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Numerische Simulation auf massiv parallelen Rechnern

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Preconditioning for the p-version of the FEM by bilinear elements

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Abstract

Finding a fast solver for the inner problem in a DD preconditioner for the p-version of the FEM is a difficult question. We discovered, that the system matrix for the inner problem in any dimension has a similar structure to matrices resulting from discretizations of $-y^2 u_{xx} - x^2 u_{yy}$ in the unit square using h-version of the FEM and bilinear elements.

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1 Introduction

We consider the problem

$$\begin{aligned} -\triangle u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^2$ is a domain which can be decomposed into quadrilaterals. We solve this problem with the p-version of the finite element method. We divide Ω in a mesh of quadrilaterals R_s. Let $\mathcal{R} = (-1,1)^2$ the reference element and let $\Phi_s : \mathcal{R} \to R_s$ the bilinear mapping to the element R_s. We define now the space

$$\mathbb{M} = \{u \in H_0^1(\Omega), u\mid_{R_s} = u(\Phi_s(\xi,\eta)) = \tilde{u}(\xi,\eta), \tilde{u} \in P_p\},$$

where P_p is the space of all polynomials $p(\xi,\eta) = p_1(\xi)p_2(\eta)$ of maximal degree p in each variable. So, we can formulate the discretized problem: Find $u_p \in \mathbb{M}$ such that

$$a_\triangle(u_p, v_p) := \int_\Omega \nabla u_p \cdot \nabla v_p = \int_\Omega f v_p \; \forall v_p \in \mathbb{M}$$

holds. Let $(\psi_1, \dots, \psi_{n_p})$ a basis of $\mathbb{M}$. Then, this problem is equivalent to solving

$$A\underline{u}_p = \underline{f}_p,$$

where

$$\begin{aligned} A &= (a(\psi_i, \psi_j))_{i,j=1}^{n_p}, \\ \underline{f}_p &= (\int_\Omega f\psi_p)_{i=1}^{n_p}. \end{aligned}$$

Now, we specify the choice of the basis and divide the shape functions into 3 groups,

- the vertex functions, which are the usual piecewise bilinear functions,
- the edge bubble functions,
- the interior bubbles, which are nonzero only on one element.

The edge bubble functions correspond to an edge e of the mesh. Their support contains that two elements which have this edge e in common. Corresponding to the division of the shape functions, we split the matrix A as follows

$$A = \begin{pmatrix} A_v & A_{v,e} & A_{v,i} \\ A_{e,v} & A_e & A_{e,i} \\ A_{i,v} & A_{i,e} & A_i \end{pmatrix}.$$

The indices v, e and i denote the blocks of the vertex, edge bubble and interior bubble function, respectively. Jensen/Korneev [13] and Ivanov/Korneev [11],[12] developed preconditioners for the $p-$version of the FEM in a two-dimensional domain using domain decomposition techniques.
They considered the matrix

$$C = \begin{pmatrix} A_v & & \\ & A_e & A_{e,i} \\ & A_{i,e} & A_i \end{pmatrix}$$

and proved that the condition number $\kappa(C^{-1}A)$ grows as $1+\log p$, cf. Lemma 2.3 in [11] or [4]. Therefore, the vertex unknowns can be determined separately. Computing the other unknowns, we factorize the remaining 2 by 2 block as follows

$$\begin{pmatrix} A_e & A_{e,i} \\ A_{i,e} & A_i \end{pmatrix} = \begin{pmatrix} I & A_{e,i}A_i^{-1} \\ \mathbf{0} & I \end{pmatrix} \begin{pmatrix} S & \mathbf{0} \\ \mathbf{0} & A_i \end{pmatrix} \begin{pmatrix} I & \mathbf{0} \\ A_i^{-1}A_{i,e} & I \end{pmatrix} \tag{1.1}$$

with the Schur-complement

$$S = A_e - A_{e,i}A_i^{-1}A_{i,e}.$$

Computing the interior unknowns, A_i is a block diagonal matrix, one block corresponds to one element. Therefore, for computing the interior unknowns, we solve a Dirichlet problem on each quadrilateral. The edge unknowns are computed via the Schur-complement S.
We need 3 tools for solving (1.1), a preconditioner for the interior problem, a preconditioner for the Schur-complement and an extension operator from the edges of a quadrilateral to its interior. Ivanov/Korneev [11],[12] derived

3 types $C_{i,S}$, $i = 1, 2, 3$, of preconditioning the Schur-complement. The condition number for $C_{i,S}^{-1}S$ is $\mathcal{O}(1 + \log^2 p)$ in the worst case, where p is the polynomial degree. The solution of $C_{i,S}\underline{x} = \underline{y}$ can be done fast.
Jensen/Korneev [13] considered a scaled version of the integrated Legendre polynomials as basis. They found a spectral equivalent preconditioner for the interior problem, which has $\mathcal{O}(p^2)$ nonzero entries. In the case of parallelogram elements, the element stiffness matrix has $\mathcal{O}(p^2)$ nonzero entries, too. But, the suggested methods compute the solution in $\mathcal{O}(p^3)$ arithmetical operations. Finding a fast solver for the preconditioner was an open question. This paper is concerned to the construction of a more efficient preconditioner for the interior problem.
We derive in [5],[6],[7] preconditioners for the interior problem resulting from several kinds of discretizations of an elliptic problem resulting from the h-version of the FEM or the method of finite differences. This preprint considers the case of bi-/trilinear elements in 2D and 3D.
The paper is organized as follows. In section 2, we consider the stiffness matrix for the model problem and their most important properties. In section 3, we introduce and modify the preconditioner of Jensen/Korneev. Section 4 shows that the modified preconditioner can be obtained by discretizing ellpitic problems with variable coefficients using bilinear finite elements. These resulting problems can be effectively solved by multi-grid algorithms, [14], [15], [16], [9], [8], [10] or AMLI-preconditioners [2], [3], [1] in the case of linear finite elements.
Throughout this paper, $\mathcal{R}$ will denote the unit rectangle $(-1, 1)^2$, Ω_1 the rectangle $(0, 1)^2$. The integer p is the polynomial degree, $\hat{L}_i$ the $i-$th integrated Legendre polynomial. The real number $\lambda_{max}(A)$ will denote the largest eigenvalue of a matrix A and $\lambda_{min}(A)$ the smallest eigenvalue of A. The parameter c_i will describe a constant, which is independent of p or h.

2 Origin and properties of the stiffness matrix

We consider the model problem

$$-\triangle u = f \text{ in } \mathcal{R} = (-1, 1)^2, \tag{2.1}$$
$$u = 0 \text{ on } \partial\mathcal{R}. \tag{2.2}$$

We solve (2.1,2.2) using the $p-$version of the FEM with only one element $\mathcal{R}$. Problem (2.1,2.2) is the typical model problem for solving a linear system with the matrix A_i. As finite element space, we choose

$$\mathbb{M} = \{u \in H_0^1(\mathcal{R}), u \mid_{\mathcal{R}} \in P^p\}$$

where P^p is the space of all polynomials of degree p in both variables. The discretized problem is: find $u_p \in \mathbb{M}$

$$\int_{\mathcal{R}} \nabla u_p \cdot \nabla v_p \mathrm{d}(x,y) = \int_{\mathcal{R}} f v_p \mathrm{d}(x,y)$$

for all $v_p \in \mathbb{M}$. As basis in $\mathbb{M}$, we choose the integrated Legendre polynomials, which we define below.
Let for $i = 0, 1, \ldots$

$$L_i(x) = \frac{1}{2^i i!} \frac{\mathrm{d}^i}{\mathrm{d}x^i}(x^2-1)^i$$

the i-th Legendre polynomial,

$$\hat{L}_i(x) = \gamma_i \int_{-1}^{x} L_{i-1}(s) \, \mathrm{d}s \text{ for i} \geq 2$$

the i-th integrated Legendre polynomial with

$$\gamma_i = \sqrt{\frac{(2i-3)(2i-1)(2i+1)}{4}}. \tag{2.3}$$

By definition,

$$\begin{aligned} \hat{L}_0(x) &= \frac{1+x}{2}, \\ \hat{L}_1(x) &= \frac{1-x}{2}. \end{aligned}$$

The properties

$$\int_{-1}^{1} L_i(x) L_j(x) \, \mathrm{d}x = \delta_{ij} \frac{2}{2i+1}, \tag{2.4}$$

$$\hat{L}_i(x) = \sqrt{\frac{(2i+1)(2i-3)}{4(2i-1)}} (L_i(x) - L_{i-2}(x)), \tag{2.5}$$

$$\hat{L}_i(1) = 0, \tag{2.6}$$

$$\hat{L}_i(-1) = 0, \tag{2.7}$$

$$(i+1)L_{i+1}(x) + iL_{i-1}(x) = (2i+1)xL_i(x). \tag{2.8}$$

are true for $i \geq 2$, [18].
As basis in $\mathbb{M}$, we choose

$$\hat{L}_{ij}(x,y) = \hat{L}_i(x)\hat{L}_j(y), \tag{2.9}$$

with $2 \leq i,j \leq p$. For satisfying (2.2), the polynomials $\hat{L}_0$ and $\hat{L}_1$ are not used, compare (2.6,2.7). The stiffness matrix $K = A_i$ for (2.2) is determined by

$$K = (a_{ij,kl})_{i,j=2;k,l=2}^{p} = \int_{\mathcal{R}} \nabla \hat{L}_{ij}(x,y) \cdot \nabla \hat{L}_{kl}(x,y) \,\mathrm{d}(x,y).$$

We get

$$a_{ij,kl} \quad = \quad d_{ik} f_{jl} + f_{ik} d_{jl} \tag{2.10}$$

with

$$(f_{ij})_{i,j=2}^{p} \quad = \quad \int_{-1}^{1} \hat{L}_i(x)\hat{L}_j(x) \,\mathrm{d}x,\ i,j = 2,\ldots,p, \tag{2.11}$$

$$(d_{ij})_{i,j=2}^{p} \quad = \quad \int_{-1}^{1} \frac{\mathrm{d}}{\mathrm{d}x}\hat{L}_i(x) \frac{\mathrm{d}}{\mathrm{d}x}\hat{L}_j(x) \,\mathrm{d}x,\ i,j = 2,\ldots,p. \tag{2.12}$$

In the following, we need matrices $F = (f_{ij})_{i,j=2}^{p}$ and $D = (d_{ij})_{i,j=2}^{p}$. Using (2.4,2.5), the entries of the one-dimensional mass matrix F are determined by

$$F = \begin{pmatrix} 1 & 0 & -c_2 & 0 & \cdots \\ & 1 & 0 & -c_3 & \ddots \\ & & 1 & 0 & \ddots \\ & \text{SYM} & \ddots & \ddots & \ddots \\ & & & & 1 \end{pmatrix}$$

and of the one-dimensional stiffness matrix D by

$$D = \operatorname{diag}(d_i)_{i=2}^{p} = \begin{pmatrix} d_2 & 0 & \cdots \\ 0 & d_3 & \ddots \\ 0 & 0 & \ddots \end{pmatrix}$$

with the coefficients

$$c_i = \sqrt{\frac{(2i-3)(2i+5)}{(2i-1)(2i+3)}},$$
$$d_i = \frac{(2i-3)(2i+1)}{2}$$

[13]. The stiffness matrix for the two-dimensional Laplace can be written using the matrices F and D by

$$K = F \otimes D + D \otimes F, \tag{2.13}$$

compare (2.10). Applying a permutation P of rows and columns, we get

$$PKP^{-1} = \begin{pmatrix} K_1 & & & \\ & K_2 & & \\ & & K_3 & \\ & & & K_4 \end{pmatrix}. \tag{2.14}$$

The first block results from the discretization using the polynomials $\hat{L}_{2i,2j}$, the second $\hat{L}_{2i+1,2j}$, the third $\hat{L}_{2i,2j+1}$ and the fourth $\hat{L}_{2i+1,2j+1}$. If p is odd, all four blocks have the same size. We wish to find a fast solver for a system of linear equations with the matrix K or equivalently, K_i. This solver should perform the solution in not more than $\mathcal{O}(p^2 \log p)$ arithmetical operations.

3 Deriving a preconditioner for K

3.1 Preconditioner of Jensen/Korneev

Jensen/Korneev [11] defined the following preconditioner for K. Let

$$D_1 = \mathrm{diag}(i^2)_{i=2}^{p}, \tag{3.1}$$

$$T_1 = D_1^{-1} + \frac{1}{2} \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & \cdots \\ & 2 & 0 & -1 & 0 & \cdots \\ \mathrm{SYM} & & 2 & 0 & -1 & \ddots \\ & & & 2 & \ddots & \ddots \\ & & & & \ddots & \\ & & & & & 2 \end{pmatrix} \tag{3.2}$$

and

$$C_1 = D_1 \otimes T_1 + T_1 \otimes D_1. \tag{3.3}$$

LEMMA 3.1 *The statements*

$$\begin{aligned} c_1(D_1\underline{v},\underline{v}) &\le (D\underline{v},\underline{v}) \le & c_2(D_1\underline{v},\underline{v}), && (3.4)\\ c_3(T_1\underline{v},\underline{v}) &\le (F\underline{v},\underline{v}) \le & c_4(T_1\underline{v},\underline{v}), && (3.5)\\ c_1c_3(C_1\underline{v},\underline{v}) &\le (K\underline{v},\underline{v}) \le & c_2c_4(C_1\underline{v},\underline{v}) && (3.6) \end{aligned}$$

are valid for all $\underline{v}$.

Proof: (3.4) is trivial, (3.5) is proved in [11]. (3.6) follows immediately from (3.4,3.5). □

C_1 is simpler than K, but we still need a fast solver for C_1.

3.2 Modification of the preconditioner

Our aim is to modify the preconditioners in such a way that they can be interpreted as a system matrix resulting from other kinds of discretization. So, we change in several steps the preconditioner (3.1–3.3). Let

$$D_2 = \mathrm{diag}(4\,[\frac{i}{2}]^2)_{i=2}^{p} = \mathrm{diag}(4,4,16,16,36,36,\dots),$$

$$T_2 = T_1 - D_1^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & \cdots \\ & 2 & 0 & -1 & 0 & \cdots \\ \text{SYM} & & 2 & 0 & -1 & \ddots \\ & & & 2 & \ddots & \ddots \\ & & & \vdots & & \\ & & & -1 & 0 & 2 \end{pmatrix},$$

and

$$C_2 = D_2 \otimes T_2 + T_2 \otimes D_2.$$

LEMMA 3.2 *The inequalities*

$$\begin{aligned}
(D_2\underline{v},\underline{v}) &\le (D_1\underline{v},\underline{v}) \le \frac{9}{4}(D_2\underline{v},\underline{v}), && (3.7)\\
(T_2\underline{v},\underline{v}) &\le (T_1\underline{v},\underline{v}) \le c_0(1+\log p)(T_2\underline{v},\underline{v}), && (3.8)\\
(C_2\underline{v},\underline{v}) &\le (C_1\underline{v},\underline{v}) \le \frac{9}{4}c_0(1+\log p)(C_2\underline{v},\underline{v}) && (3.9)
\end{aligned}$$

are true for all $\underline{v}$.

Proof: (3.7) and the left inequality of (3.8) are trivial, (3.9) is a corollary of (3.7,3.8). The right inequality of (3.8) is proved in [5], [6]. □
In the following, we assume p is odd. We introduce $n = [\frac{p-1}{2}] + 1$. Applying a basis-transformation using the permutation P from (2.14), C_2 is a block diagonal matrix of 4 identical blocks C_3, where

$$C_3 = D_3 \otimes T_3 + T_3 \otimes D_3, \qquad (3.10)$$

with

$$\begin{aligned}
D_3 &= \mathrm{diag}(4i^2)_{i=1}^{n-1}, && (3.11)\\
T_3 &= \frac{1}{2}\mathrm{tridiag}(-1,2,-1). && (3.12)
\end{aligned}$$

The next modification is helpful for bilinear elements. Let

$$M_2 = \mathrm{tridiag}(\mathbf{a},\mathbf{b},\mathbf{a}) = \begin{pmatrix} b_1 & a_1 & 0 & \dots & 0 \\ a_1 & b_2 & a_2 & 0 & \dots \\ 0 & a_2 & b_3 & a_3 & \\ \vdots & & & \ddots & \\ 0 & \dots & 0 & a_{n-1} & b_n \end{pmatrix} \qquad (3.13)$$

where

$$\begin{aligned}
\mathbf{a} &= (a_i)_{i=1}^{n-1} &&= i^2 + i + \frac{3}{10},\\
\mathbf{b} &= (b_i)_{i=1}^{n} &&= 4i^2 + \frac{2}{5}.
\end{aligned}$$

LEMMA 3.3 *We have*

$$c_{13}(D_3\underline{v},\underline{v}) \le (M_2\underline{v},\underline{v}) \le c_{14}(D_3\underline{v},\underline{v}).$$

.

Proof: An easy calculation shows

$$T_4 = (t_{ij}^{(4)})_{i,j=1}^n = D_3^{-\frac{1}{2}} M_2 D_3^{-\frac{1}{2}} = \text{tridiag}(\mathbf{c}, \mathbf{d}, \mathbf{c})$$

where

$$\begin{aligned} \mathbf{c} &= (c_i)_{i=1}^{n-1} &&= 1 + \frac{3}{10(i^2+i)}, \\ \mathbf{d} &= (d_i)_{i=1}^{n} &&= 4 + \frac{2}{5i^2}. \end{aligned}$$

We take now Gerschgorins-disks. We have

$$\min_i \left(t_{ii}^{(4)} - \sum_{j \neq i} \mid t_{ij}^{(4)} \mid \right) \leq \lambda_{min}(T_4) \leq \lambda_{max}(T_4) \leq \max_i \left(t_{ii}^{(4)} + \sum_{j \neq i} \mid t_{ij}^{(4)} \mid \right).$$

Using the structure of the c_i and d_i, this leads to

$$\begin{aligned} \min_i \left(t_{ii}^{(4)} - \sum_{j \neq i} \mid t_{ij}^{(4)} \mid \right) &\geq 2, \\ \max_i \left(t_{ii}^{(4)} + \sum_{j \neq i} \mid t_{ij}^{(4)} \mid \right) &\leq \frac{63}{10}. \end{aligned}$$

Hence, the assertion follows.□
Now, we introduce the matrix

$$C_7 = T_3 \otimes M_2 + M_2 \otimes T_3. \tag{3.14}$$

Using Lemma 3.3, we have

THEOREM 3.4 . *Let $K_i, i = 1, \ldots, 4$ are the 4 blocks of K. The following statement is valid $\forall \underline{v}$ and $i = 1, \ldots, 4$:*

$$c_{15}(C_7\underline{v}, \underline{v}) \quad \leq (K_i\underline{v}, \underline{v}) \leq \quad c_{16}(1 + \log p)(C_7\underline{v}, \underline{v}).$$

4 h-Version of the FEM, bilinear elements

4.1 The one-dimensional case

We consider the following problem. Find $u \in H_0^1(0,1)$, such that

$$a_1(u, v) = a_s(u, v) + a_m(u, v) = \langle g, v \rangle \tag{4.1}$$

holds for all $v \in H_0^1(0,1)$. The bilinear forms $a_s(\cdot,\cdot)$ and $a_m(\cdot,\cdot)$ are defined as follows

$$\begin{aligned} a_s(u,v) &= \int_0^1 u'v' \, \mathrm{d}x, \\ a_m(u,v) &= \int_0^1 x^2 uv \, \mathrm{d}x. \end{aligned}$$

We discretize this one-dimensional problem (4.1) by using linear elements on the simple mesh

$$T_k = \bigcup_{i=0}^{n-1} \tau_i^k,$$

where

$$\tau_i^k = \left(\frac{i}{n}, \frac{i+1}{n}\right).$$

The parameter $k = 2^n$ induces the level number. On this mesh we introduce the one-dimensional hat-functions

$$\phi_i^{(1)} = \begin{cases} nx - (i-1) & \text{on} \quad \tau_i^k \\ (i+1) - nx & \text{on} \quad \tau_{i+1}^k \\ 0 & \text{else} \end{cases}.$$

Then, we obtain

$$(a_s(\phi_i^{(1,k)}, \phi_j^{(1,k)}))_{i,j=1}^{n-1} = \frac{n}{2} T_3 = n \ \mathrm{tridiag}(-1, 2, -1). \tag{4.2}$$

An easy calculation shows

$$(a_s(\phi_i^{(1,k)}, \phi_j^{(1,k)}))_{i,j=1}^{n-1} = c \, M_2 \tag{4.3}$$

with some constant c depending on n. So, we see the reason for introducing the matrices T_3 (3.12) and M_2 (3.13).

4.2 The two-dimensional case

We consider the following problem: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) := \int_\Omega y^2 u_x v_x + x^2 u_y v_y \, \mathrm{d}x\mathrm{d}y = \int_\Omega gv \, \mathrm{d}x\mathrm{d}y =: \langle g, v \rangle \tag{4.4}$$

$\forall v \in H_0^1(\Omega)$ holds. The domain Ω is the unit square $(0,1)^2$.
We want to find a numerical solution of (4.4) using finite elements. For this purpose, we introduce some notation. Let k be the level of approximation and $n = 2^k$. Let us introduce $x_{ij}^k = (\frac{i}{n}, \frac{j}{n})$, where $i, j = 0, \ldots, n$. We divide Ω into congruent quadrilaterals $\mathcal{E}_{ij}^k$. Let $\mathcal{E}_{ij}^k = \overline{\tau_{ij}^{1,k} \cup \tau_{ij}^{2,k}}$ be the square

$$\left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right].$$

On the mesh

$$\mathcal{E}^k = \left\{\mathcal{E}_{ij}^k\right\}_{i,j=0}^{n-1}$$

we introduce the shape functions $\phi_{b,ij}$ as tensor products of the one-dimensional shape functions $\phi_i^{(1,k)}$

$$\phi_{b,ij}^k = \phi_i^{(1,k)} \phi_j^{(1,k)} \text{ for } i, j = 1, \ldots, n-1.$$

We set $\mathbb{V}_k^{(b)} = \operatorname{span}(\phi_{b,ij}^k)_{i,j=1}^{n-1}$. Now, we can formulate the discrete problem. Find $u_k \in \mathbb{V}_k^{(b)}$, such that

$$a(u^k, v^k) = \langle g, v^k \rangle \ \forall v \in \mathbb{V}_k^{(b)} \tag{4.5}$$

holds. Problem (4.5) is equivalent to solving

$$K_{b,k} \underline{u}_b = \underline{g}_b, \tag{4.6}$$

where

$$\begin{aligned} K_{b,k} &= a(\phi_{b,ij}^k, \phi_{b,lm}^k)_{i,j,l,m=1}^{n-1}, \\ \underline{g}_b &= \langle g, \phi_{b,lm}^k \rangle_{l,m=1}^{n-1}, \\ u_b &= \sum_{i,j=1}^{n-1} u_{b,ij} \phi_{b,ij}^k. \end{aligned}$$

From (4.2), (4.3) we can conclude

$$\begin{aligned} K_{b,k} &= c(n) \frac{n}{2} (T_3 \otimes M_2 + M_2 \otimes T_3), \\ &= c(n) C_7. \end{aligned} \tag{4.7}$$

Hence, we have found an interpretation of C_7 (3.14).

5 Concluding Remarks

This approach can be extended as the finite-difference case to the three-dimensional case. For details, see [7].

References

[1] O. Axelsson and A. Padiy. On the additive version of the algebraic multilevel iteration for anisotropic elliptic problems. *SIAM J. Sci. Comp.*, 20(5):1807–1830, 1999.

[2] O. Axelsson and P.S. Vassilevski. Algebraic multilevel preconditioning methods i. *Numer. Math.*, 56:157–177, 1989.

[3] O. Axelsson and P.S. Vassilevski. Algebraic multilevel preconditioning methods ii. *SIAM J. Numer. Anal.*, 27(6):1569–1590, 1990.

[4] I. Babuska, A. Craig, J. Mandel, and J. Pitkäranta. Efficent precontitioning for the p-version finite element method in two dimensions. *SIAM J.Numer.Anal.*, 28(3):624–661, 1991.

[5] S. Beuchler. Lösungsmethoden bei der p-version der f.e.m. Diplomarbeit, TU Chemnitz, December 1999.

[6] S. Beuchler. A preconditioner for solving the inner problem of the p-version of the f.e.m. Technical Report SFB393 00-25, Technische Universität Chemnitz, May 2000.

[7] S. Beuchler. A preconditioner for solving the inner problem of the p-version of the f.e.m, part ii- algebraic multi-grid proof. Technical Report SFB393 01-07, Technische Universität Chemnitz, March 2001.

[8] D. Braess. The contraction number of a multigrid method for solving the poisson equation. *Numer. Math*, 37:387–404, 1981.

[9] J. Bramble and X. Zhang. Uniform convergence of the multigrid v-cycle for an anisotropic problem. *Math. Comp.*, 70(234):453–470, 2001.

[10] W. Hackbusch. *Multigrid Methods and Applications*. Springer-Verlag. Heidelberg, 1985.

[11] S.A. Ivanov and V.G. Korneev. On the preconditioning in the domain decomposition technique for the p-version finite element method. part i. Technical Report SPC 95-35, Technische Universität Chemnitz-Zwickau, December 1995.

[12] S.A. Ivanov and V.G. Korneev. On the preconditioning in the domain decomposition technique for the p-version finite element method. part ii. Technical Report SPC 95-36, Technische Universität Chemnitz-Zwickau, December 1995.

[13] S. Jensen and V.G. Korneev. On domain decomposition preconditioning in the hierarchical $p-$version of the finite element method. *Comput. Methods. Appl. Mech. Eng.*, 150(1–4):215–238, 1997.

[14] Ch. Pflaum. Fast and robust multilevel algorithms. Habilitationsschrift, Universität Würzburg, 1998.

[15] Ch. Pflaum. Robust convergence of multilevel algorithms for convection-diffusion equations. *Num. Lin. Alg. Appl.*, 6:701–728, 1999.

[16] N. Schieweck. A multigrid convergence proof by a strengthened cauchy-inequality for symmetric elliptic boundary value problems. In G. Telschow, editor, *Second multigrid seminar, Garzau 1985*, number 08-86 in Report R-Math, pages 49–62, Berlin, 1986. Karl-Weierstraß-Insitut für Mathematik.

[17] G. Telschow, editor. *Second multigrid seminar, Garzau 1985*, number 08-86 in Report R-Math, Berlin, 1986. Karl-Weierstraß-Insitut für Mathematik.

[18] F.G. Tricomi. *Vorlesungen über Orthogonalreihen.* Springer. Berlin-Göttingen-Heidelberg, 1955.