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Generalized Lyapunov Equations for Descriptor Systems: Stability and Inertia Theorems

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Abstract

We study generalized Lyapunov equations and present generalizations of Lyapunov stability theorems and some matrix inertia theorems for matrix pencils. We discuss applications of generalized Lyapunov equations with special right-hand sides in stability theory and control problems for descriptor systems.

Key words. generalized Lyapunov equations, inertia, descriptor systems, asymptotic stability, controllability and observability Gramians.

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1 Introduction

Generalized continuous-time Lyapunov equations

$$E^*XA + A^*XE = -G \tag{1.1}$$

and generalized discrete-time Lyapunov equations

$$A^*XA - E^*XE = -G \tag{1.2}$$

with given matrices E, A, G and unknown matrix X arise naturally in control problems [2, 12], stability theory for the differential and difference equations [13, 14, 24, 31], problems of spectral dichotomy [15, 20, 21] and numerical solution of algebraic Riccati equations [19, 22].

Equations (1.1) and (1.2) with $E = I$ are the standard continuous-time and discrete-time Lyapunov equations (the latter is also known as the Stein equation). The theoretical analysis, numerical solution and perturbation theory for these equations has been the topic of numerous publications, see [1, 13, 16, 17, 27] and the references therein. The case of nonsingular E has been considered in [3, 28]. However, many applications in singular systems or descriptor systems [9] lead to generalized Lyapunov equations with a singular matrix E, see [2, 21, 24, 32, 31].

The solvability of the generalized Lyapunov equations (1.1) and (1.2) can be described in terms of the generalized eigenstructure of the matrix pencil $\alpha E - \beta A$. The pencil $\alpha E - \beta A$ is called *regular* if E and A are square and $\det(\alpha E - \beta A) \neq 0$ for some $(\alpha, \beta) \in \mathbb{C}^2$. Otherwise, $\alpha E - \beta A$ is called *singular.* A pair $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$ is said to be *generalized eigenvalue* of the regular pencil $\alpha E - \beta A$ if $\det(\alpha E - \beta A) = 0$. If $\beta \neq 0$, then $\lambda = \alpha/\beta$ is *a finite eigenvalue* of the pencil $\lambda E - A$. The pair $(\alpha, 0)$ represents *an infinite eigenvalue.* Clearly, the pencil $\lambda E - A$ has infinite eigenvalues if and only if the matrix E is singular.

A regular matrix pencil $\lambda E - A$ with a singular matrix E can be reduced to *the Weierstrass (Kronecker) canonical form* [30]. There exist nonsingular matrices W and T such that

$$E = W \begin{pmatrix} I_m & 0 \\ 0 & N \end{pmatrix} T \qquad \text{and} \qquad A = W \begin{pmatrix} J & 0 \\ 0 & I_{n-m} \end{pmatrix} T, \tag{1.3}$$

where I_m is the identity matrix of order m and N is nilpotent. The block J corresponds to the finite eigenvalues of the pencil $\lambda E - A$, the block N corresponds to the infinite eigenvalues. The index of nilpotency of N is called *index* of the pencil $\lambda E - A$. The spaces spanned by the first m columns of W and T^{-1} are, respectively, *the left and right deflating subspaces* of $\lambda E - A$ corresponding to the finite eigenvalues, whereas the spans of the last $n - m$ columns of W and T^{-1} form *the left and right deflating subspaces* corresponding to the infinite eigenvalues, respectively. For simplicity, the deflating subspaces of $\lambda E - A$ corresponding to the finite (infinite) eigenvalues we will call *the finite (infinite) deflating subspaces.* The matrices

$$P_l = W \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} W^{-1}, \qquad P_r = T^{-1} \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} T, \tag{1.4}$$

are *the spectral projections* onto the left and right finite deflating subspaces of the pencil $\lambda E - A$ along the left and right infinite deflating subspaces, respectively.

In this paper we study the existence and uniqueness of solutions of generalized Lyapunov equations with general and special right-hand sides. Our main focus are generalized continuous-time Lyapunov equations

$$E^* X A + A^* X E = -P_r^* G P_r, \tag{1.5}$$

and generalized discrete-time Lyapunov equations

$$A^* X A - E^* X E = -P_r^* G P_r \pm (I - P_r)^* G (I - P_r). \tag{1.6}$$

Under some assumptions on the finite spectrum of $\lambda E - A$, equations (1.5) and (1.6) have solutions that are, in general, not unique. We are interested in the solution X of (1.5) satisfying $X = XP_l$ and the solution X of (1.6) satisfying $P_l^* X = XP_l$. Such solutions are uniquely defined and have some useful properties. We discuss applications of equations (1.5) and (1.6) in the study of the asymptotic behaviour of solutions of singular systems, the distribution of the generalized eigenvalues of a pencil in the complex plane with respect to the imaginary axis and the unit circle, as well as the controllability and observability properties of descriptor systems.

This paper is organized as follows. In Section 2 we study generalized Lyapunov equations with general and special right-hand sides and extend classical Lyapunov stability theorems to these equations. Section 3 contains generalizations of matrix inertia theorems with respect to the imaginary axis and the unit circle for matrix pencils. In Section 4 we establish a connection between the controllability and observability Gramians for descriptor systems and partial solutions of the generalized Lyapunov equations (1.5) and (1.6).

Throughout the paper $\mathbb{F}$ denotes the field of real ($\mathbb{F} = \mathbb{R}$) or complex ($\mathbb{F} = \mathbb{C}$) numbers, $\mathbb{F}^{n,m}$ is the space of $n \times m$-matrices over $\mathbb{F}$. The matrix $A^* = A^T$ denotes the transpose of a real matrix A, $A^* = A^H$ denotes the complex conjugate transpose of complex A and $A^{-*} = (A^{-1})^*$. The matrix A is *Hermitian* if $A = A^*$. The matrix A is *positive definite* (*positive semidefinite*) if $x^* A x > 0$ ($x^* A x \geq 0$) for all nonzero $x \in \mathbb{F}^n$, and A is *positive definite on a subspace* $\mathcal{X} \subset \mathbb{F}^n$ if $x^* A x > 0$ for all nonzero $x \in \mathcal{X}$. We will denote by $\|\cdot\|$ the spectral matrix norm and the Euclidean vector norm.

2 Generalized Lyapunov equations

In this section we present some general results concerning the solution of the generalized continuous-time algebraic Lyapunov equation (GCALE)

$$E^* X A + A^* X E = -G, \tag{2.1}$$

and the generalized discrete-time algebraic Lyapunov equation (GDALE)

$$A^* X A - E^* X E = -G, \tag{2.2}$$

where E, A, $G \in \mathbb{F}^{n,n}$ are given matrices and $X \in \mathbb{F}^{n,n}$ is the unknown matrix.

2.1 General case

A *continuous-time Lyapunov operator* $\mathcal{L}_c : \mathbb{F}^{n,n} \to \mathbb{F}^{n,n}$ has the form

$$\mathcal{L}_c(X) := E^*XA + A^*XE \tag{2.3}$$

and a *discrete-time Lyapunov operator* $\mathcal{L}_d : \mathbb{F}^{n,n} \to \mathbb{F}^{n,n}$ has the form

$$\mathcal{L}_d(X) := A^*XA - E^*XE. \tag{2.4}$$

Let $x = vec(X)$ and $g = vec(G)$ be vectors of order n^2 obtained by stacking the columns of the matrices X and G, respectively. Then we can rewrite the GCALE (2.1) in the equivalent form of the linear system

$$L_c x = -g, \tag{2.5}$$

where the $n^2 \times n^2$-matrix

$$L_c = E^T \otimes A^* + A^T \otimes E^* \tag{2.6}$$

is the matrix representation of the continuous-time Lyapunov operator $\mathcal{L}_c$, see, e.g., [18]. Here $\otimes$ denotes the Kronecker product. Analogously, the GDALE (2.2) can be rewritten as the linear system

$$L_d x = -g, \tag{2.7}$$

where the $n^2 \times n^2$-matrix

$$L_d = A^T \otimes A^* - E^T \otimes E^* \tag{2.8}$$

is the matrix representation of the discrete-time Lyapunov operator $\mathcal{L}_d$, see [18]. Thus, we may apply the theory of linear systems [13] to determine conditions for the existence and uniqueness of solutions of the generalized Lyapunov equations (2.1) and (2.2).

Theorem 2.9 *Let L_c and L_d be as in (2.6) and (2.8) and let $x = vec(X)$, $g = vec(G)$.*

1. *The GCALE (2.1) has a solution if and only if* $\operatorname{rank}[\,L_c, g\,] = \operatorname{rank} L_c$. *There exists a unique solution of (2.1) if and only if the matrix L_c is nonsingular.*
2. *The GDALE (2.2) has a solution if and only if* $\operatorname{rank}[\,L_d, g\,] = \operatorname{rank} L_d$. *There exists a unique solution of (2.2) if and only if the matrix L_d is nonsingular.*

Note that already for moderately large n the matrices L_c and L_d are very large. Therefore, the equivalent formulations (2.5) and (2.7) for the generalized Lyapunov equations are only of theoretical interest.

The generalized Lyapunov equations (2.1) and (2.2) are special cases of the generalized Sylvester equation

$$BXE - FXA = -G \tag{2.10}$$

which has a unique solution if and only if the matrix pencils $\lambda F - B$ and $\lambda E - A$ are regular and have no common eigenvalues [8]. As a consequence we have the following necessary and sufficient conditions for the existence and uniqueness of solutions of the generalized Lyapunov equations (2.1) and (2.2) in terms of the spectrum of the pencil $\lambda E - A$.

Theorem 2.11 *Let $\lambda E - A$ be a regular pencil with eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ counted according to their multiplicities.*

1. *The GCALE (2.1) has a unique solution for every matrix G if and only if all eigenvalues of the pencil $\lambda E - A$ are finite and $\overline{\lambda}_j + \lambda_k \neq 0$ for all j, $k = 1, \ldots, n$.*

2. *The GDALE (2.2) has a unique solution for every matrix G if and only if $\lambda_j \overline{\lambda}_k \neq 1$ for all j, $k = 1, \ldots, n$.*

If the GCALE (2.1) is uniquely solvable, then the finiteness of the eigenvalues of $\lambda E - A$ implies the nonsingularity of E, while the condition $\overline{\lambda}_j + \lambda_k \neq 0$ implies that the pencil $\lambda E - A$ has no eigenvalues on the imaginary axis and, hence, the matrix A is nonsingular. The GCALE (2.1) is called *non-degenerate* if both matrices E and A are nonsingular.

If the GDALE (2.2) has a unique solution, then it follows from the condition $\lambda_j \overline{\lambda}_k \neq 1$ that the pencil $\lambda E - A$ has no eigenvalues on the unit circle and the singularity of one of the matrices E and A implies the nonsingularity of the other. Thus, the GDALE (2.2) will be called *non-degenerate* if one of the matrices E and A is nonsingular.

Non-degenerate generalized Lyapunov equations (2.1) and (2.2) are equivalent to standard continuous-time Lyapunov equations

$$XAE^{-1} + (AE^{-1})^* X = -E^{-*} G E^{-1} \quad \text{and} \quad (EA^{-1})^* X + XEA^{-1} = -A^{-*} G A^{-1}, \quad (2.12)$$

and standard discrete-time Lyapunov equations

$$(AE^{-1})^* X A E^{-1} - X = -E^{-*} G E^{-1} \quad \text{or} \quad X - (EA^{-1})^* X E A^{-1} = -A^{-*} G A^{-1}, \quad (2.13)$$

respectively. In this case the classical Lyapunov theorems [13] on the existence and uniqueness of positive definite solutions of (2.12) and (2.13) can be generalized to equations (2.1) and (2.2).

Theorem 2.14 *Let $\lambda E - A$ be a regular matrix pencil.*

1. *If all eigenvalues of $\lambda E - A$ are finite and lie in the open left half-plane, then for every Hermitian, positive (semi)definite matrix G, the GCALE (2.1) has a unique Hermitian, positive (semi)definite solution X. Conversely, if there exist Hermitian, positive definite matrices X and G satisfying (2.1), then all eigenvalues of the pencil $\lambda E - A$ are finite and lie in the open left half-plane.*

2. *If all eigenvalues of $\lambda E - A$ are finite and lie inside the unit circle, then for every Hermitian, positive (semi)definite matrix G, the GDALE (2.2) has a unique Hermitian, positive (semi)definite solution X. Conversely, if there exist Hermitian, positive definite matrices X and G satisfying (2.2), then all eigenvalues of the pencil $\lambda E - A$ are finite and lie inside the unit circle.*

If at least one of the matrices E and A is singular, then the GCALE (2.1) will be called *degenerate*. Such an equation is *singular* in the sense that it may have no solution even if all finite eigenvalues of the pencil $\lambda E - A$ have negative real part. Since E and A play a symmetric role in (2.1), in the sequel we will assume that the matrix E is singular.

Example 2.15 The GCALE (2.1) with

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A = -I_2, \qquad G = I_2$$

has no solution.

Unlike the GCALE (2.1), the GDALE (2.2) with singular E and positive definite G has a unique negative definite solution X if and only if the matrix A is nonsingular and all eigenvalues of the pencil $\lambda E - A$ lie outside the unit circle or, equivalently, the eigenvalues of the reciprocal pencil $E - \mu A$ are finite and lie inside the unit circle. The GDALE (2.2) will be called *degenerate* if both the matrices E and A are singular. The degenerate GDALE (2.2) may have no solution although all finite eigenvalues of $\lambda E - A$ lie inside the unit circle.

Example 2.16 The GDALE (2.2) with

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad G = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

is not solvable.

But even if solutions of the degenerate generalized Lyapunov equations (2.1) or (2.2) exist, they are not unique. Indeed, if X is a solution of the degenerate GCALE (2.1), then for any nonzero vector $y \in \mathit{ker}\, E^*$, the matrix $X + yy^*$ satisfies (2.1) as well. Assume now that X satisfies the degenerate GDALE (2.2). Then for any nonzero vectors $y \in \mathit{ker}\, E^*$ and $z \in \mathit{ker}\, A^*$, the matrix $X + yz^*$ also satisfies (2.2).

We now introduce the notion of c-stability and d-stability for a matrix pencil.

Definition 2.17 A regular matrix pencil $\lambda E - A$ is *c-stable* if all finite eigenvalues of $\lambda E - A$ lie in the open left half-plane.

Definition 2.18 A regular matrix pencil $\lambda E - A$ is *d-stable* if all finite eigenvalues of $\lambda E - A$ lie inside the unit circle.

The following theorem gives sufficient conditions for the pencil $\lambda E - A$ to be c-stable and d-stable.

Theorem 2.19 *Let P_l and P_r be the spectral projections onto the left and right finite deflating subspaces of a regular pencil $\lambda E - A$ and let G be a matrix that is Hermitian, positive definite on the subspace $\operatorname{im} P_r$.*

1. *If the GCALE (2.1) has a solution X which is Hermitian, positive definite on the subspace $\operatorname{im} P_l$, then the pencil $\lambda E - A$ is c-stable.*

2. *If the GDALE (2.2) has a solution X which is Hermitian, positive definite on the subspace* $\mathrm{im}\, P_l$, *then the pencil $\lambda E - A$ is d-stable.*

PROOF. Let the pencil $\lambda E - A$ be in Weierstrass canonical form (1.3) and let the Hermitian matrix

$$X = W^{-*} \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{pmatrix} W^{-1} \tag{2.20}$$

satisfy the GCALE (2.1) or the GDALE (2.2). If X is positive definite on $\mathit{im}\, P_l$, then X_{11} in (2.20) is positive definite, and, hence, the matrix

$$E^* X E = T^* \begin{pmatrix} X_{11} & X_{12} N \\ N^* X_{12}^* & N^* X_{22} N \end{pmatrix} T \tag{2.21}$$

is Hermitian, positive definite on the subspace $\mathit{im}\, P_r$.

Let $z \neq 0$ be an eigenvector of the pencil $\lambda E - A$ corresponding to a finite eigenvalue λ, i.e., $\lambda E z = A z$ and $z \in \mathit{im}\, P_r$. Multiplication of (2.1) on the right and left by z and z^*, respectively, gives

$$-z^* G z = z^*(E^* X A + A^* X E) z = \lambda\, z^* E^* X E z + \overline{\lambda}\, z^* E^* X E z = 2(\Re e\, \lambda) z^* E^* X E z. \tag{2.22}$$

Since G and $E^* X E$ are positive definite on $\mathit{im}\, P_r$, we obtain that $\Re e\, \lambda < 0$, i.e., all finite eigenvalues of the pencil $\lambda E - A$ lie in the open left half-plane.

Analogously, multiplying the GDALE (2.1) by z and z^* we obtain from

$$-z^* G z = z^*(A^* X A - E^* X E) z = \lambda \overline{\lambda}\, z^* E^* X E z - z^* E^* X E z = (|\lambda|^2 - 1) z^* E^* X E z \tag{2.23}$$

that $|\lambda| < 1$, i.e., all finite eigenvalues of the pencil $\lambda E - A$ lie inside the unit circle. $\square$

Remark 2.24 From (2.20) and (2.21) we have that a matrix X is positive definite on $\mathit{im}\, P_l$ if and only if the matrix $E^* X E$ is positive definite on $\mathit{im}\, P_r$. Moreover, it follows from (2.22) and (2.23) that the condition for X to be positive definite on $\mathit{im}\, P_l$ can be replaced by the assumption that X is positive semidefinite on $\mathbb{F}^n$. Thus, we obtain the following theorem.

Theorem 2.25 *Let P_r be the spectral projection onto the right finite deflating subspace of a regular pencil $\lambda E - A$ and let G be a matrix that is Hermitian, positive definite on* $\mathrm{im}\, P_r$.

1. *If the GCALE (2.1) has a solution X which is Hermitian, positive semidefinite, then the pencil $\lambda E - A$ is c-stable.*

2. *If the GDALE (2.2) has a solution X which is Hermitian, positive semidefinite, then the pencil $\lambda E - A$ is d-stable.*

Examples 2.15 and 2.16 demonstrate that the c-stability and d-stability of the pencil $\lambda E - A$ does not imply the existence of solutions of the degenerate generalized Lyapunov equations (2.1) and (2.2).

The GCALE (2.1) is closely related to the study of the asymptotic properties of solutions of the homogeneous continuous-time descriptor system

$$E\dot{x}(t) = Ax(t), \tag{2.26}$$

with a singular matrix E, see, e.g., [9, 24, 26].

Definition 2.27 System (2.26) is called *stable in the sense of Lyapunov* if for all $x^0 \in \mathbb{F}^n$, the initial value problem

$$\begin{array}{rcl} E\dot{x}(t) - Ax(t) & = & 0, \\ P_r(x(0) - x^0) & = & 0 \end{array} \tag{2.28}$$

has a unique solution $x(t, x^0) \in im\, P_r$ which is bounded for all $t \in [\,0, \infty)$. System (2.26) is called *asymptotically stable* if it is stable and $\lim\limits_{t\to\infty} x(t, x^0) = 0$ for the solution $x(t, x^0)$ of (2.28).

It is well-known that the continuous-time descriptor system (2.26) is asymptotically stable if and only if the pencil $\lambda E - A$ is regular and all its finite eigenvalues lie in the open half-plane, i.e., $\lambda E - A$ is c-stable, e.g., [9].

Let G be a matrix that is Hermitian, positive definite on the subspace $im\, P_r$ and let X be a solution of (2.1) that is Hermitian, positive definite on the subspace $im\, P_l$. Then by Remark 2.24 we obtain that for all nonzero solutions $x(t) \in im\, P_r$ of equation (2.26),

$$v(t) := x^*(t)E^*XEx(t) > 0$$

for all $t \in [\,0, \infty)$. Moreover, we have

$$\dot{v}(t) = x^*(t)(A^*XE + E^*XA)x(t) = -x^*(t)Gx(t) < 0.$$

The quadratic form $v(t)$ is the *Lyapunov function* for the continuous-time descriptor system (2.26).

Analogous to the continuous-time case, the GDALE (2.2) can be used to investigate the asymptotic behaviour of solutions of the homogeneous discrete-time descriptor system

$$Ex_{k+1} = Ax_k \tag{2.29}$$

with singular E, see [9, 32].

Definition 2.30 System (2.29) is called *stable in the sense of Lyapunov* if for all $x^0 \in \mathbb{F}^n$ the initial value problem

$$\begin{array}{rcl} Ex_{k+1} - Ax_k & = & 0, \\ P_r(x_0 - x^0) & = & 0 \end{array} \tag{2.31}$$

has a unique solution $x_k \in im\, P_r$ that is bounded for all $k = 0, 1, \ldots$. System (2.29) is called *asymptotically stable* if it is stable and $\lim\limits_{k\to\infty} x_k = 0$ for the solution x_k of (2.31).

Similar to the continuous-time case, it can be proved that the discrete-time descriptor system (2.29) is asymptotically stable if and only if the pencil $\lambda E - A$ is d-stable [9]. Note that d-stability includes regularity of $\lambda E - A$.

Let G be a matrix that is Hermitian, positive definite on the subspace $\mathit{im}\, P_r$ and let X be a matrix that is Hermitian, positive definite on the subspace $\mathit{im}\, P_l$ and satisfies the GDALE (2.2). Consider a quadratic form

$$v_k := x_k^* E^* X E x_k$$

which presents the *Lyapunov function* for the discrete-time descriptor system (2.29). By Remark 2.24 we obtain that $v_k > 0$ for all nonzero solutions $x_k \in \mathit{im}\, P_r$ of equation (2.29) and

$$v_{k+1} - v_k = x_k^*(A^* X A - E^* X E) x_k = -x_k^* G x_k < 0.$$

As a consequence of Theorems 2.19 and 2.25 we have the following sufficient conditions for the descriptor systems (2.26) and (2.29) to be asymptotically stable.

Corollary 2.32 *Let $\lambda E - A$ be a regular pencil and let G be an Hermitian matrix that is positive definite on the subspace* $\mathrm{im}\, P_r$.

1. *The continuous-time descriptor system (2.26) is asymptotically stable if there exists an Hermitian solution X of the GCALE (2.1) that is positive definite on the subspace* $\mathrm{im}\, P_l$ *or positive semidefinite on $\mathbb{F}^n$.*

2. *The discrete-time descriptor system (2.29) is asymptotically stable if there exists an Hermitian solution X of the GDALE (2.2) that is positive definite on the subspace* $\mathrm{im}\, P_l$ *or positive semidefinite on $\mathbb{F}^n$.*

It is well-known that standard continuous-time and discrete-time Lyapunov equations are related via a *Cayley transformation* for matrices defined by $\mathcal{C}(A) = (A - I)^{-1}(A + I)$, see, e.g., [27]. A *generalized Cayley transformation* for matrix pencils given by

$$\mathcal{C}(E, A) = \lambda(A - E) - (E + A) \tag{2.33}$$

allows us to state a similar connection between generalized Lyapunov equations in continuous-time and discrete-time cases [23]. Indeed, X is a solution of the GCALE (2.1) if and only if X satisfies the GDALE

$$\mathcal{A}^* X \mathcal{A} - \mathcal{E}^* X \mathcal{E} = -2G,$$

where $\lambda\mathcal{E} - \mathcal{A} = \lambda(A - E) - (E + A)$ is the Cayley-transformed pencil.

The following theorem gives a relationship between the eigenvalues of the pencils $\lambda E - A$ and $\lambda\mathcal{E} - \mathcal{A}$, see [23] for details.

Theorem 2.34 *1. Consider the generalized Cayley transformation (2.33) for the pencil $\lambda E - A$ associated with the GCALE (2.1). Then*

(a) *the finite eigenvalues of $\lambda E - A$ in the open left and right half-plane are mapped to eigenvalues inside and outside the unit circle, respectively, and the eigenvalue $\lambda = 1$ is mapped to ∞;*

(b) *the finite eigenvalues on the imaginary axis are mapped to eigenvalues on the unit circle;*

(c) *the infinite eigenvalues of $\lambda E - A$ are mapped to $\lambda = 1$.*

2. *Consider the generalized Cayley transformation (2.33) for the pencil $\lambda E - A$ associated with the GDALE (2.2). Then*

   (a) *the finite eigenvalues of $\lambda E - A$ inside and outside the unit circle are mapped to eigenvalues in the open left and right half-plane, respectively;*

   (b) *the finite eigenvalues on the unit circle except $\lambda = 1$ are mapped to eigenvalues on the imaginary axis and the eigenvalue $\lambda = 1$ is mapped to ∞;*

   (c) *the infinite eigenvalues of $\lambda E - A$ are mapped to $\lambda = 1$.*

Thus, in the case of the nonsingular matrix E we obtain from Theorem 2.34 that the matrix pencil $\lambda E - A$ is c-stable (d-stable) if and only if the Cayley-transformed pencil $\lambda \mathcal{E} - \mathcal{A}$ is d-stable (c-stable). However, if E is singular, then this assertion does not hold any more, since infinite eigenvalues of a c-stable pencil are mapped under the generalized Cayley transformation to an eigenvalue $\lambda = 1$ on the unit circle and infinite eigenvalues of a d-stable pencil are mapped to an eigenvalue $\lambda = 1$ in the right half-plane. This leads to a discrepancy in the solvability theory for the degenerate continuous-time and discrete-time Lyapunov equations related via the generalized Cayley transformation. Therefore, in the sequel we will consider the generalized Lyapunov equations in the continuous-time and discrete-time case separately.

2.2 Special right-hand side: case 1

Consider the generalized continuous-time Lyapunov equation

$$E^*XA + A^*XE = -E^*GE. \tag{2.35}$$

The following theorems give necessary and sufficient conditions for the existence of solutions of this equation.

Theorem 2.36 *Let $\lambda E - A$ be a regular matrix pencil of index at most two. If $\lambda E - A$ is c-stable, then for every matrix G, the GCALE (2.35) has a solution. For all solutions X of (2.35) the matrix E^*XE is unique. Moreover, if G is positive definite, then every solution X of (2.35) is positive definite on $\operatorname{im} P_l$ and E^*XE is positive definite on $\operatorname{im} P_r$ and positive semidefinite on $\mathbb{F}^n$.*

PROOF. Let the pencil $\lambda E - A$ be in Weierstrass canonical form (1.3), where the eigenvalues of J lie in the open left half-plane and $N^2 = 0$ by assumption. Let the matrices

$$W^*GW = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \qquad \text{and} \qquad W^*XW = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \tag{2.37}$$

be partitioned in blocks conformally to E and A. Then from (2.35) we have

$$\begin{aligned} X_{11}J + J^*X_{11} &= -W_{11}, && (2.38)\\ X_{12} + J^*X_{12}N &= -W_{12}N, && (2.39)\\ N^*X_{21}J + X_{21} &= -N^*W_{21}, && (2.40)\\ N^*X_{22} + X_{22}N &= -N^*W_{22}N. && (2.41) \end{aligned}$$

Since all eigenvalues of J have negative real part, the standard Lyapunov equation (2.38) has a unique solution X_{11} for every matrix W_{11} [13].

Equations (2.39) and (2.40) are equivalent to the Sylvester equations

$$J^{-*}X_{12} + X_{12}N = -J^{-*}W_{12}N, \tag{2.42}$$

$$N^*X_{21} + X_{21}J^{-1} = -N^*W_{21}J^{-1}, \tag{2.43}$$

respectively. Since the matrices J^{-*} and $-N$ have disjoint spectra, equations (2.42) and (2.43) are uniquely solvable [27] and their solutions are given by

$$X_{12} = -W_{12}N, \qquad\qquad X_{21} = -N^*W_{21}.$$

Equation (2.41) is also solvable and has the solution $X_{22} = -\frac{1}{2}(N^*W_{22} + W_{22}N)$.

Thus, every solution of the GCALE (2.35) has the form

$$X = W^{-*}\begin{pmatrix} X_{11} & -W_{12}N \\ -N^*W_{21} & X_{22} \end{pmatrix} W^{-1}, \tag{2.44}$$

with X_{11} and X_{22} satisfying equations (2.38) and (2.41), respectively. Multiplying equation (2.41) on the right by the matrix N we obtain that $N^*X_{22}N = 0$ holds for every solution X_{22} of (2.41). Since equation (2.38) has a unique solution X_{11}, the matrix

$$E^*XE = T^*\begin{pmatrix} X_{11} & -W_{12}N^2 \\ -(N^*)^2W_{21} & N^*X_{22}N \end{pmatrix} T = T^*\begin{pmatrix} X_{11} & 0 \\ 0 & 0 \end{pmatrix} T. \tag{2.45}$$

is uniquely defined for all solutions X of (2.35). If G is positive definite, then also W_{11} is positive definite and, hence, the solution X_{11} of (2.38) is positive definite. Then X in (2.44) is positive definite on $\mathit{im}\, P_l$. Moreover, E^*XE in (2.45) is positive definite on $\mathit{im}\, P_r$ and positive semidefinite on $\mathbb{F}^n$. □

Note that the assumption for $\lambda E - A$ to be of index two is important, since otherwise the GCALE (2.35) may have no solution at all even if the pencil $\lambda E - A$ is c-stable. To understand better what happens if the index of the pencil is increased from two to three, consider the following example.

Example 2.46 Let $A = -I_n$, $G = I_n$, $X = [x_{ij}]_{i,j=1}^n$ and let $E = N_n$ be a nilpotent Jordan block of order n. Taking these matrices with $n = 2$ in (2.35), we have the equation

$$\begin{pmatrix} 0 & x_{11} \\ x_{11} & x_{12}+x_{21} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

which has the solution set

$$\left\{ X = \begin{pmatrix} 0 & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \quad : \quad x_{12}+x_{21} = 1 \right\}.$$

For $n = 3$ we obtain the equation

$$\begin{pmatrix} 0 & x_{11} & x_{12} \\ x_{11} & x_{12}+x_{21} & x_{13}+x_{22} \\ x_{21} & x_{22}+x_{31} & x_{23}+x_{32} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

which has no solution.

Remark 2.47 Note that if G is Hermitian, then (2.35) has Hermitian as well as non-Hermitian solutions, see Example 2.46. But in this case the matrix E^*XE will be Hermitian for every solution X of (2.35).

Remark 2.48 Theorem 2.36 still holds if the matrix G in the GCALE (2.35) is positive definite only on the subspace $\operatorname{im} P_l$.

The converse of Theorem 2.36 also holds.

Theorem 2.49 *Let $\lambda E - A$ be a regular matrix pencil and let G be an Hermitian, positive definite matrix. If there exists a matrix X that is Hermitian, positive definite on the subspace $\operatorname{im} P_l$ and satisfies the GCALE (2.35), then the index of $\lambda E - A$ is at most two and the pencil $\lambda E - A$ is c-stable.*

PROOF. Let the pencil $\lambda E - A$ be in Weierstrass canonical form (1.3) and let the matrices G and X as in (2.37) satisfy the GCALE (2.35). Then equations (2.38)–(2.41) are fulfilled.

We may assume without loss of generality that the nilpotent matrix N in (1.3) has the block diagonal form

$$N = \begin{pmatrix} N_1 & & 0 \\ & \ddots & \\ 0 & & N_h \end{pmatrix}, \tag{2.50}$$

where N_j is a nilpotent Jordan block of order n_j. Clearly, $N_j^{n_j} = 0$ and the size of the largest block in (2.50) is the index of the pencil $\lambda E - A$. Let the matrices

$$X_{22} = \begin{pmatrix} \check{X}_{11} & \cdots & \check{X}_{1h} \\ \vdots & \ddots & \vdots \\ \check{X}_{h1} & \cdots & \check{X}_{hh} \end{pmatrix} \qquad \text{and} \qquad W_{22} = \begin{pmatrix} \check{W}_{11} & \cdots & \check{W}_{1h} \\ \vdots & \ddots & \vdots \\ \check{W}_{h1} & \cdots & \check{W}_{hh} \end{pmatrix}$$

be partitioned in blocks conformally to N. Since G is Hermitian, positive definite, also all W_{jj} are Hermitian, positive definite. In this case equation (2.41) is equivalent to the system of matrix equations

$$N_p^*\check{X}_{pq} + \check{X}_{pq}N_q = -N_q^*\check{W}_{pq}N_q, \qquad p = 1,\dots,h, \quad q = p, p+1,\dots,h. \tag{2.51}$$

Assume that the index of the pencil $\lambda E - A$ is larger than two. Then there exists a block N_k of order $n_k > 2$. Let $\check{X}_{kk} = [x_{ij}]_{i,j=1}^{n_k}$ and $\check{W}_{kk} = [w_{ij}]_{i,j=1}^{n_k}$. It is easy to verify that

$$\begin{array}{rcll} (N_k^*\check{X}_{kk})_{ij} & = & x_{i-1,j}, & \qquad i,j = 1,2,\dots,n_k, \\ (\check{X}_{kk}N_k)_{ij} & = & x_{i,j-1}, & \qquad i,j = 1,2,\dots,n_k, \\ (N_k^*\check{W}_{kk}N_k)_{ij} & = & w_{i-1,j-1}, & \qquad i,j = 1,2,\dots,n_k, \end{array}$$

where we have set

$$x_{0j} = x_{j0} = w_{0j} = w_{j0} = w_{00} = 0, \qquad j = 1,2,\dots,n_k. \tag{2.52}$$

It follows from (2.51) for $p = q = k$ that

$$x_{i-1,j} + x_{i,j-1} = -w_{i-1,j-1}, \qquad i,j = 1,2,\dots,n_k. \tag{2.53}$$

Hence, by (2.52) we obtain $x_{1,j-1} = x_{j-1,1} = 0$ for all $j = 2,\dots,n_k$. Then it follows from (2.53) that $w_{11} = -x_{12} - x_{21} = 0$ which contradicts the positive definiteness of $\check{W}_{kk}$. Thus, the index of the pencil $\lambda E - A$ is at most two.

Taking into account that E^*GE is positive definite on $\mathit{im}\, P_r$ and X is positive definite on $\mathit{im}\, P_l$, we have from Theorem 2.19 that all finite eigenvalues of the pencil $\lambda E - A$ lie in the open left half-plane. □

Analogous to the continuous-time case, we consider the generalized discrete-time Lyapunov equation

$$A^*XA - E^*XE = -E^*GE. \tag{2.54}$$

The following theorem gives sufficient conditions for the existence of solutions of equation (2.54), where both the matrices E and A are singular.

Theorem 2.55 *Let $\lambda E - A$ be a d-stable matrix pencil. If $\lambda E - A$ is of index one or if the zero eigenvalues of $\lambda E - A$ are simple, then for every matrix G, the degenerate GDALE (2.54) has a solution X. If G is Hermitian, then (2.54) has an Hermitian solution.*

PROOF. We may assume without loss of generality that the pencil $\lambda E - A$ is in Weierstrass canonical form

$$E = W\begin{pmatrix} I & & \\ & I & \\ & & N \end{pmatrix}T, \qquad A = W\begin{pmatrix} J_1 & & \\ & J_2 & \\ & & I \end{pmatrix}T,$$

where the matrix J_1 is nonsingular with all eigenvalues inside the unit circle and the matrix J_2 has zero eigenvalues only. Let the matrices

$$W^*GW = \begin{pmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{pmatrix} \qquad \text{and} \qquad W^*XW = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} \tag{2.56}$$

be partitioned in blocks conformally to E and A. Then from (2.54) we have

$$J_p^* X_{pq} J_q - X_{pq} = -W_{pq}, \qquad p,q = 1,2, \tag{2.57}$$
$$J_p^* X_{p3} - X_{p3} N = -W_{p3} N, \qquad p = 1,2, \tag{2.58}$$
$$X_{3q} J_q - N^* X_{3q} = -N^* W_{3q}, \qquad q = 1,2, \tag{2.59}$$
$$X_{33} - N^* X_{33} N = -N^* W_{33} N. \tag{2.60}$$

Since all eigenvalues of J_1 lie inside the unit circle and J_2, N are nilpotent, the standard Lyapunov equations (2.57) and (2.60) have unique solutions for every right-hand side, see [27]. Equations (2.58) with $p = 1$ and (2.59) with $q = 1$ are uniquely solvable for every W_{13} and W_{31} since J_1 and N have no common eigenvalues. Moreover, if $W_{31} = W_{13}^*$, then $X_{31} = X_{13}^*$.

Consider equations (2.58) with $p = 2$ and (2.59) with $q = 2$. If the index of $\lambda E - A$ is one, i.e., $N = 0$, then these equations have the trivial solutions for every W_{23} and W_{32}. If $N \neq 0$ but the zero eigenvalues of $\lambda E - A$ are simple, i.e., $J_2 = 0$, then these equations have solutions $X_{23} = W_{23}$ and $X_{32} = W_{32}$, respectively. Clearly, if G is Hermitian, then the GDALE (2.54) has an Hermitian solution. $\square$

Note that if the index of the pencil $\lambda E - A$ is larger than one and $\lambda E - A$ has a zero eigenvalue which is not simple, then as the following example shows, the GDALE (2.54) may have no solution.

Example 2.61 For $X = [x_{ij}]_{i,j=1}^4$, $G = [g_{ij}]_{i,j=1}^4$ and

$$E = \begin{pmatrix} I_2 & 0 \\ 0 & N_2 \end{pmatrix}, \qquad A = \begin{pmatrix} N_2 & 0 \\ 0 & I_2 \end{pmatrix},$$

we have

$$\begin{aligned} A^*XA - E^*XE &= \begin{bmatrix} -x_{11} & -x_{12} & 0 & -x_{13} \\ -x_{21} & x_{11} - x_{22} & x_{13} & x_{14} - x_{23} \\ 0 & x_{31} & x_{33} & x_{34} \\ -x_{31} & x_{41} - x_{32} & x_{43} & x_{44} - x_{33} \end{bmatrix} = \\ &= -E^*GE = -\begin{bmatrix} g_{11} & g_{12} & 0 & g_{13} \\ g_{21} & g_{22} & 0 & g_{23} \\ 0 & 0 & 0 & 0 \\ g_{31} & g_{32} & 0 & g_{33} \end{bmatrix}. \end{aligned}$$

If $g_{13} \neq 0$ or $g_{31} \neq 0$, then this equation has no solution.

The following theorem gives necessary and sufficient conditions for the GDALE (2.54) to have an Hermitian, positive semidefinite solution.

Theorem 2.62 *Let $\lambda E - A$ be a regular matrix pencil and let G be an Hermitian, positive definite matrix. The GDALE (2.54) has an Hermitian, positive semidefinite solution X if and only if the pencil $\lambda E - A$ is of index at most one and it is d-stable.*

PROOF. From the proof of Theorem 2.55 we have that if the d-stable pencil $\lambda E - A$ is of index at most one, then the matrix

$$X = W^{-*} \begin{pmatrix} X_{11} & X_{12} & 0 \\ X_{21} & X_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} W^{-1}$$

satisfies the GCALE (2.54), where

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = \sum_{j=0}^{\infty} \begin{pmatrix} J_1^* & 0 \\ 0 & J_2^* \end{pmatrix}^j \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}^j.$$

Since G is Hermitian, positive definite, X is Hermitian, positive semidefinite.

Conversely, let G and X as in (2.56) satisfy equation (2.54) and let G be Hermitian, positive definite and X be Hermitian, positive semidefinite. Then X_{33} is an Hermitian, positive semidefinite solution of equation (2.60). This solution is given by

$$X_{33} = -\sum_{j=1}^{\nu-1} (N^*)^j W_{33} N^j,$$

where ν is the index of the pencil $\lambda E - A$. Then for every nonzero vector z we have

$$0 \leq z^* X_{33} z = -\sum_{j=1}^{\nu-1} z^* (N^*)^j W_{33} N^j z \leq 0.$$

Hence $z^* N^* W_{33} N z = 0$. Since W_{33} is Hermitian and positive definite, we obtain that $Nz = 0$ for all z, i.e., $N = 0$.

Since G is Hermitian, positive definite, the matrix $E^* G E$ is Hermitian, positive definite on $\mathit{im}\, P_r$. Then by Theorem 2.25 we obtain that the pencil $\lambda E - A$ is d-stable. $\square$

Remark 2.63 Note that if the d-stable pencil $\lambda E - A$ is of index at most one and if G is Hermitian, positive definite only on $\mathit{im}\, P_l$, then equation (2.54) has an Hermitian solution which is positive definite on $\mathit{im}\, P_l$ and positive semidefinite on $\mathbb{F}^n$. In this case for all solution X of (2.54), the matrix $E^* X E$ is unique, positive definite on $\mathit{im}\, P_r$ and positive semidefinite on $\mathbb{F}^n$.

The generalized Lyapunov equations (2.35) and (2.54) arise in the stability analysis for the continuous-time descriptor system (2.26) of index at most two [21] and the discrete-time descriptor system (2.29) of index at most one [32], respectively. From Theorems 2.36, 2.49 and 2.62 one immediately obtains necessary and sufficient conditions for the descriptor systems (2.26) and (2.29) to be asymptotically stable.

Corollary 2.64 *Let $\lambda E - A$ be a regular pencil and let G be an Hermitian, positive definite matrix.*

1. *The continuous-time descriptor system (2.26) is asymptotically stable and has index at most two if and only if there exists a matrix X which is Hermitian, positive definite on $\operatorname{im} P_l$ and satisfies the GCALE (2.35).*

2. *The discrete-time descriptor system (2.29) is asymptotically stable and has index at most one if and only if there exists an Hermitian, positive semidefinite matrix X satisfying the GDALE (2.54).*

2.3 Special right-hand side: case 2

Consider the generalized continuous-time Lyapunov equation

$$E^*XA + A^*XE = -P_r^*GP_r, \tag{2.65}$$

where P_r is as in (1.4). An analogue of the classical Lyapunov theorem [13] can be proved for this equation.

Theorem 2.66 *Let $\lambda E - A$ be a regular matrix pencil and let P_r be the spectral projection onto the right finite deflating subspace of $\lambda E - A$. If there exist an Hermitian, positive definite matrix G and an Hermitian, positive semidefinite matrix X satisfying the GCALE (2.65), then the pencil $\lambda E - A$ is c-stable.*

PROOF. See, [31, Theorem 6].

The converse of Theorem 2.66 also holds.

Theorem 2.67 *Let $\lambda E - A$ be a regular matrix pencil and let P_r and P_l be the spectral projections onto the right and left finite deflating subspaces of $\lambda E - A$, respectively. If the pencil $\lambda E - A$ is c-stable, then for every matrix G, the GCALE (2.65) has a solution. Moreover, if a solution X of (2.65) satisfies $X = XP_l$, then it is unique and given by*

$$X = \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi E - A)^{-*}P_r^*GP_r(i\xi E - A)^{-1}d\xi.$$

If G is Hermitian, then this solution X is Hermitian. If G is positive definite or positive semidefinite, then X is positive semidefinite.

PROOF. See, [31, Theorem 7].

Remark 2.68 *Note that if G is positive definite, then every solution X of the GCALE (2.65) is positive definite on the subspace* $\operatorname{im} P_l$ *and the matrix E^*XE is positive definite on the subspace* $\operatorname{im} P_r$.

It has been shown in [31] that if the matrix pencil $\lambda E - A$ is in Weierstrass canonical form (1.3) and the matrix

$$T^{-*}GT^{-1} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

is partitioned conformally to E and A, then every solution X of the GCALE (2.65) has the form

$$X = W^{-*} \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} W^{-1}, \tag{2.69}$$

where X_{11} satisfies the standard Lyapunov equation

$$J^*X_{11} + X_{11}J = -T_{11} \tag{2.70}$$

and X_{22} satisfies the homogeneous Lyapunov equation

$$N^*X_{22} + X_{22}N = 0. \tag{2.71}$$

Since equation (2.71) has many solutions, the GCALE (2.65) is not unique solvable. In fact, if we constrain the solution of (2.65) to satisfy $X = XP_l$, we choose the nonunique part X_{22} to be zero. In the following a system of matrix equations

$$\begin{aligned} &E^*XA + A^*XE = -P_r^*GP_r, \\ &X = XP_l \end{aligned} \tag{2.72}$$

will be called *constrained generalized continuous-time Lyapunov equation.*

Consider now the generalized discrete-time Lyapunov equation

$$A^*XA - E^*XE = -P_r^*GP_r + s(I - P_r)^*G(I - P_r), \tag{2.73}$$

where $s = -1$, 0 or 1. Note that, unlike the GCALE (2.65), equation (2.73) has in the right-hand side two terms. The sign s of the second one, as we will see later, depends on different applications. We will study all three cases simultaneously. Analogous to Theorems 2.66 and 2.67 we can prove the following stability theorem for the GDALE (2.73).

Theorem 2.74 *Let $\lambda E - A$ be a regular matrix pencil and let P_r and P_l be the spectral projections onto the right and left finite deflating subspaces of $\lambda E - A$, respectively. For every Hermitian, positive definite matrix G, the GDALE (2.73) has an Hermitian solution X which is positive definite on* $\operatorname{im} P_l$ *if and only if the pencil $\lambda E - A$ is d-stable. Moreover, if a solution of (2.73) satisfies $P_l^*X = XP_l$, then it is unique and given by*

$$X = \frac{1}{2\pi} \int_0^{2\pi} (e^{i\varphi}E - A)^{-*} \Big(P_r^*GP_r + s(I - P_r)^*G(I - P_r) \Big) (e^{i\varphi}E - A)^{-1} d\varphi.$$

PROOF. Let the pencil $\lambda E - A$ be in Weierstrass canonical form (1.3) and let the matrices

$$T^{-*}GT^{-1} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \qquad \text{and} \qquad W^*XW = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \tag{2.75}$$

satisfy the GDALE (2.73). Since the matrix X is positive definite on the subspace $im\, P_l$ and $P_r^*GP_r - s(I - P_r)^*G(I - P_r)$ for $s = -1, 0, 1$ is positive definite on $im\, P_r$, by Theorem 2.19 the pencil $\lambda E - A$ is d-stable.

Assume now that $\lambda E - A$ is d-stable. Using (1.3), (1.4) and (2.75) we obtain from the GDALE (2.73) the system of matrix equations

$$\begin{aligned} J^*X_{11}J - X_{11} &= -T_{11}, & (2.76)\\ J^*X_{12} - X_{12}N &= 0, & (2.77)\\ X_{21}J - N^*X_{21} &= 0, & (2.78)\\ X_{22} - N^*X_{22}N &= sT_{22}. & (2.79) \end{aligned}$$

Since all eigenvalues of J lie inside the unit circle and N is nilpotent, equations (2.76) and (2.79) have unique solutions for every T_{11} and T_{22} and the solutions are given by

$$X_{11} = \frac{1}{2\pi} \int_0^{2\pi} (e^{i\varphi} I - J)^{-*} T_{11} (e^{i\varphi} I - J)^{-1} d\varphi$$

and

$$X_{22} = \frac{s}{2\pi} \int_0^{2\pi} (e^{i\varphi} I - N)^{-*} T_{22} (e^{i\varphi} I - N)^{-1} d\varphi = \frac{s}{2\pi} \int_0^{2\pi} (e^{i\varphi} N - I)^{-*} T_{22} (e^{i\varphi} N - I)^{-1} d\varphi,$$

see [15]. Clearly, if G is Hermitian, positive definite, then X_{11} and X_{22} are Hermitian and X_{11} is positive definite.

Equations (2.77) and (2.78) are solvable and have, for example, trivial solutions. It follows from $P_l^*X = XP_l$ that

$$P_l^*X = W^{-*} \begin{pmatrix} X_{11} & X_{12} \\ 0 & 0 \end{pmatrix} W^{-1} = XP_l = W^{-*} \begin{pmatrix} X_{11} & 0 \\ X_{21} & 0 \end{pmatrix} W^{-1},$$

i.e., $X_{12} = X_{21} = 0$. Thus, the matrix

$$\begin{aligned} X &= W^{-*} \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} W^{-1} = \\ &= \frac{1}{2\pi} \int_0^{2\pi} (e^{i\varphi} E - A)^{-*} \Big(P_r^*GP_r + s(I - P_r)^*G(I - P_r) \Big) (e^{i\varphi} E - A)^{-1} d\varphi \end{aligned}$$

is the unique Hermitian solution of the GDALE (2.73) together with $P_l^*X = XP_l$. Clearly, this solution is positive definite on $im\, P_l$. □

In the following a system of matrix equations of the form

$$\begin{aligned} &A^*XA - E^*XE = -P_r^*GP_r + s(I - P_r)^*G(I - P_r), \\ &P_l^*X = XP_l \end{aligned} \tag{2.80}$$

is called *constrained generalized discrete-time Lyapunov equation.*

Remark 2.81 Note that if the pencil $\lambda E - A$ is d-stable and G is positive definite, then the solution X of the constrained GDALE (2.80) is positive definite on $im\, P_l$ and negative definite on $ker\, P_l$ for $s = -1$, positive semidefinite on $\mathbb{F}^n$ for $s = 0$ and positive definite on $\mathbb{F}^n$ for $s = 1$.

Like (2.1) and (2.2), the generalized Lyapunov equations (2.65) and (2.73) can be used in the stability analysis for the descriptor systems (2.26) and (2.29). From Theorems 2.66, 2.67 and 2.74 we have the following necessary and sufficient conditions for (2.26) and (2.29) to be asymptotically stable.

Corollary 2.82 *Let $\lambda E - A$ be regular and let G be Hermitian, positive definite.*

1. *The continuous-time descriptor system (2.26) is asymptotically stable if and only if there exists an Hermitian, positive semidefinite matrix X that satisfies the GCALE (2.65).*

2. *The discrete-time descriptor system (2.29) is asymptotically stable if and only if there exists an Hermitian, positive definite matrix X satisfying the GDALE (2.73) with $s = 1$ or an Hermitian, positive semidefinite matrix X satisfying the GDALE (2.73) with $s = 0$.*

Remark 2.83 Note that the assertions of Theorems 2.66, 2.67, 2.74, Remark 2.68 and part 1 of Corollary 2.82 remain valid if the matrix G is positive definite only on the subspace $im\, P_r$.

In Table 1 we review the generalized continuous-time and discrete-time Lyapunov equations with different right-hand sides discussed in this section.

3 Inertia theorems

The constrained generalized Lyapunov equations can be used to generalize some inertia theorems for matrices, e.g., [7, 10, 27, 25, 33], to matrix pencils. A brief survey of matrix inertia theorems and their applications has been presented in [11].

3.1 Inertia with respect to the imaginary axis

First we recall some facts from the inertia theory for matrices, see [6, 7, 25] and the references therein.

Definition 3.1 The *inertia of a matrix A with respect to the imaginary axis* (*c-inertia*) is defined by the triplet of integers

$$In_c(A) = \{\ \pi_-(A),\ \pi_+(A),\ \pi_0(A)\ \},$$

where $\pi_-(A)$, $\pi_+(A)$ and $\pi_0(A)$ denote the number of eigenvalues of A with negative, positive and zero real part, respectively, counting multiplicities.

Case G	$E^*XA + A^*XE = -G$		$A^*XA - E^*XE = -G$	
	$X = X^* > 0$ on $im\, P_l$	$X = X^* \geq 0$ on $\mathbb{F}^n$	$X = X^* > 0$ on $im\, P_l$	$X = X^* \geq 0$ on $\mathbb{F}^n$
$G = G^* > 0$ on $im\, P_r$	$\Longrightarrow$ c-stable	$\Longrightarrow$ c-stable	$\Longrightarrow$ d-stable	$\Longrightarrow$ d-stable
Case E^*GE	$E^*XA + A^*XE = -E^*GE$		$A^*XA - E^*XE = -E^*GE$	
	$X = X^* > 0$ on $im\, P_l$	$X = X^* \geq 0$ on $\mathbb{F}^n$	$X = X^* > 0$ on $im\, P_l$	$X = X^* \geq 0$ on $\mathbb{F}^n$
$G = G^* > 0$ on $\mathbb{F}^n$	$\Longleftrightarrow$ c-stable index at most 2	$\Longrightarrow$ c-stable index at most 2	$\Longleftarrow$ d-stable index at most 1	$\Longleftrightarrow$ d-stable index at most 1
$G = G^* > 0$ on $im\, P_l$	$\Longrightarrow$ c-stable	$\Longrightarrow$ c-stable	$\Longrightarrow$ d-stable	$\Longrightarrow$ d-stable
	$\Longleftarrow$ c-stable index at most 2		$\Longleftarrow$ d-stable index at most 1	$\Longleftarrow$ d-stable index at most 1
Case $P_r^*GP_r$	$E^*XA + A^*XE = -P_r^*GP_r$ $X = XP_l$		$A^*XA - E^*XE = -P_r^*GP_r$ $X = XP_l$	
	$X = X^* > 0$ on $im\, P_l$, unique	$X = X^* \geq 0$ on $\mathbb{F}^n$, unique	$X = X^* > 0$ on $im\, P_l$, unique	$X = X^* \geq 0$ on $\mathbb{F}^n$, unique
$G = G^* > 0$ on $\mathbb{F}^n$	$\Longleftrightarrow$ c-stable	$\Longleftrightarrow$ c-stable	$\Longleftrightarrow$ d-stable	$\Longleftrightarrow$ d-stable
$G = G^* > 0$ on $im\, P_r$	$\Longleftrightarrow$ c-stable	$\Longleftrightarrow$ c-stable	$\Longleftrightarrow$ d-stable	$\Longleftrightarrow$ d-stable
$G = G^* \geq 0$ on $\mathbb{F}^n$		$\Longleftarrow$ c-stable		$\Longleftarrow$ d-stable
	$A^*XA - E^*XE = -P_r^*GP_r + s(I - P_r)^*G(I - P_r)$ $P_l^*X = XP_l$			
	$s = -1$		$s = 1$	
	$X = X^* > 0$ on $im\, P_l$, unique	$X = X^* < 0$ on $ker\, P_l$, unique	$X = X^* > 0$ on $im\, P_l$, unique	$X = X^* > 0$ on $\mathbb{F}^n$, unique
$G = G^* > 0$ on $\mathbb{F}^n$	$\Longleftrightarrow$ d-stable	$\Longleftarrow$ d-stable	$\Longleftrightarrow$ d-stable	$\Longleftrightarrow$ d-stable
$G = G^* > 0$ on $im\, P_r$	$\Longleftrightarrow$ d-stable		$\Longleftrightarrow$ d-stable	$\Longrightarrow$ d-stable

Table 1: Generalized Lyapunov equations

Theorem 3.2 (Sylvester law of inertia) *[6] Let A be an Hermitian matrix and let Q be a nonsingular matrix. Then $\mathrm{In}_c(A) = \mathrm{In}_c(QAQ^*)$.*

Theorem 3.3 *[25] If X is an Hermitian solution of $XA + A^*X = -G$, where the matrix G is Hermitian, positive definite, then*

$$\pi_-(A) = \pi_+(X), \quad \pi_+(A) = \pi_-(X), \quad \pi_0(A) = \pi_0(X) = 0. \tag{3.4}$$

Conversely, if $\pi_0(A) = 0$, then there exists an Hermitian matrix X such that the matrix $G = -XA - A^*X$ is Hermitian, positive definite and the c-inertia identities (3.4) hold.

Theorem 3.5 *[7] Let G be an Hermitian, positive semidefinite matrix and let X be an Hermitian solution of the Lyapunov equation $XA + A^*X = -G$.*
1. *If $\pi_0(A) = 0$, then $\pi_-(X) \leq \pi_+(A)$ and $\pi_+(X) \leq \pi_-(A)$.*
2. *If $\pi_0(X) = 0$, then $\pi_+(A) \leq \pi_-(X)$ and $\pi_-(A) \leq \pi_+(X)$.*

We define now the c-inertia of a regular matrix pencil $\lambda E - A$, where E may be singular.

Definition 3.6 *The c-inertia of a regular matrix pencil $\lambda E - A$* is defined by the quadruple of integers

$$\mathrm{In}_c(E, A) = \{\, \pi_-(E, A),\ \pi_+(E, A),\ \pi_0(E, A),\ \pi_\infty(E, A) \,\},$$

where $\pi_-(E, A)$, $\pi_+(E, A)$ and $\pi_0(E, A)$ denote the numbers of the finite eigenvalues of $\lambda E - A$ counted with their algebraic multiplicities with negative, positive and zero real part, respectively, and $\pi_\infty(E, A)$ denotes the number of infinite eigenvalues of $\lambda E - A$.

Clearly, $\pi_-(E, A) + \pi_+(E, A) + \pi_0(E, A) + \pi_\infty(E, A) = n$ is the size of E and A. If E is nonsingular, then $\pi_\infty(E, A) = 0$. A c-stable matrix pencil $\lambda E - A$ has the c-inertia $\mathrm{In}_c(E, A) = \{\, m, 0, 0, n - m \,\}$, where m is the numbers of finite eigenvalues of $\lambda E - A$ counting their multiplicities.

The following theorems give connections between the c-inertia of the matrix pencil $\lambda E - A$ and the c-inertia of the Hermitian solution X of the the constrained GCALE (2.72).

Theorem 3.7 *Let $\lambda E - A$ be a regular pencil. If there exist an Hermitian matrix X which satisfies the constrained GCALE (2.72) with Hermitian, positive definite G, then*

$$\pi_-(E, A) = \pi_+(X), \quad \pi_+(E, A) = \pi_-(X), \quad \pi_0(E, A) = 0, \quad \pi_\infty(E, A) = \pi_0(X). \tag{3.8}$$

Conversely, if $\pi_0(E, A) = 0$, then there exists an Hermitian matrix X and an Hermitian, positive definite matrix G such that the GCALE in (2.72) is fulfilled and the c-inertia identities (3.8) hold.

PROOF. Since the Hermitian solution X of the constrained GCALE (2.72) has the form

$$X = W^{-*}\begin{pmatrix} X_{11} & 0 \\ 0 & 0 \end{pmatrix} W^{-1}, \tag{3.9}$$

where the Hermitian matrix X_{11} satisfies the Lyapunov equation (2.70) with the Hermitian, positive definite matrix T_{11}, it follows from Theorems 3.2 and 3.3 that

$$\begin{aligned}
&\pi_-(E,A) = \pi_-(J) = \pi_+(X_{11}) = \pi_+(X),\\
&\pi_+(E,A) = \pi_+(J) = \pi_-(X_{11}) = \pi_-(X),\\
&\pi_0(E,A) = \pi_0(J) = \pi_0(X_{11}) = 0,\\
&\pi_\infty(E,A) = \pi_\infty(E,A) + \pi_0(X_{11}) = \pi_0(X).
\end{aligned}$$

Assume now that $\pi_0(E,A) = 0$. Then $\pi_0(J) = 0$ and by Theorem 3.3 there exists an Hermitian matrix X_{11} such that $T_{11} = -(X_{11}J + J^*X_{11})$ is Hermitian, positive definite and

$$\pi_-(J) = \pi_+(X_{11}), \qquad \pi_+(J) = \pi_-(X_{11}), \qquad \pi_0(J) = \pi_0(X_{11}) = 0.$$

In this case the Hermitian matrices

$$X = W^{-*}\begin{pmatrix} X_{11} & 0 \\ 0 & 0 \end{pmatrix} W^{-1}, \qquad G = T^*\begin{pmatrix} T_{11} & 0 \\ 0 & I \end{pmatrix} T,$$

satisfy the GCALE in (2.72), G is positive definite and the c-inertia identities (3.8) hold.
□

Consider now the case when the matrix G is Hermitian, positive semidefinite. There is a generalization of Theorem 3.5 for matrix pencils.

Theorem 3.10 *Let $\lambda E - A$ be a regular pencil and let X be an Hermitian solution of the constrained GCALE (2.72) with an Hermitian, positive semidefinite matrix G.*

1. *If $\pi_0(E,A) = 0$, then $\pi_-(X) \le \pi_+(E,A)$ and $\pi_+(X) \le \pi_-(E,A)$.*
2. *If $\pi_0(X) = \pi_\infty(E,A)$, then $\pi_+(E,A) \le \pi_-(X)$ and $\pi_-(E,A) \le \pi_+(X)$.*

PROOF. Let the pencil $\lambda E - A$ be in Weierstrass canonical form (1.3). The Hermitian solution X of the constrained GCALE (2.72) has the form (3.9), where the Hermitian matrix X_{11} satisfies the Lyapunov equation (2.70) with Hermitian, positive semidefinite T_{11}.

Since $\pi_0(J) = \pi_0(E,A) = 0$, by Theorem 3.2 and part 1 of Theorem 3.5 we have

$$\pi_-(X) = \pi_-(X_{11}) \le \pi_+(J) = \pi_+(E,A), \qquad \pi_+(X) = \pi_+(X_{11}) \le \pi_-(J) = \pi_-(E,A).$$

Since $\pi_0(X_{11}) = \pi_0(X) - \pi_\infty(E,A) = 0$, it follows from Theorem 3.2 and part 1 of Theorem 3.5 that

$$\pi_-(E,A) = \pi_-(J) \le \pi_+(X_{11}) = \pi_+(X), \qquad \pi_+(E,A) = \pi_+(J) \le \pi_-(X_{11}) = \pi_-(X).$$

□

As an immediate consequence of Theorem 3.10 we obtain a generalization of Theorem 3.7 for the case that G is Hermitian, positive semidefinite.

Theorem 3.11 *Let $\lambda E - A$ be regular and let G be Hermitian, positive semidefinite. Assume that $\pi_0(E, A) = 0$. If there exists an Hermitian matrix X which satisfies the constrained GCALE (2.72) and $\pi_0(X) = \pi_\infty(E, A)$, then the c-inertia identities (3.8) hold.*

3.2 Controllability and observability

Similar to the matrix case [27, 33], the c-inertia identities (3.8) for Hermitian, positive semidefinite G can be also derived using controllability and observability conditions for the linear continuous-time descriptor system

$$\begin{array}{rcl} E\dot{x}(t) & = & Ax(t) + Bu(t), \quad x(t) = x_0, \\ y(t) & = & Cx(t), \end{array} \tag{3.12}$$

where E, $A \in \mathbb{F}^{n,n}$, $B \in \mathbb{F}^{n,q}$, $C \in \mathbb{F}^{p,n}$, $x(t) \in \mathbb{F}^n$ is the state, $u(t) \in \mathbb{F}^q$ is the control input and $y(t) \in \mathbb{F}^p$ is the output. We will assume that the pencil $\lambda E - A$ is regular, $\mathit{rank}\, B = q \le n$ and $\mathit{rank}\, C = p \le n$.

For descriptor systems there are various concepts of controllability and observability [4, 5, 9, 34]. We consider the following conditions

$$\begin{array}{lll} \mathbf{C0}: & \mathit{rank}\,[\,\alpha E - \beta A,\ B] = n & \text{for all } (\alpha, \beta) \in \mathbb{C}^2 \backslash \{(0,0)\}; \\ \mathbf{C1}: & \mathit{rank}\,[\,\lambda E - A,\ B] = n & \text{for all finite } \lambda \in \mathbb{C}; \\ \mathbf{C2}: & \mathit{rank}\,[\,E,\ AK_E,\ B] = n, & \text{where the columns of } K_E \text{ span } \ker E\ . \end{array} \tag{3.13}$$

The descriptor system (3.12) is called *completely controllable* (*C-controllable*) or, equivalently, the triplet (E, A, B) is *C-controllable* if condition **C0** holds. System (3.12) and the triplet (E, A, B) are called *controllable on the reachable set* (*R-controllable*) if **C1** is satisfied. System (3.12) and the triplet (E, A, B) that satisfy **C2** are called *controllable at infinity* (*I-controllable*). If conditions **C1** and **C2** together hold, then the descriptor system (3.12) and the triplet (E, A, B) are called *strongly controllable* (*S-controllable*).

Note that **C0** implies **C1** and **C2**. Moreover, condition **C0** holds if and only if condition **C1** together with the condition

$$\mathit{rank}[\,E,\ B] = n \tag{3.14}$$

is fulfilled. Clearly, **C2** is weaker than (3.14).

Observability is a dual property of controllability. We consider the following conditions

$$\begin{array}{lll} \mathbf{O0}: & \mathit{rank} \begin{bmatrix} \alpha E - \beta A \\ C \end{bmatrix} = n & \text{for all } (\alpha, \beta) \in \mathbb{C}^2 \backslash \{(0,0)\}; \\ \mathbf{O1}: & \mathit{rank} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = n & \text{for all finite } \lambda \in \mathbb{C}; \\ \mathbf{O2}: & \mathit{rank} \begin{bmatrix} E \\ K_{E^*}^* A \\ C \end{bmatrix} = n, & \text{where the columns of } K_{E^*} \text{ span } \ker E^*\ . \end{array} \tag{3.15}$$

The continuous-time descriptor system (3.12) and the triplet (E, A, B) are called *completely observable* (*C-observable*) if the condition **O0** holds. System (3.12) and the triplet (E, A, B)

are called *observable on the reachable set* (*R-observable*) if the condition **O1** holds and are called *observable at infinity* (*I-observable*) if **O2** holds. If conditions **O1** and **O2** together are satisfied, then the descriptor system (3.12) and the triplet (E, A, B) are called *strongly observable* (*S-observable*).

Condition **O0** implies **O1** and **O2**. Moreover, the conditions **O1** and

$$\operatorname{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n \tag{3.16}$$

together are equivalent to condition **O0**.

Consider the constrained GCALE

$$E^* X A + A^* X E = -P_r^* C^* C P_r, \qquad X = X P_l \tag{3.17}$$

and its dual equation

$$E X A^* + A X E^* = -P_l B B^* P_l^*, \qquad X = P_r X. \tag{3.18}$$

The following theorem shows that in the case of an Hermitian, positive definite matrix $G = C^* C$, the conditions $\pi_0(E, A) = 0$ and $\pi_\infty(E, A) = \pi_0(X)$ in Theorem 3.11 may be replaced by the condition for the triplet (E, A, C) to be R-observable.

Theorem 3.19 *Consider system (3.12) with a regular pencil $\lambda E - A$. If there exists an Hermitian matrix X satisfying the constrained GCALE (3.17) and if the triplet (E, A, C) is R-observable, then the c-inertia identities (3.8) hold.*

Proof. Let $\lambda E - A$ be in Weierstrass canonical form (1.3) and let

$$C T^{-1} = [\, C_1 \;\; C_2 \,]$$

be partitioned in blocks conformally to E and A. Then the Hermitian solution of the constrained GCALE (3.17) has the form (3.9), where X_{11} satisfies the Lyapunov equation

$$X_{11} J + J^* X_{11} = -C_1^* C_1. \tag{3.20}$$

Moreover, we have from the R-observability condition that

$$n = \operatorname{rank} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = \operatorname{rank} \begin{bmatrix} W^{-1}(\lambda E - A) T^{-1} \\ C T^{-1} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \lambda I - J & 0 \\ 0 & \lambda N - I \\ C_1 & C_2 \end{bmatrix}. \tag{3.21}$$

Since $\lambda N - I$ is nonsingular for all $\lambda \in \mathbb{C}$, then the matrix

$$\begin{bmatrix} \lambda I - J \\ C_1 \end{bmatrix}$$

has full column rank. Then the solution X_{11} of (3.20) is nonsingular and the matrix J has no eigenvalues on the imaginary axis, e.g., [27]. Hence, $\pi_0(X) = \pi_0(X_{11}) + \pi_\infty(E, A) = \pi_\infty(E, A)$. The remaining relations in (3.8) immediately follow from Theorem 3.11. □

By duality of controllability and observability conditions we have the following result.

Theorem 3.22 *Consider system (3.12) with a regular pencil $\lambda E - A$. If there exists an Hermitian matrix X satisfying the constrained GCALE (3.18) and if the triplet (E, A, C) is R-controllable, then the c-inertia identities (3.8) hold.*

The following theorem gives connections between c-stability of the pencil $\lambda E - A$, the R-observability of the triplet (E, A, C) and the existence of an Hermitian solution of the constrained GCALE (3.17).

Theorem 3.23 *Consider the statements*

1. *the pencil $\lambda E - A$ is c-stable,*
2. *the triplet (E, A, C) is R-observable,*
3. *the constrained GCALE (3.17) has a unique solution X which is Hermitian, positive definite on the subspace $\operatorname{im} P_l$.*

Any two of these statements together imply the third.

PROOF. '2 and 3 $\Rightarrow$ 1'. Since the solution X of (3.17) is positive definite on $im\, P_l$ and $X = XP_l$, it follows that X is positive semidefinite. It follows from the R-observability of (E, A, C) by Theorem 3.19 that $\pi_+(E, A) = \pi_-(X) = 0$ and $\pi_0(E, A) = 0$, i.e., all finite eigenvalues of the pencil $\lambda E - A$ lie in the open left half-plane.

'1 and 3 $\Rightarrow$ 2'. Let the pencil $\lambda E - A$ be c-stable and let X be the solution of (3.17) that is positive definite on $im\, P_l$. Suppose that (E, A, C) is not R-observable. Then there exists $\lambda_0 \in \mathbb{C}$ with $\Re e\, \lambda_0 \neq 0$ and a vector $z \neq 0$ such that

$$\begin{bmatrix} \lambda_0 E - A \\ C \end{bmatrix} z = 0.$$

We obtain that z is the eigenvector of the pencil $\lambda E - A$ corresponding to the finite eigenvalue λ_0 and, hence, $z \in im\, P_r$. Moreover, we have $Cz = 0$. On the other hand, it follows from the Lyapunov equation in (3.17) that

$$-\|Cz\|^2 = -z^* P_r^* C^* C P_r z = z^*(E^*XA + A^*XE)z = 2(\Re e\, \lambda_0) z^* E^* X E z. \tag{3.24}$$

Since X is positive definite on the subspace $im\, P_l$, the matrix E^*XE is positive definite on the subspace $im\, P_r$. Then from (3.24) we obtain that $Cz \neq 0$. Thus, the triplet (E, A, C) is R-observable.

'1 and 2 $\Rightarrow$ 3'. Assume that the pencil $\lambda E - A$ be c-stable and the triplet (E, A, C) is R-observable. Then by Theorem 2.67 the constrained GCALE (3.17) has an Hermitian solution X for every matrix C. This solution is given by (3.9). In this case by Theorem 3.19 we obtain that $\pi_-(X_{11}) = \pi_-(X) = \pi_+(E, A) = 0$, i.e., X is positive definite on the subspace $im\, P_l$. □

Theorem 3.23 generalizes the stability theorems (see Theorems 2.66, 2.67) to the case that $G = C^*C$ is Hermitian, positive semidefinite. We see, that weakening the assumption for G to be positive semidefinite requires the additional R-observability condition. Moreover, Theorem 3.23 gives necessary and sufficient conditions for the triplet (E, A, C) to be R-observable.

By duality an analogous result can be proved for the constrained GCALE (3.18).

Theorem 3.25 *Consider the statements*

1. *the pencil $\lambda E - A$ is c-stable,*
2. *the triplet (E, A, B) is R-controllable,*
3. *the constrained GCALE (3.18) has a unique solution X that is Hermitian, positive definite on $\operatorname{im} P_r^*$.*

Any two of these statements together imply the third.

3.3 Inertia with respect to the unit circle

We recall that the *inertia of a matrix A with respect to the unit circle* (*d-inertia*) is defined by the triplet of integers

$$In_d(A) = \{ \pi_{<1}(A),\ \pi_{>1}(A),\ \pi_1(A) \},$$

where $\pi_{<1}(E, A)$, $\pi_{>1}(E, A)$ and $\pi_1(E, A)$ denote the numbers of the eigenvalues of A counted with their algebraic multiplicities inside, outside and on the unit circle, respectively.

Before extending the d-inertia for matrix pencils, it should be noted that in some problems it is necessary to distinguish the finite eigenvalues of a matrix pencil of modulus larger that 1 and the infinite eigenvalues although the latter also lie outside the unit circle. For example, the presence of infinite eigenvalues of $\lambda E - A$, in contrast to the finite eigenvalues outside the unit circle, does not affect the behaviour at infinity of solutions of the discrete-time descriptor system (2.29).

Definition 3.26 *The inertia of a regular matrix pencil $\lambda E - A$ with respect to the unit circle* (*d-inertia*) is defined by the quadruple of integers

$$In_d(E, A) = \{ \pi_{<1}(E, A),\ \pi_{>1}(E, A),\ \pi_1(E, A),\ \pi_\infty(E, A) \},$$

where $\pi_{<1}(E, A)$, $\pi_{>1}(E, A)$ and $\pi_1(E, A)$ denote the numbers of the finite eigenvalues of $\lambda E - A$ counted with their algebraic multiplicities inside, outside and on the unit circle, respectively, and $\pi_\infty(E, A)$ denotes the number of infinite eigenvalues of $\lambda E - A$.

If E is nonsingular, then $\pi_\infty(E, A) = 0$. A d-stable pencil $\lambda E - A$ has the d-inertia $In_d(E, A) = \{ m, 0, 0, n - m \}$, where m is the number of finite eigenvalues of $\lambda E - A$ counting their multiplicities. There is a unit circle analogue of Theorem 3.3.

Theorem 3.27 *[27] There exist an Hermitian, positive definite matrix G and an Hermitian matrix X satisfying the Lyapunov equation $A^*XA - X = -G$ if and only if $\pi_1(A) = 0$. In this case $\pi_{<1}(A) = \pi_+(X)$, $\pi_{>1}(A) = \pi_-(X)$ and $\pi_1(A) = \pi_0(X) = 0$.*

The following theorem generalizes Theorem 3.27 and gives a connection between the d-inertia of the matrix pencil $\lambda E - A$ and the c-inertia of the Hermitian solution of the constrained GDALE

$$\begin{aligned} &A^*XA - E^*XE = -P_r^*GP_r - (I - P_r)^*G(I - P_r), \\ &P_l^*X = XP_l. \end{aligned} \tag{3.28}$$

Theorem 3.29 *Let $\lambda E - A$ be a regular pencil. If there exists an Hermitian matrix X that satisfies the constrained GDALE (3.28) with Hermitian, positive definite G, then*

$$\pi_{<1}(E,A) = \pi_+(X), \quad \pi_{>1}(E,A) + \pi_\infty(E,A) = \pi_-(X), \quad \pi_1(E,A) = \pi_0(X) = 0. \quad (3.30)$$

Conversely, if $\pi_1(E,A) = 0$, then there exist an Hermitian matrix X and an Hermitian, positive definite matrix G such that the GDALE in (3.28) is satisfied and the d-inertia identities (3.30) hold.

Proof. Every Hermitian solution X of (3.28) has the form

$$X = W^{-*} \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} W^{-1},$$

where the Hermitian matrix X_{11} satisfies the Lyapunov equation

$$J^* X_{11} J - X_{11} = -T_{11} \quad (3.31)$$

and the Hermitian matrix X_{22} satisfies the Lyapunov equation

$$X_{22} - N^* X_{22} N = -T_{22}. \quad (3.32)$$

Clearly, equation (3.32) has a unique solution

$$X_{22} = -\sum_{j=0}^{\nu-1} (N^*)^j T_{22} N^j$$

that is negative definite if T_{22} is positive definite.

It follows from Theorems 3.2 and 3.27 that

$$\begin{aligned} &\pi_{<1}(E,A) = \pi_{<1}(J) = \pi_+(X_{11}) = \pi_+(X) - \pi_+(X_{22}) = \pi_+(X), \\ &\pi_{>1}(E,A) = \pi_{>1}(J) = \pi_-(X_{11}) = \pi_-(X) - \pi_-(X_{22}) = \pi_-(X) - \pi_\infty(E,A), \\ &\pi_1(E,A) = \pi_1(J) = \pi_0(X_{11}) = 0. \end{aligned}$$

Moreover, $\pi_0(X) = \pi_0(X_{11}) + \pi_0(X_{22}) = 0$.

Suppose that $\pi_1(E,A) = 0$. Then by Theorem 3.27 there exists Hermitian matrices X_{11}, X_{22} and Hermitian, positive definite matrices T_{11}, T_{22} such that (3.31) and (3.32) are satisfied and

$$\begin{aligned} &\pi_{<1}(J) = \pi_+(X_{11}), \qquad \pi_{>1}(J) = \pi_-(X_{11}), \qquad \pi_1(J) = \pi_0(X_{11}) = 0, \\ &\pi_\infty(E,A) = \pi_-(X_{22}), \qquad \pi_+(X_{22}) = \pi_0(X_{22}) = 0. \end{aligned}$$

Thus, the Hermitian matrices

$$X = W^{-*} \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} W^{-1}, \qquad G = T^* \begin{pmatrix} T_{11} & 0 \\ 0 & T_{22} \end{pmatrix} T$$

satisfy the GDALE in (3.28), G is positive definite and the d-inertia identities (3.30) hold.

□

There are also unit circle analogues of Theorems 3.10 and 3.11 that can be established in the same way.

Theorem 3.33 *Let $\lambda E - A$ be a regular pencil and let X be an Hermitian matrix that satisfy the constrained GDALE (3.28) with Hermitian, positive semidefinite G.*

1. *If $\pi_1(E, A) = 0$, then $\pi_+(X) \le \pi_{<1}(E, A)$ and $\pi_-(X) \le \pi_{>1}(E, A) + \pi_\infty(E, A)$.*
2. *If $\pi_0(X) = 0$, then $\pi_+(X) \ge \pi_{<1}(E, A)$ and $\pi_-(X) \ge \pi_{>1}(E, A) + \pi_\infty(E, A)$.*

Corollary 3.34 *Let $\lambda E - A$ be regular and let G be an Hermitian, positive semidefinite. Assume that $\pi_1(E, A) = 0$. If there exists a nonsingular Hermitian matrix X that satisfies the constrained GDALE (3.28), then the inertia identities (3.30) hold.*

Like the continuous-time case, the inertia identities (3.30) for Hermitian, positive semidefinite G can be obtained from controllability and observability conditions for the linear discrete-time descriptor system

$$\begin{array}{rcl} Ex_{k+1} & = & Ax_k + Bu_k, \quad x_0 = x^0, \\ y_k & = & Cx_k, \end{array} \tag{3.35}$$

where $E,\ A \in \mathbb{F}^{n,n}$, $B \in \mathbb{F}^{n,q}$, $C \in \mathbb{F}^{p,n}$, $x_k \in \mathbb{F}^n$ is the state, $u_k \in \mathbb{F}^q$ is the control input and $y_k \in \mathbb{F}^p$ is the output, see [9].

The discrete-time descriptor system (3.35) is C-controllable (R-controllable, I-controllable, S-controllable) if the triplet (E, A, B) is C-controllable (R-controllable, I-controllable, S-controllable) and (3.35) is C-observable (R-observable, I-observable, S-observable) if the triplet (E, A, C) is C-observable (R-observable, I-observable, S-observable).

Consider the constrained GDALE

$$\begin{array}{l} A^*XA - E^*XE = -P_r^*C^*CP_r - (I - P_r)^*C^*C(I - P_r), \\ P_l^*X = XP_l. \end{array} \tag{3.36}$$

A dual of (3.36) has the form

$$\begin{array}{l} AXA^* - EXE^* = -P_lBB^*P_l^* - (I - P_l)BB^*(I - P_l)^*, \\ P_rX = XP_r^*. \end{array} \tag{3.37}$$

We will show that the condition for the pencil $\lambda E - A$ to have no eigenvalues of modulus 1 and the condition for the solution of (3.36) (or (3.37)) to be nonsingular together are equivalent to the property for the triplet (E, A, C) to be C-observable (or for (E, A, B) to be C-controllable).

Theorem 3.38 *Consider system (3.35) with a regular matrix pencil $\lambda E - A$.*

1. *Let X be an Hermitian solution of the constrained GDALE (3.36). The triplet (E, A, C) is C-observable if and only if $\pi_1(E, A) = 0$ and X is nonsingular.*

2. *Let X be an Hermitian solution of the constrained GDALE (3.37). The triplet (E, A, B) is C-controllable if and only if $\pi_1(E, A) = 0$ and X is nonsingular.*

PROOF. 1. Let the matrix pencil $\lambda E - A$ be in Weierstrass canonical form (1.3) and let the matrix $CT^{-1} = [\, C_1,\, C_2 \,]$ be partitioned conformally to E and A. The solution of the constrained GDALE (3.36) has the form

$$X = W^{-*} \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} W^{-1},$$

where X_{11} is the solution of the Lyapunov equation

$$J^* X_{11} J - X_{11} = -C_1^* C_1 \tag{3.39}$$

and X_{22} is the solution of the Lyapunov equation

$$X_{22} - N^* X_{22} N = -C_2^* C_2. \tag{3.40}$$

Since the triplet (E, A, C) is C-observable, conditions **O1** and (3.16) hold. Taking into account (3.21) we obtain from **O1** that the matrix

$$\begin{bmatrix} \lambda I - J \\ C_1 \end{bmatrix}$$

has full column rank for all $\lambda \in \mathbb{C}$. Then the solution X_{11} of (3.39) is nonsingular and J has no eigenvalues on the unit circle [27, p.453].

From (3.16) we have that

$$n = rank \begin{bmatrix} E \\ C \end{bmatrix} = rank \begin{bmatrix} W^{-1}ET^{-1} \\ CT^{-1} \end{bmatrix} = rank \begin{bmatrix} I & 0 \\ 0 & N \\ C_1 & C_2 \end{bmatrix} = rank \begin{bmatrix} N \\ C_2 \end{bmatrix} + m.$$

and, hence, for all $\lambda \in \mathbb{C}$, the matrix

$$\begin{bmatrix} \lambda I - N \\ C_2 \end{bmatrix}$$

has full column rank for all $\lambda \in \mathbb{C}$. Then the solution X_{22} of (3.40) is nonsingular, since equation (3.40) is a special case of (3.39). Thus, the solution X of the constrained GDALE (3.36) is nonsingular and $\pi_1(E, A) = 0$.

Conversely, let z be a right eigenvector of $\lambda E - A$ corresponding to a finite eigenvalue λ with $|\lambda| \neq 1$. We have

$$-\|Cz\|^2 = -z^* C^* C z = z^*(A^* X A - E^* X E) z = (|\lambda|^2 - 1) z^* E^* X E z.$$

Since X is nonsingular, $Ez \neq 0$ and $\pi_1(E, A) = 0$, then $Cz \neq 0$, i.e., (E, A, C) satisfies the condition **O1** in (3.15).

Let $z \in ker\, E$. Then $-\|Cz\|^2 = -z^* C^* C z = z^* A^* X A z \neq 0$, i.e., (3.16) holds. Thus, the triplet (E, A, C) is C-observable.

Part 2 follows analogously. □

Remark 3.41 It follows from Theorem 3.38 that if $\pi_1(E,A)=0$ and an Hermitian solution X of (3.36) (or (3.37)) is nonsingular, then the triplet (E,A,C) is S-observable (or the triplet (E,A,B) is S-controllable). However, S-observability of (E,A,C) and S-controllability of (E,A,B) do not imply that the solutions of (3.36) and (3.37), respectively, are nonsingular.

Example 3.42 The constrained GDALE (3.37) with

$$E=\begin{pmatrix}1&0\\0&0\end{pmatrix},\qquad A=\begin{pmatrix}2&0\\0&1\end{pmatrix},\qquad B=\begin{pmatrix}1\\0\end{pmatrix}$$

has the solution

$$X=\begin{pmatrix}1/3&0\\0&0\end{pmatrix}$$

which is singular although $rank\,[\,\lambda E-A,\ B\,]=2$ and $rank\,[\,E,\ AK_E,\ B\,]=2$.

As immediate consequence of Corollary 3.34 and Theorem 3.38 we obtain the following results.

Theorem 3.43 *Consider system (3.35) with a regular matrix pencil $\lambda E-A$.*

1. *Let the triplet (E,A,C) be C-observable. If an Hermitian matrix X satisfies the constrained GDALE (3.36), then the inertia identities (3.30) hold.*
2. *Let the triplet (E,A,B) be C-controllable. If an Hermitian matrix X satisfies the constrained GDALE (3.37), then the inertia identities (3.30) hold.*

Moreover, from Theorem 3.38 we have the following necessary and sufficient conditions for the triplet (E,A,C) to be C-observable and for the triplet (E,A,B) to be C-controllable.

Corollary 3.44 *Let $\lambda E-A$ be a regular d-stable pencil.*

1. *The triplet (E,A,C) is C-observable if and only if the constrained GDALE (3.36) has a unique solution X which is Hermitian, positive definite on $\operatorname{im} P_l$ and negative definite on $\ker P_l$.*
2. *The triplet (E,A,B) is C-controllable if and only if the constrained GDALE (3.37) has a unique solution X which is Hermitian, positive definite on $\operatorname{im} P_r^*$ and negative definite on $\ker P_r^*$.*

Similar to Theorem 2.74 it can be shown that the solution of the constrained GDALE (3.36) has the form

$$X=\frac{1}{2\pi}\int_0^{2\pi}(e^{i\varphi}E-A)^{-*}\Big(P_r^*C^*CP_r-(I-P_r)^*C^*C(I-P_r)\Big)(e^{i\varphi}E-A)^{-1}d\varphi \quad (3.45)$$

and the solution of (3.37) is given by

$$X = \frac{1}{2\pi}\int_0^{2\pi}(e^{i\varphi}E - A)^{-1}\Big(P_lBB^*P_l^* - (I-P_l)BB^*(I-P_l)^*\Big)(e^{i\varphi}E - A)^{-*}d\varphi. \quad (3.46)$$

From Theorem 3.43 and Corollary 3.44 we immediately obtain necessary and sufficient conditions for the pencil $\lambda E - A$ to be d-stable.

Corollary 3.47 *Consider system (3.35) with a regular matrix pencil $\lambda E - A$.*

1. *Assume that the triplet (E, A, C) is C-observable. Then the matrix pencil $\lambda E - A$ is d-stable if and only if the constrained GDALE (3.36) has a unique solution which is Hermitian, positive definite on $\operatorname{im} P_l$ and negative definite on $\ker P_l$.*

2. *Assume that the triplet (E, A, B) is C-controllable. Then the matrix pencil $\lambda E - A$ is d-stable if and only if the constrained GDALE (3.36) has a unique solution which is Hermitian, positive definite on $\operatorname{im} P_r^*$ and negative definite on $\ker P_r^*$.*

Remark 3.48 Note that Corollaries 3.44 and 3.47 still hold if we replace the C-observability and C-controllability conditions by the conditions for (E, A, C) and (E, A, B) to be R-observable and R-controllable, respectively, and if we require for solutions of (3.36) and (3.37) to be positive definite on $im\, P_l$ and $im\, P_r^*$, respectively.

Remark 3.49 All results of this subsection can be reformulated for the constrained generalized discrete-time Lyapunov equation

$$\begin{array}{l} A^*XA - E^*XE = -P_r^*GP_r + s(I-P_r)^*G(I-P_r), \\ P_l^*X = XP_l \end{array} \quad (3.50)$$

and its dual

$$\begin{array}{l} AXA^* - EXE^* = -P_lGP_l^* + s(I-P_l)G(I-P_l)^*, \\ P_rX = XP_r^*, \end{array} \quad (3.51)$$

where s is 0 or 1. For these equations we must consider instead of (3.30) the d-inertia identities

$$\pi_{<1}(E,A) + = \pi_+(X), \qquad \pi_{>1}(E,A) = \pi_-(X), \qquad \pi_1(E,A) = 0, \qquad \pi_\infty(E,A) = \pi_0(X)$$

for the case $s = 0$ and

$$\pi_{<1}(E,A) + \pi_\infty(E,A) = \pi_+(X), \qquad \pi_{>1}(E,A) = \pi_-(X), \qquad \pi_1(E,A) = \pi_0(X) = 0$$

for the case $s = 1$.

4 Controllability and observability Gramians

In this section we establish relationships among solutions of constrained generalized Lyapunov equations and the controllability and observability Gramians for descriptor systems induced in [2]. Since the results for the continuous-time case are partly related to the discrete-time case, we begin our discussions with the discrete-time problem.

4.1 The discrete-time case

Let W, T, J and N be as in (1.3). Consider a sequence of matrices

$$F_k = \begin{cases} T^{-1}\begin{pmatrix} J^k & 0 \\ 0 & 0 \end{pmatrix} W^{-1}, & k = 0, 1, 2\dots, \\ T^{-1}\begin{pmatrix} 0 & 0 \\ 0 & -N^{-k-1} \end{pmatrix} W^{-1}, & k = -1, -2, \dots \end{cases} \tag{4.52}$$

that satisfies the difference matrix equation

$$EF_k = AF_{k-1} + \delta_{0,k}I, \tag{4.53}$$

where $\delta_{i,j}$ is the Kronecker delta.

Assume that the pencil $\lambda E - A$ is d-stable. Then there exist matrices

$$G_{dcc} = \sum_{k=0}^{\infty} F_k BB^* F_k^* \qquad \text{and} \qquad G_{dco} = \sum_{k=0}^{\infty} F_k^* C^* C F_k \tag{4.54}$$

that are called the *causal controllability Gramian* and the *causal observability Gramian*, respectively, for the discrete-time descriptor system (3.35) [2]. The matrices

$$G_{dnc} = -\sum_{k=-\nu}^{-1} F_k BB^* F_k^* \qquad \text{and} \qquad G_{dno} = -\sum_{k=-\nu}^{-1} F_k^* C^* C F_k \tag{4.55}$$

are called the *noncausal controllability Gramian* and the *noncausal observability Gramian* for (3.35). In summary, the *controllability Gramian* for the discrete-time descriptor system (3.35) is defined by

$$G_{cd} = G_{dcc} + G_{dnc} \tag{4.56}$$

and the *observability Gramian* for the discrete-time descriptor system (3.35) is defined by

$$G_{od} = G_{dco} + G_{dno}. \tag{4.57}$$

If $E = I$, then the causal controllability and observability Gramians are the usual controllability and observability Gramians for the discrete-time state-space system [35].

The following lemma gives integral representations for the controllability and observability Gramians for the descriptor system (3.35).

Lemma 4.58 *Consider system (3.35). Let the pencil $\lambda E - A$ be d-stable.*

1. *The controllability Gramian for the discrete-time descriptor system (3.35) can be represented as*

$$G_{cd} = -\frac{1}{2\pi}\int_0^{2\pi} (e^{i\varphi}E - A)^{-1}\Big(P_l BB^* P_l^* - (I - P_l)BB^*(I - P_l)^*\Big)(e^{i\varphi}E - A)^{-*} d\varphi. \tag{4.59}$$

2. The observability Gramian for the discrete-time descriptor system (3.35) can be represented as

$$G_{od} = -\frac{1}{2\pi}\int_0^{2\pi} (e^{i\varphi}E - A)^{-*}\Big(P_r^*C^*CP_r - (I-P_r)^*C^*C(I-P_r)\Big)(e^{i\varphi}E - A)^{-1}d\varphi. \tag{4.60}$$

Proof. Since all finite eigenvalues of $\lambda E - A$ lie inside the unit circle, the sequence $\|F_k\|$ is uniformly bounded for all integers k. Then the Fourier series

$$\sum_{k=-\infty}^{\infty} F_k e^{ik\varphi}$$

converges [29]. Using (4.53) we have

$$(E - e^{i\varphi}A)\sum_{k=-\infty}^{\infty} F_k e^{ik\varphi} = \sum_{k=-\infty}^{\infty}(EF_k - AF_{k-1})e^{ik\varphi} = I$$

and, hence,

$$(E - e^{i\varphi}A)^{-1} = \sum_{k=-\infty}^{\infty} F_k e^{ik\varphi} = \sum_{k=-\nu}^{\infty} F_k e^{ik\varphi} \tag{4.61}$$

is the Fourier expansion of the matrix-valued function $(E-e^{i\varphi}A)^{-1}$. It immediately follows from the Parseval identity [29] that

$$\begin{aligned}
G_{dcc} &= \sum_{k=0}^{\infty} F_k BB^*F_k^* = \sum_{k=-\infty}^{\infty} F_kP_lBB^*P_l^*F_k^* = \\
&= \frac{1}{2\pi}\int_0^{2\pi}(E - e^{i\varphi}A)^{-1}P_lBB^*P_l^*(E - e^{i\varphi}A)^{-*}d\varphi = \\
&= \frac{1}{2\pi}\int_0^{2\pi}(e^{i\varphi}E - A)^{-1}P_lBB^*P_l^*(e^{i\varphi}E - A)^{-*}d\varphi, \qquad (4.62)\\
G_{dnc} &= -\sum_{k=-\nu}^{-1} F_k BB^*F_k^* = -\sum_{k=-\infty}^{\infty} F_k(I-P_l)BB^*(I-P_l)^*F_k^* = \\
&= -\frac{1}{2\pi}\int_0^{2\pi}(e^{i\varphi}E - A)^{-1}(I-P_l)BB^*(I-P_l)^*(e^{i\varphi}E - A)^{-*}d\varphi. \qquad (4.63)
\end{aligned}$$

Then

$$\begin{aligned}
G_{cd} &= G_{dcc} + G_{dnc} = \\
&= \frac{1}{2\pi}\int_0^{2\pi}(e^{i\varphi}E - A)^{-1}\Big(P_lBB^*P_l^* - (I-P_l)BB^*(I-P_l)^*\Big)(e^{i\varphi}E - A)^{-*}d\varphi.
\end{aligned}$$

The integral representation (4.60) for G_{od} can be proved analogously. □

If we compare the controllability Gramian in (4.59) with the solution of the constrained GDALE (3.37) which is given by (3.46), then from Corollary 3.44 and Remark 3.48 we obtain the following result.

Corollary 4.64 *Consider the discrete-time descriptor system (3.35) and let the pencil $\lambda E - A$ be d-stable.*

1. *The causal controllability Gramian G_{dcc} for (3.35) exists and is a unique Hermitian solution of the constrained GDALE*

$$AXA^* - EXE^* = -P_l BB^* P_l^*, \qquad X = P_r X.$$

Moreover, G_{dcc} is positive definite on the subspace $\operatorname{im} P_r^*$ if and only if the triplet (E, A, B) is R-controllable.

2. *The noncausal controllability Gramian G_{dnc} for (3.35) is a unique Hermitian solution of the constrained GDALE*

$$AXA^* - EXE^* = -(I - P_l)BB^*(I - P_l)^*, \qquad X = (I - P_r)X.$$

Moreover, G_{dnc} is negative definite on $\ker P_r^*$ if and only if (3.14) holds.

3. *The controllability Gramian G_{cd} for (3.35) exists and is a unique solution of the constrained GDALE (3.37). Moreover, G_{cd} is positive definite on $\operatorname{im} P_r^*$ and negative definite on $\ker P_r^*$ if and only if the triplet (E, A, B) is C-controllable.*

For the observability Gramians we have analogously the following results.

Corollary 4.65 *Consider the discrete-time descriptor system (3.35) and let the pencil $\lambda E - A$ be d-stable.*

1. *The causal observability Gramian G_{dco} for (3.35) exists and is a unique Hermitian solution of the constrained GDALE*

$$A^*XA - E^*XE = -P_r^* C^* C P_r, \qquad X = XP_l. \tag{4.66}$$

Moreover, G_{dco} is positive definite on the subspace $\operatorname{im} P_l$ if and only if the triplet (E, A, C) is R-observable.

2. *The noncausal observability Gramian G_{dno} for (3.35) is a unique Hermitian solution of the constrained GDALE*

$$A^*XA - E^*XE = -(I - P_r)^* C^* C(I - P_r), \qquad X = X(I - P_l). \tag{4.67}$$

Moreover, G_{dno} is negative definite on $\ker P_l$ if and only if (3.16) holds.

3. *The observability Gramian G_{od} for (3.35) exists and is a unique solution of the constrained GDALE (3.36). Moreover, G_{od} is positive definite on $\operatorname{im} P_l$ and negative definite on $\ker P_l$ if and only if the triplet (E, A, C) is C-observable.*

4.2 The continuous-time case

We now define the controllability and observability Gramians for the continuous-time descriptor system (3.12), see [2]. Assume that the pencil $\lambda E - A$ is c-stable. Then there exist the integrals

$$G_{cpc} = \int_0^\infty F_0 e^{tAF_0} BB^* e^{tF_0^* A^*} F_0^* dt \tag{4.68}$$

and

$$G_{cpo} = \int_0^\infty F_0^* e^{tA^* F_0^*} C^* C e^{tF_0 A} F_0 dt \tag{4.69}$$

that are called the *proper controllability Gramian* and the *proper observability Gramian*, respectively, for the continuous-time descriptor system (3.12). The *nonproper controllability Gramian* and the *nonproper observability Gramian* for (3.12) are defined by

$$G_{cnc} = -\sum_{k=-\nu}^{-1} F_k BB^* F_k^* \qquad \text{and} \qquad G_{cno} = -\sum_{k=-\nu}^{-1} F_k^* C^* C F_k, \tag{4.70}$$

respectively. Here F_k are as in (4.52). The *controllability Gramian* for the continuous-time descriptor system (3.12) is given by

$$G_{cc} = G_{cpc} + G_{cnc} \tag{4.71}$$

and the *observability Gramian* for the continuous-time descriptor system (3.12) is given by

$$G_{oc} = G_{cpo} + G_{cno}. \tag{4.72}$$

In the case $E = I$ the proper controllability and observability Gramians are the usual controllability and observability Gramians for the continuous-time state-space system [35].

It follows from (4.55) and (4.70) that the noncausal and nonproper controllability (observability) Gramians for the discrete-time descriptor system (3.35) and the continuous-time descriptor system (3.12) coincide. Therefore, in the sequel we are concerned only with the proper controllability and observability Gramians for (3.12).

Similarly to the classical state-space case [14], we define a fundamental solution matrix $\mathcal{F}(t)$ for the continuous-time descriptor system (3.12) as a unique solution of the initial value problem

$$E\mathcal{F}'(t) = A\mathcal{F}(t), \qquad \mathcal{F}(0) = P_r. \tag{4.73}$$

It was shown in [31] that there exists a unique fundamental solution matrix $\mathcal{F}(t)$ which has the form

$$\mathcal{F}(t) = T^{-1} \begin{pmatrix} e^{tJ} & 0 \\ 0 & 0 \end{pmatrix} T. \tag{4.74}$$

The following lemma gives integral representations for G_{cpc} and G_{cpo} in terms of the fundamental solution matrix $\mathcal{F}(t)$ and the generalized resolvent $(\lambda E - A)^{-1}$.

Lemma 4.75 *Consider the continuous-time descriptor system (3.12). Let the matrix pencil $\lambda E - A$ be c-stable.*

1. *The controllability Gramian of (3.12) can be represented as*

$$\begin{aligned} G_{cpc} &= \int_0^\infty \mathcal{F}(t)F_0BB^*F_0^*\mathcal{F}^*(t)dt = &(4.76)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi E - A)^{-1}P_lBB^*P_l^*(i\xi E - A)^{-*}d\xi. &(4.77)\end{aligned}$$

2. *The observability Gramian of (3.12) can be represented as*

$$\begin{aligned} G_{cpo} &= \int_0^\infty F_0^*\mathcal{F}^*(t)C^*C\mathcal{F}(t)F_0dt = &(4.78)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi E - A)^{-*}P_r^*C^*CP_r(i\xi E - A)^{-1}d\xi. &(4.79)\end{aligned}$$

PROOF. Using (1.3) and (4.74) we obtain that

$$F_0e^{tAF_0} = e^{tF_0A}F_0 = T^{-1}\begin{pmatrix} e^{tJ} & 0 \\ 0 & 0 \end{pmatrix}W^{-1} = \mathcal{F}(t)F_0. \tag{4.80}$$

Thus, the integral representations (4.76) and (4.78) hold.

Since all finite eigenvalues of the pencil $\lambda E - A$ lie in the open left half-plane, $(i\xi I - J)^{-1}$ exists for all $\xi \in \mathbb{R}$. Then, using the integral representation for the matrix exponential

$$e^{tJ} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi t}(i\xi I - J)^{-1}d\xi$$

see, e.g., [14], we obtain from (4.53) and (4.80) that

$$\begin{aligned} \mathcal{F}(t)F_0 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} T^{-1}\begin{pmatrix} e^{i\xi t}(i\xi I - J)^{-1} & 0 \\ 0 & 0 \end{pmatrix}W^{-1}d\xi = \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi t}P_r(i\xi E - A)^{-1}d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi t}(i\xi E - A)^{-1}P_ld\xi.\end{aligned}$$

Thus, the entries of the matrices $P_r(i\xi E - A)^{-1}$ and $(i\xi E - A)^{-1}P_l$ are the Fourier transformations of the entries of $\mathcal{F}(t)F_0$. Then the integrals (4.77) and (4.79) immediately follow from the Parseval identity [29]. □

If we compare the integrals (4.77) and (4.79) with the solutions of the GCALEs (3.18) and (3.17), respectively, then from Theorem 3.23 and Corollary 3.25 we obtain the following result.

Corollary 4.81 *Consider the the continuous-time descriptor system (3.12). Let the pencil $\lambda E - A$ be c-stable.*

1. *The proper controllability Gramian G_{cpc} of (3.12) exists and is a unique Hermitian solution of the constrained GCALE (3.18). Moreover, G_{cpc} is positive definite on $\mathrm{im}\, P_r^*$ if and only if the triplet (E, A, B) is R-controllable.*

2. *The proper observability Gramian G_{cpo} of (3.12) exists and is a unique Hermitian solution of the constrained GCALE (3.17). Moreover, G_{cpo} is positive definite on $\mathrm{im}\, P_l$ if and only if the triplet (E, A, C) is R-observable.*

Remark 4.82 Corollaries 4.64, 4.65 and 4.81 imply the following conditions.

1. The controllability Gramian G_{cc} of (3.12) is positive definite on $im\, P_r^*$ and negative definite on $ker\, P_r^*$ if and only if the pencil $\lambda E - A$ is c-stable and the triplet (E, A, B) is C-controllable.

2. The observability Gramian G_{oc} of (3.12) is positive definite on $im\, P_l$ and negative definite on $ker\, P_l$ if and only if the pencil $\lambda E - A$ is c-stable and the triplet (E, A, C) is C-observable.

Since the proper controllability (observability) Gramian of (3.12) is defined via the constrained generalized continuous-time Lyapunov equation and the nonproper controllability (observability) Gramian of (3.12) is defined via the constrained generalized discrete-time Lyapunov equation, it is impossible, in contrast to the discrete-time case (see, part 3 of Corollary 4.65), to write one generalized Lyapunov equation for the controllability (observability) Gramian in the continuous-time case.

5 Conclusions

We have studied generalized continuous-time and discrete-time Lyapunov equations and presented generalizations of Lyapunov stability theorems and matrix inertia theorems for matrix pencils. We also have shown that the stability, controllability and observability properties of descriptor systems can be characterized in terms of solutions of generalized Lyapunov equations with special right-hand sides.

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