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Optimal Multilevel Extension Operators

Abstract

In the present paper we suggest the norm-preserving explicit operator for the extension of finite-element functions from boundaries of domains into the inside. The construction of this operator is based on the multilevel decomposition of functions on the boundaries and on the equivalent norm for this decomposition. The cost of the action of this operator is proportional to the number of nodes.

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Let Ω be a bounded, polygonal domain and Γ be its boundary. Let us consider a coarse grid triangulation of Ω

$$\Omega_0^h = \bigcup_{i=1}^{M_0} \overline{\tau}_i^{(0)}, \qquad \mathrm{diam}(\tau_i^{(0)}) = O(1)$$

and we refine Ω_0^h several times. This results in a sequence of nested triangulations $\Omega_0^h,\ \Omega_1^h,\ \ldots,\ \Omega_J^h$ such that

$$\overline{\Omega}_k^h = \bigcup_{i=1}^{M_k} \overline{\tau}_i^{(k)}, \quad k = 0, 1, \ldots, J,$$

where the triangles $\tau_i^{(k+1)}$ are generated by subdividing triangles $\tau_i^{(k)}$ into four congruent subtriangles by connecting the midpoints of the edges. Introduce the spaces W_k and V_k of finite–element functions. The space consists of real-valued functions which are continuous on Ω and linear on the triangles in Ω_k^h. The space V_k is the space of traces on Γ of functions from W_k:

$$V_k = \{\varphi^h | \varphi^h = u^h|_\Gamma, \ \text{ with } u^h \in W_k\}$$

We consider W_k and V_k as the subspaces of the Sobolev spaces $H^1(\Omega)$ and $H^{\frac{1}{2}}(\Gamma)$, respectively, with corresponding norms [2]. The main goal is the construction of some norm–preserving explicit extension operator t from V_J to W_J:

$$t : V_J \to W_J$$

This construction is based on the idea from [3] but instead of Yserentant's hierarchical decomposition [8,9] of the space V_J we use some analogue of the so–called BPX–decomposition of V_J [1]. Denote by $\varphi_i^{(k)}, \ \ i = 1, 2, \ldots, N_k$, the nodal basis of V_k and denote by $\Phi_i^{(k)}$ the one–dimensional subspace spanned by this function $\varphi_i^{(k)}$. Define

$$Q_i^{(k)} : L_2(\Gamma) \to \Phi_i^{(k)}$$

the L_2–orthoprojection from $L_2(\Gamma)$ onto $\Phi_i^{(k)}$ and denote

$$\widetilde{Q}_k = \sum_{i=1}^{N_k} Q_i^{(k)}, \quad k = 0, 1, \ldots, J-1.$$

For $k = J, J+1, J+2, \ldots$ we define $\widetilde{Q}_k$ as the L_2–orthoprojection from $L_2(\Omega)$ onto V_k.

Lemma 1 *There exist positive constants c_1, c_2, independent of h, such that*

$$\begin{aligned} c_1\,\|\varphi^h\|^2_{H^{\frac{1}{2}}(\Gamma)} &\le \|\tilde{Q}_0\,\varphi^h\|^2_{L_2(\Gamma)} + \sum_{k=1}^{J} 2^k \|(\tilde{Q}_k - \tilde{Q}_{(k-1)})\,\varphi^h\|^2_{L_2(\Gamma)} \\ &\le c_2\,\|\varphi^h\|^2_{H^{\frac{1}{2}}(\Gamma)}. \end{aligned}$$

Proof It is easy to see that $\widetilde{Q}_k$ is the linear projection onto V_k and there exists a positive constant c_3, independent of h, such that

$$\|\widetilde{Q}_k\,\varphi\|_{L_2(\Gamma)} \le c_3\,\|\varphi\|_{L_2(\Gamma)},\ \forall\,\varphi\,\in\,L_2(\Omega).$$

Since

$$\|\widetilde{Q}_0\,\varphi^h\|^2_{L_2(\Gamma)} + \sum_{k=1}^{J} 2^k\,\|(\widetilde{Q}_k - \widetilde{Q}_{k-1})\,\varphi^h\|^2_{L_2(\Gamma)} =$$

$$\|\widetilde{Q}_0\,\varphi^h\|^2_{L_2(\Gamma)} + \sum_{k=1}^{\infty} 2^k\,\|(\widetilde{Q}_k - \widetilde{Q}_{k-1})\,\varphi^h\|^2_{L_2(\Gamma)},$$

then we get from [7] the equivalence of these two norms. Denote by $x_i^{(k)}$, i=1,2, $\ldots$, L_k the nodes of the triangulation Ω_k^h (we assume that nodes $x_i^{(k)}$ are enumerated first on Γ and then inside Ω) and define the extension operator t in the following way. For any $\varphi^h \in V_J$ set

$$\begin{aligned} \psi_0^h &= \widetilde{Q}_0\,\varphi^h, \\ \psi_k^h &= (\widetilde{Q}_k - \widetilde{Q}_{k-1})\,\varphi^h, \quad k = 1, 2, \ldots, J. \end{aligned} \tag{1}$$

Then

$$\varphi^h = \psi_1^h + \psi_2^h + \ldots + \psi_J^h$$

Define the extension $u_k^h \in W_k$ of the function ψ_k^h according to [3]:

$$\begin{aligned} u_0^h(x_i^{(0)}) &= \begin{cases} \psi_0^h(x_i^{(0)}) & ,\ x_i^{(0)}\,\in\,\Gamma, \\ \overline{\psi} & ,\ x_i^{(0)}\,\notin\Gamma, \end{cases} \\ u_k^h(x_i^{(k)}) &= \begin{cases} \psi_k^h(x_i^{(0)}) & ,\ x_i^{(k)}\,\in\,\Gamma, \\ 0 & ,\ x_i^{(k)}\,\notin\Gamma, \end{cases} \end{aligned} \tag{2}$$

Here $\overline{\psi}$ is, for instance, the mean value of the function ψ_0^h on Γ:

$$\overline{\psi} = \frac{1}{N_0}\sum_{i=1}^{N_0} \psi_0^h(x_i^{(0)}).$$

Define

$$t\,\varphi^h = u^h \equiv u_0^h + u_1^h + \ldots + u_J^h \tag{3}$$

Remark 1 *We can use the L_2–orthoprojection from $L_2(\Omega)$ onto V_k instead of $\widetilde{Q}_k$, $k = 0, 1, \ldots, J-1$. But in this case the cost of the decomposition (1) is expensive (especially for three dimensional problems).*

Lemma 2 *There exists a positive constant c_4, independent of h, such that*

$$\|u_k^h\|_{H^1(\Omega)} \leq c_4\, 2^k\, \|\psi_k^h\|_{L_2(\Gamma)},\ k = 0, 1, \ldots, J.$$

Proof of this lemma is obvious and was done in [3].

By the Friedrichs inequality there exists a positive constant c_5, independent of h, such that

$$\begin{aligned} \|t\,\varphi^h\|_{H^1(\Omega)} &\equiv \|u^h\|_{H^1(\Omega)} \\ &\leq c_5\,(\|\varphi^h\|_{L_2(\Gamma)} + \|\nabla u^h\|_{L_2(\Omega)}). \end{aligned}$$

Then to estimate the norm of the operator t from (3), we need to estimate

$$\sum_{i=1}^{J}\sum_{j=1}^{J}(\nabla u_i^h,\, \nabla u_j^h)_{L_2(\Omega)}.$$

Let us consider the following representation of the function ψ_k^h:

$$\psi_k^h = \sum_{i=1}^{N_k} \alpha_i^{(k)} \varphi_i^{(k)}, \qquad \alpha_i^{(k)} \in \mathbb{R} \tag{4}$$

Then the function u_k^h from (2) has the representation

$$u_k^h = \sum_{i=1}^{N_k} \alpha_i^{(k)} u_i^{(k)}, \quad k = 1, 2, \cdots, J,$$

where $u_i^{(k)}$ is the nodal basis function which corresponds to the node $x_i^{(k)} \in \Gamma$.

Lemma 3 *Let $k_2 > k_1$. Then*

$$\left|(\nabla u_{i_1}^{(k_1)}, \nabla u_{i_2}^{(k_2)})_{L_2(\Omega)}\right| \leq \begin{cases} 0 & , \textit{ if } x_{i_2}^{(k_2)} \notin \ \textit{supp}(\varphi_{i_1}^{(k_1)}), \\ c_6 \cdot 2^{k_1-k_2} & , \textit{ if } x_{i_2}^{(k_2)} \in \ \textit{supp}(\varphi_{i_1}^{(k_1)}). \end{cases}$$

Here c_6 is independent of h.

Proof This is a trivial consequence of the following obvious estimates:

$$\begin{aligned} \left|\nabla u_{i_1}^{(k_1)}\right| &\le c_7 \cdot 2^{k_1}, \\ \left|\nabla u_{i_2}^{(k_2)}\right| &\le c_7 \cdot 2^{k_2}, \\ \text{meas} \ (\ \text{supp} \ u_{i_2}^{(k_2)}) &\le c_7 \cdot (2^{-k_2})^2 \end{aligned}$$

where c_7 is independent of h.

The following lemma is valid.

Lemma 4 *There exists a positive constant c_8, independent of h, such that*

$$\sum_{k_1=1}^{J} \sum_{k_2=k_1+1}^{J} \left|(\nabla u_{k_1}^h, \nabla u_{k_2}^h)_{L_2(\Omega)}\right| \le c_8 \sum_{k=1}^{J} \sum_{i=1}^{N_k} (\alpha_i^{(k)})^2.$$

Here $\alpha_i^{(k)}$ is from (4).

Proof We have

$$\sum_{k_1=1}^{J} \sum_{k_2=k_1+1}^{J} \left|(\nabla u_{k_1}^h, \nabla u_{k_2}^h)_{L_2(\Omega)}\right| =$$

$$= \sum_{k_1=1}^{J} \sum_{k_2=k_1+1}^{J} \left| \Big(\sum_{i_1=1}^{N_{k_1}} \alpha_{i_1}^{(k_1)} \nabla u_{i_1}^{(k_1)}, \sum_{i_2=1}^{N_{k_2}} \alpha_{i_2}^{(k_2)} \nabla u_{i_2}^{(k_2)} \Big)_{L_2(\Omega)} \right|.$$

Using the Lemma 3 and the Cauchy inequality, we have:

$$\left| \Big(\alpha_{i_1}^{(k_1)} \nabla u_{i_1}^{k_1)}, \sum_{i_2=1}^{N_{k_2}} \alpha_{i_2}^{(k_2)} \nabla u_{i_2}^{(k_2)} \Big)_{L_2(\Omega)} \right| \le$$

$$\le c_6 \sum_{x_{i_2}^{(k_2)} \in \ \text{supp} \ (\varphi_{i_1}^{(k_1)})} 2^{k_1-k_2} |\alpha_{i_1}^{(k_1)}| \ \ |\alpha_{i_2}^{(k_2)}|$$

$$\le c_9 \left(\sqrt{2^{k_1-k_2}} \ |\alpha_{i_1}^{(k_1)}| \sqrt{ \left(\sum_{x_{i_2}^{(k_2)} \in \ \text{supp}(\varphi_{i_1}^{(k_1)})} (\alpha_{i_2}^{(k_2)})^2 \right) } \right)$$

$$\le \frac{1}{2} c_9 \left(\sqrt{2^{k_1-k_2}} (\alpha_{i_1}^{(k_1)})^2 + \sqrt{2^{k_1-k_2}} \left(\sum_{x_{i_2}^{(k_2)} \in \ \text{supp}(\varphi_{i_1}^{(k_1)})} (\alpha_{i_2}^{(k_2)})^2 \right) \right).$$

Here we use the fact that the number of nodes $x_{i_2}^{(k_2)}$ satisfying $x_{i_2}^{(k_2)} \in \operatorname{supp}(\varphi_{i_1}^{(k_1)})$ is $O(2^{k_2-k_1})$. Summing up these estimates, we have

$$\sum_{i_1=1}^{J} \left| \left(\alpha_{i_1}^{(k_1)} \nabla u_{i_1}^{(k_1)}, \sum_{i_2=1}^{N_{k_2}} \alpha_{i_2}^{(k_2)} \nabla u_{i_2}^{(k_2)} \right)_{L_2(\Omega)} \right| \leq$$

$$\leq c_{10} \left(\sqrt{2^{k_1-k_2}} \sum_{i_1=1}^{N_{k_1}} (\alpha_{i_1}^{(k_1)})^2 + \sqrt{2^{k_1-k_2}} \sum_{i_2=1}^{N_{k_2}} (\alpha_{i_2}^{(k_2)})^2 \right),$$

$$\sum_{k_1=1}^{J} \sum_{k_2=k_1+1}^{J} \sum_{i_1=1}^{N_{k_1}} \left| \left(\alpha_{i_1}^{(k_1)} \nabla u_{i_1}^{(k_1)}, \sum_{i_2=1}^{N_{k_2}} \alpha_{i_2}^{(k_2)} \nabla u_{i_2}^{(k_2)} \right)_{L_2(\Omega)} \right| \leq$$

$$\leq c_{10} \sum_{k_1=1}^{J} \sum_{k_2=k_1+1}^{J} \left(\sqrt{2^{k_1-k_2}} \sum_{i_1=1}^{N_{k_1}} (\alpha_{i_1}^{(k_1)})^2 + \sqrt{2^{k_1-k_2}} \sum_{i_2=1}^{N_{k_2}} (\alpha_{i_2}^{(k_2)})^2 \right) \leq$$

$$\leq c_8 \left(\sum_{k=1}^{J} \sum_{i=1}^{N_k} (\alpha_i^{(k)})^2 \right).$$

Here the constants c_9, c_{10} are independent of h.

Theorem 1 *There exists a positive constant c_{11}, independent of h, such that*

$$\|t\varphi^h\|_{H^1(\Omega)} \leq c_{11} \|\varphi^h\|_{H^{\frac{1}{2}}(\Gamma)} \quad \forall \varphi^h \in V_J.$$

Here the operator t is from (3).

Proof of this theorem follows from the Lemma 1, the Lemma 2, and the Lemma 4.

Remark 2 *The construction of the extension operator t for three dimensional problems can be done in the same way. The Theorem 1 is valid too. Indeed, it's obvious that the Lemma 1 and the Lemma 2 are valid. Instead of the Lemma 3 we have the following lemma.*

Lemma 3′ *Let $k_2 > k_1$. Then*

$$\left| (\nabla u_{i_1}^{(k_1)}, \nabla u_{i_2}^{(k_2)})_{L_2(\Omega)} \right| \leq \begin{cases} 0 & , \text{ if } x_{i_2}^{(k_2)} \notin \ supp(\varphi_{i_1}^{(k_1)}), \\ c_6' \cdot 2^{k_1-2k_2} & , \text{ if } x_{i_2}^{(k_2)} \in \ supp(\varphi_{i_1}^{(k_1)}). \end{cases}$$

Here c_6' is independent of h.

Proof This is a trivial consequence of the following obvious estimates:

$$\begin{aligned}\left|\nabla u_{i_1}^{(k_1)}\right| &\leq c_7' \cdot 2^{k_1},\\ \left|\nabla u_{i_2}^{(k_2)}\right| &\leq c_7' \cdot 2^{k_2},\\ \text{meas}(\text{ supp } u_{i_2}^{(k_2)}) &\leq c_7' \cdot (2^{-k_2})^3.\end{aligned}$$

where c_7' is independent of h. The Lemma 4 is transformed to the following lemma:

Lemma 4$'$ *There exists a positive constant c_8', independent of h, such that*

$$\sum_{k_1=1}^{J}\sum_{k_2=k_1+1}^{J}\left|(\nabla u_{k_1}^h, \nabla u_{k_2}^h)_{L_2(\Omega)}\right| \leq c_8' \sum_{k=1}^{J}\sum_{i=1}^{N_k} 2^{-k}(\alpha_i^{(k)})^2.$$

Here $\alpha_i^{(k)}$ is from (4).

Proof We have

$$\sum_{k_1=1}^{J}\sum_{k_2=k_1+1}^{J}\left|(\nabla u_{k_1}^h, \nabla u_{k_2}^h)_{L_2(\Omega)}\right| =$$

$$= \sum_{k_1=1}^{J}\sum_{k_2=k_1+1}^{J}\left|\Bigl(\sum_{i_1=1}^{N_{k_1}}\alpha_{i_1}^{(k_1)}\nabla u_{i_1}^{(k_1)}, \sum_{i_2=1}^{N_{k_2}}\alpha_{i_2}^{(k_2)}\nabla u_{i_2}^{(k_2)}\Bigr)_{L_2(\Omega)}\right|.$$

Using the Lemma 3' and the Cauchy inequality, we have:

$$\left|\Bigl(\alpha_{i_1}^{(k_1)}\nabla u_{i_1}^{k_1)}, \sum_{i_2=1}^{N_{k_2}}\alpha_{i_2}^{(k_2)}\nabla u_{i_2}^{(k_2)}\Bigr)_{L_2(\Omega)}\right|$$

$$\leq c_6' \sum_{x_{i_2}^{(k_2)}\in\ \text{supp}(\varphi_{i_1}^{(k_1)})} 2^{k_1-2k_2}|\alpha_{i_1}^{(k_1)}|\ |\alpha_{i_2}^{(k_2)}|$$

$$\leq c_6' \sum_{x_{i_2}^{(k_2)}\in\ \text{supp}(\varphi_{i_1}^{(k_1)})} (2^{k_1-\frac{3}{2}k_2}|\alpha_{i_1}^{(k_1)}|)(2^{-\frac{1}{2}k_2}|\alpha_{i_2}^{(k_2)}|)$$

$$\leq c_9'\sqrt{2^{k_1-k_2}}\ \sqrt{2^{-k_1}}\ |\alpha_{i_1}^{(k)}|\Bigl(\sum_{x_{i_2}^{(k_2)}\in\ \text{supp}(\varphi_{i_1}^{(k_1)})} 2^{-k_2}(\alpha_{i_2}^{(k_2)})^2\Bigr)^{\frac{1}{2}}$$

$$\leq \frac{1}{2}c_9'\left[\sqrt{2^{k_1-k_2}}(2^{-k_1}(\alpha_{i_1}^{(k_1)})^2) + \sqrt{2^{k_1-k_2}}\left(\sum_{x_{i_2}^{(k_2)}\in\ \text{supp}(\varphi_{i_1}^{(k_1)})} 2^{-k_2}(\alpha_{i_2}^{(k_2)})^2\right)\right].$$

Here we use the fact that the number of nodes $x_{i_2}^{(k_2)}$ satisfying $x_{i_2}^{(k_2)} \in \ \mathrm{supp}(\varphi_{i_1}^{(k_1)})$ is $O(2^{2(k_2-k_1)})$. Summing up these estimates, we obtain

$$\sum_{i_1=1}^{N_{k_1}} \left| \left(\alpha_{i_1}^{(k_1)} \nabla u_{i_1}^{(k_1)}, \sum_{i_2=1}^{N_{k_2}} \alpha_{i_2}^{(k_2)} \nabla u_{i_2}^{(k_2)} \right)_{L_2(\Omega)} \right| \leq$$

$$\leq c'_{10} (\sqrt{2^{k_1-k_2}} \sum_{i_1=1}^{N_{k_1}} (2^{-k_1} (\alpha_{i_1}^{(k_1)})^2) + \sqrt{2^{k_1-k_2}} \sum_{i_2=1}^{N_{k_2}} \left(2^{-k_2} (\alpha_{i_2}^{(k_2)})^2 \right).$$

Then, repeating the estimates from the proof of the Lemma 4, we get the statement of the Lemma 4'.

Remark 3 *The cost of the action of the extention operator t is proportional to the number of nodes of the grid domain.*

If the original domain is splitted into many subdomains in domain decomposition methods [5], then the diameters of the subdomains depend on some small parameter ε and we need the extension operator t such that the constant c_{11} from the Theorem 1 is independent of ε. To do this, let us assume that by making the change of variables

$$x = \varepsilon \cdot s, \quad x \in \Omega \tag{5}$$

the domain Ω is transformed into the domain Ω' with the boundary Γ' and that the properties of Ω' are independent of ε. From [5,6] we have the following.

Lemma 5 *There exists a positive constant c_{12}, independent of h and ε, such that*

$$c_{12} \|\varphi^h\|_{H_\varepsilon^{\frac{1}{2}}(\Gamma)} \leq \|u^h\|_{H^1(\Omega)}$$

for any function $u^h \in W_J$, where $\varphi^h \in V_J$ is the trace of u^h at the boundary Γ. And there exists a positive constant c_{13}, independent of h and ε, such that for any $\varphi^h \in V_J$ there exists $u^h \in W_J$:

$$\begin{aligned} u^h(x) &= \varphi^h(x), \quad x \in \Gamma, \\ \|u^h\|_{H^1(\Omega)} &\leq c_{13} \|\varphi^h\|_{H_\varepsilon^{\frac{1}{2}}(\Gamma)}. \end{aligned}$$

Here

$$\begin{aligned} \|\varphi^h\|^2_{H_\varepsilon^{\frac{1}{2}}(\Gamma)} &= \varepsilon \|\varphi^h\|^2_{L_2(\Gamma)} + |\varphi^h|^2_{H^{\frac{1}{2}}(\Gamma)}, \\ \|\varphi^h\|^2_{L_2(\Gamma)} &= \int_\Gamma (\varphi^h(x))^2 dx, \\ |\varphi^h|^2_{H^{\frac{1}{2}}(\Gamma)} &= \int_\Gamma \int_\Gamma \frac{(\varphi^h(x) - \varphi^h)y))^2}{|x-y|^2} dx\, dy. \end{aligned}$$

Lemma 6 *There exists a positive constant* c_{14}*, independent of* h *and* ε*, such that for any* $\varphi^h \in V_J$

$$\|\varphi_0^h\|^2_{H_\epsilon^{\frac{1}{2}}(\Gamma)} + \frac{1}{\varepsilon}\|\varphi_1^h\|^2_{L_2(\Gamma)} + |\varphi_1^h|^2_{H^{\frac{1}{2}}(\Gamma)} \le c_{14}\|\varphi^h\|^2_{H_\varepsilon^{\frac{1}{2}}(\Gamma)}.$$

Here

$$\varphi_0^h = \widetilde{Q}_0\varphi^h, \quad \varphi_1^h = \varphi^h - \varphi_0^h.$$

The following lemma is valid.

Lemma 7 *There exists a positive constant* c_{15}*, independent of* h *and* ε*, such that*

$$\|\varphi_0^h\|^2_{H_\varepsilon^{\frac{1}{2}}(\Gamma)} + \frac{1}{\varepsilon}\left(\|\widetilde{Q}_0\varphi_1^h\|^2_{L_2(\Gamma)} + \sum_{k=1}^{J} 2^k\|(\widetilde{Q}_k - \widetilde{Q}_{k-1})\varphi_1^h\|^2_{L_2(\Gamma)}\right) \le c_{15}\|\varphi^h\|^2_{H_\epsilon^{\frac{1}{2}}(\Gamma)}$$

Here φ_0^h, φ_1^h*, are from (6).*

Proof Using (5) and the Lemma 1, we have

$$\frac{1}{\varepsilon}\|\varphi_1^h\|^2_{L_2(\Gamma)} + |\varphi_1^h|^2_{H^{\frac{1}{2}}(\Gamma)}$$

$$= \|\varphi_1^h\|^2_{L_2(\Gamma')} + |\varphi_1^h|^2_{H^{\frac{1}{2}}(\Gamma')}$$

$$\le \frac{1}{c_1}(\|\widetilde{Q}_0'\varphi_1^h\|^2_{L_2(\Gamma')} + \sum_{k=1}^{J} 2^k\|(\widetilde{Q}_k' - \widetilde{Q}_{k-1}')\varphi_1^h\|^2_{L_2(\Gamma')})$$

$$= \frac{1}{\varepsilon}\frac{1}{c_1}(\|\widetilde{Q}_0\varphi_1^h\|^2_{L_2(\Gamma)} + \sum_{k=1}^{J} 2^k\|(\widetilde{Q}_k - \widetilde{Q}_{k-1})\varphi_1^h\|^2_{L_2(\Gamma)}.$$

Here $\widetilde{Q}_k'$ is the projection which corresponds to $\widetilde{Q}_k$ with the change of variables.

Theorem 2 *There exists a positive constant* c_{16}*, independent of* h *and* ε*, such that*

$$\|t\varphi^h\|_{H^1(\Omega)} \le c_{16}\|\varphi^h\|_{H_\varepsilon^{\frac{1}{2}}(\Gamma)} \quad \forall\varphi^h \in V_J.$$

Here the operator t *is from (3).*

Proof For φ_0^h, φ_1^h from (6) we have

$$
\begin{aligned}
\|\widetilde{Q}_0\varphi^h\|^2_{H_\varepsilon^{\frac{1}{2}}(\Gamma)} &+ \sum_{k=1}^{J} 2^k \|(\widetilde{Q}_k - \widetilde{Q}_{k-1})\varphi^h\|^2_{L_2(\Gamma)} \le \\
\le \|\widetilde{Q}_0\varphi^h\|^2_{H_\varepsilon^{\frac{1}{2}}(\Gamma)} &+ \sum_{k=1}^{J} 2^k \|(\widetilde{Q}_k - \widetilde{Q}_{k-1})\varphi_1^h\|^2_{L_2(\Gamma)} + \\
&+ \sum_{k=1}^{J} 2^k \|(\widetilde{Q}_k - \widetilde{Q}_{k-1})\varphi_0^h\|^2_{L_2(\Gamma)}.
\end{aligned}
$$

For the function φ_0^h let us consider the following decomposition:

$$
\begin{aligned}
\varphi_0^h &= \varphi_{0,0}^h + \varphi_{0,1}^h, \\
\varphi_{0,0}^h &= \text{const} = \frac{1}{\text{meas}(\Gamma)} \int\limits_\Gamma \varphi_0^h(x)\, dx \\
\varphi_{0,1}^h &= \varphi_0^h - \varphi_{0,0}^h.
\end{aligned}
$$

It is easy to see that

$$
(\widetilde{Q}_k - \widetilde{Q}_{k-1})\varphi_{0,0}^h = 0, \quad k = 1, 2, \cdots, J.
$$

Then we can use the evident trick from [4] with the Poincare inequality in $H^{\frac{1}{2}}(\Gamma')$:

$$
\sum_{k=1}^{J} 2^k \|(\widetilde{Q}_k - \widetilde{Q}_{k-1})\varphi_0^h\|^2_{L_2(\Gamma)} = \sum_{k=1}^{J} 2^k \|(\widetilde{Q}_k - \widetilde{Q}_{k-1})\varphi_{0,1}^h\|^2_{L_2(\Gamma)} =
$$

$$
= \varepsilon \sum_{k=1}^{J} 2^k \|(\widetilde{Q}'_k - \widetilde{Q}'_{k-1})\varphi_{0,1}^h\|^2_{L_2(\Gamma')} \le c_2 \varepsilon \|\varphi_{0,1}^h\|^2_{H^{\frac{1}{2}}(\Gamma')} \le
$$

$$
\le c_{17}\, \varepsilon |\varphi_{0,1}^h|^2_{H^{\frac{1}{2}}(\Gamma^1)} = c_{17}\, \varepsilon |\varphi_{0,1}^h|^2_{H^{\frac{1}{2}}(\Gamma)} = c_{17}\, \varepsilon |\varphi_0^h|^2_{H^{\frac{1}{2}}(\Gamma)}.
$$

Here c_{17} is from the Poincare inequality. It is easy to see that there exists a positive constant c_{18}, independent of h and ε, such that

$$
\|u_0^h\|_{H^1(\Omega)} \le c_{18} \|\psi_0^h\|_{H_\varepsilon^{\frac{1}{2}}(\Gamma)},
$$

where $\psi_0^h = \varphi_0^h = \widetilde{Q}_0\varphi^h$, and $u_0^h \in W_0$ is from (2). The rest of the estimates for the Theorem 2 and the Theorem 1 is the same.

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