mathrm{Si}(y)$ exists.

From equation (43) with $x = \pi$ it follows that

$$\frac{\pi}{2} = \int_0^{\pi/2} \frac{\sin((2n+1)t)}{t}\,\mathrm{d}t + \int_0^{\pi/2} h(t)\,\sin((2n+1)t)\,\mathrm{d}t\,.$$

By the Lemma 24 of Riemann–Lebesgue we conclude for $k \to \infty$ that

$$\frac{\pi}{2} = \lim_{k\to\infty} \int_0^{\pi/2} \frac{\sin((2k+1)t)}{t}\,\mathrm{d}t = \lim_{k\to\infty} \int_0^{(k+\frac{1}{2})\pi} \frac{\sin x}{x}\,\mathrm{d}x\,.$$

Consequently,

$$\sum_{k=0}^{\infty} a_k = \frac{\pi}{2}\,, \quad \mathrm{Si}(n\pi) = \sum_{k=0}^{n-1} a_k\,, \quad n \in \mathbb{N}\,.$$

The function Si defined on $[0,\,\infty)$ is continuous, bounded and non-negative. Further Si increases monotonously on $[2k\pi,\,(2k+1)\pi]$ and decreases monotonously on $[(2k+1)\pi,\,(2k+2)\pi]$ for all $k \in \mathbb{N}_0$. Thus we have

$$\max\{\mathrm{Si}(y) :\, y \in [0,\,\infty)\} = \mathrm{Si}(\pi) \approx 1.8519\,. \quad \blacksquare$$

For $x = \frac{2\pi}{2n+1}$, we obtain by (44) and Lemma 37 that

$$(S_n s)\big(\frac{2\pi}{2n+1}\big) = \frac{1}{\pi}\,\mathrm{Si}(\pi) - \frac{1}{2n+1} + \mathcal{O}(n^{-1})\,, \quad n \to \infty\,,$$

where $\frac{1}{\pi}\,\mathrm{Si}(\pi)$ is the maximum value of $\frac{1}{\pi}\,\mathrm{Si}\big((n+\frac{1}{2})x\big)$ for all $x > 0$.
Ignoring the term $-\frac{1}{2n+1} + \mathcal{O}(n^{-1})$ for large n, we conclude that

$$\begin{aligned} &\lim_{n\to\infty} (S_n s)\big(\frac{2\pi}{2n+1}\big) = \frac{1}{\pi}\,\mathrm{Si}(\pi) \\ = \quad & s(0+0) + \big(\frac{1}{\pi}\mathrm{Si}(\pi) - \frac{1}{2}\big)\big(s(0+0) - s(0-0)\big), \end{aligned}$$

where $\frac{1}{\pi}\,\mathrm{Si}(\pi) - \frac{1}{2} \approx 0.08949$. Since the sawtooth function $s : \mathbb{T} \to \mathbb{C}$ is odd, we obtain that

$$\begin{aligned} &\lim_{n\to\infty} (S_n s)\big(-\frac{2\pi}{2n+1}\big) \\ = \quad & -\frac{1}{\pi}\,\mathrm{Si}(\pi) \\ = \quad & s(0-0) - \big(\frac{1}{\pi}\,\mathrm{Si}(\pi) - \frac{1}{2}\big)\,\big(s(0+0) - s(0-0)\big)\,. \end{aligned}$$

Thus for large n, we observe an overshooting and undershooting of $S_n s$ at both sides of the jump discontinuity of approximately 9% of the jump height $s(0+0) - s(0-0)$. This behavior does not change with growing n und is typical for the convergence of $S_n s$ near a jump discontinuity. Figure 9 illustrates this behavior.

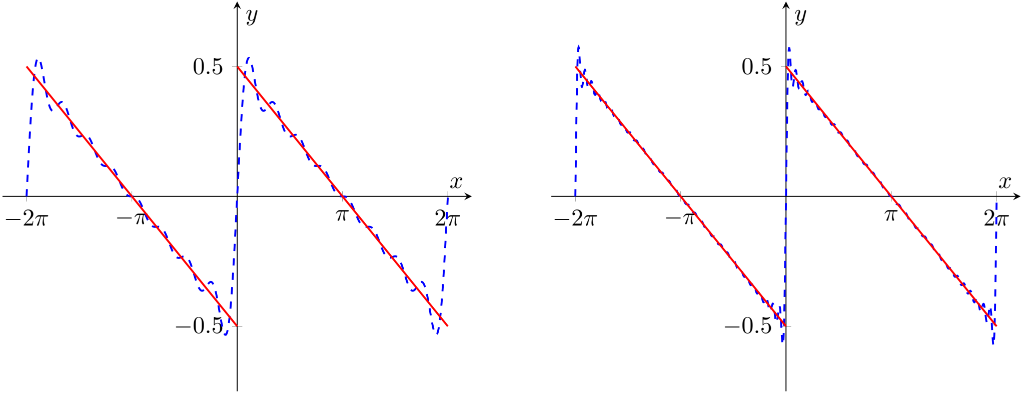


Figure 9: Gibbs phenomenon for the Fourier partial sums $S_8 s$ (blue, left) and $S_{16} s$ (blue,right), where s is the 2π-periodic sawtooth function (red).

A general description of the Gibbs phenomenon is given by the following

Theorem 38 (Gibbs phenomenon)

Let $f : \mathbb{T} \to \mathbb{C}$ be a piecewise continuously differentiable function with a jump discontinuity at $x_0 \in \mathbb{R}$. Assume that $f(x_0) = \frac{1}{2}\big(f(x_0 - 0) + f(x_0 + 0)\big)$. Then it holds

$$\begin{aligned} & \lim_{n\to\infty} (S_n f)(x_0 + \frac{2\pi}{2n+1}) \\ = \quad & f(x_0 + 0) + (\frac{1}{\pi}\,\mathrm{Si}(\pi) - \frac{1}{2})\,(f(x_0 + 0) - f(x_0 - 0)) \end{aligned}$$

$$\begin{aligned} & \lim_{n\to\infty} (S_n f)(x_0 - \frac{2\pi}{2n+1}) \\ = \quad & f(x_0 - 0) - (\frac{1}{\pi}\,\mathrm{Si}(\pi) - \frac{1}{2})\,(f(x_0 + 0) - f(x_0 - 0))\,. \end{aligned}$$

Proof: Let $s : \mathbb{T} \to \mathbb{C}$ denote the sawtooth function of Example 9. We consider the function

$$g := f - \big(f(x_0+0) - f(x_0-0)\big)\, s(\cdot - x_0)\,.$$

Then $g : \mathbb{T} \to \mathbb{C}$ is also piecewise continuously differentiable and continuous in an interval $[x_0 - \delta,\, x_0 + \delta]$ with $\delta > 0$. Further we have $g(x_0) = f(x_0) = \frac{1}{2}\,(f(x_0-0) + f(x_0+0))$. By the Theorem 31 of Dirichlet–Jordan, the Fourier series of g converges uniformly to g in $[x_0 - \delta,\, x_0 + \delta]$. By

$$(S_n f)(x) = (S_n g)(x) + \big(f(x_0+0) - f(x_0-0)\big) \sum_{k=1}^{n} \frac{1}{\pi k}\, \sin(k(x - x_0))$$

it follows for $x = x_0 \pm \frac{2\pi}{2n+1}$ and $n \to \infty$ that

$$\begin{aligned}
\lim_{n\to\infty} (S_n f)(x_0 + \frac{2\pi}{2n+1}) &= g(x_0) + \frac{1}{\pi}\, \mathrm{Si}(\pi)\, (f(x_0+0) - f(x_0-0)) \\
\lim_{n\to\infty} (S_n f)(x_0 - \frac{2\pi}{2n+1}) &= g(x_0) - \frac{1}{\pi}\, \mathrm{Si}(\pi)\, (f(x_0+0) - f(x_0-0))\,.
\end{aligned}$$

This completes the proof. ■

For large n, the Fourier partial sum $S_n f$ of a piecewise continuously differentiable function $f : \mathbb{T} \to \mathbb{C}$ exhibits the overshoot and undershoot at each point of discontinuity. If f is continuous at x_0, then $S_n f$ converges uniformly to f as $n \to \infty$ in a certain neighborhood of x_0 and the Gibbs phenomenon is absent.

Remark 39

Assume that $f : \mathbb{T} \to \mathbb{C}$ *is a piecewise continuously differentiable function. By the Gibbs phenomenon, the truncation of Fourier series to* $S_n f$ *causes ripples in a neighborhood of each point of jump discontinuity. These ripples can be removed by the use of properly weighted Fourier coefficients such as by Fejér summation or Lanczos smoothing.*
By the Fejér summation , we take the arithmetic mean $\sigma_n f$ *of all Fourier partial sums* $S_k f$, $k = 0, \ldots, n$, *i.e.*

$$\sigma_n f = \frac{1}{n+1} \sum_{k=0}^{n} S_k f \in \mathcal{T}_n \,.$$

Then $\sigma_n f$ *is the nth Fejér sum of* f.

With the Fejér kernel

$$F_n = \frac{1}{n+1} \sum_{k=0}^{n} D_k \in \mathcal{T}_n$$

of Example 15 and by $S_k f = f * D_k$, $k = 0, \ldots, n$, we obtain the representation $\sigma_n f = f * F_n$. Since

$$S_k f = \sum_{j=-k}^{k} c_j(f)\, \mathrm{e}^{\mathrm{i}j\cdot}\,,$$

then it follows that

$$\begin{aligned}\sigma_n f &= \frac{1}{n+1} \sum_{k=0}^{n} \sum_{j=-k}^{k} c_j(f)\, \mathrm{e}^{\mathrm{i}j\cdot} \\ &= \sum_{\ell=-n}^{n} \Big(1 - \frac{|\ell|}{n+1}\Big)\, c_\ell(f)\, \mathrm{e}^{\mathrm{i}\ell\cdot}\,.\end{aligned}$$

Note that the positive weights

$$\omega_\ell := 1 - \frac{|\ell|}{n+1}\,, \quad \ell = -n, \ldots, n$$

decay linearly from $\omega_0 = 1$ to $\omega_n = \omega_{-n} = (n+1)^{-1}$ as $|\ell|$ increases from 0 to n. In contrast to the Fejér summation, the *Lanczos smoothing* uses the means of the function $S_n f$ over the intervals $[x - \frac{\pi}{n}, x - \frac{\pi}{n}]$ for each $x \in \mathbb{T}$, i.e., we form

$$(\Lambda_n f)(x) := \frac{n}{2\pi} \int_{x-\pi/n}^{x+\pi/n} (S_n f)(u)\,\mathrm{d}u\,.$$

By

$$S_n f = \sum_{k=-n}^{n} c_k(f)\, \mathrm{e}^{\mathrm{i}k\cdot}\,,$$

we obtain the weighted Fourier partial sum

$$\begin{aligned}(\Lambda_n f)(x) &= \frac{n}{2\pi} \sum_{k=-n}^{n} c_k(f) \int_{x-\pi/n}^{x+\pi/n} \mathrm{e}^{\mathrm{i}ku}\, \mathrm{d}u \\ &= \sum_{k=-n}^{n} \Big(\operatorname{sinc} \frac{k\pi}{n}\Big)\, c_k(f)\, \mathrm{e}^{\mathrm{i}kx}\,,\end{aligned}$$

where the non-negative weights $\omega_k := \operatorname{sinc} \frac{k\pi}{n}$, $k = -n, \ldots, n$, decay from $\omega_0 = 1$ to $\omega_n = \omega_{-n} = 0$ as $|\ell|$ increases from 0 to n. If we arrange that $\omega_k := 0$ for all $k \in \mathbb{Z}$ with $|k| > n$, then we obtain a so-called window sequence which will be considered in the next section. $\square$

Fourier transform

Let $C_0(\mathbb{R})$ denote the Banach space of continuous functions vanishing as $|x| \to \infty$ with norm

$$\|f\|_{C_0(\mathbb{R})} := \max_{x \in \mathbb{R}} |f(x)|$$

and let $C_c(\mathbb{R})$ be the subspace of continuous functions with compact support. By $C^r(\mathbb{R})$, $r \in \mathbb{N}$, we denote the r-times continuously differentiable functions on $\mathbb{R}$. Accordingly $C_0^r(\mathbb{R})$ and $C_c^r(\mathbb{R})$ are defined.

For $1 \leq p \leq \infty$, let $L_p(\mathbb{R})$ denote the Banach space of all (equivalence classes of) measurable functions $f : \mathbb{R} \to \mathbb{C}$ with finite norm

$$\|f\|_{L_p(\mathbb{R})} := \begin{cases} \left(\int_{\mathbb{R}} |f(x)|^p \, \mathrm{d}x \right)^{1/p} & 1 \leq p < \infty\,, \\ \operatorname{ess\,sup}\{|f(x)| : x \in \mathbb{R}\} & p = \infty\,. \end{cases}$$

In particular, we are interested in the Hilbert space $L_2(\mathbb{R})$ with inner product and norm

$$\langle f, g \rangle_{L_2(\mathbb{R})} := \int\limits_{\mathbb{R}} f(x)\, \overline{g(x)} \, \mathrm{d}x\,, \quad \|f\|_{L_2(\mathbb{R})} := \Big(\int\limits_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x \Big)^{1/2}.$$

If it is clear from the context which inner product resp. norm is addressed, we abbreviate $\langle f, g \rangle := \langle f, g \rangle_{L_2(\mathbb{R})}$ and $\|f\| := \|f\|_{L_2(\mathbb{R})}$.

Note that in contrast to the periodic setting there is no continuous embedding of the $L_p(\mathbb{R})$ spaces. We neither have $L_1(\mathbb{R}) \subseteq L_2(\mathbb{R})$ nor $L_1(\mathbb{R}) \supseteq L_2(\mathbb{R})$. For example $f(x) := \frac{1}{x} 1_{[1,\infty)}(x)$, where $1_{[1,\infty)}$ denotes the *characteristic function* of the interval $[1,\infty)$, is in $L_2(\mathbb{R})$ but not in $L_1(\mathbb{R})$. On the other hand, $f(x) := \frac{1}{\sqrt{x}} 1_{(0,1]}(x)$ is in $L_1(\mathbb{R})$ but not in $L_2(\mathbb{R})$.

Remark 40

Note that a continuous function in $C_0(\mathbb{R})$ is uniformly continuous by the following reason: For an arbitrary fixed $\varepsilon > 0$ there exists $L = L(\varepsilon)$ such that $|f(x)| \le \varepsilon/3$ if $|x| \ge L$. If $x, y \in [-L, L]$, then there exists $\delta > 0$ such that $|f(x) - f(y)| \le \varepsilon/3$ whenever $|x - y| \le \delta$. If $x, y \in \mathbb{R}\backslash[-L, L]$, then $|f(x) - f(y)| \le |f(x)| + |f(y)| \le 2\varepsilon/3$. If $x \in [-L, L]$ and $y \in \mathbb{R}\backslash[-L, L]$, say $y > L$ with $|x - y| \le \delta$, then $|f(x) - f(y)| \le |f(x) - f(L)| + |f(L) - f(y)| \le \varepsilon$. In summary we have that $|f(x) - f(y)| \le \varepsilon$ whenever $|x - y| \le \delta$.

Fourier transform in $L_1(\mathbb{R})$

The (continuous) *Fourier transform* $\hat{f} = \mathcal{F}f$ of a function $f \in L_1(\mathbb{R})$ is defined by

$$\hat{f}(\omega) = (\mathcal{F}f)(\omega) := \int_{\mathbb{R}} f(x)\,\mathrm{e}^{-\mathrm{i}x\omega}\,\mathrm{d}x\,, \quad \omega \in \mathbb{R}\,. \tag{45}$$

Since $|f(x)\mathrm{e}^{-\mathrm{i}x\omega}| = |f(x)|$ and $f \in L_1(\mathbb{R})$, the integral (45) is well defined. In practice, the variable x denotes mostly the time or the space and the variable ω is the frequency. Therefore the domain of the Fourier transform is called *time domain* or *space domain*. The range of the Fourier transform is called *frequency domain*. Roughly spoken, the Fourier transform (45) measures how much oscillations around the frequency ω are contained $f \in L_1(\mathbb{R})$. The function $\hat{f} = |f|\,\mathrm{e}^{\mathrm{i}\arg f}$ is also called *spectrum* of f with *modulus* $|\hat{f}|$ and *phase* $\arg f$.

Remark 41

In the literature, the Fourier transform is not consistently defined. For instance, other frequently applied definitions are

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-\mathrm{i}\omega x}\, \mathrm{d}x\,, \quad \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-2\pi\mathrm{i}\omega x}\, \mathrm{d}x\,. \qquad \square$$

Example 42

Let $L > 0$. The *rectangle function*

$$f(x) := \begin{cases} 1 & x \in (-L,\ L)\,, \\ \frac{1}{2} & x \in \{-L,\ L\}\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

has the Fourier transform

$$\begin{aligned} \hat{f}(\omega) &= \int_{-L}^{L} \mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x = \frac{-\mathrm{e}^{-\mathrm{i}\omega L} + \mathrm{e}^{\mathrm{i}L\omega}}{\mathrm{i}\omega} = \frac{2\mathrm{i}L\sin(\omega L)}{\mathrm{i}L\omega} \\ &= \frac{2L\sin(L\omega)}{L\omega} = 2L\mathrm{sinc}(L\omega) \end{aligned}$$

with the *cardinal sine function* or sinc *function*

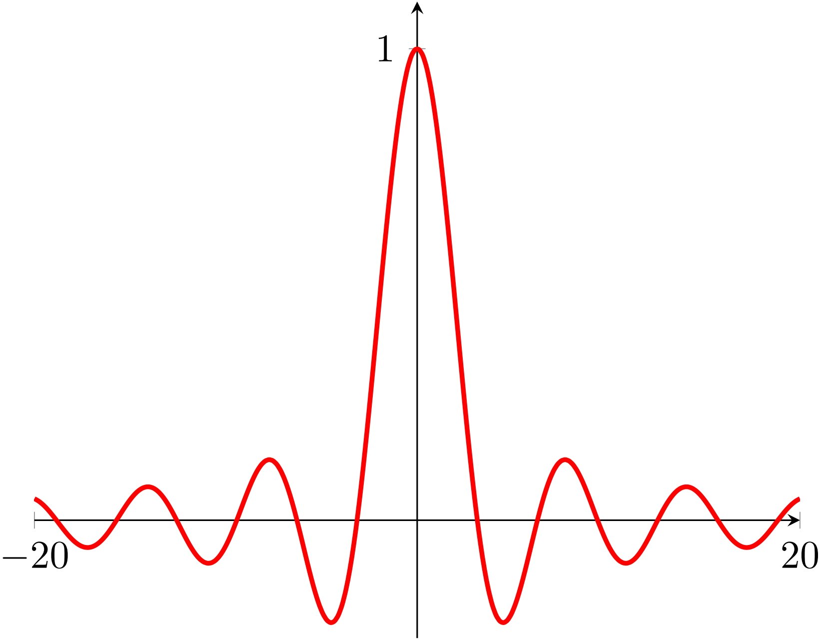
$$\mathrm{sinc}(x) = \begin{cases} \frac{\sin x}{x} & x \in \mathbb{R} \setminus \{0\}\,, \\ 1 & x = 0 \end{cases}$$

Example 42 (continue)

While $\operatorname{supp} f = [-L, L]$ is bounded, this is not the case for $\hat{f}$. Even worse, $\hat{f} \notin L_1(\mathbb{R})$, since

$$\begin{aligned}
\int_0^{n\pi} |\operatorname{sinc}(x)| \,\mathrm{d}x &= \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin(x)|}{|x|} \,\mathrm{d}x \\
&\geq \sum_{k=1}^{n} \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin(x)| \mathrm{d}x \\
&= \frac{2}{\pi} \sum_{k=1}^{n} \frac{1}{k}
\end{aligned}$$

and the last sum becomes infinitely large as $n \to \infty$. Thus the Fourier transform does not map $L_1(\mathbb{R})$ into itself. $\square$


1
−20
20

Example 43

For given $L > 0$, the *hat function*

$$f(x) := \begin{cases} 1 - \frac{|x|}{L} & x \in [-L,\, L]\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

has the Fourier transform

$$\begin{aligned} \hat{f}(\omega) &= 2 \int_0^L \left(1 - \frac{x}{L}\right) \cos(\omega x)\,\mathrm{d}x = \frac{2}{L\omega} \int_0^L \sin(\omega x)\,\mathrm{d}x \\ &= \frac{2}{L\omega^2} \big(1 - \cos(L\omega)\big) = L \left(\operatorname{sinc}\frac{L\omega}{2}\right)^2 \end{aligned}$$

for $\omega \in \mathbb{R} \setminus \{0\}$. In the case $\omega = 0$, we obtain

$$\hat{f}(0) = 2 \int_0^L \left(1 - \frac{x}{L}\right) \mathrm{d}x = L\,. \qquad \square$$

Theorem 44 (Properties of the Fourier transform)

Let $f \in L_1(\mathbb{R})$. Then the following properties holds true:

1. *Translation and modulation: For each $x_0, \omega_0 \in \mathbb{R}$,*

$$\big(f(\cdot - x_0)\big)\hat{}(\omega) = \mathrm{e}^{-\mathrm{i}x_0\omega}\,\hat{f}(\omega),$$
$$\big(\mathrm{e}^{-\mathrm{i}\omega_0\cdot} f\big)\hat{}(\omega) = \hat{f}(\omega_0 + \omega)\,.$$

2. *Differentiation and multiplication: For an absolutely continuous function $f \in L_1(\mathbb{R})$ with $f' \in L_1(\mathbb{R})$,*

$$(f')\hat{}(\omega) = \mathrm{i}\omega\,\hat{f}(\omega)\,.$$

If $g(x) := x\,f(x)$, $x \in \mathbb{R}$, is absolutely integrable, then

$$\hat{g}(\omega) = \mathrm{i}\,(\hat{f})'(\omega)\,.$$

3. *Scaling: For $\alpha \neq 0$,*

$$\big(f(\alpha\cdot)\big)\hat{}(\omega) = \frac{1}{|\alpha|}\hat{f}\left(\alpha^{-1}\omega\right).$$

Applying these properties we can calculate the Fourier transforms of some special functions.

We consider the *normalized Gaussian function*

$$f(x) := \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-\frac{x^2}{2\sigma^2}}, \quad x \in \mathbb{R}, \tag{46}$$

with standard deviation $\sigma > 0$. Note that $\int_{\mathbb{R}} f(x)\,\mathrm{d}x = 1$, since for $a > 0$ we obtain using polar coordinates r and φ that

$$\begin{aligned}
\Big(\int_{\mathbb{R}} \mathrm{e}^{-ax^2}\,\mathrm{d}x\Big)^2 &= \Big(\int_{\mathbb{R}} \mathrm{e}^{-ax^2}\,\mathrm{d}x\Big)\Big(\int_{\mathbb{R}} \mathrm{e}^{-ay^2}\,\mathrm{d}y\Big) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{-a(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^{2\pi} \Big(\int_0^{\infty} r\,\mathrm{e}^{-ar^2}\,\mathrm{d}r\Big)\,\mathrm{d}\varphi = \frac{\pi}{a}\,.
\end{aligned}$$

Now we compute the Fourier transform

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{x^2}{2\sigma^2}} \, \mathrm{e}^{-\mathrm{i}\omega x} \, \mathrm{d}x \,. \tag{47}$$

This integral can be calculated by Cauchy's integral theorem of complex function theory. Here we use another technique. Obviously, the Gaussian function (46) satisfies the differential equation

$$f'(x) + \frac{x}{\sigma^2} \, f(x) = 0 \,.$$

Applying Fourier transform to this differential equation, we obtain by the differentiation–multiplication property of Theorem 44

$$\mathrm{i}\omega \, \hat{f}(\omega) + \frac{\mathrm{i}}{\sigma^2} \, (\hat{f})'(\omega) = 0 \,.$$

This differential equation has the general solution

$$\hat{f}(\omega) = C\,\mathrm{e}^{-\frac{1}{2}\sigma^2\omega^2}\,,$$

with an arbitrary constant C. From (47) it follows that

$$\hat{f}(0) = C = \int_{\mathbb{R}} f(x)\,\mathrm{d}x = 1$$

and hence

$$\hat{f}(\omega) = \mathrm{e}^{-\frac{1}{2}\sigma^2\omega^2} \tag{48}$$

is a non-normalized Gaussian function with standard deviation $1/\sigma$. The smaller the standard deviation is in the space domain the larger it is in the frequency domain. In particular for $\sigma = 1$, the Gaussian function (46) coincides with its Fourier transform $\hat{f}$ up to the factor $1/\sqrt{2\pi}$. Note that the Gaussian function is the only function with this behavior. $\square$

Example 45

Let $a > 0$ and $b \in \mathbb{R} \setminus \{0\}$ be given. We consider the *Gaussian chirp*

$$f(x) := \mathrm{e}^{-(a-\mathrm{i}b)x^2}\,. \tag{49}$$

The Fourier transform of (49) reads as follows

$$\hat{f}(\omega) = \sqrt{\frac{\pi}{a-\mathrm{i}b}}\,\exp\,\frac{-(a+\mathrm{i}b)\omega^2}{4\,(a^2+b^2)}\,,$$

which can be calculated by a similar differential equation as above.

In Example 42 we have seen that the Fourier transformed function of an L_1 functions is not necessarily in L_1. By the following theorem it is a continuous function which vanishes at infinity.

Theorem 46

The Fourier transform $\mathcal{F}$ defined by (45) *is a linear, continuous operator from $L_1(\mathbb{R})$ into $C_0(\mathbb{R})$ with operator norm $\|\mathcal{F}\|_{L_1(\mathbb{R})\to C_0(\mathbb{R})} = 1$.*

More precisely $\mathcal{F}$ maps onto a dense subspace of $C_0(\mathbb{R})$.

Proof: The linearity of $\mathcal{F}$ follows from those of the integral operator. Let $f \in L_1(\mathbb{R})$. For any ω, $h \in \mathbb{R}$ we can estimate

$$|\hat{f}(\omega + h) - \hat{f}(\omega)| = \Big|\int_{\mathbb{R}} f(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,(\mathrm{e}^{-\mathrm{i}hx} - 1)\,\mathrm{d}x\Big| \leq \int_{\mathbb{R}} |f(x)|\,|\mathrm{e}^{-\mathrm{i}hx} - 1|\,\mathrm{d}x\,.$$

Since $|f(x)|\,|\mathrm{e}^{-\mathrm{i}hx} - 1| \leq 2\,|f(x)| \in L_1(\mathbb{R})$ and

$$|\mathrm{e}^{-\mathrm{i}hx} - 1| = \big((\cos(hx) - 1)^2 + (\sin(hx))^2\big)^{\frac{1}{2}} = \big(2 - 2\cos(hx)\big)^{\frac{1}{2}} \to 0$$

as $h \to 0$, we obtain by the convergence theorem of Lebesgue

$$\begin{aligned}\lim_{h\to 0} |\hat{f}(\omega + h) - \hat{f}(\omega)| &\leq \lim_{h\to 0} \int_{\mathbb{R}} |f(x)|\,\big|\mathrm{e}^{-\mathrm{i}hx} - 1\big|\,\mathrm{d}x \\ &= \int_{\mathbb{R}} |f(x)|\,\big(\lim_{h\to 0} |\mathrm{e}^{-\mathrm{i}hx} - 1|\big)\,\mathrm{d}x = 0\,.\end{aligned}$$

Hence $\hat{f}$ is continuous. Further, we know by Lemma 24 of Riemann – Lebesgue that $\lim_{|\omega|\to\infty} \hat{f}(\omega) = 0$. Thus $\hat{f} = \mathcal{F}f \in C_0(\mathbb{R})$.
For $f \in L_1(\mathbb{R})$ we have

$$|\hat{f}(\omega)| \leq \int_{\mathbb{R}} |f(x)| \,\mathrm{d}x = \|f\|_{L_1(\mathbb{R})} \,,$$

so that

$$\|\mathcal{F}f\|_{C_0(\mathbb{R})} = \|\hat{f}\|_{C_0(\mathbb{R})} \leq \|f\|_{L_1(\mathbb{R})}$$

and consequently $\|\mathcal{F}\|_{L_1(\mathbb{R})\to C_0(\mathbb{R})} \leq 1$. In particular we obtain for $g(x) := \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}$, $x \in \mathbb{R}$, that $\|g\|_{L_1(\mathbb{R})} = 1$ and $\hat{g}(\omega) = \mathrm{e}^{-\omega^2/2}$, $\omega \in \mathbb{R}$, see the Fourier transform of the Gaussian.
Hence $\|\hat{g}\|_{C_0(\mathbb{R})} = 1$ and $\|\mathcal{F}\|_{L_1(\mathbb{R})\to C_0(\mathbb{R})} = 1$. $\blacksquare$

Using the Theorem 46 we obtain following result.

Lemma 47

Let $f, g \in L_1(\mathbb{R})$. *Then we have* $\hat{f}g, \hat{g}f \in L_1(\mathbb{R})$ *and*

$$\int_{\mathbb{R}} f(x)\hat{g}(x)\,\mathrm{d}x = \int_{\mathbb{R}} \hat{f}(x)g(x)\,\mathrm{d}x\,. \tag{50}$$

Proof: By Theorem 46 we know that $\hat{g}$ is bounded so that $f\hat{g} \in L_1(\mathbb{R})$. Taking into account that $f(x)g(y)\mathrm{e}^{-\mathrm{i}xy} \in L_1(\mathbb{R}^2)$, equality (50) follows as a direct application of Fubini's theorem

$$\begin{aligned}
\int_{\mathbb{R}} f(x)\hat{g}(x)\,\mathrm{d}x &= \int_{\mathbb{R}} f(x) \int_{\mathbb{R}} g(y)\mathrm{e}^{-\mathrm{i}xy}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{\mathbb{R}} g(y) \int_{\mathbb{R}} f(x)\mathrm{e}^{-\mathrm{i}xy}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}} g(y)\hat{f}(y)\,\mathrm{d}y.
\end{aligned}$$

■

Next we examine under which assumptions on $f \in L_1(\mathbb{R})$ the Fourier inversion formula

$$f(x) = (\hat{f})\check{}(x) := \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\omega)\, \mathrm{e}^{\mathrm{i}\omega x}\, \mathrm{d}\omega \tag{51}$$

holds true. Note that this is the same formula as those for $\hat{f}$ in terms of f, except of the plus sign in the exponential and the factor $\frac{1}{2\pi}$.

Theorem 48 (Fourier inversion formula for $L_1(\mathbb{R})$ functions))

Let $f \in L_1(\mathbb{R})$ and $\hat{f} \in L_1(\mathbb{R})$. Then the Fourier inversion formula (51) holds true for almost every $x \in \mathbb{R}$. If f is in addition continuous, then the inversion formula is pointwise true for all $x \in \mathbb{R}$.

In the following we give the proof for $f \in L_1(\mathbb{R}) \cap C_0(\mathbb{R})$ with $\hat{f} \in L_1(\mathbb{R})$. For the general setting we refer, e.g. to [2, p. 38–44].

Proof: For any $n > 0$ we use the function $g_n(x) := \frac{1}{2\pi}\mathrm{e}^{-|x|/n}$ which has by straightforward computation the Fourier transform

$$\hat{g}_n(\omega) = \frac{n}{\pi(1 + n^2\omega^2)}\,.$$

Both functions g_n and $\hat{g}_n$ are in $L_1(\mathbb{R})$. By (50) and Theorem 44ii) we deduce for the functions f and $g_n\mathrm{e}^{\mathrm{i}xy}$ the relation

$$\int_{\mathbb{R}} \hat{f}(x)g_n(x)\mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}x = \int_{\mathbb{R}} f(\omega)\hat{g}_n(\omega - y)\;\mathrm{d}\omega.$$

We examine this equation as $n \to \infty$. We have $\lim_{n\to\infty} g_n(x) = \frac{1}{2\pi}$. For the left-hand side, since $|\hat{f}(x)g_n(x)\mathrm{e}^{\mathrm{i}xy}| \leq |\hat{f}(x)|$ and $\hat{f} \in L_1(\mathbb{R})$, we can pass to the limit under the integral

$$\lim_{n\to\infty}\int_{\mathbb{R}} \hat{f}(x)g_n(x)\mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}x = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(x)\mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}x = \check{\hat{f}}(y).$$

It remains to show that the limit on the right-hand side is $\lim_{n\to\infty}\int_{\mathbb{R}} f(\omega)\hat{g}_n(\omega - y)\,\mathrm{d}\omega = f(y)$. First we note that since $\hat{g}_n \in L_1(\mathbb{R})$ the relation

$$\int_{\mathbb{R}} \hat{g}_n(\omega)\ \mathrm{d}\omega = \lim_{L\to\infty}\int_{-L}^{L} \hat{g}_n(\omega)\ \mathrm{d}\omega = \frac{2}{\pi}\lim_{L\to\infty}\arctan(nL) = 1$$

holds true. Then we get

$$\begin{aligned}
\int_{\mathbb{R}} f(\omega)\hat{g}_n(\omega - y)\,\mathrm{d}\omega - f(y) &= \int_{\mathbb{R}} \left(f(\omega + y) - f(y)\right)\hat{g}_n(\omega)\,\mathrm{d}\omega \\
&= \int_{|\omega|\le\delta} \left(f(\omega + y) - f(y)\right)|\hat{g}_n(\omega)|\,\mathrm{d}\omega \\
&\quad + \int_{|\omega|>\delta} \left(f(\omega + y) - f(y)\right)|\hat{g}_n(\omega)|\,\mathrm{d}\omega.
\end{aligned}$$

By assumption $f \in L_1(\mathbb{R}) \cap C_0(\mathbb{R})$. Then f is also uniformly continuous, i.e., for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $|f(x) - f(y)| < \varepsilon$ if $|x - y| \leq \delta$. For all $n > 0$, we obtain

$$\int_{|\omega|\leq\delta} (f(\omega + y) - f(y))\, |\hat{g}_n(\omega)|\, \mathrm{d}\omega \leq \varepsilon \int_{|\omega|\leq\delta} |\hat{g}_n(\omega)|\, \mathrm{d}\omega \leq \varepsilon.$$

Next we see

$$|f(y) \int_{|\omega|>\delta} \hat{g}_n(\omega)\mathrm{d}\omega| \leq |f(y)| \left(1 - \frac{2}{\pi} \arctan(n\delta)\right), \tag{52}$$

and since $\hat{g}_n$ is decreasing on $\mathbb{R}_{\geq 0}$ further

$$|\int_{|\omega|>\delta} f(\omega + y)\hat{g}_n(\omega)\, \mathrm{d}\omega| \leq \hat{g}_n(\delta)\, \|f\|_{L_1(\mathbb{R})}. \tag{53}$$

As $n \to \infty$ the right-hand sides in (52) and (53) go to zero which finishes the proof.
■

As a corollary we obtain that the Fourier transform is one-to-one.

Corollary 49

For $f \in L_1(\mathbb{R})$ let $\hat{f} = 0$. Then $f = 0$ almost everywhere on $\mathbb{R}$.

We have seen that a 2π-periodic function can be reconstructed from its Fourier coefficients by the Fourier series in the $L_2(\mathbb{T})$ sense and that pointwise and uniform convergence requires additional assumptions on the function. Now we consider a corresponding problem and ask for the convergence of Cauchy principal value (of an improper integral)

$$\lim_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} \hat{f}(\omega)\mathrm{e}^{\mathrm{i}\omega x}\mathrm{d}\omega\,.$$

Note that for Lebesgue integrable functions f on $\mathbb{R}$ Cauchy's mean value coincides with the integral of f over $\mathbb{R}$.

Similar to Riemann's localization principle in Theorem 25 in the 2π-periodic setting we have the following result.

Theorem 50 (Riemann's localization principle)

Let $f \in L_1(\mathbb{R})$ and $x_0 \in \mathbb{R}$. Further let $\varphi(t) := f(x_0 + t) + f(x_0 - t) - 2f(x_0)$, $t \in \mathbb{R}$. Assume that for some $\delta > 0$

$$\int_0^\delta \frac{|\varphi(t)|}{t} \, \mathrm{d}t < \infty \, .$$

Then it holds

$$f(x_0) = \lim_{L \to \infty} \frac{1}{2\pi} \int_{-L}^{L} \hat{f}(\omega) \mathrm{e}^{\mathrm{i}\omega x_0} \mathrm{d}\omega.$$

Proof: It follows

$$I_L(x_0) := \frac{1}{2\pi} \int_{-L}^{L} \hat{f}(\omega)\, \mathrm{e}^{\mathrm{i}\omega x_0}\, \mathrm{d}\omega = \frac{1}{2\pi} \int_{-L}^{L} \int_{\mathbb{R}} f(u)\, \mathrm{e}^{-\mathrm{i}\omega u}\, \mathrm{d}u\, \mathrm{e}^{\mathrm{i}\omega x_0}\, \mathrm{d}\omega$$

$$= \frac{1}{2\pi} \int_{-L}^{L} \int_{\mathbb{R}} f(u)\, \mathrm{e}^{\mathrm{i}\omega(x_0-u)}\, \mathrm{d}u\, \mathrm{d}\omega.$$

Since $|f(u)\, \mathrm{e}^{\mathrm{i}\omega(x_0-u)}| = |f(u)|$ and $f \in L_1(\mathbb{R})$, we can change the order of integration in I_L by Fubini's theorem which results in

$$I_L(x_0) = \frac{1}{2\pi} \int_{\mathbb{R}} f(u) \int_{-L}^{L} \mathrm{e}^{\mathrm{i}\omega(x_0-u)}\, \mathrm{d}\omega\, \mathrm{d}u = \frac{1}{\pi} \int_{\mathbb{R}} f(u)\, \frac{\sin(L(x_0-u))}{x_0-u}\, \mathrm{d}u$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \big(f(x_0+t) + f(x_0-t)\big)\, \frac{\sin(Lt)}{t} \mathrm{d}t.$$

Since we have by Lemma 37 that

$$\int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t = \int_0^\infty \frac{\sin(Lt)}{t}\,\mathrm{d}t = \frac{\pi}{2}\,, \tag{54}$$

we conclude

$$\begin{aligned} I_L(x_0) - f(x_0) &= \frac{1}{\pi}\int_0^\infty \frac{\varphi(t)}{t}\,\sin(Lt)\,\mathrm{d}t \\ &= \frac{1}{\pi}\int_0^\delta \frac{\varphi(t)}{t}\,\sin(Lt)\,\mathrm{d}t \\ &\quad + \frac{1}{\pi}\int_\delta^\infty \frac{f(x_0+t)+f(x_0-t)}{t}\,\sin(Lt)\,\mathrm{d}t \\ &\quad - \frac{2}{\pi}\,f(x_0)\int_\delta^\infty \frac{\sin(Lt)}{t}\,\mathrm{d}t. \end{aligned}$$

Since $\varphi(t)/t \in L_1([0,\delta])$ by assumption, the first integral converges to zero as $L \to \infty$ by Lemma 24 of Riemann – Lebesgue. The same holds true for the second integral. Concerning the third integral we use

$$\frac{\pi}{2} = \int_0^\infty \frac{\sin(Lt)}{t}\,\mathrm{d}t = \int_0^\delta \frac{\sin(Lt)}{t}\,\mathrm{d}t + \int_\delta^\infty \frac{\sin(Lt)}{t}\,\mathrm{d}t$$
$$= \int_0^{L\delta} \frac{\sin(t)}{t}\,\mathrm{d}t + \int_\delta^\infty \frac{\sin(Lt)}{t}\,\mathrm{d}t.$$

Since the first summand converges to $\frac{\pi}{2}$ as $L \to \infty$, the third integral converges to zero as $L \to \infty$. This finishes the proof. ∎

A function $f : \mathbb{R} \to \mathbb{C}$ is called *piecewise continuously differentiable* on $\mathbb{R}$, if there exists a finite partition of $\mathbb{R}$ determined by $-\infty < x_0 < x_1 < \ldots < x_n < \infty$ of $\mathbb{R}$ such that f is continuously differentiable on each interval $(-\infty, x_0)$, (x_0, x_1), $\ldots$, (x_{n-1}, x_n), (x_n, ∞) and the one-sided limits $\lim_{x\to x_j\pm 0} f(x)$ and $\lim_{x\to x_j\pm 0} f'(x)$, $j = 0, \ldots, n$ exist. Similarly as in the proof of Theorem 31 of Dirichlet – Jordan the previous theorem can be used to prove that for a piecewise continuously differentiable function $f \in L_1(\mathbb{R})$ it holds

$$\frac{1}{2}(f(x_0+0) + f(x_0-0)) = \lim_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} \hat{f}(\omega) \mathrm{e}^{\mathrm{i}\omega x_0} \mathrm{d}\omega$$

for all $x_0 \in \mathbb{R}$.

The Fourier transform is again closely related to the convolution of functions. If $f : \mathbb{R} \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{C}$ are given functions, then their *convolution* $f * g$ is defined by

$$(f * g)(x) := \int_{\mathbb{R}} f(y)\, g(x-y)\, \mathrm{d}y\,, \quad x \in \mathbb{R}\,, \tag{55}$$

provided that this integral (55) exists. Note that the convolution is a commutative, associative, and distributive operation. Various conditions can be imposed on f and g to ensure that (55) exists. For instance, if f and g are both in $L_1(\mathbb{R})$, then $(f * g)(x)$ exists for almost every $x \in \mathbb{R}$ and further $f * g \in L_1(\mathbb{R})$. In the same way as for 2π-periodic functions we can prove the following theorem.

Theorem 51

1. *Let* $f \in L_p(\mathbb{R})$ *with* $1 \le p \le \infty$ *and* $g \in L_1(\mathbb{R})$ *be given. Then* $f * g$ *exists almost everywhere and* $f * g \in L_p(\mathbb{R})$. *Further we have the Young inequality*

$$\|f * g\|_{L_p(\mathbb{R})} \le \|f\|_{L_p(\mathbb{R})} \|g\|_{L_1(\mathbb{R})} .$$

2. *Let* $f \in L_p(\mathbb{R})$ *and* $g \in L_q(\mathbb{R})$, *where* $1 \le p, q \le \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then* $f * g$ *is a bounded, uniformly continuous function fulfilling*

$$\|f * g\|_{C(\mathbb{R})} \le \|f\|_{L_p(\mathbb{R})} \|g\|_{L_q(\mathbb{R})} .$$

   Furthermore, $\lim_{|x| \to \infty} (f * g)(x) = 0$ *if* $p \in (1, \infty)$.

3. *Let* $f \in L_p(\mathbb{R})$ *and* $g \in L_q(\mathbb{R})$, *where* $1 \le p,\ q,\ r \le \infty$ *and* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. *Then* $f * g \in L_r(\mathbb{R})$ *and we have the generalized Young inequality*

$$\|f * g\|_{L_r(\mathbb{R})} \le \|f\|_{L_p(\mathbb{R})} \|g\|_{L_q(\mathbb{R})} .$$

Differentiation and convolution are related by the following lemma.

Corollary 52

Let $f \in L_1(\mathbb{R})$ and $g \in C^r(\mathbb{R})$, where $g^{(k)}$ is bounded for $k = 0, \ldots, r$. Then $f * g \in C^r(\mathbb{R})$ and

$$(f * g)^{(k)} = f * g^{(k)}, \quad k = 1, \ldots, r.$$

Proof: Since $g^{(k)} \in L_\infty(\mathbb{R})$, the first assertion follows by the second part of Theorem 51. The function $x \mapsto f(y)g(x-y)$ is r-times differentiable, and for $k = 0, \ldots, r$ we have

$$|f(y)g^{(k)}(x-y)| \leq |f(y)| \sup_{t \in \mathbb{R}} |g^{(k)}(t)|.$$

Since $f \in L_1(\mathbb{R})$ we can differentiate under the integral sign, see [7, Proposition 14.2.2] which results in

$$(f * g)^{(k)}(x) = \int_{\mathbb{R}} f(y)g^{(k)}(x-y)\,\mathrm{d}y = f * g^{(k)}(x).$$

■

The relation between convolution and Fourier transform in the following theorem resemble their behavior in the periodic setting.

Theorem 53 (Convolution and Fourier transform)

Let $f, g \in L_1(\mathbb{R})$. Then we have

$$(f * g)\hat{}= \hat{f}\,\hat{g}\,.$$

Proof: For f, $g \in L_1(\mathbb{R})$ we have $f * g \in L_1(\mathbb{R})$ by Theorem 51. Using Fubini's theorem, we obtain for all $\omega \in \mathbb{R}$

$$\begin{aligned}
(f * g)\hat{}(\omega) &= \int_{\mathbb{R}} (f * g)(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f(y)\,g(x-y)\,\mathrm{d}y \Big)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\
&= \int_{\mathbb{R}} f(y) \Big(\int_{\mathbb{R}} g(x-y)\,\mathrm{e}^{-\mathrm{i}\omega (x-y)}\mathrm{d}x \Big)\,\mathrm{e}^{-\mathrm{i}\omega y}\,\mathrm{d}y \\
&= \int_{\mathbb{R}} f(y) \Big(\int_{\mathbb{R}} g(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t \Big)\,\mathrm{e}^{-\mathrm{i}\omega y}\,\mathrm{d}y = \hat{f}(\omega)\,\hat{g}(\omega)\,.
\end{aligned}$$

Applying these properties we can calculate the Fourier transforms of some special functions. Let $N_1 : \mathbb{R} \to \mathbb{R}$ denote the *cardinal B-spline of order* 1 defined by

$$N_1(x) := \begin{cases} 1 & x \in (0,\, 1)\,, \\ 1/2 & x \in \{0,\, 1\}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

For $m \in \mathbb{N}$, the convolution

$$N_{m+1}(x) := (N_m * N_1)(x) = \int_0^1 N_m(x - t)\,\mathrm{d}t\,,$$

is the *cardinal B-spline of order* $m + 1$.

Especially, for $m = 1$ we obtain the linear cardinal B-spline

$$N_2(x) := \begin{cases} x & x \in [0,\, 1)\,, \\ 2 - x & x \in [1,\, 2)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Note that the support of N_m is the interval $[0,\, m]$. By

$$\hat{N}_1(\omega) = \int_0^1 \mathrm{e}^{-\mathrm{i}x\omega}\,\mathrm{d}x = \frac{1 - \mathrm{e}^{-\mathrm{i}\omega}}{\mathrm{i}\omega}\,,$$

and $\hat{N}_1(0) = 1$, we obtain

$$\hat{N}_1(\omega) = \mathrm{e}^{-\mathrm{i}\omega/2}\ \mathrm{sinc}\,\frac{\omega}{2}\,, \quad \omega \in \mathbb{R}\,.$$

By the convolution property of Theorem 53, we obtain

$$\hat{N}_{m+1}(\omega) = \hat{N}_m(\omega)\,\hat{N}_1(\omega) = \big(\hat{N}_1(\omega)\big)^{m+1}\,.$$

Hence the Fourier transform of the cardinal B-spline N_m reads as follows

$$\hat{N}_m(\omega) = \mathrm{e}^{-\mathrm{i}m\omega/2}\ \big(\mathrm{sinc}\,\frac{\omega}{2}\big)^m\,.$$

For the *centered cardinal B-spline of order* $m \in \mathbb{N}$ defined by

$$M_m(x) := N_m(x + \frac{m}{2}),$$

we obtain by the translation property of Theorem 44 that

$$\hat{M}_m(\omega) = \left(\operatorname{sinc} \frac{\omega}{2}\right)^m. \qquad \square$$

The space $L_1(\mathbb{R})$ with the addition and convolution of functions is a Banach algebra. As for periodic functions there is no identity element with respect to the convolution. A remedy is again to work with approximate identities.

Theorem 54 (approximate identity)

Let $\varphi \in L_1(\mathbb{R})$ *with* $\int\limits_{\mathbb{R}} \varphi(x)\,\mathrm{d}x = 1$ *and*

$$\varphi_\varepsilon(x) := \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right).$$

Then the following relations hold true:

i) *For* $f \in L_p(\mathbb{R})$, $1 \leq p < \infty$, *we have*

$$\lim_{\varepsilon \to 0} \|f * \varphi_\varepsilon - f\|_{L_p(\mathbb{R})} = 0.$$

ii) *For a continuous function* f *with compact support, the sequence* $f * \varphi_\varepsilon$ *converges uniformly on* $\operatorname{supp} f$ *to* f *as* $\varepsilon \to 0$.

Proof: i) With $\int\limits_{\mathbb{R}} \varphi_\varepsilon(x)\,\mathrm{d}x = 1$ we obtain

$$\begin{aligned}
\|f * \varphi_\varepsilon - f\|_{L_p(\mathbb{R})}^p &= \int\limits_{\mathbb{R}} |\int\limits_{\mathbb{R}} f(x-y)\,\varphi_\varepsilon(y)\,\mathrm{d}y - f(x)|^p\,\mathrm{d}x \\
&= \int\limits_{\mathbb{R}} |\int\limits_{\mathbb{R}} (f(x-y) - f(x))\,\varphi_\varepsilon(y)\,\mathrm{d}y|^p\,\mathrm{d}x \\
&= \int\limits_{\mathbb{R}} |\int\limits_{\mathbb{R}} (f(x-\varepsilon\,y) - f(x))\,\varphi(y)\,\mathrm{d}y|^p\,\mathrm{d}x.
\end{aligned}$$

By Hölder's inequality we further conclude with $\varphi = \varphi^{\frac{1}{p}}\,\varphi^{\frac{1}{q}}$, $\frac{1}{p} + \frac{1}{q} = 1$ that

$$\begin{aligned}
\|f * \varphi_\varepsilon - f\|_{L_p(\mathbb{R})}^p \quad \leq \quad & \int\limits_{\mathbb{R}} \Big(\int\limits_{\mathbb{R}} |f(x-\varepsilon\,y) - f(x)|^p\,|\varphi(y)|\,\mathrm{d}y\Big) \\
& \Big(\int\limits_{\mathbb{R}} |\varphi(y)|\,\mathrm{d}y\Big)^{p/q}\,\mathrm{d}x.
\end{aligned}$$

Applying the Theorem of Fubini we get

$$\|f * \varphi_\varepsilon - f\|_{L_p(\mathbb{R})}^p \le \|\varphi\|_{L_1(\mathbb{R})}^{p/q} \int_{\mathbb{R}} |\varphi(y)| \int_{\mathbb{R}} |f(x - \varepsilon\, y) - f(x)|^p \,\mathrm{d}x\,\mathrm{d}y. \tag{56}$$

Now

$$\int_{\mathbb{R}} |f(x - \varepsilon\, y) - f(x)|^p \,\mathrm{d}x = \|f(\cdot - \varepsilon\, y) - f\|_{L_p(\mathbb{R})}^p \le 2^p \|f\|_{L_p(\mathbb{R})}^p$$

and $\varphi \in L_1(\mathbb{R})$ so that the sequence in (56) has an integrable upper bound. By Lebesgue's theorem and continuity of the norm we obtain

$$\lim_{\varepsilon \to 0} \|f * \varphi_\varepsilon - f\|_{L_p(\mathbb{R})} \le \|\varphi\|_{L_1(\mathbb{R})}^{p/q} \int_{\mathbb{R}} |\varphi(y)| \lim_{\varepsilon \to 0} \|f(\cdot - \varepsilon\, y) - f\|_{L_p(\mathbb{R})}^p \,\mathrm{d}y = 0.$$

ii) We have

$$
\begin{aligned}
|f * \varphi_\varepsilon(x) - f(x)| &= |\int_{\mathbb{R}} (f(x-y) - f(x))\varphi_\varepsilon(y)\,\mathrm{d}y| \\
&\leq \int_{\mathbb{R}} |(f(x - \varepsilon y) - f(x))|\,|\varphi(y)|\,\mathrm{d}y.
\end{aligned}
$$

Since $\varphi \in L_1(\mathbb{R})$, there exists a compact set W such that $\int_{\mathbb{R}\setminus W} |\varphi(y)|\,\mathrm{d}y < \delta$. Then we get

$$
\sup_{x \in \operatorname{supp} f} |f * \varphi_\varepsilon(x) - f(x)| \leq \sup_{\substack{x \in \operatorname{supp} f, (x - \varepsilon\tilde{y}) \in \operatorname{supp} f \\ \tilde{y} \in W}} |f(x - \varepsilon\tilde{y}) - f(x)| \int_W |\varphi(y)|\,\mathrm{d}y + 2\|f\|_{L_\infty}\delta.
$$

■

Another result on approximate identities is the following lemma.

Lemma 55

Let $f \in L_1(\mathbb{R})$ *and*

$$g_\sigma(x) := \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-x^2/2\sigma^2}.$$

Then it holds in each point x *where* f *is continuous*

$$\lim_{\sigma\to 0+} (f * g_\sigma)(x) = f(x).$$

Proof: Let f be continuous in x. Then, for any $\varepsilon > 0$, there exists $h > 0$ such that for all $|t| \le h$,

$$|f(x-t) - f(x)| < \varepsilon.$$

Since $\int\limits_{\mathbb{R}} g_\sigma(t)\,\mathrm{d}t = 1$ we get

$$(f * g_\sigma)(x) - f(x) = \int\limits_{-\infty}^{\infty} (f(x-t) - f(x)) g_\sigma(t)\,\mathrm{d}t$$

and consequently

$$\begin{aligned}
|(f * g_\sigma)(x) - f(x)| \leq & \int\limits_{|t|\leq h} |f(x-t) - f(x)| g_\sigma(t) \mathrm{d}t \\
& + \int\limits_{|t|\geq h} (|f(x-t)| + |f(x)|) g_\sigma(t) \mathrm{d}t \\
& \leq \varepsilon \int\limits_{-h}^{h} g_\sigma(t) \mathrm{d}t + \|f\|_{L_1} g_\sigma(h) + |f(x)| \int\limits_{|t|\geq h} g_\sigma(t) \mathrm{d}t.
\end{aligned}$$

The first summand is smaller than ε and the other two summands go to zero as $\sigma \to 0$. ■

Fourier transform in $L_2(\mathbb{R})$

Up to now we have considered the Fourier transform of functions in $L_1(\mathbb{R})$. Next we want to establish a Fourier transform in the Hilbert space $L_2(\mathbb{R})$, where the Fourier integral

$$\int_{\mathbb{R}} f(x)\, \mathrm{e}^{-\mathrm{i}x\omega}\, \mathrm{d}x$$

may not exist, i.e., it does not take a finite value for some $\omega \in \mathbb{R}$. Therefore we define the Fourier transform of an $L_2(\mathbb{R})$ function in a different way based on the following lemma.

Lemma 56

Let f, $g \in L_1(\mathbb{R})$, such that $\hat{f}$, $\hat{g} \in L_1(\mathbb{R})$. Then the following Parseval equality is valid

$$2\pi \langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle .$$

Note that $f, \hat{f} \in L_1(\mathbb{R})$ implies that $(\hat{f})^{\check{}} = f$ almost everywhere and $(\hat{f})^{\check{}} \in C_0(\mathbb{R})$. Thus,

$$\int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x = \int_{\mathbb{R}} |(\hat{f})^{\check{}}(x)| \, |f(x)| \, \mathrm{d}x \leq \|(\hat{f})^{\check{}}\|_{C_0(\mathbb{R})} \, \|f\|_{L_1(\mathbb{R})} .$$

Proof: Using Fubini's theorem and Fourier's integral formula we obtain

$$\begin{aligned}\int_{\mathbb{R}} \hat{f}(\omega)\,\overline{\hat{g}(\omega)}\,\mathrm{d}\omega &= \int_{\mathbb{R}} \hat{f}(\omega)\,\overline{\int_{\mathbb{R}} g(x)\,\mathrm{e}^{-\mathrm{i}x\omega}\,\mathrm{d}x}\,\mathrm{d}\omega \\ &= \int_{\mathbb{R}} \overline{g(x)} \int_{\mathbb{R}} \hat{f}(\omega)\,\mathrm{e}^{\mathrm{i}x\omega}\,\mathrm{d}\omega\;\mathrm{d}x \\ &= 2\pi \int_{\mathbb{R}} \overline{g(x)}\,f(x)\,\mathrm{d}x\,. \qquad \blacksquare\end{aligned}$$

For any function $f \in L_2(\mathbb{R})$ there exists a sequence $\{f_j\}_{j\in\mathbb{N}}$ of functions in $C_c^1(\mathbb{R})$ such that

$$\lim_{j\to\infty} \|f - f_j\|_{L_2(\mathbb{R})} = 0\,.$$

Thus $\{f_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $L_2(\mathbb{R})$, i.e., for every $\varepsilon > 0$ there exists an index $N(\varepsilon) \in \mathbb{N}$ so that for all $j,\, k \geq N(\varepsilon)$

$$\|f_k - f_j\|_{L_2(\mathbb{R})} < \varepsilon\,.$$

Clearly, $f_j,\, \hat{f}_j \in L_1(\mathbb{R})$. By Parseval's equality we obtain for all $j,\, k \geq N(\varepsilon)$

$$\|f_k - f_j\|_{L_2(\mathbb{R})} = \frac{1}{\sqrt{2\pi}}\, \|\hat{f}_k - \hat{f}_j\|_{L_2(\mathbb{R})} < \varepsilon\,,$$

so that $\{\hat{f}_j\}_{j\in\mathbb{N}}$ is also a Cauchy sequence in $L_2(\mathbb{R})$. Since $L_2(\mathbb{R})$ is complete, this Cauchy sequence converges to some function in $L_2(\mathbb{R})$.

We define the *Fourier transform* $\hat{f} = \mathcal{F}f \in L_2(\mathbb{R})$ of $f \in L_2(\mathbb{R})$ as

$$\hat{f} = \mathcal{F}f := \lim_{j \to \infty} \hat{f}_j.$$

In this way the domain of the Fourier transform is extended to include all of $L_2(\mathbb{R})$. By the continuity of the inner product we obtain also the Parseval equality in $L_2(\mathbb{R})$. We summarize:

Theorem 57 (Plancherel)

The Fourier transform truncated on $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ can be uniquely extended to a bounded linear operator of $L_2(\mathbb{R})$ onto itself which satisfies the Parseval equality

$$2\pi \langle f, g \rangle_{L_2(\mathbb{R})} = \langle \hat{f}, \hat{g} \rangle, \quad \sqrt{2\pi}\, \|f\|_{L_2(\mathbb{R})} = \|\hat{f}\|_{L_2(\mathbb{R})}$$

for all $f, g \in L_2(\mathbb{R})$.

Note that Theorem 44 is also true for $L_2(\mathbb{R})$ functions. Moreover, we have the following inversion formula.

Theorem 58 (Fourier inversion formula for $L_2(\mathbb{R})$ functions)

Let $f \in L_2(\mathbb{R})$ and $\hat{f} \in L_1(\mathbb{R})$. Then the Fourier inversion formula

$$f(x) = \frac{1}{2\pi} \int\limits_{\mathbb{R}} \hat{f}(\omega)\, \mathrm{e}^{\mathrm{i}\omega x}\, \mathrm{d}\omega \tag{57}$$

holds true for almost every $x \in \mathbb{R}$. If f is in addition continuous, then the inversion formula holds pointwise for all $x \in \mathbb{R}$.

Remark 59

Often the integral notation

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-\mathrm{i}x\omega}\, \mathrm{d}x$$

is also used for the Fourier transform of L_2 functions although the integral may not converge pointwise. But it may be interpreted by a limiting process. For $\varepsilon > 0$ and $f \in L_2(\mathbb{R})$, the function $g_\varepsilon : \mathbb{R} \to \mathbb{C}$ is defined by

$$g_\varepsilon(\omega) := \int_{\mathbb{R}} \mathrm{e}^{-\varepsilon^2 x^2} f(x)\, \mathrm{e}^{-\mathrm{i}x\omega}\, \mathrm{d}x\,, \quad \omega \in \mathbb{R}\,.$$

Then g_ε converges in the $L_2(\mathbb{R})$ norm and pointwise almost everywhere to $\hat{f}$ for $\varepsilon \to 0$.
$\square$

Finally we introduce an orthogonal basis of $L_2(\mathbb{R})$ which elements are eigenfunctions of the Fourier operator. For $n \in \mathbb{N}_0$, the n-th *Hermite polynomial* H_n is defined by

$$H_n(x) := (-1)^n \,\mathrm{e}^{x^2} \, \frac{\mathrm{d}^n}{\mathrm{d}x^n} \mathrm{e}^{-x^2}\,, \quad x \in \mathbb{R}\,.$$

In particular we have

$$H_0(x) = 1\,, \quad H_1(x) = 2x\,, \quad H_2(x) = 4x^2 - 2\,, \quad H_3(x) = 8x^3 - 12x\,.$$

The Hermite polynomials fulfill the three term relation

$$H_{n+1}(x) = 2x\, H_n(x) - 2n\, H_{n-1}(x)\,, \tag{58}$$

and the recursion

$$H_n'(x) = 2n\, H_{n-1}(x)\,. \tag{59}$$

For $n \in \mathbb{N}_0$, the *n-th Hermite function* h_n is given by

$$h_n(x) := H_n(x)\,\mathrm{e}^{-x^2/2} = (-1)^n\,\mathrm{e}^{x^2/2}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{e}^{-x^2}, \quad x \in \mathbb{R}\,.$$

In particular, we have $h_0(x) = \mathrm{e}^{-x^2/2}$ which has the Fourier transform $\hat{h}_0(\omega) = \sqrt{2\pi}\,\mathrm{e}^{-\omega^2/2}$. The Hermite functions fulfill the differential equation

$$h_n''(x) - (x^2 - 2n - 1)\,h_n(x) = 0 \tag{60}$$

and can be computed recursively by

$$h_{n+1}(x) = x\,h_n(x) - h_n'(x)\,.$$

Theorem 60

The Hermite functions h_n, $n \in \mathbb{N}_0$, with

$$\langle h_n, h_n \rangle = \sqrt{\pi}\, 2^n n!$$

form a complete orthogonal system in $L_2(\mathbb{R})$. The Fourier transforms of the Hermite functions are given by

$$\hat{h}_n(\omega) = \sqrt{2\pi}\, (-\mathrm{i})^n\, h_n(\omega)\,, \quad \omega \in \mathbb{R}\,. \tag{61}$$

In other words, the functions h_n are the eigenfunctions of the Fourier operator $\mathcal{F} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ with eigenvalues $\sqrt{2\pi}\, (-\mathrm{i})^n$ for all $n \in \mathbb{N}_0$.

By the Theorem 60 we see that the Hermite polynomials are orthogonal polynomials in the weighted space $L_{2,w}(\mathbb{R})$ with $w(x) := \mathrm{e}^{-x^2}$, $x \in \mathbb{R}$, i.e., they are orthogonal with respect to the weighted Lebesgue measure $\mathrm{e}^{-x^2}\,\mathrm{d}x$.
Proof: 1. We show that $\langle h_m, h_n \rangle = 0$ for $m \neq n$. By the differential equation (60) we obtain

$$(h_m'' - x^2 h_m)\, h_n = -(2m+1)\, h_m\, h_n\,,$$
$$(h_n'' - x^2 h_n)\, h_m = -(2n+1)\, h_m\, h_n\,.$$

Subtraction yields

$$h_m''\, h_n - h_n''\, h_m = (h_m' h_n - h_n' h_m)' = 2(n-m)\, h_m\, h_n\,,$$

which results after integration in

$$\begin{aligned} 2(n-m)\,\langle h_m, h_n \rangle &= 2(m-n) \int_{\mathbb{R}} h_m(x)\, h_n(x)\,\mathrm{d}x \\ &= \big(h_m'(x)\, h_n(x) - h_n'(x)\, h_m(x)\big)\big|_{-\infty}^{\infty} = 0\,. \end{aligned}$$

2. Next we prove for $n \in \mathbb{N}_0$ that

$$\langle h_n, h_n \rangle = \sqrt{\pi}\, 2^n n!\,. \tag{62}$$

For $n = 0$ the relation holds true by (48). We show the recursion

$$\langle h_{n+1}, h_{n+1} \rangle = 2(n+1)\, \langle h_n, h_n \rangle \tag{63}$$

which implies (62). Using (59), integration by parts, and step 1 of this proof, we obtain

$$\begin{aligned}
\langle h_{n+1}, h_{n+1} \rangle &= \int_{\mathbb{R}} \mathrm{e}^{-x^2}\, (H_{n+1}(x))^2\, \mathrm{d}x \\
&= \int_{\mathbb{R}} \big(2x\, \mathrm{e}^{-x^2}\big)\, \big(H_n(x)\, H_{n+1}(x)\big)\, \mathrm{d}x \\
&= \int_{\mathbb{R}} \mathrm{e}^{-x^2}\, \big(H_n'(x)\, H_{n+1}(x) + H_n(x)\, H_{n+1}'(x)\big)\, \mathrm{d}x \\
&= 2(n+1)\, \int_{\mathbb{R}} \mathrm{e}^{-x^2}\, \big(H_n(x)\big)^2\, \mathrm{d}x = 2(n+1)\, \langle h_n, h_n \rangle\,.
\end{aligned}$$

3. To verify the completeness of the orthogonal system $\{h_n : n \in \mathbb{N}_0\}$ we prove that $f \in L_2(\mathbb{R})$ with $\langle f, h_n \rangle = 0$ for all $n \in \mathbb{N}_0$ implies $f = 0$ almost everywhere. To this end, we consider the complex function $g \colon \mathbb{C} \to \mathbb{C}$ defined by

$$g(z) := \int_{\mathbb{R}} h_0(x)\, f(x)\, \mathrm{e}^{-\mathrm{i}xz}\, \mathrm{d}x\,.$$

This is the holomorphic continuation of the Fourier transform of $h_0\, f$ onto whole $\mathbb{C}$. For every $m \in \mathbb{N}_0$ it holds

$$g^{(m)}(z) = (-\mathrm{i})^m \int_{\mathbb{R}} x^m\, h_0(x)\, f(x)\, \mathrm{e}^{-\mathrm{i}xz}\, \mathrm{d}x\,, \quad z \in \mathbb{C}\,.$$

Since $g^{(m)}(0)$ is a certain linear combination of $\langle f, h_n \rangle$, $n = 0, \ldots, m$, we conclude that $g^{(m)}(0) = 0$ for all $m \in \mathbb{N}_0$. Thus, $g = 0$ and $(h_0 f)^{\check{}} = 0$. By Corollary 49 we have $h_0 f = 0$ almost everywhere and consequently $f = 0$ almost everywhere.

4. By the Fourier transform of the Gaussian we know that

$$\hat{h}_0(\omega) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}x\omega - x^2/2}\,\mathrm{d}x = \sqrt{2\pi}\,\mathrm{e}^{-\omega^2/2}\,, \quad \omega \in \mathbb{R}\,.$$

We compute the Fourier transform of h_n and obtain after n times integration by parts

$$\begin{aligned}
\hat{h}_n(\omega) &= \int_{\mathbb{R}} h_n(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\
&= (-1)^n \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\omega x + x^2/2} \left(\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,\mathrm{e}^{-x^2}\right)\mathrm{d}x \\
&= \int_{\mathbb{R}} \mathrm{e}^{-x^2} \left(\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,\mathrm{e}^{-\mathrm{i}\omega x + x^2/2}\right)\mathrm{d}x \\
&= \mathrm{e}^{\omega^2/2} \int_{\mathbb{R}} \mathrm{e}^{-x^2} \left(\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,\mathrm{e}^{(x-\mathrm{i}\omega)^2/2}\right)\mathrm{d}x\,.
\end{aligned}$$

By symmetry reasons we have

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,\mathrm{e}^{(x-\mathrm{i}\omega)^2/2} = \mathrm{i}^n\,\frac{\mathrm{d}^n}{\mathrm{d}\omega^n}\,\mathrm{e}^{(x-\mathrm{i}\omega)^2/2}\,,$$

so that

$$\begin{aligned}\hat{h}_n(\omega) &= \mathrm{i}^n\,\mathrm{e}^{\omega^2/2}\int_{\mathbb{R}}\mathrm{e}^{-x^2}\left(\frac{\mathrm{d}^n}{\mathrm{d}\omega^n}\,\mathrm{e}^{(x-\mathrm{i}\omega)^2/2}\right)\mathrm{d}x\\ &= \mathrm{i}^n\,\mathrm{e}^{\omega^2/2}\,\frac{\mathrm{d}^n}{\mathrm{d}\omega^n}\left(\mathrm{e}^{-\omega^2/2}\int_{\mathbb{R}}\mathrm{e}^{-\mathrm{i}x\omega-x^2/2}\,\mathrm{d}x\right)\\ &= \sqrt{2\pi}\,\mathrm{i}^n\,\mathrm{e}^{\omega^2/2}\,\frac{\mathrm{d}^n}{\mathrm{d}\omega^n}\,\mathrm{e}^{-\omega^2} = \sqrt{2\pi}\,(-\mathrm{i})^n\,h_n(\omega)\,.\end{aligned}$$

This completes the proof. ■

Poisson's summation formula and Shannon's sampling theorem

Poisson's summation formula establishes an interesting relation between Fourier series and Fourier transforms. For $n \in \mathbb{N}$ and $f \in L_1(\mathbb{R})$ we consider the functions

$$\varphi_n(x) := \sum_{k=-n}^{n} |f(x + 2k\pi)|$$

which fulfill

$$\begin{aligned}\int_{-\pi}^{\pi} \varphi_n(x)\,\mathrm{d}x &= \int_{-\pi}^{\pi} \sum_{k=-n}^{n} |f(x+2k\pi)|\,\mathrm{d}x = \sum_{k=-n}^{n} \int_{-\pi}^{\pi} |f(x+2k\pi)|\,\mathrm{d}x \\ &= \sum_{k=-n}^{n} \int_{2k\pi-\pi}^{2k\pi+\pi} |f(x)|\,\mathrm{d}x = \int_{-2n\pi-\pi}^{2n\pi+\pi} |f(x)|\,\mathrm{d}x\,.\end{aligned}$$

Since $\{\varphi_n\}_{n\in\mathbb{N}}$ is a monotone increasing sequence of nonnegative functions, we obtain by the monotone convergence theorem of B. Levi that the function $\varphi(x) := \lim_{n\to\infty} \varphi_n(x)$, $x \in \mathbb{R}$, is measurable and fulfills

$$\int_{-\pi}^{\pi} \varphi(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_{-\pi}^{\pi} \varphi_n(x)\,\mathrm{d}x = \|f\|_{L_1(\mathbb{R})}\,.$$

We introduce the 2π-periodic function

$$\tilde{f}(x) := \sum_{k\in\mathbb{Z}} f(x+2k\pi)\,. \tag{64}$$

The 2π–periodic function $\tilde{f}$ is called 2π–*periodization* of f. Since

$$|\tilde{f}(x)| = \Big|\sum_{k\in\mathbb{Z}} f(x+2k\pi)\Big| \le \sum_{k\in\mathbb{Z}} |f(x+2k\pi)| = \varphi(x)\,,$$

we obtain

$$\int_{\mathbb{T}} |\tilde{f}(x)|\,\mathrm{d}x \le \int_{\mathbb{T}} |\varphi(x)|\,\mathrm{d}x = \|f\|_{L_1(\mathbb{R})}$$

so that $\tilde{f} \in L_1(\mathbb{T})$. After these preparations we can formulate the Poisson summation formula.

Theorem 61 (Poisson summation formula)

Assume that $f \in L_1(\mathbb{R}) \cap C_0(\mathbb{R})$ *fulfills the conditions*

1. $\sum_{k\in\mathbb{Z}} \max\limits_{x\in[-\pi,\pi]} |f(x+2k\pi)| < \infty$,
2. $\sum_{k\in\mathbb{Z}} |\hat{f}(k)| < \infty$.

Then for all $x \in \mathbb{R}$, *the following relation is fulfilled*

$$2\pi\,\tilde{f}(x) = 2\pi \sum_{k\in\mathbb{Z}} f(x+2k\pi) = \sum_{k\in\mathbb{Z}} \hat{f}(k)\,\mathrm{e}^{\mathrm{i}kx} .$$

For $x = 0$ *this implies the Poisson summation formula*

$$2\pi \sum_{k\in\mathbb{Z}} f(2k\pi) = \sum_{k\in\mathbb{Z}} \hat{f}(k) . \tag{65}$$

Proof: By the first assumption the convergence of the series defining $\tilde{f}$ is uniformly by the known criterium of Weierstrass. Since f is continuous, also $\tilde{f}$ is continuous. Its Fourier coefficient can be written using Fubini's theorem as

$$\begin{aligned} 2\pi\, c_k(\tilde{f}) &= \int_{-\pi}^{\pi} \sum_{l\in\mathbb{Z}} f(x+2l\pi)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \sum_{l\in\mathbb{Z}} \int_{-\pi}^{\pi} f(x+2l\pi)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\ &= \int_{\mathbb{R}} f(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \hat{f}(k)\,. \end{aligned}$$

Thus,

$$\tilde{f}(x) = \sum_{k\in\mathbb{Z}} c_k(\tilde{f})\,\mathrm{e}^{\mathrm{i}kx} = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \hat{f}(k)\,\mathrm{e}^{\mathrm{i}kx}$$

where the series converges uniformly by the second assumption. ∎

Remark 62

It can be shown that the Poisson summation formula is fulfilled pointwise absolute convergence of both series for any function satisfying

$$f(x) = \mathcal{O}\left(\frac{1}{1+|x|^{1+\varepsilon}}\right), \; \hat{f}(\omega) = \mathcal{O}\left(\frac{1}{1+|\omega|^{1+\varepsilon}}\right), \quad \varepsilon > 0,$$

see, e.g., [9, 17]. The Poisson summation formula was generalized for slowly growing functions in [14].

We illustrate the performance of Poisson summation formula (65) by an example.

Example 63

For fixed $\alpha > 0$, we consider the function $f(x) := \mathrm{e}^{-\alpha|x|}$, $x \in \mathbb{R}$. Simple calculation shows that its Fourier transform reads

$$\hat{f}(\omega) = \int_0^\infty \left(\mathrm{e}^{(-\alpha-\mathrm{i}\omega)x} + \mathrm{e}^{(-\alpha+\mathrm{i}\omega)x}\right) \mathrm{d}x = \frac{2\alpha}{\alpha^2+\omega^2} \,.$$

Note that by Fourier's inversion formula in Theorem 48, the function $g(x) := (x^2+\alpha^2)^{-1}$ has the Fourier transform $\hat{g}(\omega) = \frac{\pi}{\alpha}\,\mathrm{e}^{-\alpha|\omega|}$.
The function f is contained in $L_1(\mathbb{R}) \cap C_0(\mathbb{R})$ and fulfills both conditions of Theorem 61. Since

$$\sum_{k\in\mathbb{Z}} f(2\pi k) = 1 + 2\sum_{k=1}^{\infty} \left(\mathrm{e}^{-2\pi\alpha}\right)^k = \frac{1+\mathrm{e}^{-2\pi\alpha}}{1-\mathrm{e}^{-2\pi\alpha}} \,,$$

we obtain by the Poisson summation formula (65) that

$$\sum_{k\in\mathbb{Z}} \frac{1}{\alpha^2+k^2} = \frac{\pi}{\alpha}\,\frac{1+\mathrm{e}^{-2\pi\alpha}}{1-\mathrm{e}^{-2\pi\alpha}} \,. \qquad \square$$

The following sampling theorem in various generalizations goes back to Whittaker [21], Kotelnikov [12] and Shannon [16], see also [6, 19]. It answers the question how to sample a function f by its values $f(nT)$, $n \in \mathbb{Z}$, for an appropriate $T > 0$ while keeping the whole information contained in f. The distance T between two successive sample points is called *sampling period*. In other words, we want to find a convenient sampling period T such that f can be recovered from its samples $f(nT)$, $n \in \mathbb{Z}$. The *sampling rate* is defined as the reciprocal value $\frac{1}{T}$ of the sampling period T. Indeed this question can be only answered for a certain class of functions.

A function $f \in L_2(\mathbb{R})$ is called *bandlimited* on $[-L, L]$ with some $L > 0$, if $\operatorname{supp} \hat{f} \subseteq [-L, L]$, i.e., if $\hat{f}(\omega) = 0$ for all $|\omega| > L$. The positive number L is the *bandwidth* of f. A typical bandlimited function on $[-L, L]$ is

$$h(x) = \frac{L}{\pi} \operatorname{sinc}(Lx)\,.$$

Note that $h \in L_2(\mathbb{R}) \setminus L_1(\mathbb{R})$. Its Fourier transform $\hat{h}$ can be determined by the theory of Section 6. Then we obtain that

$$\hat{h}(\omega) = \begin{cases} 1 & x \in (-L, L)\,, \\ \frac{1}{2} & x \in \{-L, L\}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{66}$$

Theorem 64 (Shannon – Whittaker – Kotelnikov)

Let $f \in L_1(\mathbb{R}) \cap C_0(\mathbb{R})$ be bandlimited on $[-L, L]$. Let $M \geq L$. Then f is completely determined by its values $f\left(\frac{k\pi}{M}\right)$, $k \in \mathbb{Z}$, and further f can be represented in the form

$$f(x) = \sum_{k\in\mathbb{Z}} f\left(\frac{k\pi}{M}\right) \operatorname{sinc}(Mx - k\pi)\,, \tag{67}$$

where the series converges absolutely and uniformly on $\mathbb{R}$.

Proof: 1. We prove that the formula holds pointwise. First note that from $f \in L_1(\mathbb{R}) \cap C_0(\mathbb{R})$ it follows that $f \in L_2(\mathbb{R})$, since

$$\|f\|_{L_2(\mathbb{R})} \leq \sqrt{\|f\|_{C_0(\mathbb{R})} \, \|f\|_{L_1(\mathbb{R})}} < \infty \, .$$

Since $\hat{f} \in C_0(\mathbb{R})$ by Theorem 46 and since $\operatorname{supp} \hat{f} \subseteq [-L, L]$ by assumption, we have $\hat{f} \in L_1(\mathbb{R})$ so that the Fourier inversion formula $(\hat{f})^{\vee}(x) = f(x)$ is valid almost everywhere by Theorem 48. Because $f \in C_0(\mathbb{R})$, the Fourier inversion formula holds for all $x \in \mathbb{R}$. Then for $M \geq L$ we have

$$f(x) = \frac{1}{2\pi} \int_{-M}^{M} \hat{f}(\omega) \, \hat{h}(\omega) \, \mathrm{e}^{\mathrm{i}\omega x} \, \mathrm{d}\omega = \frac{1}{2\pi} \int_{-M}^{M} \hat{f}(\omega) \, \overline{\hat{g}_x(\omega)} \, \mathrm{d}\omega \, , \tag{68}$$

where $\hat{h}$ is given by (66) and

$$\hat{g}_x(\omega) := \hat{h}(\omega) \, \mathrm{e}^{-\mathrm{i}\omega x} \, .$$

In the case $M = L$, the function (66) is the only possible choice for $\hat{h}$, while for $M > L$ also smoother functions with $\hat{h}(\omega) = 1$ for $\omega \in [-L, L]$ can be used in the above equation.
We form the $2M$–periodic functions

$$(\hat{f})^{\sim}(\omega) := \sum_{r\in\mathbb{Z}} \hat{f}(\omega + 2Mr)\,,$$
$$(\hat{g}_x)^{\sim}(\omega) := \sum_{r\in\mathbb{Z}} \hat{h}(\omega + 2Mr)\,\mathrm{e}^{-\mathrm{i}(\omega+2Mr)x}\,.$$

Clearly $(\hat{f})^{\sim}(\omega) = \hat{f}(\omega)$ and $(\hat{g}_x)^{\sim}(\omega) = \hat{g}_x(\omega)$ for $|\omega| < M$. Further we have $(\hat{f})^{\sim}$, $(\hat{g}_x)^{\sim} \in L_2([-M, M))$. Applying the Parseval equation for $2M$–periodic Fourier series (see Remark 4) and (68), we obtain

$$f(x) = \frac{1}{2\pi} \int_{-M}^{M} (\hat{f})^{\sim}(\omega)\,\overline{(\hat{g}_x)^{\sim}(\omega)}\,\mathrm{d}\omega = \frac{M}{\pi} \sum_{k\in\mathbb{Z}} c_k^{(2M)}\big((\hat{f})^{\sim}\big)\,\overline{c_k^{(2M)}\big((\hat{g}_x)^{\sim}\big)}\,.$$

In a similar way as in the proof of Theorem 61, we see that the Fourier coefficients of $(\hat{f})^{\sim}$ and $(\hat{g}_x)^{\sim}$ have the following forms

$$\begin{aligned} c_k^{(2M)}\big((\hat{f})^{\sim}\big) &:= \frac{1}{2M} \int\limits_{-M}^{M} (\hat{f})^{\sim}(\omega)\, \mathrm{e}^{-\pi \mathrm{i}\omega k/M}\, \mathrm{d}\omega \\ &= \frac{1}{2M} \int_{\mathbb{R}} \hat{f}(\omega)\, \mathrm{e}^{-\pi \mathrm{i}\omega k/M}\, \mathrm{d}\omega \\ &= \frac{\pi}{M}\, f\big(-\frac{k\pi}{M}\big)\,, \quad k \in \mathbb{Z}\,, \end{aligned}$$

and

$$\begin{aligned} c_k^{(2M)}\big((\hat{g}_x)^{\sim}\big) &:= \frac{1}{2M} \int\limits_{-M}^{M} (\hat{g}_x)^{\sim}(\omega)\, \mathrm{e}^{-\pi \mathrm{i}\omega k/M}\, \mathrm{d}\omega \\ &= \frac{1}{2M} \int\limits_{\mathbb{R}} \hat{h}(\omega)\, \mathrm{e}^{-\mathrm{i}\omega(x+k\pi/M)}\, \mathrm{d}\omega \\ &= \frac{1}{2M} \int_{-M}^{M} \mathrm{e}^{-\mathrm{i}\omega(x+k\pi/M)}\, \mathrm{d}\omega = \mathrm{sinc}\big(Mx + k\pi\big)\,, \quad k \in \mathbb{Z}\,. \end{aligned}$$

Hence we obtain for all $x \in \mathbb{R}$ that

$$f(x) = \sum_{k\in\mathbb{Z}} f\Big(\frac{k\pi}{M}\Big)\operatorname{sinc}(Mx - k\pi)\,.$$

Note that each summand of the above series has the following interpolation property

$$f\left(\frac{k\pi}{M}\right)\operatorname{sinc}(Mx - k\pi) = \begin{cases} f\left(\frac{k\pi}{M}\right) & x = \frac{k\pi}{M}\,, \\ 0 & x \in \frac{\pi}{M}\,(\mathbb{Z}\setminus\{k\})\,. \end{cases}$$

2. We show that the sum in (67) converges uniformly. We obtain

$$
\begin{aligned}
&2\pi \left| f(x) - \sum_{k=-n}^{n} f\Big(-\frac{k\pi}{M}\Big)\operatorname{sinc}(Mx + k\pi) \right| \\
&= \left| \int_{-M}^{M} \hat{f}(\omega)\mathrm{e}^{i\omega x}\mathrm{d}\omega - \sum_{k=-n}^{n} c_k^{(2M)}(\hat{f}^{\sim}) \int_{-M}^{M} \hat{g}_x^{\sim}(\omega)\mathrm{e}^{-\pi\mathrm{i}\omega k/(M)}\mathrm{d}\omega \right| \\
&= \left| \int_{-M}^{M} \left(\hat{f}^{\sim}(\omega) - \sum_{k=-n}^{n} c_k^{(2M)}(\hat{f}^{\sim})\mathrm{e}^{-\pi\mathrm{i}\omega k/(M)} \right) e^{-\mathrm{i}\omega x}\mathrm{d}\omega \right| \\
&\leq \int_{-M}^{M} \left| \hat{f}^{\sim}(\omega) - S_n(\hat{f}^{\sim}) \right| \mathrm{d}\omega \\
&\leq \sqrt{2M}\, \| \hat{f}^{\sim}(\omega) - S_n(\hat{f}^{\sim}) \|_{L_2}.
\end{aligned}
$$

The last expression becomes arbitrary small as $n \to \infty$ independently of x.

3. Finally we see absolute convergence of the sampling sum by the following computation:

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}} \left| f\Big(-\frac{k\pi}{M}\Big)\right| \left|\operatorname{sinc}\bigg(M\Big(x+\frac{k\pi}{M}\Big)\bigg)\right| \\
&= \frac{\pi}{M}\sum_{k\in\mathbb{Z}} |c_k^{(2M)}(\hat{f}^{\sim})||c_k^{(2M)}(\hat{g}_x^{\sim})| \\
&\le \frac{\pi}{M}\|\big(c_k^{(2M)}(\hat{f}^{\sim})\big)_k\|_2 \| \Big(c_k^{(2M)}(\hat{g}_x^{\sim})\Big)_k \|_2 \\
&< \infty.
\end{aligned}
$$

This finishes the proof. This completes the proof. ■

By the sampling Theorem 64, the bandlimited f with $\operatorname{supp} \hat{f} \subseteq [-L, L]$ can be reconstructed from its equispaced samples $f\left(\frac{k\pi}{M}\right)$, $k \in \mathbb{Z}$, with $M \geq L > 0$. Then the sampling period $T = \frac{\pi}{L}$ is the largest and the sampling rate $\frac{L}{\pi}$ is the smallest possible one. This sampling rate is called *Nyquist rate* after Nyquist [15]. The sinc function decreases only slightly as $|x| \to \infty$ so that we have to incorporate many summands in a truncated series (67) to get a good approximation of f.

One can obtain a better approximation of f by the choice of a higher sampling rate $\frac{L(1+\lambda)}{\pi}$ with some $\lambda > 0$ and corresponding sample values $f\left(\frac{k\pi}{L(1+\lambda)}\right)$, $k \in \mathbb{Z}$. This so-called *oversampling* allows a smoother choice of $\hat{h}$ in the above proof. The smoother $\hat{h}$ the faster decays h, so that (67) converges fast for such h. The choice of a lower sampling rate $\frac{L(1-\lambda)}{\pi}$ with some $\lambda \in (0, 1)$ is called *undersampling* which results in a reconstruction of a function f° where higher frequency parts of f appear in lower frequency parts of f°. This effect is called *aliasing* or *Moiré effect* in imaging.

Heisenberg's uncertainty principle

In this section, we consider nonzero functions $f \in L_2(\mathbb{R})$. A signal is often measured in time. We keep the spatial variable x instead of t also when speaking about time-dependent signals. In the following, we investigate the time–frequency locality of f and $\hat{f}$.

It is impossible to construct a nonzero compactly supported function $f \in L_2(\mathbb{R})$ whose Fourier transform $\hat{f}$ has a compact support too. More generally we show the following lemma.

Lemma 65

If the Fourier transform $\hat{f}$ of a nonzero function $f \in L_2(\mathbb{R})$ has compact support, then f cannot be zero on a whole interval. If a nonzero function $f \in L_2(\mathbb{R})$ has compact support, then $\hat{f}$ cannot be zero on a whole interval.

Proof: We consider $f \in L_2(\mathbb{R})$ with supp $\hat{f} \subseteq [-L, L]$ with some $L > 0$. By the Fourier inversion formula of Theorem 58 we have almost everywhere

$$f(x) = \frac{1}{2\pi} \int_{-L}^{L} \hat{f}(\omega)\, \mathrm{e}^{\mathrm{i}\omega x}\, \mathrm{d}\omega\,,$$

where the function on the right-hand side is infinitely differentiable. Since we identify almost everywhere equal functions in $L_2(\mathbb{R})$, we can assume that f is smooth.

Assume that $f(x) = 0$ for all $x \in [a, b]$ with $a < b$. For $x_0 = \frac{a+b}{2}$ we obtain by repeated differentiation with respect to x that

$$f^{(n)}(x_0) = \frac{1}{2\pi} \int_{-L}^{L} \hat{f}(\omega)\, (\mathrm{i}\omega)^n\, \mathrm{e}^{\mathrm{i}\omega x_0}\, \mathrm{d}\omega = 0\,, \quad n \in \mathbb{N}_0\,.$$

Expressing the exponential $\mathrm{e}^{\mathrm{i}\omega(x-x_0)}$ as power series, we see that for all $x \in \mathbb{R}$,

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-L}^{L} \hat{f}(\omega)\, \mathrm{e}^{\mathrm{i}\omega(x-x_0)}\, \mathrm{e}^{\mathrm{i}\omega x_0}\, \mathrm{d}\omega \\ &= \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} \int_{-L}^{L} \hat{f}(\omega)\, (\mathrm{i}\omega)^n\, \mathrm{e}^{\mathrm{i}\omega x_0}\, \mathrm{d}\omega = 0\,. \end{aligned}$$

This contradicts the assumption that $f \neq 0$. Analogously, we can show the second assertion. $\blacksquare$

By Lemma 65 gives a special aspect of a general principle that says that both f and $\hat{f}$ cannot be highly localized, i.e., if $|f|^2$ vanishes or is very small outside some small interval, then $|\hat{f}|^2$ spreads out, and conversely. We measure the dispersion of f about the time $x_0 \in \mathbb{R}$ by

$$\Delta_{x_0} f := \frac{1}{\|f\|^2} \int_{\mathbb{R}} (x - x_0)^2 \, |f(x)|^2 \mathrm{d}x > 0\,.$$

Note that if $x\, f(x)$, $x \in \mathbb{R}$, is not in $L_2(\mathbb{R})$, then $\Delta_{x_0} f = \infty$ for any $x_0 \in \mathbb{R}$. The positive number $\Delta_{x_0} f$ measures how much $|f(x)|^2$ spreads out in a neighborhood of x_0. If $|f(x)|^2$ is very small outside a small neighborhood of x_0, then the factor $(x - x_0)^2$ makes the numerator of $\Delta_{x_0} f$ small in comparison to the denominator $\|f\|^2$. Otherwise, if $|f(x)|^2$ is large far away from x_0, then the factor $(x - x_0)^2$ makes the numerator of $\Delta_{x_0} f$ large in comparison to the denominator $\|f\|^2$.

Analogously, we measure the dispersion of $\hat{f}$ about the frequency $\omega_0 \in \mathbb{R}$ by

$$\Delta_{\omega_0} \hat{f} := \frac{1}{\|\hat{f}\|^2} \int_{\mathbb{R}} (\omega - \omega_0)^2 \, |\hat{f}(\omega)|^2 \mathrm{d}\omega > 0 \, .$$

By the Parseval equation $\|\hat{f}\|^2 = 2\pi \, \|f\|^2 > 0$ we obtain

$$\Delta_{\omega_0} \hat{f} = \frac{1}{2\pi \, \|f\|^2} \int_{\mathbb{R}} (\omega - \omega_0)^2 \, |\hat{f}(\omega)|^2 \mathrm{d}\omega \, .$$

If $\omega \, f(\omega)$, $\omega \in \mathbb{R}$, is not in $L_2(\mathbb{R})$, then $\Delta_{\omega_0} f = \infty$ for any $\omega_0 \in \mathbb{R}$.

We consider the normalized Gaussian function

$$f(x) := \frac{1}{\sqrt{2\pi\sigma^2}}\, \mathrm{e}^{-x^2/(2\sigma^2)} \tag{69}$$

with standard deviation $\sigma > 0$. Then f has $L_1(\mathbb{R})$ norm one, but the energy

$$\|f\|^2 = \frac{1}{2\pi\sigma^2} \int_{\mathbb{R}} \mathrm{e}^{-x^2/\sigma^2}\, \mathrm{d}x = \frac{1}{2\sigma\sqrt{\pi}}\,.$$

Further f has the Fourier transform

$$\hat{f}(\omega) = \mathrm{e}^{-\sigma^2\omega^2/2}$$

with the energy

$$\|\hat{f}\|^2 = \int_{\mathbb{R}} \mathrm{e}^{-\sigma^2\omega^2}\, \mathrm{d}\omega = \frac{\sqrt{\pi}}{\sigma}\,.$$

For small deviation σ we observe that f is highly localized near zero, but its Fourier transform $\hat{f}$ has the large deviation $\frac{1}{\sigma}$ and is not concentrated near zero. Now we measure the *dispersion* of f around the time $x_0 \in \mathbb{R}$ by

$$\begin{aligned}
\Delta_{x_0} f &= \frac{1}{2\pi\sigma^2 \, \|f\|^2} \int_{\mathbb{R}} (x - x_0)^2 \, \mathrm{e}^{-x^2/\sigma^2} \, \mathrm{d}x \\
&= \frac{1}{2\pi\sigma^2 \, \|f\|^2} \int_{\mathbb{R}} x^2 \, \mathrm{e}^{-x^2/\sigma^2} \, \mathrm{d}x + x_0^2 = \frac{\sigma^2}{2} + x_0^2 \, .
\end{aligned}$$

For the dispersion of $\hat{f}$ about the frequency $\omega_0 \in \mathbb{R}$, we obtain

$$\begin{aligned}
\Delta_{\omega_0} \hat{f} &= \frac{1}{\|\hat{f}\|^2} \int_{\mathbb{R}} (\omega - \omega_0)^2 \, \mathrm{e}^{-\sigma^2\omega^2} \, \mathrm{d}\omega \\
&= \frac{1}{\|\hat{f}\|^2} \int_{\mathbb{R}} \omega^2 \, \mathrm{e}^{-\sigma^2\omega^2} \, \mathrm{d}\omega + \omega_0^2 = \frac{1}{2\sigma^2} + \omega_0^2 \, .
\end{aligned}$$

Thus for each $\sigma > 0$ we get the inequality

$$\left(\Delta_{x_0} f\right) \left(\Delta_{\omega_0} \hat{f}\right) = \left(\frac{\sigma^2}{2} + x_0^2\right) \left(\frac{1}{2\sigma^2} + \omega_0^2\right) \geq \frac{1}{4}$$

with equality for $x_0 = \omega_0 = 0$. $\square$

Heisenberg's uncertainty principle says that for any $x_0, \omega_0 \in \mathbb{R}$, both functions f and $\hat{f}$ cannot be localized around time $x_0 \in \mathbb{R}$ and frequency $\omega_0 \in \mathbb{R}$, respectively.

Theorem 66 (Heisenberg's uncertainty principle)

For any nonzero function $f \in L_2(\mathbb{R})$, the inequality

$$\left(\Delta_{x_0} f\right)\left(\Delta_{\omega_0} \hat{f}\right) \geq \frac{1}{4} \tag{70}$$

is fulfilled for each x_0, $\omega_0 \in \mathbb{R}$. The equality in (70) holds if and only if

$$f(x) = C\, \mathrm{e}^{\mathrm{i}\omega_0 x}\, \mathrm{e}^{-a(x-x_0)^2/2}, \quad x \in \mathbb{R}, \tag{71}$$

with some $a > 0$ and complex constant $C \neq 0$.

Proof: 1. Without lost of generality, we can assume that both functions $x\,f(x)$, $x \in \mathbb{R}$, and $\omega\,\hat{f}(\omega)$, $\omega \in \mathbb{R}$ are contained in $L_2(\mathbb{R})$ too, since otherwise we have $\left(\Delta_{x_0} f\right)\left(\Delta_{\omega_0} \hat{f}\right) = \infty$ and the inequality (70) is true.
2. In the special case $x_0 = \omega_0 = 0$, we obtain by the definitions that

$$\left(\Delta_{x_0} f\right)\left(\Delta_{\omega_0} \hat{f}\right) = \frac{1}{2\pi\,\|f\|^4}\Big(\int_{\mathbb{R}} |x\,f(x)|^2\,\mathrm{d}x\Big)\Big(\int_{\mathbb{R}} |\omega\,\hat{f}(\omega)|^2\,\mathrm{d}\omega\Big)\,.$$

From $\omega\hat{f}(\omega) \in L_2(\mathbb{R})$ it follows by Theorems 44 and 57 that $f' \in L_2(\mathbb{R})$. Thus we get by $(\mathcal{F}f')(\omega) = \mathrm{i}\omega\hat{f}(\omega)$ and the Parseval equation that

$$\begin{aligned}\left(\Delta_{x_0} f\right)\left(\Delta_{\omega_0} \hat{f}\right) &= \frac{1}{2\pi\,\|f\|^4}\Big(\int_{\mathbb{R}} |x\,f(x)|^2\,\mathrm{d}x\Big)\Big(\int_{\mathbb{R}} |(\mathcal{F}f')(\omega)|^2\,\mathrm{d}\omega\Big) \\ &= \frac{1}{\|f\|^4}\Big(\int_{\mathbb{R}} |x\,f(x)|^2\,\mathrm{d}x\Big)\Big(\int_{\mathbb{R}} |f'(x)|^2\,\mathrm{d}x\Big)\,. \qquad (72)\end{aligned}$$

By integration by parts we obtain

$$\int_{\mathbb{R}} \left(x\,\overline{f(x)}\right) f'(x)\,\mathrm{d}x = \underbrace{x\,|f(x)|^2\Big|_{-\infty}^{\infty}}_{=0} - \int_{\mathbb{R}} |f(x)|^2 + x\,f(x)\,\overline{f'(x)}\,\mathrm{d}x$$

and hence

$$\|f\|^2 = -2\ \mathrm{Re} \int_{\mathbb{R}} \overline{x\,f(x)}\,f'(x)\,\mathrm{d}x\,.$$

By the Cauchy–Schwarz inequality in $L_2(\mathbb{R})$ it follows that

$$\begin{aligned} \|f\|^4 &= 4\left(\mathrm{Re}\int_{\mathbb{R}} \overline{x\,f(x)}\,f'(x)\,\mathrm{d}x\right)^2 \\ &\le 4\left|\int_{\mathbb{R}} \overline{x\,f(x)}\,f'(x)\,\mathrm{d}x\right|^2 \\ &\le 4\left(\int_{\mathbb{R}} x^2\,|f(x)|^2\,\mathrm{d}x\right)\left(\int_{\mathbb{R}} |f'(x)|^2\,\mathrm{d}x\right). \end{aligned} \tag{73}$$

Then by (72) and (73) we obtain the inequality (70) for $x_0 = \omega_0 = 0$.

3. As known, the equality in the Cauchy–Schwarz inequality

$$\Big|\int_{\mathbb{R}} \overline{x\,f(x)}\,f'(x)\,\mathrm{d}x\Big|^2 = \Big(\int_{\mathbb{R}} x^2\,|f(x)|^2\,\mathrm{d}x\Big)\,\Big(\int_{\mathbb{R}} |f'(x)|^2\,\mathrm{d}x\Big)$$

holds if and only if the functions $f'(x)$ and $x\,f(x)$ are linearly dependent. Thus the equality in (73) holds if and only if $f'(x)$ and $x\,f(x)$ are linearly dependent and $\int_{\mathbb{R}} \overline{x\,f(x)}\,f'(x)\,\mathrm{d}x$ is real. Therefore for some $a + b\,\mathrm{i} \in \mathbb{C} \setminus \{0\}$, we obtain the differential equation

$$f'(x) + (a + b\,\mathrm{i})\,x\,f(x) = 0\,, \quad x \in \mathbb{R}\,,$$

which has the general (nonzero) solution $f(x) = C\,\mathrm{e}^{-(a+b\,\mathrm{i})\,x^2/2}$, $x \in \mathbb{R}$, with an arbitrary complex constant $C \neq 0$. By $f \in L_2(\mathbb{R})$ we have $a > 0$. Since

$$-\int_{\mathbb{R}} \overline{x\,f(x)}\,f'(x)\,\mathrm{d}x = |C|^2\,(a + b\,\mathrm{i})\,\int_{\mathbb{R}} x^2\,\mathrm{e}^{-a\,x^2/2}\,\mathrm{d}x$$

is real, we obtain $b = 0$. Consequently, we have equality in (70) with $x_0 = \omega_0 = 0$ only for $f(x) = C\,\mathrm{e}^{-a\,x^2/2}$, $x \in \mathbb{R}$.

4. In the general case with any x_0, $\omega_0 \in \mathbb{R}$, we introduce the function

$$g(x) := \mathrm{e}^{-\mathrm{i}\omega_0 (x+x_0)}\, f(x+x_0)\,, \quad x \in \mathbb{R}\,. \tag{74}$$

Obviously, $g \in L_2(\mathbb{R})$ is nonzero. By Theorem 44, this function g has the Fourier transform

$$\hat{g}(\omega) = \mathrm{e}^{\mathrm{i}(\omega+\omega_0)x_0}\, \hat{f}(\omega+\omega_0)\,, \quad \omega \in \mathbb{R}\,,$$

such that

$$\begin{aligned} \Delta_0 g &= \int_{\mathbb{R}} x^2\, |f(x+x_0)|^2\, \mathrm{d}x = \Delta_{x_0} f\,, \\ \Delta_0 \hat{g} &= \int_{\mathbb{R}} \omega^2\, |\hat{f}(\omega+\omega_0)|^2\, \mathrm{d}\omega = \Delta_{\omega_0} \hat{f}\,. \end{aligned}$$

Thus we obtain by step 2 that

$$\big(\Delta_{x_0} f\big)\,\big(\Delta_{\omega_0} \hat{f}\big) = \big(\Delta_0 g\big)\,\big(\Delta_0 \hat{g}\big) \ge \frac{1}{4}\,.$$

5. From the equality $\left(\Delta_0 g\right)\left(\Delta_0 \hat{g}\right) = \frac{1}{4}$ it follows by step 3 that $g(x) = C\,\mathrm{e}^{-ax^2/2}$ with $C \in \mathbb{C}$ and $a > 0$. By the substitution (74) we see that the equality in (70) means that f has the form (71). $\blacksquare$

The *average time* of a nonzero function $f \in L_2(\mathbb{R})$ is defined by

$$x^* := \frac{1}{\|f\|^2} \int_{\mathbb{R}} x\,|f(x)|^2\,\mathrm{d}x\,.$$

This value exists and is a real number, if $\int_{\mathbb{R}} |x|\,|f(x)|^2\,\mathrm{d}x < \infty$. For a nonzero function $f \in L_2(\mathbb{R})$ with $x^* \in \mathbb{R}$, the quantity $\Delta_{x^*} f$ is the so-called *temporal variance* of f.
Analogously, the *average frequency* of the Fourier transform $\hat{f} \in L_2(\mathbb{R})$ is defined by

$$\omega^* := \frac{1}{\|\hat{f}\|^2} \int_{\mathbb{R}} \omega\,|\hat{f}(\omega)|^2\,\mathrm{d}\omega\,.$$

For a Fourier transform $\hat{f}$ with $\omega^* \in \mathbb{R}$, the quantity $\Delta_{\omega^*} \hat{f}$ is the so-called *frequency variance* of $\hat{f}$.

Example 67

The normalized Gaussian function in (69) has the average time zero and the temporal variance $\Delta_0 f = \frac{\sigma^2}{2}$, where $\sigma > 0$ denotes the standard deviation. Its Fourier transform has the average frequency zero and the frequency variance $\Delta_0 \hat{f} = \frac{1}{2\sigma^2}$. $\square$

Lemma 68

For each nonzero function $f \in L_2(\mathbb{R})$ with finite average time x^*, it holds the estimate

$$\Delta_{x_0} f = \Delta_{x^*} f + (x^* - x_0)^2 \geq \Delta_{x^*} f$$

for any $x_0 \in \mathbb{R}$.
Similarly, for each nonzero function $f \in L_2(\mathbb{R})$ with finite average frequency ω^* of $\hat{f}$ it holds the estimate

$$\Delta_{\omega_0} \hat{f} = \Delta_{\omega^*} \hat{f} + (\omega^* - \omega_0)^2 \geq \Delta_{\omega^*} \hat{f}$$

for any $\omega_0 \in \mathbb{R}$.

Proof: From

$$(x - x_0)^2 = (x - x^*)^2 + 2\,(x - x^*)(x^* - x_0) + (x^* - x_0)^2$$

it follows immediately that

$$\int_{\mathbb{R}} (x - x_0)^2\,|f(x)|^2\,\mathrm{d}x = \int_{\mathbb{R}} (x - x^*)^2\,|f(x)|^2\,\mathrm{d}x + 0 + (x^* - x_0)^2\,\|f\|^2$$

and hence

$$\Delta_{x_0} f = \Delta_{x^*} f + (x^* - x_0)^2 \geq \Delta_{x^*} f\ .$$

Analogously, one can show the second result. ■

Applying Theorem 66 in the special case $x_0 = x^*$ and $\omega_0 = \omega^*$, we obtain the following corollary.

Corollary 69

For any nonzero function $f \in L_2(\mathbb{R})$ with finite average time x^* and finite average frequency ω^*, the inequality

$$\left(\Delta_{x^*} f\right) \left(\Delta_{\omega^*} \hat{f}\right) \geq \frac{1}{4}$$

is fulfilled. The equality in above inequality holds if and only if

$$f(x) = C\, \mathrm{e}^{\mathrm{i}\omega^* x}\, \mathrm{e}^{-a(x-x^*)^2/2}\,, \quad x \in \mathbb{R}\,,$$

with some $a > 0$ and complex constant $C \neq 0$.

Windowed Fourier transform

The Fourier transform $\hat{f}$ contains frequency information of the whole function $f \in L_2(\mathbb{R})$. Now we are interested in simultaneous information about time and frequency of a given function $f \in L_2(\mathbb{R})$. In time–frequency analysis we ask for frequency information of f near certain time. Analogously, we are interested in the time information of the Fourier transform $\hat{f}$ near certain frequency. Therefore we localize the function f and its Fourier transform $\hat{f}$ by using windows.
A real, even nonzero function $\psi \in L_2(\mathbb{R})$, where ψ and $\hat{\psi}$ are localized near zero, is called a *window function* or *window*. Thus $\hat{\psi}$ is a window too.

Example 70

Let $L > 0$ be fixed. Frequently applied window functions are the

rectangular window $\psi(x) = 1_{[-L,\,L]}(x)$,

the *triangular window* $\psi(x) = \left(1 - \frac{|x|}{L}\right) 1_{[-L,\,L]}(x)$,

the *Gaussian window* with deviation $\sigma > 0$

$$\psi(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, \mathrm{e}^{-x^2/(2\sigma^2)}\,,$$

the *Hanning window* $\psi(x) = \frac{1}{2}\left(1 + \cos\frac{\pi x}{L}\right) 1_{[-L,\,L]}(x)$,

and the *Hamming window*

$$\psi(x) = \left(0.54 + 0.46\,\cos\frac{\pi x}{L}\right) 1_{[-L,\,L]}(x)\,. \qquad \square$$

Using the shifted window $\psi(\cdot - b)$, we consider the product $f\,\psi(\cdot - b)$ which is localized in some neighborhood of $b \in \mathbb{R}$. Then we form the Fourier transform of the localized function $f\,\psi(\cdot - b)$. The mapping $\mathcal{F}_\psi \colon L_2(\mathbb{R}) \to L_2(\mathbb{R}^2)$ defined by

$$(\mathcal{F}_\psi f)(b, \omega) := \int_{\mathbb{R}} f(x)\,\psi(x - b)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x = \langle f,\, \Psi_{b,\omega}\rangle_{L_2(\mathbb{R})} \tag{75}$$

with the *time–frequency atom*

$$\Psi_{b,\omega}(x) := \psi(x - b)\,\mathrm{e}^{\mathrm{i}\omega x}\,, \quad x \in \mathbb{R}\,,$$

is called *windowed Fourier transform* or *short time Fourier transform* (STFT) .

Note that the time–frequency atom $\Psi_{b,\omega}$ is concentrated in time b and in frequency ω.
A special case of the windowed Fourier transform is the *Gabor transform* (Gabor 1946) which uses a Gaussian window. The squared magnitude $|(\mathcal{F}_\psi f)(b,\omega)|^2$ of the windowed Fourier transform is called *spectrogram* of f with respect to ψ.
The windowed Fourier transform $\mathcal{F}_\psi f$ can be interpreted as a joint time–frequency information of f. Thus $(\mathcal{F}_\psi f)(b,\omega)$ can be considered as a measure for the amplitude of a frequency band near ω at time b.

We choose the Gaussian window ψ with deviation $\sigma = 1$, i.e.

$$\psi(x) := \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}, \quad x \in \mathbb{R},$$

and consider the $L_2(\mathbb{R})$ function $f(x) := \psi(x) \mathrm{e}^{\mathrm{i}\omega_0 x}$ with fixed frequency $\omega_0 \in \mathbb{R}$. We show that the frequency ω_0 can be detected by windowed Fourier transform $\mathcal{F}_\psi f$ which reads as follows

$$(\mathcal{F}_\psi f)(b, \omega) = \frac{1}{2\pi} \mathrm{e}^{-b^2/2} \int_{\mathbb{R}} \mathrm{e}^{-x^2} \, \mathrm{e}^{bx + \mathrm{i}(\omega_0 - \omega)x} \, \mathrm{d}x .$$

From the Fourier transform of the Gaussian we know that

$$\int_{\mathbb{R}} \mathrm{e}^{-x^2} \mathrm{e}^{\mathrm{i}\omega x} \,\mathrm{d}x = \sqrt{\pi}\, \mathrm{e}^{-\omega^2/4}$$

and hence we obtain by substitution that

$$(\mathcal{F}_\psi f)(b,\omega) = \frac{1}{2\sqrt{\pi}} \mathrm{e}^{-b^2/4} \mathrm{e}^{-(\omega_0-\omega)^2/4} \mathrm{e}^{\mathrm{i}b(\omega_0-\omega)/2} .$$

Thus the spectrogram is given by

$$\left|(\mathcal{F}_\psi f)(b,\omega)\right|^2 = \frac{1}{4\pi} \mathrm{e}^{-b^2/2} \mathrm{e}^{-(\omega_0-\omega)^2/2} .$$

For each time $b \in \mathbb{R}$, the spectrogram has its maximum at the frequency $\omega = \omega_0$. In practice, one can detect ω_0 only, if $|b|$ is not too large. $\square$
The following identity combines f and $\hat{f}$ in a joint time–frequency representation.

Lemma 71

Let ψ be a window. Then for all time–frequency locations $(b,\omega) \in \mathbb{R}^2$ we have

$$2\pi\,(\mathcal{F}_{\psi} f)(b,\omega) = \mathrm{e}^{-\mathrm{i}b\omega}\,(\mathcal{F}_{\hat{\psi}} \hat{f})(\omega,-b)\,.$$

Proof: Since ψ is real and even by definition, its Fourier transform $\hat{\psi}$ is real and even too. Thus $\hat{\psi}$ is a window too. By Theorem 44 and Parseval's equality we obtain

$$2\pi\,\langle f,\, \psi(\cdot - b)\,\mathrm{e}^{\mathrm{i}\omega\,\cdot}\rangle_{L_2(\mathbb{R})} = \langle \hat{f},\, \hat{\psi}(\cdot - \omega)\,\mathrm{e}^{-\mathrm{i}b(\cdot-\omega)}\rangle_{L_2(\mathbb{R})}\,.$$

and hence

$$\begin{aligned} 2\pi \int_{\mathbb{R}} f(x)\,\psi(x-b)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x &= \int_{\mathbb{R}} \hat{f}(u)\,\hat{\psi}(u-\omega)\,\mathrm{e}^{\mathrm{i}b(u-\omega)}\,\mathrm{d}u \\ &= \mathrm{e}^{-\mathrm{i}b\omega} \int_{\mathbb{R}} \hat{f}(u)\,\hat{\psi}(u-\omega)\,\mathrm{e}^{\mathrm{i}bu}\,\mathrm{d}u\,. \end{aligned}$$

This completes the proof. ■

Let ψ be a window function, where the functions $x\,\psi(x)$ and $\omega\,\hat{\psi}(\omega)$ are in $L_2(\mathbb{R})$ too. For all time–frequency locations $(b,\,\omega) \in \mathbb{R}^2$, the time–frequency atoms $\Psi_{b,\omega} = \psi(\cdot - b)\,\mathrm{e}^{\mathrm{i}\omega\cdot}$ and their Fourier transforms $\hat{\Psi}_{b,\omega} = \hat{\psi}(\cdot - \omega)\,\mathrm{e}^{-\mathrm{i}b\,(\cdot-\omega)}$ have constant energies $\|\Psi_{b,\omega}\|^2 = \|\psi\|^2$ and $\|\hat{\Psi}_{b,\omega}\|^2 = \|\hat{\psi}\|^2 = 2\pi\,\|\psi\|^2$, respectively. Then the atom $\Psi_{b,\omega}$ has the average time $x^* = b$ and $\hat{\Psi}_{b,\omega}$ has the average frequency $\omega^* = \omega$, since

$$\begin{aligned}
x^* &= \frac{1}{\|\psi\|^2}\int_{\mathbb{R}} x\,|\Psi_{b,\omega}(x)|^2\,\mathrm{d}x = \frac{1}{\|\psi\|^2}\int_{\mathbb{R}} (x+b)\,|\psi(x)|^2\,\mathrm{d}x = b\,,\\
\omega^* &= \frac{1}{\|\hat{\psi}\|^2}\int_{\mathbb{R}} u\,|\hat{\Psi}_{b,\omega}(u)|^2\,\mathrm{d}u = \frac{1}{\|\hat{\psi}\|^2}\int_{\mathbb{R}} (u+\omega)\,|\hat{\psi}(u)|^2\,\mathrm{d}u = \omega\,.
\end{aligned}$$

Further, the temporal variance of the time–frequency atom $\Psi_{b,\omega}$ is invariant for all time–frequency locations $(b, \omega) \in \mathbb{R}^2$, because

$$\begin{aligned}\Delta_b \Psi_{b,\omega} &= \frac{1}{\|\psi\|^2} \int_{\mathbb{R}} (x-b)^2 \, |\Psi_{b,\omega}(x)|^2 \, \mathrm{d}x \\ &= \frac{1}{\|\psi\|^2} \int_{\mathbb{R}} x^2 \, |\psi(x)|^2 \, \mathrm{d}x = \Delta_0 \psi \, .\end{aligned}$$

Analogously, the frequency variance of $\hat{\Psi}_{b,\omega}$ is constant for all time–frequency locations $(b, \omega) \in \mathbb{R}^2$, because

$$\begin{aligned}\Delta_\omega \hat{\Psi}_{b,\omega} &= \frac{1}{\|\hat{\psi}\|^2} \int_{\mathbb{R}} (u-\omega)^2 \, |\hat{\Psi}_{b,\omega}(u)|^2 \, \mathrm{d}u \\ &= \frac{1}{\|\hat{\psi}\|^2} \int_{\mathbb{R}} u^2 \, |\hat{\psi}(u)|^2 \, \mathrm{d}u = \Delta_0 \hat{\psi} \, .\end{aligned}$$

For arbitrary $f \in L_2(\mathbb{R})$, we obtain by Parseval's equality

$$2\pi \, (\mathcal{F}_\psi)(b, \omega) = 2\pi \, \langle f, \, \Psi_{b,\omega} \rangle_{L_2(\mathbb{R})} = \langle \hat{f}, \, \hat{\Psi}_{b,\omega} \rangle_{L_2(\mathbb{R})} \, .$$

Hence the value $(\mathcal{F}_{\psi})(b, \omega)$ contains information on f in the *time–frequency window* resp. *Heisenberg box*

$$\left[b - \sqrt{\Delta_0 \psi},\ b + \sqrt{\Delta_0 \psi}\right] \times \left[\omega - \sqrt{\Delta_0 \hat{\psi}},\ \omega + \sqrt{\Delta_0 \hat{\psi}}\right],$$

since the deviation is the square root of the variance. Note that the area of the Heisenberg box cannot become arbitrary small, i.e., it holds by Heisenberg's uncertainty principle (see Corollary 69) that

$$\left(2\sqrt{\Delta_0 \psi}\right)\left(2\sqrt{\Delta_0 \hat{\psi}}\right) \geq 2.$$

The size of the Heisenberg box is independent of the time–frequency location $(b, \omega) \in \mathbb{R}^2$. This means that a windowed Fourier transform has the same resolution across the whole time–frequency plane $\mathbb{R}^2$.

Theorem 72

Let ψ be a window function. Then for $f, g \in L_2(\mathbb{R})$ the following relation holds true:

$$\langle \mathcal{F}_\psi f,\, \mathcal{F}_\psi g \rangle_{L_2(\mathbb{R}^2)} = 2\pi \, \|\psi\|^2_{L_2(\mathbb{R})} \, \langle f, g \rangle_{L_2(\mathbb{R})} \,.$$

In particular, for $\|\psi\|_{L_2(\mathbb{R})} = 1$ the energies of $\mathcal{F}_\psi f$ and f are equal up to the factor 2π,

$$\|\mathcal{F}_\psi f\|^2_{L_2(\mathbb{R}^2)} = 2\pi \, \|f\|^2_{L_2(\mathbb{R})} \,.$$

Proof: 1. First, let $\psi \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$. Then we have

$$\langle \mathcal{F}_\psi f \,, \mathcal{F}_\psi g \rangle_{L_2(\mathbb{R}^2)} = \int_{\mathbb{R}} \int_{\mathbb{R}} (\mathcal{F}_\psi f)(b,\omega)\, \overline{(\mathcal{F}_\psi g)(b,\omega)}\, \mathrm{d}\omega\, \mathrm{d}b\,.$$

We consider the inner integral

$$\int_{\mathbb{R}} (\mathcal{F}_\psi f)(b,\omega)\, \overline{(\mathcal{F}_\psi g)(b,\omega)}\, \mathrm{d}\omega = \int_{\mathbb{R}} (f\, \psi(\cdot - b))\hat{}(\omega)\, \overline{(g\, \psi(\cdot - b))\hat{}(\omega)}\, \mathrm{d}\omega\,.$$

By

$$\int_{\mathbb{R}} |f(x)\, \psi(x-b)|^2\, \mathrm{d}x \le \|\psi\|^2_{L_\infty(\mathbb{R})}\, \|f\|^2_{L_2(\mathbb{R})} < \infty$$

we see that $f\psi \in L_2(\mathbb{R})$ such that we can apply Parseval's equality

$$\int_{\mathbb{R}} (\mathcal{F}_\psi f)(b,\omega)\, \overline{(\mathcal{F}_\psi g)(b,\omega)}\, \mathrm{d}\omega = 2\pi \int_{\mathbb{R}} f(x)\, \overline{g(x)}\, |\psi(x-b)|^2\, \mathrm{d}x\,.$$

Using this in the above inner product results in

$$\langle \mathcal{F}_\psi f,\, \mathcal{F}_\psi g \rangle_{L_2(\mathbb{R}^2)} = 2\pi \int_{\mathbb{R}} \int_{\mathbb{R}} f(x)\, \overline{g(x)}\, |\psi(x-b)|^2\, \mathrm{d}x\, \mathrm{d}b\,.$$

Since $f,\ g \in L_2(\mathbb{R})$, we see as in the above argumentation that the absolute integral exists. Hence we can change the order of integration by Fubini's theorem which results in

$$\begin{aligned}\langle \mathcal{F}_\psi f,\, \mathcal{F}_\psi g\rangle_{L_2(\mathbb{R}^2)} &= 2\pi \int_{\mathbb{R}} f(x)\,\overline{g(x)} \int_{\mathbb{R}} |\psi(x-b)|^2 \,\mathrm{d}b\,\mathrm{d}x \\ &= 2\pi\, \|\psi\|^2_{L_2(\mathbb{R})} \langle f, g\rangle_{L_2(\mathbb{R})}\,.\end{aligned}$$

2. Let $f,\ g \in L_2(\mathbb{R})$ be fixed. By $\psi \mapsto \langle \mathcal{F}_\psi f, \mathcal{F}_\psi g\rangle_{L_2(\mathbb{R}^2)}$ a continuous functional is defined on $L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$. Now $L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$ is a dense subspace of $L_2(\mathbb{R})$. By the Hahn–Banach theorem this can be uniquely extended to a functional on $L_2(\mathbb{R})$, where $\langle f, g\rangle_{L_2(\mathbb{R})}$ is kept. ■

Remark 73

By Theorem 72 we know that

$$\int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} |(\mathcal{F}_\psi f)(b, \omega)|^2 \, \mathrm{d}b \, \mathrm{d}\omega \, .$$

Hence the spectrogram $|(\mathcal{F}_\psi f)(b, \omega)|^2$ *can be interpreted as an energy density, i.e., the time–frequency rectangle* $[b,\ b + \Delta b] \times [\omega,\ \omega + \Delta\omega]$ *corresponds to the energy*

$$\frac{1}{2\pi} \, |(\mathcal{F}_\psi f)(b, \omega)|^2 \, \Delta b \, \Delta\omega \, . \quad \square$$

By Theorem 72 the windowed Fourier transform represents a univariate signal $f \in L_2(\mathbb{R})$ by a bivariate function $\mathcal{F}_\psi f \in L_2(\mathbb{R}^2)$. Conversely, from given windowed Fourier transform $\mathcal{F}_\psi f$ one can recover the function f:

Corollary 74

Let ψ be a window function with $\|\psi\|_{L_2(\mathbb{R})} = 1$. Then for all $f \in L_2(\mathbb{R})$ it holds the representation formula

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} (\mathcal{F}_\psi f)(b,\omega)\, \psi(x-b)\, \mathrm{e}^{\mathrm{i}\omega x}\, \mathrm{d}b\, \mathrm{d}\omega\,,$$

where the integral is meant in the weak sense.

Proof: Let

$$\tilde{f}(x) := \int_{\mathbb{R}} \int_{\mathbb{R}} (\mathcal{F}_\psi f)(b,\omega)\, \psi(x-b)\, \mathrm{e}^{\mathrm{i}\omega x}\, \mathrm{d}b\, \mathrm{d}\omega\,, \quad x \in \mathbb{R}\,.$$

By Theorem 72 we obtain

$$\begin{aligned}\langle \tilde{f},\, h\rangle_{L_2(\mathbb{R})} &= \int_{\mathbb{R}} \int_{\mathbb{R}} (\mathcal{F}_\psi f)(b,\omega)\, \langle \psi(\cdot - b)\, \mathrm{e}^{\mathrm{i}\cdot\omega},\, h\rangle_{L_2(\mathbb{R})}\, \mathrm{d}b\, \mathrm{d}\omega \\ &= \langle \mathcal{F}_\psi f,\, \mathcal{F}_\psi h\rangle_{L_2(\mathbb{R}^2)} = 2\pi\, \langle f,\, h\rangle_{L_2(\mathbb{R})}\end{aligned}$$

for all $h \in L_2(\mathbb{R})$ so that $\tilde{f} = 2\pi\, f$ in $L_2(\mathbb{R})$. ■

A typical application of this time–frequency analysis consists in the following three steps:
1. For a given (noisy) signal $f \in L_2(\mathbb{R})$ compute the windowed Fourier transform $\mathcal{F}_\psi f$ with respect to a suitable window ψ.
2. Then $(\mathcal{F}_\psi f)(b, \omega)$ is transformed into a new function $g(b, \omega)$ by so-called *signal compression*. Usually, $(\mathcal{F}_\psi f)(b, \omega)$ is truncated to a region of interest where $|(\mathcal{F}_\psi f)(b, \omega)|$ is larger than a given threshold.
3. By the compressed function g compute an approximate signal $\tilde{f}$ (of the given signal f) by a modified reconstruction formula of Corollary 74

$$\tilde{f}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} g(b, \omega)\, \varphi(x - b)\, \mathrm{e}^{\mathrm{i}\omega x}\, \mathrm{d}b\, \mathrm{d}\omega\,,$$

where φ is a convenient window. Note that distinct windows ψ and φ may be used in steps 1 and 3.
For application of the windowed Fourier transform in music analysis we refer to paperes of C. Févotte, see, e.g. [4].

Motivations for discrete Fourier transforms

We start with introducing the discrete Fourier transform.
For a given vector $\mathbf{a} = (a_j)_{j=0}^{N-1} \in \mathbb{C}^N$ we call the vector $\hat{\mathbf{a}} = (\hat{a}_k)_{k=0}^{N-1} \in \mathbb{C}^N$ the discrete Fourier transform of $\mathbf{a}$ if

$$\hat{a}_k = \sum_{j=0}^{N-1} a_j \mathrm{e}^{-2\pi \mathrm{i} jk/N} = \sum_{j=0}^{N-1} a_j w_N^{jk}, \qquad k = 0, \ldots, N-1, \tag{76}$$

where

$$w_N := \mathrm{e}^{-2\pi \mathrm{i}/N} = \cos \frac{2\pi}{N} - \mathrm{i} \sin \frac{2\pi}{N}. \tag{77}$$

Obviously, $w_N \in \mathbb{C}$ is a *primitive* N*th root of unity*, because $w_N^N = 1$ and $w_N^k \neq 1$ for $k = 1, \ldots, N-1$. Since

$$(w_N^k)^N = \left(\mathrm{e}^{-2\pi \mathrm{i} k/N}\right)^N = \mathrm{e}^{-2\pi \mathrm{i} k} = 1,$$

all numbers w_N^k, $k = 0, \ldots, N-1$ are Nth roots of unity and form the vertices of a regular N-gon inscribed in the complex unit circle.

In this section we will show that the discrete Fourier transform naturally comes into play for the numerical solution of following fundamental problems:

- computation of Fourier coefficients of a function $f \in C(\mathbb{T})$,
- computation of the values of a trigonometric polynomial on a uniform grid of the interval $[0, 2\pi)$,
- calculation of the continuous Fourier transform of a function $f \in L_1(\mathbb{R}) \cap C(\mathbb{R})$ on a uniform grid of an interval $[-n\pi, n\pi)$ with certain $n \in \mathbb{N}$,
- interpolation by trigonometric polynomials on a uniform grid.

Approximation of Fourier coefficients and aliasing formula

First we describe a numerical approach to compute the Fourier coefficients $c_k(f)$, $k \in \mathbb{Z}$, of a given function $f \in C(\mathbb{T})$, where f is given by its values sampled on the uniform grid

$$\{\frac{2\pi j}{N} : j = 0, \dots, N-1\}.$$

Assume that $N \in \mathbb{N}$ is even. Using the trapezoidal rule for numerical integration, we can compute $c_k(f)$ for each $k \in \mathbb{Z}$ approximately.

By $f(0) = f(2\pi)$ we find that

$$
\begin{aligned}
c_k(f) &= \frac{1}{2\pi} \int_0^{2\pi} f(t)\, \mathrm{e}^{-\mathrm{i}kt}\, \mathrm{d}t \\
&\approx \frac{1}{2N} \sum_{j=0}^{N-1} \Big[f\big(\frac{2\pi j}{N}\big)\, \mathrm{e}^{-2\pi\mathrm{i}jk/N} + f\big(\frac{2\pi(j+1)}{N}\big)\, \mathrm{e}^{-2\pi\mathrm{i}(j+1)k/N} \Big] \\
&= \frac{1}{2N} \sum_{j=0}^{N-1} f\big(\frac{2\pi j}{N}\big)\, \mathrm{e}^{-2\pi\mathrm{i}jk/N} + \frac{1}{2N} \sum_{j=1}^{N} f\big(\frac{2\pi j}{N}\big)\, \mathrm{e}^{-2\pi\mathrm{i}jk/N} \\
&= \frac{1}{N} \sum_{j=0}^{N-1} f\big(\frac{2\pi j}{N}\big)\, \mathrm{e}^{-2\pi\mathrm{i}jk/N}, \quad k \in \mathbb{Z}\,.
\end{aligned}
$$

Thus we obtain

$$
\hat{f}_k := \frac{1}{N} \sum_{j=0}^{N-1} f\big(\frac{2\pi j}{N}\big)\, w_N^{jk} \tag{78}
$$

as approximate values of $c_k(f)$.

If f is real-valued, then we observe the symmetry relation

$$\hat{f}_k = \overline{\hat{f}}_{-k}\,, \quad k \in \mathbb{Z}\,.$$

Obviously, the values $\hat{f}_k$ are N-periodic, i.e. $\hat{f}_{k+N} = \hat{f}_k$ for all $k \in \mathbb{Z}$, since $w_N^N = 1$. However, by the Lemma 24 of Riemann–Lebesgue we know that $c_k(f) \to 0$ as $|k| \to \infty$. Therefore, $\hat{f}_k$ is only an acceptable approximation of $c_k(f)$ for small $|k|$, i.e.,

$$\hat{f}_k \approx c_k(f)\,, \quad k = -\frac{N}{2}, \ldots, \frac{N}{2} - 1\,.$$

Example 75

Let f be the 2π-periodic extension of the pulse function

$$f(x) := \begin{cases} 1 & x \in \left(-\frac{\pi}{2},\, \frac{\pi}{2}\right), \\ \frac{1}{2} & x \in \left\{-\frac{\pi}{2},\, \frac{\pi}{2}\right\}, \\ 0 & x \in \left[-\pi,\, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2},\, \pi\right). \end{cases}$$

Note that f is even. Then its Fourier coefficients read for $k \in \mathbb{Z} \setminus \{0\}$ as follows

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x = \frac{1}{\pi} \int_0^{\pi/2} \cos(kx)\, \mathrm{d}x = \frac{1}{\pi k} \sin \frac{\pi k}{2}$$

and $c_0(f) = \frac{1}{2}$.

Example 75 (continue)

For fixed $N \in 4\,\mathbb{N}$, we obtain the related approximate values

$$\begin{aligned} \hat{f}_k &= \frac{1}{N} \sum_{j=-N/2}^{N/2-1} f\big(\frac{2\pi j}{N}\big)\, w_N^{jk} \\ &= \frac{1}{N} \Big(\cos\frac{\pi k}{2} + 1 + 2 \sum_{j=1}^{N/4-1} \cos\frac{2\pi jk}{N} \Big) \quad k \in \mathbb{Z}\,. \end{aligned}$$

Hence we have $\hat{f}_k = \frac{1}{2}$ for $k \in N\,\mathbb{Z}$. Using the Dirichlet kernel $D_{N/4-1}$ with (13), it follows that for $k \in \mathbb{Z} \setminus (N\,\mathbb{Z})$

$$\hat{f}_k = \frac{1}{N} \Big(\cos\frac{\pi k}{2} + D_{N/4-1}\big(\frac{2\pi k}{N}\big) \Big) = \frac{1}{N}\, \sin\frac{\pi k}{2}\, \cot\frac{\pi k}{N}\,.$$

This example illustrates the different asymptotic behavior of the Fourier coefficients $c_k(f)$ and its approximate values $\hat{f}_k$ for $|k| \to \infty$. $\square$

To see this effect more clearly, we will derive a so-called aliasing formula for Fourier coefficients. To this end we use the following notations. As usual, δ_j, $j \in \mathbb{Z}$, denotes the *Kronecker symbol* with

$$\delta_j := \begin{cases} 1 & j = 0\,, \\ 0 & j \neq 0\,. \end{cases}$$

For $j \in \mathbb{Z}$, we denote the *nonnegative residue modulo* $N \in \mathbb{N}$ by $(j \bmod N)$, where $(j \bmod N) \in \{0, \ldots, N-1\}$ and N is a divisor of $j - (j \bmod N)$. Note that we have for all $j,\ k \in \mathbb{Z}$

$$(j\,k) \bmod N = \big((j \bmod N)\,k\big) \bmod N\,. \tag{79}$$

Lemma 76

Let $N \in \mathbb{N}$ be given. For each $j \in \mathbb{Z}$, the primitive Nth root of unity w_N has the property

$$\sum_{k=0}^{N-1} w_N^{jk} = N\,\delta_{j \bmod N}\,, \tag{80}$$

where

$$\delta_{j \bmod N} := \begin{cases} 1 & j \bmod N = 0\,, \\ 0 & j \bmod N \neq 0 \end{cases}$$

denotes the N-periodic Kronecker symbol.

Proof: In the case $j \bmod N = 0$ we have $j = \ell\, N$ with certain $\ell \in \mathbb{Z}$ and hence $w_N^j = (w_N^N)^\ell = 1$. This yields (80) for $j \bmod N = 0$.
In the case $j \bmod N \neq 0$ we have $j = \ell\, N + m$ with certain $\ell \in \mathbb{Z}$ and $m \in \{1, \ldots, N-1\}$ such that $w_N^j = (w_N^N)^\ell\, w_N^m = w_N^m \neq 1$. For arbitrary $x \neq 1$, it holds

$$\sum_{k=0}^{N-1} x^k = \frac{x^N - 1}{x - 1}\,.$$

For $x = w_N^j$ we obtain (80) for $j \bmod N \neq 0$. $\blacksquare$

Lemma 76 can be used to prove the following aliasing formula, which describes the relation between the Fourier coefficients $c_k(f)$ and their approximate values $\hat{f}_k$.

Theorem 77 (Aliasing formula for Fourier coefficients)

Let $f \in C(\mathbb{T})$ be given. Assume that the Fourier coefficients of f satisfy the condition $\sum_{k\in\mathbb{Z}}|c_k(f)| < \infty$. Then the aliasing formula

$$\hat{f}_k = \sum_{\ell\in\mathbb{Z}} c_{k+\ell N}(f)\,, \quad k \in \mathbb{Z} \tag{81}$$

holds.

Proof: Using Theorem 34, the Fourier series of f converges uniformly to f. Hence for each $x \in \mathbb{T}$,

$$f(x) = \sum_{\ell\in\mathbb{Z}} c_\ell(f)\,\mathrm{e}^{\mathrm{i}\ell x}\,.$$

For $x = \frac{2\pi j}{N}$, $j = 0, \ldots, N-1$, we obtain that

$$f\Big(\frac{2\pi j}{N}\Big) = \sum_{\ell\in\mathbb{Z}} c_\ell(f)\,\mathrm{e}^{2\pi\mathrm{i}j\ell/N} = \sum_{\ell\in\mathbb{Z}} c_\ell(f)\,w_N^{-\ell j}\,.$$

Hence due to (78) and the convergence of the Fourier series

$$\hat{f}_k = \frac{1}{N} \sum_{j=0}^{N-1} \left(\sum_{\ell \in \mathbb{Z}} c_\ell(f)\, w_N^{-j\ell} \right) w_N^{jk} = \frac{1}{N} \sum_{\ell \in \mathbb{Z}} c_\ell(f) \sum_{j=0}^{N-1} w_N^{j(\ell-k)}$$

which yields by (80) the aliasing formula (81). ■

By Theorem 77 we have no aliasing effect if f is a trigonometric polynomial of degree $< \frac{N}{2}$, i.e. for

$$f = \sum_{k=-N/2+1}^{N/2-1} c_k(f)\, \mathrm{e}^{2\pi \mathrm{i} k \cdot}$$

we have $\hat{f}_k = c_k(f)$, $k = -N/2+1, \dots, N/2-1$.

Corollary 78

With the assumptions of Theorem 77, the error estimate

$$|\hat{f}_k - c_k(f)| \leq \sum_{\ell \neq 0} |c_{k+\ell N}(f)| \tag{82}$$

holds for $k = -\frac{N}{2}, \ldots, \frac{N}{2} - 1$. *Especially for* $f \in C^r(\mathbb{T})$, $r \in \mathbb{N}$, *with the property*

$$|c_k(f)| \leq \frac{c}{|k|^{r+1}}\,, \quad k \neq 0\,, \tag{83}$$

where $c > 0$ *is a constant, we have the error estimate*

$$|\hat{f}_k - c_k(f)| \leq \frac{c}{r\,N^{r+1}} \Big(\big(\frac{1}{2} + \frac{k}{N}\big)^{-r} + \big(\frac{1}{2} - \frac{k}{N}\big)^{-r} \Big) \tag{84}$$

for $|k| < \frac{N}{2}$.

Proof: The estimate (82) immediately follows from the aliasing formula (81) by triangle inequality. With the assumption (83), formula (82) implies that

$$|\hat{f}_k - c_k(f)| \leq \sum_{\ell=1}^{\infty} \Big(|c_{k+\ell N}(f)| + |c_{k-\ell N}(f)| \Big)$$
$$\leq \frac{c}{N^{r+1}} \sum_{\ell=1}^{\infty} \Big(\big|\ell + \frac{k}{N}\big|^{-r-1} + \big|\ell - \frac{k}{N}\big|^{-r-1} \Big) .$$

For $|s| < \frac{1}{2}$ and $\ell \in \mathbb{N}$, it can be simply checked that

$$\big(\ell + s\big)^{-r-1} < \int_{\ell-1/2}^{\ell+1/2} (x+s)^{-r-1} \, \mathrm{d}x,$$

since the function $g(x) = (x+s)^{-r-1}$ is convex and monotonically decreasing. Hence

$$\sum_{\ell=1}^{\infty} \big(\ell + s\big)^{-r-1} < \int_{1/2}^{\infty} (x+s)^{-r-1} \, \mathrm{d}x = \frac{1}{r} \, \big(\frac{1}{2} + s\big)^{-r} ,$$

since for $s = \pm\frac{k}{N}$ with $|k| < \frac{N}{2}$ we have $|s| < \frac{1}{2}$. This completes the proof of (84). ■

Computation of Fourier series and the continuous Fourier transform

First we study the computation of a trigonometric polynomial $p \in \mathcal{T}_n$, $n \in \mathbb{N}$, on a uniform grid of $[0, 2\pi)$. Choosing $N \in \mathbb{N}$ with $N \geq 2n+1$, we want to calculate the value of $p = \sum_{j=-n}^{n} c_j \, \mathrm{e}^{\mathrm{i}j\cdot}$ at all grid points $\frac{2\pi k}{N}$ for $k = 0, \ldots, N-1$, where the coefficients $c_j \in \mathbb{C}$ are given. Using (77) we have

$$
\begin{aligned}
p\Big(\frac{2\pi k}{N}\Big) &= \sum_{j=-n}^{n} c_j \, \mathrm{e}^{2\pi \mathrm{i} jk/N} = \sum_{j=-n}^{n} c_j \, w_N^{-jk} \\
&= \sum_{j=0}^{n} c_{-j} \, w_N^{jk} + \sum_{j=1}^{n} c_j \, w_N^{(N-j)k} \\
&= \sum_{j=0}^{n} c_{-j} \, w_N^{jk} + \sum_{j=N-n}^{N-1} c_{N-j} \, w_N^{jk} \,.
\end{aligned}
\tag{85}
$$

Introducing the entries

$$d_j := \begin{cases} c_{-j} & j = 0, \ldots, n\,, \\ 0 & j = n+1, \ldots, N-n-1\,, \\ c_{N-j} & j = N-n, \ldots, N-1\,, \end{cases}$$

we obtain

$$p\Big(\frac{2\pi k}{N}\Big) = \sum_{j=0}^{N-1} d_j\, w_N^{jk}\,, \quad k = 0, \ldots, N-1\,, \tag{86}$$

which can be interpreted as a discrete Fourier transform of length N.
Now, in order to evaluate a Fourier series on a uniform grid of an interval of length 2π, we use their partial sum $p = S_n f$ as an approximation. For smooth functions, the Fourier series converges rapidly, see Theorem 36, such that we can approximate the Fourier series arbitrarily exactly by choosing the polynomial degree n properly.

Next we sketch the computation of the continuous Fourier transform $\hat{f}$ of a given function $f \in L_1(\mathbb{R}) \cap C(\mathbb{R})$. Since $f(x) \to 0$ for $|x| \to \infty$, we obtain for sufficiently large $n \in \mathbb{N}$ that

$$\hat{f}(v) = \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-\mathrm{i}xv}\, \mathrm{d}x \approx \int_{-n\pi}^{n\pi} f(x)\, \mathrm{e}^{-\mathrm{i}xv}\, \mathrm{d}x\,, \quad v \in \mathbb{R}\,.$$

Using the uniform grid $\{\frac{2\pi j}{N} : j = -\frac{nN}{2}, \ldots, \frac{nN}{2} - 1\}$ of the interval $[-n\pi,\, n\pi)$ for even $n \in \mathbb{N}$, we approximate the integral by the rectangle rule,

$$\int_{-n\pi}^{n\pi} f(x)\, \mathrm{e}^{-\mathrm{i}xv}\, \mathrm{d}x \approx \frac{2\pi}{N} \sum_{j=-nN/2}^{nN/2-1} f\big(\frac{2\pi j}{N}\big)\, \mathrm{e}^{-2\pi \mathrm{i} j v/N}\,.$$

For $v = \frac{k}{n}$ with $k = -\frac{nN}{2}, \ldots, \frac{nN}{2} - 1$ we find the following approximate value of $\hat{f}\big(\frac{k}{n}\big)$,

$$\hat{f}\big(\frac{k}{n}\big) \approx \frac{2\pi}{N} \sum_{j=-nN/2}^{nN/2-1} f\big(\frac{2\pi j}{N}\big)\, w_{nN}^{jk}\,. \tag{87}$$

This is indeed a discrete Fourier transform of length nN when we shift the summation index similarly as in (85). Here, as before when evaluating the Fourier coefficients, the approximation is only acceptable for the $|k| \leq \frac{nN}{2}$ since the values $\hat{f}(\frac{k}{n})$ are periodic while the Fourier transform decays with $\lim_{\nu\to\infty} |\hat{f}(\nu)| = 0$.

Finally we consider the *interpolation by a trigonometric polynomial* on a uniform grid of $[0, 2\pi)$. First we discuss the trigonometric interpolation with an *odd* number of equidistant nodes

$$x_k := \frac{2\pi k}{2n+1} \in [0, 2\pi), \quad k = 0, \dots, 2n.$$

Lemma 79

Let $n \in \mathbb{N}$ be given and $N = 2n + 1$. For arbitrary $p_k \in \mathbb{C}$, $k = 0, \ldots, N - 1$, there exists a unique trigonometric polynomial of degree n,

$$p = \sum_{\ell=-n}^{n} c_\ell \, \mathrm{e}^{\mathrm{i}\ell\cdot} \in \mathcal{T}_n \tag{88}$$

satisfying the interpolation conditions

$$p(x_k) = p\Big(\frac{2\pi k}{2n+1}\Big) = p_k \,, \quad k = 0, \ldots, 2n \,. \tag{89}$$

The coefficients $c_\ell \in \mathbb{C}$ of (88) are given by

$$c_\ell = \frac{1}{2n+1} \sum_{k=0}^{2n} p_k \, w_N^{\ell k} \,, \quad \ell = -n, \ldots, n \,. \tag{90}$$

Lemma 79 (continue)

Using the Dirichlet kernel D_n, the interpolating trigonometric polynomial (88) *can be written in the form*

$$p = \frac{1}{2n+1} \sum_{k=0}^{2n} p_k \, D_n(\cdot - x_k) \,. \tag{91}$$

Proof: 1. From the interpolation conditions (89) it follows by (77) that solving the trigonometric interpolation problem is equivalent to solving the system of linear equations

$$p(x_k) = \sum_{\ell=-n}^{n} c_\ell \, w_N^{-\ell k} = p_k \,, \quad k = 0, \ldots, 2n \,. \tag{92}$$

Assume that $c_\ell \in \mathbb{C}$ solve (92). Then by Lemma 76 we obtain

$$\sum_{k=0}^{2n} p_k \, w_N^{jk} \; = \; \sum_{k=0}^{2n} \Big(\sum_{\ell=-n}^{n} c_\ell \, w_N^{-k\ell} \Big) \, w_N^{jk} = \sum_{\ell=-n}^{n} c_\ell \, \Big(\sum_{k=0}^{2n} w_N^{(j-\ell)k} \Big) = (2n+1) \, c_j \,.$$

Hence any solution of (92) has to be of the form (90).
On the other hand, for c_ℓ given by (90) we find by Lemma 76 that for $k = 0, \ldots, 2n$

$$\begin{aligned} p\Big(\frac{2\pi k}{2n+1}\Big) &= p(x_k) = \sum_{\ell=-n}^{n} c_\ell \, w_N^{-\ell k} = \frac{1}{2n+1} \sum_{\ell=-n}^{n} \Big(\sum_{j=0}^{2n} p_j \, w_N^{j\ell} \Big) \, w_N^{-\ell k} \\ &= \frac{1}{2n+1} \sum_{j=0}^{2n} p_j \Big(\sum_{\ell=-n}^{n} w_N^{(j-k)\ell} \Big) = p_k \,. \end{aligned}$$

Thus the linear system (92) is uniquely solvable.
2. From (88) and (90) it follows by $c_{-\ell} = c_{N-\ell}$, $\ell = 1, \ldots, n$, that

$$\begin{aligned} p(x) &= c_0 + \sum_{\ell=1}^{n} \big(c_\ell \, \mathrm{e}^{\mathrm{i}\ell x} + c_{N-\ell} \, \mathrm{e}^{-\mathrm{i}\ell x} \big) \\ &= \frac{1}{2n+1} \sum_{k=0}^{2n} p_k \Big(1 + \sum_{\ell=1}^{n} \big(\mathrm{e}^{\mathrm{i}\ell(x-x_k)} + \mathrm{e}^{-\mathrm{i}\ell(x-x_k)} \big) \Big) \end{aligned}$$

and we conclude (91) by the definition (12) of the Dirichlet kernel D_n. ■

Formula (91) particularly implies that the *trigonometric Lagrange polynomials* $\ell_k \in \mathcal{T}_n$ with respect to the uniform grid $\{x_k = \frac{2\pi k}{2n+1} : k = 0, \ldots, 2n\}$ are given by

$$\ell_k := \frac{1}{2n+1} D_n(\cdot - x_k), \quad k = 0, \ldots, 2n.$$

By Lemma 79 the trigonometric Lagrange polynomials ℓ_k, $k = 0, \ldots, N-1$, form a basis of $\mathcal{T}_n$ and satisfy the interpolation conditions

$$\ell_k(x_j) = \delta_{j-k}, \quad j, k = 0, \ldots, 2n.$$

Further, the trigonometric Lagrange polynomials generate a *partition of unity*, since (91) yields for $p = 1$ that

$$1 = \frac{1}{2n+1} \sum_{k=0}^{2n} p_k D_n(\cdot - x_k) = \sum_{k=0}^{2n} \ell_k. \tag{93}$$

Now we consider the trigonometric interpolation for an *even* number of equidistant nodes $x_k^* := \frac{\pi k}{n} \in [0, 2\pi)$, $k = 0, \ldots, 2n-1$.

Lemma 80

Let $n \in \mathbb{N}$ be given and $N = 2n$. For arbitrary $p_k^* \in \mathbb{C}$, $k = 0, \ldots, 2n-1$, there exists a unique trigonometric polynomial of the special form

$$p^* = \sum_{\ell=1-n}^{n-1} c_\ell^* \, \mathrm{e}^{\mathrm{i}\ell\cdot} + \frac{1}{2} \, c_n^* \left(\mathrm{e}^{\mathrm{i}n\cdot} + \mathrm{e}^{-\mathrm{i}n\cdot} \right) \in \mathcal{T}_n \tag{94}$$

satisfying the interpolation conditions

$$p^*\left(\frac{2\pi k}{2n}\right) = p_k^* \,, \quad k = 0, \ldots, 2n-1 \,. \tag{95}$$

The coefficients $c_\ell^* \in \mathbb{C}$ of (94) are given by

$$c_\ell^* = \frac{1}{2n} \sum_{k=0}^{2n-1} p_k^* \, w_N^{\ell k} \,, \quad \ell = 1-n, \ldots, n \,. \tag{96}$$

The interpolating trigonometric polynomial (94) can be written in the form

Lemma 80 (continue)

$$p^* = \frac{1}{2n} \sum_{k=0}^{2n-1} p_k^* \, D_n^*(\cdot - x_k^*)\,, \tag{97}$$

where $D_n^* := D_n - \cos(n\,\cdot)$ *denotes the modified nth Dirichlet kernel.*

A proof of Lemma 80 is omitted here, since this result can be similarly shown as Lemma 79.

Remark 81

By $\sin(nx_k^*) = \sin(\pi k) = 0$ *for* $k = 0, \ldots, 2n-1$, *each trigonometric polynomial* $p^* + c\,\sin(n\,\cdot)$ *with arbitrary* $c \in \mathbb{C}$ *is a solution of the trigonometric interpolation problem* (95). *Therefore the restriction to trigonometric polynomials of the special form* (94) *is essential for the unique solvability of the trigonometric interpolation problem* (95). $\square$

Formula (97) implies that the *trigonometric Lagrange polynomials* $\ell_k^* \in \mathcal{T}_n$ with respect to the uniform grid $\{x_k^* = \frac{\pi k}{n} : \; k = 0, \dots, 2n-1\}$ are given by

$$\ell_k^* := \frac{1}{2n} \, D_n^*(\cdot - x_k^*) \,, \quad k = 0, \dots, 2n-1 \,.$$

By Lemma 80 the $2n$ trigonometric Lagrange polynomials ℓ_k^* are linearly independent, but they do not form a basis of $\mathcal{T}_n$, since dim $\mathcal{T}_n = 2n+1$.

Fourier matrix and discrete Fourier transform

For fixed $N \in \mathbb{N}$, we consider the vectors $\mathbf{a} = (a_j)_{j=0}^{N-1}$ and $\mathbf{b} = (b_j)_{j=0}^{N-1}$ with components a_j, $b_j \in \mathbb{C}$. As usual, the inner product and the Euclidean norm in the vector space $\mathbb{C}^N$ are defined by

$$\langle \mathbf{a}, \mathbf{b} \rangle := \mathbf{a}^\top \overline{\mathbf{b}} = \sum_{j=0}^{N-1} a_j \, \overline{b}_j \,, \quad \|\mathbf{a}\|_2 := \sqrt{\sum_{j=0}^{N-1} |a_j|^2} \,.$$

Lemma 82

Let $N \in \mathbb{N}$ *be given and* $w_N := \mathrm{e}^{-2\pi \mathrm{i}/N}$. *Then the set of the exponential vectors* $\mathbf{e}_k := \left(w_N^{jk}\right)_{j=0}^{N-1}$, $k = 0, \ldots, N-1$, *forms an orthogonal basis of* $\mathbb{C}^N$, *where* $\|\mathbf{e}_k\|_2 = \sqrt{N}$ *for each* $k = 0, \ldots, N-1$. *Any* $\mathbf{a} \in \mathbb{C}^N$ *can be represented in the form*

$$\mathbf{a} = \frac{1}{N} \sum_{k=0}^{N-1} \langle \mathbf{a}, \mathbf{e}_k \rangle \, \mathbf{e}_k \,. \tag{98}$$

The set of complex conjugate exponential vectors $\overline{\mathbf{e}}_k = \left(w_N^{-jk}\right)_{j=0}^{N-1}$, $k = 0, \ldots, N-1$, *forms also an orthogonal basis of* $\mathbb{C}^N$.

Proof: For $k, \ell \in \{0, \ldots, N-1\}$, the inner product $\langle \mathbf{e}_k, \mathbf{e}_\ell \rangle$ can be calculated by Lemma 76 such that

$$\langle \mathbf{e}_k, \mathbf{e}_\ell \rangle = \sum_{j=0}^{N-1} w_N^{(k-\ell)j} = N \, \delta_{(k-\ell) \bmod N} \cdot$$

Thus $\{\mathbf{e}_k : k = 0, \ldots, N-1\}$ is an orthogonal basis of $\mathbb{C}^N$, because the N exponential vectors $\mathbf{e}_k$ are linearly independent and $\dim \mathbb{C}^N = N$. Consequently, each vector $\mathbf{a} \in \mathbb{C}^N$ can be expressed in the form (98). Analogously, the vectors $\overline{\mathbf{e}}_k$, $k = 0, \ldots, N-1$, form an orthogonal basis of $\mathbb{C}^N$. $\blacksquare$

The N-by-N *Fourier matrix* is defined by

$$\mathbf{F}_N := \left(w_N^{jk}\right)_{j,k=0}^{N-1} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & w_N & \dots & w_N^{N-1} \\ \vdots & \vdots & & \vdots \\ 1 & w_N^{N-1} & \dots & w_N \end{pmatrix} .$$

Due to the properties of the primitive Nth root of unity w_N, the Fourier matrix $\mathbf{F}_N$ consists of only N distinct entries. Obviously, $\mathbf{F}_N$ is symmetric, $\mathbf{F}_N = \mathbf{F}_N^\top$, but not Hermitian for $N > 2$. The columns of $\mathbf{F}_N$ are the vectors $\mathbf{e}_k$ of the orthogonal basis of $\mathbb{C}^N$ such that by Lemma 82

$$\mathbf{F}_N^\top \overline{\mathbf{F}}_N = N\, \mathbf{I}_N \,, \tag{99}$$

where $\mathbf{I}_N$ denotes the N-by-N identity matrix. Hence the scaled Fourier matrix $\frac{1}{\sqrt{N}}\, \mathbf{F}_N$ is unitary.

The linear map from $\mathbb{C}^N$ to $\mathbb{C}^N$, which is represented as the matrix vector product

$$\hat{\mathbf{a}} = \mathbf{F}_N\, \mathbf{a} = \big(\langle \mathbf{a},\, \overline{\mathbf{e}}_k\rangle\big)_{k=0}^{N-1}\,, \quad \mathbf{a} \in \mathbb{C}^N\,,$$

is called *discrete Fourier transform of length N* and abbreviated by DFT(N). The transformed vector $\hat{\mathbf{a}} = \big(\hat{a}_k\big)_{k=0}^{N-1}$ is called the discrete Fourier transform (DFT) of $\mathbf{a} = \big(a_j\big)_{j=0}^{N-1}$ and we have

$$\hat{a}_k = \langle \mathbf{a},\, \overline{\mathbf{e}}_k\rangle = \sum_{j=0}^{N-1} a_j\, w_N^{jk}\,, \quad k = 0, \ldots, N-1\,. \tag{100}$$

In practice, one says that the DFT(N) maps from *time domain* $\mathbb{C}^N$ to *frequency domain* $\mathbb{C}^N$.

The main importance of the DFT arises from the fact that there exist fast and numerically stable algorithms for its computation.

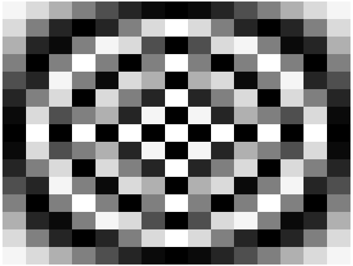
Example 83

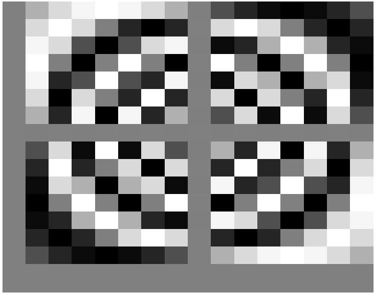
For $N \in \{2, 3, 4\}$ we obtain the Fourier matrices

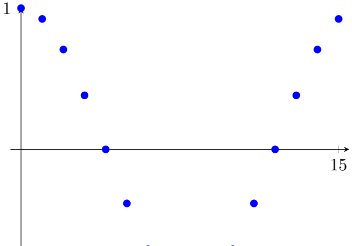
$$\mathbf{F}_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{F}_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & w_3 & \overline{w}_3 \\ 1 & \overline{w}_3 & w_3 \end{pmatrix}, \quad \mathbf{F}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \end{pmatrix}$$

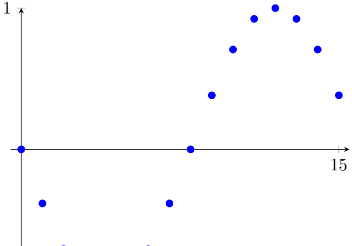
with $w_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$. Figure 11 displays both real and imaginary part of the Fourier matrix $\mathbf{F}_{16}$ and a plot of the second row of both below. In the grayscale images, white corresponds to the value 1 and black corresponds to -1. $\square$

$$\operatorname{Re}\mathbf{F}_{16} = \Big(\cos\frac{\pi jk}{8}\Big)_{j,k=0}^{15}, \quad \operatorname{Im}\mathbf{F}_{16} = -\Big(\sin\frac{\pi jk}{8}\Big)_{j,k=0}^{15}.$$









Remark 84

Let $N \in \mathbb{N}$ with $N > 1$ be given. Obviously we can compute the values

$$\hat{a}_k = \sum_{j=0}^{N-1} a_j \, w_N^{jk} \tag{101}$$

for all $k \in \mathbb{Z}$. From

$$w_N^{j(k+N)} = w_N^{jk} \cdot 1 = w_N^{jk} \,, \quad k \in \mathbb{Z} \,,$$

we observe that the resulting sequence $(\hat{a}_k)_{k\in\mathbb{Z}}$ is N-periodic. The same is true for the inverse DFT(N). For a given vector $(\hat{a}_k)_{k=0}^{N-1}$ the sequence $(a_j)_{j\in\mathbb{Z}}$ with

$$a_j = \frac{1}{N} \sum_{k=0}^{N-1} \hat{a}_k \, w_N^{-jk} \,, \quad j \in \mathbb{Z} \,,$$

is an N-periodic sequence, since

Remark 84 (continue)

$$w_N^{-(j+N)k} = w_N^{-jk} \cdot 1 = w_N^{-jk}\,, \quad j \in \mathbb{Z}\,.$$

Thus, the DFT(N) can be extended, mapping an N-periodic sequence $(a_j)_{j\in\mathbb{Z}}$ to an N-periodic sequence $(\hat{a}_k)_{k=0}^{N-1}$. A consequence of this property is the fact that the DFT(N) of even length N of a complex N-periodic sequence $(a_j)_{j\in\mathbb{Z}}$ can be formed by any N-dimensional subvector of $(a_j)_{j\in\mathbb{Z}}$. For instance, if we choose $(a_j)_{j=-N/2}^{N/2-1}$, then we obtain the same transformed sequence, since

$$\begin{aligned} \sum_{j=-N/2}^{N/2-1} a_j\, w_N^{jk} &= \sum_{j=1}^{N/2} a_{N-j}\, w_N^{(N-j)k} + \sum_{j=0}^{N/2-1} a_j\, w_N^{jk} \\ &= \sum_{j=0}^{N-1} a_j\, w_N^{jk}\,, \quad k \in \mathbb{Z}\,. \qquad \square \end{aligned}$$

Example 85

For given $N \in 2\,\mathbb{N}$, we consider the vector $\mathbf{a} = (a_j)_{j=0}^{N-1}$ with

$$a_j = \begin{cases} 0 & j \in \{0,\, \frac{N}{2}\}\,, \\ 1 & j = 1, \ldots, \frac{N}{2} - 1\,, \\ -1 & j = \frac{N}{2} + 1, \ldots, N - 1\,. \end{cases}$$

We determine the DFT(N) of $\mathbf{a}$, i.e., $\hat{\mathbf{a}} = (\hat{a}_k)_{k=0}^{N-1}$. Obviously, we have $\hat{a}_0 = 0$. For $k \in \{1, \ldots, N-1\}$ we obtain

$$\hat{a}_k = \sum_{j=1}^{N/2-1} w_N^{jk} - \sum_{j=N/2+1}^{N-1} w_N^{jk} = \big(1 - (-1)^k\big) \sum_{j=1}^{N/2-1} w_N^{jk}$$

and hence $\hat{a}_k = 0$ for even k.

Example 85 (continue)

Using

$$\sum_{j=1}^{N/2-1} x^j = \frac{x - x^{N/2}}{1-x}\,, \quad x \neq 1\,,$$

it follows for $x = w_N^k$ with odd k that

$$\hat{a}_k = 2\,\frac{w_N^k - w_N^{kN/2}}{1 - w_N^k} = 2\,\frac{w_N^k + 1}{1 - w_N^k} = 2\,\frac{w_{2N}^k + w_{2N}^{-k}}{w_{2N}^{-k} - w_{2N}^k} = -2\mathrm{i}\,\cot\frac{\pi k}{N}\,.$$

Thus we receive

$$\hat{a}_k = \begin{cases} 0 & k = 0,\,2,\,\ldots,\,N-2\,, \\ -2\mathrm{i}\,\cot\frac{\pi k}{N} & k = 1,\,3,\,\ldots,\,N-1\,. \end{cases} \qquad \square$$

Example 86

For given $N \in \mathbb{N} \setminus \{1\}$, we consider the vector $\mathbf{a} = (a_j)_{j=0}^{N-1}$ with

$$a_j = \begin{cases} \frac{1}{2} & j = 0\,, \\ \frac{j}{N} & j = 1, \ldots, N-1\,. \end{cases}$$

Note that the related N-periodic sequence $(a_j)_{j \in \mathbb{Z}}$ with $a_j = a_{j \bmod N}$, $j \in \mathbb{Z}$, is a sawtooth sequence. Now we calculate the DFT(N) of $\mathbf{a}$, i.e., $\hat{\mathbf{a}} = (\hat{a}_k)_{k=0}^{N-1}$. Obviously, we have

$$\hat{a}_0 = \frac{1}{2} + \frac{1}{N} \sum_{j=1}^{N-1} j = \frac{1}{2} + \frac{N\,(N-1)}{2\,N} = \frac{N}{2}\,.$$

Example 86 (continue)

Using the sum formula

$$\sum_{j=1}^{N-1} j\, x^j = -\frac{(N-1)\, x^N}{1-x} + \frac{x - x^N}{(1-x)^2}\,, \quad x \neq 1\,,$$

we obtain for $x = w_N^k$ with $k \in \{1, \ldots, N-1\}$ that

$$\sum_{j=1}^{N-1} j\, w_N^{jk} = \frac{-(N-1)}{1-w_N^k} + \frac{w_N^k - 1}{(1-w_N^k)^2} = -\frac{N}{1-w_N^k}$$

and hence

$$\hat{a}_k = \frac{1}{2} + \frac{1}{N} \sum_{j=1}^{N-1} j\, w_N^{jk} = \frac{1}{2} - \frac{1}{1-w_N^k} = -\frac{1+w_N^k}{2\,(1-w_N^k)} = \frac{\mathrm{i}}{2}\, \cot \frac{\pi k}{N}\,. \quad \square$$

Remark 87

In the literature, the Fourier matrix is not consistently defined. In particular, the normalization constants differ and one finds for example $\left(w_N^{-jk}\right)_{j,k=0}^{N-1}$, $\frac{1}{\sqrt{N}}\left(w_N^{jk}\right)_{j,k=0}^{N-1}$, $\frac{1}{N}\left(w_N^{jk}\right)_{j,k=0}^{N-1}$, *and* $\left(w_N^{jk}\right)_{j,k=1}^{N}$. *Consequently, there exist different forms of the DFT(N). For the sake of clarity, we emphasize that the DFT(N) is differently defined in the respective package documentations. For instance, Mathematica uses the DFT(N) of the form*

$$\hat{a}_k = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} a_j \, w_N^{-(j-1)(k-1)}, \quad k = 1, \ldots, N.$$

In Matlab, the DFT(N) is defined by

$$\hat{a}_{k+1} = \sum_{j=0}^{N-1} a_{j+1} \, w_N^{jk}, \quad k = 0, \ldots, N-1.$$

Theorem 88 (Properties of the Fourier matrix)

The Fourier matrix $\mathbf{F}_N$ *is invertible and its inverse reads as follows*

$$\mathbf{F}_N^{-1} = \frac{1}{N}\,\overline{\mathbf{F}}_N = \frac{1}{N}\left(w_N^{-jk}\right)_{j,k=0}^{N-1}. \tag{102}$$

The corresponding DFT *is a bijective map on* $\mathbb{C}^N$. *The inverse* DFT *of length* N *is given by the matrix-vector product*

$$\mathbf{a} = \mathbf{F}_N^{-1}\,\hat{\mathbf{a}} = \frac{1}{N}\left(\langle\hat{\mathbf{a}},\,\mathbf{e}_k\rangle\right)_{k=0}^{N-1}, \quad \hat{\mathbf{a}} \in \mathbb{C}^N$$

such that

$$a_j = \frac{1}{N}\,\langle\hat{\mathbf{a}},\,\mathbf{e}_k\rangle = \frac{1}{N}\sum_{k=0}^{N-1}\hat{a}_k\,w_N^{-jk}, \quad j = 0, \dots, N-1. \tag{103}$$

Proof: Relation (102) follows immediately from (99). Consequently, the DFT(N) is bijective on $\mathbb{C}^N$. ■

Lemma 89

The Fourier matrix $\mathbf{F}_N$ *satisfies*

$$\mathbf{F}_N^2 = N\,\mathbf{J}'_N\,, \quad \mathbf{F}_N^4 = N^2\,\mathbf{I}_N\,, \tag{104}$$

with the flip matrix

$$\mathbf{J}'_N := \left(\delta_{(j+k)\ \mathrm{mod}\ N}\right)_{j,k=0}^{N-1} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 1 & \ldots & 0 \end{pmatrix}.$$

Further we have

$$\mathbf{F}_N^{-1} = \frac{1}{N}\,\mathbf{J}'_N\,\mathbf{F}_N = \frac{1}{N}\,\mathbf{F}_N\,\mathbf{J}'_N\,. \tag{105}$$

Proof: Let $\mathbf{F}_N^2 = \big(c_{j,\ell}\big)_{j,\ell=0}^{N-1}$. Using Lemma 76, we find

$$c_{j,\ell} = \sum_{k=0}^{N-1} w_N^{jk}\, w_N^{k\ell} = \sum_{k=0}^{N-1} w_N^{(j+\ell)k} = N\, \delta_{(j+\ell) \bmod N} \cdot$$

and hence $\mathbf{F}_N^2 = N\,\mathbf{J}'_N$. From $(\mathbf{J}'_N)^2 = \mathbf{I}_N$ it follows that

$$\mathbf{F}_N^4 = \mathbf{F}_N^2\,\mathbf{F}_N^2 = (N\,\mathbf{J}'_N)\,(N\,\mathbf{J}'_N) = N^2\,(\mathbf{J}'_N)^2 = N^2\,\mathbf{I}_N\,.$$

By $N\,\mathbf{F}_N\,\mathbf{J}'_N = N\,\mathbf{J}'_N\,\mathbf{F}_N = \mathbf{F}_N^3$ and $\mathbf{F}_N^4 = N^2\,\mathbf{I}_N$ we finally obtain

$$\mathbf{F}_N^{-1} = \frac{1}{N^2}\,\mathbf{F}_N^3 = \frac{1}{N}\,\mathbf{F}_N\,\mathbf{J}'_N = \frac{1}{N}\,\mathbf{J}'_N\,\mathbf{F}_N\,.$$

This completes the proof. ■

Using (105), the inverse DFT(N) can be computed by the *same* algorithm as the DFT(N) employing a reordering and a scaling.

Remark 90

The application of the flip matrix $\mathbf{J}'_N$ *to a vector* $\mathbf{a} = (a_k)_{k=0}^{N-1}$ *provides the vector*

$$\mathbf{J}'_N\,\mathbf{a} = \big(a_{(-j) \bmod N}\big)_{j=0}^{N-1} = (a_0,\, a_{N-1},\, \ldots,\, a_1)^\top ,$$

i.e., the components of $\mathbf{a}$ *are "flipped".* $\square$

Now we want to study the spectral properties of the Fourier matrix in a more detailed manner. For that purpose, let the *counter-identity matrix* $\mathbf{J}_N$ be defined by

$$\mathbf{J}_N := \big(\delta_{(j+k+1) \bmod N}\big)_{j,k=0}^{N-1} = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$$

having nonzero entries 1 only on the main counter-diagonal. Then $\mathbf{J}_N\mathbf{a}$ provides the reversed vector

$$\mathbf{J}_N\mathbf{a} = \big(a_{(-j-1) \bmod N}\big)_{j=0}^{N-1} = (a_{N-1},\, a_{N-2},\, \ldots,\, a_1,\, a_0)^\top .$$

DFT and cyclic convolutions

The *cyclic convolution* of the vectors $\mathbf{a} = (a_k)_{k=0}^{N-1}$, $\mathbf{b} = (b_k)_{k=0}^{N-1} \in \mathbb{C}^N$ is defined as the vector $\mathbf{c} = (c_n)_{n=0}^{N-1} := \mathbf{a} * \mathbf{b} \in \mathbb{C}^N$ with the components

$$c_n = \sum_{k=0}^{N-1} a_k \, b_{(n-k) \bmod N} = \sum_{k=0}^{n} a_k \, b_{n-k} + \sum_{k=n+1}^{N-1} a_k \, b_{N+n-k} \,, \quad n = 0, \ldots, N-1 \,.$$

The cyclic convolution in $\mathbb{C}^N$ is a commutative, associative, and distributive operation with the unity $\mathbf{b}_0 = \big(\delta_{j \bmod N}\big)_{j=0}^{N-1} = (1, 0, \ldots, 0)^\top$ which is the so-called *pulse vector*. The *forward-shift matrix* $\mathbf{V}_N$ is defined by

$$\mathbf{V}_N := \big(\delta_{(j-k-1) \bmod N}\big)_{j,k=0}^{N-1} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix} .$$

The application of $\mathbf{V}_N$ to a vector $\mathbf{a} = (a_k)_{k=0}^{N-1}$ provides the forward-shifted vector

$$\mathbf{V}_N\mathbf{a} = \big(a_{(j-1) \bmod N}\big)_{j=0}^{N-1} = (a_{N-1},\, a_0,\, a_1,\, \ldots,\, a_{N-2})^\top .$$

Hence we obtain

$$\mathbf{V}_N^2 := \big(\delta_{(j-k-2) \bmod N}\big)_{j,k=0}^{N-1} = \begin{pmatrix} 0 & \ldots & 0 & 1 & 0 \\ 0 & \ldots & 0 & 0 & 1 \\ 1 & \ldots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \ldots & 1 & 0 & 0 \end{pmatrix}$$

and

$$\mathbf{V}_N^2\mathbf{a} = \big(a_{(j-2) \bmod N}\big)_{j=0}^{N-1} = (a_{N-2},\, a_{N-1},\, a_0,\, \ldots,\, a_{N-3})^\top .$$

Further we have $\mathbf{V}_N^N = \mathbf{I}_N$ and

$$\mathbf{V}_N^\top = \mathbf{V}_N^{-1} = \mathbf{V}_N^{N-1} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix},$$

which is called *backward-shift matrix*, since

$$\mathbf{V}_N^{-1}\mathbf{a} = \big(a_{(j+1) \bmod N}\big)_{j=0}^{N-1} = (a_1,\, a_2,\, \ldots,\, a_{N-1},\, a_0)^\top .$$

is the backward-shifted vector of $\mathbf{a}$.
The matrix $\mathbf{I}_N - \mathbf{V}_N$ is the *cyclic difference matrix*, since

$$\begin{aligned} (\mathbf{I}_N - \mathbf{V}_N)\,\mathbf{a} &= \big(a_j - a_{(j-1) \bmod N}\big)_{j=0}^{N-1} \\ &= (a_0 - a_{N-1},\, a_1 - a_0,\, \ldots, a_{N-1} - a_{N-2})^\top . \end{aligned}$$

We observe that

$$\mathbf{I}_N + \mathbf{V}_N + \mathbf{V}_N^2 + \ldots + \mathbf{V}_N^{N-1} = \big(1\big)_{j,k=0}^{N-1} .$$

We want to characterize all linear maps $\mathbf{H}_N$ from $\mathbb{C}^N$ to $\mathbb{C}^N$ which are *shift-invariant*, i.e., satisfy

$$\mathbf{H}_N(\mathbf{V}_N\mathbf{a}) = \mathbf{V}_N(\mathbf{H}_N\mathbf{a})$$

for all $\mathbf{a} \in \mathbf{C}^N$. Thus we have $\mathbf{H}_N\,\mathbf{V}_N^k = \mathbf{V}_N^k\,\mathbf{H}_N$, $k = 0, \ldots, N-1$. Shift-invariant maps play an important role for signal filtering. We show that a shift-invariant map $\mathbf{H}_N$ can be represented by the cyclic convolution.

Lemma 91

Each shift-invariant, linear map $\mathbf{H}_N$ from $\mathbb{C}^N$ to $\mathbb{C}^N$ can be represented in the form

$$\mathbf{H}_N\,\mathbf{a} = \mathbf{a} * \mathbf{h}\,, \quad \mathbf{a} \in \mathbb{C}^N\,,$$

where $\mathbf{h} := \mathbf{H}_N\,\mathbf{b}_0$ is the impulse response vector of the pulse vector $\mathbf{b}_0$.

Proof: Let $\mathbf{b}_k := \big(\delta_{(j-k) \bmod N}\big)_{j=0}^{N-1}$, $k = 0, \ldots, N-1$, be the standard basis vectors of $\mathbb{C}^N$. Then $\mathbf{b}_k = \mathbf{V}_N^k\, \mathbf{b}_0$, $k = 0, \ldots, N-1$. An arbitrary vector $\mathbf{a} = (a_k)_{k=0}^{N-1} \in \mathbb{C}^N$ can be represented in the standard basis as

$$\mathbf{a} = \sum_{k=0}^{N-1} a_k\, \mathbf{b}_k = \sum_{k=0}^{N-1} a_k\, \mathbf{V}_N^k\, \mathbf{b}_0\,.$$

Applying the linear, shift-invariant map $\mathbf{H}_N$ to this vector $\mathbf{a}$, we get

$$\begin{aligned}
\mathbf{H}_N\, \mathbf{a} &= \sum_{k=0}^{N-1} a_k\, \mathbf{H}_N(\mathbf{V}_N^k\, \mathbf{b}_0) = \sum_{k=0}^{N-1} a_k\, \mathbf{V}_N^k\, (\mathbf{H}_N\, \mathbf{b}_0) = \sum_{k=0}^{N-1} a_k\, \mathbf{V}_N^k\, \mathbf{h} \\
&= \big(\mathbf{h} \,|\, \mathbf{V}_N\, \mathbf{h} \,|\, \ldots \,|\, \mathbf{V}_N^{N-1}\, \mathbf{h}\big)\, \mathbf{a}
\end{aligned}$$

that means

$$
\begin{aligned}
\mathbf{H}_N\,\mathbf{a} &= \begin{pmatrix} h_0 & h_{N-1} & \dots & h_2 & h_1 \\ h_1 & h_0 & \dots & h_3 & h_2 \\ \vdots & \vdots & & \vdots & \vdots \\ h_{N-1} & h_{N-2} & \dots & h_1 & h_0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{pmatrix} \\
&= \left(\sum_{k=0}^{N-1} a_k\, h_{(n-k) \bmod N} \right)_{n=0}^{N-1} = \mathbf{a} * \mathbf{h}\,.
\end{aligned}
$$

This completes the proof. ■

Now we present the basic properties of DFT(N) and start with an example.

Example 92

Let $\mathbf{b}_k = \big(\delta_{(j-k) \bmod N}\big)_{j=0}^{N-1}$, $k = 0, \ldots, N-1$, be the standard basis vectors of $\mathbb{C}^N$ and let $\mathbf{e}_k = \big(w_N^{jk}\big)_{j=0}^{N-1}$, $k = 0, \ldots, N-1$, be the exponential vectors in Lemma 82 that form the columns of $\mathbf{F}_N$. Then we obtain for $k = 0, \ldots, N-1$ that

$$\mathbf{F}_N \, \mathbf{b}_k = \mathbf{e}_k \,, \quad \mathbf{F}_N \, \mathbf{e}_k = \mathbf{F}_N^2 \, \mathbf{b}_k = N \, \mathbf{J}_N' \, \mathbf{b}_k = N \, \mathbf{b}_{(-k) \bmod N} \,.$$

In particular we observe that the sparse vectors $\mathbf{b}_k$ are transformed into non-sparse vectors $\mathbf{e}_k$.

Example 92 (continue)

Further we obtain that for all $k = 0, \dots, N-1$

$$\mathbf{F}_N \, \mathbf{V}_N \, \mathbf{b}_k = \mathbf{F}_N \, \mathbf{b}_{(k+1) \bmod N} = \mathbf{e}_{(k+1) \bmod N} = \mathbf{M}_N \, \mathbf{F}_N \, \mathbf{b}_k \,,$$

where $\mathbf{M}_N := \operatorname{diag} \mathbf{e}_1$ is the so-called *modulation matrix* which generates a modulation or frequency shift by the property $\mathbf{M}_N \, \mathbf{e}_k = \mathbf{e}_{(k+1) \bmod N}$. Consequently, we have

$$\mathbf{F}_N \, \mathbf{V}_N = \mathbf{M}_N \, \mathbf{F}_N \tag{106}$$

and more generally $\mathbf{F}_N \, \mathbf{V}_N^k = \mathbf{M}_N^k \, \mathbf{F}_N$, $k = 1, \dots, N-1$. Transposing the last equation for $k = N-1$, we obtain

$$\mathbf{V}_N^\top \, \mathbf{F}_N = \mathbf{V}_N^{-1} \, \mathbf{F}_N = \mathbf{F}_N \, \mathbf{M}_N, \quad \mathbf{V}_N \, \mathbf{F}_N = \mathbf{F}_N \, \mathbf{M}_N^{-1} \,. \quad \square \tag{107}$$

Theorem 93

The $\mathrm{DFT}(N)$ *possesses the following properties.*

1. *Linearity: For all* $\mathbf{a}$, $\mathbf{b} \in \mathbb{C}^N$ *and* $\alpha \in \mathbb{C}$ *we have*

$$(\mathbf{a} + \mathbf{b})\hat{} = \hat{\mathbf{a}} + \hat{\mathbf{b}}\,, \quad (\alpha\,\mathbf{a})\hat{} = \alpha\,\hat{\mathbf{a}}\,.$$

2. *Inversion: For all* $\mathbf{a} \in \mathbb{C}^N$ *we have*

$$\mathbf{a} = \mathbf{F}_N^{-1}\,\hat{\mathbf{a}} = \frac{1}{N}\,\overline{\mathbf{F}}_N\,\hat{\mathbf{a}} = \frac{1}{N}\,\mathbf{J}_N'\,\mathbf{F}_N\,\hat{\mathbf{a}}\,.$$

3. *Flipping property: For all* $\mathbf{a} \in \mathbb{C}^N$ *we have*

$$(\mathbf{J}_N'\,\mathbf{a})\hat{} = \mathbf{J}_N'\,\hat{\mathbf{a}}\,, \quad (\overline{\mathbf{a}})\hat{} = \mathbf{J}_N'\,\overline{\hat{\mathbf{a}}}\,.$$

4. *Shifting in time and frequency domain: For all* $\mathbf{a} \in \mathbb{C}^N$ *we have*

$$(\mathbf{V}_N\,\mathbf{a})\hat{} = \mathbf{M}_N\,\hat{\mathbf{a}}\,, \quad (\mathbf{M}_N^{-1}\,\mathbf{a})\hat{} = \mathbf{V}_N\,\hat{\mathbf{a}}\,.$$

Theorem 93 (continue)

5. *Cyclic convolution in time and frequency domain: For all* $\mathbf{a}, \mathbf{b} \in \mathbb{C}^N$ *we have*

$$(\mathbf{a} * \mathbf{b})\hat{} = \hat{\mathbf{a}} \circ \hat{\mathbf{b}}\,, \quad N\,(\mathbf{a} \circ \mathbf{b})\hat{} = \hat{\mathbf{a}} * \hat{\mathbf{b}}\,,$$

   where $\mathbf{a} \circ \mathbf{b} := (a_k\, b_k)_{k=0}^{N-1}$ *denotes the componentwise product of the vectors* $\mathbf{a} = (a_k)_{k=0}^{N-1}$ *and* $\mathbf{b} = (b_k)_{k=0}^{N-1}$.

6. *Parseval identity: For all* $\mathbf{a}, \mathbf{b} \in \mathbb{C}^N$ *we have*

$$\frac{1}{N}\,\langle \hat{\mathbf{a}}, \hat{\mathbf{b}} \rangle = \langle \mathbf{a}, \mathbf{b} \rangle\,, \quad \frac{1}{N}\,\|\hat{\mathbf{a}}\|_2^2 = \|\mathbf{a}\|_2^2\,.$$

7. *Difference property in time and frequency domain: For all* $\mathbf{a} \in \mathbb{C}^N$ *we have*

$$\big((\mathbf{I}_N - \mathbf{V}_N)\,\mathbf{a}\big)\hat{} = (\mathbf{I}_N - \mathbf{M}_N)\,\hat{\mathbf{a}}\,, \quad \big((\mathbf{I}_N - \mathbf{M}_N^{-1})\,\mathbf{a}\big)\hat{} = (\mathbf{I}_N - \mathbf{V}_N)\,\hat{\mathbf{a}}\,.$$

Theorem 93 (continue)

8. *Permutation property: Let $p \in \mathbb{Z}$ and N be relatively prime. Assume that $q \in \mathbb{Z}$ satisfies the condition $(p\,q) \bmod N = 1$ and that the $\mathrm{DFT}(N)$ of $(a_j)_{j=0}^{N-1} \in \mathbb{C}^N$ is equal to $(\hat{a}_k)_{k=0}^{N-1}$. Then the $\mathrm{DFT}(N)$ of the permuted vector $\left(a_{(pj) \bmod N}\right)_{j=0}^{N-1}$ is equal to the permuted vector $\left(\hat{a}_{(qk) \bmod N}\right)_{k=0}^{N-1}$.*

Proof: 1. The linearity follows from the definition of the DFT(N).
2. The second property is obtained from (102) and (105).
3. By (102) and (105) we have $\mathbf{F}_N\,\mathbf{J}'_N = \mathbf{J}'_N\,\mathbf{F}_N = \overline{\mathbf{F}}_N$ and hence

$$\begin{aligned}(\mathbf{J}'_N\mathbf{a})\hat{} \quad &= \quad \mathbf{F}_N\,\mathbf{J}'_N\,\mathbf{a} = \mathbf{J}'_N\,\mathbf{F}_N\,\mathbf{a} = \mathbf{J}'_N\,\hat{\mathbf{a}}\,,\\ (\overline{\mathbf{a}})\hat{} \quad &= \quad \mathbf{F}_N\,\overline{\mathbf{a}} = \overline{\overline{\mathbf{F}}_N\,\mathbf{a}} = \overline{\mathbf{J}'_N\,\mathbf{F}_N\,\mathbf{a}} = \mathbf{J}'_N\,\overline{\hat{\mathbf{a}}}\,.\end{aligned}$$

4. From (106) and (107) it follows that

$$\begin{aligned}(\mathbf{V}_N\,\mathbf{a})^\wedge &= \mathbf{F}_N\,\mathbf{V}_N\,\mathbf{a} = \mathbf{M}_N\,\mathbf{F}_N\,\mathbf{a} = \mathbf{M}_N\,\hat{\mathbf{a}}\,,\\ (\mathbf{M}_N^{-1}\,\mathbf{a})^\wedge &= \mathbf{F}_N\,\mathbf{M}_N^{-1}\,\mathbf{a} = \mathbf{V}_N\,\mathbf{F}_N\,\mathbf{a} = \mathbf{V}_N\,\hat{\mathbf{a}}\,.\end{aligned}$$

5. Let $\mathbf{c} = \mathbf{a} * \mathbf{b}$ be the cyclic convolution of $\mathbf{a}$ and $\mathbf{b}$ with the components

$$c_j = \sum_{n=0}^{N-1} a_n\, b_{(j-n)\bmod N}\,, \quad j = 0, \ldots, N-1\,.$$

We calculate the components of $\hat{\mathbf{c}} = \left(\hat{c}_k\right)_{k=0}^{N-1}$ and obtain for $k = 0, \ldots, N-1$

$$\begin{aligned}
\hat{c}_k &= \sum_{j=0}^{N-1} \Big(\sum_{n=0}^{N-1} a_n \, b_{(j-n) \bmod N} \Big) \, w_N^{jk} \\
&= \sum_{n=0}^{N-1} a_n \, w_N^{nk} \Big(\sum_{j=0}^{N-1} b_{(j-n) \bmod N} \, w_N^{((j-n) \bmod N)\, k} \Big) \\
&= \Big(\sum_{n=0}^{N-1} a_n \, w_N^{nk} \Big) \, \hat{b}_k = \hat{a}_k \, \hat{b}_k \, .
\end{aligned}$$

Now let $\mathbf{c} = \mathbf{a} \circ \mathbf{b} = (a_j \, b_j)_{j=0}^{N-1}$. Using the second property, we get

$$a_j = \frac{1}{N} \sum_{k=0}^{N-1} \hat{a}_k \, w_N^{-jk} \, , \quad b_j = \frac{1}{N} \sum_{\ell=0}^{N-1} \hat{b}_\ell \, w_N^{-j\ell} \, , \quad j = 0, \ldots, N-1 \, .$$

Thus we obtain that for $j = 0, \ldots, N-1$

$$\begin{aligned} c_j &= a_j\, b_j = \frac{1}{N^2} \Big(\sum_{k=0}^{N-1} \hat{a}_k\, w_N^{-jk} \Big) \Big(\sum_{\ell=0}^{N-1} \hat{b}_\ell\, w_N^{-j\ell} \Big) \\ &= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} \hat{a}_k\, \hat{b}_\ell\, w_N^{-j(k+\ell)} \\ &= \frac{1}{N^2} \sum_{n=0}^{N-1} \Big(\sum_{k=0}^{N-1} \hat{a}_k\, \hat{b}_{(n-k) \bmod N} \Big)\, w_N^{-jn}\,, \end{aligned}$$

i.e., $\mathbf{c} = \frac{1}{N}\, \mathbf{F}_N^{-1}(\hat{\mathbf{a}} * \hat{\mathbf{b}})$ and hence $N\, \hat{\mathbf{c}} = \hat{\mathbf{a}} * \hat{\mathbf{b}}$.

6. For arbitrary $\mathbf{a}$, $\mathbf{b} \in \mathbb{C}^N$ we conclude

$$\langle \hat{\mathbf{a}},\, \hat{\mathbf{b}} \rangle = \mathbf{a}^\top\, \mathbf{F}_N\, \overline{\mathbf{F}}_N\, \overline{\mathbf{b}} = N\, \mathbf{a}^\top\, \overline{\mathbf{b}} = N\, \langle \mathbf{a},\, \mathbf{b} \rangle\,.$$

7. The difference properties follow directly from the shift properties.

8. Since $p \in \mathbb{Z}$ and N are relatively prime, the greatest common divisor of p and N is one. Then there exist q, $M \in \mathbb{Z}$ with $p\,q + M\,N = 1$ (see [1, p. 21]). By the Euler–Fermat theorem (see [1, p. 114]) the (unique modulo N) solution of the linear congruence $p\,q \equiv 1 \pmod N$ is given by $q \equiv p^{\varphi(N)-1} \pmod N$, where $\varphi(N)$ denotes the Euler totient function.
Now we compute the DFT(N) of the permuted vector $\big(a_{(pj) \bmod N}\big)_{j=0}^{N-1}$. Then the kth component of the transformed vector reads

$$\sum_{j=0}^{N-1} a_{(pj) \bmod N}\, w_N^{jk}\,. \tag{108}$$

The value (108) does not change if the sum is reordered and the summation index $j = 0, \ldots, N-1$ is replaced by $(q\,\ell) \bmod N$ with $\ell = 0, \ldots, N-1$.

Indeed, by $p\,q \equiv 1 \pmod N$ and (79) we have

$$\ell = (p\,q\,\ell) \bmod N = \Big[\big((q\,\ell) \bmod N\big)\,p\Big] \bmod N$$

and furthermore

$$w_N^{\big((q\,\ell) \bmod N\big)\,k} = w_N^{q\,\ell\,k} = w_N^{\ell\,\big((q\,k) \bmod N\big)}\,.$$

Thus we obtain

$$\begin{aligned}
\sum_{j=0}^{N-1} a_{(pj) \bmod N}\, w_N^{jk} &= \sum_{\ell=0}^{N-1} a_{(p\,q\,\ell) \bmod N}\, w_N^{q\,\ell\,k} \\
&= \sum_{j=0}^{N-1} a_\ell\, w_N^{\ell\,\big((q\,k) \bmod N\big)} = \hat{a}_{(qk) \bmod N}\,.
\end{aligned}$$

For example, in the special case $p = q = -1$, the flipped vector $\big(a_{(-j) \bmod N}\big)_{j=0}^{N-1}$ is transformed to the flipped vector $\big(\hat{a}_{(-k) \bmod N}\big)_{k=0}^{N-1}$. ■

Now we analyze the symmetry properties of DFT(N). A vector $\mathbf{a} = (a_j)_{j=0}^{N-1} \in \mathbb{C}^N$ is called *even*, if $\mathbf{a} = \mathbf{J}'_N\,\mathbf{a}$, i.e. $a_j = a_{(N-j) \bmod N}$ for all $j = 0, \ldots, N-1$, and it is called *odd*, if $\mathbf{a} = -\mathbf{J}'_N\,\mathbf{a}$, i.e. $a_j = -a_{(N-j) \bmod N}$ for all $j = 0, \ldots, N-1$. For $N = 6$ the vector $(a_0,\, a_1,\, a_2,\, a_3,\, a_2,\, a_1)^\top$ is even and $(0,\, a_1,\, a_2,\, 0,\, -a_2,\, -a_1)^\top$ is odd.

Corollary 94

For $\mathbf{a} \in \mathbb{R}^N$ *and* $\hat{\mathbf{a}} = \mathbf{F}_N \mathbf{a} = \left(\hat{a}_j\right)_{j=0}^{N-1}$ *we have*

$$\overline{\hat{\mathbf{a}}} = \mathbf{J}'_N\,\hat{\mathbf{a}}\,,$$

i.e., $\overline{\hat{a}}_j = \hat{a}_{(N-j) \bmod N}$, $j = 0, \ldots, N-1$. *In other words,* $\mathrm{Re}\;\hat{\mathbf{a}}$ *is even and* $\mathrm{Im}\;\hat{\mathbf{a}}$ *is odd.*

Proof: By $\mathbf{a} = \overline{\mathbf{a}} \in \mathbb{R}^N$ and $\overline{\mathbf{F}}_N = \mathbf{J}'_N \, \mathbf{F}_N$ it follows that

$$\mathbf{J}'_N \, \hat{\mathbf{a}} = \mathbf{J}'_N \, \mathbf{F}_N \, \mathbf{a} = \overline{\mathbf{F}}_N \, \mathbf{a} = \overline{\mathbf{F}}_N \, \overline{\mathbf{a}} = \overline{\hat{\mathbf{a}}} \, .$$

For $\hat{\mathbf{a}} = \mathrm{Re}\ \hat{\mathbf{a}} + \mathrm{i}\ \mathrm{Im}\ \hat{\mathbf{a}}$ we obtain

$$\overline{\hat{\mathbf{a}}} = \mathrm{Re}\ \hat{\mathbf{a}} - \mathrm{i}\ \mathrm{Im}\ \hat{\mathbf{a}} = \mathbf{J}'_N \, \hat{\mathbf{a}} = \mathbf{J}'_N \, (\mathrm{Re}\ \hat{\mathbf{a}}) + \mathrm{i} \, \mathbf{J}'_N \, (\mathrm{Im}\ \hat{\mathbf{a}})$$

and hence $\mathrm{Re}\ \hat{\mathbf{a}} = \mathbf{J}'_N \, (\mathrm{Re}\ \hat{\mathbf{a}})$ and $\mathrm{Im}\ \hat{\mathbf{a}} = -\mathbf{J}'_N \, (\mathrm{Im}\ \hat{\mathbf{a}})$. ■

Corollary 95

If $\mathbf{a} \in \mathbb{C}^N$ *is even/odd, then* $\hat{\mathbf{a}} = \mathbf{F}_N \mathbf{a}$ *is even/odd.*
If $\mathbf{a} \in \mathbb{R}^N$ *is even, then* $\hat{\mathbf{a}} = \mathrm{Re}\, \hat{\mathbf{a}} \in \mathbb{R}^N$ *is even.*
If $\mathbf{a} \in \mathbb{R}^N$ *is odd, then* $\hat{\mathbf{a}} = \mathrm{i}\, \mathrm{Im}\, \hat{\mathbf{a}} \in \mathrm{i}\,\mathbb{R}^N$ *is odd.*

Proof: From $\mathbf{a} = \pm \mathbf{J}'_N\, \mathbf{a}$ it follows that

$$\hat{\mathbf{a}} = \mathbf{F}_N\, \mathbf{a} = \pm \mathbf{F}_N\, \mathbf{J}'_N\, \mathbf{a} = \pm \mathbf{J}'_N\, \mathbf{F}_N\, \mathbf{a} = \pm \mathbf{J}'_N\, \hat{\mathbf{a}}\,.$$

For even $\mathbf{a} \in \mathbb{R}^N$ we obtain by Corollary 94 that $\overline{\hat{\mathbf{a}}} = \mathbf{J}'_N\, \hat{\mathbf{a}} = \hat{\mathbf{a}}$, i.e., $\hat{\mathbf{a}} \in \mathbb{R}^N$ is even. Analogously we can show the assertion for odd $\mathbf{a} \in \mathbb{R}^N$. ■

Circulant matrices

An N-by-N matrix

$$\operatorname{circ} \mathbf{a} := \left(a_{(j-k) \bmod N}\right)_{j,k=0}^{N-1} = \begin{pmatrix} a_0 & a_{N-1} & \dots & a_2 & a_1 \\ a_1 & a_0 & \dots & a_3 & a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{N-1} & a_{N-2} & \dots & a_1 & a_0 \end{pmatrix} \tag{109}$$

is called *circulant matrix* generated by $\mathbf{a} = (a_k)_{k=0}^{N-1} \in \mathbb{C}^N$. The first column of circ $\mathbf{a}$ is equal to $\mathbf{a}$. A circulant matrix is a special Toeplitz matrix in which the diagonals wrap around. Remember that a *Toeplitz matrix* is a structured matrix $\left(a_{j-k}\right)_{j,k=0}^{N-1}$ for given $\left(a_k\right)_{k=1-N}^{N-1} \in \mathbb{C}^{2N-1}$ such that the entries along each diagonal are constant.

Example 96

If $\mathbf{b}_k = (\delta_{j-k})_{j=0}^{N-1}$, $k = 0, \ldots, N-1$, denote the standard basis vectors of $\mathbb{C}^N$, then the forward shift matrix $\mathbf{V}_N$ is a circulant matrix, since $\mathbf{V}_N = \operatorname{circ} \mathbf{b}_1$. More generally, we obtain that

$$\mathbf{V}_N^k = \operatorname{circ} \mathbf{b}_k \,, \quad k = 0, \ldots, N-1 \,.$$

with $\mathbf{V}_N^0 = \operatorname{circ} \mathbf{b}_0 = \mathbf{I}_N$ and $\mathbf{V}_N^{N-1} = \mathbf{V}_N^{-1} = \operatorname{circ} \mathbf{b}_{N-1}$. The cyclic difference matrix is also a circulant matrix, since $\mathbf{I}_N - \mathbf{V}_N = \operatorname{circ}(\mathbf{b}_0 - \mathbf{b}_1)$. $\square$

Circulant matrices and cyclic convolutions of vectors in $\mathbb{C}^N$ are closely related. From Lemma 91 it follows that for arbitrary vectors $\mathbf{a}, \mathbf{b} \in \mathbb{C}^N$

$$(\operatorname{circ} \mathbf{a})\, \mathbf{b} = \mathbf{a} * \mathbf{b}\,.$$

Using the cyclic convolution property of DFT(N) (see property 5 of Theorem 93), we obtain that a circulant matrix can be diagonalized by Fourier matrices.

Theorem 97

For each $\mathbf{a} \in \mathbb{C}^N$*, the circulant matrix* circ $\mathbf{a}$ *can be diagonalized by the Fourier matrix* $\mathbf{F}_N$*. We have*

$$\operatorname{circ} \mathbf{a} = \mathbf{F}_N^{-1} \left(\operatorname{diag} (\mathbf{F}_N \mathbf{a}) \right) \mathbf{F}_N\,. \tag{110}$$

Proof: For any $\mathbf{b} \in \mathbb{C}^N$ we form the cyclic convolution of $\mathbf{a}$ and $\mathbf{b}$. Then by the convolution property of Theorem 93 we obtain that

$$\mathbf{F}_N\,\mathbf{c} = (\mathbf{F}_N\,\mathbf{a}) \circ (\mathbf{F}_N\,\mathbf{b}) = \big(\operatorname{diag}(\mathbf{F}_N\,\mathbf{a})\big)\,\mathbf{F}_N\,\mathbf{b}\,.$$

and hence

$$\mathbf{c} = \mathbf{F}_N^{-1}\,\big(\operatorname{diag}(\mathbf{F}_N\,\mathbf{a})\big)\,\mathbf{F}_N\,\mathbf{b}\,.$$

On the other hand we have $\mathbf{c} = (\operatorname{circ}\,\mathbf{a})\,\mathbf{b}$ such that for all $\mathbf{b} \in \mathbb{C}^N$

$$(\operatorname{circ}\,\mathbf{a})\,\mathbf{b} = \mathbf{F}_N^{-1}\,\big(\operatorname{diag}(\mathbf{F}_N\,\mathbf{a})\big)\,\mathbf{F}_N\,\mathbf{b}\,.$$

This completes the proof of (110). $\blacksquare$

Remark 98

Using the decomposition (110), the matrix-vector product $(\operatorname{circ}\mathbf{a})\,\mathbf{b}$ *can be realized by employing three DFT(N) and one componentwise vector multiplication. We compute*

$$(\operatorname{circ}\mathbf{a})\,\mathbf{b} = \mathbf{F}_N^{-1}\left(\operatorname{diag}\left(\mathbf{F}_N\mathbf{a}\right)\right)\mathbf{F}_N\,\mathbf{b} = \mathbf{F}_N^{-1}\left(\operatorname{diag}\,\hat{\mathbf{a}}\right)\hat{\mathbf{b}} = \mathbf{F}_N^{-1}\left(\hat{\mathbf{a}}\circ\hat{\mathbf{b}}\right).$$

As we will see later, one DFT(N) can be realized by $\mathcal{O}(N\log N)$ *arithmetical operations.* □

Corollary 99

For arbitrary $\mathbf{a} \in \mathbb{C}^N$*, the eigenvalues of* $\operatorname{circ} \mathbf{a}$ *coincide with the components* $\hat{a}_j$*,* $j = 0, \ldots, N-1$*, of* $(\hat{a}_j)_{j=0}^{N-1} = \mathbf{F}_N \, \mathbf{a}$*. A right eigenvector related to the eigenvalue* $\hat{a}_j$*,* $j = 0, \ldots, N-1$*, is the complex conjugate exponential vector* $\overline{\mathbf{e}}_j = (w_N^{-jk})_{k=0}^{N-1}$ *and a left eigenvector of* $\hat{a}_j$ *is* $\mathbf{e}_j^\top$*, i.e.,*

$$(\operatorname{circ} \mathbf{a})\, \overline{\mathbf{e}}_j = \hat{a}_j \, \overline{\mathbf{e}}_j \,, \quad \mathbf{e}_j^\top \, (\operatorname{circ} \mathbf{a}) = \hat{a}_j \, \mathbf{e}_j^\top \,. \tag{111}$$

Proof: Using (110), we obtain that

$$\big(\operatorname{circ}\mathbf{a}\big)\,\mathbf{F}_N^{-1} = \mathbf{F}_N^{-1}\,\operatorname{diag}\,(\hat{a}_j)_{j=0}^{N-1}\,, \quad \mathbf{F}_N\,\operatorname{circ}\mathbf{a} = \big(\operatorname{diag}\,(\hat{a}_j)_{j=0}^{N-1}\big)\,\mathbf{F}_N$$

with

$$\mathbf{F}_N = \big(\mathbf{e}_0\,|\,\mathbf{e}_1\,|\,\ldots\,|\,\mathbf{e}_{N-1}\big) = \begin{pmatrix} \mathbf{e}_0^\top \\ \mathbf{e}_1^\top \\ \vdots \\ \mathbf{e}_{N-1}^\top \end{pmatrix}, \tag{112}$$

$$\mathbf{F}_N^{-1} = \frac{1}{N}\big(\overline{\mathbf{e}}_0\,|\,\overline{\mathbf{e}}_1\,|\,\ldots\,|\,\overline{\mathbf{e}}_{N-1}\big) = \frac{1}{N}\begin{pmatrix} \overline{\mathbf{e}}_0^\top \\ \overline{\mathbf{e}}_1^\top \\ \vdots \\ \overline{\mathbf{e}}_{N-1}^\top \end{pmatrix}.$$

Hence we conclude (111) holds. Note that the eigenvalues $\hat{a}_j$ of circ **a** need not be distinct. ■

By the definition of the forward-shift matrix $\mathbf{V}_N$, each circulant matrix (109) can be written in the form

$$\operatorname{circ} \mathbf{a} = \sum_{k=0}^{N-1} a_k \, \mathbf{V}_N^k \,, \tag{113}$$

where $\mathbf{V}_N^0 = \mathbf{V}_N^N = \mathbf{I}_N$. Therefore, $\mathbf{V}_N$ is called *basic circulant matrix*.
The representation (113) reveals that N-by-N circulant matrices form a commutative algebra. Linear combinations and products of circulant matrices are also circulant matrices, and products of any two circulant matrices commute. The inverse of a nonsingular circulant matrix is again a circulant matrix. The following result is very useful for the computation with circulant matrices:

Theorem 100

For arbitrary $\mathbf{a}$, $\mathbf{b} \in \mathbb{C}^N$ *and* $\alpha \in \mathbb{C}$ *we have*

1. $(\operatorname{circ} \mathbf{a})^\top = \operatorname{circ}(\mathbf{J}'_N \, \mathbf{a})$,
2. $(\operatorname{circ} \mathbf{a}) + (\operatorname{circ} \mathbf{b}) = \operatorname{circ}(\mathbf{a} + \mathbf{b})$, $\quad \alpha (\operatorname{circ} \mathbf{a}) = \operatorname{circ}(\alpha \, \mathbf{a})$,
3. $(\operatorname{circ} \mathbf{a})(\operatorname{circ} \mathbf{b}) = (\operatorname{circ} \mathbf{b})(\operatorname{circ} \mathbf{a}) = \operatorname{circ}(\mathbf{a} * \mathbf{b})$,
4. $\operatorname{circ} \mathbf{a}$ *is a normal matrix with the spectral decomposition* (110),
5. $\det(\operatorname{circ} \mathbf{a}) = \prod_{j=0}^{N-1} \hat{a}_j$ *with* $\left(\hat{a}_j\right)_{j=0}^{N-1} = \mathbf{F}_N \, \mathbf{a}$.
6. *The Moore–Penrose pseudo-inverse of* $\operatorname{circ} \mathbf{a}$ *has the form*

$$(\operatorname{circ} \mathbf{a})^+ = \mathbf{F}_N^{-1} \left(\operatorname{diag}(\hat{a}_j^+)_{j=0}^{N-1}\right) \mathbf{F}_N ,$$

   where $\hat{a}_j^+ := \hat{a}_j^{-1}$ *if* $\hat{a}_j \neq 0$ *and* $\hat{a}_j^+ := 0$ *if* $\hat{a}_j = 0$.

7. $\operatorname{circ} \mathbf{a}$ *is invertible if and only if* $\hat{a}_j \neq 0$ *for all* $j = 0, \ldots, N-1$. *Under this condition,* $(\operatorname{circ} \mathbf{a})^{-1}$ *is the circulant matrix*

$$(\operatorname{circ} \mathbf{a})^{-1} = \mathbf{F}_N^{-1} \left(\operatorname{diag}(\hat{a}_j^{-1})_{j=0}^{N-1}\right) \mathbf{F}_N .$$

Proof: 1. Using $\mathbf{V}_N^\top = \mathbf{V}_N^{-1}$ and $\mathbf{V}_N^N = \mathbf{I}_N$, we obtain for $\mathbf{a} = (a_k)_{k=0}^{N-1} \in \mathbb{C}^N$ by (113) that

$$\begin{aligned}
(\mathrm{circ}\ \mathbf{a})^\top &= \sum_{k=0}^{N-1} a_k \left(\mathbf{V}_N^k\right)^\top = \sum_{k=0}^{N-1} a_k \left(\mathbf{V}_N^\top\right)^k \\
&= \sum_{k=0}^{N-1} a_k\, \mathbf{V}_N^{-k} = \sum_{k=0}^{N-1} a_k\, \mathbf{V}_N^{N-k} \\
&= a_0\, \mathbf{I}_N + a_{N-1}\, \mathbf{V}_N + \ldots + a_1\, \mathbf{V}_N^{N-1} = \mathrm{circ}\left(\mathbf{J}_N'\, \mathbf{a}\right).
\end{aligned}$$

2. The two relations follow from the definition (109).

3. Let $\mathbf{a} = (a_k)_{k=0}^{N-1}$, $\mathbf{b} = (b_\ell)_{\ell=0}^{N-1} \in \mathbb{C}^N$ be given. Using $\mathbf{V}_N^N = \mathbf{I}_N$, we conclude that by (113)

$$(\mathrm{circ}\,\mathbf{a})\,(\mathrm{circ}\,\mathbf{b}) = \Big(\sum_{k=0}^{N-1} a_k\,\mathbf{V}_N^k\Big)\,\Big(\sum_{\ell=0}^{N-1} b_\ell\,\mathbf{V}_N^\ell\Big) = \sum_{n=0}^{N-1} c_n\,\mathbf{V}_N^n$$

with the entries

$$c_n = \sum_{j=0}^{N-1} a_j\, b_{(n-j)\ \mathrm{mod}\ N}\,, \quad n = 0, \ldots, N-1\,.$$

By $(c_n)_{n=0}^{N-1} = \mathbf{a} * \mathbf{b}$ we obtain $(\mathrm{circ}\,\mathbf{a})\,(\mathrm{circ}\,\mathbf{b}) = \mathrm{circ}\,(\mathbf{a} * \mathbf{b})$. Since the cyclic convolution is commutative, the product of circulant matrices is also commutative.

4. By property 1, the conjugate transpose of circ $\mathbf{a}$ is again a circulant matrix. Since circulant matrices commute by property 3, circ $\mathbf{a}$ is a normal matrix. By (110) we obtain the spectral decomposition of the normal matrix

$$\operatorname{circ} \mathbf{a} = \frac{1}{\sqrt{N}} \overline{\mathbf{F}}_N \left(\operatorname{diag} (\mathbf{F}_N \mathbf{a}) \right) \frac{1}{\sqrt{N}} \mathbf{F}_N \,, \tag{114}$$

because $\frac{1}{\sqrt{N}} \mathbf{F}_N$ is unitary.

5. The determinant $\det(\operatorname{circ} \mathbf{a})$ of the matrix product (110) can be computed by

$$\det(\operatorname{circ} \mathbf{a}) = \left(\det \mathbf{F}_N \right)^{-1} \left(\prod_{j=0}^{N-1} \hat{a}_j \right) \det \mathbf{F}_N = \prod_{j=0}^{N-1} \hat{a}_j \,.$$

6. The *Moore–Penrose pseudo-inverse* $\mathbf{A}_N^+$ of an N-by-N matrix $\mathbf{A}_N$ is uniquely determined by the properties

$$\mathbf{A}_N \, \mathbf{A}_N^+ \, \mathbf{A}_N = \mathbf{A}_N \,, \quad \mathbf{A}_N^+ \, \mathbf{A}_N \, \mathbf{A}_N^+ = \mathbf{A}_N^+ \,,$$

where $\mathbf{A}_N \, \mathbf{A}_N^+$ and $\mathbf{A}_N^+ \, \mathbf{A}_N$ are Hermitian. From the spectral decomposition (114) of circ $\mathbf{a}$ it follows that

$$(\text{circ}\ \mathbf{a})^+ = \frac{1}{\sqrt{N}} \, \overline{\mathbf{F}}_N \left(\text{diag}\,(\hat{a}_j)_{j=0}^{N-1} \right)^+ \frac{1}{\sqrt{N}} \, \mathbf{F}_N = \mathbf{F}_N^{-1} \left(\text{diag}\,(\hat{a}_j^+)_{j=0}^{N-1} \right) \mathbf{F}_N \,.$$

7. The matrix circ $\mathbf{a}$ is invertible if and only if $\det(\text{circ}\ \mathbf{a}) \neq 0$, i.e., if $\hat{a}_j \neq 0$ for all $j = 0, \ldots, N-1$. In this case,

$$\mathbf{F}_N^{-1} \left(\text{diag}\,(\hat{a}_j^{-1})_{j=0}^{N-1} \right) \mathbf{F}_N$$

is the inverse of circ $\mathbf{a}$. $\blacksquare$

Circulant matrices can be characterized by the following property.

Lemma 101

An N-by-N matrix $\mathbf{A}_N$ is a circulant matrix if and only if $\mathbf{A}_N$ and the basic circulant matrix $\mathbf{V}_N$ commute, i.e.,

$$\mathbf{V}_N \, \mathbf{A}_N = \mathbf{A}_N \, \mathbf{V}_N \, . \tag{115}$$

Each circulant matrix circ $\mathbf{a}$ with $\mathbf{a} \in \mathbb{C}^N$ can be represented in the form (113). Hence circ $\mathbf{a}$ commutes with $\mathbf{V}_N$.
Let $\mathbf{A}_N = \big(a_{j,k}\big)_{j,k=0}^{N-1}$ be an arbitrary N-by-N matrix with the property (115) such that $\mathbf{V}_N\,\mathbf{A}_N\,\mathbf{V}_N^{-1} = \mathbf{A}_N$. From

$$\mathbf{V}_N\,\mathbf{A}_N\,\mathbf{V}_N^{-1} = \big(a_{(j-1)\bmod N,(k-1)\bmod N}\big)_{j,k=0}^{N-1}$$

it follows for all $j,\ k = 0, \ldots, N-1$

$$a_{(j-1)\bmod N,(k-1)\bmod N} = a_{j,k}\,.$$

Setting $a_j := a_{j,0}$ for $j = 0, \ldots, N-1$, we conclude that $a_{j,k} = a_{(j-k)\bmod N}$ for $j,\ k = 0, \ldots, N-1$, i.e., $\mathbf{A}_N = \mathrm{circ}\,(a_j)_{j=0}^{N-1}$. ■

Remark 102

For arbitrarily given $t_k \in \mathbb{C}$, $k = 1-N, \ldots, N-1$, we consider the N-by-N Toeplitz matrix

$$\mathbf{T}_N := \left(t_{j-k}\right)_{j,k=0}^{N-1} = \begin{pmatrix} t_0 & t_{-1} & \ldots & t_{2-N} & t_{1-N} \\ t_1 & t_0 & \ldots & t_{3-N} & t_{2-N} \\ \vdots & \vdots & & \vdots & \vdots \\ t_{N-1} & t_{N-2} & \ldots & t_1 & t_0 \end{pmatrix}.$$

In general, $\mathbf{T}_N$ is not a circulant matrix. But $\mathbf{T}_N$ can be extended to a circulant matrix $\mathbf{C}_{2N}$ of order $2N$. We define

$$\mathbf{C}_{2N} := \begin{pmatrix} \mathbf{T}_N & \mathbf{E}_N \\ \mathbf{E}_N & \mathbf{T}_N \end{pmatrix}$$

Remark 102 (continue)

with

$$\mathbf{E}_N := \begin{pmatrix} 0 & t_{N-1} & \ldots & t_2 & t_1 \\ t_{1-N} & 0 & \ldots & t_3 & t_2 \\ \vdots & \vdots & & \vdots & \vdots \\ t_{-1} & t_{-2} & \ldots & t_{1-N} & 0 \end{pmatrix}.$$

Then, $\mathbf{C}_{2N} = \mathrm{circ}\, \mathbf{c}$ *with the vector*

$$\mathbf{c} := (t_0,\, t_1,\, \ldots,\, t_{N-1},\, 0,\, t_{1-N}, \ldots,\, t_{-1})^\top \in \mathbb{C}^{2N}.$$

Thus for an arbitrary vector $\mathbf{a} \in \mathbf{C}^N$*, the matrix vector product* $\mathbf{T}_N\, \mathbf{a}$ *can be computed using the circulant matrix vector product*

$$\mathbf{C}_{2N} \begin{pmatrix} \mathbf{a} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{T}_N\, \mathbf{a} \\ \mathbf{E}_N\, \mathbf{a} \end{pmatrix},$$

where $\mathbf{0} \in \mathbb{C}^N$ *denotes the zero vector. Applying a fast Fourier transform of Chapter 4, this matrix-vector product can therefore be realized with only* $\mathcal{O}(N \log N)$ *arithmetical*

Summary: The 4 Fourier transforms

freq. \ time	continuous	discrete
continuous	Fourier transform	"semidiscrete" Fourier transform
discrete	Fourier series	discrete Fourier transform

Summary: The 4 Fourier transforms

Fourier transform on $\mathbb{R}$

forward:	$\hat{f}(v) = \int\limits_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}vx}\mathrm{d}x$
inverse:	$f(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \hat{f}(v)\mathrm{e}^{\mathrm{i}vx}\mathrm{d}x$
periodicity:	none

Summary: The 4 Fourier transforms

"semidiscrete" Fourier transform

forward: $\hat{f}(v) = \sum\limits_{j=-\infty}^{\infty} f(j)\mathrm{e}^{-\mathrm{i}vj}$

inverse: $f(j) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \hat{f}(v)\mathrm{e}^{\mathrm{i}vj}\mathrm{d}v$

periodicity: $\hat{f}(v) = \hat{f}(v + 2\pi)$

Summary: The 4 Fourier transforms

Fourier series

forward: $c_k(f) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} f(x) \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}x$

inverse: $f(x) = \sum\limits_{k=-\infty}^{\infty} c_k(f) \mathrm{e}^{\mathrm{i}kx} \mathrm{d}x$

periodicity: $f(x) = f(x + 2\pi)$

Summary: The 4 Fourier transforms

discrete Fourier transform (DFT)

forward: $\hat{f}_k = \sum_{j=0}^{N-1} f_j \mathrm{e}^{-2\pi \mathrm{i} jk/N}$

inverse: $f_j = \frac{1}{N} \sum_{k=0}^{N-1} \hat{f}_k \mathrm{e}^{2\pi \mathrm{i} jk/N}$

periodicity: $\hat{f}_k = \hat{f}_{k+rN}$; $f_j = f_{j+rN}$

Fast Fourier transforms

Any application of Fourier methods leads to the evaluation of a discrete Fourier transform of length N (DFT(N)). Thus the efficient computation of DFT(N) is very important. Therefore this chapter is devoted to fast Fourier transforms. A *fast Fourier transform* (FFT) is an algorithm for computing the DFT(N) which needs only a relatively low number of arithmetic operations.

Fast FFT's considerably reduce the computational cost for computing the DFT(N) from $2\,N^2$ to $\mathcal{O}(N \log N)$ arithmetic operations. We will study the numerical stability of the derived FFT. Note there exists no linear algorithm that can realize the DFT(N) with a smaller computational cost than $\mathcal{O}(N \log N)$ (see [13]). Faster algorithms can be only derived if some a priori information on the resulting vector are available.

Construction principles of fast algorithms

One of the main reasons for the great importance of Fourier methods is the existence of fast algorithms for its implementation of DFT. Nowadays, the FFT is one of the most well-known and mostly applied fast algorithms. Many applications in physics, engineering and signal processing were just not possible without FFT.
A frequently applied FFT is due to J.W. Cooley and J.W. Tuckey [3]. Indeed an earlier fast algorithm by I.J. Good [8] used for statistical computations did not find further attention. Early ideas for efficient computation of DFT(N) for $N = 12$ and $N = 36$ go even back to C.F. Gauss. In 1805, he derived a special algorithm to determine the orbit of Pallas, the second largest asteroid in our solar system. Being interested in trigonometric interpolation problems, C. Runge developed in 1903 fast methods for discrete sine transforms of certain lengths.

But only the development of the computer technology heavily enforced the development of fast algorithms. After deriving the Cooley–Tukey FFT in 1965, many further FFT's emerged being mostly based on similar strategies. We especially mention the Sande–Tukey FFT as a second radix–2 FFT, the radix–4 FFT, and the split–radix FFT. While these FFT methods are only suited for length $N = 2^t$ or even $N = 4^t$, other approaches employ cyclic convolutions and can be generalized to other lengths N. For the history of FFT see [10].

First we want to present five aspects being important for the evaluation and comparison of fast algorithms, namely computational cost, storage cost, numerical stability, suitability for parallel programming, and needed number of data rearrangements.

1. *Computational cost*

The *computational cost* of an algorithm is determined by the number of floating point operations (flops), i.e., of single (real/complex) additions and (real/complex) multiplications to perform the algorithm. For the considered FFT we will separately give the number of required additions and multiplications.

Usually, one is only interested in the order of magnitude of the computational cost of an algorithm in dependence of the number of input values and uses the big O notation. For two functions $f, g : \mathbb{N} \to \mathbb{R}$ with $f(N) \neq 0$ for all $N \in \mathbb{N}$, we write $g(N) = \mathcal{O}(f(N))$ for $N \to \infty$, if there exists a constant $c > 0$ such that $|g(N)/f(N)| \leq c$ holds for all $N \in \mathbb{N}$. By

$$\log_a N = (\log_a b)(\log_b N)\,, \qquad a,\, b > 1\,,$$

we have

$$\mathcal{O}(\log_a N) = \mathcal{O}(\log_b N)\,.$$

Therefore it is usual to write simply $\mathcal{O}(\log N)$ without fixing the base of the logarithm.

2. *Storage cost*

While memory capacities got tremendously cheaper within the last years, it is desired that algorithms require only a memory capacity being in the same order as the size of input data. Therefore we prefer so-called *in-place algorithms*, where the needed intermediate and final results can be stored by overwriting the input data. Clearly, these algorithms have to be carefully derived, since a later access to the input data or intermediate data is then impossible. Most algorithms that we consider in this chapter can be written as in-place algorithms.

3. *Numerical stability*

Since the evaluations are performed in floating point arithmetic, rounding errors can accumulate essentially during a computation leading to an inaccurate result. We will show that the FFT's accumulate smaller rounding errors than the direct computation of the DFT using a matrix vector multiplication.

4. *Parallel programming*

In order to increase the speed of computation it is of great interest to decompose the algorithm into independent subprocesses such that execution can be carried out simultaneously using multiprocessor systems. The results of these independent evaluations have to be combined afterwards upon completion.

The FFT has been shown to be suitable for parallel computing. One approach to efficiently implement the FFT and to represent the decomposition of the FFT into subprocesses is to use signal flow graphs.

5. *Rearrangements of data*
The computation time of an algorithm mainly depends on the computational cost of the algorithm but also on the data structure as e.g. the number and complexity of needed data rearrangements.
In practical applications the simplicity of the implementation of an algorithm plays an important role. Therefore FFT's with a simple and clear data structure are preferred to FFT's with slightly smaller computational cost but requiring more complex data arrangements.

Basic principles for the construction of fast algorithms are

- the application of recursions,
- the divide–and–conquer technique, and
- parallel programming.

All three principles are applied for the construction of FFT's.

Recursions can be used, if the computation of the final result can be decomposed into consecutive steps, where in the nth step only the intermediate results from the previous r steps are required. Optimally, we need only the information of one previous step to perform the next intermediate result such that an in-place processing is possible.

The *divide–and–conquer technique* is a suitable tool to reduce the execution time of an algorithm. The original problem is decomposed into several subproblems of smaller size but with the same structure. This decomposition is then iteratively applied to decrease the subproblems even further. Obviously, this technique is closely related to the recursion approach. In order to apply the divide–and–conquer technique to construct FFT's a suitable indexing of the data is needed.

The FFT's can be described in different forms. We will especially consider the sum representation, the representation based on polynomials, and the matrix representation. The original derivation of the FFT by J.W. Cooley and J.W. Tukey [3] applied the sum representation of the DFT(N). For a vector $\mathbf{a} = (a_j)_{j=0}^{N-1} \in \mathbb{C}^N$ the DFT is given by $\hat{\mathbf{a}} = (\hat{a}_k)_{k=0}^{N-1} \in \mathbb{C}^N$ with the *sum representation*

$$\hat{a}_k := \sum_{j=0}^{N-1} a_j \, w_N^{jk} \,, \quad k = 0, \ldots, N-1 \,, \quad w_N := \mathrm{e}^{-2\pi\mathrm{i}/N}. \tag{116}$$

The FFT performs the above summation using the iterative evaluation of partial sums applying the divide–and–conquer technique.

Employing the *polynomial representation* of the FFT, we interpret the DFT(N) as evaluation of the polynomial

$$a(z) := a_0 + a_1 z + \ldots + a_{N-1} z^{N-1} \in \mathbb{C}[z]$$

at the N knots w_N^k, $k = 0, \ldots, N-1$, i.e.,

$$\hat{a}_k := a(w_N^k)\,, \quad k = 0, \ldots, N-1\,. \tag{117}$$

This approach to the DFT is connected with trigonometric interpolation. The FFT is now based on the fast polynomial evaluation by reducing it to the evaluation of polynomials of smaller degrees.
Besides the polynomial arithmetic, the *matrix representation* has been shown to be appropriate for representing fast DFT algorithms. Starting with the matrix representation of the DFT

$$\hat{\mathbf{a}} := \mathbf{F}_N \mathbf{a}\,, \tag{118}$$

the Fourier matrix $\mathbf{F}_N := (w_N^{jk})_{j,k=0}^{N-1}$ is factorized into a product of sparse matrices.

Then the FFT is performed by successive matrix vector multiplications. This method requires essentially less arithmetical operations than a direct multiplication with the full matrix $\mathbf{F}_N$. The obtained algorithm is recursive, where at the nth step, only the intermediate vector obtained in the previous step is employed.

Beside the three possibilities to describe the FFT's, one tool to show the data structures of the algorithm and to simplify the programming is the signal flow graph. The *signal flow graph* is a directed graph whose vertices represent the intermediate results and whose edges illustrate the arithmetical operations. In this chapter, all signal flow graphs are composed of butterfly forms as presented in Figure 12.

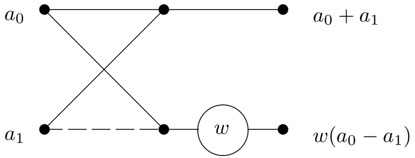


Figure 12: Butterfly signal flow graph.

The direction of evaluation in signal flow graphs is always from left to right. In particular, the factorization of the Fourier matrix into sparse matrices with at most two nonzero entries per row and per column can be simply transferred to a signal flow graph. For example, the matrix vector multiplications

$$\mathbf{F}_2\mathbf{a} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$$

can be transferred to the signal flow graphs in Figure 13.

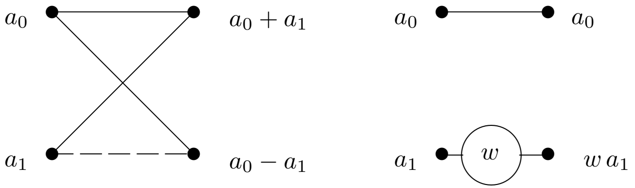


Figure 13: Signal flow graphs of $\mathbf{F}_2\mathbf{a}$ and $\mathrm{diag}\,(1, w)\,\mathbf{a}$.

Most applications use beside the DFT also the inverse DFT, such that we need also a fast algorithm for the inverse transform. However, since

$$\mathbf{F}_N^{-1} = \frac{1}{N}\,\mathbf{J}'_N\,\mathbf{F}_N$$

with the flip matrix $\mathbf{J}'_N := (\delta_{(j+k) \mod N})_{j,k=0}^{N-1}$ in Lemma 89, each fast algorithm for the DFT(N) also provides a fast algorithm for the inverse DFT(N), and we need not to consider this case separately.

Radix–2 FFT's

Radix–2 FFT's are based on the iterative divide–and–conquer technique for computing the DFT(N), if N is a power of 2. The most well-known radix–2 FFT's are the Cooley–Tukey FFT and the Sande–Tukey FFT, [3]. These algorithms can be also adapted for parallel processing. The two radix–2 FFT's only differ regarding the order of components of the input and output vector and the order of the multiplication with twiddle factors. As we will see from the corresponding factorization of the Fourier matrix into a product of sparse matrices, the one algorithm is derived from the other by using the transpose of the matrix product. In particular, the two algorithms possess the same computational cost. Therefore we also speak about variants of only one radix–2 FFT.

We start with deriving the Sande–Tukey FFT using the sum representation. Then we develop the Cooley–Tukey FFT in polynomial form. Finally we show the close relation between the two algorithms by examining the corresponding factorization of the Fourier matrix. This representation will be also applied to derive an implementation that is suitable for parallel programming.

Sande–Tukey FFT in summation form

Assume that $N = 2^t$, $t \in \mathbb{N} \setminus \{1\}$, is given. Then (116) implies

$$\begin{aligned} \hat{a}_k &= \sum_{j=0}^{N/2-1} a_j \, w_N^{jk} + \sum_{j=0}^{N/2-1} a_{N/2+j} \, w_N^{(N/2+j)k} \\ &= \sum_{j=0}^{N/2-1} \left(a_j + (-1)^k a_{N/2+j}\right) w_N^{jk} \, , \; k = 0, \ldots, N-1 \, . \end{aligned} \tag{119}$$

Considering the components of the output vector with even and odd indices, respectively, we obtain

$$\hat{a}_{2k} = \sum_{j=0}^{N/2-1} \left(a_j + a_{N/2+j}\right) w_{N/2}^{jk} \, , \tag{120}$$

$$\hat{a}_{2k+1} = \sum_{j=0}^{N/2-1} \left(a_j - a_{N/2+j}\right) w_N^{j} \, w_{N/2}^{jk} \, , \; k = 0, \ldots, \frac{N}{2} - 1 \, . \tag{121}$$

DFT(N) takes N additions, $N/2$ multiplications and 2 DFT($N/2$).
Thus, using the divide–and–conquer technique, the DFT(N) is obtained by computing

- $N/2\times$ DFT(2) of the vectors $(a_j, a_{N/2+j})^\top$, $j = 0, \dots, N/2-1$,
- $N/2$ multiplications with the twiddle factors w_N^j, $j = 0, \dots, N/2-1$,
- 2 DFT($N/2$) of the vectors $(a_j + a_{N/2+j})_{j=0}^{N/2-1}$ and $\left((a_j - a_{N/2+j})\, w_N^j\right)_{j=0}^{N/2-1}$.

However, we do not evaluate the two DFT($N/2$) directly but apply the decomposition in (119) again to the two sums. We iteratively continue this procedure and obtain the desired output vector after t decomposition steps. At each iteration step we require $N/2\times$ DFT(2) and $N/2$ multiplications with twiddle factors. As we will show in Subsection 4.2.5, this procedure reduces the computational cost to perform the DFT(N) to $\mathcal{O}(N \log N)$. This is an essential improvement! For example, for $N = 512 = 2^9$ the computation cost are reduced by more than 50 times.
The above algorithm is called *Sande–Tukey* FFT. In Figure 15 we show the corresponding signal flow graph of the DFT(8).

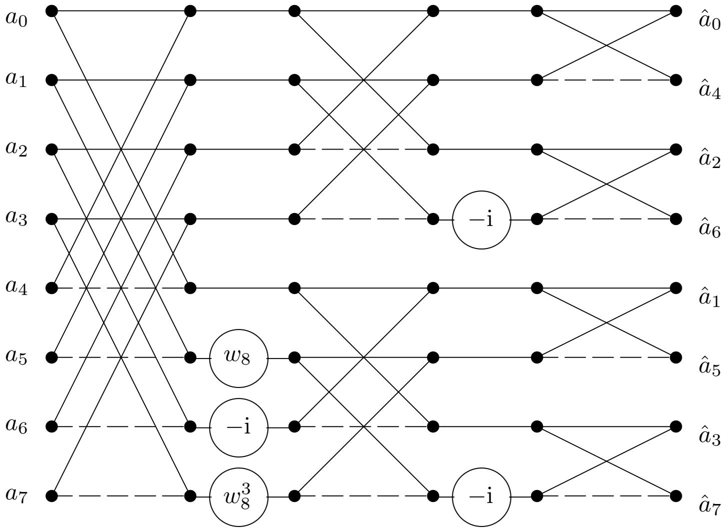


Figure 14: Sande–Tukey algorithm for DFT(8) with input values in natural order.

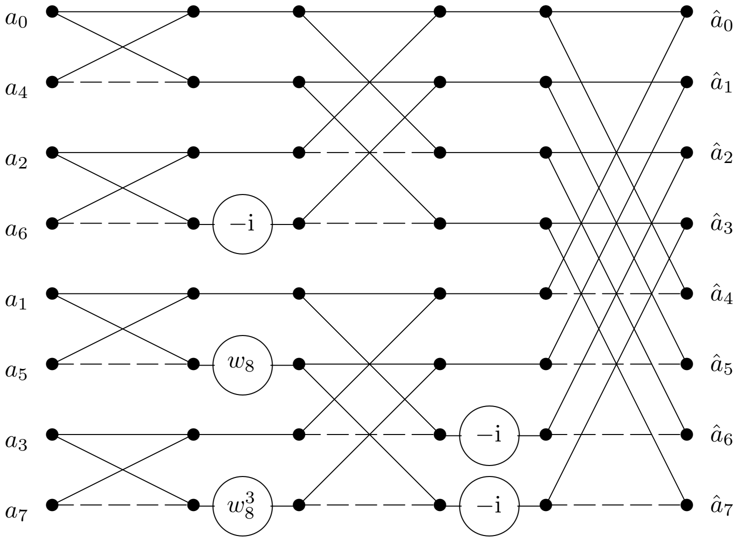


Figure 15: Sande–Tukey algorithm for DFT(8) with input values in bit reversal order.

The evaluation of $\hat{a}_0 = \sum_{j=0}^{N-1} a_j$ in the Sande–Tukey FFT is obviously executed by cascade summation. The signal flow graph well illustrates how to implement an in-place algorithm. Note that the output components are obtained in a different order, which can be described by a permutation of indices.
All indices are in the set

$$J_N := \{0, \ldots, N-1\} = \{0, \ldots, 2^t - 1\}$$

and can be written as t–digit binary numbers,

$$k = (k_{t-1}, \ldots, k_1, k_0)_2 := k_{t-1} 2^{t-1} + \ldots + k_1 2 + k_0\,, \quad k_j \in \{0, 1\}\,.$$

The permutation $\varrho : J_N \to J_N$ with

$$\varrho(k) = (k_0, k_1, \ldots, k_{t-1})_2 = k_0 2^{t-1} + \ldots + k_{t-2} 2 + k_{t-1}$$

is called *bit reversal* or *bit-reversed permutation* of J_N.

Let $\mathbf{R}_N := (\delta_{\varrho(j)-k})_{j,k=0}^{N-1}$ be the permutation matrix corresponding to ϱ. Since we have $\varrho^2(k) = k$ for all $k \in J_N$, it follows that

$$\mathbf{R}_N^2 = \mathbf{I}_N \,, \quad \mathbf{R}_N = \mathbf{R}_N^{-1} = \mathbf{R}_N^\top \,. \tag{122}$$

Table 1 shows the bit reversal for $N = 8 = 2^3$.

k	$k_2 k_1 k_0$	$k_0 k_1 k_2$	$\varrho(k)$
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

Table 1: Bit reversal for $N = 8 = 2^3$.

The comparison with Figure 15 demonstrates that $\varrho(k)$ indeed determines the order of output components. In general we can show the following:

Lemma 103

For an input vector with natural order of components, the Sande–Tukey FFT *computes the output components in bit-reversed order.*

Proof: We show by induction with respect to t that for $N = 2^t$ $t \in \mathbb{N} \setminus \{1\}$ the kth value of the output vector is $\hat{a}_{\varrho(k)}$.
For $t = 1$, the assertion is obviously correct. Assuming that the assertion holds for $N = 2^t$, we consider now the DFT of length $2N = 2^{t+1}$.

The first step of the algorithm decomposes the DFT($2N$) into two DFT(N), where for $k = 0, \ldots, N-1$ the values $\hat{a}_{2k}$ are computed at the kth position and $\hat{a}_{2k+1}$ at the $(N+k)$th position of the output vector. Afterwards the two DFT(N) are independently computed using further decomposition steps of the Sande–Tukey FFT. By induction assumption, we thus obtain after executing the complete algorithm the values $\hat{a}_{2\varrho(k)}$ at the kth position, and $\hat{a}_{2\varrho(k)+1}$ at the $(N+k)$th position of the output vector. The permutation $\pi : J_{2N} \to J_{2N}$ with

$$\pi(k) = 2\,\varrho(k)\,, \quad \pi(k+N) = 2\,\varrho(k) + 1\,, \quad k = 0, \ldots, N-1\,,$$

is by

$$\begin{aligned} \pi(k) &= \pi((0, k_{t-1}, \ldots, k_0)_2) \\ &= 2\,(0, k_0, \ldots, k_{t-2}, k_{t-1})_2 = (k_0, \ldots, k_{t-1}, 0)_2\,, \\ \pi(N+k) &= \pi((1, k_{t-1}, \ldots, k_0)_2) \\ &= 2\,(0, k_0, \ldots, k_{t-2}, k_{t-1})_2 + 1 = (k_0, \ldots, k_{t-1}, 1)_2 \end{aligned}$$

indeed equivalent to the bit reversal of J_{2N}. Thus the assertion of the lemma is true.
■

We summarize the pseudo-code for the Sande–Tukey FFT as follows.
Input: $N = 2^t$ with $t \in \mathbb{N} \setminus \{1\}$, $a_j \in \mathbb{C}$ for $j = 0, \ldots, N-1$.

```
for n := 1 to t do
 begin m := 2^(t-n+1)
  for l := 0 to 2^(n-1) - 1 do
   begin
    for r := 0 to m/2 - 1 do
     begin  j := r + lm;
            s := a_j + a_{m/2+j};
            a_{m/2+j} := (a_j - a_{m/2+j}) w_m^r;
            a_j := s
     end
   end
 end.
```

Output: $\hat{a}_k := a_{\varrho(k)} \in \mathbb{C}$, $k = 0, \ldots, N-1$.

Next, we derive the *Cooley–Tukey* FFT in polynomial form. In the presentation of the algorithm we use multi-indices for a better illustration of the order of data. We consider the polynomial $a(z) := a_0 + a_1 z + \ldots + a_{N-1} z^{N-1}$ that has to be evaluated at the N knots $z = w_N^k$, $k = 0, \ldots, N-1$. We decompose the polynomial $a(z)$ as follows

$$a(z) = \sum_{j=0}^{N/2-1} a_j\, z^j + \sum_{j=0}^{N/2-1} a_{N/2+j}\, z^{N/2+j} = \sum_{j=0}^{N/2-1} (a_j + z^{N/2}\, a_{N/2+j})\, z^j \,.$$

By $w_N^{kN/2} = (-1)^k = (-1)^{k_0}$ for all $k \in \{0, \ldots, N-1\}$ with

$$k = (k_{t-1}, \ldots, k_0)_2 \,, \quad k_j \in \{0,\, 1\} \,,$$

the term $z^{N/2}$ can be only 1 or -1.

Thus the evaluation of $a(z)$ at $z = w_N^k,\ k = 0, \ldots, N-1$, can be reduced to the evaluation of the two polynomials

$$a^{(i_0)}(z) := \sum_{j=0}^{N/2-1} a_j^{(i_0)} z^j\,, \quad i_0 = 0,\, 1\,,$$

with the coefficients

$$a_j^{(i_0)} := a_j + (-1)^{i_0}\, a_{N/2+j}\,, \quad j = 0, \ldots, N/2-1\,,$$

at the $N/2$ knots w_N^k with $k = (k_{t-1}, \ldots, k_1, i_0)_2$. In the first step of the algorithm, we compute the coefficients of the new polynomials $a^{(i_0)}(z)$, $i_0 = 0,\, 1$.

Then we apply the method again separately to the two polynomials $a^{(i_0)}(z)$, $i_0 = 0, 1$. By

$$a^{(i_0)}(z) := \sum_{j=0}^{N/4-1} \left(a_j^{(i_0)} + z^{N/4} a_{N/4+j}^{(i_0)}\right) z^j$$

and $w_N^{kN/4} = (-1)^{k_1} (-\mathrm{i})^{k_0}$, this polynomial evaluation is equivalent to the evaluating the four polynomials

$$a^{(i_0,i_1)}(z) := \sum_{j=0}^{N/4-1} a_j^{(i_0,i_1)} z^j\,, \quad i_0,\, i_1 \in \{0,\, 1\}\,,$$

with the coefficients

$$a_j^{(i_0,i_1)} := a_j^{(i_0)} + (-1)^{i_1} (-\mathrm{i})^{i_0} a_{N/4+j}^{(i_0)}\,, \quad j = 0, \ldots, N/4-1\,,$$

at the $N/4$ knots w_N^k with $k = (k_{t-1}, \ldots, k_2, i_1, i_0)_2$.

Therefore, at the second step we compute the coefficients of $a^{(i_0,i_1)}(z)$, i_0, $i_1 \in \{0, 1\}$. We iteratively continue the method and obtain after t steps N polynomials of degree 0, i.e., constants that yield the desired output values. At the $(i_0, \ldots, i_{t-1})_2$th position of the output vector we get

$$a^{(i_0,\ldots,i_{t-1})}(z) = a_0^{(i_0,\ldots,i_{t-1})} = a(w_N^k) = \hat{a}_k, \qquad i_0, \ldots, i_{t-1} \in \{0, 1\},$$

with $k = (i_{t-1}, \ldots, i_0)_2$. Thus, the output values are again in bit-reversed order. Figure 17 shows the signal flow graph of the described Cooley–Tukey FFT for $N = 8$.

Remark 104

In the Sande–Tukey FFT, the number of output values that can be independently computed doubles at each iteration step, i.e., the sampling rate is iteratively reduced in frequency domain. Therefore this algorithm is also called decimation-in-frequency FFT, see Figure 15. Analogously, the Cooley–Tukey FFT corresponds to reduction of sampling rate in time and is therefore called decimation-in-time FFT, see Figure 17. □

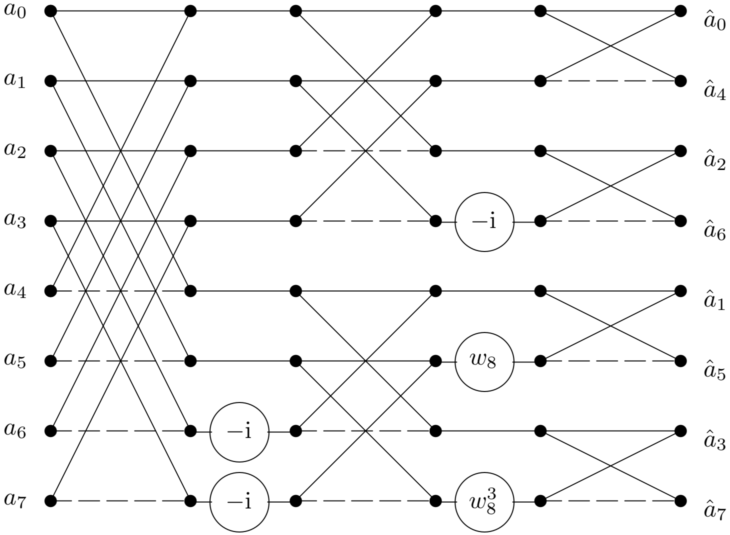


Figure 16: Cooley–Tukey FFT for $N = 8$ with input values in natural order.

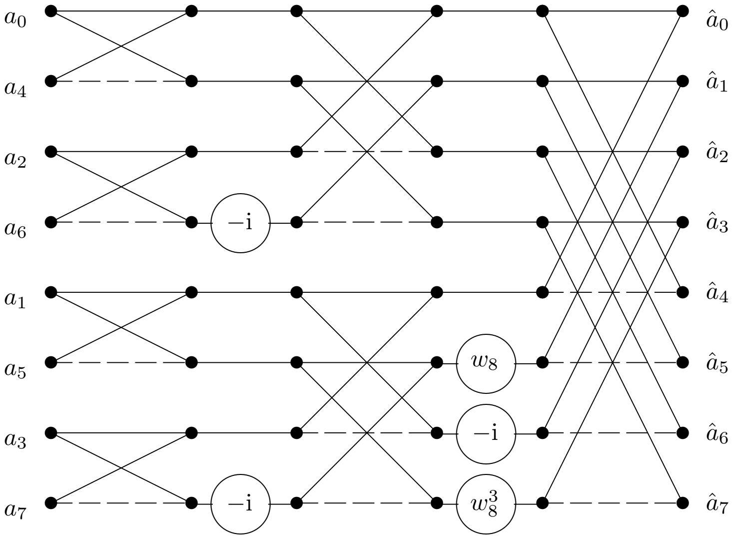


Figure 17: Cooley–Tukey FFT for $N = 8$ with input values in in bit reversal order.

Radix–2 FFT's in matrix form

The close connection between the two radix–2 FFT's can be well illustrated using the matrix representation. For this purpose we consider first the permutation matrices that yield the occurring index permutations when executing the algorithms. Beside the bit reversal, we introduce the *perfect shuffle* $\pi_N : J_N \to J_N$ by

$$\begin{aligned}
\pi_N(k) &= \pi_N((k_{t-1}, \ldots, k_0)_2) \\
&= (k_{t-2}, \ldots, k_0, k_{t-1})_2 \\
&= \begin{cases} 2k & k = 0, \ldots, N/2 - 1\,, \\ 2k + 1 - N & k = N/2, \ldots, N - 1\,. \end{cases}
\end{aligned}$$

The perfect shuffle realizes the cyclic shift of binary representation of the numbers $0, \ldots, N-1$. Then the t-times repeated cyclic shift π_N^t yields again the original order of the coefficients.

Let $\mathbf{P}_N := \left(\delta_{\pi_N(k)-j}\right)_{j,k=0}^{N-1}$ denote the corresponding permutation matrix, then

$$(\mathbf{P}_N)^t = \mathbf{I}_N\,, \quad (\mathbf{P}_N)^{t-1} = \mathbf{P}_N^{-1} = \mathbf{P}_N^\top\,. \tag{123}$$

Obviously

$$\mathbf{P}_N\,\mathbf{a} = \left(a_0, a_{N/2}, a_1, a_{N/2+1}, \ldots, a_{N/2-1}, a_{N-1}\right)^\top.$$

The cyclic shift of $(k_0, k_{t-1}, \ldots, k_1)_2$ provides the original number $(k_{t-1}, \ldots, k_0)_2$, i.e.,

$$\begin{aligned} \pi_N^{-1}(k) &= \pi_N^{-1}((k_{t-1}, \ldots, k_0)_2) = (k_0, k_{t-1}, \ldots, k_1)_2 \\ &= \begin{cases} k/2 & k \equiv 0 \mod 2\,, \\ N/2 + (k-1)/2 & k \equiv 1 \mod 2\,. \end{cases} \end{aligned}$$

Hence, at the first step of the algorithm, $\mathbf{P}_N^{-1} = \mathbf{P}_N^\top$ yields the desired rearrangement of output components $\hat{a}_k$ taking first all even and then all odd indices.

Example 105

For $N = 8$, i.e. $t = 3$, we obtain

$$\mathbf{P}_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{P}_8 \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{pmatrix} = \begin{pmatrix} c_0 \\ c_4 \\ c_1 \\ c_5 \\ c_2 \\ c_6 \\ c_3 \\ c_7 \end{pmatrix}$$

Example 105 (continue)

and

$$\mathbf{P}_8^\top = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{P}_8^\top \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{pmatrix} = \begin{pmatrix} c_0 \\ c_2 \\ c_4 \\ c_6 \\ c_1 \\ c_3 \\ c_5 \\ c_7 \end{pmatrix}.$$

$\square$

The first step of the Sande–Tukey FFT is now by (120) und (121) equivalent to the matrix factorization

$$\mathbf{F}_N = \mathbf{P}_N\,(\mathbf{I}_2 \otimes \mathbf{F}_{N/2})\,\mathbf{D}_N\,(\mathbf{F}_2 \otimes \mathbf{I}_{N/2}) \tag{124}$$

with the diagonal matrices

$$\mathbf{D}_N := \mathrm{diag}\,(\mathbf{I}_{N/2}, \mathbf{W}_{N/2})\,, \quad \mathbf{W}_{N/2} := \mathrm{diag}\,(w_N^j)_{j=0}^{N/2-1}\,.$$

At the second step of the decomposition the factorization is again applied to $\mathbf{F}_{N/2}$. Thus we obtain

$$\mathbf{F}_N = \mathbf{P}_N\left(\mathbf{I}_2 \otimes \left[\mathbf{P}_{N/2}\,(\mathbf{I}_2 \otimes \mathbf{F}_{N/4})\,\mathbf{D}_{N/2}\,(\mathbf{F}_2 \otimes \mathbf{I}_{N/4})\right]\right)\mathbf{D}_N\,(\mathbf{F}_2 \otimes \mathbf{I}_{N/2})$$

with the diagonal matrices

$$\mathbf{D}_{N/2} := \mathrm{diag}\,(\mathbf{I}_{N/4}, \mathbf{W}_{N/4})\,, \quad \mathbf{W}_{N/4} := \mathrm{diag}\,(w_{N/2}^j)_{j=0}^{N/4-1}\,.$$

Application of properties of Kronecker products yields

$$\mathbf{F}_N = \mathbf{P}_N\,(\mathbf{I}_2 \otimes \mathbf{P}_{N/2})\,(\mathbf{I}_4 \otimes \mathbf{F}_{N/4})\,(\mathbf{I}_2 \otimes \mathbf{D}_{N/2})\,(\mathbf{I}_2 \otimes \mathbf{F}_2 \otimes \mathbf{I}_{N/4})\,\mathbf{D}_N\,(\mathbf{F}_2 \otimes \mathbf{I}_{N/2})\,.$$

After t steps we thus obtain the factorization of the Fourier matrix $\mathbf{F}_N$ into sparse matrices for the *Sande–Tukey* FFT *with natural order of input components*

$$\begin{aligned} \mathbf{F}_N &= \mathbf{R}_N\,(\mathbf{I}_{N/2} \otimes \mathbf{F}_2)\,(\mathbf{I}_{N/4} \otimes \mathbf{D}_4)\,(\mathbf{I}_{N/4} \otimes \mathbf{F}_2 \otimes \mathbf{I}_2)\,(\mathbf{I}_{N/8} \otimes \mathbf{D}_8)\ \ldots \\ &\quad \ldots\,\mathbf{D}_N\,(\mathbf{F}_2 \otimes \mathbf{I}_{N/2}) \\ &= \mathbf{R}_N\,\prod_{n=1}^{t} \mathbf{T}_n\,(\mathbf{I}_{N/2^n} \otimes \mathbf{F}_2 \otimes \mathbf{I}_{2^{n-1}}) \end{aligned} \tag{125}$$

with the permutation matrix $\mathbf{R}_N = \mathbf{P}_N\,(\mathbf{I}_2 \otimes \mathbf{P}_{N/2})\ \ldots\ (\mathbf{I}_{N/4} \otimes \mathbf{P}_4)$ and the diagonal matrices

$$\begin{aligned} \mathbf{T}_n &:= \mathbf{I}_{N/2^n} \otimes \mathbf{D}_{2^n}\,, \\ \mathbf{D}_{2^n} &:= \operatorname{diag}\,(\mathbf{I}_{2^{n-1}}, \mathbf{W}_{2^{n-1}})\,, \quad \mathbf{W}_{2^{n-1}} := \operatorname{diag}\,(w_{2^n}^j)_{j=0}^{2^{n-1}-1}\,. \end{aligned}$$

Note that $\mathbf{T}_1 = \mathbf{I}_N$. From Lemma 103 and by (122) we know already that $\mathbf{R}_N$ in (125) is the permutation matrix corresponding to the bit reversal. We illustrate this fact taking a different view.

Remark 106

For distinct indices $j_1, \ldots, j_n \in J_t := \{0, \ldots, t-1\}$ *let* $(j_1, j_2, \ldots, j_n)$ *with* $1 \le n < t$ *be that permutation of* J_t *that maps* j_1 *onto* j_2, j_2 *onto* j_3, ... , j_{n-1} *onto* j_n, *and* j_n *onto* j_1. *Such a permutation is called* n*–cycle. For* $N = 2^t$, *the permutations of the index set* J_N *occurring in a radix–2 FFT can be represented by permutations of the indices in its binary presentation, i.e.,* $\pi : J_N \to J_N$ *can be written as*

$$\pi(k) = \pi((k_{t-1}, \ldots, k_0)_2) = (k_{\pi_t(k-1)}, \ldots, k_{\pi_t(0)})_2$$

with a certain permutation $\pi_t : J_t \to J_t$. *The perfect shuffle* $\pi_N : J_N \to J_N$ *corresponds to the* t*–cycle*

$$\pi_{N,t} := (0, \ldots, t-1)$$

and the bit reversal $\varrho : J_N \to J_N$ *to the permutation*

$$\varrho_t := \begin{cases} (0, t-1)(1, t-2) \ldots (t/2-1, t/2) & t \equiv 0 \mod 2\,, \\ (0, t-1)(1, t-2) \ldots ((t-1)/2) & t \equiv 1 \mod 2\,. \end{cases}$$

Remark 106 (continue)

Let $\pi_{N,n} : J_t \to J_t$ *with* $1 \leq n \leq t$ *be given by the n-cycle*

$$\pi_{N,n} := (0, \ldots, n-1).$$

Then we can prove by induction that

$$\varrho_t = \pi_{N,t}\, \pi_{N,t-1} \, \cdots \, \pi_{N,2} \,.$$

Using the matrix representation we obtain now the desired relation

$$\mathbf{R}_N = \mathbf{P}_N \left(\mathbf{I}_2 \otimes \mathbf{P}_{N/2}\right) \, \ldots \, \left(\mathbf{I}_{N/4} \otimes \mathbf{P}_4\right) .$$

□

Example 107

The factorization of $\mathbf{F}_8$ in (125) has the form

$$\mathbf{F}_8 = \mathbf{R}_8 \left(\mathbf{I}_4 \otimes \mathbf{F}_2\right) \left(\mathbf{I}_2 \otimes \mathbf{D}_4\right) \left(\mathbf{I}_2 \otimes \mathbf{F}_2 \otimes \mathbf{I}_2\right) \mathbf{D}_8 \left(\mathbf{F}_2 \otimes \mathbf{I}_4\right),$$

i.e., $\mathbf{F}_8 =$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

$$\cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & w_8 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

Corollary 108

Let $N = 2^t$. Then we have

$$\begin{aligned} \mathbf{P}_N^n \left(\mathbf{I}_{N/2} \otimes \mathbf{F}_2\right) \mathbf{P}_N^{-n} &= \mathbf{I}_{N/2^{n+1}} \otimes \mathbf{F}_2 \otimes \mathbf{I}_{2^n}\,, \quad n = 0, \ldots, t-1\,, \\ \mathbf{R}_N \left(\mathbf{I}_{N/2^n} \otimes \mathbf{F}_2 \otimes \mathbf{I}_{2^{n-1}}\right) \mathbf{R}_N &= \mathbf{I}_{2^{n-1}} \otimes \mathbf{F}_2 \otimes \mathbf{I}_{N/2^n}\,, \quad n = 1, \ldots, t\,. \end{aligned}$$

From (125) and Corollary 108 we conclude the factorization of $\mathbf{F}_N$ corresponding to the Sande–Tukey FFT with bit reversed order of input values,

$$\mathbf{F}_N = \left(\prod_{n=1}^{t} \mathbf{T}_n^{\varrho} \left(\mathbf{I}_{2^{n-1}} \otimes \mathbf{F}_2 \otimes \mathbf{I}_{N/2^n}\right)\right) \mathbf{R}_N\,, \quad \mathbf{T}_n^{\varrho} := \mathbf{R}_N\, \mathbf{T}_n\, \mathbf{R}_N\,.$$

The matrix factorization corresponding to the Cooley–Tukey FFT is obtained from (125) by taking the transpose.

From $\mathbf{F}_N = \mathbf{F}_N^\top$ it follows that

$$\mathbf{F}_N = \Big(\prod_{n=1}^{t} (\mathbf{I}_{2^{n-1}} \otimes \mathbf{F}_2 \otimes \mathbf{I}_{N/2^n})\, \mathbf{T}_{t-n+1} \Big)\, \mathbf{R}_N \,.$$

This factorization equates the Cooley–Tukey FFT with bit reversal order of input values. By Corollary 108 we finally observe that

$$\mathbf{F}_N = \mathbf{R}_N \prod_{n=1}^{t} (\mathbf{I}_{N/2^n} \otimes \mathbf{F}_2 \otimes \mathbf{I}_{2^{n-1}})\, \mathbf{T}^{\varrho}_{t-n+1}$$

is the matrix factorization of the Cooley–Tukey FFT with natural order of input values.

Nonequispaced FFT's
NFFT of type I

The aim is the fast and stable computation of

$$f(x_j) = \sum_{k=-N/2}^{N/2-1} \hat{f}_k \, \mathrm{e}^{\mathrm{i}\, k\, x_j}\,, \quad j = -M/2, \ldots, M/2-1\,,$$

for given nonequispaced nodes $x_j \in [-\pi,\, \pi)$ and given data $\hat{f}_k \in \mathbb{C}$ at equispaced frequencies $k = -N/2, \ldots, N/2-1$.

Remark 109

In the case of equispaced nodes $x_j := 2\pi j/N$ with $j = -N/2, \ldots, N/2-1$ and $M = N$, the FFT requires $\mathcal{O}(N \log N)$ arithmetical operations.

We introduce the 2π-periodic trigonometric polynomial

$$f(x) = \sum_{k=-N/2}^{N/2-1} \hat{f}_k \, \mathrm{e}^{\mathrm{i}kx} \, .$$

First we approximate f by a linear combination s_1 of translates of a 2π-periodic function $\tilde{\varphi}$. Let $\varphi \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ be a convenient window function such that its periodization

$$\tilde{\varphi}(x) := \sum_{r \in \mathbb{Z}} \varphi(x + 2\pi r)$$

has a uniformly convergent Fourier series. Then $\tilde{\varphi}$ can be represented as Fourier series

$$\tilde{\varphi}(x) := \sum_{k \in \mathbb{Z}} c_k(\tilde{\varphi}) \, \mathrm{e}^{\mathrm{i}\,k\,x}$$

with Fourier coefficients

$$\begin{aligned} c_k(\tilde{\varphi}) &:= \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\varphi}(x) \, \mathrm{e}^{-\mathrm{i}\,k\,x} \, \mathrm{d}x \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(x) \, \mathrm{e}^{-\mathrm{i}\,k\,x} \, \mathrm{d}x = \frac{1}{2\pi} \, \hat{\varphi}(k) \, . \end{aligned} \tag{126}$$

Example 110

A popular window function is the centered cardinal B-spline of order $m \in \mathbb{N}$

$$M_1(x) := \begin{cases} 1 & x \in [-1/2, 1/2), \\ 0 & \text{otherwise}, \end{cases}$$

$$M_{m+1}(x) := \int_{-1/2}^{1/2} M_m(x-t)\,\mathrm{d}t = (M_m * M_1)(x)\,.$$

The support of M_m is the interval $[-m/2,\, m/2]$. The Fourier transform of M_m is equal to

$$\hat{M}_m(v) = \big(\mathrm{sinc}(\pi v)\big)^m.$$

Choose an *oversampling factor* $\sigma \geq 1$ such that σN is even. Now we determine coefficients $g_\ell \in \mathbb{C}$, $\ell = -\sigma N/2, \ldots, \sigma N/2 - 1$ of the linear combination

$$s_1(x) := \sum_{\ell=-\sigma N/2}^{\sigma N/2-1} g_\ell \, \tilde{\varphi}\big(x - \frac{2\pi\ell}{\sigma N}\big)$$

such that s_1 approximates f.
Then we have

$$s_1(x) = \sum_{k\in\mathbb{Z}} c_k(s_1)\, \mathrm{e}^{\mathrm{i}\,k\,x} = \sum_{k\in\mathbb{Z}} \hat{g}_k\, c_k(\tilde{\varphi})\, \mathrm{e}^{\mathrm{i}\,k\,x}$$

with DFT

$$\hat{g}_k := \sum_{\ell=-\sigma N/2}^{\sigma N/2-1} g_\ell\, \mathrm{e}^{-2\pi\,\mathrm{i}\,k\ell/(\sigma N)}\,.$$

Hence we obtain

$$s_1(x) := \sum_{k=-\sigma N/2}^{\sigma N/2-1} \hat{g}_k\, c_k(\tilde{\varphi})\, \mathrm{e}^{\mathrm{i}\,k\,x}$$

$$+ \sum_{r \in \mathbb{Z}\setminus\{0\}} \sum_{k=-\sigma N/2}^{\sigma N/2-1} \hat{g}_k\, c_{k+\sigma N r}(\tilde{\varphi})\, \mathrm{e}^{\mathrm{i}\,(k+\sigma N r)\,x}\,. \qquad (127)$$

If $|c_k(\tilde{\varphi})| \ll 1$ for $|k| \geq \sigma N - \frac{N}{2}$ and if $c_k(\tilde{\varphi}) \neq 0$ for $k = -N/2, \ldots, N/2-1$, then a comparison of the Fourier series of f and s_1 shows that

$$\hat{g}_k = \hat{f}_k / c_k(\tilde{\varphi}) \qquad k = -N/2, \ldots, N/2-1, \qquad (128)$$

and $\hat{g}_k := 0$ for $k = -\sigma N/2, \ldots, -N/2-1; N/2, \ldots, \sigma N/2 - 1$. We compute the coefficients g_ℓ of s_1 by inverse DFT(σN)

$$g_\ell = \frac{1}{\sigma N} \sum_{k=-\sigma N/2}^{\sigma N/2-1} \hat{g}_k\, \mathrm{e}^{2\pi\mathrm{i}\,k\,\ell/(\sigma N)}, \quad \ell = -\sigma N/2, \ldots, \sigma N/2-1\,.$$

Further we assume that φ is well-localized (cf. Example 110) such that φ can be approximated by its truncation

$$\psi(x) := \begin{cases} \varphi(x) & x \in [-2\pi m/(\sigma N), 2\pi m/(\sigma N)], \\ 0 & \text{otherwise} \end{cases}$$

with $2m \ll \sigma N$. Now we form the 2π-periodic function

$$\tilde{\psi}(x) := \sum_{r \in \mathbb{Z}} \psi(x + 2\pi r) \in L_2(\mathbb{T})$$

and approximate s_1 by

$$s(x) := \sum_{\ell=-\sigma N/2}^{\sigma N/2-1} g_\ell \, \tilde{\psi}\big(x - \frac{2\pi\ell}{\sigma N}\big) \,. \tag{129}$$

Since the support of ψ is bounded, we introduce the set $I_{\sigma N,m}(x)$ of all indices $\ell \in \{-\sigma N/2, \ldots, \sigma N/2 - 1\}$ with the property

$$\frac{\sigma N}{2\pi} x - m \le \ell \le \frac{\sigma N}{2\pi} x + m\,.$$

For each fixed knot $x_j \in [-\pi,\, \pi)$, the sum (129) has at most $2m + 1$ nonzero terms. Thus we obtain

$$f(x_j) \approx s_1(x_j) \approx s(x_j)\,.$$

We can *approximately* compute the trigonometric polynomial f for all $x_j \in [-\pi,\, \pi)$, $j = -M/2, \ldots, M/2 - 1$ with a computational cost of $\mathcal{O}(N \log N + m\, M)$ operations. Note that the computational cost of an algorithm is measured in the number of arithmetical operations, where all operations are counted equally.

Algorithm for NFFT of type I

Input: N, $M \in \mathbb{N}$, $\sigma > 1$, $m \in \mathbb{N}$,
$x_j \in [-\pi, \pi)$ for $j = -M/2, \ldots, M/2$,
$\hat{f}_k \in \mathbb{C}$ for $k = -N/2, \ldots, N/2 - 1$.

Precomputation: (i) Compute the Fourier coefficients $c_k(\tilde{\varphi})$ for all $k = -N/2, \ldots, N/2 - 1$.
(ii) Compute the values $\tilde{\psi}(x_j - \frac{2\pi\ell}{\sigma N})$ for $j = -M/2, \ldots, M/2 - 1$ and $\ell \in I_{\sigma N,m}(x_j)$.

1. Let $\hat{g}_k := \hat{f}_k / c_k(\tilde{\varphi})$ for $k = -N/2, \ldots, N/2 - 1$.
2. By FFT compute the values

$$g_\ell := \frac{1}{\sigma N} \sum_{k=-N/2}^{N/2-1} \hat{g}_k \, \mathrm{e}^{2\pi\mathrm{i}\, k\, \ell/(\sigma N)}\,, \quad \ell \in I_{\sigma N}^d\,.$$

3. Compute

$$s(x_j) := \sum_{\ell \in I_{\sigma N, m}(x_j)} g_\ell \, \tilde{\psi}\big(x_j - \frac{2\pi\ell}{\sigma N}\big)\,, \quad j = -M/2, \ldots, M/2-1\,.$$

Output: $s(x_j)$, $j = -M/2, \ldots, M/2-1$, approximate values of $f(x_j)$.

Remark 111

The NFFT of type II *reads as follows*

$$h(k) := \sum_{j=-M/2}^{M/2-1} f_j \,\mathrm{e}^{\mathrm{i}\,k\,x_j}\,, \quad k = -N/2, \ldots, N/2-1\,,$$

with nonequispaced nodes $x_j \in [-\pi, \pi)$ and given data $f_j \in \mathbb{C}$, $j = -M/2, \ldots, M/2-1$. Here we introduce the 2π-periodic function

$$\tilde{g}(x) := \sum_{j=-M/2}^{M/2-1} f_j\, \tilde{\varphi}(x + x_j)$$

which has the Fourier coefficients $c_k(\tilde{g}) = h(k)\, c_k(\tilde{\varphi})$. By the trapezoidal rule, approximate value of $c_k(\tilde{g})$ is

$$\frac{1}{\sigma N} \sum_{\ell=-\sigma N/2}^{\sigma N/2-1} \sum_{j=-M/2}^{M/2-1} f_j\, \tilde{\varphi}\big(x_j - \frac{2\pi\ell}{\sigma N}\big)\, \mathrm{e}^{2\pi\mathrm{i}k\ell/(\sigma N)}\,.$$

4.4.2. Error estimates for some window functions

In contrast to FFT, the above algorithm for NFFT is an *approximate algorithm*.
Hence we have to estimate the *approximation error* $E(x_j) := |f(x_j) - s(x_j)|$.
Introducing the *aliasing error*

$$E_{\mathrm{a}}(x_j) := |f(x_j) - s_1(x_j)|$$

and the *truncation error*

$$E_{\mathrm{t}}(x_j) := |s_1(x_j) - s(x_j)|\,,$$

we have by triangle inequality

$$E(x_j) \; \leq \; E_{\mathrm{a}}(x_j) + E_{\mathrm{t}}(x_j)\,.$$

Lemma 112

For $\|\hat{\mathbf{f}}\|_1 := \sum_{k=-N/2}^{N/2-1} |\hat{f}_k|$, *the aliasing and truncation errors can be estimated by*

$$E_{\mathrm{a}}(x_j) \le \|\hat{\mathbf{f}}\|_1 \max_{k=-N/2,\dots,N/2-1} \sum_{r\in\mathbb{Z}\setminus\{0\}} \frac{|\hat{\varphi}(k+\sigma Nr)|}{|\hat{\varphi}(k)|}, \tag{130}$$

$$E_{\mathrm{t}}(x_j) \le \frac{\|\hat{\mathbf{f}}\|_1}{\sigma N} \max_{k=-N/2,\dots,N/2-1} \frac{1}{|\hat{\varphi}(k)|} \sum_{|x_j+\frac{2\pi r}{\sigma N}|>\frac{2\pi m}{\sigma N}} \left|\varphi(x_j+\frac{2\pi r}{\sigma N})\right|. \tag{131}$$

Proof: For simplicity, we estimate here only the aliasing error $E_{\mathrm{a}}(x_j) = |f(x_j) - s_1(x_j)|$. By (127) and (128), we have

$$E_{\mathrm{a}}(x_j) = \Big| \sum_{k=-\sigma N/2}^{\sigma N/2-1} \sum_{r\neq 0} \hat{g}_k\, c_{k+\sigma Nr}(\tilde{\varphi})\, \mathrm{e}^{\mathrm{i}\,(k+\sigma Nr)\,x} \Big|.$$

From (128) and (126), it follows by triangle inequality

$$E_{\rm a}(x_j) \le \sum_{k=-N/2}^{N/2-1} |\hat{f}_k| \sum_{r\neq 0} \frac{|\hat{\varphi}(k+\sigma N r)|}{|\hat{\varphi}(k)|}$$
$$\le \|\hat{\mathbf{f}}\|_1 \max_{k=-N/2,\ldots,N/2-1} \sum_{r\in\mathbb{Z}\setminus\{0\}} \frac{|\hat{\varphi}(k+\sigma N r)|}{|\hat{\varphi}(k)|}. \quad \blacksquare$$

Now we estimate the approximation errors of NFFT for special window functions φ with good localizations in time and frequency domain. First we consider

$$\varphi(x) := M_{2m}\big(\frac{\sigma N}{2\pi}\, x\big), \tag{132}$$

where $\sigma \ge 1$ and $2m \ll \sigma N$. Then φ is supported on

$$\big[-\frac{2\pi m}{\sigma N},\, \frac{2\pi m}{\sigma N}\big] \subset [-\pi,\, \pi].$$

We compute the Fourier transform

$$\begin{aligned}\hat{\varphi}(v) &= \int_{\mathbb{R}} \varphi(x)\,\mathrm{e}^{-\mathrm{i}\,vx}\,\mathrm{d}x = \int_{\mathbb{R}} M_{2m}\big(\frac{\sigma N}{2\pi}\,x\big)\,\mathrm{e}^{-\mathrm{i}\,vx}\,\mathrm{d}x \\ &= \frac{2\pi}{\sigma N}\int_{\mathbb{R}} M_{2m}(t)\,\mathrm{e}^{-2\pi\mathrm{i}\,vt/(\sigma N)}\,\mathrm{d}t\,.\end{aligned}$$

By Example 110, we have $M_{2m}(t) = (M_1 * M_1 * \ldots * M_1)(t)$. The convolution property of the Fourier transform yields

$$\hat{\varphi}(v) = \frac{2\pi}{\sigma N}\Big(\int_{-1/2}^{1/2} \mathrm{e}^{-2\pi\mathrm{i}vt/(\sigma N)}\,\mathrm{d}t\Big)^{2m} = \frac{2\pi}{\sigma N}\big(\operatorname{sinc}\frac{v\pi}{\sigma N}\big)^{2m}$$

with the sinc function $\operatorname{sinc} x := \frac{\sin x}{x}$ for $x \in \mathbb{R}\setminus\{0\}$ and $\operatorname{sinc} 0 := 1$. Note that $\hat{\varphi}(k) > 0$ for all $k = -N/2, \ldots, N/2 - 1$. Since $\varphi(x)$ is supported on $[-2\pi m/(\sigma N),\, 2\pi m/(\sigma N)]$, we have $\psi = \varphi$.

For arbitrary knots $x_j \in [-\pi, \pi)$, $j = -M/2, \dots M/2 - 1$, and each data vector $\hat{\mathbf{f}} = \left(\hat{f}_k\right)_{k=-N/2}^{N/2-1}$, we obtain by (130) the approximation error

$$E(x_j) = E_{\mathrm{a}}(x_j) \le \|\hat{\mathbf{f}}\|_1 \max_{k=-N/2,\dots,N/2-1} \sum_{r \neq 0} \frac{|\hat{\varphi}(k + \sigma N r)|}{|\hat{\varphi}(k)|} \tag{133}$$

with

$$\frac{|\hat{\varphi}(k + \sigma N r)|}{|\hat{\varphi}(k)|} = \left(\frac{k}{k + \sigma N r}\right)^{2m}. \tag{134}$$

Lemma 113

Assume that $\sigma > 1$ and $2m \ll \sigma N$. Then for the window function φ in (132) with $\psi = \varphi$, the approximation error of the NFFT can be estimated by

$$E(x_j) \le \frac{4}{(2\sigma - 1)^{2m}} \|\hat{\mathbf{f}}\|_1 , \tag{135}$$

where $x_j \in [-\pi, \pi)$, $j = -M/2, \dots, M/2 - 1$, are arbitrary knots and $\hat{\mathbf{f}} \in \mathbb{C}^N$ is an arbitrary data vector.

Proof: By (133) and (134) we have

$$E(x_j) \le \|\hat{\mathbf{f}}\|_1 \max_{k=-N/2,\ldots,N/2-1} \sum_{r\neq 0} \Big(\frac{k/(\sigma N)}{r+k/(\sigma N)}\Big)^{2m} . \tag{136}$$

Setting $u = \frac{k}{\sigma N}$ for $k = -N/2, \ldots, N/2-1$, we have $|u| \le \frac{1}{2\sigma} < 1$. Now we show that

$$\sum_{r\neq 0} \big(\frac{u}{u+r}\big)^{2m} \le \frac{4}{(2\sigma-1)^{2m}} . \tag{137}$$

For $0 \le u \le \frac{1}{2\sigma} < 1$ we have

$$\sum_{r\neq 0} \big(\frac{u}{u+r}\big)^{2m} = \big(\frac{u}{u-1}\big)^{2m} + \big(\frac{u}{u+1}\big)^{2m} + \sum_{r=2}^{\infty} \Big[\big(\frac{u}{u-r}\big)^{2m} + \big(\frac{u}{u+r}\big)^{2m}\Big] .$$

By $u+r > |u-r|$ for $r \in \mathbb{N}$ we have $\left(\frac{u}{u+r}\right)^{2m} \le \left(\frac{u}{u-r}\right)^{2m}$ and hence

$$\begin{aligned}\sum_{r\neq 0}\Big(\frac{u}{u+r}\Big)^{2m} &\le 2\,\Big(\frac{u}{u-1}\Big)^{2m} + 2\sum_{r=2}^{\infty}\Big(\frac{u}{u-r}\Big)^{2m}\\ &\le 2\,\Big(\frac{u}{u-1}\Big)^{2m} + 2\int_{1}^{\infty}\Big(\frac{u}{u-x}\Big)^{2m}\mathrm{d}x\\ &\le 2\,\Big(\frac{u}{u-1}\Big)^{2m}\Big(1+\frac{1-u}{2m-1}\Big) < 4\,\Big(\frac{u}{u-1}\Big)^{2m}.\end{aligned}$$

Since the function $\left(\frac{u}{u-1}\right)^{2m}$ increases in $[0,\, \frac{1}{2\sigma}]$, the above sum has the upper bound $\frac{4}{(2\sigma-1)^{2m}}$ for each $m \in \mathbb{N}$.

In the case $-1 \le -\frac{1}{2\sigma} < u < 0$, we replace u by $-u$ and obtain the same upper bound. Now, the estimate (135) follows from (136) and (137). ■

Another popular window function is the (dilated) *Gaussian function*

$$\varphi(x) := \frac{1}{\sqrt{\pi b}}\, \mathrm{e}^{-(\frac{\sigma N}{2\pi}x)^2/b}, \quad x \in \mathbb{R}, \tag{138}$$

with the parameter $b := \frac{2\sigma m}{(2\sigma-1)\pi}$ which determines the localization of (138) in time and frequency domain. As shown in Example 48, the Fourier transform of (138) reads

$$\hat{\varphi}(\omega) = \frac{2\pi}{\sigma N}\, \mathrm{e}^{-(\frac{\pi\omega}{\sigma N})^2 b}. \tag{139}$$

Lemma 114

Assume that $\sigma > 1$ *and* $2m \ll \sigma N$. *Then for the Gaussian function* (138) *and the truncated function* $\psi = \varphi|\,[-\frac{2\pi m}{\sigma N}, \frac{2\pi m}{\sigma N}]$, *the approximation error of the* NFFT *can be estimated by*

$$E(x_j) \le 4\,\mathrm{e}^{-m\pi(1-1/(2\sigma-1))}\,\|\hat{\mathbf{f}}\|_1\,, \tag{140}$$

where $x_j \in [-\pi, \pi)$, $j = -M/2, \ldots, M/2-1$, *is an arbitrary knot and* $\hat{\mathbf{f}} \in \mathbb{C}^N$ *is an arbitrary data vector.*

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