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Numerische Simulation auf massiv parallelen Rechnern

Sven Beuchler

The MTS-BPX-preconditioner for the p-Version of the fem

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Abstract

Finding a fast solver for the inner problem in a DD preconditioner for the p-version of the FEM is a difficult question. We discovered, that the system matrix for the inner problem in any dimension has a similar structure to matrices resulting from discretizations of $-y^2u_{xx} - x^2u_{yy}$ in the unit square using h-version of the FEM. Numerical experiments show that the MTS-BPX-preconditioner with a tridiagonal scaling brings good results.

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Author's addresses:

Sven Beuchler
TU Chemnitz
Fakultät für Mathematik
D-09107 Chemnitz
Germany

email: sven.beuchler@mathematik.tu-chemnitz.de

`http://www.tu-chemnitz.de/sfb393/`

1 Introduction

1.1 Origin of the problem from the p-version

Jensen/Korneev [15] and Ivanov/Korneev [13],[14] developed preconditioniers for the $p-$version of the FEM in a two-dimensional domain. They used DD-methods. The unknowns are splitted into 3 groups, the interior, the edge and vertex unknowns

$$A = \begin{pmatrix} A_{vert} & A_{vert,edg} & A_{vert,int} \\ A_{edg,vert} & A_{edg} & A_{edg,int} \\ A_{int,vert} & A_{int,edg} & A_{int} \end{pmatrix}.$$

The vertex unknowns can be solved separately, cf. Lemma 2.3 [13], or [3], using

$$\tilde{C} = \begin{pmatrix} A_{vert} & & \\ & A_{edg} & A_{edg,int} \\ & A_{int,edg} & A_{int} \end{pmatrix}.$$

Computing the other unknowns, we factorize the remaining stiffness matrix as follows

$$\begin{aligned} \begin{pmatrix} A_{edg} & A_{edg,int} \\ A_{int,edg} & A_{int} \end{pmatrix} &= \begin{pmatrix} I & A_{edg,int}A_{int}^{-1} \\ & I \end{pmatrix} \\ & \quad \begin{pmatrix} S & \\ & A_{int} \end{pmatrix} \begin{pmatrix} I & \\ A_{int}^{-1}A_{int,edg} & I \end{pmatrix} \end{aligned}$$

with the Schur-komplement

$$S = A_{edg} - A_{edg,int}A_{int}^{-1}A_{int,edg}.$$

Computing the interior unknowns, we solve a Dirichlet problem on each quadrangle. The vertex unknowns are computed via the Schur-complement S. A_{int} is a block diagonal matrix, one block corresponds to one element.
Jensen/Korneev [15] considered as basis a scaled version of the integrated Legendre polynomials. They found a spectral equivalent preconditioner C for each block of A_{int}, which has $\mathcal{O}(p^2)$ nonzero entries, where p is the polynomial degree. In the case of parallelogram elements, the element stiffness matrix has $\mathcal{O}(p^2)$ nonzero entries, too. But, the suggested methods compute the solution in $\mathcal{O}(p^3)$ arithmetical operations. Finding a fast solver for the preconditioner was an open

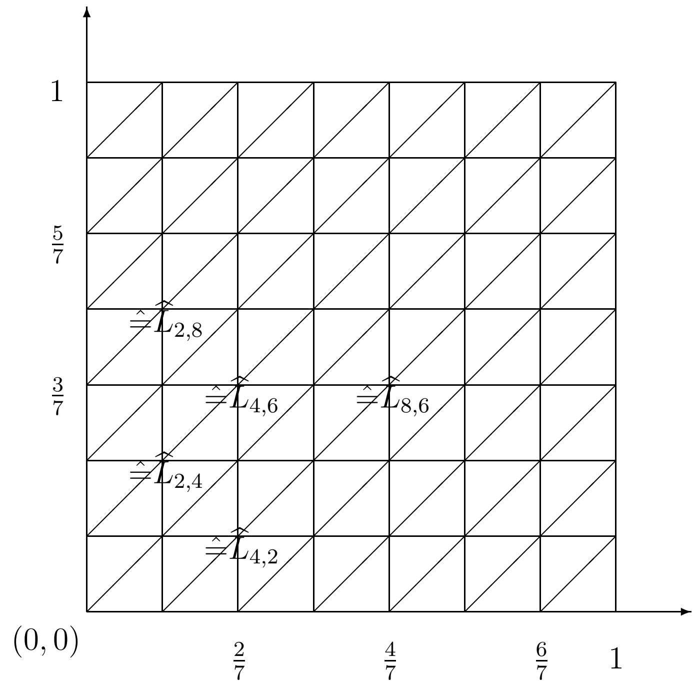

Figure 1: Mesh for h-Version.

question. This paper is concerned to the construction of a **more efficient preconditioner** for the interior problem. The matrix C is a block diagonal matrix of 4 blocks $C^{(i)}$. In [5], we derived a new preconditioner C_4 for each block of C. This matrix is defined via the matrices D_3 and T_3,

$$D_3 = \mathrm{diag}(4(i^2 + \frac{1}{6}))_{i=1}^{n-1}, \tag{1.1}$$

$$T_3 = \frac{1}{2}\mathrm{tridiag}(-1, 2, -1), \tag{1.2}$$

$$C_4 = D_4 \otimes T_3 + T_3 \otimes D_4. \tag{1.3}$$

We have proved in [5], that the condition number of matrix $C_4^{-1}C^{(i)}$ grows as $(1 + \log p)$. But, the matrix C_4 has an another origin, which we will see now.

1.2 Formulation of the elliptic problem

We consider the following problem: Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) := \int_\Omega y^2 u_x v_x + x^2 u_y v_y \,\mathrm{d}x\mathrm{d}y = \int_\Omega gv \,\mathrm{d}x\mathrm{d}y =: \langle g, v\rangle \tag{1.4}$$

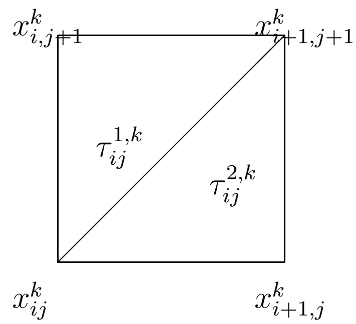

Figure 2: Notation.

$\forall v \in H^1_0(\Omega)$ holds. The domain Ω is the unit square $(0,1)^2$.

We want to find a numerical solution of (1.4) using finite elements. For this purpose, we introduce some notation. Let k be the level of approximation and $n = 2^k$. Let us introduce $x^k_{ij} = (\frac{i}{n}, \frac{j}{n})$, where $i, j = 0, \dots, n$. We divide Ω into congruent, isosceles, orthogonal triangles $\tau^{s,k}_{ij}$, where $0 \le i, j < n$ and $s = 1, 2$, compare Figure 1. The triangle $\tau^{1,k}_{ij}$ has the three vertices $x^k_{ij}, x^k_{i+1,j+1}$ and $x^k_{i,j+1}$, $\tau^{2,k}_{ij}$ has the three vertices $x^k_{ij}, x^k_{i+1,j+1}$ and $x^k_{i+1,j}$, see Figure 2. We use linear finite elements on the mesh

$$T_k = \{\tau^{s,k}_{ij}\}^{n-1,n-1,2}_{i,j=0,s=1}$$

and denote by $\mathbb{V}_k$ the subspace of piecewise linear functions ϕ^k_{ij} with

$$\phi^k_{ij} \in H^1_0(\Omega),\ \phi^k_{ij}\mid_{\tau^{sk}_{lm}} \in P^1(\tau^{sk}_{lm}),$$

where P^1 is the space of polynomials of degree ≤ 1. A basis of $\mathbb{V}_k$ is the system of functions $\{\phi^k_{ij}\}^{n-1}_{i,j=1}$ uniquely defined by

$$\phi^k_{ij}(x^k_{lm}) = \delta_{il}\delta_{jm},$$

where δ_{il} is the Kronecker delta.
Now, we can formulate the discretized problem. Find $u^k \in \mathbb{V}_k$ such that

$$a(u^k, v^k) = \langle g, v^k \rangle \; \forall v \in \mathbb{V}_k \tag{1.5}$$

holds. Problem (1.5) is equivalent to solving

$$K_{h,k}\underline{u}_h = \underline{g}_h, \tag{1.6}$$

where

$$
\begin{aligned}
K_{h,k} &= a(\phi^k_{ij}, \phi^k_{lm})^{n-1}_{i,j,l,m=1}, \\
\underline{g}_h &= \langle g, \phi^k_{lm} \rangle^{n-1}_{l,m=1}, \\
u_h &= \sum_{i,j=1}^{n-1} u_{ij} \phi^k_{ij}.
\end{aligned}
$$

We obtain by an easy calculation [6], [7] and (1.3)

$$
K_{h,k} = \frac{1}{2n^2} C_4, \tag{1.7}
$$

if we insert the boundary condition and choose a proper permutation of the unknowns. For differential equations as $-u_{xx} - u_{yy} = f$ efficient solution techniques are found, the BPX-preconditioner [9], the HB-preconditioner [19], or multi-grid methods [11], [12]. But, we consider problems with variable coefficients which tend to 0, if $x \to 0$ or $y \to 0$. The paper of Bramble and Zhang [10] considers multi-grid methods in a more general case as for Laplace.
The paper [8] deals with the solution of (1.6) by multi-grid using special line-smoothers in $\mathcal{O}(n^2)$ arithmetical operations. This proof is a multi-grid proof of the projection type, [18], [16] or [17].
An another method is the AMLI-method derived by Axelsson and Vassilevski [1], [2], where the idea of the smoother is used. Both methods analyze a strengthened Cauchy-inequality.
In this preprint, we give numerical examples for an improved BPX-preconditioner, a MTS-BPX-preconditioner.

2 The definition of the preconditioner

We are interested in a fast preconditioner for $K_{h,k}$. This preconditioner is an BPX-like [9] preconditioner with a multiple tridiagonal scaling (MTS). In [4], numerical experiments show an increasing of the number of iterations of the PCG-method by increasing the number of unknowns using the multiple diagonal scaling-BPX (MDS) preconditioner. But this preconditioner can be improved by the following modification. Let

$$
C^{-1}_{h,k} = \sum_{l=0}^{k} Q_l T_l^{-1} Q_l^t,
$$

Level	MDS-BPX	MTS-BPX		
	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-9}$	$\varepsilon = 10^{-16}$
2	9	8	9	9
3	16	11	18	27
4	24	14	23	37
5	33	15	26	44
6	44	16	28	49
7	58	17	30	52
8	76	17	31	56
9	97	18	32	58

Table 1: Number of iterations of the PCG-method for solving $K_p \underline{u}_p = \underline{f}_p$ with the preconditioner $C_{h,k}$.

where Q_l, $l = 0, \ldots, k$ is the basis transformation matrix from the basis $\{\phi_{ij}^l\}_{i,j=1}^{n_l} \in \mathbb{V}_l$ to $\{\phi_{ij}^k\}_{i,j=1}^{n_k} \in \mathbb{V}_k$, where $n_j = 2^j - 1$. The matrix T_l is a tridiagonal matrix in which the absolute smaller entries of the off-diagonals of the matrix $K_{h,l} = (a_{ij})_{i,j=1}^{n_l}$ are omitted. More precisely, all off-diagonals a_{ij} with

$$4 \mid a_{ij} \mid < \max\{a_{ii}, a_{jj}\}$$

are omitted. For more details, see [6], subsection 5.4. In this preprint, this matrix is denoted by $\tilde{K}_h$.

3 Numerical results

3.1 Results for $-y^2 u_{xx} - x^2 u_{yy} = f$

We give now results for solving

$$K_{h,k} \underline{u}_h = \underline{f}_h.$$

We solve this linear system of equations with PCG-method and the preconditioner $C_{h,k}$. The right-hand side $\underline{f}_h = (1, \ldots, 1)^t$ is chosen. Table 1 displays the number of iterations for several relative accuracies ε in the preconditioned energy norm. We see in all cases constant number of iterations.

p	$\varepsilon = 10^{-5}$		$\varepsilon = 10^{-9}$		$\varepsilon = 10^{-16}$	
	It	time [sec]	It	time [sec]	It	time [sec]
7	12	0.002	17	0.002	22	0.004
15	15	0.013	24	0.023	38	0.031
31	18	0.071	28	0.102	47	0.176
63	20	0.327	32	0.523	54	0.889
127	22	1.672	37	2.773	62	4.987
255	25	8.324	42	13.688	70	22.916
511	27	37.956	46	63.922	78	108.561
1023	29	163.822	50	281.523	85	478.944

Table 2: Number of iterations of the PCG-method for solving $K_p\underline{u}_p = \underline{f}_p$ with the preconditioner $C_{h,k}$.

3.2 Results for the p-version

A typical reference example as preconditioner for the matrix A_{int} is the matrix K_p, which is the element stiffness matrix on the unit square $(-1,1)^2$. Each element stiffness matrix is spectral equivalent to K_p with respect to the polynomial degree p, [15].
Now, we can apply this preconditioner as preconditioner for each of the 4 blocks of the matrix K_p. Results for solving

$$K_p\underline{u}_p = \underline{f}_p$$

are given. This linear system of equations is solved with PCG-method and the preconditioner $C_{h,k}$. We choose $\underline{f}_p = (1, \dots, 1)^t$. Table 2 displays the number of iterations for several relative accuracies ε in the preconditioned energy norm. We see in all 3 cases a growing as $1 + \log p$. This preconditioner is nearly as fast as the preconditioner $M_1^{S_1}$, which is a multi-grid preconditioner involving the tridiagonal matrices T_l in the smoother $S_l = I - \omega T_l^{-1} K_{h,l}$ on level l.

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