Anisotropic mesh refinement in stabilized Galerkin methods*

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Abstract. The numerical solution of the convection-diffusion-reaction problem is considered in two and three dimensions. A stabilized finite element method of Galerkin/Least-squares type accomodates diffusion-dominated as well as convection- and/or reaction-dominated situations. The resolution of boundary layers occuring in the singularly perturbed case is accomplished using anisotropic mesh refinement in boundary layer regions. In this paper, the standard analysis of the stabilized Galerkin method on isotropic meshes is extended to more general meshes with boundary layer refinement. Simplicial Lagrangian elements of arbitrary order are used.

Key Words. Elliptic boundary value problem, convection-diffusion-reaction problem, finite element method, Galerkin/Least-squares, mesh refinement, anisotropic finite elements

AMS(MOS) subject classification. 65N30, 65N50

Contents

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1 Introduction

This paper is concerned with the finite element solution of the following elliptic boundary value problem in a bounded polyhedral domain $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$, with Lipschitz boundary $\partial\Omega$:

$$\begin{aligned} L_\varepsilon u \equiv -\varepsilon \Delta u + b \cdot \nabla u + cu &= f \quad \text{in } \Omega, && (1.1) \\ u &= 0 \quad \text{on } \partial\Omega, && (1.2) \end{aligned}$$

$\varepsilon \in (0, 1]$ is a parameter. (1.1) (1.2) is a linear(ized) diffusion-convection-reaction model. In particular, arbitrary ratios $P(x) \equiv \varepsilon^{-1} \|b(x); R^d\|$ (Peclet number) and $\Gamma(x) \equiv \varepsilon^{-1}|c(x)|$ will be considered. Hence the whole range from (locally) diffusion-dominated ($P, \Gamma \ll 1$) to (locally) convection- and/or reaction-dominated problems ($P \gg 1$ and/or $\Gamma \gg 1$) is of interest. In case of $P \gg 1$ and/or $\Gamma \gg 1$, (1.1) (1.2) is of singularly perturbed type and the solution u may generate sharp boundary or interior layers where the solution of the limit problem with $\varepsilon = 0$ is not smooth or cannot satisfy the boundary condition (1.2). The resolution of such layers is often the main interest in applications and will be considered in this paper.

Standard Galerkin finite element solutions may suffer from numerical instabilities which are generated by dominant convection and/or reaction terms unless the mesh is sufficiently refined. As a remedy, stabilized Galerkin methods have been proposed: the streamline diffusion method (SD) [7, 12, 17], the Galerkin/Least-squares method (GLS), see for example [13], and shock-capturing variants of them, see for example [9, 10, 14, 15]. In contrast to standard methods of upwind type, stabilized Galerkin methods have the advantage to be consistent with the weak formulation of (1.1) (1.2). We will focus on the (GLS)-method.

Up to now, stabilized Galerkin methods were analyzed for *isotropic* meshes, that means $h_e/\varrho_e = \mathcal{O}(1)$ for $\varepsilon \to 0$, $h \to 0$, where h_e and ϱ_e denote the diameter of the finite element e and the diameter of the largest inscribed ball in e, respectively. But a resolution of boundary and interior layers with isotropic elements leads to an overrefinement. An *anisotropic* mesh refinement in the sense $\lim\limits_{\varepsilon \to +0} h_e/\varrho_e = \infty$ is much more efficient in such thin layers.

We remark that the permission of $\varrho_e = o(h_e)$ for $h \to 0$ was already discussed in [6, 16, 18, 19] but they did not derive an advantage (from the point of view of numerical analysis) of using different element diameters in different directions. This remedy was removed in [3, 4, 5] by proving sharper estimates on the reference element, and the improved estimates were applied to establish a-priori mesh refinement near geometrical singularities (edges) in the case of diffusion-dominated equations (Poisson type problems) [3, 5]. In this case anisotropy was used in a slightly different sense than we do here, namely $\lim\limits_{h \to +0} h_e/\varrho_e = \infty$. But this makes no difference for the anisotropic local estimates. — We note that anisotropic elements were also considered from other points of view in [20, 23, 24, 25, 26, 28, 29, 30].

In this paper, we extend the numerical analysis of the Galerkin/Least-squares method to meshes which are anisotropically refined at least in boundary layers. The aim is to derive error estimates in the energy norm uniformly with respect to $\varepsilon \in (0, 1]$. Such an approach is theoretically possible also in interior layers. But, unfortunately, it turns out that the elements in the layer have to be oriented with respect to the manifold where the layer is located; in general this cannot be done a-priori. A numerical localization procedure for interior layers is described in [30]. — We remark that ε-uniform estimates were also derived using exponentially fitted Galerkin methods [1, 22] or finite difference methods on certain orthogonal meshes [27].

The outline of the paper is as follows: In Section 2 we consider Lagrangian interpolation on simplicial elements and review local inequalities in the anisotropic case. In Section 3 we introduce the stabilized Galerkin method (GLS) for problem (1.1) (1.2). Under weak conditions to the mesh (maximal angle condition instead of minimal angle condition) we prove existence and stability of the discrete solution, as well as convergence to the weak

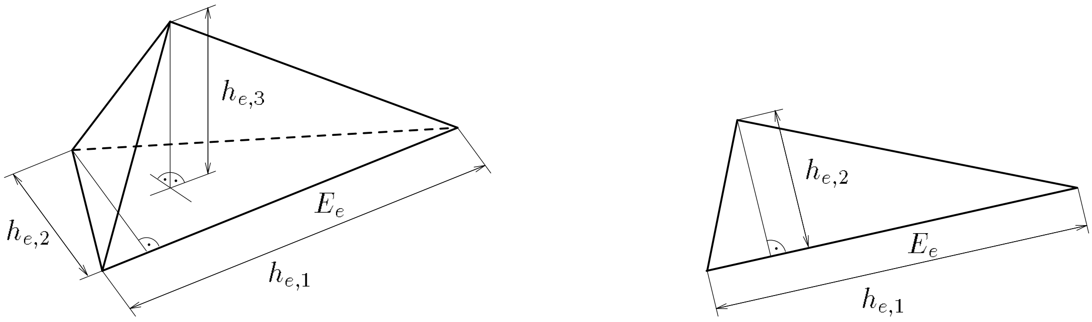

Figure 2.1: Illustration of the definition of the element related mesh sizes.

solution $u \in W^{1,2}(\Omega)$ and to regular solutions $u \in W^{r+1,2}(\Omega)$, $r \geq 1$. Moreover we derive the optimal choice of the numerical damping parameters.

Section 4 is devoted to the analysis of anisotropic mesh refinement in boundary layers of problem (1.1) (1.2) using the results of Section 2. However, the critical point is an assumption on the Sobolev norms of u which are hard to prove in general cases. For this reason we apply these quite general results to a special class of problems where such a priori knowledge on u is available and thus an almost optimal refinement strategy can be proposed. This is done in Section 5 by considering domains of channel type. In particular, the actual choice of the element diameters in the refinement zone and the determination of the numerical damping parameters is addressed. The final error estimate is almost uniform with respect to the small parameter ε.

2 Notation and local estimates for general finite elements

We consider Lagrangian elements on simplices $e \subset \mathbb{R}^3$, $d = 2, 3$, with spaces $\mathcal{P}_k$ of polynomials of maximal degree $k \geq 1$. The interpolant of a continuous function v is uniquely determined by $(I_h^{(k)} v)(x^{(i)}) = v(x^{(i)})$ $(i = 1, \ldots, n,\ n = \dim(\mathcal{P}_k) = \binom{k+d}{d})$, where $x^{(i)}$ are the nodal points of the element e. In this section, we summarize the local inequalities and a density result which were proved in [4].

For exploring the different sizes of the element e in different directions we introduce the following notation, compare Figure 2.1. For $e \subset \mathbb{R}^2$ let E_e be the longest edge of e. Then we denote by $h_{1,e} \equiv \mathrm{meas}_1(E_e)$ its length and by $h_{2,e} \equiv 2\,\mathrm{meas}_2(e)/h_{1,e}$ the diameter of e perpendicularly to E_e. In the three-dimensional case, we proceed by analogy. Let again E_e be the longest edge of e, and let F_e be the larger of the two faces of e with $E_e \subset \overline{F}_e$. Then we denote by $h_{1,e} \equiv \mathrm{meas}_1(E_e)$ the length of E_e, by $h_{2,e} \equiv 2\,\mathrm{meas}_2(F_e)/h_{1,e}$ the diameter of F_e perpendicularly to E_e, and by $h_{3,e} \equiv 6\,\mathrm{meas}_3(e)/(h_{1,e}h_{2,e})$ the diameter of e perpendicularly to F_e. Note that for the element sizes the relation

$$h_{1,e} \geq \ldots \geq h_{d,e}, \tag{2.1}$$

holds.

Introduce further a Cartesian coordinate system $(x_{1,e}, x_{2,e}, x_{3,e})$ such that $(0,0,0)$ is a vertex of $\hat{e}$, E_e is part of the $x_{1,e}$–axis, and F_e is part of the $x_{1,e}, x_{2,e}$–plane. The two-dimensional case is treated by analogy. Subsequently, this system will be called *element related coordinate system.* By contrast we consider a *discretization independent coordinate system* (x_1, x_2) or (x_1, x_2, x_3) which may be global or related to the boundary or it may be problem related in any other sense but independent of the finite element mesh.

For anisotropic interpolation error estimates we have to assume that the elements fulfill a maximal angle condition.

Maximal angle condition (2D): There is a constant $\gamma_* < \pi$ (independent of h and $e \in \mathcal{T}_h$) such that the maximal interior angle γ_e of any element e is bounded by γ_* : $\gamma_e \leq \gamma_*$.

Maximal angle condition (3D): There is a constant $\gamma_* < \pi$ (independent of h and $e \in \mathcal{T}_h$) such that the maximal interior angle $\gamma_{f,e}$ of the four faces as well as the maximal angle $\gamma_{E,e}$ between two faces of any element e is bounded by $\gamma_* :\ \gamma_{f,e} \le \gamma_*,\ \gamma_{E,e} \le \gamma_*$.

Moreover, we need for all anisotropic estimates the coordinate system condition.

Coordinate system condition (2D): The element related coordinate system $(x_{1,e}, x_{2,e})$ can be transformed into the discretization independent coordinate system (x_1, x_2) via a translation and a rotation by an angle ψ_e, where $|\sin\psi_e| \le Ch_{2,e}/h_{1,e}$.

Coordinate system condition (3D): The transformation of the element related coordinate system $(x_{1,e}, x_{2,e}, x_{3,e})$ into the discretization independent system (x_1, x_2, x_3) can be determined as a translation and three rotations around the $x_{j,e}$-axes by angles $\psi_{j,e}\,(j = 1, 2, 3)$, where

$$|\sin\psi_{1,e}| \le Ch_3/h_2, \quad |\sin\psi_{2,e}| \le Ch_3/h_1, \quad |\sin\psi_{3,e}| \le Ch_2/h_1. \tag{2.2}$$

Note that we use the symbol C for a generic positive constant, that means, C may be of different value at each occurrence. But C is always independent of the function under consideration, of the finite element mesh, and particularly of ε. On the contrary, some constants are indexed with a letter for later reference to them.

Let $W^{m,2}(e)$, $m \in I\!N$, be the usual Sobolev spaces with the norm and the special seminorm

$$\|v; W^{m,2}(e)\| \equiv \left\{ \sum_{|\alpha|\le m} \int_e |D^\alpha v|^2 dx \right\}^{1/2}, \qquad |v; W^{m,2}(e)| \equiv \left\{ \sum_{|\alpha| = m} \int_e |D^\alpha v|^2 dx \right\}^{1/2}.$$

We use a multi-index notation with

$$\alpha = (\alpha_1, \dots, \alpha_d), \quad |\alpha| = \alpha_1 + \ldots + \alpha_d, \quad D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}, \quad h_e^\alpha = h_{1,e}^{\alpha_1} \cdots h_{d,e}^{\alpha_d},$$

the numbers α_i $(i = 1, \dots, d)$ are non-negative integers.

Lemma 2.1 (Inverse inequality) *Assume that for the element e the coordinate system condition holds. Then for $v \in \mathcal{P}_k$, $k \in I\!N$ arbitrary, the estimate*

$$\|\Delta v; L^2(e)\| \le C \left(\sum_{i=1}^d h_{i,e}^{-2} \left\| \frac{\partial v}{\partial x_i}; L^2(e) \right\|^2 \right)^{1/2} \tag{2.3}$$

holds. The particular result

$$\|\Delta v; L^2(e)\| \le C_s h_{d,e}^{-1} |v; W^{1,2}(e)| \tag{2.4}$$

is valid without the coordinate system condition. Moreover, there is $C_s = 0$ for $k = 1$.

Lemma 2.2 (Anisotropic interpolation error estimates) *Assume that for an element e the maximal angle condition as well as the coordinate system condition hold. Then for $v \in W^{k+1,2}(e)$ and $m = 0, \dots, k$ the estimate*

$$|v - I_h^{(k)} v; W^{m,2}(e)|^2 \le C \sum_{|\alpha| = k+1-m} h_e^{2\alpha} |D^\alpha v; W^{m,2}(e)|^2 \tag{2.5}$$

holds, if $d = 2$ or $m < k$. If $v \in W^{k+2,2}(e)$, there holds

$$|v - I_h^{(k)} v; W^{m,2}(e)|^2 \le C \sum_{k+1-m \le |\alpha| \le k+2-m} h_e^{2\alpha} |D^\alpha v; W^{m,2}(e)|^2 \tag{2.6}$$

for $d = 2, 3$, $m = 0, \dots, k$.

Remark 2.3 The size of the constants C in the coordinate system condition influences the size of the constants in (2.5) and (2.6). Without the coordinate system condition we can only prove estimates without deriving advantage of the different element diameters, see the following lemma.

Lemma 2.4 *Assume that the element e fulfills the maximal angle condition. Then for $v \in W^{k+1,2}(e)$ and $m = 0, \ldots, k$ the estimate*

$$|v - I_h^{(k)} v; W^{m,2}(e)| \leq C h_{1,e}^{k+1-m} |v; W^{k+1,2}(e)| \tag{2.7}$$

holds, if $d = 2$ or $m < k$. If $v \in W^{k+2,2}(e)$ there holds

$$|v - I_h^{(k)} v; W^{m,2}(e)| \leq C \sum_{\ell=k+1}^{k+2} h_{1,e}^{\ell-m} |v; W^{\ell,2}(e)| \tag{2.8}$$

for $d = 2, 3$, $m = 0, \ldots, k$.

Note that the coordinate system condition is not necessary for Lemma 2.4.

Let $\mathcal{T}_h = \{e\}$ be an admissible triangulation of $\overline{\Omega} = \bigcup_e \overline{e}$, that means, let properties ($\mathcal{T}_h 1$) $\cdots$ ($\mathcal{T}_h 5$) of [8, Chapter 2] be fulfilled. Assume that $\mathcal{T}_h$ satisfies the maximal angle condition. Moreover, introduce the spaces V and V_h by

$$\begin{aligned} V &\equiv W_0^{1,2}(\Omega) \equiv \{v \in W^{1,2}(\Omega) : v|_{\partial\Omega} = 0\}, && (2.9) \\ V_h &\equiv \{v \in V : v|_e \in \mathcal{P}_k(e) \quad \forall e \in \mathcal{T}_h\}. && (2.10) \end{aligned}$$

The index h indicates that we are considering a family of spaces for $h \to +0$, h itself characterizes the mesh size; we can for example think of $h = \max_{e \in \mathcal{T}_h} h_{1,e}$.

Lemma 2.5 (Density of V_h in V) *Let $u \in V$ be an arbitrary function, then*

$$\lim_{h \to +0} \inf_{v_h \in V_h} \|u - v_h; W^{1,2}(\Omega)\| = 0.$$

Remark 2.6 If v has the property $v \in W^{r+1,2}(e)$ with $r > k$ (or $r > k+1$) then the estimates (2.5) and (2.7) (or (2.6) and (2.8), respectively) hold true. If $r < k$ (or $r < k+1$) we should use $I_h^{(r)}$ for interpolation. Note that $I_h^{(r)} u \in V_h$, too.

3 A stabilized Galerkin method on general meshes

3.1 Statement of the problem

We consider the second order elliptic boundary value problem

$$\begin{aligned} L_\varepsilon u \equiv -\varepsilon \Delta u + b \cdot \nabla u + cu &= f \quad \text{in } \Omega \subset I\!R^d,\ d = 2, 3, && (3.1) \\ u &= 0 \quad \text{on } \partial\Omega, && (3.2) \end{aligned}$$

with the basic assumptions

(H.1) $0 < \varepsilon \leq 1$, $b \in [W^{1,\infty}(\Omega)]^d$, $c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$,

(H.2) $\nabla \cdot b = 0$, $c \geq 0$ almost everywhere in Ω.

The variational formulation of (3.1) (3.2) reads

$$\textit{Find } u \in V, \textit{ such that } B_G(u, v) = L_G(v) \quad \forall v \in V. \tag{3.3}$$

where

$$B_G(u,v) \equiv \varepsilon(\nabla u, \nabla v)_\Omega + \tfrac{1}{2}\{(b\cdot\nabla u, v)_\Omega - (b\cdot\nabla v, u)_\Omega\} + (cu, v)_\Omega, \tag{3.4}$$

$$L_G(v) \equiv (f, v)_\Omega, \tag{3.5}$$

and $(.\,,.)_G$ denotes the inner product in $L^2(G)$, $G \subseteq \Omega$. Moreover, the standard Galerkin method (G) of (3.3) is introduced by

$$\textit{Find } u_h \in V_h, \textit{ such that } B_G(u_h, v_h) = L_G(v_h) \quad \forall v_h \in V_h. \tag{G}$$

V and V_h are introduced in Section 2.

We remind the well-known fact that the solution u_h of (G) on isotropic meshes may suffer from non-physical oscillations unless the elementwise numbers

$$P_e \equiv \varepsilon^{-1} h_{1,e} \|b; [L^\infty(e)]^d\|, \quad \Gamma_e \equiv \varepsilon^{-1} h_{1,e}^2 \|c; L^\infty(e)\| \tag{3.6}$$

are sufficiently small. As a remedy, we consider the following stabilized method of Galerkin/ Least-squares type:

$$\textit{Find } U_h \in V_h, \textit{ such that } B_{SG}(U_h, v_h) = L_{SG}(v_h) \quad \forall v_h \in V_h. \tag{GLS}$$

with

$$B_{SG}(u,v) \equiv B_G(u,v) + \sum_e \delta_e\,(L_\varepsilon u, L_\varepsilon v)_e, \tag{3.7}$$

$$L_{SG}(v) \equiv L_G(v) + \sum_e \delta_e\,(f, L_\varepsilon v)_e, \tag{3.8}$$

and a set $\{\delta_e\}$ of non-negative numerical diffusion parameters to be determined below.

3.2 Existence and stability of discrete solutions

First of all, we state lower and upper bounds of the bilinear form $B_{SG}(.\,,.)$.

Lemma 3.1 *Under the assumptions (H.1), (H.2), there holds for $v \in V$ with $\Delta v|_e \in L^2(e)$ $\forall e \in \mathcal{T}_h$ that*

$$B_{SG}(v,v) = |||\, v \,|||_{\varepsilon,\delta}^2$$

with

$$|||\, v \,|||_{\varepsilon,\delta}^2 \equiv \varepsilon\|\nabla v; L^2(\Omega)\|^2 + \|\sqrt{c}v; L^2(\Omega)\|^2 + \sum_e \delta_e\,\|L_\varepsilon v; L^2(e)\|^2. \tag{3.9}$$

Proof Set $u = v$ in (3.7). □

Lemma 3.2 *For $v_h \in V_h$ and $u \in V$ with $\Delta u|_e \in L^2(e)$ $\forall e \in \mathcal{T}_h$ there holds*

$$|B_{SG}(u, v_h)| \le |||\, v_h \,|||_{\varepsilon,\delta} \left\{ |||\, u \,|||_{\varepsilon,\delta} + \left(\sum_e Z_e^2 \|u; L^2(e)\|^2 \right)^{1/2} \right\} \tag{3.10}$$

with

$$Z_e^2 \equiv \min\{B_e^2 \varepsilon^{-1};\ \delta_e^{-1} + \varepsilon C_s^2 h_{d,e}^{-2} + C_e\}, \tag{3.11}$$

$$B_e \equiv \|b; [L^\infty(e)]^d\|, \quad C_e \equiv \|c; L^\infty(e)\|. \tag{3.12}$$

C_s is the constant from (2.4).

Proof Integration by parts of the non-symmetric part of $B_{SG}(.\,,.)$ together with (H.2) yields for all $u, v \in V$

$$\tfrac{1}{2}\{(b\cdot\nabla u, v)_\Omega - (b\cdot\nabla v, u)_\Omega\} = -(b\cdot\nabla v, u)_\Omega,$$

hence with (3.4), (3.7), and (3.9)

$$
\begin{aligned}
|B_{SG}(u,v)| &\le \varepsilon\|\nabla u;[L^2(\Omega)]^d\|\,\|\nabla v;[L^2(\Omega)]^d\| + \|\sqrt{c}\,u;L^2(\Omega)\|\,\|\sqrt{c}\,v;L^2(\Omega)\| + \\
&\quad + \left(\sum_e \delta_e \|L_\varepsilon u;L^2(e)\|^2\right)^{1/2}\left(\sum_e \delta_e \|L_\varepsilon v;L^2(e)\|^2\right)^{1/2} + |(b\cdot\nabla v,u)_\Omega| \\
&\le |\!|\!|\, u \,|\!|\!|_{\varepsilon,\delta}\, |\!|\!|\, v \,|\!|\!|_{\varepsilon,\delta} + \sum_e |(b\cdot\nabla v,u)_e|. \qquad (3.13)
\end{aligned}
$$

Consider the last term at the right hand side. We get for $v_h \in V_h$ via inverse inequality (2.3)

$$
\begin{aligned}
|(b\cdot\nabla v_h,u)_e| &\le \|u;L^2(e)\|\min\{\|b\cdot\nabla v_h;L^2(e)\|; \\
&\qquad \|-\varepsilon\Delta v_h + b\cdot\nabla v_h + cv_h;L^2(e)\| + \|\varepsilon\Delta v_h;L^2(e)\| + \|cv_h;L^2(e)\|\} \\
&\le \|u;L^2(e)\|\min\{B_e\|\nabla v_h;[L^2(e)]^d\|; \\
&\qquad \|L_\varepsilon v_h;L^2(e)\| + \varepsilon C_s h_{d,e}^{-1}\|\nabla v_h;L^2(e)\| + \sqrt{C_e}\,\|\sqrt{c}\,v_h;L^2(e)\|\} \\
&\le \min\{B_e\varepsilon^{-1/2};\ \max\{\delta_e^{-1/2};\ \varepsilon^{1/2}C_s h_{d,e}^{-1};\ C_e^{1/2}\}\}\ |\!|\!|\, v_h \,|\!|\!|_e \|u;L^2(e)\| \quad (3.14)
\end{aligned}
$$

where $|\!|\!|\, v_h \,|\!|\!|_e$ is defined in analogy to (3.9) by

$$
|\!|\!|\, v_h \,|\!|\!|_e^2 \equiv \varepsilon\|\nabla v_h;L^2(e)\|^2 + \|\sqrt{c}v_h;L^2(e)\|^2 + \delta_e\|L_\varepsilon v_h;L^2(e)\|^2. \qquad (3.15)
$$

Using (3.13) – (3.15) we get the assertion by standard inequalities. □

Furthermore, we find the following a-priori stability estimate.

Lemma 3.3 *For the solution $U_h \in V_h$ and the residual $L_\varepsilon U_h - f$ of scheme (GLS) there holds*

$$
|\!|\!|\, U_h \,|\!|\!|_{\varepsilon,\delta}^2 + \sum_e \delta_e\|L_\varepsilon U_h - f;L^2(e)\|^2 \le D^2 \equiv C(\min\{C_F^2\varepsilon^{-1};\ \gamma^{-1}\} + \delta)\|f;L^2(\Omega)\|^2 \quad (3.16)
$$

with $\delta \equiv \max_e \delta_e$, $\gamma \equiv \inf_\Omega c(x)$ and Friedrichs' constant C_F.

Proof Set $v = U_h$ in (GLS). Lemma 3.1, together with Hölder's and Friedrichs' inequalities, implies

$$
\begin{aligned}
|\!|\!|\, U_h \,|\!|\!|_{\varepsilon,\delta}^2 &\le B_{SG}(U_h,U_h) \ = \ L_{SG}(U_h) \\
&\le \|f;L^2(\Omega)\|\,\|U_h;L^2(\Omega)\| + \left(\sum_e \delta_e\|f;L^2(e)\|^2\right)^{1/2}\left(\sum_e \delta_e\|L_\varepsilon U_h;L^2(e)\|^2\right)^{1/2} \\
&\le \|f;L^2(\Omega)\|\ \min\{\gamma^{-1/2}\|\sqrt{c}U_h;L^2(\Omega)\|;\ C_F\|\nabla U_h;[L^2(\Omega)]^d\|\} + \\
&\qquad + \sqrt{\delta}\|f;L^2(\Omega)\|\left(\sum_e \delta_e\|L_\varepsilon U_h;L^2(e)\|^2\right)^{1/2} \\
&\le \frac{1}{2}|\!|\!|\, U_h \,|\!|\!|_{\varepsilon,0}^2 + \frac{1}{2}\min\{\gamma^{-1};\ \varepsilon^{-1}C_F^2\}\|f;L^2(\Omega)\|^2 + \\
&\qquad + \frac{1}{2}\sum_e \delta_e\|L_\varepsilon U_h;L^2(e)\|^2 + \frac{\delta}{2}\|f;L^2(\Omega)\|^2,
\end{aligned}
$$

hence

$$
|\!|\!|\, U_h \,|\!|\!|_{\varepsilon,\delta}^2 \le (\min\{\gamma^{-1};\ \varepsilon^{-1}C_F^2\} + \delta)\|f;L^2(\Omega)\|^2
$$

A slight modification of the proof yields the weighted control of the discrete residual. □

Lemma 3.3 implies uniqueness and stability of the (GLS)-solution on an general admissible mesh (including anisotropic mesh refinement).

Theorem 3.4 *Under assumptions (H.1), (H.2) there exists one and only one solution $U_h \in V_h$ of scheme (GLS) which additionally satisfies (3.16).*

3.3 Convergence towards the weak solution

Let us consider now the strong convergence of the family $\{U_h\}$ of solutions of (GLS) to the weak solution $u \in V$ of (3.3). Note that we use only data under the assumptions (H.1), (H.2) and a technical condition (H.3) on the parameter set $\{\delta_e\}$:

$$\text{(H.3)} \qquad \lim_{h\to+0} \max_e \{\delta_e(\varepsilon C_s^2 h_{d,e}^{-2} + B_e^2 \varepsilon^{-1} + C_e)\} = 0$$

Theorem 3.5 *Assume that (H.1) – (H.3) are valid. Then the solution $U_h \in V_h$ of (GLS) converges strongly in V to the solution $u \in V$ of (3.3) according to*

$$\lim_{h\to+0} |||\, u - U_h \,|||_{\varepsilon,0} = 0. \tag{3.17}$$

Proof We split the error $u - U_h$ as follows:

$$u - U_h = (u - u_h) + (u_h - U_h) \equiv w_1 + w_2 \tag{3.18}$$

with the Galerkin solution $u_h \in V_h$ of (G), that means of (GLS) with $\delta_e = 0\ \forall e$. Let $\Pi_h v$ be the best approximate of v in V_h:

$$|||\, v - \Pi_h v \,|||_{\varepsilon,0} = \min_{v_h \in V_h} |||\, v - v_h \,|||_{\varepsilon,0}.$$

Denoting by $\tilde{\eta} \equiv u - \Pi_h u$ the approximation error, there holds via (3.3) – (3.5) and (3.10) with $\delta_e = 0\ \forall e$:

$$\begin{aligned} |||\, u_h - \Pi_h u \,|||_{\varepsilon,0}^2 &\leq B_G(u_h - \Pi_h u, u_h - \Pi_h u) = B_G(u - \Pi_h u, u_h - \Pi_h u) \\ &\leq |||\, u_h - \Pi_h u \,|||_{\varepsilon,0} \left[|||\, \tilde{\eta} \,|||_{\varepsilon,0} + \left(\sum_e B_e^2 \varepsilon^{-1} \|\tilde{\eta}; L^2(e)\|^2 \right)^{1/2} \right], \end{aligned}$$

hence

$$|||\, u_h - \Pi_h u \,|||_{\varepsilon,0} \leq |||\, \tilde{\eta} \,|||_{\varepsilon,0} + \left(\sum_e B_e^2 \varepsilon^{-1} \|\tilde{\eta}; L^2(e)\|^2 \right)^{1/2} \equiv F(\tilde{\eta}).$$

Lemma 2.5 yields that for $u \in V$ $\lim_{h\to+0} F(\tilde{\eta}) = 0$ and

$$\lim_{h\to+0} |||\, w_1 \,|||_{\varepsilon,0} \leq \lim_{h\to+0} (|||\, u - \Pi_h u \,|||_{\varepsilon,0} + |||\, \Pi_h u - u_h \,|||_{\varepsilon,0}) = 0. \tag{3.19}$$

For $w_2 = u_h - U_h \in V_h$ we have by (3.4), (3.7), (3.8), (3.9), (3.18), (G), (GLS), and (3.16)

$$\begin{aligned} |||\, w_2 \,|||_{\varepsilon,0}^2 &= B_G(w_2, w_2) = \sum_e \delta_e (L_\varepsilon U_h - f, L_\varepsilon w_2)_e \\ &\leq \left(\sum_e \delta_e \, \|L_\varepsilon U_h - f; L^2(e)\|^2 \right)^{1/2} \left(\sum_e \delta_e \, \|L_\varepsilon w_2; L^2(e)\|^2 \right)^{1/2} \\ &\leq D \left(\sum_e \delta_e \| -\varepsilon \Delta w_2 + b \cdot \nabla w_2 + c w_2; L^2(e)\|^2 \right)^{1/2} \equiv DT. \end{aligned} \tag{3.20}$$

The inverse inequality (2.3) yields

$$\begin{aligned} T^2 &\leq \sum_e \delta_e \left(\varepsilon^2 C_s^2 h_{d,e}^{-2} \|\nabla w_2; [L^2(e)]^d\|^2 + B_e^2 \|\nabla w_2; [L^2(e)]^d\|^2 + C_e \|\sqrt{c} w_2; L^2(e)\|^2 \right) \\ &\leq \max_e \{\delta_e(\varepsilon C_s^2 h_{d,e}^{-2} + B_e^2 \varepsilon^{-1} + C_e)\} \; |||\, w_2 \,|||_{\varepsilon,0}. \end{aligned} \tag{3.21}$$

A sufficient condition for $\lim_{h\to+0} |||\, w_2 \,|||_{\varepsilon,0} = 0$ is then (H.3), hence via (3.18), (3.19) we arrive at (3.17). $\square$

Remark 3.6 We conjecture that the asymptotic error estimate (3.17) remains valid under the weaker condition

(H.3') $\quad |\delta_e \varepsilon C_s^2 h_{d,e}^{-2}| \le 1 \;\; \forall e \in \mathcal{T}_h \quad$ and $\quad \lim_{h\to+0} \max_e \{\delta_e (B_e^2 \varepsilon^{-1} + C_e)\} = 0.$

This assertion is clear in the case of piecewise linear, simplicial elements ($k = 1$) because there holds $C_s = 0$. In the general case $k \ge 1$ we were not able to prove this, even if we assumed the weak regularity assumption $L_\varepsilon u = f$ in $L^2(e)$ $\forall e \in \mathcal{T}_h$. We found only a simplification of the proof of (3.17) but we were not able to avoid (H.3).

3.4 Convergence towards regular solutions

We consider now the case of smooth solutions of (3.3) according to

(H.4) $\quad u \in V \cap W^{r+1,2}(\Omega)$ for some $r \in I\!N$, $r \ge 1$.

Consequently, we have

$$B_{SG}(u - U_h, v) = 0 \quad \forall v \in V_h. \tag{3.22}$$

Note that (H.4) is valid with $r = 1$ if Ω is convex. — In order to simplify the forthcoming analysis we assume that the following modification of (H.3') holds:

(H.3") $\quad \delta_e^{-1} \ge \varepsilon C_s^2 h_{d,e}^{-2} + C_e$, which is with $\kappa_e \equiv h_{1,e}^{-1} h_{d,e}$ and $\Gamma_e \equiv \varepsilon^{-1} h_{1,e}^2 C_e$ equivalent to

$$\delta_e \le \frac{h_{1,e}^2}{\varepsilon(\kappa_e^2 C_s^2 + \Gamma_e)}.$$

Hence we replace Z_e^2 in (3.11) by

$$\tilde{Z}_e^2 \equiv \min\{B_e^2 \varepsilon^{-1};\; 2\delta_e^{-1}\}. \tag{3.23}$$

Theorem 3.7 *Let (H.1), (H.2), (H.3"), (H.4), as well as the* maximal angle condition *be satisfied. Then there hold for the error $u - U_h$*

$$|\!|\!| u - U_h |\!|\!|_{\varepsilon,\delta}^2 \le C_A \sum_e E_e^2 h_{1,e}^{2\min\{k;r\}} |u; W^{1+\min\{k;r\},2}(e)|^2 \tag{3.24}$$

if $\min\{k;r\} \ge 3$ *or* $d = 2$, *and*

$$|\!|\!| u - U_h |\!|\!|_{\varepsilon,\delta}^2 \le C_A \sum_e E_e^2 \sum_{\ell=\min\{k;r-1\}-1}^{\min\{k;r-1\}} h_{1,e}^{2\ell} |u; W^{\ell+2,2}(e)|^2, \tag{3.25}$$

without these conditions. E_e^2 is defined by

$$E_e^2 \equiv \varepsilon + C_e h_{1,e}^2 + \delta_e(\varepsilon^2 h_{1,e}^{-2} + B_e^2 + C_e^2 h_{1,e}^2) + h_{1,e}^2 \min\{\varepsilon^{-1} B_e^2;\; \delta_e^{-1}\}. \tag{3.26}$$

Proof Using the error splitting

$$u - U_h = (u - I_h^{(\min\{k;r\})} u) + (I_h^{(\min\{k;r\})} u - U_h) \equiv \eta + \chi,$$

we conclude from Lemmata 3.1, 3.2 and (3.22), (3.23) that

$$\begin{aligned} |\!|\!| \chi |\!|\!|_{\varepsilon,\delta}^2 &\le B_{SG}(\chi,\chi) \;=\; B_{SG}(e - \eta, \chi) \;=\; -B_{SG}(\eta,\chi) \\ &\le |\!|\!| \chi |\!|\!|_{\varepsilon,\delta} \left\{ |\!|\!| \eta |\!|\!|_{\varepsilon,\delta} + \left(\sum_e \tilde{Z}_e^2 \|\eta; L^2(e)\|^2 \right)^{1/2} \right\}, \end{aligned} \tag{3.27}$$

$$|\!|\!| u - U_h |\!|\!|_{\varepsilon,\delta}^2 \le 2|\!|\!| \eta |\!|\!|_{\varepsilon,\delta} + \left(\sum_e \tilde{Z}_e^2 \|\eta; L^2(e)\|^2 \right)^{1/2}. \tag{3.28}$$

The local interpolation error estimate (2.7) yields (3.24). Note that in the two-dimensional case $m = k$ is allowed. Furthermore, for $k = 1$ there is $\Delta v_h|_e = 0\ \forall v_h \in V_h$, that means, (2.7) is used only for $m = 0, 1$.

For (3.25) we use $I_h^{(\min\{k;r-1\})}$ instead of $I_h^{(\min\{k;r\})}$ and (2.8) instead of (2.7) in order to be able to treat also linear and quadratic elements in the three-dimensional case. □

3.5 Choice of the numerical damping parameters

A suitable strategy is to choose the numerical damping parameters δ_e in such a way, that the terms E_e^2 in (3.24) and (3.25) are minimized with respect to δ_e.

Lemma 3.8 *The term E_e^2 defined in (3.26) is minimal for*

$$\delta_e = \frac{h_{1,e}^2}{\varepsilon\sqrt{1 + P_e^2 + \Gamma_e^2}} \quad \textit{if } P_e^2 \geq \tilde{P}_e^2 \equiv \sqrt{1 + P_e^2 + \Gamma_e^2} \tag{3.29}$$

(convection-reaction dominated case), and

$$\delta_e = \min\left\{\frac{\varepsilon}{B_e^2};\ \frac{h_{1,e}^2}{\varepsilon} \cdot \frac{1 + P_e^2 + \Gamma_e}{1 + P_e^2 + \Gamma_e^2}\right\} \quad \textit{if } 0 \leq P_e \leq \tilde{P}_e \tag{3.30}$$

(diffusion dominated case). Hence there holds

$$E_e^2 \leq C\varepsilon(1 + P_e + \Gamma_e) = C(\varepsilon + h_{1,e}B_e + h_{1,e}^2 C_e), \tag{3.31}$$

$d = 2, 3,\ 1 \leq r \leq k$. For the definition of P_e and Γ_e see (3.6).

Proof Let first be $P_e^2 \geq \varepsilon^{-1}\delta_e^{-1}h_{1,e}^2$ such that $\min\{\varepsilon^{-1}B_e^2;\ \delta_e^{-1}\} = \delta_e^{-1}$. Then E_e^2 is minimized for

$$\delta_e = \frac{h_{1,e}^2}{\varepsilon\sqrt{1 + P_e^2 + \Gamma_e^2}}. \tag{3.32}$$

Then the condition $P_e^2 \geq \varepsilon^{-1}\delta_e^{-1}h_{1,e}^2$ is equivalent by (3.32) to $P_e^2 \geq \sqrt{1 + P_e^2 + \Gamma_e^2}$, that means $P_e^2 \geq (1 + \sqrt{5 + 4\Gamma_e^2})/2$, and we have $E_e^2 = \varepsilon(1 + \Gamma_e + 2\sqrt{1 + P_e^2 + \Gamma_e^2})$ and thus (3.31).

Consider now the case that $h_{1,e}^2 \min\{\varepsilon^{-1}B_e^2;\ \delta_e^{-1}\} = \varepsilon P_e^2$. If we demand that the term $\delta_e(\varepsilon^2 h_{1,e}^{-2} + B_e^2 + C_e^2 h_{1,e}^2)$ in (3.26) is not greater than the other term $\varepsilon + C_e h_{1,e}^2 + h_{1,e}^2\varepsilon^{-1}B_e^2$ then we find for δ_e the inequality

$$\delta_e \leq \frac{\varepsilon + C_e h_{1,e}^2 + h_{1,e}^2 B_e^2 \varepsilon^{-1}}{\varepsilon h_{1,e}^{-2} + B_e^2 + C_e^2 h_{1,e}^{-2}} = \frac{h_{1,e}^2}{\varepsilon} \cdot \frac{1 + P_e^2 + \Gamma_e}{1 + P_e^2 + \Gamma_e^2}. \tag{3.33}$$

A simple calculation gives via (3.26) that $E_e^2 \leq C\varepsilon(1 + P_e^2 + \Gamma_e) \leq C\varepsilon(1 + \sqrt{1 + P_e^2 + \Gamma_e^2} + \Gamma_e)$, hence (3.31). □

Remark 3.9 Note that assumption (H.3) in Subsection 3.3 is not guaranteed by (3.29) (3.30) if $k \geq 2$, see also Remark 3.6.

Remark 3.10 The analysis of Lemma 3.8 is valid only modulo multiplicative constants in (3.24) (3.25) which are independent of ε, $h_{1,e}$, and δ_e. Therefore it is possible to improve formulae (3.29) (3.30). Let us consider piecewise linear elements ($k = 1$) and the case $c = 0$. In case of $d = 1$ with constant coefficients ε and b, we have the well-known superconvergence result of nodally exact solutions for

$$\delta_e = \frac{h_{1,e}}{2}\left(\coth\frac{P_e}{2} - \frac{2}{P_e}\right). \tag{3.34}$$

The proposed tuning approach results in

$$\delta_e = \frac{h_{1,e}}{2B_e} \cdot \frac{P_e}{\sqrt{P_e^2+36}} \approx \begin{cases} \frac{h_{1,e}}{2B_e} & \text{if } P_e \gg 1 \\ \frac{P_e h_{1,e}}{12B_e} & \text{if } P_e \ll 1 \end{cases} \tag{3.35}$$

which reflects the asymptotic behaviour of (3.34) for both $P_e \to +\infty$ and $P_e \to +0$.

4 Anisotropic refinement

4.1 Necessity of boundary and interior layer refinement

A critical inspection shows that error estimates (3.24)–(3.26) may be useless in boundary or interior layer regions $\mathcal{R}_\varepsilon$ unless the mesh is sufficiently fine:

$$\begin{aligned} |u; W^{r+1,2}(e)|^2 &= \mathcal{O}(1) \quad \text{for } \varepsilon \to +0 \quad \text{if } e \notin \mathcal{R}_\varepsilon, \\ |u; W^{r+1,2}(e)|^2 &\to +\infty \quad \text{for } \varepsilon \to +0 \quad \text{if } e \in \mathcal{R}_\varepsilon. \end{aligned}$$

Practical calculations underline this and show the occurrence of so-called wiggles in the case of large numbers $P_e \gg 1$ and/or $\Gamma_e \gg 1$, for their definition see (3.6). Typically, they occur globally in Ω for the standard Galerkin method, but they are restricted to a numerical layer region $\mathcal{R}_h$ of width $\mathcal{O}(h^{\kappa_2}|\ln h|)$ for the (GLS)-scheme. It turns out that the layers $\mathcal{R}_h$ are in general larger than the boundary and interior layers $\mathcal{R}_\varepsilon$ of width $\mathcal{O}(\varepsilon^{\kappa_1}|\ln\varepsilon|)$. The sizes of κ_1 and κ_2 depend on the problem and characterize the layer, for κ_1 see the example below, κ_2 depends on the discretization and is not known in general. Nevertheless, a resolution of sharp layer gradients is often the main interest in applications, and improved methods are necessary. Usually this is accomplished by using exponentially fitted methods [1] or isotropic mesh refinement. We try to resolve $\mathcal{R}_\varepsilon$ by means of anisotropic mesh refinement in order to decrease the complexity of the discrete problem.

The anisotropic mesh in the boundary layer should give uniform bounds for $|||\, u - U_h \,|||_{\varepsilon,\delta}$ with respect to ε and contain a minimal number of finite elements. In order to exploit the anisotropic interpolation results, see Section 2, we need sharp local Sobolev norm estimates of u. Such estimates depend strongly on the asymptotic structure of u for $0 < \varepsilon \ll 1$, for example on the type of the boundary and interior layers, on the existence of turning points with $\|b\| = 0$, or on periodic characteristics. In the case of sufficiently smooth data we can take advantage of asymptotic expansions, see [27]. Unfortunately, such estimates are rare in the literature for the case of Lipschitzian domains $\Omega \subset \mathbb{R}^d$, $d \geq 2$, and less regular data, see [2] for the problems appearing. Future research should extend the knowledge about the solutions.

The first task is to detect the location of the manifolds where boundary and interior layers emanate. This could be accomplished in an adaptive method, see [30]. Nevertheless, we focus here on incompressible flow fields b. In contrast to compressible flow problems, interior layers (as shocks) are rare, and the location of boundary layers is well-known.

To get an example we consider a simple but typical boundary layer problem for the diffusion-convection-reaction model (3.1) (3.2) in a square or cube $\Omega = (0,1)^d$:

$$L_\varepsilon u \equiv -\varepsilon \Delta u - \sum_{i=1}^{d} \cos(\alpha_i) \frac{\partial u}{\partial x_i} + cu \;=\; f \quad \text{in } \Omega, \tag{4.1}$$

$$u \;=\; g \quad \text{on } \partial\Omega, \tag{4.2}$$

with $\alpha_i \in [0, \frac{\pi}{2}]$. In case of $\alpha_i \in (0, \frac{\pi}{2})$ there occur only *ordinary* (or *outflow*) *boundary layers* of thickness $\mathcal{O}(\varepsilon \ln \frac{1}{\varepsilon})$ at $x_i = 0$, $i = 1, \ldots, d$. In the case of $\alpha_i = \frac{\pi}{2}$, $i = 1, \ldots, d$, (no convection) and $c > 0$ there exists a boundary layer of thickness $\mathcal{O}(\sqrt{\varepsilon} \ln \frac{1}{\varepsilon})$ along the boundary $\partial\Omega$. For $\alpha_1 = 0$ and $\alpha_2 (= \alpha_3) = \frac{\pi}{2}$ *parabolic* (or *characteristic*) *layers* of thickness

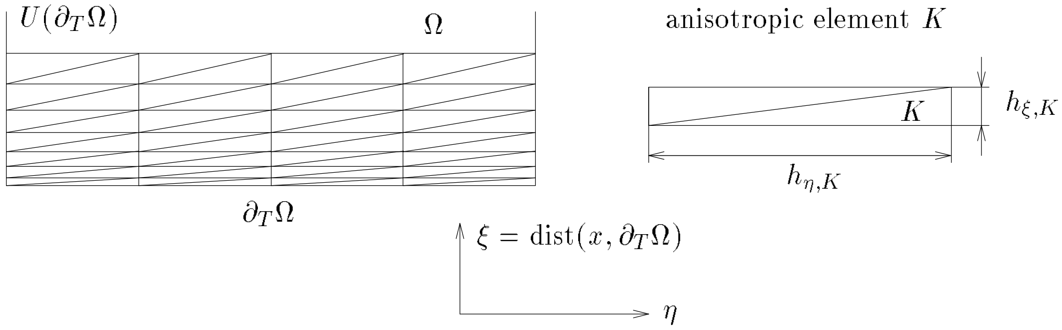

Figure 4.1: Anisotropic mesh in the boundary layer region

$\mathcal{O}(\sqrt{\varepsilon}\ln\frac{1}{\varepsilon})$ are located at $\partial\Omega$ with the exception of the *inflow* boundary part at $x_1 = 1$ (no layer) and the *outflow* boundary part at $x_1 = 0$ where again ordinary boundary layers of thickness $\mathcal{O}(\varepsilon\ln\frac{1}{\varepsilon})$ occur [21].

In Section 5 we consider a more general type of domain, but only in the two-dimensional case.

4.2 Mesh generation with anisotropic boundary layer refinement

The idea is now

- to construct a fixed mesh in the boundary layer region with anisotropic refinement and
- to use an isotropic mesh away from the boundary layers, possibly constructed by an advancing front technique and (isotropically) refined via standard adaptive methods (including interior layer refinement).

Without loss of generality we assume that a boundary layer of thickness $\mathcal{O}(\varepsilon^\kappa\ln\frac{1}{\varepsilon})$ is located at some line or plane $\partial_T\Omega \subset \partial\Omega$. We have $\kappa = \frac{1}{2}$ or $\kappa = 1$ in example above but it can be more general.

We introduce local coordinates (ξ,η) or (ξ,η,ζ) with $\xi = 0$ at $\partial_T\Omega$. As a starting point, we generate an orthogonal mesh via lines (planes) $\xi = \xi_i$, $\eta = \eta_j$, $(\zeta = \zeta_k)$ with real numbers ξ_i, η_j, ζ_k $(i = 0,\ldots,M,\ j = 0,\ldots,j_0,\ k = 0,\ldots,k_0)$ and particularly $\xi_0 = 0$, $\xi_M = d(\varepsilon) \equiv \varepsilon^\kappa\ln\frac{1}{\varepsilon}$. We assume that for a boundary layer rectangle (rectangular cube) $K = [\xi_i,\xi_{i+1}]\times[\eta_i,\eta_{i+1}]$ or $K = [\xi_i,\xi_{i+1}]\times[\eta_i,\eta_{i+1}]\times[\zeta_i,\zeta_{i+1}]$ the following relation holds close to the boundary:

$$h_{\xi,K} \equiv \xi_{i+1} - \xi_i \ll h_K \equiv \max\{h_{\eta,K}; h_{\zeta,K}\} \equiv \max\{\eta_{j+1} - \eta_j; \zeta_{k+1} - \zeta_k\}. \qquad (4.3)$$

The exceptions are geometric singularities (corners, edges) of the boundary $\partial\Omega$ where possibly different boundary layer parts intersect. Note that our approach guarantees an stronger refinement there.

The elements K are split into simplicial elements e (2 triangles or 6 tetrahedra) which satisfy the maximal angle condition and the coordinate system condition with respect to the boundary fitted coordinate system, see Figure 4.1. The mesh outside the (fixed) boundary layer regions should be of isotropic type. The results of Section 2 on inverse and interpolation error estimates are then applicable.

Note that an isotropic mesh refinement is possible via standard error estimators in the region away from the boundary layers. This is even desirable in the case of interior layers. Because of the difficulty with the coordinate system condition, no attempt will be made here to resolve interior layers (which are in general located at characteristic lines or surfaces) with anisotropic elements. However, this problem was attacked experimentally in [30]. We refer also to the test in [4] where a numerical example is given for the sensibility of the solution with respect to the coordinate system condition.

4.3 Modified error estimate on anisotropic elements

We try to refine the error analysis of Theorem 3.7 on anisotropic boundary elements $e \in \mathcal{R}_h$. In order to apply the anisotropic interpolation results of Section 2, it is essential that each element $e \in \mathcal{R}_h$ satisfies the maximum angle condition and the coordinate system condition with respect to the boundary fitted system, see figure 4.1.

Starting again from (3.28) and using Lemma 2.2, we find

$$||| u - U_h |||_{\varepsilon,\delta}^2 \leq \sum_e I_e(u)$$

with

$$\begin{aligned}
I_e(u) &\equiv 2\varepsilon\|\nabla\eta; L^2(e)\|^2 + 2C_e\|\eta; L^2(e)\|^2 + \\
&\quad +2\delta_e\left(\varepsilon^2\|\Delta\eta; L^2(e)\|^2 + B_e^2\|\nabla\eta; L^2(e)\|^2 + C_e^2\|\eta; L^2(e)\|^2\right) + \\
&\quad + \min\{B_e^2\varepsilon^{-1};\ 2\delta_e^{-1}\}\|\eta; L^2(e)\|^2 \qquad (4.4)\\
&\leq C\Big\{\varepsilon^2\delta_e \sum_{|\alpha|=r-1} h_e^{2\alpha}|D^\alpha u; W^{2,2}(e)|^2 + (\varepsilon + \delta_e B_e^2)\sum_{|\beta|=r} h_e^{2\beta}|D^\beta u; W^{1,2}(e)|^2 + \\
&\quad + \left[C_e + \delta_e C_e^2 + \min\left\{B_e^2\varepsilon^{-1};\ \delta_e^{-1}\right\}\right]\sum_{|\gamma|=r+1} h_e^{2\gamma}|D^\gamma u; L^2(e)|^2\Big\}, \\
&\leq C \sum_{|\alpha|=r-1}\sum_{|\beta|=1}\sum_{|\gamma|=1} E_{e,\beta,\gamma}^{\mathrm{an}} h_e^{2(\alpha+\beta)}\|D^{\alpha+\beta+\gamma}u; L^2(e)\|^2 \qquad (4.5)\\
E_{e,\beta,\gamma}^{\mathrm{an}} &\equiv \varepsilon + C_e h_e^{2\gamma} + \delta_e(\varepsilon^2 h_e^{-2\beta} + B_e^2 + C_e^2 h_e^{2\gamma}) + h_e^{2\gamma}\min\{\varepsilon^{-1}B_e^2;\ \delta_e^{-1}\} \qquad (4.6)
\end{aligned}$$

provided that $u \in W^{r+1,2}(\Omega)$ and $d = 2$, $1 \leq r \leq k$, or $d = 3$, $3 \leq r \leq k$. In the other case $d = 3$, $k = 1, 2$, we conclude from (4.4) and (2.6)

$$\begin{aligned}
I_e(u) &\leq C \sum_{|\alpha|=r-1}\sum_{|\beta|=1}\sum_{|\gamma|=1} E_{e,\beta,\gamma}^{\mathrm{an}} h_e^{2(\alpha+\beta)}\Big(\|D^{\alpha+\beta+\gamma}u; L^2(e)\|^2 + \\
&\quad + \sum_{s=1} h_e^s\|D^{\alpha+\beta+\gamma+s}u; L^2(e)\|^2\Big) \qquad (4.7)
\end{aligned}$$

A suitable strategy is now to generate the anisotropic mesh (via choice of $h_{\xi,e} = h_{d,e}$) and to choose the numerical damping parameters δ_e in such a way, that the error term $I_e(u)$ is minimized. That means, that the task is to minimize the different terms $E_{e,\beta,\gamma}^{\mathrm{an}}$, but the problem is that there is only one free parameter δ_e. On account of the presumably largest derivative $\frac{\partial^{r+1}u}{\partial\xi^{r+1}} = \frac{\partial^{r+1}u}{\partial x_d^{r+1}}$, we propose as a first attempt to minimize $E_{e,\beta,\gamma}^{\mathrm{an}}$ in the case $\beta = \gamma = (0,1)$ for $d = 2$ and $\beta = \gamma = (0,0,1)$ for $d = 3$, respectively. Considering

$$E_e^{\mathrm{an}} \equiv \varepsilon + C_e h_{d,e}^2 + \delta_e(\varepsilon^2 h_{d,e}^{-2} + B_e^2 + C_e^2 h_{d,e}^2) + h_{d,e}^2\min\left\{B_e^2\varepsilon^{-1};\ \delta_e^{-1}\right\}, \qquad (4.8)$$

we proceed as in the proof of Lemma 3.8 and choose δ_e according to (3.29), (3.30) with h_e, P_e and Γ_e replaced by $h_{d,e}$, P_e^{an} and Γ_e^{an},

$$P_e^{\mathrm{an}} \equiv \frac{h_{d,e}\|b; [L^\infty(e)]^d\|}{\varepsilon}, \quad \Gamma_e^{\mathrm{an}} \equiv \frac{h_{d,e}^2\|c; L^\infty(e)\|}{\varepsilon}, \qquad (4.9)$$

respectively. But for a conclusive error estimate we must also consider the terms

$$h_e^{2(\alpha+\beta)}\|D^{\alpha+\beta+\gamma}u; L^2(e)\|^2.$$

That is why we give a more refined analysis for a special case in the next section.

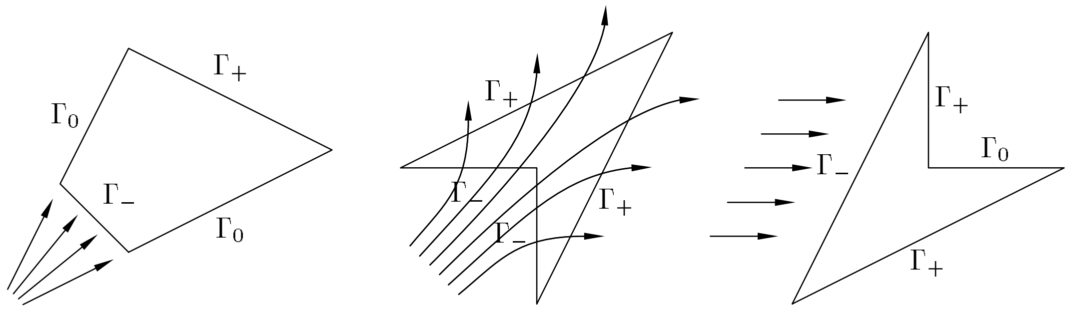

Figure 5.1: Problems of channel type.

5 Application to problems of channel type

5.1 Definition and properties of the solution

In view of the difficulties to get a priori information on the solution u we restrict our consideration in this section to a certain class of problems which should be introduced in the following. The main point is a correspondence of the domain Ω and the flow field b considered.

Given a subdomain $G \subseteq \Omega$ and a flow field b we denote by $(\partial G)_-$, $(\partial G)_+$, and $(\partial G)_0$ the inflow, outflow and characteristic parts of ∂G; the index denotes the sign of $(b \cdot \nu_G)(x)$ where ν_G is the outward unit normal on ∂G. Let $\xi_x(\tau)$ be the solution of

$$\dot{\xi}(\tau) = b(\xi(\tau)), \quad \xi(0) = x \in \overline{\Omega},$$

the streamline of b passing through $x \in \overline{\Omega}$. Denoting for any point $x \in G \cup \overline{(\partial G)_-}$ by

$$\tau_+^G(x) \equiv \inf\{\tau > 0 : \xi_x(\tau) \notin G\}$$

the first exit time of $\xi_x(\tau)$ from G, we define the domain of influence of any $\Gamma' \subset \partial G$ by

$$E(\Gamma') \equiv \{\xi_x(\tau) \in G : x \in \Gamma', 0 \leq \tau \leq \tau_+^G(x)\}.$$

We say now that a domain G is of *channel type* with respect to a flow field b if the following three conditions are satisfied:

(i) $\overline{G} = E(\overline{(\partial G)_-})$,

(ii) $E((\partial G)_0) \subseteq (\partial G)_0$, $E(\overline{(\partial G)_-} \cap \overline{(\partial G)_+}) \cap G = \emptyset$,

(iii) $|(b \cdot \nu_G)(x)| \geq \beta > 0$ on $(\partial G)_- \cup (\partial G)_+$.

In particular, this implies that all streamlines $\xi_x(\tau)$, $x \in \overline{G}$, leave $\overline{G}$ in finite time. Hence turning points with $\|b; \mathbb{R}^d\| = 0$ and periodic characteristics are excluded. For an illustration of channel type problems see Figure 5.1, whereas Figure 5.2 shows some situations not allowed.

On the other hand, boundary layers will appear in the case $G = \Omega$ at $(\partial\Omega)_+$ and $(\partial\Omega)_0$. We have the following result of [21, Theorem 2.3] which gives a localization of the boundary layers $\mathcal{R}_\varepsilon$.

Lemma 5.1 *Let Ω and Ω_i, $i = 1, 2$, be simply connected domains of channel type, and assume that $\Omega_1 \subseteq \Omega_2 \subseteq \Omega$, $\overline{(\partial\Omega_1)_-} \subseteq \overline{(\partial\Omega_2)_-} \subseteq \overline{(\partial\Omega)_-}$ and that $(\partial\Omega_2)_-$ is sufficiently smooth. For given numbers $s > 0$ and $r \in \mathbb{N}_0$ there exist $C_i(r, s, \Omega_i)$, $i = 1, 2$, and $C(\Omega_2, \Omega)$ such that if*

$$\operatorname{dist}(\Omega_1, (\partial\Omega_2)_+) \geq C_1 \varepsilon |\ln \varepsilon|, \quad \operatorname{dist}(\Omega_1, (\partial\Omega_2)_0) \geq C_2 \sqrt{\varepsilon} |\ln \varepsilon|,$$

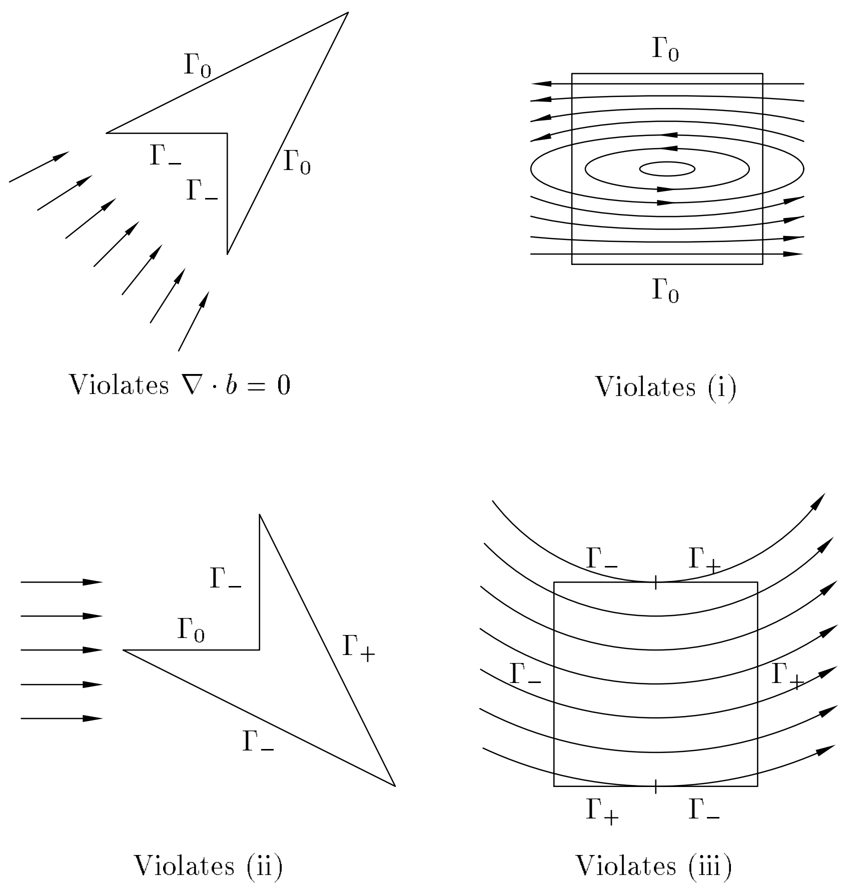

Figure 5.2: Not allowed situations.

then

$$\|u; W^{r+1,2}(\Omega_1)\| \leq C\left\{\|f; W^{r+1,2}(\Omega_2)\| + \varepsilon^s\|f; L^2(\Omega)\|\right\}.$$

So we denote by

$$\mathcal{R}_\varepsilon \equiv \{x \in \Omega : \mathrm{dist}(x, (\partial\Omega)_+) \leq C_1\varepsilon|\ln\varepsilon|,\ \mathrm{dist}(x, (\partial\Omega)_0) \leq C_2\sqrt{\varepsilon}|\ln\varepsilon|\} \tag{5.1}$$

the boundary layer region of a domain Ω of channel type. Furthermore we define

$$\begin{aligned} \mathcal{R}_\varepsilon^+ &\equiv \{x \in \mathcal{R}_\varepsilon : \mathrm{dist}(x, (\partial\Omega)_+) \leq C_1\varepsilon|\ln\varepsilon|\} \\ \mathcal{R}_\varepsilon^0 &\equiv \{x \in \mathcal{R}_\varepsilon : \mathrm{dist}(x, (\partial\Omega)_0) \leq C_2\sqrt{\varepsilon}|\ln\varepsilon|\} \\ \mathcal{R}_\varepsilon^c &\equiv \mathcal{R}_\varepsilon^+ \cap \mathcal{R}_\varepsilon^0 \end{aligned} \tag{5.2}$$

5.2 Generation of the anisotropic mesh in the boundary layer

The meshes are constructed as introduced in Subsection 4.2. We choose

$$h_{1,e} = h_{\eta,e} = g_1(\varepsilon)h, \quad \text{and} \quad h_{2,e} = h_{\xi,e} = g_2(\varepsilon)h, \tag{5.3}$$

with

$$\begin{aligned} g_1(\varepsilon) &= \mathcal{O}(1), & g_2(\varepsilon) &= \varepsilon & &\text{if} \quad e \subset \mathcal{R}_\varepsilon^+ \setminus \mathcal{R}_\varepsilon^c, \\ g_1(\varepsilon) &= \mathcal{O}(1), & g_2(\varepsilon) &= \sqrt{\varepsilon} & &\text{if} \quad e \subset \mathcal{R}_\varepsilon^0 \setminus \mathcal{R}_\varepsilon^c, \\ g_1(\varepsilon) &= \sqrt{\varepsilon}, & g_2(\varepsilon) &= \varepsilon & &\text{if} \quad e \subset \mathcal{R}_\varepsilon^c, \end{aligned} \tag{5.4}$$

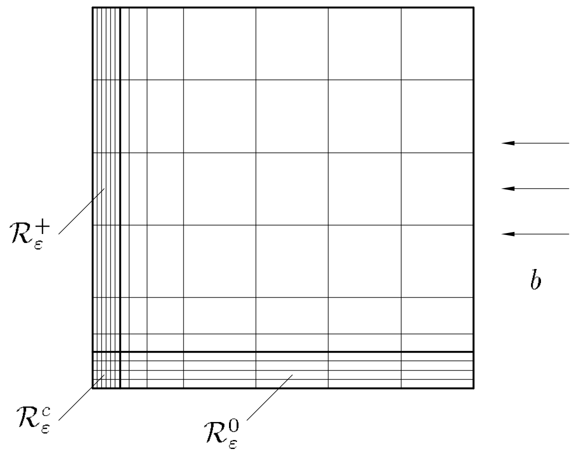

Figure 5.3: Generation of the anisotropic mesh, see example (4.1) (4.2) with $\alpha_1 = 0$, $\alpha_2 = \frac{\pi}{2}$.

and observe that

$$g_2(\varepsilon) = o(g_1(\varepsilon)) \quad \text{and} \quad g_1(\varepsilon) \leq \mathcal{O}(1). \tag{5.5}$$

Note that by construction a condensed mesh occurs around the corners of $(\partial\Omega)_+ \cup (\partial\Omega)_0$ where the layers intersect, compare Figure 5.3. Outside $\mathcal{R}_\varepsilon$ we double $h_{\xi,e}$ in ξ-direction (perpendicularly to $(\partial\Omega)_+$ and $(\partial\Omega)_0$, respectively) until $h_{\xi,e} \sim h$. We see easily that the number of elements is of the order $h^{-2}|\ln\varepsilon|^{-1}$.

In regard of lacking Sobolev norm estimates of u in $\mathcal{R}_\varepsilon$, we assume the following hypothesis to be satisfied:

(H.6) $\quad \|D^\alpha_{\eta,\xi}u; L^2(e)\| \leq \sqrt{\text{meas}(e)}\,[(g_1(\varepsilon))^{-\alpha_1} + (g_2(\varepsilon))^{-\alpha_2} + (g_1(\varepsilon))^{-\alpha_1}(g_2(\varepsilon))^{-\alpha_2}]\,K(f)$

with g_1, g_2 as in (5.4). The manifold with $\xi = 0$ corresponds to $(\partial\Omega)_+$ for $e \subset \mathcal{R}_\varepsilon^+ \setminus \mathcal{R}_\varepsilon^c$ and to $(\partial\Omega)_0$ elsewhere in $\mathcal{R}_\varepsilon$.

Remark 5.2 As in Shishkin meshes [11, 27] we could omit the transition layer where we double the previous mesh sizes; our forthcoming analysis is not affected. However, we expect a more regular behaviour of the discrete solution and better algebraic properties of the related system of equations with our approach.

5.3 Error estimates

With $I_e(u)$ as in (4.4), we split the error as follows:

$$|\!|\!| u - U_h |\!|\!|^2_{\varepsilon,\delta} \leq \sum_e I_e(u) = \sum_{e\subset\Omega\setminus\mathcal{R}_\varepsilon} I_e(u) + \sum_{e\subset\overline{\mathcal{R}_\varepsilon}} I_e(u). \tag{5.6}$$

In view of Lemma 5.1 we can consider the elements in the first sum as in Section 3 and it remains to treat the anisotropic elements $e \subset \overline{\mathcal{R}_\varepsilon}$.

Lemma 5.3 *The error term $I_e(u)$ for $e \subset \overline{\mathcal{R}_\varepsilon}$ is minimal (up to multiplicative constants) for the following choice of δ_e:*

$$\delta_e = \frac{h^2_{2,e}}{\varepsilon\sqrt{1 + (P_e^{\text{an}})^2 + (\Gamma_e^{\text{an}})^2}} \quad \textit{if } (P_e^{\text{an}})^2 \geq (\tilde{P}_e^{\text{an}})^2 \equiv \sqrt{1 + (P_e^{\text{an}})^2 + (\Gamma_e^{\text{an}})^2}, \tag{5.7}$$

$$\delta_e = \min\left\{\frac{\varepsilon}{B_e^2};\ \frac{h^2_{2,e}}{\varepsilon}\cdot\frac{1 + (P_e^{\text{an}})^2 + \Gamma_e^{\text{an}}}{1 + (P_e^{\text{an}})^2 + (\Gamma_e^{\text{an}})^2}\right\} \quad \textit{if } 0 \leq P_e^{\text{an}} \leq \tilde{P}_e^{\text{an}}, \tag{5.8}$$

with P_e^{an} and Γ_e^{an} defined in (4.9). Hence there holds

$$\begin{aligned} I_e(u) &\leq Ch^{2r+2}K^2(f)h_{1,e}h_{2,e}^{-1}\varepsilon(1 + P_e^{\text{an}} + \Gamma_e^{\text{an}}) \\ &\sim Ch^{2r+2}K^2(f)(\varepsilon h_{1,e}h_{2,e}^{-1} + C_e h_{1,e}h_{2,e} + B_e h_{1,e}) \end{aligned} \tag{5.9}$$

Proof The relations (4.5), (5.3), (5.5) as well as assumption (H.6) imply

$$\begin{aligned}
I_e(u) &= C \sum_{|\alpha|=r-1} \sum_{|\beta|=1} \sum_{|\gamma|=1} E^{\mathrm{an}}_{e,\beta,\gamma}\, h_e^{2(\alpha+\beta)} \|D^{\alpha+\beta+\gamma} u; L^2(e)\|^2 \\
&\le C \sum_{|\alpha|=r-1} \sum_{|\beta|=1} \sum_{|\gamma|=1} E^{\mathrm{an}}_{e,\beta,\gamma}\, (g_1(\varepsilon)\, h)^{2(\alpha_1+\beta_1)+1} (g_2(\varepsilon)\, h)^{2(\alpha_2+\beta_2)+1} \times \\
&\quad \times \left[g_1^{-2(\alpha_1+\beta_1+\gamma_1)} + g_2^{-2(\alpha_2+\beta_2+\gamma_2)} + g_1^{-2(\alpha_1+\beta_1+\gamma_1)} g_2^{-2(\alpha_2+\beta_2+\gamma_2)} \right] K^2(f) \\
&\le C h^{2r+2} K^2(f) \sum_{|\beta|=1} \sum_{|\gamma|=1} E^{\mathrm{an}}_{e,\beta,\gamma}\, g_{e,\beta,\gamma}, & (5.10) \\
g_{e,\beta,\gamma} &\equiv g_1^{-2\gamma_1+1}\, g_2^{2\beta_2+1} + g_1^{2\beta_1+1}\, g_2^{-2\gamma_2+1} + g_1^{-2\gamma_1+1}\, g_2^{-2\gamma_2+1}. & (5.11)
\end{aligned}$$

Expressing $g_{e,\beta,\gamma}$ via (5.3) in terms of h, $h_{1,e}$, and $h_{2,e}$, and using $h_{2,e} = o(h_{1,e})$, $h_{1,e} \le \mathcal{O}(1)$, we find

$$\sum_{|\beta|=1} \sum_{|\gamma|=1} g_{e,\beta,\gamma} \sim h_{1,e} h_{2,e}^{-1}, \quad \sum_{|\beta|=1} \sum_{|\gamma|=1} h_e^{2\gamma} g_{e,\beta,\gamma} \sim h_{1,e} h_{2,e}, \quad \sum_{|\beta|=1} \sum_{|\gamma|=1} h_e^{-2\beta} g_{e,\beta,\gamma} \sim h_{1,e} h_{2,e}^{-3},$$

that means with (4.6) that

$$I_e(u) \le C h^{2r+2} K^2(f) h_{1,e} h_{2,e}^{-1} E_e^{\mathrm{an}}$$

with E_e^{an} from (4.8), and thus we get with the same arguments as in Subsection 4.3 the expressions (5.7) (5.8) for δ_e and (5.9) for $I_e(u)$. □

Note that this result does not hold for general anisotropic meshes or general convection-diffusion-reaction problems because the assertion is mainly based on assumption (H.6) and a mesh satisfying (5.3).

As a result of the analysis in Lemmata 3.8 and 5.3 we propose the design of the numerical damping parameters δ_e as in (5.7) (5.8) in all cases. That means, δ_e as well as the local numbers P_e and Γ_e are dependent only on $h_{2,e}$, which is equivalent to the radius of the inner circle.

Using (5.6) we can summarize the error estimates as follows.

Corollary 5.4 *Under the assumptions (H.1) ... (H.6), $u \in H^{r+1,2}(\Omega)$, $1 \le r \le k$, and using the anisotropically refined boundary layer mesh (5.3) (5.4) and the parameter design (5.7) (5.8) we get the almost uniform (with respect to ε) error estimate*

$$||| \, u - U_h \, |||_{\varepsilon,\delta} \le C h^{2r} |\ln \varepsilon| K^2(f)(1 + C_e h_{1,e} h_{2,e} + B_e h_{1,e}). \tag{5.12}$$

Remark 5.5 The parameters δ_e are very small in the boundary layer, but $\delta_e = 0$ does not give the optimal result: In parabolic boundary layers we would get instead

$$||| \, u - U_h \, |||_{\varepsilon,\delta} \le C h^{2r} |\ln \varepsilon| K^2(f) \sqrt{\varepsilon}(1 + h^2(C_e + \varepsilon^{-1} B_e).$$

Remark 5.6 We conjecture that the analysis of this section can be refined in order to avoid the factor $|\ln \varepsilon|$ in (5.12) if we used a sharper estimate on the exponential decay of the solution than in Lemma 5.1 and Assumption (H.6).

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