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Anti-triangular and anti-m-Hessenberg forms for Hermitian matrices and pencils

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Abstract

Hermitian pencils, i.e., pairs of Hermitian matrices, arise in many applications, such as linear quadratic optimal control or quadratic eigenvalue problems. We derive conditions which anti-triangular and anti-m-Hessenberg forms for general (including singular) Hermitian pencils can be obtained under unitary equivalence transformations.

Key Words: Hermitian pencil, Hermitian matrix, Schur-form

AMS subject classification: 15A21, 15A22, 15A57, 93B25

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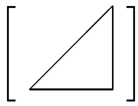
1 Introduction

In this paper, we discuss necessary and sufficient conditions for the existence of particular condensed forms for Hermitian matrices and pencils from which eigenvalues and nested sets of invariant subspaces can be obtained.

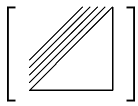
Definition 1 *Let $X = (x_{jk}) \in \mathbb{C}^{n\times n}$ and $m \in \mathbb{N}$.*

1. *We say that X is* **lower anti-triangular** *if $x_{jk} = 0$ for $j + k \leq n$, i.e.,*

   $X \hat{=}$  .

   Analogously we say that X is upper anti-triangular if $x_{j,k} = 0$ for $j + k \geq n$.

2. *We say that X is* **lower anti-m-Hessenberg** *if $x_{jk} = 0$ for $j + k \leq n - m$, i.e.,*

   $X \hat{=}$  .

   Analogously we define upper anti-m-Hessenberg matrices. If X is lower anti-1-Hessenberg, we also say that X is **anti-Hessenberg**.

As long as it is not stated otherwise, 'anti-triangular' and 'anti-m-Hessenberg' always means 'lower anti-triangular' and 'lower anti-m-Hessemberg', respectively. Analogous to the matrix case, we define anti-triangular and anti-m-Hessenberg forms for pencils.

In this paper we will discuss the reduction of Hermitian pencils to anti-m-Hessenberg forms and anti-triangular forms via unitary equivalence transformations that preserve the Hermitian structure.

The motivation for this research arises from structured eigenvalue problems in control theory and in the numerical simulation of mechanical systems.

The first application is the linear quadratic optimal control problem, see [12, 13, 18] and the references therein.

This is the problem of minimizing the cost functional

$$\frac{1}{2}\int_{t_0}^{\infty}\Big(x(t)^*Qx(t) + u(t)^*Ru(t) + u(t)^*S^*x(t) + x(t)^*Su(t)\Big)dt$$

subject to the dynamics

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \qquad t_0 < t \qquad (1)\\ x(t_0) &= x_0, \qquad (2) \end{aligned}$$

where $A, E, Q \in \mathbb{C}^{n\times n}, B, S \in \mathbb{C}^{n\times m}, R \in \mathbb{C}^{m\times m}$, Q and R Hermitian, $x_0, x(t), u(t) \in \mathbb{C}^n$, and $t_0, t \in \mathbb{R}$. It is known that solutions of (1) can be obtained via the solution of a boundary value problem, see [17, 18] and the references therein. For the solution of this boundary value problem one has to compute deflating subspaces of the matrix pencil

$$\lambda \begin{bmatrix} E & 0 & 0 \\ 0 & -E^* & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} A & 0 & B \\ Q & A^* & S \\ S^* & B^* & R \end{bmatrix}. \tag{3}$$

Multiplying both matrices with

$$P = \begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

from the left, we see that the pencil (3) is equivalent to the pencil

$$\lambda\mathcal{A} - \mathcal{B} = \lambda \begin{bmatrix} 0 & -E^* & 0 \\ E & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} Q & A^* & S \\ A & 0 & B \\ S^* & B^* & R \end{bmatrix}. \tag{4}$$

Multiplying $\mathcal{A}$ by i, we find that $\lambda i\mathcal{A} - \mathcal{B}$ is a Hermitian pencil, i.e., both $i\mathcal{A}$ and $\mathcal{B}$ are Hermitian. Clearly, both pencils $\lambda\mathcal{A} - \mathcal{B}$ and $\lambda i\mathcal{A} - \mathcal{B}$ have the same right deflating subspaces and the eigenvalues of $\lambda i\mathcal{A} - \mathcal{B}$ coincide with the eigenvalues of $\lambda\mathcal{A} - \mathcal{B}$ multiplied by i. Therefore, to analyze and compute eigenvalues and deflating subspaces, it is sufficient to consider the Hermitian pencil $\lambda i\mathcal{A} - \mathcal{B}$. It should be noted, however, that if the original problem is real, then we have obtained an Hermitian nonreal problem in this way. For the real case one has to discuss 'skew-Hermitian/Hermitian' pencils $\lambda\mathcal{S} - \mathcal{H}$, i.e., pencils where $\mathcal{S}$ is skew Hermitian and $\mathcal{H}$ is Hermitian. This case is more complicated, because one has to deal with an additional symmetry. It is well known that the spectra of skew-Hermitian/Hermitian pencils are symmetric with respect to the imaginary axis (see [23]). In the real case, the spectra have an additional symmetry with respect to the real axis. In this paper we only consider the complex case. The real case is referred to a later discussion.

Other applications of Hermitian pencils arise in the numerical treatment of quadratic eigenvalue problems in mechanics. In quadratic eigenvalue problems one is interested in computing $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^n\backslash\{0\}$ such that

$$(A + \lambda B + \lambda^2 C)x = 0,$$

where typically $A, C \in \mathbb{C}^{n\times n}$ are Hermitian and B is Hermitian or skew Hermitian. Hermitian quadratic eigenvalue problems arise for example in the analysis of geometrical nonlinear buckling structures with finite element methods (see [3, 9]) or in the theory of damped oscillatory systems (see [6, 11]). With the substitution $\mu = \frac{1}{\lambda}$ for $\lambda \neq 0$, the problem can be linearized such that it reduces to the generalized Hermitian eigenvalue problem

$$\mu \begin{bmatrix} B & A \\ A & 0 \end{bmatrix} \begin{bmatrix} \lambda x \\ x \end{bmatrix} = \begin{bmatrix} -C & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} \lambda x \\ x \end{bmatrix}, \tag{5}$$

see, e.g., [9]. Quadratic eigenvalue problems with B skew Hermitian arise in numerical simulation of the deformation of anisotropic materials (see [14]) and the acoustic simulation of poroelastic materials (see [20]). In this case the substitution $\mu = i\lambda$ leads to the linearized eigenvalue problem

$$\mu \begin{bmatrix} 0 & iC \\ -iC & -iB \end{bmatrix} \begin{bmatrix} \lambda x \\ x \end{bmatrix} = \begin{bmatrix} -C & 0 \\ 0 & -A \end{bmatrix} \begin{bmatrix} \lambda x \\ x \end{bmatrix}, \tag{6}$$

For a detailed study of Hermitian quadratic eigenvalue problems and, more general, of matrix polynomials see [6].

Canonical forms for Hermitian pencils or for related pairs of quadratic or Hermitian forms are well known and have been widely discussed in literature, starting with results of Weierstraß for the regular case (see [24]) and results of Kronecker for the singular case (see [10]). For a complete discussion of canonical forms for Hermitian pencils see [22], and for a large list of references see [23].

But for the sake of numerical stability, we are interested in finding condensed forms for Hermitian pencils under unitary transformations. In other words, we try to reduce both matrices of the pencil via a simultaneous unitary similarity transformation. Anti-triangular forms for Hermitian pencils seem to be good forms to chase for. We say that a Hermitian pencil $\lambda A - B$ is congruent to a pencil in anti-triangular form if there exists a nonsingular matrix P such that

$$P^*(\lambda A - B)P \,\hat{=}\, \left[\quad \right]. \tag{7}$$

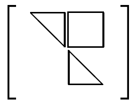
Indeed, if one is interested in finding condensed forms under unitary transformations, it does not make sense to look for classical Schur forms for Hermitian pencils, because this reduces to the problem of diagonalizing two Hermitian matrices simultaneously. It is well-known that this is possible if and only if the matrices commute (see, e.g., [21]). On the other hand, if (7) holds then P can be chosen to be unitary. This follows easily by applying the QR-decomposition on P, see also Lemma 2 in the following section. Hence, both A and B are simultaneously unitarily similar to anti-triangular matrices.

Anti-triangular forms for Hermitian pencils are related to Schur-like forms for skew-Hamiltonian/Hamiltonian pencils that are discussed in [16]. A skew-Hamiltonian/Hamiltonian pencil is a pencil $\lambda S - H$ such that S is skew-Hamiltonian, that is $SJ - JS^* = 0$, and such that H is Hamiltonian, that is $HJ + JH^* = 0$, where

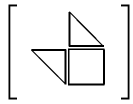
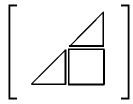
$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Thus, skew-Hamiltonian/Hamiltonian pencils are structured with respect to an indefinite (skew) scalar product, defined by the matix J. Condensed forms for matrices and pencils that are structured with respect to indefinite scalar products have been widely discussed in the literature, see [4, 5, 7, 12, 15, 19, 25], to name a few.

If $\lambda S - H$ is a skew-Hamiltonian/Hamiltonian pencil, then the pencil $\lambda iJS - JH$ is Hermitian. Furthermore, if $\lambda S - H$ is in Schur-like form, i.e.,

$\lambda S - H \hat{=}$  ,

then the corresponding Hermitian pencil $\lambda iJS - JH$ is congruent to a pencil in anti-triangular form and has the structure

$\lambda iJS - JH \hat{=}$  $\sim_c$  .

From this point of view, it seems that anti-triangular forms for Hermitian pencils are the natural forms to look for if one is interested in obtaining condensed forms under unitary transformations.

In [16] it was shown that not every regular skew-Hamiltonian/Hamiltonian pencil can be reduced to Schur-like form. This generalizes a result on Hamiltonian matrices (see [15]). The reason why a Schur-like form does not always exist is because certain conditions on the purely imaginary eigenvalues have to be satisfied. This comes from the fact that purely imaginary eigenvalues of Hamiltonian matrices have signs $\varepsilon = \pm 1$ that are invariant under structure-preserving transformations, see [15], or [6] and [12] for a more general setting. An analogous situation holds in the pencil case (see [16] and [22]).

However, the consideration of Hermitian pencils is more general than the consideration of skew-Hamiltonian/Hamiltonian pencils, since the case of odd-sized pencils is included in the context of Hermitian pencils. Furthermore, only the case of regular pencils is discussed in [16] and it is the purpose of this paper to include the singular case. This case is of interest as well; see for example [18] for applications when the pencil (4) is singular.

It will turn out that the existence of anti-triangular forms for singular Hermitian pencils is equivalent to the existence of anti-m-Hessenberg forms for certain regular Hermitian pencils. But besides this, anti-m-Hessenberg forms of Hermitian pencils are of interest themselves. During the numerical computation of the Schur form of a matrix, the matrix is usually reduced to Hessenberg form in the first step (see, e.g., [8]). Hessenberg-like forms for Hamiltonian matrices have been discussed in, e.g., [1] and [4]. Anti-Hessenberg forms for Hermitian matrices correspond to Hessenberg-like forms for Hamiltonian matrices.

In section 2 we will discuss basic properties of Hermitian anti-triangular and anti-m-Hessenberg matrices and in section 3 we discuss corresponding forms for the case of regular Hermitian pencils. In section 3 another important condensed form for Hermitian pencils is derived, the so-called sign condensed form. In a certain sense, this form displays 'how far away' a Hermitian pencil is from being congruent to anti-triangular or anti-m-Hessenberg form. The case of singular pencils will be discussed in section 4.

Throughout the paper we use the following notation.

1. Given two square matrices A, B, we define the *direct sum* $A \oplus B$ of A and B by

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

   Analogously we define the direct sum of square pencils.

2. By Z_p we denote the $p \times p$ zip matrix $Z_p = [\delta_{i+j,p+1}]_{i,j=1}^{p}$ with ones on the anti-diagonal and zeros elsewhere.

3. By $\sigma(\lambda)$ we denote the sign of $\lambda \in \mathbb{R}$, that is

$$\sigma(\lambda) = \begin{cases} 1 & \text{if } \lambda > 0, \\ 0 & \text{if } \lambda = 0, \\ -1 & \text{if } \lambda < 0. \end{cases}$$

4. By $A \sim_c B$ we denote that the matrices A and B are congruent.

5. By $\operatorname{spec}(A)$ we denote the spectrum of a square matrix A.

6. By e_j we denote the jth unit vector.

2 Anti-triangular and anti-m-Hessenberg forms

In this section we discuss conditions when Hermitian matrices can be transformed to anti-triangular and anti-m-Hessenberg matrices via unitary congruence transformations. It turns out that the conditions for unitary congruence are the same as for congruence.

Lemma 2 *Let $A \in \mathbb{C}^{n \times n}$ be Hermitian and congruent to an anti-m-Hessenberg matrix for some $m \in \mathbb{N}$. Then A is unitarily similar to an anti-m-Hessenberg matrix.*

Proof. Let $\tilde{A}$ be in anti-m-Hessenberg form and let $\tilde{A}$ and A be congruent, i.e., there exists a nonsingular matrix $P \in \mathbb{C}^{n \times n}$, such that $P^*AP = \tilde{A}$. Let $P = QR$ be the QR-decomposition (see [8]) of P. Then $Q^*AQ = R^{-*}\tilde{A}R^{-1}$ is still anti-m-Hessenberg. □

Let us recall that the **inertia index** of a Hermitian matrix G is

$$\operatorname{Ind}(G) = (\nu_+, \nu_-, \nu_0),$$

where ν_+, ν_-, ν_0 are the numbers of positive, negative and zero eigenvalues of G, respectively. Conditions for the existence of both anti-triangular and anti-m-Hessenberg forms will be based on the following lemma.

Lemma 3 *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and let* $\operatorname{Ind}(A) = (\nu_+, \nu_-, \nu_0)$. *Then A is congruent to a matrix of the form*

$$\left[\begin{array}{cc} 0 & A_2 \\ A_2^* & A_3 \end{array}\right], \tag{8}$$

where $A_3 \in \mathbb{C}^{k\times k}$, $A_2 \in \mathbb{C}^{(n-k)\times k}$ if and only if $|\nu_+ - \nu_-| \le 2k + \nu_0 - n$.

Proof. '⇒': Let A be in the form (8). Then there exist nonsingular matrices $S \in \mathbb{C}^{(n-k)\times(n-k)}$ and $T \in \mathbb{C}^{k\times k}$ such that

$$SA_2T = \left[\begin{array}{cc} I_m & 0 \\ 0 & 0 \end{array}\right],$$

where $m \le k, n-k$. From this we obtain that

$$\left[\begin{array}{cc} S & 0 \\ 0 & T^* \end{array}\right] A \left[\begin{array}{cc} S^* & 0 \\ 0 & T \end{array}\right] = \begin{array}{c} \begin{array}{cccc} m & n-k-m & m & k-m \end{array} \\ \left[\begin{array}{cccc} 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 \\ I_m & 0 & A_{31} & A_{32} \\ 0 & 0 & A_{32}^* & A_{33} \end{array}\right] \end{array},$$

where $T^*A_3T = \left[\begin{array}{cc} A_{31} & A_{32} \\ A_{32}^* & A_{33} \end{array}\right]$. Furthermore, we obtain

$$\left[\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ -\frac{1}{2}A_{31} & 0 & I & 0 \\ -A_{32}^* & 0 & 0 & I \end{array}\right] \left[\begin{array}{cccc} 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 \\ I_m & 0 & A_{31} & A_{32} \\ 0 & 0 & A_{32}^* & A_{33} \end{array}\right] \left[\begin{array}{cccc} I & 0 & -\frac{1}{2}A_{31} & -A_{32} \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right]$$
$$= \left[\begin{array}{cccc} 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 \\ I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{33} \end{array}\right].$$

This implies $\operatorname{Ind}(A) = (m, m, n-k-m) + \operatorname{Ind}(A_{33})$ and since A_{33} is a $(k-m)\times(k-m)$ matrix, we obtain from $n-k-m \le \nu_0$ that

$$|\nu_+ - \nu_-| \le k - m = 2k + n - k - m - n \le 2k + \nu_0 - n.$$

'⇐': Assume w.l.o.g. that $\nu_+ - \nu_- \ge 0$; otherwise consider $-A$. Then the matrix

$$\tilde{A} = \left[\begin{array}{cccc} 0 & 0 & I_{\nu_-} & 0 \\ 0 & O_{\nu_0} & 0 & 0 \\ I_{\nu_-} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\nu_+-\nu_-} \end{array}\right],$$

is congruent to A, since $\mathrm{Ind}(\tilde{A}) = (\nu_+, \nu_-, \nu_0)$. Here O_{ν_0} denotes the $\nu_0 \times \nu_0$ zero matrix. It remains to show that $\nu_- + \nu_0 \geq n - k$ and this follows by

$$\begin{aligned} \nu_- + \nu_0 &= n - \nu_+ = n - \nu_- - (\nu_+ - \nu_-) \\ &\geq n - \nu_- - (2k + \nu_0 - n) = 2(n-k) - (\nu_- + \nu_0). \quad \square \end{aligned}$$

Corollary 4 *Let $A \in \mathbb{C}^{n \times n}$ be Hermitian,* $\mathrm{Ind}(A) = (\nu_+, \nu_-, \nu_0)$*, and $n > m \in \mathbb{N}$.*

1. *If $n - m$ is even, then A is congruent to an anti-m-Hessenberg matrix if and only if*
$$|\nu_+ - \nu_-| \leq \nu_0 + m.$$
2. *If $n - m$ is odd, then A is congruent to an anti-m-Hessenberg matrix if and only if*
$$|\nu_+ - \nu_-| \leq \nu_0 + m + 1.$$

Proof. Let us first consider the case that $n - m$ is even. If A is congruent to an anti-m-Hessenberg matrix, then in particular A is congruent to a matrix of the form

$$\begin{bmatrix} 0 & A_2 \\ A_2^* & A_3 \end{bmatrix},$$

where $A_3 \in \mathbb{C}^{(\frac{n+m}{2}) \times (\frac{n+m}{2})}$ and $A_2 \in \mathbb{C}^{(\frac{n-m}{2}) \times (\frac{n+m}{2})}$. Hence, Lemma 3 implies that

$$|\nu_+ - \nu_-| \leq 2\frac{n+m}{2} + \nu_0 - n = \nu_0 + m.$$

Conversely assume that $|\nu_+ - \nu_-| \leq \nu_0 + m$. Then Lemma 3 implies that A is congruent to a matrix of the form

$$\begin{bmatrix} 0 & A_2 \\ A_2^* & A_3 \end{bmatrix},$$

where $A_2 \in \mathbb{C}^{(\frac{n-m}{2}) \times (\frac{n+m}{2})}$ and $A_3 \in \mathbb{C}^{(\frac{n+m}{2}) \times (\frac{n+m}{2})}$. Let $S \in \mathbb{C}^{(\frac{n-m}{2}) \times (\frac{n-m}{2})}$ and $T \in \mathbb{C}^{(\frac{n+m}{2}) \times (\frac{n+m}{2})}$ be nonsingular, such that

$$SA_2T = \begin{bmatrix} 0 & \tilde{A}_2 \end{bmatrix},$$

where $\tilde{A}_2 \in \mathbb{C}^{(\frac{n-m}{2}) \times (\frac{n-m}{2})}$ is anti-triangular. Clearly such matrices always exist. It follows that

$$\begin{bmatrix} S & 0 \\ 0 & T^* \end{bmatrix} \begin{bmatrix} 0 & A_2 \\ A_2^* & A_3 \end{bmatrix} \begin{bmatrix} S^* & 0 \\ 0 & T \end{bmatrix} = \begin{bmatrix} 0 & SA_2T \\ (SA_2T)^* & T^*A_3T \end{bmatrix}$$

is anti-triangular and thus, A is congruent to an anti-triangular matrix. The case that $n - m$ is odd follows in an analogous way, noting that in this case an anti-m-Hessenberg form of A has the structure

$$\begin{bmatrix} 0 & A_2 \\ A_2^* & A_3 \end{bmatrix},$$

where $A_3 \in \mathbb{C}^{(\frac{n+m+1}{2}) \times (\frac{n+m+1}{2})}$ and $A_2 \in \mathbb{C}^{(\frac{n-m-1}{2}) \times (\frac{n+m+1}{2})}$. $\square$

Corollary 5 *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and let* $\mathrm{Ind}(A) = (\nu_+, \nu_-, \nu_0)$.

1. *If n is even, then A is congruent to an anti-triangular matrix if and only if*

$$|\nu_+ - \nu_-| \leq \nu_0.$$

2. *If n is odd, then A is congruent to an anti-triangular matrix if and only if*

$$|\nu_+ - \nu_-| \leq \nu_0 + 1.$$

We see from these results that the inertia indices of Hermitian matrices play a key role in the discussion of anti-triangular and anti-m-Hessenberg forms. The following lemma establishes an auxiliary result for the computation of the inertia index of some special Hermitian matrices.

Lemma 6 *Let $A \in \mathbb{C}^{n\times n}$ be an Hermitian matrix of the form*

$$A = \begin{bmatrix} 0 & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13}^* & A_{23}^* & A_{33}, \end{bmatrix},$$

where $A_{13} \in \mathbb{C}^{m\times k}$ and $A_{22} \in \mathbb{C}^{(n-m-k)\times(n-m-k)}$.

1. *If $m = k$ and A_{13} is invertible, then*

$$\mathrm{Ind}(A) = (m, m, 0) + \mathrm{Ind}(A_{22}).$$

2. *If $A_{22} \in \mathbb{C}^{(n-m-k)\times(n-m-k)}$ is invertible, then*

$$\mathrm{Ind}(A) = \mathrm{Ind}\left(\begin{bmatrix} 0 & A_{13} \\ A_{13}^* & \tilde{A}_{33} \end{bmatrix}\right) + \mathrm{Ind}(A_{22}),$$

where $\tilde{A}_{33} = A_{33} - A_{23}^* A_{22} A_{23}$.

Proof. 1. If $m = k$ and A_{13} is invertible, we find that

$$\begin{bmatrix} A_{13}^{-1} & 0 & 0 \\ -A_{23}A_{13}^{-1} & I & 0 \\ -\frac{1}{2}A_{33}A_{13}^{-1} & 0 & I \end{bmatrix} \begin{bmatrix} 0 & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13}^* & A_{23}^* & A_{33} \end{bmatrix} \begin{bmatrix} A_{13}^{-*} & -A_{13}^{-*}A_{23}^* & -\frac{1}{2}A_{13}^{-*}A_{33}^* \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & I \\ 0 & A_{22} & 0 \\ I & 0 & 0 \end{bmatrix} \sim_c \begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & A_{22}. \end{bmatrix}$$

This implies that $\mathrm{Ind}(A) = (m, m, 0) + \mathrm{Ind}(A_{22})$.

2. If A_{22} is invertible, we find that

$$\begin{bmatrix} I_m & 0 & 0 \\ 0 & I & 0 \\ 0 & -A_{23}^* A_{22}^{-1} & I_k \end{bmatrix} \begin{bmatrix} 0 & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13}^* & A_{23}^* & A_{33} \end{bmatrix} \begin{bmatrix} I_m & 0 & 0 \\ 0 & I & -A_{22}A_{23} \\ 0 & 0 & I_k \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & A_{13} \\ 0 & A_{22} & 0 \\ A_{13}^* & 0 & \tilde{A}_{33} \end{bmatrix} \sim_c \begin{bmatrix} 0 & A_{13} & 0 \\ A_{13}^* & \tilde{A}_{33} & 0 \\ 0 & 0 & A_{22}, \end{bmatrix}$$

where $\tilde{A}_{33} = A_{33} - A_{23}^* A_{22} A_{23}$. $\square$

3 Condensed forms for regular Hermitian pencils

In this section we discuss condensed forms for regular Hermitian pencils, i.e., pencils $\lambda G - H \in \mathbb{C}^{n\times n}$ such that both G and H are Hermitian. These forms are the canonical form, anti-triangular forms that can be obtained via a unitary similarity transformation that operates simultaneously on G and H, anti-m-Hessenberg forms, and the so-called sign condensed form. First let us recall the well-known canonical form for Hermitian pencils (see [22]).

Theorem 7 *Let $\lambda G - H$ be a regular Hermitian pencil. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$P^*(\lambda G - H)P = (\lambda G_1 - H_1) \oplus \ldots \oplus (\lambda G_l - H_l), \tag{9}$$

where the blocks $\lambda G_j - H_j$ have one and only one of the following forms.

1. *Blocks associated with paired nonreal eigenvalues λ_0, λ_0^*:*

$$\lambda \begin{bmatrix} 0 & Z_r \\ Z_r & 0 \end{bmatrix} - \begin{bmatrix} 0 & Z_r \mathcal{J}_r(\lambda_0) \\ \mathcal{J}_r(\lambda_0)^* Z_r & 0 \end{bmatrix}.$$

2. *Blocks associated with real eigenvalues λ_0 and sign $\varepsilon \in \{1, -1\}$:*

$$\lambda \varepsilon Z_r - \varepsilon Z_r \mathcal{J}_r(\lambda_0) = \varepsilon \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix} - \varepsilon \begin{bmatrix} 0 & & & \lambda_0 \\ & & \lambda_0 & 1 \\ & \iddots & \iddots & \\ \lambda_0 & 1 & & 0 \end{bmatrix}.$$

3. Blocks associated with the eigenvalue ∞ and sign $\varepsilon \in \{1,-1\}$:

$$\lambda \varepsilon Z_r \mathcal{J}_r(0) - \varepsilon Z_r = \varepsilon \begin{bmatrix} 0 & & & 0 \\ & & 0 & 1 \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ 0 & 1 & & 0 \end{bmatrix} - \varepsilon \begin{bmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{bmatrix}.$$

Proof. See [22]. □

Definition 8 *Let $\lambda G - H$ be a regular Hermitian pencil and let $\lambda G_j - H_j$ be a single block of the canonical form (9) of $\lambda G - H$. If $\lambda G_j - H_j$ is a block of type 2) or 3) then the parameter ε that appears in the canonical form (9) is called the* **sign** *associated with the block $\lambda G_j - H_j$.*

Besides the eigenvalues of a Hermitian pencil, the signs associated with blocks to real eigenvalues or the eigenvalue ∞ are invariants under congruence. The collection of these signs is sometimes referred to as the **sign characteristic** (see, e.g., [7] and [12] for related work on H-selfadjoint matrices, where H is a nonsingular Hermitian matrix). It will turn out that especially the signs of odd-sized blocks play a key role in our investigation of condensed forms. This motivates the following definition of the sign sum.

Definition 9 *Let $\lambda G - H \in \mathbb{C}^{n \times n}$ be a regular Hermitian pencil and let $\lambda_0 \in \mathbb{R} \cup \{\infty\}$ be a real eigenvalue of $\lambda G - H$ with partial multiplicities $(p_1, \ldots, p_r, p_{r+1}, \ldots, p_m)$, where $p_1, \ldots, p_r$ are odd and $p_{r+1}, \ldots, p_m$ are even.*

1. *The tupel $(\varepsilon_1, \ldots, \varepsilon_m)$ is called the* **sign characteristic** *of λ_0, where ε_j is the sign associated with the block in the canonical form (9) that corresponds to λ_0 and p_j.*
2. *The integer* $\mathrm{Signsum}(\lambda_0, G, H) := \varepsilon_1 + \ldots + \varepsilon_r$ *is called the* **sign sum** *of λ_0 with respect to $\lambda G - H$. If there is no risk of confusion we write* $\mathrm{Signsum}(\lambda_0)$ *instead of* $\mathrm{Signsum}(\lambda_0, G, H)$.

In addition, we set $\mathrm{Signsum}(\lambda_0, G, H) = 0$, whenever $\lambda_0 \in \mathbb{R} \cup \{\infty\}$ is not an eigenvalue of $\lambda G - H$. We note that if in the canonical form (9) there are only even sized blocks associated with λ_0 then $\mathrm{Signsum}(\lambda_0) = 0$, since the sign sum is obtained by the sum of the signs that correspond to odd sized blocks. The following theorem allows to 'split' a regular Hermitian pencil into an anti-triangular part and a diagonal part. Furthermore, all the information on the sign sum, i.e., all the information on the signs that are needed in the following, can be read off the diagonal part. For the proof of this result, we first state the following auxiliary remark.

Remark 10 Let $A \in \mathbb{C}^{n \times n}$ be Hermitian.

1. If $A = \begin{bmatrix} 0 & A_{12} & 0 \\ A_{12}^* & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}$, then A is congruent to $\begin{bmatrix} 0 & 0 & A_{12} \\ 0 & A_{33} & 0 \\ A_{12}^* & 0 & A_{22} \end{bmatrix}$.

2. If $A = \begin{bmatrix} 0 & 0 & A_{13} & 0 \\ 0 & A_{22} & A_{23} & 0 \\ A_{13}^* & A_{23}^* & A_{33} & 0 \\ 0 & 0 & 0 & A_{44} \end{bmatrix}$, then A is congruent to $\begin{bmatrix} 0 & 0 & 0 & A_{13} \\ 0 & A_{22} & 0 & A_{23} \\ 0 & 0 & A_{44} & 0 \\ A_{13}^* & A_{23}^* & 0 & A_{33} \end{bmatrix}$.

Theorem 11 (Sign condensed form) *Let $\lambda G - H \in \mathbb{C}^{n\times n}$ be a regular Hermitian pencil. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ and $m \in \mathbb{N}$, such that*

$$P^*(\lambda G - H)P = \lambda \begin{bmatrix} 0 & 0 & G_{13} \\ 0 & G_{22} & G_{23} \\ G_{13}^* & G_{23}^* & G_{33} \end{bmatrix} - \begin{bmatrix} 0 & 0 & H_{13} \\ 0 & H_{22} & H_{23} \\ H_{13}^* & H_{23}^* & H_{33} \end{bmatrix}, \tag{10}$$

where $G_{13}, H_{13} \in \mathbb{C}^{m\times m}$ are anti-triangular and

$$\lambda G_{22} - H_{22} = \lambda \begin{bmatrix} \varepsilon_1 I_{p_1} & & & 0 \\ & \ddots & & \\ & & \varepsilon_k I_{p_k} & \\ 0 & & & 0 \end{bmatrix} - \begin{bmatrix} \varepsilon_1\lambda_1 I_{p_1} & & & 0 \\ & \ddots & & \\ & & \varepsilon_k\lambda_k I_{p_k} & \\ 0 & & & \varepsilon_{k+1} I_{p_{k+1}} \end{bmatrix}, \tag{11}$$

where $\lambda_1 < \ldots < \lambda_k$ and $\varepsilon_1, \ldots, \varepsilon_{k+1} \in \{1, -1\}$. Furthermore, we have for all $\lambda_0 \in \mathbb{R}\cup\{\infty\}$ that

$$\operatorname{Signsum}(\lambda_0, G, H) = \operatorname{Signsum}(\lambda_0, G_{22}, H_{22}).$$

Proof. Assume, w.l.o.g., that $\lambda G - H$ is in the canonical form (9). The proof now proceeds by induction on the number l of distinct real eigenvalues, including the eigenvalue ∞.

'$l = 0$': If $\lambda G - H$ has neither real eigenvalues nor the eigenvalue ∞, then clearly all the blocks in the canonical form (9) have even sizes. Thus, applying Remark 10 part 1 repeatedly, we find that $\lambda G - H$ is congruent to a pencil in form (10), where the block $\lambda G_{22} - H_{22}$ does not appear.

'$l \Rightarrow l + 1$': Let us pick an eigenvalue $\lambda_0 \in \mathbb{R} \cup \{\infty\}$ of $\lambda G - H$. For the sake of briefness of notation, we consider only the case $\lambda_0 \in \mathbb{R}$. The case $\lambda_0 = \infty$ can be proved analogously. (This can be seen easily by interchanging the roles of G and H.) After an eventual reordering of blocks, we may assume that

$$\lambda G - H = \lambda \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} - \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix},$$

where $\lambda G_1 - H_1$ contains all the blocks associated with λ_0 and $\lambda G_2 - H_2$ contains all the other blocks. We assume furthermore that $\lambda G_1 - H_1$ contains p_+ odd sized blocks with sign ε and p_- odd sized blocks with sign $-\varepsilon$, where $\varepsilon \in \{1, -1\}$, i.e., in particular we have $\operatorname{Signsum}(\lambda_0) = \varepsilon(p_+ - p_-)$. Then, applying Remark 10 various times on $\lambda G_1 - H_1$ and eventually reordering some blocks, we find that

$$\begin{aligned} &\lambda G - H \\ \sim_c\ &\lambda \begin{bmatrix} 0 & 0 & 0 & 0 & \hat{G}_{15} \\ 0 & \varepsilon I_{p_+} & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon I_{p_-} & 0 & 0 \\ 0 & 0 & 0 & G_2 & 0 \\ \hat{G}_{15}^* & 0 & 0 & 0 & \hat{G}_{55} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & \hat{H}_{15} \\ 0 & \varepsilon\lambda_0 I_{p_+} & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon\lambda_0 I_{p_-} & 0 & 0 \\ 0 & 0 & 0 & H_2 & 0 \\ \hat{H}_{15}^* & 0 & 0 & 0 & \hat{H}_{55} \end{bmatrix}, \end{aligned}$$

where $\hat{G}_{15}$ and $\hat{H}_{15}$ are anti-triangular. Let us assume, w.l.o.g., that $p_+ \geq p_-$. Setting

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} I_{p_+ - p_-} & 0 & 0 \\ 0 & \varepsilon I_{p_-} & \varepsilon I_{p_-} \\ 0 & -\varepsilon I_{p_-} & \varepsilon I_{p_-} \end{bmatrix}$$

and noting that

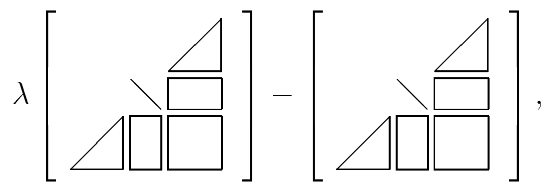
$$\begin{aligned} &P^* \left(\lambda \begin{bmatrix} \varepsilon I_{p_+ - p_-} & 0 & 0 \\ 0 & \varepsilon I_{p_-} & 0 \\ 0 & 0 & -\varepsilon I_{p_-} \end{bmatrix} - \begin{bmatrix} \varepsilon\lambda_0 I_{p_+ - p_-} & 0 & 0 \\ 0 & \varepsilon\lambda_0 I_{p_-} & 0 \\ 0 & 0 & -\varepsilon\lambda_0 I_{p_-} \end{bmatrix} \right) P \\ = \ &\lambda \begin{bmatrix} \varepsilon I_{p_+ - p_-} & 0 & 0 \\ 0 & 0 & \varepsilon I_{p_-} \\ 0 & \varepsilon I_{p_-} & 0 \end{bmatrix} - \begin{bmatrix} \varepsilon\lambda_0 I_{p_+ - p_-} & 0 & 0 \\ 0 & 0 & \varepsilon\lambda_0 I_{p_-} \\ 0 & \varepsilon\lambda_0 I_{p_-} & 0 \end{bmatrix}, \end{aligned}$$

we obtain by applying Remark 10 that

$$\lambda G - H \sim_c \lambda \begin{bmatrix} 0 & 0 & 0 & \check{G}_{14} \\ 0 & \varepsilon I_{p_+ - p_-} & 0 & 0 \\ 0 & 0 & G_2 & 0 \\ \check{G}_{14}^* & 0 & 0 & \check{G}_{44} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & \check{H}_{14} \\ 0 & \varepsilon\lambda_0 I_{p_+ - p_-} & 0 & 0 \\ 0 & 0 & H_2 & 0 \\ \check{H}_{14}^* & 0 & 0 & \check{H}_{44} \end{bmatrix},$$

where $\check{G}_{14}$ and $\check{H}_{14}$ are anti-triangular and the block $\varepsilon\lambda I_{p_+ - p_-} - \varepsilon\lambda_0 I_{p_+ - p_-}$ displays the sign sum of λ_0. Using the induction hypothesis on $\lambda G_2 - H_2$, the result follows by one more application of Remark 10. $\square$

Remark 12 The pencil $P^*(\lambda G - H)P$ has the pattern



and the sign sum of each real eigenvalue or the eigenvalue ∞ of $\lambda G - H$ can be easily read off the subpencil $\lambda G_{22} - H_{22}$, since obviously we have

$$\operatorname{Signsum}(\lambda_\alpha, G_{22}, H_{22}) = \varepsilon_\alpha p_\alpha \quad \text{for } \alpha = 1, \ldots, k.$$

Remark 13 In [16], it was shown how to obtain an analogue of form (10) for skew-Hamiltonian/Hamiltonian pencils. This method can be easily adapted to Hermitian pencils. Doing so, one can see that in a step-wise reduction, *the reduction to the blocks G_{13} and H_{13} can be executed via unitary transformations.*

In the following we will deduce necessary and sufficient conditions for the existence of anti-triangular forms and anti-m-Hessenberg forms for Hermitian pencils. Given a Hermitian pencil $\lambda G - H$, we note that for every $t \in \mathbb{R}$, we have a Hermitian matrix $tG - H$. It is clear that if the pencil $\lambda G - H$ is in anti-triangular form then so is the Hermitian matrix $tG - H$. It will turn out that also the converse is true - at least in the case that the size of the pencil is even. Therefore, the results of section 2 imply that the existence of anti-triangular forms for the Hermitian pencil $\lambda G - H$ is linked to conditions on the indices of the matrices $tG - H$, where t is real.

Moreover, we will see that these conditions on indices can be interpreted as conditions on the sign sums of the real eigenvalues and the eigenvalue ∞ of the pencil $\lambda G - H$. Since we may assume that the pencil is in sign condensed form and since the blocks G_{13} and H_{13} in (10) are already in anti-triangular form, it remains to consider the block (11) that inherits all information on the sign sums. The following lemma examines this block and will be applied repeatedly.

Lemma 14 *Consider the pencil $\lambda G_{22} - H_{22}$ in the form (11). Furthermore, let $t_1, t_2 \in \mathbb{R}$ such that*

$$\lambda_1 < \ldots < \lambda_{\alpha-1} < t_1 < \lambda_\alpha < \ldots < \lambda_{\alpha+\beta} < t_2 < \lambda_{\alpha+\beta+1} < \ldots < \lambda_k.$$

Setting $\operatorname{Ind}(tG_{22} - H_{22}) = \Big(\nu_+(t), \nu_-(t), \nu_0(t)\Big)$, *we obtain that*

$$\Big(\nu_+(t_2) - \nu_-(t_2)\Big) - \Big(\nu_+(t_1) - \nu_-(t_1)\Big) = 2 \sum_{j=\alpha}^{\alpha+\beta} \varepsilon_j p_j.$$

$$\Big(\nu_+(t_2) - \nu_-(t_2)\Big) + \Big(\nu_+(t_1) - \nu_-(t_1)\Big) = 2 \left(\sum_{j=1}^{\alpha-1} \varepsilon_j p_j\right) - 2 \left(\sum_{j=\alpha+\beta+1}^{k} \varepsilon_j p_j\right) - 2\varepsilon_{k+1} p_{k+1}.$$

Proof. We obtain that

$$\begin{aligned} \nu_+(t_1) - \nu_-(t_1) &= \left(\sum_{j=1}^{\alpha-1} \varepsilon_j p_j\right) - \left(\sum_{j=\alpha}^{\alpha+\beta} \varepsilon_j p_j\right) - \left(\sum_{j=\alpha+\beta+1}^{k} \varepsilon_j p_j\right) - \varepsilon_{k+1} p_{k+1} \\ \text{and} \quad \nu_+(t_2) - \nu_-(t_2) &= \left(\sum_{j=1}^{\alpha-1} \varepsilon_j p_j\right) + \left(\sum_{j=\alpha}^{\alpha+\beta} \varepsilon_j p_j\right) - \left(\sum_{j=\alpha+\beta+1}^{k} \varepsilon_j p_j\right) - \varepsilon_{k+1} p_{k+1}. \end{aligned}$$

This implies the assertion. $\square$

We are now able to discuss necessary and sufficient conditions for the existence of anti-triangular forms for regular Hermitian pencils. We start with a result for the case that the size of the pencil is even.

Theorem 15 *Let $\lambda G - H \in \mathbb{C}^{2n\times 2n}$ be a regular Hermitian pencil and for $t \in \mathbb{R}$ let $Ind(tG-H) = \Big(\nu_+(t), \nu_-(t), \nu_0(t)\Big)$. Then the following statements are equivalent.*

1. *$\lambda G - H$ is congruent to a pencil in anti-triangular form.*
2. *For all $t \in \mathbb{R}$ we have that $|\nu_+(t) - \nu_-(t)| \leq \nu_0(t)$.*
3. *For almost all $t \in \mathbb{R}$ we have that $|\nu_+(t) - \nu_-(t)| \leq \nu_0(t)$.*
4. *If $\lambda_0 \in \mathbb{R} \cup \{\infty\}$ is an eigenvalue of $\lambda G - H$ then* $\mathrm{Signsum}(\lambda_0) = 0$.

Proof. '1) $\Rightarrow$ 2)': Let $P \in \mathbb{C}^{2n\times 2n}$ be a nonsingular matrix such that $P^*(\lambda G - H)P$ is in anti-triangular form. Then clearly $P^*(tG-H)P$ is Hermitian anti-triangular for all $t \in \mathbb{R}$. Thus, 2) follows from Corollary 5.

'2) $\Rightarrow$ 3)': is trivial.

'3) $\Rightarrow$ 4)': W.l.o.g. we may assume that $\lambda G - H$ is in sign condensed form (10). If λ_0 is not an eigenvalue of $\lambda G_{22} - H_{22}$ then trivially $\mathrm{Signsum}(\lambda_0) = 0$. Thus, let us consider an eigenvalue λ_α of $\lambda G_{22} - H_{22}$. There are two possible cases.

Case (1) Assume that $\lambda_\alpha \in \mathbb{R}$, that is $\alpha \in \{1, \ldots, k\}$, where $\lambda_1, \ldots, \lambda_k$ are as in (11).

Choose $t_1, t_2 \in \mathbb{R}$ such that

$$\lambda_1 < \ldots < \lambda_{\alpha-1} < t_1 < \lambda_\alpha < t_2 < \lambda_{\alpha+1} < \ldots < \lambda_k,$$

and furthermore, such that $|\nu_+(t_j) - \nu_-(t_j)| \leq \nu_0(t_j)$ holds for $j = 1, 2$ and that $t_1G - H$ and $t_2G - H$ are nonsingular. This is possible, since the pencil $\lambda G - H$ is regular, i.e., $tG - H$ is nonsingular for almost all $t \in \mathbb{R}$, and, in addition, condition 3) holds for almost all $t \in \mathbb{R}$. Then, we obtain from (10) and Lemma 6 that

$$\Big(\nu_+(t_j), \nu_-(t_j), \nu_0(t_j)\Big) = (m, m, 0) + \mathrm{Ind}(t_jG_{22} - H_{22}) \quad \text{for } j = 1, 2.$$

Since $t_1G - H$ and $t_2G - H$ are nonsingular, we have $\nu_0(t_1) = \nu_0(t_2) = 0$. Therefore, we obtain from Lemma 14 that

$$\begin{aligned} 0 &= \nu_0(t_2) + \nu_0(t_1) \geq |\nu_+(t_2) - \nu_-(t_2)| + |\nu_+(t_1) - \nu_-(t_1)| \\ &\geq \Big|\Big(\nu_+(t_2) - \nu_-(t_2)\Big) - \Big(\nu_+(t_1) - \nu_-(t_1)\Big)\Big| = 2 \cdot \mathrm{Signsum}(\lambda_\alpha). \end{aligned}$$

This implies $\mathrm{Signsum}(\lambda_\alpha) = 0$.

Case (2) If the assumption of Case (1) does not hold, then $\lambda_\alpha = \infty$.

In this case, we choose $t_1, t_2 \in \mathbb{R}$ such that

$$t_1 < \lambda_1 < \ldots < \lambda_k < t_2,$$

and furthermore such that $|\nu_+(t_j) - \nu_-(t_j)| \leq \nu_0(t)$ holds for $j = 1, 2$ and that $t_1G - H$ and $t_2G - H$ are nonsingular. Then we obtain from Lemma 14 that

$$0 \geq \left| \Big(\nu_+(t_2) - \nu_-(t_2) \Big) - \Big(\nu_+(t_1) - \nu_-(t_1) \Big) \right| = 2\,\mathrm{Signsum}(\lambda_\infty).$$

'4) $\Rightarrow$ 1)': This follows directly from Theorem 10, since 4) implies that the subpencil $\lambda G_{22} - H_{22}$ does not appear. □

Remark 16 The condition $\mathrm{Signsum}(\lambda_0) = 0$ means that in the canonical form (9) the odd-sized blocks associated with λ_0 occur in pairs with opposite signs $+1$ and -1, respectively. (The pairing applies only to the signs, but not to the sizes of the blocks!)

Our next result gives necessary and sufficient conditions for the existence of anti-Hessenberg forms for a Hermitian pencil $\lambda G - H$. Again, we will consider the indices of the Hermitian matrices $tG - H$, where $t \in \mathbb{R}$, and then interprete these conditions in terms of the sign sums of the real eigenvalues and the eigenvalue ∞. First, we consider the case that the size of the pencil is odd.

Theorem 17 *Let $\lambda G - H \in \mathbb{C}^{(2n+1)\times(2n+1)}$ be a regular Hermitian pencil and for $t \in \mathbb{R}$ let* $\mathrm{Ind}(tG - H) = \Big(\nu_+(t), \nu_-(t), \nu_0(t) \Big)$. *Then the following statements are equivalent.*

1. *$\lambda G - H$ is congruent to a pencil in anti-Hessenberg form.*
2. *For all $t \in \mathbb{R}$ we have that $|\nu_+(t) - \nu_-(t)| \leq \nu_0(t) + 1$.*
3. *For almost all $t \in \mathbb{R}$ we have that $|\nu_+(t) - \nu_-(t)| \leq \nu_0(t) + 1$.*
4. *For every real eigenvalue $\lambda_0 \in \mathbb{R} \cup \{\infty\}$ we have that $|\mathrm{Signsum}(\lambda_0)| \leq 1$ and if $\lambda_1 < \ldots < \lambda_r \leq \infty$ denote the real eigenvalues (including ∞) with nonzero sign sum then $\lambda_1, \ldots, \lambda_r$ satisfy the property*

$$\mathrm{Signsum}(\lambda_\alpha) = -\mathrm{Signsum}(\lambda_{\alpha+1}), \quad \alpha = 1, \ldots, r - 1. \tag{12}$$

Proof. '1) $\Rightarrow$ 2)': Let $P \in \mathbb{C}^{2n\times 2n}$ be a nonsingular matrix such that $P^*(\lambda G - H)P$ is in anti-Hessenberg form. Then $P^*(tG - H)P$ is Hermitian anti-Hessenberg for all $t \in \mathbb{R}$. Thus, 2) follows from Corollary 4.

’2) $\Rightarrow$ 3)’: is trivial.

’3) $\Rightarrow$ 4)’: W.l.o.g. we may assume that $\lambda G - H$ is in sign condensed form (10). Again, it is sufficient to consider the subpencil $\lambda G_{22} - H_{22}$ that has the form (11). Let us consider an eigenvalue λ_α of $\lambda G_{22} - H_{22}$.

Case (1) Assume that $\lambda_\alpha \in \mathbb{R}$, that is $\lambda_\alpha \in \{\lambda_1, \ldots, \lambda_k\}$. Choose $t_1, t_2 \in \mathbb{R}$ such that

$$\lambda_1 < \ldots < \lambda_{\alpha-1} < t_1 < \lambda_\alpha < t_2 < \lambda_{\alpha+1} < \ldots < \lambda_k,$$

and such that $t_j G - H$ is nonsingular and $|\nu_+(t_j) - \nu_-(t)| \leq \nu_0(t_j) + 1$ for $j = 1, 2$. Then we obtain from Lemma 14 and $\nu_0(t_1) = \nu_0(t_2) = 0$ that

$$\begin{aligned} 2 &\geq |\nu_+(t_1) - \nu_-(t_1)| + |\nu_+(t_2) - \nu_-(t_2)| \\ &\geq \Big| \Big(\nu_+(t_1) - \nu_-(t_1)\Big) - \Big(\nu_+(t_2) - \nu_-(t_2)\Big) \Big| = |2\mathrm{Signsum}(\lambda_\alpha)|. \end{aligned}$$

This implies $|\mathrm{Signsum}(\lambda_\alpha)| \leq 1$.

Case (2) If the assumption of Case (1) does not hold then $\lambda_\alpha = \infty$.

In this case, we choose $t_1, t_2 \in \mathbb{R}$ such that

$$t_1 < \lambda_1 < \ldots < \lambda_k < t_2,$$

and, furthermore, such that $t_j G - H$ is nonsingular and $|\nu_+(t_j) - \nu_-(t_j)| \leq \nu_0(t_j) + 1$ for $j = 1, 2$. Applying Lemma 14 once more, we conclude that

$$2 \geq 2|\mathrm{Signsum}(\lambda_\infty)|.$$

For the second part of 3) we first note that $|\mathrm{Signsum}(\lambda_\beta)| = 1$ for all the eigenvalues λ_β of $\lambda G_{22} - H_{22}$, since this subpencil does not contain eigenvalues with sign sum zero. We pick an $\alpha \in \{1, \ldots, k\}$ and distinguish two cases.

Case (a) Assume $\alpha < k$. Then choose $t_1, t_2 \in \mathbb{R}$ such that $t_j G - H$ is nonsingular, $|\nu_+(t_j) - \nu_-(t)| \leq \nu_0(t_j) + 1$ for $j = 1, 2$, and such that

$$\lambda_1 < \ldots < \lambda_{\alpha-1} < t_1 < \lambda_\alpha < \lambda_{\alpha+1} < t_2 < \lambda_{\alpha+2} < \ldots < \lambda_k.$$

Applying Lemma 14 again, we obtain that

$$2 \geq 2|\mathrm{Signsum}(\lambda_\alpha) + \mathrm{Signsum}(\lambda_{\alpha+1})|.$$

This implies $\mathrm{Signsum}(\lambda_\alpha) = -\mathrm{Signsum}(\lambda_{\alpha+1})$, since both terms do not vanish.

Case (b) If the assumption of Case (a) does not hold, then $\alpha = k$. If $\lambda G_{22} - H_{22}$ does not have the eigenvalue ∞ then λ_α is already the eigenvalue of maximal modulus and the proof of (12) proceeds as in Case (a). Otherwise, choose $t_1, t_2 \in \mathbb{R}$ such that $t_j G - H$ is nonsingular, $|\nu_+(t_j) - \nu_-(t)| \leq \nu_0(t_j) + 1$ for $j = 1, 2$, and such that

$$t_1 < \lambda_1 < \ldots < \lambda_{k-1} < t_2 < \lambda_k.$$

Then we obtain from Lemma 14 that

$$\begin{aligned} 2 &\geq |\nu_+(t_2) - \nu_-(t_2)| + |\nu_+(t_1) - \nu_-(t_1)| \\ &\geq |\nu_+(t_2) - \nu_-(t_2) + \nu_+(t_1) - \nu_-(t_1)| \\ &= 2|\mathrm{Signsum}(\lambda_k) + \mathrm{Signsum}(\lambda_\infty)|. \end{aligned}$$

This implies $\mathrm{Signsum}(\lambda_k) = -\mathrm{Signsum}(\lambda_\infty)$.

'4) $\Rightarrow$ 1)': Again, we may assume that the pencil is in sign condensed form (10). It remains to show that the subpencil $\lambda G_{22} - H_{22}$ of the form (11) is congruent to anti-Hessenberg form. From 4) we find in particular that all the eigenvalues of $\lambda G_{22} - H_{22}$ are simple. Again, we consider two different cases.

Case (1) Assume that $\lambda G_{22} - H_{22}$ does not have the eigenvalue ∞.

This implies in particular that $k = 2q+1$ is odd, since the size of $\lambda G_{22} - H_{22}$ is necessarily odd and all its eigenvalues are simple. Let us assume, w.l.o.g., that the sign ε_1 of λ_1 is equal to one. Otherwise, we may consider the pencil $-(\lambda G - H)$. Then, the property (12) implies that the eigenvalues with sign $+1$ interlace the eigenvalues with sign -1. We visualize that by the following formula.

$$\begin{array}{cccccccccc} \lambda_1 < & & \lambda_3 < & & \ldots & < \lambda_{2q-1} & & < \lambda_{2q+1} & & \text{sign } 1 \\ & \lambda_2 < & & \lambda_4 < & \ldots & & < \lambda_{2q} & & & \text{sign } -1 \end{array} \tag{13}$$

By row and column permutations we find that

$$\lambda G_{22} - H_{22} \sim_c \lambda \begin{bmatrix} -I_q & 0 \\ 0 & I_{q+1} \end{bmatrix} - \begin{bmatrix} -\tilde{H}_1 & 0 \\ 0 & \tilde{H}_2 \end{bmatrix},$$

where $\mathrm{spec}(\tilde{H}_1) = \{\lambda_2, \lambda_4, \ldots, \lambda_{2q}\}$ and $\mathrm{spec}(\tilde{H}_2) = \{\lambda_1, \lambda_3, \ldots, \lambda_{2q+1}\}$.

The interlacing property (13) allows us to solve an inverse eigenvalue problem (see [2] or [8]). There, it is shown that (13) is sufficient for the existence of a unitary matrix $Q \in \mathbb{C}^{(q+1)\times(q+1)}$ such that

$$Q^* \tilde{H}_2 Q = \begin{bmatrix} \tilde{H}_{21} & \tilde{H}_{22} \\ \tilde{H}_{22}^* & \tilde{H}_{23} \end{bmatrix},$$

where $\tilde{H}_{23} \in \mathbb{R}$ and $\mathrm{spec}(\tilde{H}_{21}) = \mathrm{spec}(\tilde{H}_1)$. From this, we see that

$$\lambda G_{22} - H_{22} \ \sim_c \ \lambda \begin{bmatrix} -I_q & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -\tilde{H}_1 & 0 & 0 \\ 0 & \tilde{H}_{21} & \tilde{H}_{22} \\ 0 & \tilde{H}_{22}^* & \tilde{H}_{23} \end{bmatrix}.$$

Note that we obtain from $\operatorname{spec}(\tilde{H}_{21}) = \operatorname{spec}(\tilde{H}_1)$ that every eigenvalue of the upper principal subpencil

$$\lambda \begin{bmatrix} -I_q & 0 \\ 0 & I_q \end{bmatrix} - \begin{bmatrix} -\tilde{H}_1 & 0 \\ 0 & \tilde{H}_{21} \end{bmatrix}$$

occurs with algebraic multiplicity 2 and opposite signs. Hence, the pencil satisfies condition 4) of Theorem 15 and there exists a nonsingular $P \in \mathbb{C}^{2q\times 2q}$ such that

$$P^* \left(\lambda \begin{bmatrix} -I_q & 0 \\ 0 & I_q \end{bmatrix} - \begin{bmatrix} \tilde{H}_1 & 0 \\ 0 & \tilde{H}_{21} \end{bmatrix} \right) P$$

is in anti-triangular form. This implies that

$$\begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix}^* \left(\lambda \begin{bmatrix} -I_q & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -\tilde{H}_1 & 0 & 0 \\ 0 & \tilde{H}_{21} & \tilde{H}_{22} \\ 0 & \tilde{H}_{22}^* & \tilde{H}_{23} \end{bmatrix} \right) \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix}$$

is in anti-Hessenberg form.

Case (2) If the assumption of Case (1) does not hold then $\lambda G_{22} - H_{22}$ has the eigenvalue ∞.

This implies that $k = 2q$ is even. Again, property (12) implies that the eigenvalues with sign $+1$ interlace the eigenvalues with sign -1, where we assume again that $\varepsilon_1 = 1$. Thus, we have the following situation.

$$\begin{array}{llllll} \lambda_1 < & & \lambda_3 < & & \ldots < \lambda_{2q-1} & & \text{with sign } +1 \\ & \lambda_2 < & & \lambda_4 < & \ldots & < \lambda_{2q} & \text{with sign } -1 \end{array} \tag{14}$$

Furthermore, the eigenvalue ∞ has the sign $+1$. By row and column permutations we find that

$$\lambda G_{22} - H_{22} \sim_c \lambda \begin{bmatrix} I_q & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \tilde{H}_1 & 0 & 0 \\ 0 & -\tilde{H}_2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $\operatorname{spec}(\tilde{H}_1) = \{\lambda_1, \lambda_3, \ldots, \lambda_{2q-1}\}$ and $\operatorname{spec}(\tilde{H}_2) = \{\lambda_2, \lambda_4, \ldots, \lambda_{2q}\}$.

The interlacing property (14) allows us to solve another inverse eigenvalue problem. In [26] it is shown that (14) is sufficient for the existence of a rank-one updating with a vector $x \in \mathbb{R}^q$ such that $\operatorname{spec}(\tilde{H}_1 + xx^*) = \operatorname{spec}(\tilde{H}_2)$. From this, we see that

$$\begin{aligned} &\begin{bmatrix} I_q & 0 & x \\ 0 & I_q & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\lambda \begin{bmatrix} I_q & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \tilde{H}_1 & 0 & 0 \\ 0 & -\tilde{H}_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} I_q & 0 & 0 \\ 0 & I_q & 0 \\ x^* & 0 & 1 \end{bmatrix} \\ = \ &\lambda \begin{bmatrix} I_q & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \tilde{H}_1 + xx^* & 0 & x \\ 0 & -\tilde{H}_2 & 0 \\ x^* & 0 & 1 \end{bmatrix}. \end{aligned}$$

Again, we see from Theorem 15 that the upper principal $2q \times 2q$ subpencil is congruent to a pencil in anti-triangular form and thus, $\lambda G_{22} - H_{22}$ is congruent to a pencil in anti-Hessenberg form. $\square$

Theorem 15 and Theorem 17 are special cases of a more general result for anti-m-Hessenberg forms. This general result can be shown by induction on m. For the induction step, we need the following lemma.

Lemma 18 *Let* $\lambda G_{22} - H_{22} \in \mathbb{C}^{(n-2m)\times(n-2m)}$ *be a pencil in form (11). Furthermore, let* $\mathrm{Ind}(tG_{22} - H_{22}) = \Big(\nu_+(t), \nu_-(t), \nu_0(t)\Big)$*, and assume that*

$$|\nu_+(t) - \nu_-(t)| \leq \nu_0(t) + m + 1 \quad \textit{for almost all } t \in \mathbb{R}.$$

Then there exists a nonsingular matrix $P \in \mathbb{C}^{(n-2m)\times(n-2m)}$ *such that*

$$P^*(\lambda G_{22} - H_{22})P = \lambda \begin{bmatrix} G' & 0 \\ 0 & G'' \end{bmatrix} - \begin{bmatrix} H' & 0 \\ 0 & H'' \end{bmatrix},$$

where the size of $\lambda G'' - H''$ *is odd and such that the following conditions are satisfied.*

1. *Setting* $\mathrm{Ind}(tG' - H') = \Big(\mu_+(t), \mu_-(t), \mu_0(t)\Big)$*, we have that*

$$|\mu_+(t) - \mu_-(t)| \leq \mu_0(t) + m \quad \textit{for almost all } t \in \mathbb{R}.$$

2. *Setting* $\mathrm{Ind}(tG'' - H'') = \Big(\pi_+(t), \pi_-(t), \pi_0(t)\Big)$*, we have that*

$$|\pi_+(t) - \pi_-(t)| \leq \pi_0(t) + 1 \quad \textit{for almost all } t \in \mathbb{R}.$$

Proof. Let $s_1, \ldots, s_{k+1} \in \mathbb{R}$ be arbitrary with the condition that for $j = 1, \ldots, k+1$ we have that $|\nu_+(s_j) - \nu_-(s_j)| \leq \nu_0(s_j) + m + 1$, and such that

$$s_1 < \lambda_1 < s_2 < \ldots < s_k < \lambda_k < s_{k+1}.$$

This implies in particular that $\nu_0(s_j) = 0$. Applying Lemma 14, we find the recursive formula

$$\Big(\nu_+(s_{\alpha+1}) - \nu_-(s_{\alpha+1})\Big) - \Big(\nu_+(s_\alpha) - \nu_-(s_\alpha)\Big) = 2p_\alpha\varepsilon_\alpha.$$

Thus, the map $\alpha \mapsto \Big(\nu_+(s_\alpha) - \nu_-(s_\alpha)\Big)$ is increasing whenever ε_α is positive and decreasing whenever ε_α is negative. Hence, 'extremal points' such that $|\nu_+(s_\alpha) - \nu_-(s_\alpha)| = m+1$, can only be reached for an α such that $\varepsilon_\alpha \neq \varepsilon_{\alpha-1}$. Therefore, let us combine adjacent blocks with equal signs. W.l.o.g., we may assume that $\varepsilon_1 = 1$. Hence, we find that $\lambda G_{22} - H_{22}$ is in the form

$$\lambda G_{22} - H_{22} = \lambda \begin{bmatrix} (-1)^2 E_{q_1} & & 0 \\ & \ddots & \\ 0 & & (-1)^{l+1} E_{q_l} \end{bmatrix} - \begin{bmatrix} (-1)^2 D_{q_1} & & 0 \\ & \ddots & \\ 0 & & (-1)^{l+1} D_{q_l} \end{bmatrix},$$

where $E_{q_\alpha} = I_{q_\alpha}$ for $\alpha = 1, \ldots, l-1$ and $E_{q_l} = \begin{bmatrix} I_{q_l - p_{k+1}} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{q_l \times q_l}$, and the matrices $D_{q_\alpha} \in \mathbb{C}^{q_\alpha \times q_\alpha}$, $\alpha = 1, \ldots, l$ are diagonal. Choose $t_\alpha \in \{s_1, \ldots, s_{k+1}\}$ such that t_α is smaller than all the eigenvalues displayed by the blocks $\lambda E_\beta - D_\beta$ for $\beta \geq \alpha$ and larger than all the eigenvalues displayed by the blocks $\lambda E_\beta - D_\beta$ for $\beta < \alpha$. We will distinguish two cases.

Case (1) Assume that l is odd.

We construct $\lambda G' - H'$ and $\lambda G'' - H''$ as follows. Writing

$$E_\alpha = \begin{bmatrix} e_\alpha & 0 \\ 0 & \tilde{E}_\alpha \end{bmatrix} \quad \text{and} \quad D_\alpha = \begin{bmatrix} d_\alpha & 0 \\ 0 & \tilde{D}_\alpha \end{bmatrix}, \tag{15}$$

where $e_\alpha, d_\alpha \in \mathbb{C}$, let us define

$$\begin{aligned} \lambda G' - H' &= \lambda \begin{bmatrix} (-1)^2 \tilde{E}_1 & & \\ & \ddots & \\ & & (-1)^{l+1} \tilde{E}_l \end{bmatrix} - \begin{bmatrix} (-1)^2 \tilde{D}_1 & & \\ & \ddots & \\ & & (-1)^{l+1} \tilde{D}_l \end{bmatrix}, \\ \lambda G'' - H'' &= \lambda \begin{bmatrix} (-1)^2 e_1 & & \\ & \ddots & \\ & & (-1)^{l+1} e_l \end{bmatrix} - \begin{bmatrix} (-1)^2 d_1 & & \\ & \ddots & \\ & & (-1)^{l+1} d_l \end{bmatrix}. \end{aligned}$$

From the construction, we see immediately that the eigenvalues of $\lambda G'' - H''$ have sign sum equal to one and satisfy the interlacing property (12) of Theorem 17. This implies in particular

$$|\pi_+(t) - \pi_-(t)| \leq \pi_0(t) + 1.$$

It remains to show that $|\mu_+(t) - \mu_-(t)| \leq \mu_0(t) + m$. For this, we note that from Lemma 14 we obtain that

$$\begin{aligned} \Big(\nu_+(t_{\alpha+1}) - \nu_-(t_{\alpha+1})\Big) - \Big(\nu_+(t_\alpha) - \nu_-(t_\alpha)\Big) &= 2(-1)^{\alpha+1} q_\alpha, &(16) \\ \Big(\mu_+(t_{\alpha+1}) - \mu_-(t_{\alpha+1})\Big) - \Big(\mu_+(t_\alpha) - \mu_-(t_\alpha)\Big) &= 2(-1)^{\alpha+1} (q_\alpha - 1). &(17) \end{aligned}$$

Furthermore, for $\alpha = 1$ we obtain that

$$\begin{aligned} \nu_+(t_1) - \nu_-(t_1) &= -q_1 + q_2 - + \ldots + q_{l-1} - q_l, \\ \mu_+(t_1) - \mu_-(t_1) &= -(q_1 - 1) + (q_2 - 1) - + \ldots + (q_{l-1} - 1) - (q_l - 1) \\ &= \nu_+(t_1) - \nu_-(t_1) + 1. \end{aligned}$$

From this and from (16) and (17), we conclude that

$$\Big(\mu_+(t_\alpha) - \mu_-(t_\alpha)\Big) - \Big(\nu_+(t_\alpha) - \nu_-(t_\alpha)\Big) = (-1)^{\alpha+1}. \tag{18}$$

Formulas (16) and (17) imply in particular that the map $\alpha \mapsto \Big(\nu_+(t_\alpha) - \nu_-(t_\alpha)\Big)$ reaches a maximum only for an even α and a minimum only for an odd α. This, together with (18) implies that

$$\max_\alpha |\nu_+(t_\alpha) - \nu_-(t_\alpha)| = \max_\alpha |\mu_+(t_\alpha) - \mu_-(t_\alpha)| + 1,$$

that is

$$|\mu_+(t_\alpha) - \mu_-(t_\alpha)| \le m \quad \text{for all } \alpha. \tag{19}$$

By the choice of the t_α, it is now clear that we have

$$|\mu_+(t) - \mu_-(t)| \le \mu_0(t) + m$$

for almost all $t \in \mathbb{R}$.

Case (2) If l is even, then the situation is more complicated, since we want the pencil $\lambda G'' - H''$ to have odd size. Therefore, we have to change the construction of Case (1). We consider two different subcases.

Subcase (2a) Assume that $|\nu_+(t_l) - \nu_-(t_l)| < |\nu_+(t_1) - \nu_-(t_1)|$.

With the notation (15), let us define

$$\begin{aligned} &\lambda G' - H' \\ = \;& \lambda \begin{bmatrix} (-1)^2\tilde{E}_1 & & & \\ & \ddots & & \\ & & (-1)^l\tilde{E}_{l-1} & \\ & & & -E_l \end{bmatrix} - \begin{bmatrix} (-1)^2\tilde{D}_1 & & & \\ & \ddots & & \\ & & (-1)^l\tilde{D}_{l-1} & \\ & & & -D_l \end{bmatrix}, \end{aligned}$$

$$\text{and} \quad \lambda G'' - H'' = \lambda \begin{bmatrix} (-1)^2 e_1 & & \\ & \ddots & \\ & & (-1)^l e_{l-1} \end{bmatrix} - \begin{bmatrix} (-1)^2 d_1 & & \\ & \ddots & \\ & & (-1)^l d_{l-1} \end{bmatrix},$$

i.e., we left out e_l and d_l in the construction of $\lambda G'' - H''$. Analogous to Case 91) we find that

$$\Big(\mu_+(t_\alpha) - \mu_-(t_\alpha)\Big) - \Big(\nu_+(t_\alpha) - \nu_-(t_\alpha)\Big) = (-1)^{\alpha+1}$$

for $\alpha = 1, \ldots, l-1$, but

$$\Big(\mu_+(t_l) - \mu_-(t_l)\Big) - \Big(\nu_+(t_l) - \nu_-(t_l)\Big) = 1.$$

For $\alpha = 1, \ldots, l-1$, we can proceed as in Case (1) such that we find

$$|\mu_+(t_\alpha) - \mu_-(t_\alpha)| \le m \quad \text{for } \alpha = 1, \ldots, l-1.$$

Thus, the only case that may cause problems is the case that $\nu_+(t_l) - \nu_-(t_l) \geq m$, since then $\mu_+(t_l) - \mu_-(t_l) \geq m+1$. We show that this case does not occur.

Assume $\nu_+(t_l) - \nu_-(t_l) \geq m$. From the assumption of the Subcase (2a) and the fact that for odd α the term $\nu_+(t_\alpha) - \nu_-(t_\alpha)$ may only reach a minimum, but not a maximum, we obtain that

$$\nu_+(t_1) - \nu_-(t_1) = -(m+1).$$

It follows from Lemma 14 that

$$2q_l = \nu_+(t_1) - \nu_-(t_1) + \nu_+(t_l) - \nu_-(t_l) \leq m - (m+1) = -1.$$

This is a contradiction. Therefore, we have $\nu_+(t_l) - \nu_-(t_l) < m$ and thus, we obtain analogous to Case (1) for almost all $t \in \mathbb{R}$ that

$$|\mu_+(t) - \mu_-(t)| \leq \mu_0(t) + m.$$

Subcase (2b) If the assumption of (2a) does not hold then $|\nu_+(l) - \nu_-(l)| \geq |\nu_+(1) - \nu_-(1)|$.

In this case we leave out e_1 and d_1 in the construction of $\lambda G'' - H''$, i.e., we set

$$\begin{aligned}
\lambda G' - H' &= \lambda \begin{bmatrix} E_1 & & & \\ & -\tilde{E}_2 & & \\ & & \ddots & \\ & & & (-1)^{l+1}\tilde{E}_l \end{bmatrix} - \begin{bmatrix} D_1 & & & \\ & -\tilde{D}_2 & & \\ & & \ddots & \\ & & & (-1)^{l+1}\tilde{D}_l \end{bmatrix}, \\
\lambda G'' - H'' &= \lambda \begin{bmatrix} -e_2 & & \\ & \ddots & \\ & & (-1)^{l+1}e_l \end{bmatrix} - \begin{bmatrix} -d_2 & & \\ & \ddots & \\ & & (-1)^{l+1}d_l \end{bmatrix}.
\end{aligned}$$

Analogous to Case (1) we find that

$$\Big(\mu_+(t_\alpha) - \mu_-(t_\alpha)\Big) - \Big(\nu_+(t_\alpha) - \nu_-(t_\alpha)\Big) = (-1)^{\alpha+1}$$

for $\alpha = 2, \ldots, l$, but

$$\Big(\mu_+(t_1) - \mu_-(t_1)\Big) - \Big(\nu_+(t_1) - \nu_-(t_1)\Big) = -1.$$

Analogous to Subcase (2a) it remains to show that $\nu_+(t_1) - \nu_-(t_1) > -m$.

Assume $\nu_+(t_1) - \nu_-(t_1) \leq -m$. Then

$$2q_l = \nu_+(t_1) - \nu_-(t_1) + \nu_+(t_l) - \nu_-(t_l) \leq -m + (m+1) = 1,$$

and this is a contradiction. The rest of the proof proceeds analogous to Subcase (2a). $\square$

Theorem 19 *Let $\lambda G - H \in \mathbb{C}^{n\times n}$ be a regular Hermitian pencil and let $m \le n$ be such that $n-m$ is even. Furthermore, let Ind$(tG-H) = \Big(\nu_+(t), \nu_-(t), \nu_0(t)\Big)$ for $t \in \mathbb{R}$. Then the following statements are equivalent.*

1. *$\lambda G - H$ is congruent to a pencil in anti-m-Hessenberg form.*
2. *For all $t \in \mathbb{R}$ we have that $|\nu_+(t) - \nu_-(t)| \le \nu_0(t) + m$.*
3. *For almost all $t \in \mathbb{R}$ we have that $|\nu_+(t) - \nu_-(t)| \le \nu_0(t) + m$.*

Proof. '1) $\Rightarrow$ 2)': Let $P \in \mathbb{C}^{n\times n}$ be nonsingular such that $P^*(\lambda G - H)P$ is in anti-m-Hessenberg form. Then 2) follows from Corollary 4.

'2) $\Rightarrow$ 3)': is trivial.

'3) $\Rightarrow$ 1)': We proceed by induction on m.

'$m = 0$' and '$m = 1$' have already been proved, see Theorem 15 and Theorem 17.

'$m \Rightarrow (m+1)$': Once again we may assume that $\lambda G - H$ is in sign condensed form (10) and it is sufficient to consider the subpencil $\lambda G_{22} - H_{22}$ that has the form (11). By Lemma 18 we find that there exists a nonsingular matrix $\tilde{P} \in \mathbb{C}^{n\times n}$ such that

$$\tilde{P}^*(\lambda G_{22} - H_{22})\tilde{P} = \lambda \begin{bmatrix} G' & 0 \\ 0 & G'' \end{bmatrix} - \begin{bmatrix} H' & 0 \\ 0 & H'' \end{bmatrix},$$

where the size of $\lambda G'' - H''$ is odd and, setting $\mathrm{Ind}(tG' - H') = \Big(\mu_+(t), \mu_-(t), \mu_0(t)\Big)$ and $\mathrm{Ind}(tG'' - H'') = \Big(\pi_+(t), \pi_-(t), \pi_0(t)\Big)$, the following conditions are satisfied for almost all $t \in \mathbb{R}$.

$$\begin{aligned} |\mu_+(t) - \mu_-(t)| &\le \mu_0(t) + m, \\ |\pi_+(t) - \pi_-(t)| &\le \pi_0(t) + 1. \end{aligned}$$

Let n' and n'' denote the sizes of $\lambda G' - H'$ and $\lambda G'' - H''$, respectively. By assumption, $n - (m+1)$ is even and thus, so is $n' - m$, since $n - n' = n''$ is odd. Therefore, by the induction hypothesis and by Theorem 17, the pencil $\lambda G' - H'$ is congruent to a pencil in anti-m-Hessenberg form and $\lambda G'' - H''$ is congruent to a pencil in anti-Hessenberg form, i.e.,

$$\begin{aligned} &\lambda G_{22} - H_{22} \\ \sim_c\ &\lambda \begin{bmatrix} 0 & \hat{G}_{12} & \hat{G}_{13} & 0 & 0 & 0 \\ \hat{G}_{12}^* & \hat{G}_{22} & \hat{G}_{23} & 0 & 0 & 0 \\ \hat{G}_{13}^* & \hat{G}_{23}^* & \hat{G}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \check{G}_{12} & \check{G}_{13} \\ 0 & 0 & 0 & \check{G}_{12}^* & \check{G}_{22} & \check{G}_{23} \\ 0 & 0 & 0 & \check{G}_{13}^* & \check{G}_{23}^* & \check{G}_{33} \end{bmatrix} - \begin{bmatrix} 0 & \hat{H}_{12} & \hat{H}_{13} & 0 & 0 & 0 \\ \hat{H}_{12}^* & \hat{H}_{22} & \hat{H}_{23} & 0 & 0 & 0 \\ \hat{H}_{13}^* & \hat{H}_{23}^* & \hat{H}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \check{H}_{12} & \check{H}_{13} \\ 0 & 0 & 0 & \check{H}_{12}^* & \check{H}_{22} & \check{H}_{23} \\ 0 & 0 & 0 & \check{H}_{13}^* & \check{H}_{23}^* & \check{H}_{33} \end{bmatrix} \end{aligned}$$

$$
\sim_c \quad \lambda \begin{bmatrix} 0 & 0 & 0 & 0 & \hat{G}_{12} & \hat{G}_{13} \\ 0 & 0 & \check{G}_{12} & \check{G}_{13} & 0 & 0 \\ 0 & \check{G}_{12}^* & \check{G}_{22} & \check{G}_{23} & 0 & 0 \\ 0 & \check{G}_{13}^* & \check{G}_{23}^* & \check{G}_{33} & 0 & 0 \\ \hat{G}_{12}^* & 0 & 0 & 0 & \hat{G}_{22} & \hat{G}_{23} \\ \hat{G}_{13}^* & 0 & 0 & 0 & \hat{G}_{23}^* & \hat{G}_{33} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & \hat{H}_{12} & \hat{H}_{13} \\ 0 & 0 & \check{H}_{12} & \check{H}_{13} & 0 & 0 \\ 0 & \check{H}_{12}^* & \check{H}_{22} & \check{H}_{23} & 0 & 0 \\ 0 & \check{H}_{13}^* & \check{H}_{23}^* & \check{H}_{33} & 0 & 0 \\ \hat{H}_{12}^* & 0 & 0 & 0 & \hat{H}_{22} & \hat{H}_{23} \\ \hat{H}_{13}^* & 0 & 0 & 0 & \hat{H}_{23}^* & \hat{H}_{33} \end{bmatrix} \tag{20}
$$

where the submatrices have the following forms.

$$
\begin{aligned}
\hat{G}_{12}, \hat{H}_{12} &\in \mathbb{C}^{(\frac{n'-m}{2})\times(\frac{n'-m}{2})} \quad \text{are anti-triangular,} \\
\hat{G}_{13}, \hat{H}_{13} &\in \mathbb{C}^{(\frac{n'-m}{2})\times m}, \\
\check{G}_{12}, \check{H}_{12} &\in \mathbb{C}^{(\frac{n''-1}{2})\times(\frac{n''-1}{2})} \quad \text{are anti-triangular,} \\
\check{G}_{13}, \check{H}_{13} &\in \mathbb{C}^{(\frac{n''-1}{2})\times 1},
\end{aligned}
$$

and the other blocks have corresponding sizes. Hence, the pencil (20) is in anti-(m+1)-Hessenberg form. □

In Theorem 19 we did not give conditions on the sign sums as in the Theorems 15 and 17. In principle, this is also possible for the case $m > 1$. But then the conditions become very complicated, since we have to consider many subcases. Therefore, we prefer the conditions given in Theorem 19.

Clearly, Theorem 19 does not hold in the case that $n - m$ is odd. For example, let us consider the case $m = 0$ and $n = 3$. The Hermitian pencil

$$
\lambda \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}
$$

is in anti-triangular form, but we immediately obtain $\mathrm{Signsum}(2) = 1$. We see from this example that the eigenvalue that is displayed in the middle of the anti-diagonal plays an exceptional role and has to be treated differently from the rest of the eigenvalues. In fact, we may omit the eigenvalue that is displayed in the middle of the anti-diagonal, and its tribute to the sign sum may also be omitted, such that we can use the fact that $n - 1 - m$ is even and apply Theorem 19. This is done in the proof of the next theorem.

Theorem 20 *Let $\lambda G - H \in \mathbb{C}^{n\times n}$ be a regular Hermitian pencil and let $m \leq n$ be such that $n - m$ is odd. Furthermore, let $Ind(tG - H) = \Big(\nu_+(t), \nu_-(t), \nu_0(t)\Big)$ for $t \in \mathbb{R}$. Then the following statements are equivalent.*

1. *$\lambda G - H$ is congruent to a pencil in anti-m-Hessenberg form.*

2. There exists $t_0 \in \mathbb{R} \cup \{\infty\}$ *and* $\varepsilon \in \{1, -1\}$ *such that*

$$\begin{aligned} & |\nu_+(t) - \nu_-(t) + \varepsilon| \leq \nu_0(t) + m \quad \textit{for all} \quad t < t_0 \\ \textit{and} \quad & |\nu_+(t) - \nu_-(t) - \varepsilon| \leq \nu_0(t) + m \quad \textit{for all} \quad t > t_0. \end{aligned}$$

3. There exists $t_0 \in \mathbb{R} \cup \{\infty\}$ *and* $\varepsilon \in \{1, -1\}$ *such that*

$$\begin{aligned} & |\nu_+(t) - \nu_-(t) + \varepsilon| \leq \nu_0(t) + m \quad \textit{for almost all} \quad t < t_0 \\ \textit{and} \quad & |\nu_+(t) - \nu_-(t) - \varepsilon| \leq \nu_0(t) + m \quad \textit{for almost all} \quad t > t_0. \end{aligned}$$

Proof. '1) $\Rightarrow$ 2)': Let $P \in \mathbb{C}^{n\times n}$ be nonsingular such that $P^*(\lambda G - H)P$ is in anti-m-Hessenberg form. Thus, $P^*(tG - H)P$ is Hermitian anti-m-Hessenberg for all $t \in \mathbb{R}$. This means in particular that

$$P^*(tG-H)P = \begin{bmatrix} 0 & 0 & tG_{13} - H_{13} \\ 0 & tg_{22} - h_{22} & tG_{23} - H_{23} \\ tG_{13}^* - H_{13}^* & tG_{23}^* - H_{23}^* & tG_{33}^* - H_{33}^* \end{bmatrix},$$

where $tG_{13} - H_{13} \in \mathbb{C}^{(\frac{n-m-1}{2})\times(\frac{n+m-1}{2})}$, and $tg_{22} - h_{22} \in \mathbb{C}$, and where the other blocks have corresponding sizes. If $g_{22} \neq 0$ then let $t_0 = \frac{h_{22}}{g_{22}}$, otherwise set $t_0 = \infty$. Then Lemma 6 for $t \neq t_0$ implies that

$$\operatorname{Ind}(tG - H) = \operatorname{Ind}(t\tilde{G} - \tilde{H}) + \operatorname{Ind}(tg_{22} - h_{22}), \tag{21}$$

where

$$t\tilde{G} - \tilde{H} = \begin{bmatrix} 0 & tG_{13} - H_{13} \\ tG_{13}^* - H_{13}^* & * \end{bmatrix}.$$

For some $\tilde{t} < t_0$, set $\varepsilon = -\sigma(\tilde{t}g_{22} - h_{22})$ and $\Big(\mu_+(t), \mu_-(t), \mu_0(t)\Big) = \operatorname{Ind}(t\tilde{G} - \tilde{H})$, and note that $t\tilde{G} - \tilde{H}$ is in anti-m-Hessenberg form with size $n-1$. Thus, since $n-1-m$ is even, we can apply Theorem 19 and we obtain from (21) for $t > t_0$ that

$$|\nu_+(t) - \nu_-(t) + \varepsilon| = |\mu_+(t) - \mu_-(t)| \leq \mu_0(t) + m = \nu_0(t) + m,$$

since $\mu_0(t) = \nu_0(t)$ for $t \neq t_0$. Analogously we obtain for $t > t_0$ that

$$|\nu_+(t) - \nu_-(t) - \varepsilon| = |\mu_+(t) - \mu_-(t)| \leq \mu_0(t) + m = \nu_0(t) + m.$$

'2) $\Rightarrow$ 3)': is trivial.

'3) $\Rightarrow$ 1)': W.l.o.g. we may assume that $\varepsilon = 1$. Otherwise, we may consider the pencil $-(\lambda G - H)$. Repeating our proof strategy once more, we assume that $\lambda G - H$ is in sign condensed form (10) and we consider the subpencil $\lambda G_{22} - H_{22}$. By 2) there exists $t_0 \in \mathbb{R} \cup \{\infty\}$, such that

$$\begin{array}{lll} |\nu_+(t) - \nu_-(t) + 1| \leq \nu_0(t) + m & \text{for almost all} & t < t_0 \\ |\nu_+(t) - \nu_-(t) - 1| \leq \nu_0(t) + m & \text{for almost all} & t > t_0. \end{array} \tag{22}$$

We show next that we may assume that t_0 is an eigenvalue of $\lambda G_{22} - H_{22}$. For this, let λ_α be an eigenvalue of $\lambda G_{22} - H_{22}$ such that $|\lambda_\alpha - t_0|$ is minimal. We assume that $\lambda_\alpha \leq t_0$; the case $t_0 < \lambda_\alpha$ can be proved analogously. Clearly we have

$$\begin{array}{llll} & |\nu_+(t) - \nu_-(t) + 1| \leq \nu_0(t) + m & \text{for almost all} & t < \lambda_\alpha \quad (\text{since } \lambda_\alpha \leq t_0) \\ \text{and} & |\nu_+(t) - \nu_-(t) - 1| \leq \nu_0(t) + m & \text{for almost all} & t > t_0. \end{array}$$

Thus, it remains to show that $|\nu_+(t) - \nu_-(t) - 1| \leq \nu_0(t) + m$ for almost all $t \in (\lambda_\alpha, t_0)$. But this follows from the fact that $t \mapsto \Big(\nu_+(t) - \nu_-(t) - 1\Big)$ and $\nu_0(t)$ are constant on (λ_α, t_0), since by the choice of λ_α there are no other eigenvalues in the interval (λ_α, t_0).

Hence, we may assume that $t_0 = \lambda_\alpha$ is an eigenvalue of $\lambda G_{22} - H_{22}$. Let α be chosen minimal with the property that (22) is satisfied for all $t_0 = \lambda_\beta$, where $\beta \geq \alpha$. For the rest of the proof, we distinguish two different cases.

Case (1) Assume that $\alpha > 1$. Then there exists infinitely many t_1 with $\lambda_{\alpha-1} < t_1 < \lambda_\alpha$, such that

$$|\nu_+(t_1) - \nu_-(t_1) - 1| > \nu_0(t_1) + m.$$

On the other hand, we know that (22) holds for $t_0 = \lambda_\alpha$, i.e., t_1 can be chosen, such that

$$|\nu_+(t_1) - \nu_-(t_1) + 1| \leq \nu_0(t_1) + m.$$

Both inequalities hold simultaneously only if $\nu_+(t_1) - \nu_-(t_1) - 1 < -\Big(\nu_0(t_1) + m\Big)$. Choose t_2 such that $\lambda_\alpha < t_2 < \lambda_{\alpha+1}$ and $\Big|\nu_+(t_2) - \nu_-(t_2) - 1\Big| \leq \nu_0(t_2) + m$. Then Lemma 14 implies that

$$\Big(\nu_+(t_2) - \nu_-(t_2)\Big) - \Big(\nu_+(t_1) - \nu_-(t_1)\Big) = 2\varepsilon_\alpha p_\alpha.$$

If ε_α is equal to -1, then

$$\nu_+(t_2) - \nu_-(t_2) < \nu_+(t_1) - \nu_-(t_1) \leq -\Big(\nu_0(t_1) + m\Big) = -\Big(\nu_0(t_2) + m\Big),$$

since $\nu_0(t_1) = \nu_0(t_2) = 0$, which is a contradiction to $|\nu_+(t_2) - \nu_-(t_2) - 1| \leq \nu_0(t_2) + m$. Thus, $\varepsilon_\alpha = 1$. By permuting some rows and columns, we obtain that

$$\lambda G_{22} - H_{22} \sim_c \lambda \begin{bmatrix} g & 0 \\ 0 & \tilde{G} \end{bmatrix} - \lambda \begin{bmatrix} h & 0 \\ 0 & \tilde{H} \end{bmatrix},$$

where $\lambda g - h \in \mathbb{C}$ has the eigenvalue λ_α. Defining $\Big(\mu_+(t), \mu_-(t), \mu_0(t)\Big) = \mathrm{Ind}(t\tilde{G} - \tilde{H})$, we find that

$$|\mu_+(t) - \mu_-(t)| = \begin{cases} |\nu_+(t) - \nu_-(t) + 1| & \text{for all} \quad t < \lambda_\alpha \\ |\nu_+(t) - \nu_-(t) - 1| & \text{for all} \quad t > \lambda_\alpha. \end{cases}$$

This implies that $|\mu_+(t) - \mu_-(t)| \leq \nu_0(t) + m = \mu_0(t) + m$ for almost all $t \in \mathbb{R}$. Hence, by Theorem 19 the pencil $\lambda G_{22} - H_{22}$ is congruent to a pencil

$$\lambda \begin{bmatrix} g & 0 & 0 \\ 0 & 0 & \hat{G}_{23} \\ 0 & \hat{G}_{23}^* & \hat{G}_{33} \end{bmatrix} - \begin{bmatrix} h & 0 & 0 \\ 0 & 0 & \hat{H}_{23} \\ 0 & \hat{H}_{23}^* & \hat{H}_{33} \end{bmatrix},$$

where the subpencil $\lambda \begin{bmatrix} 0 & \hat{G}_{23} \\ \hat{G}_{23}^* & \hat{G}_{33} \end{bmatrix} - \lambda \begin{bmatrix} 0 & \hat{H}_{23} \\ \hat{H}_{23}^* & \hat{H}_{33} \end{bmatrix}$ is in anti-m-Hessenberg form. Thus, we finally obtain

$$\lambda G_{22} - H_{22} \sim_c \lambda \begin{bmatrix} 0 & 0 & \hat{G}_{23} \\ 0 & g & 0 \\ \hat{G}_{23}^* & 0 & \hat{G}_{33} \end{bmatrix} - \begin{bmatrix} 0 & 0 & \hat{H}_{23} \\ 0 & h & 0 \\ \hat{H}_{23}^* & 0 & \hat{H}_{33} \end{bmatrix},$$

and this pencil is in anti-m-Hessenberg form.

Case (2) If $\alpha \not> 1$, then $\alpha = 1$. Thus, (22) holds for all $t_0 = \lambda_\beta$. This means in particular that both

$$|\nu_+(t) - \nu_-(t) + 1| \leq \nu_0(t) + m \tag{23}$$
$$\text{and} \quad |\nu_+(t) - \nu_-(t) - 1| \leq \nu_0(t) + m \tag{24}$$

hold for almost all $t \in \mathbb{R}$. Permuting some rows and columns, we obtain that

$$\lambda G_{22} - H_{22} \sim \lambda \begin{bmatrix} g & 0 \\ 0 & \tilde{G} \end{bmatrix} - \lambda \begin{bmatrix} h & 0 \\ 0 & \tilde{H} \end{bmatrix},$$

where $\lambda g - h \in \mathbb{C}$ has the eigenvalue λ_1. Setting $\Big(\mu_+(t), \mu_-(t), \mu_0(t)\Big) = \mathrm{Ind}(t\tilde{G} - \tilde{H})$, we find that

$$|\mu_+(t) - \mu_-(t)| = \begin{cases} |\nu_+(t) - \nu_-(t) + \varepsilon_1| & \text{for all} \quad t < \lambda_1 \\ |\nu_+(t) - \nu_-(t) - \varepsilon_1| & \text{for all} \quad t > \lambda_1. \end{cases}$$

Then (23) and (24) imply that

$$|\mu_+(t) - \mu_-(t)| \leq \nu_0(t) + m = \mu_0(t) + m,$$

for almost all $t \in \mathbb{R}$ and hence we may proceed as in Case (1). This completes the proof. $\square$

Analogous to the proof of Theorem 15, we obtain conditions on the sign sum for the real eigenvalues and the eigenvalue ∞. We only state this for the anti-triangular case.

Corollary 21 *Let* $\lambda G - H \in \mathbb{C}^{(2n+1)\times(2n+1)}$ *be a regular Hermitian pencil. Then the following statements are equivalent.*

1. $\lambda G - H$ *is congruent to a pencil in anti-triangular form.*
2. *There exists exactly one eigenvalue* $\lambda_0 \in \mathbb{R}$ *such that* $\mathrm{Signsum}(\lambda_0) = \pm 1$ *and for every eigenvalue* $\lambda_\alpha \in \mathbb{R} \cup \{\infty\}$ *with* $\lambda_\alpha \neq \lambda_0$ *we have that* $\mathrm{Signsum}(\lambda_\alpha) = 0$.

4 Condensed forms for singular Hermitian pencils

In this section we include the case of singular Hermitian pencils. Although in this case an anti-triangular form does not necessarily display the roots of the elementary divisors, it still displays a nested set of invariant subspaces and therefore, the consideration of condensed forms of singular Hermitian pencils does still make sense.

Analogous to the regular case, we derive a sign condensed form and then discuss the existence of anti-triangular and anti-m-Hessenberg forms. Let us first consider the canonical form (see [22]).

Theorem 22 *Let* $\lambda G - H \in \mathbb{C}^{n\times n}$ *be a Hermitian pencil. Then there exists a nonsingular matrix* $P \in \mathbb{C}^{n\times n}$, *such that*

$$P^*(\lambda G - H)P = \lambda \begin{bmatrix} G' & 0 \\ 0 & G'' \end{bmatrix} - \begin{bmatrix} H' & 0 \\ 0 & H'' \end{bmatrix}, \tag{25}$$

where the following conditions are satisfied.

1. *The subpencil* $\lambda G' - H'$ *is block diagonal with diagonal blocks of the form*

$$\lambda \begin{bmatrix} 0 & 0 & Z_r \\ 0 & 0 & 0 \\ Z_r & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & \mathcal{J}_r(0)^* Z_r \\ 0 & 0 & e_1^* \\ Z_r \mathcal{J}_r(0) & e_1 & 0 \end{bmatrix} \in \mathbb{C}^{(r+1)\times(r+1)}, \tag{26}$$

   where $r \geq 0$.

2. *The subpencil* $\lambda G'' - H''$ *is regular and in canonical form (9).*

Proof. The proof follows directly from [22], Lemma 3. □

In the following, if we speak of the sign characteristic or sign sum of $\lambda_0 \in \mathbb{R} \cup \{\infty\}$ with respect to $\lambda G - H$, we mean the sign characteristic or sign sum, respectively, of $\lambda_0 \in \mathbb{R} \cup \{\infty\}$ with respect to the regular subpencil $\lambda G'' - H''$ in the canonical form (25) of $\lambda G - H$. Next, we generalize Theorem 11 to the case of singular pencils.

Theorem 23 (Sign condensed form) *Let $\lambda G - H \in \mathbb{C}^{n\times n}$ be a Hermitian pencil. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ and $m \in \mathbb{N}$, such that*

$$P^*(\lambda G - H)P = \lambda \begin{bmatrix} 0 & 0 & G_{13} \\ 0 & G_{22} & G_{23} \\ G_{13}^* & G_{23}^* & G_{33} \end{bmatrix} - \begin{bmatrix} 0 & 0 & H_{13} \\ 0 & H_{22} & H_{23} \\ H_{13}^* & H_{23}^* & H_{33} \end{bmatrix}, \tag{27}$$

where the subpencil $\lambda \begin{bmatrix} 0 & G_{13} \\ G_{13}^* & G_{33} \end{bmatrix} - \begin{bmatrix} 0 & H_{13} \\ H_{13}^* & H_{33} \end{bmatrix}$ is regular and $G_{13}, H_{13} \in \mathbb{C}^{m\times m}$ are lower anti-triangular. Furthermore

$$\begin{aligned} &\lambda G_{22} - H_{22} \\ = \ &\lambda \begin{bmatrix} O_l & & & & 0 \\ & \varepsilon_1 I_{p_1} & & & \\ & & \ddots & & \\ & & & \varepsilon_k I_{p_k} & \\ 0 & & & & 0 \end{bmatrix} - \begin{bmatrix} O_l & & & & 0 \\ & \varepsilon_1\lambda_1 I_{p_1} & & & \\ & & \ddots & & \\ & & & \varepsilon_k\lambda_k I_{p_k} & \\ 0 & & & & \varepsilon_{k+1} I_{p_{k+1}} \end{bmatrix}, \end{aligned} \tag{28}$$

where $\lambda_1 < \ldots < \lambda_k$. In addition we have for all $\lambda_0 \in \mathbb{R} \cup \{\infty\}$ that

$$\operatorname{Signsum}(\lambda_0, G, H) = \operatorname{Signsum}(\lambda_0, G_{22}, H_{22}).$$

Proof. Let $\lambda G - H$ be in canonical form (25) and let l denote the number of singular blocks of type (26). We prove the result by induction on l.

'$l = 0$': This is Theorem 10.

'$l \Rightarrow (l+1)$': It follows from Remark 10 that

$$\begin{aligned} \lambda G - H \ &= \ \lambda \begin{bmatrix} 0 & 0 & Z_r & 0 \\ 0 & 0 & 0 & 0 \\ Z_r & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{G} \end{bmatrix} - \begin{bmatrix} 0 & 0 & \mathcal{J}_r(0)^* Z_r & 0 \\ 0 & 0 & e_1^* & 0 \\ Z_r \mathcal{J}_r(0) & e_1 & 0 & 0 \\ 0 & 0 & 0 & \tilde{H} \end{bmatrix} \\ &\sim_c \ \lambda \begin{bmatrix} 0 & 0 & 0 & Z_r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{G} & 0 \\ Z_r & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & \mathcal{J}_r(0)^* Z_r \\ 0 & 0 & 0 & e_1^* \\ 0 & 0 & \tilde{H} & 0 \\ Z_r \mathcal{J}_r(0) & e_1 & 0 & 0 \end{bmatrix}, \end{aligned}$$

where the number of blocks of type (26) of the subpencil $\lambda\tilde{G} - \tilde{H}$ is equal to l. By the induction hypothesis we find that $\lambda\tilde{G} - \tilde{H}$ is congruent to a pencil that is in sign condensed form (27). Thus, the result follows by again applying Remark 10. □

We are now able to discuss necessary and sufficient conditions for the existence of anti-triangular and anti-m-Hessenberg forms for the singular case. A condition on sign sums of real eigenvalues (including ∞) of the regular subpencil that is analogous to the condition in Theorem 15 or Theorem 21 does not hold as we can see from the following example. The Hermitian pencil

$$\lambda \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

is already in anti-triangular form, but $\mathrm{Signsum}(\infty) = 1$. The background is that the problem of reducing a singular Hermitian pencil to anti-m-Hessenberg form is basically the problem of reducing a regular subpencil to anti-$(m+l)$-Hessenberg form, where l denotes the number of singular blocks of the pencil.

Theorem 24 *Let $\lambda G - H \in \mathbb{C}^{n\times n}$ be a Hermitian pencil and let $m \leq n$ be such that $n-m$ is even. Furthermore, let Ind$(tG-H) = \Big(\nu_+(t), \nu_-(t), \nu_0(t)\Big)$ for $t \in \mathbb{R}$. Then the following statements are equivalent.*

1. *$\lambda G - H$ is congruent to a pencil in anti-m-Hessenberg form.*
2. *For all $t \in \mathbb{R}$ we have that $|\nu_+(t) - \nu_-(t)| \leq \nu_0(t) + m$.*
3. *For almost all $t \in \mathbb{R}$ we have that $|\nu_+(t) - \nu_-(t)| \leq \nu_0(t) + m$.*

Proof. '1) $\Rightarrow$ 2)': As in the regular case, this follows from Lemma 3.

'2) $\Rightarrow$ 3)': is trivial.

'3) $\Rightarrow$ 1)': Assume that $\lambda G - H$ is in sign condensed form (27), i.e.,

$$\lambda G - H = \lambda \begin{bmatrix} 0 & 0 & 0 & G_{14} \\ 0 & O_l & 0 & G_{24} \\ 0 & 0 & G_{33} & G_{34} \\ G_{14}^* & G_{24}^* & G_{34}^* & G_{44} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & H_{14} \\ 0 & O_l & 0 & H_{24} \\ 0 & 0 & H_{33} & H_{34} \\ H_{14}^* & H_{24}^* & H_{34}^* & H_{44} \end{bmatrix},$$

where $\lambda G_{14} - H_{14} \in \mathbb{C}^{k\times k}$ is regular. For all $t \in \mathbb{R}$ that are not eigenvalues of the regular pencil $\lambda \begin{bmatrix} 0 & G_{14} \\ G_{14}^* & G_{44} \end{bmatrix} - \begin{bmatrix} 0 & H_{14} \\ H_{14}^* & H_{44} \end{bmatrix}$ we have that

$$\mathrm{Ind}(tG - H) = (k, k, 0) + (0, 0, l) + \mathrm{Ind}(tG_{33} - H_{33}).$$

Setting $\Big(\mu_+(t), \mu_-(t), \mu_0(t)\Big) := \mathrm{Ind}(tG_{33} - H_{33})$, we obtain for almost all these t that

$$|\mu_+(t) - \mu_-(t)| = |\nu_+(t) - \nu_-(t)| \leq \nu_0(t) + m = \mu_0(t) + m + l.$$

The size of $\lambda G_{33} - H_{33}$ is $n-2k-l$, such that $n-2k-l-(m-l) = n-m-2k-2l$ is even. Thus, Theorem 19 can be applied and $\lambda G_{33} - H_{33}$ is congruent to a pencil $\lambda \hat{G}_{33} - \hat{H}_{33}$ in anti-$(m+l)$-Hessenberg form. Hence

$$\lambda G - H \sim_c \lambda \begin{bmatrix} 0 & 0 & 0 & G_{14} \\ 0 & O_l & 0 & G_{24} \\ 0 & 0 & \hat{G}_{33} & * \\ G_{14}^* & G_{24}^* & * & G_{44} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & H_{14} \\ 0 & O_l & 0 & H_{24} \\ 0 & 0 & \hat{H}_{33} & * \\ H_{14}^* & H_{24}^* & * & H_{44} \end{bmatrix},$$

and this pencil is in anti-m-Hessenberg form. $\square$

We have a corresponding result for the case that $n-m$ is odd. Analogous to the regular case, the entry on the middel of the leftmost nonzero anti-diagonal plays an exceptional role.

Theorem 25 *Let $\lambda G - H \in \mathbb{C}^{n\times n}$ be a Hermitian pencil and let $n \geq m \in \mathbb{N}$ such that $n-m$ is odd. Furthermore, let Ind$(tG-H) = \big(\nu_+(t), \nu_-(t), \nu_0(t)\big)$ for $t \in \mathbb{R}$. Then the following statements are equivalent.*

1. *$\lambda G - H$ is congruent to a pencil in anti-m-Hessenberg form.*
2. *There exists $t_0 \in \mathbb{R} \cup \{\infty\}$ and $\varepsilon \in \{1, -1\}$, such that*
$$\begin{aligned} &|\nu_+(t) - \nu_-(t) + \varepsilon| \leq \nu_0(t) + m \quad \text{for all} \quad t < t_0 \\ \text{and} \quad &|\nu_+(t) - \nu_-(t) - \varepsilon| \leq \nu_0(t) + m \quad \text{for all} \quad t > t_0. \end{aligned}$$
3. *There exists $t_0 \in \mathbb{R} \cup \{\infty\}$ and $\varepsilon \in \{1, -1\}$, such that*
$$\begin{aligned} &|\nu_+(t) - \nu_-(t) + \varepsilon| \leq \nu_0(t) + m \quad \text{for almost all} \quad t < t_0 \\ \text{and} \quad &|\nu_+(t) - \nu_-(t) - \varepsilon| \leq \nu_0(t) + m \quad \text{for almost all} \quad t > t_0. \end{aligned}$$

Proof. '1) $\Rightarrow$ 2)': Assume there exists a nonsingular matrix $P \in \mathbb{C}^{2n\times 2n}$ such that $P^*(\lambda G - H)P$ is in anti-m-Hessenberg form, thus, $P^*(tG-H)P$ is Hermitian anti-m-Hessenberg for all $t \in \mathbb{R}$. This means in particular that

$$P^*(tG-H)P = \begin{bmatrix} 0 & 0 & tG_{13} - H_{13} \\ 0 & tg_{22} - h_{22} & tG_{23} - H_{23} \\ tG_{13}^* - H_{13}^* & tG_{23}^* - H_{23}^* & tG_{33}^* - H_{33}^* \end{bmatrix},$$

where $tG_{13} - H_{13} \in \mathbb{C}^{(\frac{n-m-1}{2})\times(\frac{n+m-1}{2})}$, $tg_{22} - h_{22} \in \mathbb{C}$, and the other blocks have corresponding sizes. If the subpencil $\lambda g_{22} - h_{22}$ is regular we may proceed as in the proof of Theorem 20. Otherwise, $\lambda g_{22} - h_{22} \equiv 0$. Then it follows from Lemma 3 that

$$|\nu_+(t) - \nu_-(t)| \leq 2\frac{n+m-1}{2} + \nu_0(t) - n = \nu_0(t) + m - 1$$

for all $t \in \mathbb{R}$. Hence, 2) is trivially satisfied for any $t_0 \in \mathbb{R} \cup \{\infty\}$.

'2) $\Rightarrow$ 3)': is trivial.

'3) $\Rightarrow$ 1)': is proved analogous to the proof of Theorem 24. $\square$

It was our main goal to obtain necessary and sufficient conditions for the existence of anti-triangular forms for general (including singular) Hermitian pencils. This explicit result follows now directly from Theorem 24 and Theorem 25.

Corollary 26 *Let $\lambda G - H \in \mathbb{C}^{2n\times 2n}$ be a Hermitian pencil. Furthermore, for $t \in \mathbb{R}$ let Ind$(tG - H) = \Big(\nu_+(t), \nu_-(t), \nu_0(t)\Big)$. Then the following statements are equivalent.*

1. *$\lambda G - H$ is congruent to a pencil in anti-triangular form.*
2. *For all $t \in \mathbb{R}$ we have that $|\nu_+(t) - \nu_-(t)| \le \nu_0(t)$.*

Corollary 27 *Let $\lambda G - H \in \mathbb{C}^{(2n+1)\times(2n+1)}$ be a Hermitian pencil. Furthermore, for $t \in \mathbb{R}$ let Ind$(tG - H) = \Big(\nu_+(t), \nu_-(t), \nu_0(t)\Big)$. Then the following statements are equivalent.*

1. *$\lambda G - H$ is congruent to a pencil in anti-triangular form.*
2. *There exists $t_0 \in \mathbb{R} \cup \{\infty\}$ and $\varepsilon \in \{1, -1\}$, such that*

$$\begin{array}{rll} & |\nu_+(t) - \nu_-(t) + \varepsilon| \le \nu_0(t) & \textit{for all} \quad t < t_0 \\ \textit{and} & |\nu_+(t) - \nu_-(t) - \varepsilon| \le \nu_0(t) & \textit{for all} \quad t > t_0. \end{array}$$

5 Conclusions

We have obtained the so-called *sign condensed form* for general Hermitian pencils. This form is a mixture of a triangular form and a diagonal form, where the diagonal form displays all the 'singularity' and all the sign sums of the real eigenvalues of the pencil (or of the regular subpencil), including the eigenvalue ∞. We have furthermore obtained necessary and sufficient conditions for the existence of anti-triangular and anti-m-Hessenberg forms for Hermitian pencils in terms of conditions on the sign sum of the real eigenvalues and the eigenvalue ∞ and in terms of the inertia indices of certain Hermitian matrices. The latter conditions hold also in the case that the pencil is singular. If a Hermitian pencil can be transformed to anti-m-Hessenberg form via congruence, then the transformation matrices can be chosen to be unitary, i.e., in this case both matrices of the pencil are simultaneously unitarily similar to anti-m-Hessenberg forms.

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