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Robust a posteriori error estimation for a singularly perturbed reaction–diffusion equation on anisotropic tetrahedral meshes

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Abstract

We consider a singularly perturbed reaction–diffusion problem and derive and rigorously analyse an *a posteriori* residual error estimator that can be applied to *anisotropic* finite element meshes. The quotient of the upper and lower error bounds is the so–called *matching function* which depends on the anisotropy (of the mesh and the solution) but not on the small perturbation parameter. This matching function measures how well the anisotropic finite element mesh corresponds to the anisotropic problem. Provided this correspondence is sufficiently good, the matching function is $\mathcal{O}(1)$. Hence one obtains tight error bounds, i.e. the error estimator is reliable and efficient as well as robust with respect to the small perturbation parameter.

A numerical example supports the anisotropic error analysis.

Keywords: error estimator, anisotropic solution, stretched elements, reaction diffusion equation, singularly perturbed problem

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1 Introduction

Adaptive finite element algorithms have become an important tool for numerical simulations. Along with other ingredients, they usually employ *a posteriori* error estimators or indicators, cf. Ainsworth/Oden [2], Verfürth [21] and the literature cited therein.

In this work we consider a singularly perturbed reaction-diffusion model problem whose classical formulation reads: Find $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$-\varepsilon\Delta u + u = f \quad \text{in } \Omega, \qquad\qquad u = 0 \quad \text{on } \Gamma_D = \partial\Omega \tag{1}$$

in a bounded, polyhedral domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. The perturbation parameter ε is supposed to be very small, $0 < \varepsilon \ll 1$, and to have much influence on the solution. For a comprehensive discussion of its analysis and numerical treatment we refer to Roos, Stynes, Tobiska [19], and to Miller, O'Riordan, Shishkin [16], and the citations therein. Here it will suffice to remark that the singularly perturbed problem (1) usually gives rise to a solution with boundary layers when a non-vanishing right-hand side f meets homogeneous Dirichlet boundary conditions. Inside the domain Ω and sufficiently far away from the boundary, the solution is usually smooth provided f is smooth enough too. Thus the boundary layers mark the domain of interest, and their resolution requires increased numerical effort.

The knowledge of *a posteriori* error estimators for the singularly perturbed problem (1) has been unsatisfactory until recently. Most estimators yield upper and lower bounds on the error that are not asymptotically equivalent. By this we mean that the upper and lower bound differ by a factor that increases, for example, as the discretization parameter $h \to 0$, or as the perturbation parameter $\varepsilon \to 0$. The first *a posteriori* error estimate with asymptotically equivalent upper and lower bound on the error is, to our knowledge, due to Angermann [3]. He measures the error in a somewhat strange norm which seems to be mainly of theoretical interest. Only recently Verfürth [22] derived the first robust *a posteriori* error estimator for the energy norm. Ainsworth/Babuška [1] extended the 'equilibrated residual method' to the singularly perturbed problem and obtained a robust error estimate as well.

Let us now consider the finite element method and some discretization aspects in particular. Standard methods employ so–called *isotropic* meshes. That is, the elements are shape regular or, equivalently, the ratio of the diameters of the circumscribed and inscribes spheres is bounded. However, some problems (e.g. the singularly perturbed problem above) admit a solution with strong directional features such as boundary or interior layers. To approximate such an anisotropic solution, it can be advantageous to use *anisotropic* elements, i.e. elements that are no longer shape regular.

Anisotropic elements are already used in practice, see e.g. [4, 5, 7, 11, 17, 18, 23], but when commonly known (isotropic) *a posteriori* error estimators are applied to anisotropic meshes, they usually fail. The development of error estimators that are suitable for anisotropic elements is just beginning. The only mathematically founded anisotropic estimators are, as far as we know, due to [20, 14, 13, 15, 10], and will be discussed now briefly.

Siebert [20] considers a residual error estimator for the Poisson problem on cuboidal or prismatic grids. He has to impose two conditions to obtain upper and lower error bounds. Kunert [14, 13] investigates the Poisson equation on tetrahedral meshes, and a residual error estimator and a local problem based error estimator are derived. The lower error bounds there hold unconditionally, whereas the upper bounds are formulated such that the influence of the anisotropy becomes apparent. In Kunert/Verfürth [15] it is shown that anisotropic residual error estimators can be modified such that they only contain the face residuals, but they still bound the error reliably. This has been proven for the Poisson equation (H^1 and L_2 error estimators). The investigation has already been extended there to a reaction diffusion problem by using certain results that are presented in our paper here. Finally, Dobrowolski/Gräf/Pflaum [10] propose an error estimator that requires the solution of a global problem (and slightly more restrictive mesh assumptions, e.g. a maximum angle condition). The sharpness of the error bounds relies on a saturation assumption whose dependence on the anisotropy is not fully discussed.

Almost all of the aforementioned anisotropic error estimators deal with the Poisson equation. In contrast to this, we consider the singularly perturbed problem (1) and derive a robust *a posteriori* error estimator that can be applied to *anisotropic* meshes. The upper and lower error bounds involve the same terms and are asymptotically equivalent, provided that the anisotropic mesh corresponds sufficiently to the anisotropic problem. Our estimator is partially influenced by Verfürth's isotropic version [22]. The results coincide when our estimator is applied to isotropic meshes.

As a side effect and a corollary of this paper here we prove some results that have already been utilized (without proof) in [15] to investigate face residual error estimators not only for the Poisson equation but also for a reaction diffusion equation. In particular we prove now the fundamental interpolation estimates of section 3.3 and the lower error bound (22) of section 4. Note further that our error estimator here is improved by a different scaling of the gradient jump.

The paper is organized as follows. In section 2 we describe the model problem. Section 3 is devoted to some basic ingredients of the error estimation analysis. More precisely, we start by presenting the transformation technique and related lemmas, proceed with special bubble functions that will be essential for deriving lower error bounds, and conclude with specific interpolation estimates which eventually give the upper error bound. In section 4 the error estimator is defined and the main result, the error estimation, is presented and proved. A numerical experiment and the summary conclude this paper.

2 The singularly perturbed model problem

The classical formulation (1) is often too restrictive to describe real-world problems properly. So assume $f \in L_2(\Omega)$, and let $H^1_o(\Omega)$ be the usual Sobolev space of functions that vanish on Γ_D. The variational formulation is now more appropriate:

$$\left.\begin{array}{ll} \text{Find } u \in H^1_o(\Omega): \quad & a(u,v) = \langle f, v\rangle \qquad \forall\, v \in H^1_o(\Omega) \\ \text{with} \quad a(u,v) := \displaystyle\int_\Omega \varepsilon \cdot \nabla^T u \nabla v + u\, v & \langle f, v\rangle := \displaystyle\int_\Omega f\, v \quad . \end{array}\right\} \tag{2}$$

The continuous problem (2) is discretized by the finite element method which employs a family $\mathcal{F}$ of triangulations $\mathcal{T}_h$ of Ω. Then let $V_{o,h} \subset H_o^1(\Omega)$ be the space of continuous, piecewise linear functions over $\mathcal{T}_h$ that vanish on Γ_D. The finite element solution $u_h \in V_{o,h}$ is uniquely defined by

$$a(u_h, v_h) = \langle f, v_h \rangle \qquad \forall\, v_h \in V_{o,h} \quad . \tag{3}$$

Both problems (2) and (3) admit unique solutions due to the Lax–Milgram Lemma.

Our main objective is to bound the error $u - u_h$. Here we concentrate on the *energy norm*

$$|||v|||^2 := a(v, v) = \varepsilon \|\nabla v\|^2 + \|v\|^2 \qquad .$$

This energy norm is the most natural norm when considering a singularly perturbed reaction diffusion problem (in weak formulation). When applied in adaptive algorithms, this energy norm is able to produce appropriately refined meshes. This can be seen easily on some 1D model problem, e.g. for $-\varepsilon u'' + u = 0$ in $\Omega = (0, 1)$ with $u(0) = 1, u(1) = 0$. Even the optimal order of convergence can be recovered uniformly in ε.

Apart from these reasons for using the energy norm, we will here repudiate the common argument that the energy norm can not distinguish between a boundary layer and the zero function (cf. [16, pages 12f]). Such an argument can not be applied here since it would erroneously neglect boundary conditions. Furthermore, even if the error in the energy norm is small in absolute terms, this error can be large in relative terms and thus suffice to devise adaptive algorithms.

3 Notation, basic tools and Lemmas

In the following, let $\mathbb{P}^k(\omega)$ be the space of polynomials of order k or less over some domain $\omega \subset \mathbb{R}^3$ or $\omega \subset \mathbb{R}^2$. Furthermore, instead of $x \leq c \cdot y$ or $c_1 x \leq y \leq c_2 x$ (with constants independent of x, y, ε, and $\mathcal{T}_h$) we use the shorthand notation $x \lesssim y$ and $x \sim y$, respectively. The L_2 norm of a function over a domain ω is denoted by $\|\cdot\|_\omega$, and $(\cdot,\cdot)_\omega$ means the $L_2(\omega)$ scalar product. For $\omega = \Omega$ the subscript is omitted.

The next sections introduce the notation and important tools. Some basic relations and lemmas are given as well. All considerations are made for the 3D case. The application to the simpler 2D case is readily possible.

3.1 Tetrahedron – Subdomains – Mesh requirements – Transformations

Tetrahedron: The four vertices of an arbitrary tetrahedron $T \in \mathcal{T}_h$ are denoted by $P_0, \ldots, P_3$ such that P_0P_1 is the longest edge of T, $\text{meas}_2(\triangle P_0P_1P_2) \geq \text{meas}_2(\triangle P_0P_1P_3)$, and $\text{meas}_1(P_1P_2) \geq \text{meas}_1(P_0P_2)$.

Additionally define three pairwise orthogonal vectors $\mathbf{p}_i$ with lengths $h_{i,T} := |\mathbf{p}_i|$, see figure 1. Observe $h_{1,T} > h_{2,T} \geq h_{3,T}$ and set $h_{min,T} := h_{3,T}$.

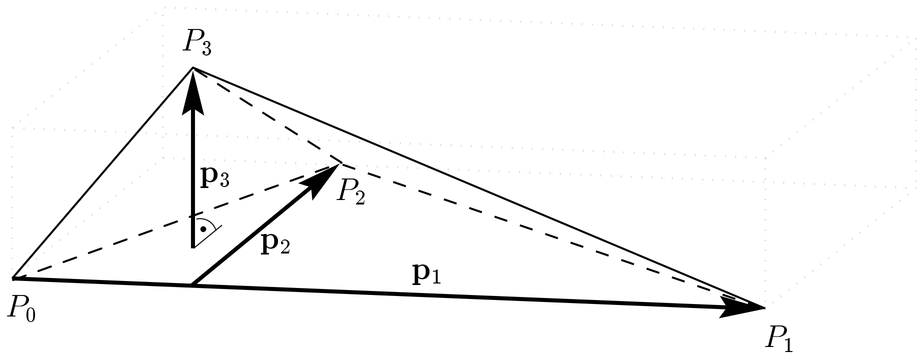

Figure 1: Notation of tetrahedron T

Tetrahedra are denoted by T, T' or T_i. Faces of a tetrahedron are denoted by E. Set $|T| = \text{meas}_3(T)$, $|E| = \text{meas}_2(E)$, and let

$$h_{E,T} := 3|T|/|E|$$

be the length of the *height* over a face E. Note that $h_{E,T}$ is *not* the diameter of E, as in the usual convention. Because of the geometrical properties of the tetrahedron one has

$$h_{E,T} > \frac{1}{2} h_{min,T} \qquad \forall\, E \subset \partial T \qquad .$$

Auxiliary subdomains: Let $T \in \mathcal{T}_h$ be an arbitrary tetrahedron. Let ω_T be that domain that is formed by T and all tetrahedra that have a common face with T. Note that ω_T consists of less than five tetrahedra if T has a boundary face.

Let E be an inner face (triangle) of $\mathcal{T}_h$, i.e. there are two tetrahedra T_1 and T_2 having the common face E. Set the domain $\omega_E := T_1 \cup T_2$. If E is a boundary face set $\omega_E := T$ with $T \supset E$.

Mesh requirements: In addition to the usual conformity conditions of the mesh (see Ciarlet [8], Chapter 2) we demand the following two assumptions.

1. The number of tetrahedra containing a node x_j is bounded uniformly.
2. The dimensions of adjacent tetrahedra must not change rapidly, i.e.

$$h_{i,T'} \sim h_{i,T} \qquad \forall\, T, T' \text{ with } T \cap T' \neq \emptyset\,,\ i = 1 \ldots d \quad .$$

The last condition also implies that terms such as $h_{min,T}$ or $h_{E,T}$ do not change rapidly across adjacent tetrahedra.

Note that the analysis of the error estimator does *not* require a maximum angle condition.

Transformations and auxiliary tetrahedra: The usual transformation technique between a tetrahedron $T \in \mathcal{T}_h$ and a standard tetrahedron plays a vital role in many proofs (cf. [8]). However, our refined analysis even shows that *two different transformations* facilitate matters considerably, see also below. Hence define the matrices $H_T := \mathrm{diag}(h_{1,T}, h_{2,T}, h_{3,T})$ and $A_T, C_T \in \mathbb{R}^{3\times 3}$ by

$$A_T := (\ \overrightarrow{P_0P_1}, \overrightarrow{P_0P_2}, \overrightarrow{P_0P_3}\ ) \qquad \text{and} \qquad C_T := (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \qquad ,$$

and introduce affine linear mappings

$$F_A(\mu) := A_T \cdot \mu + \vec{P_0} \qquad \text{and} \qquad F_C(\mu) := C_T \cdot \mu + \vec{P_0} \quad , \quad \mu \in \mathbb{R}^3.$$

These mappings implicitly define the *standard tetrahedron* $\bar{T} := F_A^{-1}(T)$ and the *reference tetrahedron* $\hat{T} := F_C^{-1}(T)$. Then $\bar{T}$ has vertices $\bar{P}_0 = (0,0,0)^T$ and $\bar{P}_i = \mathbf{e}_i^T, i = 1\ldots 3$, whereas $\hat{T}$ has vertices at $\hat{P}_0 = (0,0,0)^T$, $\hat{P}_1 = (1,0,0)^T$, $\hat{P}_2 = (\hat{x}_2, 1, 0)^T$ and $\hat{P}_3 = (\hat{x}_3, \hat{y}_3, 1)^T$. The conditions on the P_i yield immediately $0 < \hat{x}_2 \leq 1/2$, $0 < \hat{x}_3 < 1$ and $-1 < \hat{y}_3 < 1$. Figures 1 and 2 may illustrate this definition.

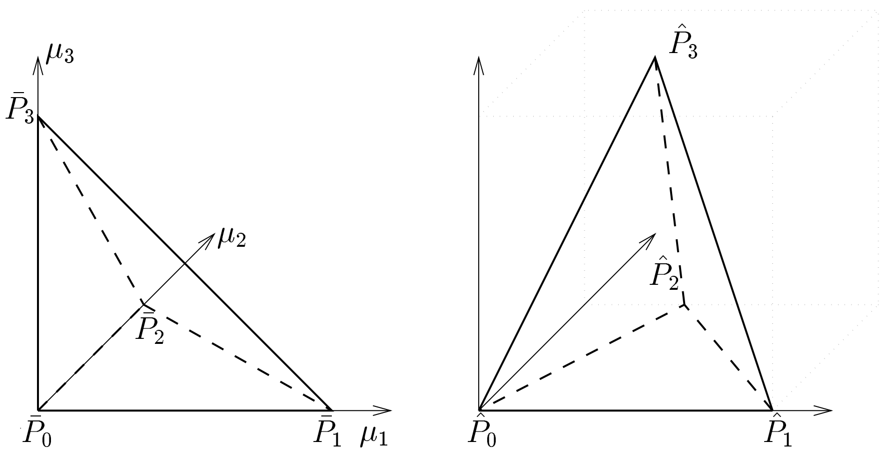

Figure 2: Standard tetrahedron $\bar{T}$ and reference tetrahedron $\hat{T}$

Variables that are related to the standard tetrahedron $\bar{T}$ and the reference tetrahedron $\hat{T}$ are referred to with a *bar* and a *hat*, respectively (e.g. $\bar{\nabla}$, $\hat{v}$). The determinants of both mappings are $|\det(A_T)| = |\det(C_T)| = 6|T|$, and the transformed derivatives satisfy $\bar{\nabla}\bar{v} = A_T^T \nabla v$ and $\hat{\nabla}\hat{v} = C_T^T \nabla v$.

Although C_T is naturally associated with our analysis, it transforms $\hat{T}$ into T. Inequality constants would thus depend on $\hat{T}$. This drawback is remedied by using the transformation via A_T in conjunction with C_T. To illustrate this principle, consider the mapping $C_T^{-1} A_T$ which maps the standard tetrahedron $\bar{T}$ onto the reference tetrahedron $\hat{T}$. Since the radii of the inscribed and circumscribed spheres of $\bar{T}$ and $\hat{T}$ are bounded from above and below, respectively, one immediately derives

$$\left\| A_T^T C_T^{-T} \right\|_{\mathbb{R}^{3\times 3}} = \left\| C_T^{-1} A_T \right\|_{\mathbb{R}^{3\times 3}} \sim 1 \qquad . \tag{4}$$

This equivalence facilitates the interaction of A_T and C_T, see e.g. the proof of the trace inequality below.

Because of the singular perturbation character of the differential equation we can favourably employ a sub–tetrahedron $T_{E,\delta} \subset T$ which depends on a face E of T and a real number $\delta \in (0,1]$. For a precise definition of $T_{E,\delta}$, let T be an arbitrary but fixed tetrahedron, and enumerate temporarily its vertices such that $E = Q_1Q_2Q_3$ and $T = OQ_1Q_2Q_3$, cf. Figure 3. Introduce barycentric coordinates such that λ_0 is related to O, and λ_1, λ_2, λ_3 correspond to Q_1, Q_2, Q_3, respectively.

Let P be that point with barycentric coordinates

$$\lambda_0(P) = \delta \qquad \text{and} \qquad \lambda_1(P) = \lambda_2(P) = \lambda_3(P) = \frac{1-\delta}{3} \qquad .$$

Then $T_{E,\delta}$ is that tetrahedron that has vertices P and Q_1, Q_2, Q_3, i.e. $T_{E,\delta}$ has the same face E as T but the fourth vertex is moved towards E with the rate δ.

An alternative description is as follows. With S_E being the midpoint (i.e. center of gravity) of face E, point P lies on the line S_EO such that $|\vec{S_EP}| = \delta \cdot |\vec{S_EO}|$. Note that for $\delta = 1$ one gets $T_{E,\delta} \equiv T$ whereas in the limiting case $\delta \to 0$ the tetrahedron $T_{E,\delta}$ collapses to the face E.

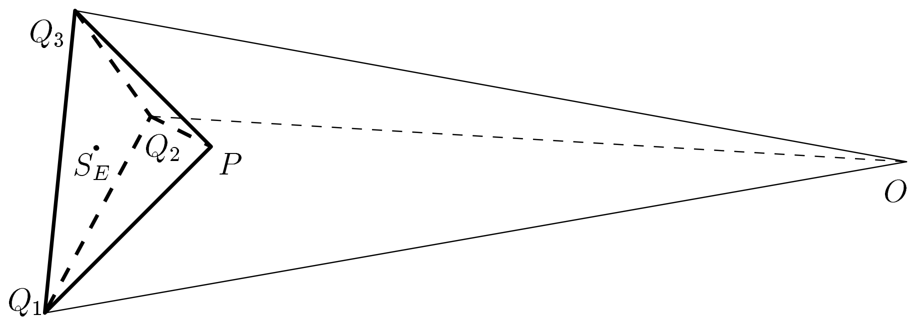

Figure 3: Tetrahedra $T = OQ_1Q_2Q_3$ and $T_{E,\delta} = PQ_1Q_2Q_3$

In order to utilize $T_{E,\delta}$ efficiently, we also require an affine linear transformation $F_{T,E,\delta}$ that maps the standard tetrahedron $\bar{T}$ onto $T_{E,\delta}$. This affine linear mapping is unique (up to permutations of the enumeration of the vertices of $\bar{T}$ and $T_{E,\delta}$).

Next we bound the transformation matrix of the affine linear mapping $F^{-1}_{T,E,\delta}$ (in a slight abuse of the notation this matrix is denoted by $F^{-1}_{T,E,\delta}$ too). Since $F^{-1}_{T,E,\delta}$ maps $T_{E,\delta}$ onto $\bar{T}$, such a bound is obtained via

$$||F^{-1}_{T,E,\delta}||_{\mathbb{R}^{3\times 3}} \le d(\bar{T})/\varrho(T_{E,\delta}) \qquad ,$$

with $d(\bar{T}) = \sqrt{2}$ being the diameter of $\bar{T}$, and $\varrho(T_{E,\delta})$ being the diameter of the largest inscribed sphere of $T_{E,\delta}$. Thus the goal of the next lemmas will be to bound that diameter.

Lemma 3.1 *Let T be an arbitrary tetrahedron with faces E_i, $i = 1 \dots 4$. The length of the height over E_i is again denoted by $h_{E_i,T}$. Then*

$$\varrho(T) \sim \min_{i=1\dots4} h_{E_i,T} \sim \frac{|T|}{\max_{i=1\dots4} |E_i|} \sim h_{min,T} \qquad .$$

Proof: The inequality $\varrho(T) < h_{E_i,T}$ for $i = 1 \dots 4$ is obvious.

To bound $\varrho(T)$ from below, consider the midpoint S_T of T (i.e. the centre of gravity). Let $l_i := \mathrm{dist}(S_T, E_i)$ be the distance between S_T and the plane that contains the face E_i. Then the sphere with centre at S_T and radius $\min_{i=1\dots4} l_i$ lies inside T; therefore

$$\varrho(T)/2 \geq \min_{i=1\dots4} l_i \qquad .$$

On the other hand $l_i = h_{E_i,T}/4$ since S_T is the midpoint of T. This gives

$$\varrho(T) \geq \frac{1}{2} \min_{i=1\dots4} h_{E_i,T} \qquad .$$

Recalling $3|T| = h_{E_i,T} \cdot |E_i|$ completes the proof. ■

Let us now investigate the sub–tetrahedron $T_{E,\delta}$, i.e. consider an arbitrary tetrahedron T and some fixed face E thereof. Enumerate both tetrahedra again as in Figure 3, and denote the three remaining faces of T (apart from E) by $E_i := OQ_iQ_{i+1}$. Indices are to be considered modulo 3 if necessary.

Lemma 3.2 *The measure of the face PQ_iQ_{i+1} of $T_{E,\delta}$ is bounded by*

$$|PQ_iQ_{i+1}| \leq \frac{1}{3} \cdot \Big[|E| + \delta \cdot (2|E_i| + |E_{i-1}| + |E_{i+1}|) \Big] \qquad .$$

Proof: Simple vector algebra yields

$$\vec{OS_E} = (\vec{OQ_1} + \vec{OQ_2} + \vec{OQ_3})/3 \qquad \text{and} \qquad \vec{OP} = (1-\delta) \cdot \vec{OS_E} \qquad .$$

The measure of some faces is computed via the vector product. This implies

$$\begin{aligned} \vec{S_EQ_i} \times \vec{OS_E} &= \frac{1}{3} \vec{OQ_i} \times \vec{OQ_{i-1}} + \frac{1}{3} \vec{OQ_i} \times \vec{OQ_{i+1}} \\ |\vec{S_EQ_i} \times \vec{OS_E}| &\leq \frac{1}{3} |\vec{OQ_i} \times \vec{OQ_{i-1}}| + \frac{1}{3} |\vec{OQ_i} \times \vec{OQ_{i+1}}| = \frac{2}{3} \Big(|E_{i-1}| + |E_i| \Big) \end{aligned}$$

since $2|E_i| = |\vec{OQ_i} \times \vec{OQ_{i+1}}|$. Using this result, one obtains

$$\begin{aligned} \vec{PQ_i} &= \vec{S_EQ_i} + \delta \cdot \vec{OS_E} \\ |\vec{PQ_i} \times \vec{PQ_{i+1}}| &\leq |\vec{S_EQ_i} \times \vec{S_EQ_{i+1}}| + \delta \cdot (|\vec{S_EQ_i} \times \vec{OS_E}| + |\vec{S_EQ_{i+1}} \times \vec{OS_E}|) \\ &\leq 2|S_EQ_iQ_{i+1}| + \frac{2}{3} \delta \cdot (|E_{i-1}| + |E_i| + |E_i| + |E_{i+1}|) \\ &= \frac{2}{3} \cdot \Big[|E| + \delta \cdot (2|E_i| + |E_{i-1}| + |E_{i+1}|) \Big] \qquad . \end{aligned}$$

Together with $2|PQ_iQ_{i+1}| = |\vec{PQ_i} \times \vec{PQ_{i+1}}|$ this proves the assertion. ■

Lemma 3.3 *The diameter of the inscribed sphere of $T_{E,\delta}$ satisfies*

$$\varrho(T_{E,\delta}) \ \sim \ \min\{\delta \cdot h_{E,T} \ , \ h_{min,T}\} \qquad .$$

Proof: Let us start to bound $\varrho(T_{E,\delta})$ from above. Obviously $h_{E,T_{E,\delta}} = \delta \cdot h_{E,T}$ and $\varrho(T_{E,\delta}) \le \varrho(T) \sim h_{min,T}$ since $T_{E,\delta} \subset T$. Using Lemma 3.1 this results in

$$\varrho(T_{E,\delta}) \ \lesssim \ \min\{\delta \cdot h_{E,T}, h_{min,T}\} \qquad .$$

In order to bound $\varrho(T_{E,\delta})$ from below, consider the faces of $T_{E,\delta}$ and apply Lemma 3.2 giving

$$\begin{aligned} |PQ_iQ_{i+1}| \ &\lesssim \ |E| \ + \ \delta \cdot (|E_1| + |E_2| + |E_3|) \ \sim \ \max\{|E|, \delta \cdot \max_{i=1,2,3} |E_i|\} \\ \max\{|E|, |PQ_1Q_2|, |PQ_2Q_3|, |PQ_3Q_1|\} \ &\lesssim \ \max\{|E|, \delta \cdot \max_{i=1,2,3} |E_i|\} \end{aligned}$$

(recall that E and E_1, E_2, E_3 are the faces of T). Employing Lemma 3.1 and $|T_{E,\delta}| = \delta \cdot |T|$ one obtains

$$\begin{aligned} \varrho(T_{E,\delta}) \ &\gtrsim \ \frac{|T_{E,\delta}|}{\max\left\{|E|, \delta \cdot \max\limits_{i=1,2,3} |E_i|\right\}} \ \ge \ \min\left\{\frac{\delta \cdot |T|}{|E|}, \frac{\delta \cdot |T|}{\delta \cdot \max\limits_{i=1,2,3} |E_i|}\right\} \\ &\gtrsim \ \min\{\delta \cdot h_{E,T}, h_{min,T}\} \end{aligned}$$

which completes the assertion. ■

The next lemma bounds the transformation matrix of $F^{-1}_{T,E,\delta}$; this will be vital to prove the inverse inequalities of Lemma 3.7 below.

Lemma 3.4 *The norm of the transformation matrix $F^{-1}_{T,E,\delta}$ is bounded by*

$$\|F^{-1}_{T,E,\delta}\|_{\mathbb{R}^{3\times 3}} \ \lesssim \ \min\{\delta \cdot h_{E,T} \ , \ h_{min,T}\}^{-1} \qquad .$$

Proof: The bound follows immediately from $\|F^{-1}_{T,E,\delta}\|_{\mathbb{R}^{3\times 3}} \le d(\bar{T})/\varrho(T_{E,\delta})$ and Lemma 3.3. ■

Trace inequality: The lemma below plays an important role when specific interpolation estimates are to be derived.

Lemma 3.5 (Trace inequality) *Let T be an arbitrary tetrahedron and E be a face of it. For $v \in H^1(T)$ the trace inequality holds:*

$$\|v\|_E^2 \ \lesssim \ h^{-1}_{E,T} \cdot \|v\|_T \cdot \left(\|v\|_T + \|C_T^T \nabla v\|_T\right) \qquad . \tag{5}$$

Proof: Consider the transformation F_A, the standard tetrahedron $\bar{T} := F_A^{-1}(T)$, the face $\bar{E} := F_A^{-1}(E)$ of $\bar{T}$, and the function $\bar{v} := v \circ F_A \in H^1(\bar{T})$. On the standard tetrahedron $\bar{T}$, the well–known (isotropic) trace inequality implies

$$\|\bar{v}\|_{\bar{E}}^2 \lesssim \|\bar{v}\|_{\bar{T}} \cdot \left(\|\bar{v}\|_{\bar{T}} + \|\bar{\nabla}\bar{v}\|_{\bar{T}}\right) \qquad ,$$

cf. [22]. The transformation onto the actual tetrahedron (via F_A) yields

$$|E|^{-1} \cdot \|v\|_E^2 \lesssim |T|^{-1} \cdot \|v\|_T \cdot \left(\|v\|_T + \|A_T^T \nabla v\|_T\right) \qquad .$$

From (4) one derives

$$\|A_T^T \nabla v\|_T = \|A_T^T C_T^{-T} \cdot C_T^T \nabla v\|_T \leq \|A_T^T C_T^{-T}\|_{\mathbb{R}^{3\times 3}} \cdot \|C_T^T \nabla v\|_T \lesssim \|C_T^T \nabla v\|_T \quad .$$

Utilizing $3\,|T| = |E| \cdot h_{E,T}$ results in the anisotropic trace inequality (5). ■

3.2 Bubble functions and inverse inequalities

As another useful and important tool we now introduce so-called bubble functions. They are used, for example, for bounding certain residual norms. The definitions below are partly as in the isotropic case, cf. [21].

Denote by $\lambda_{T,1}, \cdots, \lambda_{T,4}$ the barycentric coordinates of an arbitrary tetrahedron T. The *element bubble function* b_T is defined by

$$b_T := 256\,\lambda_{T,1} \cdot \lambda_{T,2} \cdot \lambda_{T,3} \cdot \lambda_{T,4} \in \mathbb{P}^4(T) \qquad \text{on } T \qquad . \tag{6}$$

Let $E = T_1 \cap T_2$ be an inner face (triangle) of $\mathcal{T}_h$. Enumerate the vertices of T_1 and T_2 such that the vertices of E are numbered first. Define the *face bubble function* $b_E \in C^0(\omega_E)$ on (the three–dimensional domain) $\omega_E = T_1 \cup T_2$ by

$$b_E := 27\,\lambda_{T_k,1} \cdot \lambda_{T_k,2} \cdot \lambda_{T_k,3} \qquad \text{on } T_k,\ k = 1, 2 \qquad . \tag{7}$$

For simplicity assume that b_T and b_E are extended by zero outside their original domain of definition. Note that $0 \leq b_T(x), b_E(x) \leq 1$ and $\|b_T\|_\infty = \|b_E\|_\infty = 1$.

Next we introduce an extension operator $F_{ext} : \mathbb{P}^0(E) \to C^0(\omega_E)$. For some constant function $\varphi \in \mathbb{P}^0(E)$ define

$$F_{ext}(\varphi)(x) := \varphi \qquad \text{for } x \in \omega_E \qquad . \tag{8}$$

If E is a boundary face then b_E and F_{ext} are obviously defined only on the single tetrahedron $T \supset E$.

The following anisotropic equivalences and inverse inequalities can be derived easily, cf. [12], so we only state the results.

Lemma 3.6 (Inverse inequalities I) *Assume* $\varphi_T \in \mathbb{P}^1(T)$ *and* $\varphi_E \in \mathbb{P}^0(E)$. *Then*

$$\begin{aligned}
\|b_T^{1/2} \cdot \varphi_T\|_T &\sim \|\varphi_T\|_T && (9)\\
\|\nabla(b_T \cdot \varphi_T)\|_T &\lesssim h_{min,T}^{-1} \cdot \|\varphi_T\|_T && (10)\\
\|b_E^{1/2} \cdot \varphi_E\|_E &\sim \|\varphi_E\|_E && (11)\\
\|F_{ext}(\varphi_E) \cdot b_E\|_T &\sim h_{E,T}^{1/2} \cdot \|\varphi_E\|_E \qquad \text{for } E \subset T && (12)\\
\|\nabla(F_{ext}(\varphi_E) \cdot b_E)\|_T &\lesssim h_{E,T}^{1/2} \cdot h_{min,T}^{-1} \cdot \|\varphi_E\|_E \qquad \text{for } E \subset T && (13)
\end{aligned}$$

The bubble functions above suffice to analyse the residual error estimator for the *Poisson* equation, cf. [21]. However, for the *singularly perturbed problem* considered here we have to introduce modified face bubble functions, cf. also [12, 22]. For some tetrahedron T and a face E thereof consider the sub–tetrahedron $T_{E,\delta}$ (cf. Figure 3). Define the so–called *squeezed face bubble function* $b_{E,T,\delta}$ by

$$b_{E,T,\delta} := \begin{cases} b_{\bar{E}} \circ F_{T,E,\delta}^{-1} & \text{on } T_{E,\delta} \\ 0 & \text{on } T \setminus T_{E,\delta} \end{cases} \tag{14}$$

where $b_{\bar{E}}$ is the standard face bubble function for the face $\bar{E} = F_{T,E,\delta}^{-1}(E)$ of the tetrahedron $\bar{T} = F_{T,E,\delta}^{-1}(T_{E,\delta})$. In other words, $b_{E,T,\delta}$ is the usual face bubble function for face E in the tetrahedron $T_{E,\delta}$, and it is extended by zero in $T \setminus T_{E,\delta}$.

Standard scaling arguments for the transformation $F_{T,E,\delta} : \bar{T} \to T_{E,\delta}$, together with the essential Lemma 3.4 yield now the inverse inequalities for the squeezed face bubble function.

Lemma 3.7 (Inverse inequalities II) *Let* E *be an arbitrary face of* T, *and assume* $\varphi_E \in \mathbb{P}^0(E)$. *Then*

$$\begin{aligned}
\|b_{E,T,\delta} \cdot F_{ext}(\varphi_E)\|_T &\lesssim \delta^{1/2} \cdot h_{E,T}^{1/2} \cdot \|\varphi_E\|_E && (15)\\
\|\nabla(b_{E,T,\delta} \cdot F_{ext}(\varphi_E))\|_T &\lesssim \delta^{1/2} \cdot h_{E,T}^{1/2} \cdot \min\{\delta \cdot h_{E,T} \ , \ h_{min,T}\}^{-1} \cdot \|\varphi_E\|_E \quad . && (16)
\end{aligned}$$

3.3 Matching function and interpolation estimates

When investigating interpolation error estimates on anisotropic meshes, one soon discovers that the anisotropic mesh and the anisotropic function have to correspond in some way. Hence we first discuss the relation between mesh and function before the interpolation properties are given.

From a heuristic point of view the anisotropy of the mesh should be aligned with the anisotropy of the function to provide a satisfying interpolation. Intuitively all practical applications follow this concept. For a rigorous analysis, however, we want to have some measure of the alignment of mesh and function. To this end the so-called *matching function* has been proposed by Kunert [12, 14].

Definition 3.1 (Matching function) *Let $v \in H^1(\Omega)$, and $\mathcal{T}_h \in \mathcal{F}$ be a triangulation of Ω. Define the* matching function $m_1 : H^1(\Omega) \times \mathcal{F} \mapsto \mathbb{R}$ *by*

$$m_1(v, \mathcal{T}_h) \;:=\; \Big(\sum_{T\in\mathcal{T}_h} h_{min,T}^{-2} \cdot \|C_T^T\, \nabla v\|_T^2 \Big)^{1/2} \Big/ \;\|\nabla v\| \qquad . \tag{17}$$

A comprehensive discussion is given in the literature cited above; some remarks shall suffice here. Setting $h_{max,T} := h_{1,T}$, one obtains

$$1 \;\le\; m_1(v, \mathcal{T}_h) \lesssim \max_{T\in\mathcal{T}_h} h_{max,T}/h_{min,T} \qquad .$$

The definition implies that a mesh $\mathcal{T}_h$ which is well aligned with an anisotropic function, results in a small matching function m_1. The crude upper bound of m_1 implies that, on isotropic meshes, $m_1 \sim 1$, and hence the matching function merges there with other constants. In this sense, (17) is a natural extension of the theory for isotropic meshes.

Remark 3.1 A different possibility to define a matching function consists in

$$m_{1,\varepsilon}(v, \mathcal{T}_h) \;:=\; \Big(\sum_{T\in\mathcal{T}_h} \|v\|_T^2 + \varepsilon \cdot h_{min,T}^{-2} \cdot \|C_T^T\, \nabla v\|_T^2 \Big)^{1/2} \Big/ \;|||v||| \qquad . \tag{18}$$

This definition implies $1 \le m_{1,\varepsilon}(v, \mathcal{T}_h) \le m_1(v, \mathcal{T}_h)$ while all the other inequalities below are preserved (with m_1 replaced by $m_{1,\varepsilon}$). The original definition however gives rise to a straight–forward approximation of $m_1(u - u_h, \mathcal{T}_h)$, see Remark 4.2.

Let us now switch to the interpolation of some function $v \in H^1(\Omega)$. The usual Lagrange interpolation cannot be employed. Instead, in Kunert [12] the Clément interpolation technique [9] is extended to anisotropic tetrahedral meshes. The resulting Clément like interpolation operator R_o is analysed there. Here we state the basic interpolation error estimates obtained.

Lemma 3.8 *Let $v \in H_o^1(\Omega)$. The Clément interpolation operator $R_o : H_o^1(\Omega) \mapsto V_{o,h}$ of [12] satisfies the inequalities below:*

$$\begin{aligned} \|v - R_o v\| \;&\lesssim\; \|v\| \\ \sum_{T\in\mathcal{T}_h} h_{min,T}^{-2} \cdot \|v - R_o v\|_T^2 \;&\lesssim\; m_1(v, \mathcal{T}_h)^2 \cdot \|\nabla v\|^2 \\ \sum_{T\in\mathcal{T}_h} h_{min,T}^{-2} \cdot \|C_T^T \nabla(v - R_o v)\|_T^2 \;&\lesssim\; m_1(v, \mathcal{T}_h)^2 \cdot \|\nabla v\|^2 \qquad . \end{aligned}$$

In contrast to common isotropic estimates, the additional factor $m_1(v, \mathcal{T}_h)$ in the right–hand side of the previous two inequalities is indispensable here [12].

For the analysis of the error estimator we want to obtain specific interpolation estimates that are related to the reaction–diffusion problem. More precisely, the estimates shall involve the energy norm (which is related to the differential operator but not to the interpolation operator). To shorten the notation, introduce the auxiliary term

$$\alpha_T := \min\{1, \varepsilon^{-1/2} \cdot h_{min,T}\} \qquad .$$

Lemma 3.9 *Let $v \in H_o^1(\Omega)$. The Clément interpolation operator $R_o : H_o^1(\Omega) \mapsto V_{o,h}$ satisfies the inequalities below:*

$$\sum_{T\in\mathcal{T}_h} \alpha_T^{-2} \cdot \|v - R_o v\|_T^2 \;\lesssim\; m_1(v,\mathcal{T}_h)^2 \cdot |||v|||^2 \tag{19}$$

$$\varepsilon^{1/2} \sum_{T\in\mathcal{T}_h} \sum_{E\subset\partial T\setminus\Gamma_D} \alpha_T^{-1} \cdot \|v - R_o v\|_E^2 \;\lesssim\; m_1(v,\mathcal{T}_h)^2 \cdot |||v|||^2 \qquad . \tag{20}$$

Proof: The definition of α_T implies

$$\alpha_T^{-1} \;=\; \max\left\{1, \varepsilon^{1/2} \cdot h_{min,T}^{-1}\right\} \qquad .$$

With the help of Lemma 3.8 one obtains

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h} \alpha_T^{-2} \cdot \|v - R_o v\|_T^2 \;&=\; \sum_{\substack{T\in\mathcal{T}_h \\ 1\geq\varepsilon\cdot h_{min,T}^{-2}}} \|v - R_o v\|_T^2 \;+\; \sum_{\substack{T\in\mathcal{T}_h \\ 1<\varepsilon\cdot h_{min,T}^{-2}}} \varepsilon h_{min,T}^{-2} \cdot \|v - R_o v\|_T^2 \\
&\leq\; \|v - R_o v\|^2 \;+\; \varepsilon \cdot \sum_{T\in\mathcal{T}_h} h_{min,T}^{-2} \cdot \|v - R_o v\|_T^2 \\
&\lesssim\; \|v\|^2 \;+\; \varepsilon \cdot m_1(v,\mathcal{T}_h)^2 \cdot \|\nabla v\|^2 \;\leq\; m_1(v,\mathcal{T}_h)^2 \cdot |||v|||^2
\end{aligned}$$

which proves the first inequality.

For the second estimate the trace inequality (5) is invoked giving

$$h_{E,T} \cdot \|v - R_o v\|_E^2 \lesssim \|v - R_o v\|_T \cdot \left(\|v - R_o v\|_T \;+\; \|C_T^T \nabla(v - R_o v)\|_T\right) .$$

Using the first result (19), the Cauchy–Schwarz inequality, Lemma 3.8 and the fact that $h_{min,T} \lesssim h_{E,T}$ results in

$$\begin{aligned}
&\varepsilon^{1/2} \sum_{T\in\mathcal{T}_h} \sum_{E\subset\partial T\setminus\Gamma_D} \alpha_T^{-1} \cdot \|v - R_o v\|_E^2 \;\lesssim \\
&\quad\lesssim\; \varepsilon^{1/2} \sum_{T\in\mathcal{T}_h} \left[\alpha_T^{-1} \cdot \|v - R_o v\|_T \cdot h_{min,T}^{-1} \cdot \left(\|v - R_o v\|_T \;+\; \|C_T^T \nabla(v - R_o v)\|_T\right)\right] \\
&\quad\lesssim\; \varepsilon^{1/2} \cdot \left(\sum_{T\in\mathcal{T}_h} \alpha_T^{-2} \cdot \|v - R_o v\|_T^2\right)^{1/2} \cdot \\
&\qquad\quad \cdot \left(\sum_{T\in\mathcal{T}_h} h_{min,T}^{-2} \cdot \|v - R_o v\|_T^2 \;+\; h_{min,T}^{-2} \cdot \|C_T^T \nabla(v - R_o v)\|_T^2\right)^{1/2} \\
&\quad\lesssim\; \varepsilon^{1/2} \cdot m_1(v,\mathcal{T}_h) \cdot |||v||| \cdot m_1(v,\mathcal{T}_h) \cdot \|\nabla v\| \;\leq\; m_1(v,\mathcal{T}_h)^2 \cdot |||v|||^2 \qquad .
\end{aligned}$$

Hence the second estimate is proven. ■

4 Residual error estimator

Residual error estimators bound the error $u - u_h$ by measuring the residual. However, instead of computing the norm of the residual in the dual space $[H_o^1(\Omega)]^* = H^{-1}(\Omega)$, one tries to obtain an equivalent measure by evaluating easier terms that involve the given data (e.g. f, Ω, or $\mathcal{T}_h$). The main task is to carefully calibrate the weights of the residual norms such that both an upper and lower error bound hold. The difficulties that arise from the singularly perturbed problem are here even emphasized and amplified by the anisotropic elements.

Furthermore, in order to obtain lower error bounds, we replace $f \in L_2(\Omega)$ by some approximation f_h from a finite dimensional space. In particular, f_h shall be piecewise constant over $\mathcal{T}_h$ (but otherwise arbitrary). A more comprehensive discussion is given in [21, 12].

Next, the (approximate) element residuals and the face residuals are defined with the help of f_h.

Definition 4.1 (Element and face residual) *Let $u_h \in V_{o,h}$ be the finite element solution. For an element T, define the* element residual *$r_T \in \mathbb{P}^1(T)$ by*

$$r_T := f_h \ - \ (-\varepsilon \cdot \Delta u_h + u_h) \qquad \text{on } T \quad .$$

For an interior face $E \subset \Omega$ define the face residual *$r_E \in \mathbb{P}^0(E)$ by*

$$r_E(x) \ := \ \lim_{t \to +0} \left[\frac{\partial u_h}{\partial n_E}(x + t n_E) - \frac{\partial u_h}{\partial n_E}(x - t n_E) \right] \qquad x \in E \quad .$$

Here $n_E \perp E$ is any of the two unitary normal vectors. The face residual is also known as *gradient jump* or *jump residual.* Note that the element residual r_T is clearly related to the strong form of the differential equation.

Now the error estimator is defined, and the main result is presented and proved.

Definition 4.2 (Residual error estimator) *Define the* local residual error estimator *$\eta_{\varepsilon,T}$ for a tetrahedron T by*

$$\eta_{\varepsilon,T} := \left(\alpha_T^2 \cdot \|r_T\|_T^2 \ + \ \varepsilon^{3/2} \cdot \alpha_T \cdot \sum_{E \subset \partial T \setminus \Gamma_D} \|r_E\|_E^2 \right)^{1/2} \quad . \tag{21}$$

To shorten the notation, define the local approximation term

$$\zeta_{\varepsilon,T} := \alpha_T \cdot \|f - f_h\|_{\omega_T}$$

and introduce the global *terms*

$$\eta_\varepsilon^2 := \sum_{T \in \mathcal{T}_h} \eta_{\varepsilon,T}^2 \qquad \text{and} \qquad \zeta_\varepsilon^2 := \sum_{T \in \mathcal{T}_h} \zeta_{\varepsilon,T}^2 \quad .$$

Theorem 4.1 (Residual error estimation) *Let $u \in H_o^1(\Omega)$ be the exact solution and $u_h \in V_{o,h}$ be the finite element solution. Then the error is bounded locally from below by*

$$\eta_{\varepsilon,T} \lesssim |||u - u_h|||_{\omega_T} \; + \; \zeta_{\varepsilon,T} \tag{22}$$

for all $T \in \mathcal{T}_h$. The error is bounded globally from above by

$$|||u - u_h||| \lesssim m_1(u - u_h, \mathcal{T}_h) \cdot \left[\eta_\varepsilon^2 \; + \; \zeta_\varepsilon^2\right]^{1/2} \quad . \tag{23}$$

Remark 4.1 Combining the lower and upper error bound yields

$$\eta_\varepsilon^2 - c \cdot \zeta_\varepsilon^2 \;\lesssim\; |||u - u_h||| \;\lesssim\; m_1^2(u - u_h, \mathcal{T}_h) \cdot \left[\eta_\varepsilon^2 + \zeta_\varepsilon^2\right] \qquad .$$

Assuming that the approximation term ζ_ε is negligible, one obtains sharp error bounds if the matching function $m_1(u - u_h, \mathcal{T}_h)$ is small, which in turn implies that the anisotropic mesh is well suited to the anisotropic solution.

Note that in practical applications $m_1(u - u_h, \mathcal{T}_h)$ has to be approximated, e.g. by means of a recovered gradient [12, 14].

Proof: The structure of the proofs is similar to that of known residual error estimators, cf. [21]. The lower error bound is derived with the help of bubble functions and inverse inequalities, whereas the upper bound relies on interpolation estimates. All ingredients are, of course, carefully adapted to suit our specific reaction–diffusion problem on anisotropic meshes.

Start with the lower error bound (22) for an arbitrary but fixed tetrahedron T, and consider the norm $\|r_T\|_T$ of the element residual $r_T = f_h + \varepsilon \cdot \Delta u_h - u_h$. Since we use linear ansatz functions there holds $r_T \equiv f_h - u_h \in \mathbb{P}^1(T)$. For $x \in T$ let

$$w(x) := r_T(x) \cdot b_T(x) \qquad \in \mathbb{P}^5(T) \cap H_o^1(T) \qquad ,$$

with b_T being the usual bubble functions of (6). Integration by parts yields

$$\begin{aligned}
\int_T r_T \cdot w \;&=\; \int_T (f + \varepsilon \cdot \Delta u_h - u_h) \cdot w \;+\; \int_T (f_h - f) \cdot w \\
&=\; \int_T \varepsilon \cdot \nabla^T (u - u_h) \cdot \nabla w + (u - u_h) \cdot w \;+\; \int_T (f_h - f) \cdot w \\
|(r_T, w)_T| \;&\leq\; \varepsilon \cdot \|\nabla(u - u_h)\|_T \cdot \|\nabla w\|_T \;+\; \|u - u_h\| \cdot \|w\|_T \;+\; \|f - f_h\|_T \cdot \|w\|_T.
\end{aligned}$$

The inverse inequalities (9), (10) and $0 \leq b_T \leq 1$ readily imply the bounds

$$\begin{aligned}
|(r_T, w)_T| \;&=\; \|b_T^{1/2} \cdot r_T\|_T^2 \;\sim\; \|r_T\|_T^2 \\
\|\nabla w\|_T \;&=\; \|\nabla(b_T \cdot r_T)\|_T \;\lesssim\; h_{min,T}^{-1} \cdot \|r_T\|_T \\
\|w\|_T \;&=\; \|b_T \cdot r_T\|_T \;\leq\; \|r_T\|_T \qquad .
\end{aligned}$$

Hence one obtains

$$\begin{aligned}
\|r_T\|_T^2 &\lesssim \varepsilon^2 \cdot h_{min,T}^{-2} \cdot \|\nabla(u-u_h)\|_T^2 \;+\; \|u-u_h\|_T^2 \;+\; \|f-f_h\|_T^2 \\
\text{giving} \qquad \alpha_T^2 \cdot \|r_T\|_T^2 &\lesssim \min\{\varepsilon \cdot h_{min,T}^{-2}, 1\} \cdot \varepsilon \cdot \|\nabla(u-u_h)\|_T^2 \;+ \\
&\quad + \alpha_T^2 \cdot \|u-u_h\|_T^2 \;+\; \alpha_T^2 \cdot \|f-f_h\|_T^2 \\
&\leq \varepsilon \cdot \|\nabla(u-u_h)\|_T^2 \;+\; \|u-u_h\|_T^2 \;+\; \alpha_T^2 \cdot \|f-f_h\|_T^2 \\
&= |||u-u_h|||_T^2 \;+\; \alpha_T^2 \cdot \|f-f_h\|_T^2 \qquad . \qquad (24)
\end{aligned}$$

Now we aim at a bound of the norm $\|r_E\|_E$ of the gradient jump across some inner face (triangle) $E \subset \partial T$. Since we use linear ansatz functions $r_E \in \mathbb{P}^0(E)$ holds. Let $T_1 \equiv T$ and T_2 be the two tetrahedra that E belongs to. Since $f \in L_2(\Omega)$, integration by parts yields for any function $w \in H_o^1(\omega_E)$

$$\begin{aligned}
0 &= \int_{\omega_E} \varepsilon \nabla^T u \nabla w \;+\; u \cdot w \;-\; f \cdot w \\
-\varepsilon \int_E r_E \cdot w &= \varepsilon \sum_{i=1}^{2} \int_{\partial T_i} w \cdot \frac{\partial u_h}{\partial n} = \varepsilon \sum_{i=1}^{2} \int_{T_i} \left(\nabla^T u_h \nabla w \;+\; \Delta u_h \cdot w\right) \\
&= \sum_{i=1}^{2} \int_{T_i} \left(\varepsilon \nabla^T u_h \nabla w \;+\; (r_{T_i} - f_h + u_h) \cdot w\right) \\
&= \sum_{i=1}^{2} \int_{T_i} \left(\varepsilon \nabla^T (u_h - u) \nabla w \;+\; (u_h - u) \cdot w \;+\; (r_{T_i} + f - f_h) \cdot w\right)
\end{aligned}$$

since $\varepsilon \Delta u_h = r_{T_i} - f_h + u_h$ on T_i. Let now the function w be defined by

$$w := \begin{cases} b_{E,T_1,\delta_1} \cdot F_{ext}(r_E) & \text{on } T_1 \\ b_{E,T_2,\delta_2} \cdot F_{ext}(r_E) & \text{on } T_2 \end{cases} \qquad ,$$

with F_{ext} being the extension operator of (8) and b_{E,T_i,δ_i} being the squeezed face bubble functions of (14). The real numbers δ_i will be chosen later. Note that indeed $w \in H_o^1(\omega_E)$ since $b_{E,T_1,\delta_1}\Big|_E = b_{E,T_2,\delta_2}\Big|_E = b_E\Big|_E$. Hence we conclude

$$\begin{aligned}
\varepsilon \, \|b_E^{1/2} r_E\|_E^2 \;\leq\; \sum_{i=1}^{2} \Big(& \varepsilon \, \|\nabla(u-u_h)\|_{T_i} \cdot \|\nabla w\|_{T_i} + \\
& + \Big[\|u-u_h\|_{T_i} + \|r_{T_i}\|_{T_i} + \|f-f_h\|_{T_i}\Big] \cdot \|w\|_{T_i}\Big).
\end{aligned}$$

The inverse inequalities (15) and (16) are used to bound $\|w\|_{T_i}$ and $\|\nabla w\|_{T_i}$, respectively.

Together with $\|b_E^{1/2} \cdot r_E\|_E \sim \|r_E\|_E$ from (11) this implies

$$\|r_E\|_E \lesssim \sum_{i=1}^{2} \delta_i^{1/2} \cdot h_{E,T_i}^{1/2} \cdot \Big(\min\{\delta_i \cdot h_{E,T_i} \, , \, h_{min,T_i}\}^{-1} \cdot \|\nabla(u-u_h)\|_{T_i} + \\ + \, \varepsilon^{-1} \cdot (\|u-u_h\|_{T_i} + \|r_{T_i}\|_{T_i} + \|f-f_h\|_{T_i}) \Big) \quad .$$

Now we choose

$$\delta_i := \frac{1}{2} \varepsilon^{1/2} \cdot h_{E,T_i}^{-1} \cdot \alpha_{T_i} \equiv \frac{1}{2} \min\{\varepsilon^{1/2}/h_{E,T_i} \, , \, h_{min,T_i}/h_{E,T_i}\} < 1$$

(recall $h_{E,T_i} > h_{min,T_i}/2$). This yields $\min\{\delta_i \cdot h_{E,T_i}, h_{min,T_i}\} \sim \varepsilon^{1/2} \cdot \alpha_{T_i}$. Insert the previous estimate (24) which provides a bound of $\|r_{T_i}\|_{T_i}$ to obtain

$$\begin{aligned} &\varepsilon^{3/2} \cdot \alpha_T \cdot \|r_E\|_E^2 \lesssim \\ &\quad \lesssim \sum_{i=1}^{2} \varepsilon \cdot \|\nabla(u-u_h)\|_{T_i}^2 + \alpha_{T_i}^2 \cdot \|u-u_h\|_{T_i}^2 + |||u-u_h|||_{T_i}^2 + \alpha_{T_i}^2 \cdot \|f-f_h\|_{T_i}^2 \\ &\quad \lesssim |||u-u_h|||_{\omega_E}^2 + \alpha_T^2 \cdot \|f-f_h\|_{\omega_E}^2 \end{aligned}$$

since h_{min,T_i} and α_{T_i} do not change rapidly across adjacent tetrahedra, and since $\alpha_{T_i} \leq 1$. Summing up over all faces E of T, recalling the definition of $\eta_{\varepsilon,T}$ and applying (24) finishes the proof of the lower error bound (22).

Secondly, in order to derive (23) we utilize the orthogonality property of the error

$$a(u-u_h, v_h) = 0 \qquad \forall \, v_h \in V_{o,h} \quad .$$

Integration by parts gives for all $v \in H_o^1(\Omega)$

$$\begin{aligned} a(u-u_h, v) &= a(u-u_h, v-R_o v) \\ &= \varepsilon(\nabla(u-u_h), \nabla(v-R_o v)) + (u-u_h, v-R_o v) \\ &= \sum_{T\in\mathcal{T}_h} (f + \varepsilon\Delta u_h - u_h, v-R_o v)_T + \varepsilon \sum_{E\subset\Omega\backslash\Gamma} (r_E, v-R_o v)_E \\ &= \sum_{T\in\mathcal{T}_h} \Big[(r_T + f - f_h, v-R_o v)_T + \frac{1}{2} \cdot \varepsilon \sum_{E\subset\partial T\backslash\Gamma_D} (r_E, v-R_o v)_E \Big] \\ &\leq \sum_{T\in\mathcal{T}_h} \Big[\alpha_T(\|r_T\|_T + \|f-f_h\|_T) \cdot \alpha_T^{-1} \|v-R_o v\|_T + \\ &\qquad + \frac{1}{2} \sum_{E\subset\partial T\backslash\Gamma_D} \varepsilon^{3/4} \alpha_T^{1/2} \|r_E\|_E \cdot \varepsilon^{1/4} \alpha_T^{-1/2} \|v-R_o v\|_E \Big]. \end{aligned}$$

The Cauchy-Schwarz inequality and the interpolation estimate (19) yield

$$\sum_{T\in\mathcal{T}_h} \alpha_T(\|r_T\|_T + \|f-f_h\|_T)\cdot\alpha_T^{-1}\|v-R_o v\|_T \;\le$$

$$\le \left(2\sum_{T\in\mathcal{T}_h}\alpha_T^2\Big(\|r_T\|_T^2+\|f-f_h\|_T^2\Big)\right)^{1/2}\cdot\left(\sum_{T\in\mathcal{T}_h}\alpha_T^{-2}\|v-R_o v\|_T^2\right)^{1/2}$$

$$\overset{(19)}{\lesssim} \left(\sum_{T\in\mathcal{T}_h}\alpha_T^2\Big(\|r_T\|_T^2+\|f-f_h\|_T^2\Big)\right)^{1/2}\cdot m_1(v,\mathcal{T}_h)\cdot|||v||| \qquad .$$

With the help of interpolation estimate (20) one derives analogously

$$\sum_{T\in\mathcal{T}_h}\sum_{E\subset\partial T\backslash\Gamma_D}\varepsilon^{3/4}\alpha_T^{1/2}\|r_E\|_E\;\cdot\;\varepsilon^{1/4}\alpha_T^{-1/2}\|v-R_o v\|_E\;\le$$

$$\le \left(\varepsilon^{3/2}\sum_{T\in\mathcal{T}_h}\sum_{E\subset\partial T\backslash\Gamma_D}\alpha_T\|r_E\|_E^2\right)^{1/2}\cdot\left(\varepsilon^{1/2}\sum_{T\in\mathcal{T}_h}\sum_{E\subset\partial T\backslash\Gamma_D}\alpha_T^{-1}\|v-R_o v\|_E^2\right)^{1/2}$$

$$\overset{(20)}{\lesssim} \left(\varepsilon^{3/2}\sum_{T\in\mathcal{T}_h}\sum_{E\subset\partial T\backslash\Gamma_D}\alpha_T\|r_E\|_E^2\right)^{1/2}\cdot m_1(v,\mathcal{T}_h)\cdot|||v|||\;.$$

Combining these estimates results in

$$a(u-u_h,v)\;\lesssim\;\left(\sum_{T\in\mathcal{T}_h}\Big[\alpha_T^2\Big(\|r_T\|_T^2+\|f-f_h\|_T^2\Big)+\varepsilon^{3/2}\alpha_T\sum_{E\subset\partial T\backslash\Gamma_D}\|r_E\|_E^2\Big]\right)^{1/2}$$
$$\cdot\, m_1(v,\mathcal{T}_h)\cdot|||v||| \qquad .$$

Substituting $v := u-u_h \in H_o^1(\Omega)$ finishes the proof. ■

Remark 4.2 The upper error bound (23) contains the matching function $m_1(u-u_h,\mathcal{T}_h)$. Since $u-u_h$ is not known, $m_1(u-u_h,\mathcal{T}_h)$ cannot be computed exactly. This can be remedied by using an approximation m_1^R by means of a recovered gradient $\nabla^R u_h \approx \nabla u$:

$$\begin{aligned} m_1(u-u_h,\mathcal{T}_h) &\equiv \Big(\sum_{T\in\mathcal{T}_h} h_{min,T}^{-2}\cdot\|C_T^T\,\nabla(u-u_h)\|_T^2\Big)^{1/2}\Big/\;\|\nabla(u-u_h)\| \\ &\approx \Big(\sum_{T\in\mathcal{T}_h} h_{min,T}^{-2}\cdot\|C_T^T\,(\nabla^R u_h-\nabla u_h)\|_T^2\Big)^{1/2}\Big/\;\|\nabla^R u_h-\nabla u_h\| \\ &=: m_1^R(u_h,\mathcal{T}_h) \qquad , \end{aligned} \tag{25}$$

cf. [14] for a more comprehensive discussion. All numerical experiments so far indicate that m_1^R is a robust approximation to m_1, see also Section 5 below.

5 Numerical experiments

In this section we will illustrate the error estimates of the previous section by means of numerical experiments. Here we choose the same model problem that has been employed in [15] to analyse a face–based error estimator.

Consider the three–dimensional model problem

$$-\varepsilon\Delta u + u = 0 \quad \text{in } \Omega := [0,1]^3 \qquad , \qquad u = u_0 \quad \text{on } \Gamma_D := \partial\Omega$$

with the perturbation parameter $\varepsilon = 10^{-4}$. The exact solution is prescribed to be

$$u = e^{-x/\sqrt{\varepsilon}} + e^{-y/\sqrt{\varepsilon}} + e^{-z/\sqrt{\varepsilon}} \qquad .$$

It displays typical boundary layers along the planes $x = 0$, $y = 0$, and $z = 0$. The boundary value u_0 is chosen accordingly.

The domain is discretized by a sequence of meshes, each one being the tensor product of three one–dimensional Bakhvalov–like meshes [6] with 2^k intervals in [0,1], $k = 1 \dots 6$. To describe the 1D nodal distribution properly, denote the transition point of the boundary layer by $\tau := \sqrt{\varepsilon}|\ln\sqrt{\varepsilon}|$. Then 2^{k-1} nodes are *exponentially* distributed in the boundary layer interval $[0,\tau]$ whereas the remaining interval $[\tau, 1]$ is divided into 2^{k-1} *equidistant* intervals, cf. Figure 4. More precisely, the (1D) nodal coordinate of the m-th node is

$$x_m := \begin{cases} -\beta\sqrt{\varepsilon}\ln\left[1 - \dfrac{m}{2^{k-1}}(1 - e^{-\tau/\beta/\sqrt{\varepsilon}})\right] & \text{for } m = 0 \dots 2^{k-1}, \beta = 3/2 \\ \tau + (1-\tau)\cdot\left(\dfrac{m}{2^{k-1}} - 1\right) & \text{for } m = 2^{k-1}+1 \dots 2^k \end{cases} \qquad .$$

Note that the only difference to the original Bakhvalov mesh consists in the slightly different choice of the transition point τ.

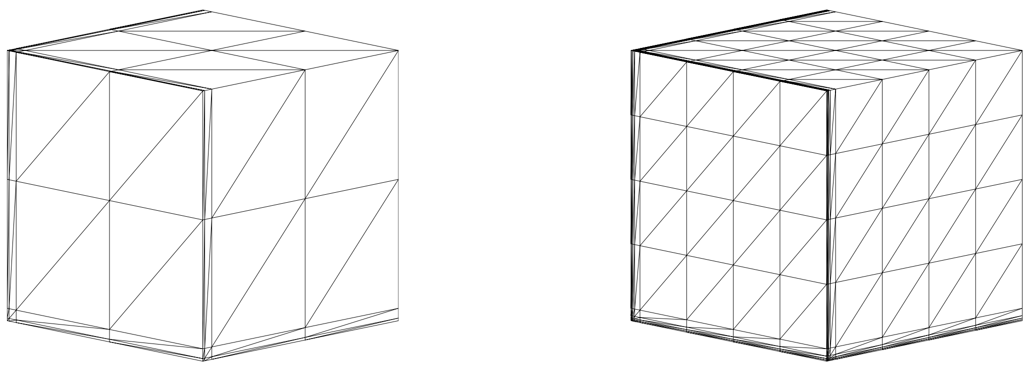

Figure 4: Mesh 2 – Mesh 3

We start by presenting the size and the maximum aspect ratio of the meshes as well as the matching function m_1.

Mesh k	# Elements	Aspect ratio	$m_1(u - u_h, \mathcal{T}_h)$
1	48	29.4	1.55
2	384	69.5	1.62
3	3 072	82.6	1.69
4	24 576	88.6	1.88
5	196 608	91.5	2.37
6	1 572 864	92.9	3.04

The size of m_1 is comparatively small and grows only mildly. This implies that the chosen meshes discretize the problem sufficiently well.

Next, we investigate the results of the error estimation of Theorem 4.1 by computing the corresponding ratios. Recall that the approximation terms $\zeta_{\varepsilon,T}$ and ζ_ε vanish here.

Mesh k	$\|\|\|u - u_h\|\|\|$	$\frac{\|\|\|u - u_h\|\|\|}{m_1 \cdot \eta_\varepsilon}$	$\max_{T \in \mathcal{T}_h} \frac{\eta_{\varepsilon,T}}{\|\|\|u - u_h\|\|\|_{\omega_T}}$
1	$0.154E + 0$	0.435	1.026
2	$0.536E - 1$	0.167	2.693
3	$0.229E - 1$	0.118	4.214
4	$0.110E - 1$	0.096	4.163
5	$0.553E - 2$	0.074	4.054
6	$0.282E - 2$	0.058	3.948

To start with, the error norm $|||u - u_h|||$ displays the optimal rate of convergence. Next consider the ratios of the third and fourth column which correspond to our main theoretical result. These ratios are bounded from above and thus confirm the predictions of Theorem 4.1. Note that from a practical point of view the moderately decreasing values of the upper error bound (third column) imply that the error is increasingly overestimated.

Finally we will investigate the matching function more closely, cf. the table below. We compare the matching function $m_1(u - u_h, \mathcal{T}_h)$ from (17) and its modification $m_{1,\varepsilon}(u - u_h, \mathcal{T}_h)$ from (18). Clearly only marginal differences can be seen, so either choice seems possible.

On the other hand we present the approximation m_1^R proposed in equation (25) of Remark 4.2. The results show a sufficient coincidence with the values of $m_1(u - u_h, \mathcal{T}_h)$. Hence the matching function and its approximation are useful tools for the theoretical analysis as well as for assessing the mesh quality in numerical computations. This topic has already been discussed for the Poisson equation in [14].

Mesh k	$m_1(u-u_h,\mathcal{T}_h)$	$m_{1,\varepsilon}(u-u_h,\mathcal{T}_h)$	$m_1^R(u_h,\mathcal{T}_h)$
1	1.55	1.23	1.68
2	1.62	1.48	1.52
3	1.69	1.64	1.69
4	1.88	1.86	1.86
5	2.37	2.34	2.03
6	3.04	3.01	2.29

6 Summary

For a singularly perturbed reaction–diffusion model problem, we have proposed and rigorously analysed a new residual error estimator that is suitable for *anisotropic* meshes. It has been shown that the error estimation is uniform in the small perturbation parameter. The analysis implies that tight error bounds are obtained as long as the anisotropic mesh is chosen according to the anisotropy of the solution. A numerical experiment confirms the theory. Hence reliable and efficient error estimation is possible on anisotropic meshes. This is a first important step towards a general adaptive anisotropic algorithm.

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