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Canonical forms for doubly structured matrices and pencils

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Abstract

In this paper we derive canonical forms under structure preserving equivalence transformations for matrices and matrix pencils that have a multiple structure, which is either an H-selfadjoint or H-skew-adjoint structure, where the matrix H is a complex nonsingular Hermitian or skew-Hermitian matrix. Matrices and pencils of such multiple structures arise for example in quantum chemistry in Hartree-Fock models or random phase approximation.

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1 Introduction

Canonical forms for matrices and matrix pencils have been studied for more than hundred years since work of Jordan, Kronecker and Weierstraß, see [5]. In recent years, motivated

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from applications in control theory as well as quantum physics and quantum chemistry, there is a revived interest in such canonical forms for matrices and pencils that have algebraic structures, like Lie groups or Lie algebras. While the possible invariants have been characterized already some time ago [2], the emphasis in the new results lies in structure preserving equivalence transformations, see e.g., [1, 13, 14, 15, 16].

In this paper we derive canonical forms under structure preserving equivalence transformations for matrices and matrix pencils with multiple structure.

Definition 1 Let $H \in \mathbb{C}^{n\times n}$ be a nonsingular Hermitian or skew-Hermitian matrix, and let $X \in \mathbb{C}^{n\times n}$.

1. X is called H-selfadjoint if $X^*H = HX$.
2. X is called H-skew-adjoint if $X^*H = -HX$.

Canonical forms for pairs (A, H), where H is Hermitian or skew-Hermitian nonsingular and A is H-selfadjoint or H-skew-adjoint are well-known in literature (see, e.g., [7, 11]). These forms are obtained under transformations of the form

$$(A, H) \mapsto (P^{-1}AP,\ P^*HP),$$

where P is nonsingular. Here, we are interested in canonical forms for matrix triples (A, H, G), where G and H are Hermitian or skew-Hermitian nonsingular and A is doubly structured with respect to G and H, i.e., A is H-selfadjoint or H-skew-adjoint and at the same time G-selfadjoint or G-skew-adjoint. We are also interested in the pencil case, i.e., we will also consider pencils $\varrho A - B$, where both A and B are doubly structured with respect to H and G.

The main motivation for our interest in these types of matrices and pencils arises from quantum chemistry. Response function models lead to the problem of solving the generalized eigenvalue problem with a matrix pencil of the form

$$\lambda\mathcal{E}_0 - \mathcal{A}_0 := \lambda \begin{bmatrix} C & Z \\ -Z & -C \end{bmatrix} - \begin{bmatrix} E & F \\ F & E \end{bmatrix}, \tag{1}$$

where $E, F, C, Z \in \mathbb{C}^{n\times n}, E = E^*, F = F^*, C = C^*, Z = -Z^*$, see [8, 17]. Furthermore, there are important special cases in which the pencil has even further structure. For example, the simplest response function model is the time-dependent Hartree-Fock model, also called the random phase approximation (RPA). In this case, C is the identity and Z is the zero matrix, see [8, 17]. Thus, the generalized eigenvalue problem (1) reduces to the problem of finding the eigenvalues of the matrix

$$\mathcal{L}_0 = \begin{bmatrix} E & F \\ -F & -E \end{bmatrix}, \tag{2}$$

where E, F are as in (1). For stable Hartree-Fock ground state wave functions, it is furthermore known that $E - F$ and $E + F$ are positive definite, see [8].

In other applications, however, like in multiconfigurational RPA [8], it is not even guaranteed that the matrix $\mathcal{E}_0$ in (1) is nonsingular.

It is easy to see that the matrices $\mathcal{E}_0$, $\mathcal{A}_0$ in (1) and $\mathcal{L}_0$ in (2) are doubly structured. With

$$G = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}, \quad H = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

we have that $\mathcal{E}_0$ is I-selfadjoint and H-skew-adjoint, $\mathcal{A}_0$ is I-selfadjoint and H-selfadjoint, while $\mathcal{L}_0$ is G-selfadjoint and J-skew-adjoint.

When designing structure preserving numerical methods for large scale structures eigenvalue problems sometimes difficulties in the convergence of the methods were observed in [3, 4] that have to do with the invariants of these pencils under structure preserving equivalence transformations, see also [1]. It is another motivation for our work to derive canonical forms that allow a better understanding of those properties of the pencils that lead to these difficulties.

We will derive the canonical form for matrix triples (A, H, G) under structure preserving transformations of the form

$$(A, H, G) \mapsto (P^{-1}AP,\ P^*HP,\ P^*GP),$$

where P is nonsingular. This preserves the (skew-)Hermitian structure of H and G and also the structure of A with respect to H and G. Based on the classical results, see Section 2, we clearly have canonical forms for (A, H), (A, G) or the pencil $\varrho H - G$, and hence the invariants of the pairs (A, H) and (A, G), as well as the invariants of the pencil $\varrho H - G$ under congruence are invariants of the triple (A, H, G).

It is our goal to obtain a canonical form that displays simultaneously the Jordan structure of A and the invariants of the canonical forms of (A, H) and (A, G). In general this is a very difficult problem, such a form may not even exist. Consider the following example.

Example 2 Consider matrices

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Then A is G-selfadjoint and H-selfadjoint. But it is impossible to simultaneously decompose A, H, and G further into smaller block diagonal forms. This follows from the obvious fact that the pencil $\varrho G - H$ can not be further decomposed. On the other hand, A has the Jordan canonical form

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, both (A, G) and (A, H) are decomposable into smaller blocks (see Theorems 9 and 11 below).

Due to this difficulty, we restrict ourselves to an important special case. In most applications, the matrices H and G, that induce the structure, are contained in the set

$$\mathcal{S} = \left\{ I_n, \quad \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}, \quad \begin{bmatrix} I_m & 0 \\ 0 & -I_{n-m} \end{bmatrix}, \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}, m, n \in \mathbb{N} \right\}. \tag{3}$$

If this is the case then the pencil $\varrho H - G$ is nondefective.

Definition 3 Let $\varrho A - B \in \mathbb{C}^{n \times n}$ be a matrix pencil. We say that $\varrho A - B$ is nondefective, if there exists nonsingular matrices $P, Q \in \mathbb{C}^{n \times n}$, such that both PAQ and PBQ are diagonal.

We will show that if G, H are Hermitian nonsingular, such that the pencil $\varrho H - G$ is nondefective, then a canonical form for the triple (A, H, G) exists, which is also unique except for the permutation of blocks. In particular, this canonical form includes the Jordan structure of A, and also the canonical forms of the pairs (A, H) and (A, G) and the pencil $\varrho H - G$ can easily be read off.

The paper is organized as follows. After providing some preliminary results in Section 2, we review canonical forms for matrices that are structured with respect to only one Hermitian matrix in Section 3. In Section 4 we then discuss doubly structured matrices and in Section 5 we discuss canonical forms for structured pencils of the form $\lambda\mathcal{A} - \mathcal{B}$, where both $\mathcal{A}$ and $\mathcal{B}$ are singly or doubly structured matrices.

2 Preliminaries

Throughout the paper, we use the following notation:

By $\sigma(A)$ we denote the spectrum of the matrix A. $\mathcal{J}_p(\lambda)$ denotes $p \times p$ upper triangular Jordan block with eigenvalue λ. By $\operatorname{sign}(t)$ we mean the sign of a real number $t \in \mathbb{R}\backslash\{0\}$. $A = A_1 \oplus \ldots \oplus A_m$ stands for the block diagonal matrix A with diagonal blocks $A_1, \ldots, A_m$.

Furthermore, we introduce the following $p \times p$ matrices.

$$Z_p := \begin{bmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{bmatrix}, \quad D_p := \begin{bmatrix} (-1)^2 & & 0 \\ & \ddots & \\ 0 & & (-1)^{p+1} \end{bmatrix},$$

$$\text{and} \quad F_p := \begin{bmatrix} 0 & & (-1)^2 \\ & \iddots & \\ (-1)^{p+1} & & 0 \end{bmatrix}$$

Note that $F_p \in \mathbb{R}^{p \times p}$ is symmetric if p is odd and skew-symmetric if p is even, whereas Z_p and D_p are symmetric for all p. We list some properties of these matrices and the matrix $\mathcal{J}_p(0)$ which can be easily verified, and will be used in the following.

Lemma 4 *Let $p \in \mathbb{N}$.*

1. $Z_p^2 = I_p, \quad D_p^2 = I_p, \quad F_p^2 = (-1)^{p+1} I_p.$
2. $F_p Z_p = D_p = (-1)^{p+1} Z_p F_p, \quad D_p F_p = Z_p = (-1)^{p+1} F_p D_p.$
3. $D_p Z_p = F_p = (-1)^{p+1} Z_p D_p, \quad F_p Z_p F_p = Z_p.$
4. $Z_p^{-1} \mathcal{J}_p(0) Z_p = \mathcal{J}_p(0)^*.$
5. $D_p^{-1} \mathcal{J}_p(0) D_p = -\mathcal{J}_p(0).$
6. $F_p^{-1} \mathcal{J}_p(0) F_p = -\mathcal{J}_p(0)^*.$

Another important result that will frequently be used throughout the paper is the following well-known result, [5].

Lemma 5 *Let A, B, X be square matrices, such that the spectra of A and B are disjoint. If $AX = XB$, then $X = 0$.*

Finally, we review the canonical forms for regular Hermitian pencils, i.e., pencils $\varrho H - G$, where both H and G are Hermitian and $\det(\varrho H - G) \not\equiv 0$. This result goes back to results from Weierstraß (see [19]) and Kronecker (see [10]).

Theorem 6 *Let $\varrho H - G$ be a regular Hermitian pencil. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n \times n}$ such that*

$$P^*(\varrho H - G)P = (\varrho H_1 - G_1) \oplus \ldots \oplus (\varrho H_l - G_l), \tag{4}$$

where the blocks $\varrho H_j - G_j$ have one and only one of the following forms:

1. *Blocks associated with paired nonreal eigenvalues λ, $\overline{\lambda}$, where $Im(\lambda) > 0$:*

$$\varrho H_j - G_j = \varrho \begin{bmatrix} 0 & I_r \\ I_r & 0 \end{bmatrix} - \begin{bmatrix} 0 & \mathcal{J}_r(\lambda) \\ \mathcal{J}_r(\lambda)^* & 0 \end{bmatrix}. \tag{5}$$

2. Blocks associated with real eigenvalues λ and sign $\varepsilon \in \{1,-1\}$:

$$\varrho H_j - G_j = \varrho\varepsilon Z_r - \varepsilon Z_r \mathcal{J}_r(\lambda) = \varrho\varepsilon \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix} - \varepsilon \begin{bmatrix} 0 & & & \lambda \\ & & \lambda & 1 \\ & \iddots & \iddots & \\ \lambda & 1 & & 0 \end{bmatrix}. \quad (6)$$

3. Blocks associated with the eigenvalue ∞ and sign $\varepsilon \in \{1,-1\}$:

$$\varrho H_j - G_j = \varrho\varepsilon Z_r \mathcal{J}_r(0) - \varepsilon Z_r = \varrho\varepsilon \begin{bmatrix} 0 & & & 0 \\ & & 0 & 1 \\ & \iddots & \iddots & \\ 0 & 1 & & 0 \end{bmatrix} - \varepsilon \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}. \quad (7)$$

Moreover, the decomposition (4) is unique up to a block permutation that exchanges blocks $\rho H_i - G_i$.

Proof. For a full proof, see [18], Lemmas 1.–4. There, the result is shown without the additional condition $\mathrm{Im}(\lambda) > 0$ for the blocks associated with nonreal eigenvalues, but applying a permutation, we may always place the block that is associated with the eigenvalue λ in the $(1,2)$-block position of the form in (5). □

If $\varrho H - G$ is nondefective then we immediately have the following corollary.

Corollary 7 *Let $\varrho H - G$ be a nondefective Hermitian pencil, where both H and G are nonsingular. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$P^*(\varrho H - G)P = (\varrho H_1 - G_1) \oplus \ldots \oplus (\varrho H_l - G_l),$$

where the spectra of $\varrho H_j - G_j$ and $\varrho H_l - G_l$ are disjoint for $j \neq l$, and where each block $\varrho H_j - G_j$ has either only one pair of complex conjugate eigenvalues or only one single real eigenvalue. Moreover, the block $\varrho H_j - G_j$ has one and only one of the following forms.

1. Blocks with nonreal eigenvalues λ, $\overline{\lambda}$, where $Im\lambda > 0$ and $r \in \mathbb{N}$:

$$\varrho H_j - G_j = \varrho \begin{bmatrix} 0 & I_r \\ I_r & 0 \end{bmatrix} - \begin{bmatrix} 0 & \lambda I_r \\ \overline{\lambda} I_r & 0 \end{bmatrix}.$$

2. Blocks with real eigenvalue λ, where $r, p \in \mathbb{N}$:

$$\varrho H_j - G_j = \begin{bmatrix} I_p & 0 \\ 0 & -I_{r-p} \end{bmatrix} - \lambda \begin{bmatrix} I_p & 0 \\ 0 & -I_{r-p} \end{bmatrix}.$$

Moreover the decomposition is unique up to a block permutation.

Proof. Since the size of every Jordan blocks equals one, the result follows directly from Theorem 6 after proper permutations such that equal eigenvalues are combined in a single block. ◻

Remark 8 Following from Theorem 6 and Corollary 7 it is obvious that if $\lambda \in \mathbb{C}\backslash\mathbb{R}$ is an eigenvalue of $\varrho H - G$ then so is $\overline{\lambda}$ and both eigenvalues have the same Jordan structures.

3 Singly structured matrices

In this section, we review the well-known canonical forms for H-selfadjoint matrices and H-skew-adjoint matrices, where H always denotes a complex nonsingular Hermitian $n \times n$ matrix.

Theorem 9 *Let $A \in \mathbb{C}^{n\times n}$ be H-selfadjoint. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$, such that*

$$P^{-1}AP = A_1 \oplus \ldots \oplus A_k \quad \text{and} \quad P^*HP = H_1 \oplus \ldots \oplus H_k, \tag{8}$$

where A_j and H_j are of the same size and the pair (A_j, H_j) has one and only one of the following forms:

1. *Blocks associated with real eigenvalues:*

$$A_j = \mathcal{J}_p(\lambda) \quad \text{and} \quad H_j = \varepsilon Z_p, \tag{9}$$

where $\lambda \in \mathbb{R}$, $p \in \mathbb{N}$, and $\varepsilon \in \{1, -1\}$.

2. *Blocks associated with a pair of nonreal eigenvalues:*

$$A_j = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 \\ 0 & \mathcal{J}_p(\overline{\lambda}) \end{bmatrix} \quad \text{and} \quad H_j = \begin{bmatrix} 0 & Z_p \\ Z_p & 0 \end{bmatrix}, \tag{10}$$

where $\lambda \in \mathbb{C}\backslash\mathbb{R}$ with $Im(\lambda) > 0$ and $p \in \mathbb{N}$.

Moreover, the form $(P^{-1}AP, P^*HP)$ of (A, H) is uniquely determined up to the permutation of blocks.

Proof. See, e.g., [7]. ◻

Even though (8) is unique only up to a permutation of blocks, we call it the **a canonical form** of the pair (A, H).

Remark 10 In some instances it will turn out be useful to use a slightly different form for the blocks of type (10) in (8). Multiplying the matrices from both sides by $I_p \oplus Z_p$, one finds that (10) takes the form

$$A_j = \begin{bmatrix} \mathcal{J}_p(\lambda_0) & 0 \\ 0 & \mathcal{J}_p(\lambda_0)^* \end{bmatrix} \quad \text{and} \quad H_j = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}. \tag{11}$$

Using the same transformation, we can also get back from the form (11) to the form (10). This transformation will frequently be used in the following and its application will be called the **Z-trick**.

Apart from the eigenvalues of an H-selfadjoint matrix A, the parameters ε that are associated with blocks to real eigenvalues are invariants of the pair (A, H). The collection of these parameters is sometimes referred to as the **sign characteristic** (see, e.g., [7] and [11]). To highlight that these parameters are related to the matrix H (we will soon have to deal with two structures), we will use the term **H-structure indices** in the following.

Theorem 11 *Let $S \in \mathbb{C}^{n\times n}$ be H-skew-adjoint. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$, such that*

$$P^{-1}SP = S_1 \oplus \ldots \oplus S_k \quad \textit{and} \quad P^*HP = H_1 \oplus \ldots \oplus H_k, \tag{12}$$

where S_j and H_j are of the same size and each pair (S_j, H_j) has one and only one of the following forms:

1. *Blocks associated with purely imaginary eigenvalues:*
$$S_j = i\mathcal{J}_p(\lambda) \quad \textit{and} \quad H_j = \varepsilon Z_p, \tag{13}$$
where $\lambda \in \mathbb{R}$, $p \in \mathbb{N}$, and $\varepsilon \in \{1, -1\}$.

2. *Blocks associated with a pair of non purely imaginary eigenvalues:*
$$S_j = \begin{bmatrix} i\mathcal{J}_p(\lambda) & 0 \\ 0 & i\mathcal{J}_p(\overline{\lambda}) \end{bmatrix} \quad \textit{and} \quad H = \begin{bmatrix} 0 & Z_p \\ Z_p & 0 \end{bmatrix}, \tag{14}$$
where $\lambda \in \mathbb{C}\backslash\mathbb{R}$ with $Im(\lambda) > 0$ and $p \in \mathbb{N}$.

Moreover, the form $(P^{-1}SP, P^*HP)$ of (S, H) is uniquely determined up to a permutation of blocks.

Proof. This follows directly from Theorem 9 considering the H-selfadjoint matrix iS. $\square$

Again, we will call the parameter ε in (13) the H-structure index of the block S_j in (13). Moreover, the form (12) will be called the canonical form of the pair (S, H).

Remark 12 From Theorems 9 and 11, it is easy to find the following symmetries in the spectra of H-selfadjoint and H-skew-adjoint matrices. If $\lambda \notin \mathbb{R}$ is an eigenvalue of the H-selfadjoint matrix A then so is $\overline{\lambda}$ and both eigenvalues have the same Jordan structure. If $\lambda \notin i\mathbb{R}$ is an eigenvalue of the H-skew-adjoint matrix A then so is $-\overline{\lambda}$ and both eigenvalues have the same Jordan structure.

4 Doubly structured matrices

In this section we give canonical forms for matrices that are doubly structured with respect to Hermitian or skew-Hermitian nonsingular matrices G and H. First, we note that by Theorem 9, Jordan blocks associated with real eigenvalues in the selfadjoint case (or purely imaginary eigenvalues in the skew-adjoint case) have structure indices with respect to G and/or H. We will call these indices the G- and H-structure indices of A, respectively.

Moreover, we may always assume that G and H are Hermitian. Otherwise, we may consider iG or iH, respectively, having in mind the following remark.

Remark 13 Let $H \in \mathbb{C}^{n\times n}$ be nonsingular and Hermitian or skew-Hermitian and let $A \in \mathbb{C}^{n\times n}$. Then the following conditions hold.

1. A is H-selfadjoint if and only if A is iH-selfadjoint.
2. A is H-skew-adjoint if and only if A is iH-skew-adjoint.
3. A is H-selfadjoint if and only if iA is H-skew-adjoint.

Remark 13 implies in particular that we may assume that the structure on A induced by one of the matrices G and H, say H, is the structure of a selfadjoint matrix. In other words, we may assume that A is H-selfadjoint. Otherwise, we may consider iA. Hence, it remains to discuss the following cases.

- Matrices that are H-selfadjoint and G-selfadjoint (Section 4.1)
- Matrices that are H-selfadjoint and G-skew-adjoint (Section 4.2)

Finally, we always assume that the pencil $\varrho H - G$ is nondefective.

Remark 14 Instead of requiring that $\varrho H - G$ is nondefective, we may as well consider the generalization of this case, that for the matrices A, G, H at least one of the three pencils $\varrho H - G$, $\varrho H - HA$ and $\varrho G - GA$ is nondefective. For example, if A is nonsingular and both H- and G-selfadjoint then we can consider the matrix triple $(H^{-1}G, H, HA)$ for which H, HA are Hermitian and $H^{-1}G$ is H and HA selfadjoint, since $(H^{-1}G)^* = GH^{-1}$. Thus, if $\varrho H - HA$ is nondefective then we can get the canonical form of this new triple. But once we have this, we can easily get the canonical form of the original triple (A, H, G). So our results will cover more general cases.

4.1 Matrices that are H-selfadjoint and G-selfadjoint

In this section we will derive a canonical form for matrices that are selfadjoint with respect to nonsingular Hermitian matrices H and G, such that the pencil $\varrho H - G$ is nondefective. For the proof of our main result, the following lemma will be needed.

Lemma 15 *Let $G, H \in \mathbb{C}^{n\times n}$ be Hermitian and nonsingular. Let $A \in \mathbb{C}^{n\times n}$ be H-selfadjoint and G-selfadjoint. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$\begin{aligned} P^{-1}AP &= A_1 \oplus \ldots \oplus A_k, \\ P^*HP &= H_1 \oplus \ldots \oplus H_k, \\ P^*GP &= G_1 \oplus \ldots \oplus G_k, \end{aligned}$$

where A_j, H_j and G_j have corresponding sizes. Moreover, each pencil $\varrho H_j - G_j$ has as spectrum either $\{\gamma_j, \overline{\gamma}_j\}$ for some $\gamma_j \in \mathbb{C}\backslash\mathbb{R}$ or $\{\gamma_j\}$ for some $\gamma_j \in \mathbb{R}$ and the spectra of two subpencils $\varrho H_j - G_j$ and $\varrho H_l - G_l$, $j \neq l$, are disjoint.

Proof. By Theorem 6, there exists a nonsingular matrix $Q \in \mathbb{C}^{n\times n}$ such that

$$Q^*GQ = \begin{bmatrix} G_1 & 0 \\ 0 & \tilde{G}_2 \end{bmatrix}, \quad Q^*HQ = \begin{bmatrix} H_1 & 0 \\ 0 & \tilde{H}_2 \end{bmatrix}, \quad \text{and} \quad Q^{-1}AQ = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where the pencil $\varrho H_1 - G_1$ has as spectrum either $\{\gamma_1, \overline{\gamma}_1\}$ for some $\gamma_1 \in \mathbb{C}\backslash\mathbb{R}$ or $\{\gamma_1\}$ for some $\gamma_1 \in \mathbb{R}$ and such that the spectra of the pencils $\varrho H_1 - G_1$ and $\varrho \tilde{H}_2 - \tilde{G}_2$ are disjoint. Since A is H-selfadjoint and G-selfadjoint, we obtain that

$$\begin{aligned} \begin{bmatrix} A_{11}^* H_1 & A_{21}^* \tilde{H}_2 \\ A_{12}^* H_1 & A_{22}^* \tilde{H}_2 \end{bmatrix} &= \begin{bmatrix} H_1 A_{11} & H_1 A_{12} \\ \tilde{H}_2 A_{21} & \tilde{H}_2 A_{22} \end{bmatrix} \\ \text{and} \quad \begin{bmatrix} A_{11}^* G_1 & A_{21}^* \tilde{G}_2 \\ A_{12}^* G_1 & A_{22}^* \tilde{G}_2 \end{bmatrix} &= \begin{bmatrix} G_1 A_{11} & G_1 A_{12} \\ \tilde{G}_2 A_{21} & \tilde{G}_2 A_{22} \end{bmatrix}. \end{aligned}$$

Since with G also $\tilde{G}_2$ is nonsingular, this implies

$$A_{21}^* \tilde{H}_2 \tilde{G}_2^{-1} = H_1 A_{12} \tilde{G}_2^{-1} = H_1 G_1^{-1} G_1 A_{12} \tilde{G}_2^{-1} = H_1 G_1^{-1} A_{21}^*.$$

Since the pencils $\varrho H_1 - G_1$ and $\varrho \tilde{H}_2 - \tilde{G}_2$ have disjoint spectra, we obtain that $A_{21}^* = 0$ and therefore $A_{12} = H_1^{-1} A_{21}^* \tilde{H}_2 = 0$. The rest of the proof now follows by induction. □

Theorem 16 *Let $G, H \in \mathbb{C}^{n\times n}$ be Hermitian and nonsingular such that the pencil $\varrho H - G$ is nondefective. Let $A \in \mathbb{C}^{n\times n}$ be H-selfadjoint and G-selfadjoint. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$\begin{aligned} P^{-1}AP &= A_1 \oplus \ldots \oplus A_k \\ P^*GP &= G_1 \oplus \ldots \oplus G_k \qquad (15) \\ P^*HP &= H_1 \oplus \ldots \oplus H_k, \end{aligned}$$

where the blocks A_j, G_j, H_j have corresponding sizes and are of one and only one of the following forms.

Type (1):

$$A_j = \mathcal{J}_p(\lambda), \quad H_j = \varepsilon Z_p, \quad \text{and} \quad G_j = \varepsilon\gamma Z_p,$$

where $\lambda \in \mathbb{R}$, $p \in \mathbb{N}$, $\varepsilon \in \{1, -1\}$, and $\gamma \in \mathbb{R}\backslash\{0\}$. The H-structure index of A_j is ε and the G-structure index of A_j is $\operatorname{sign}(\varepsilon\gamma)$.

Type (2):

$$A_j = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 \\ 0 & \mathcal{J}_p(\lambda) \end{bmatrix}, \quad H_j = \begin{bmatrix} 0 & Z_p \\ Z_p & 0 \end{bmatrix}, \quad \text{and} \quad G_j = \begin{bmatrix} 0 & \gamma Z_p \\ \overline{\gamma} Z_p & 0 \end{bmatrix},$$

where $\lambda \in \mathbb{R}$, $p \in \mathbb{N}$, and $\gamma \in \mathbb{C}$, $Im(\gamma) > 0$. The H-structure indices of A_j are $1, -1$ and the G-structure indices of A_j are $1, -1$.

Type (3):

$$A_j = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 \\ 0 & \mathcal{J}_p(\overline{\lambda}) \end{bmatrix}, \quad H_j = \begin{bmatrix} 0 & Z_p \\ Z_p & 0 \end{bmatrix}, \quad \text{and} \quad G_j = \begin{bmatrix} 0 & \gamma Z_p \\ \overline{\gamma} Z_p & 0 \end{bmatrix},$$

where $\lambda \in \mathbb{C}\backslash\mathbb{R}$, $p \in \mathbb{N}$, and $\gamma \in \mathbb{C}\backslash\{0\}$, where $Im(\gamma) \geq 0$.

Moreover, the canonical form (15) is unique up to permutation of blocks.

Proof. By Lemma 15 we may assume that the pencil $\varrho H - G$ has the eigenvalues either $\gamma, \overline{\gamma}$ for some $\gamma \in \mathbb{C}\backslash\mathbb{R}$ or γ for some $\gamma \in \mathbb{R}$.

Case 1: $\gamma \in \mathbb{R}$.

Since the pencil $\varrho H - G$ is nondefective, by Corollary 7 there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that

$$P^*(\varrho H - G)P = \varrho \begin{bmatrix} I_m & 0 \\ 0 & -I_{n-m} \end{bmatrix} - \gamma \begin{bmatrix} I_m & 0 \\ 0 & -I_{n-m} \end{bmatrix},$$

i.e., in particular that G is a scalar multiple of H. Applying Theorem 9, we find that there exists a nonsingular matrix $Q \in \mathbb{C}^{n\times n}$ such that $(Q^{-1}AQ, Q^*HQ)$ is in canonical form (8). Since $G = \gamma H$, we obtain that A, H, and G can be reduced simultaneously to block diagonal form with diagonal blocks of types 1 and 3.

Case 2: $\gamma, \overline{\gamma} \in \mathbb{C}\backslash\mathbb{R}$.

In this case, we obtain from Corollary 7 that there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that

$$P^*(\varrho H - G)P = \varrho \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} - \begin{bmatrix} 0 & \gamma I_m \\ \overline{\gamma} I_m & 0 \end{bmatrix},$$

where $2m = n$ and $\mathrm{Im}(\gamma) > 0$. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

be partitioned conformably. Then we obtain from $A^*H = HA$ and $A^*G = GA$ that

$$A_{12}^* = A_{12} \quad \text{and} \quad \gamma A_{12}^* = \overline{\gamma} A_{12}.$$

Since $\gamma \neq \overline{\gamma}$, this implies that $A_{12} = 0$. In an analogous way we show that $A_{21} = 0$, and moreover, we have $A_{22} = A_{11}^*$ by symmetry. Let Q_1 be such that $Q_1^{-1}A_{11}Q_1$ is in Jordan canonical form and set

$$Q = P \begin{bmatrix} Q_1 & 0 \\ 0 & Q_1^{-*} \end{bmatrix}.$$

Then we obtain

$$Q^{-1}AQ = \begin{bmatrix} Q_1^{-1}A_{11}Q_1 & 0 \\ 0 & Q_1^*A_{11}^*Q_1^{-*} \end{bmatrix},$$

$$Q^*HQ = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}, \quad \text{and} \quad Q^*GQ = \begin{bmatrix} 0 & \gamma I_m \\ \overline{\gamma} I_m & 0 \end{bmatrix}.$$

After a proper block permutation, we obtain that A, H, and G can be reduced simultaneously to block diagonal form with diagonal blocks of the forms

$$\tilde{A} = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 \\ 0 & \mathcal{J}_p(\lambda)^* \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}, \quad \text{and} \quad \tilde{G} = \begin{bmatrix} 0 & \gamma I_p \\ \overline{\gamma} I_p & 0 \end{bmatrix},$$

respectively, where $p \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. The result then follows by applying the Z-trick, see Remark 10.

Uniqueness: Suppose that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}, \quad G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \quad \text{and}$$

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 & 0 \\ 0 & \tilde{A}_2 \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} \tilde{H}_1 & 0 \\ 0 & \tilde{H}_2 \end{bmatrix}, \quad \tilde{G} = \begin{bmatrix} \tilde{G}_1 & 0 \\ 0 & \tilde{G}_2 \end{bmatrix},$$

are in canonical form, where H, G, $\tilde{H}$, $\tilde{G}$ are Hermitian nonsingular, A is H-selfadjoint and G-selfadjoint and $\tilde{A}$ is $\tilde{H}$-selfadjoint and $\tilde{G}$-selfadjoint and all matrices have corresponding block structures. If $P^{-1}AP = \tilde{A}$, $\sigma(A_1) = \sigma(\tilde{A}_1)$ and $\sigma(A_2) = \sigma(\tilde{A}_2)$, such that the spectra of A_1 and A_2 are disjoint, then it follows immediately that P has a corresponding block diagonal structure. Analogously, assuming that the spectra of $\varrho H_1 - G_1$ and $\varrho\tilde{H}_2 - \tilde{G}_2$ (and of $\varrho H_2 - G_2$ and $\varrho\tilde{H}_1 - \tilde{G}_1$, respectively) are disjoint and that $P^*HP = \tilde{H}$ and $P^*GP = \tilde{G}$, where P is nonsingular, we obtain again that P has a corresponding block diagonal structure. Indeed, partitioning

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \quad \text{and} \quad P^{-*} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

conformably with H, we obtain that

$$G_{11}P_{12} = Q_{12}\tilde{G}_{22} \quad \text{and} \quad H_{11}P_{12} = Q_{12}\tilde{H}_{22}.$$

This implies that

$$H_{11}^{-1}G_{11}P_{12} = P_{12}\tilde{H}_{22}^{-1}\tilde{G}_{22},$$

and from that, we obtain $P_{12} = 0$, since the spectra of $H_{11}^{-1}G_{11}$ and $\tilde{H}_{22}^{-1}\tilde{G}_{22}$ are disjoint. Analogously, we show $P_{21} = 0$.

Hence, it is sufficient to prove the uniqueness for the case that A has only one pair of eigenvalues $\lambda, \overline{\lambda}$ and that $\varrho G - H$ has only a pair of eigenvalues $\gamma, \overline{\gamma}$. But then, the uniqueness is clear, since we obtain from Theorem 9 the uniqueness of the canonical form for the pair (A, H). Note that the structure G is then uniquely defined by the invariant γ with $Im(\gamma) \geq 0$.

In both cases it is easy to verify that the H and G-structure indices of each block are as claimed in the theorem. □

4.2 Matrices that are H-selfadjoint and G-skew-adjoint

In this section we present a canonical form for a matrix A that is H-selfadjoint and G-skew-adjoint, where H and G are Hermitian nonsingular matrices, such that the pencil $\varrho H - G$ is nondefective. By Remark 12, the eigenvalues of A satisfy more symmetry properties. If $\lambda \in \mathbb{C}$ is an eigenvalue of A then, because A is G-skew-adjoint, so is $-\overline{\lambda}$ having the same Jordan structure as λ. On the other hand, A is H-selfadjoint and thus, with λ and $-\overline{\lambda}$ also $\overline{\lambda}$ and $-\lambda$ are eigenvalues of A having the same Jordan structures as λ. Thus, the eigenvalues of A occur in quadruples $\{\lambda, \overline{\lambda}, -\overline{\lambda}, -\lambda\}$, where all these eigenvalues have the same Jordan structures. If λ is real or purely imaginary, this set is equal to $\{\lambda, -\lambda\}$, and if $\lambda = 0$, this set is just $\{0\}$.

The following lemmas will be needed for constructing the canonical form.

Lemma 17 *Let $G, H \in \mathbb{C}^{n\times n}$ be Hermitian and nonsingular. Furthermore, let $A \in \mathbb{C}^{n\times n}$ be H-selfadjoint and G-skew-adjoint. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$\begin{aligned} P^{-1}AP &= A_1 \oplus \ldots \oplus A_k, \\ P^*HP &= H_1 \oplus \ldots \oplus H_k, \\ P^*GP &= G_1 \oplus \ldots \oplus G_k, \end{aligned}$$

where A_j, H_j and G_j have corresponding sizes. Moreover, each matrix A_j has the spectrum $\{\lambda_j, \overline{\lambda}_j, -\lambda_j, -\overline{\lambda}_j\}$ and the spectra of two matrices A_j and A_l, where $j \neq l$, are disjoint.

Proof. By using the eigenvalue properties of A mentioned above, one can find a matrix $Q \in \mathbb{C}^{n\times n}$ such that

$$Q^*GQ = \begin{bmatrix} G_{11} & G_{12} \\ G_{12}^* & G_{22} \end{bmatrix}, \; Q^*HQ = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix}, \quad \text{and} \quad Q^{-1}AQ = \begin{bmatrix} A_1 & 0 \\ 0 & \tilde{A}_2 \end{bmatrix},$$

where A_1 has the spectrum $\{\lambda_1, \overline{\lambda}_1, -\lambda_1, -\overline{\lambda}_1\}$ for some $\lambda_1 \in \mathbb{C}$, such that the spectra of A_1 and $\tilde{A}_2$ are disjoint. Then we obtain from $A^*H = HA$ and $A^*G = GA$ that

$$A_1^*H_{12} = H_{12}\tilde{A}_2 \quad \text{and} \quad -A_1^*G_{12} = G_{12}\tilde{A}_2$$

By construction, the spectra of $\pm A_1^*$ and $\tilde{A}_2$ are disjoint. This implies $H_{12} = 0$ and $G_{12} = 0$. The proof then follows by induction. □

Lemma 18 *Let $G, H \in \mathbb{C}^{n\times n}$ be Hermitian and nonsingular. Furthermore, let $A \in \mathbb{C}^{n\times n}$ be H-selfadjoint and G-skew-adjoint. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$\begin{aligned} P^{-1}AP &= A_1 \oplus \ldots \oplus A_k, \\ P^*HP &= H_1 \oplus \ldots \oplus H_k, \\ P^*GP &= G_1 \oplus \ldots \oplus G_k, \end{aligned}$$

where A_j, H_j and G_j have corresponding sizes. The spectrum of each pencil $\varrho H_j - G_j$ is contained in $\{\gamma_j, -\gamma_j, \overline{\gamma}_j, -\overline{\gamma}_j\}$ for some $\gamma_j \in \mathbb{C}$ and the spectrum of $\varrho H_l - G_l$ is disjoint from the set $\{\gamma_j, -\gamma_j, \overline{\gamma}_j, -\overline{\gamma}_j\}$ if $j \neq l$.

Proof. The proof proceeds analogously to the proof of Lemma 15 using the equations $A^*H = HA$ and $-A^*G = GA$. □

Note that in contrast to the eigenvalue of A, the eigenvalues of the pencil $\varrho H - G$ need not occur in quadruples $\{\gamma_j, -\gamma_j, \overline{\gamma}_j, -\overline{\gamma}_j\}$. If γ_j is an eigenvalue of $\varrho H - G$ then from Theorem 6, we only know that also $\overline{\gamma}_j$ is an eigenvalue, but $-\gamma_j$ and $-\overline{\gamma}_j$ need not be. However, to get corresponding block diagonal forms of A, G, H, we have to group γ_j and $\overline{\gamma}_j$ together with $-\gamma_j$ and $-\overline{\gamma}_j$ if they are also eigenvalues of $\varrho H - G$.

In view of Lemma 18, it is sufficient to consider pencils $\varrho H - G$ whose spectrum is contained in $\{\gamma, -\gamma, \overline{\gamma}, -\overline{\gamma}\}$. Therefore, a discussion of properties of such pencils will be helpful.

Lemma 19 *Let G, $H \in \mathbb{C}^{n\times n}$ be nonsingular and Hermitian such that the pencil $\varrho H - G$ is nondefective.*

(i) If the spectrum of $\varrho H - G$ is contained in $\{\gamma, -\gamma\}$, where $\gamma^2 \in \mathbb{R}\backslash\{0\}$, then

$$H^{-1}GH^{-1}G = \gamma^2 I_n.$$

(ii) If the spectrum of $\varrho H - G$ is contained in $\{\gamma, -\gamma, \overline{\gamma}, -\overline{\gamma}\}$, where $\gamma^2 \in \mathbb{C}\backslash\mathbb{R}$, then there exists a matrix P such that for $\tilde{H} = P^*HP$, $\tilde{G} = P^*GP$ and $\tilde{A} = P^{-1}AP$,

$$\tilde{H}^{-1}\tilde{G}\tilde{H}^{-1}\tilde{G} = \begin{bmatrix} \overline{\gamma}^2 I_m & 0 \\ 0 & \gamma^2 I_m \end{bmatrix}. \tag{16}$$

Moreover,

$$\tilde{A} = \begin{bmatrix} A_1 & 0 \\ 0 & A_1^* \end{bmatrix}.$$

Proof. (i) We consider the problem in two cases.

Case (1): $\mathrm{Im}(\gamma) = 0$.

Since the pencil $\varrho H - G$ is nondefective and has only the eigenvalues $\gamma, -\gamma \in \mathbb{R}$, by Corollary 7 there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ and numbers $p, q, r, s \in \mathbb{N}$ such that

$$H = P^* \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & -I_q & 0 & 0 \\ 0 & 0 & I_r & 0 \\ 0 & 0 & 0 & -I_s \end{bmatrix} P \quad \text{and} \quad G = P^* \begin{bmatrix} \gamma I_p & 0 & 0 & 0 \\ 0 & -\gamma I_q & 0 & 0 \\ 0 & 0 & -\gamma I_r & 0 \\ 0 & 0 & 0 & \gamma I_s \end{bmatrix} P.$$

This implies $H^{-1}GH^{-1}G = P^{-1}(\gamma^2 I_n)P = \gamma^2 I_n$.

Case (2): $\mathrm{Re}(\gamma) = 0$.

Since the pencil $\varrho H - G$ is nondefective and has only the eigenvalues $\gamma, -\gamma \in i\mathbb{R}$, by Corollary 7 there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that

$$H = P^* \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} P \quad \text{and} \quad G = P^* \begin{bmatrix} 0 & \gamma I_m \\ -\gamma I_m & 0 \end{bmatrix} P,$$

where $m = \frac{n}{2} \in \mathbb{N}$. This implies $H^{-1}GH^{-1}G = P^{-1}(\gamma^2 I_n)P = \gamma^2 I_n$.

(ii) By Corollary 7 there exists a nonsingular matrix P such that

$$\varrho\tilde{H} - \tilde{G} = \varrho P^*HP - P^*GP = \varrho \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} - \begin{bmatrix} 0 & \gamma\Sigma \\ \overline{\gamma}\Sigma & 0 \end{bmatrix},$$

where $m = \frac{n}{2} \in \mathbb{N}$ and $\Sigma = I_p \oplus (-I_{m-p})$, $0 \le p \le m$. We then obtain that

$$\tilde{H}^{-1}\tilde{G} = \begin{bmatrix} \overline{\gamma}\Sigma & 0 \\ 0 & \gamma\Sigma \end{bmatrix}, \tag{17}$$

and hence we have (16). Note that $\tilde{A}$ is $\tilde{H}$-selfadjoint and $\tilde{G}$-skew-adjoint. This implies that

$$\tilde{A}(\tilde{H}^{-1}\tilde{G}) = \tilde{H}^{-1}\tilde{A}^*\tilde{G} = -(\tilde{H}^{-1}\tilde{G})\tilde{A}.$$

Since in this case $\gamma \pm \overline{\gamma} \neq 0$, from the block form (17) we get $\tilde{A} = A_1 \oplus A_2$. Since $\tilde{A}$ is $\tilde{H}$-selfadjoint, we obtain that $A_2 = A_1^*$. $\square$

Theorem 20 *Let $G, H \in \mathbb{C}^{n\times n}$ be Hermitian nonsingular such that the pencil $\varrho H - G$ is nondefective. Furthermore, let $A \in \mathbb{C}^{n\times n}$ be H-selfadjoint and G-skew-adjoint. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$\begin{aligned} P^{-1}AP &= A_1 \oplus \ldots \oplus A_k, \\ P^*GP &= G_1 \oplus \ldots \oplus G_k, \\ P^*HP &= H_1 \oplus \ldots \oplus H_k, \end{aligned} \tag{18}$$

where, for each j, the blocks A_j, G_j, H_j have corresponding sizes and are of one and only one of the following forms.

Type (1a):

$$A_j = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 & 0 & 0 \\ 0 & -\mathcal{J}_p(\lambda) & 0 & 0 \\ 0 & 0 & \mathcal{J}_p(\overline{\lambda}) & 0 \\ 0 & 0 & 0 & -\mathcal{J}_p(\overline{\lambda}) \end{bmatrix}, \tag{19}$$

$$H_j = \begin{bmatrix} 0 & 0 & Z_p & 0 \\ 0 & 0 & 0 & Z_p \\ Z_p & 0 & 0 & 0 \\ 0 & Z_p & 0 & 0 \end{bmatrix}, \quad \text{and} \quad G_j = \begin{bmatrix} 0 & 0 & 0 & \gamma Z_p \\ 0 & 0 & \gamma Z_p & 0 \\ 0 & \overline{\gamma} Z_p & 0 & 0 \\ \overline{\gamma} Z_p & 0 & 0 & 0 \end{bmatrix}, \tag{20}$$

where $\lambda \in \mathbb{C}$ with $Re(\lambda)Im(\lambda) > 0$, $p \in \mathbb{N}$ and $\gamma^2 \in \mathbb{R}\backslash\{0\}$, $Re(\gamma), Im(\gamma) \geq 0$.

Type (1b):

$$A_j = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 \\ 0 & \mathcal{J}_p(-\lambda) \end{bmatrix}, \quad H_j = \varepsilon \begin{bmatrix} Z_m & 0 \\ 0 & (\frac{\gamma}{|\gamma|})^2 Z_m \end{bmatrix}, \quad G_j = \begin{bmatrix} 0 & \gamma Z_m \\ \overline{\gamma} Z_m & 0 \end{bmatrix}, \tag{21}$$

where $\lambda > 0$, $p \in \mathbb{N}$ and $\gamma^2 \in \mathbb{R}\backslash\{0\}$, $Re(\gamma), Im(\gamma) \geq 0$. The H_j-structure index of λ is ε and the H_j-structure index of $-\lambda$ is $\varepsilon(\frac{\gamma}{|\gamma|})^2$.

Type (1c):

$$A_j = i \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 \\ 0 & \mathcal{J}_p(-\lambda) \end{bmatrix}, \quad H_j = \begin{bmatrix} 0 & Z_m \\ Z_m & 0 \end{bmatrix}, \quad G_j = \varepsilon|\gamma| \begin{bmatrix} Z_m & 0 \\ 0 & (\frac{|\gamma|}{\gamma})^2 Z_m \end{bmatrix}, \tag{22}$$

where $\lambda > 0$, $p \in \mathbb{N}$ and $\gamma^2 \in \mathbb{R}\backslash\{0\}$, $Re(\gamma), Im(\gamma) \geq 0$. The G_j-structure index of λ is ε and the G_j-structure index of $-\lambda$ is $\varepsilon(\frac{|\gamma|}{\gamma})^2$.

Type (1d1):

$$A_j = \mathcal{J}_p(0), \quad H_j = \varepsilon Z_p, \quad \text{and} \quad G_j = \tilde{\varepsilon}\gamma F_p, \tag{23}$$

where $\gamma^2 \in \mathbb{R}\backslash\{0\}$, $Re(\gamma), Im(\gamma) \geq 0$, and $p \in \mathbb{N}$ is odd if $\gamma \in \mathbb{R}$ and even if $\gamma \in i\mathbb{R}$. Moreover, the eigenvalue $\lambda = 0$ has the H_j-structure index ε and the G_j-structure index $\operatorname{sign}(\tilde{\varepsilon}\gamma)$ *if $\gamma \in \mathbb{R}$ and* $\operatorname{sign}(-i\tilde{\varepsilon}\gamma)$ *if $\gamma \in i\mathbb{R}$.*

Type (1d2):

$$A_j = \begin{bmatrix} \mathcal{J}_p(0) & 0 \\ 0 & \mathcal{J}_p(0) \end{bmatrix}, \; H_j = \begin{bmatrix} 0 & Z_p \\ Z_p & 0 \end{bmatrix}, \quad \text{and} \;\; G_j = \begin{bmatrix} 0 & \gamma F_p \\ -\gamma F_p & 0 \end{bmatrix}, \tag{24}$$

where $\gamma^2 \in \mathbb{R}\backslash\{0\}$, $Re(\gamma), Im(\gamma) \geq 0$, and $p \in \mathbb{N}$ is even if $\gamma \in \mathbb{R}$ and odd if $\gamma \in i\mathbb{R}$. Moreover, the eigenvalue $\lambda = 0$ has the H_j-structure indices $+1, -1$ and the G_j-structure indices $+1, -1$.

Type (2a):

$$A_j = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 & 0 & 0 \\ 0 & -\mathcal{J}_p(\lambda) & 0 & 0 \\ 0 & 0 & \mathcal{J}_p(\overline{\lambda}) & 0 \\ 0 & 0 & 0 & -\mathcal{J}_p(\overline{\lambda}) \end{bmatrix}, \tag{25}$$

$$H_j = \begin{bmatrix} 0 & 0 & Z_p & 0 \\ 0 & 0 & 0 & Z_p \\ Z_p & 0 & 0 & 0 \\ 0 & Z_p & 0 & 0 \end{bmatrix}, \quad \text{and} \quad G_j = \begin{bmatrix} 0 & 0 & 0 & \gamma Z_p \\ 0 & 0 & \gamma Z_p & 0 \\ 0 & \overline{\gamma} Z_p & 0 & 0 \\ \overline{\gamma} Z_p & 0 & 0 & 0 \end{bmatrix}, \tag{26}$$

where $\lambda \in \mathbb{C}$ with $Re(\lambda)Im(\lambda) > 0$, $p \in \mathbb{N}$, and $\gamma^2 \in \mathbb{C}$ with $Re(\gamma)Im(\gamma) > 0$.

Type (2b):

$$A_j = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 & 0 & 0 \\ 0 & -\mathcal{J}_p(\lambda) & 0 & 0 \\ 0 & 0 & \mathcal{J}_p(\lambda) & 0 \\ 0 & 0 & 0 & -\mathcal{J}_p(\lambda) \end{bmatrix}, \tag{27}$$

$$H_j = \begin{bmatrix} 0 & 0 & Z_p & 0 \\ 0 & 0 & 0 & Z_p \\ Z_p & 0 & 0 & 0 \\ 0 & Z_p & 0 & 0 \end{bmatrix}, \quad \text{and} \quad G_j = \begin{bmatrix} 0 & 0 & 0 & \gamma Z_p \\ 0 & 0 & \gamma Z_p & 0 \\ 0 & \overline{\gamma} Z_p & 0 & 0 \\ \overline{\gamma} Z_p & 0 & 0 & 0 \end{bmatrix}, \tag{28}$$

where $\lambda > 0$, $p \in \mathbb{N}$, and $\gamma^2 \in \mathbb{C}$ with $Re(\gamma)Im(\gamma) > 0$. The H_j-structure indices of λ are $+1, -1$ and the H_j-structure indices of $-\lambda$ are $+1, -1$.

Type (2c):

$$A_j = i \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 & 0 & 0 \\ 0 & -\mathcal{J}_p(\lambda) & 0 & 0 \\ 0 & 0 & \mathcal{J}_p(\lambda) & 0 \\ 0 & 0 & 0 & -\mathcal{J}_p(\lambda) \end{bmatrix}, \tag{29}$$

$$H_j = \begin{bmatrix} 0 & 0 & 0 & Z_p \\ 0 & 0 & Z_p & 0 \\ 0 & Z_p & 0 & 0 \\ Z_p & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad G_j = \begin{bmatrix} 0 & 0 & \gamma Z_p & 0 \\ 0 & 0 & 0 & \gamma Z_p \\ \overline{\gamma} Z_p & 0 & 0 & 0 \\ 0 & \overline{\gamma} Z_p & 0 & 0 \end{bmatrix}, \tag{30}$$

where $\lambda > 0$, $p \in \mathbb{N}$, and $\gamma^2 \in \mathbb{C}$ with $Re(\gamma)Im(\gamma) > 0$. The G_j-structure indices of λ are $+1, -1$ and the G_j-structure indices of $-\lambda$ are $+1, -1$.

Type (2d):

$$A_j = \begin{bmatrix} \mathcal{J}_p(0) & 0 \\ 0 & \mathcal{J}_p(0) \end{bmatrix}, H_j = \begin{bmatrix} 0 & Z_p \\ Z_p & 0 \end{bmatrix}, G_j = \varepsilon \begin{bmatrix} 0 & \gamma F_p \\ (-1)^{p+1}\overline{\gamma} F_p & 0 \end{bmatrix}, \tag{31}$$

where $p \in \mathbb{N}$, $\varepsilon \in \{+1, -1\}$, and $\gamma^2 \in \mathbb{C}$ with $Re(\gamma)Im(\gamma) > 0$. Moreover, the eigenvalue $\lambda = 0$ has the H_j-structure indices $+1, -1$ and the G_j-structure indices $+1, -1$.

In the blocks of type (1a)–(1d) the subpencil $\varrho H_j - G_j$ has only real or purely imaginary eigenvalues and in the blocks (2a)–(2d) the subpencil $\varrho H_j - G_j$ has only eigenvalues that are neither real nor purely imaginary.

Moreover, the canonical form (18) is unique up to permutation of blocks.

Proof. In view of Lemma 18, we may assume that the spectrum of the pencil $\varrho H - G$ is contained in $\{\gamma, -\gamma, \overline{\gamma}, -\overline{\gamma}\}$ for some $\gamma \in \mathbb{C}\backslash\{0\}$, $\mathrm{Re}(\gamma), \mathrm{Im}(\gamma) \geq 0$, and it is sufficient to distinguish the following two cases.

Case (1): $\mathrm{Re}(\gamma)\mathrm{Im}(\gamma) = 0$.

In view of Lemma 17, we may distinguish the following four subcases.

Subcase (1a): The spectrum of A is $\{\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}\}$, where $\mathrm{Re}(\lambda)\mathrm{Im}(\lambda) > 0$.

Since A is H-selfadjoint and G-skew-adjoint, it follows from Remark 12 that λ $\overline{\lambda}$, $-\overline{\lambda}$, and $-\lambda$ have the same Jordan structures. Applying Theorem 9, the Z-trick, and a block permutation, we may assume that A and H have the following forms.

$$A = \begin{bmatrix} \mathcal{J}(\lambda) & 0 & 0 & 0 \\ 0 & -\mathcal{J}(\lambda) & 0 & 0 \\ 0 & 0 & \mathcal{J}(\lambda)^* & 0 \\ 0 & 0 & 0 & -\mathcal{J}(\lambda)^* \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & I_m \\ I_m & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \end{bmatrix}, \tag{32}$$

where $m = \frac{n}{4} \in \mathbb{N}$ and $\mathcal{J}(\lambda)$ is an $(m \times m)$ matrix in Jordan canonical form only having the eigenvalue λ. Then, the equation $-A^*G = GA$ and the fact that λ, $-\lambda$, $\overline{\lambda}$, and $-\overline{\lambda}$ are pairwise distinct, imply that G necessarily has the form

$$G = \begin{bmatrix} 0 & 0 & 0 & G_2 \\ 0 & 0 & G_3 & 0 \\ 0 & G_3^* & 0 & 0 \\ G_2^* & 0 & 0 & 0 \end{bmatrix}, \tag{33}$$

where $G_2, G_3 \in \mathbb{C}^{m \times m}$. By Lemma 19, we obtain that $H^{-1}GH^{-1}G = \gamma^2 I_n$. This implies in particular that

$$G_3 G_2 = \gamma^2 I_m. \tag{34}$$

Note that the equation $-A^*G = GA$ also implies that $\mathcal{J}(\lambda)^*G_2 = G_2\mathcal{J}(\lambda)^*$, i.e., G_2 commutes with $\mathcal{J}(\lambda)^*$. Hence, setting

$$Q := \begin{bmatrix} \overline{\gamma^{\frac{1}{2}}}G_2^{-*} & 0 & 0 & 0 \\ 0 & \overline{\gamma^{-\frac{1}{2}}}I_m & 0 & 0 \\ 0 & 0 & \gamma^{-\frac{1}{2}}G_2 & 0 \\ 0 & 0 & 0 & \gamma^{\frac{1}{2}}I_m \end{bmatrix},$$

we obtain that $Q^{-1}AQ = A$, $Q^*HQ = H$, and

$$Q^*GQ = \begin{bmatrix} 0 & 0 & 0 & \gamma I_m \\ 0 & 0 & \gamma^{-1}G_3G_2 & 0 \\ 0 & \overline{\gamma^{-1}}G_2^*G_3^* & 0 & 0 \\ \overline{\gamma}I_m & 0 & 0 & 0 \end{bmatrix}. \tag{35}$$

Then it follows from (34) and (35) that the triple (A, H, G) can be reduced to blocks of the form given by (19) and (20), by applying a proper block permutation and the Z-trick.

Subcase (1b): The spectrum of A is $\{\lambda, -\lambda\}$, where $\lambda > 0$.

Theorem 11 implies that λ and $-\lambda$ have the same Jordan structures. Moreover, applying Theorem 9, we may assume that A, H, and G have the following forms.

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & -A_1 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}, \quad \text{and} \quad G = \begin{bmatrix} G_1 & G_2 \\ G_2^* & G_3 \end{bmatrix},$$

where

$$\begin{aligned} A_1 &= \mathcal{J}_{p_1}(\lambda) \oplus \ldots \oplus \mathcal{J}_{p_k}(\lambda), \\ H_1 &= \varepsilon_1 Z_{p_1} \oplus \ldots \oplus \varepsilon_k Z_{p_k}, \\ H_2 &= \tilde{\varepsilon}_1 Z_{p_1} \oplus \ldots \oplus \tilde{\varepsilon}_k Z_{p_k} \end{aligned}$$

and $G_j \in \mathbb{C}^{m\times m}$ for $m = \frac{n}{2}$. Observing that $-A^*G = GA$, we obtain that $G_1 = G_3 = 0$, since $\lambda \neq 0$, and $A_1^*G_2 = G_2A_1$. Moreover, $H^{-1}GH^{-1}G = \gamma^2 I_n$ implies that

$$H_1^{-1}G_2H_2^{-1}G_2^* = \gamma^2 I_m = H_2^{-1}G_2^*H_1^{-1}G_2. \tag{36}$$

Setting

$$Q = \begin{bmatrix} I_m & 0 \\ 0 & \gamma^{-1}H_2^{-1}G_2^* \end{bmatrix},$$

then from (36), $A_1^*G_2 = G_2A_1$, $Z_p^{-1}\mathcal{J}_p(0)^*Z_p = \mathcal{J}_p(0)$ (Lemma 4), and the block forms of H_2 and A_1, we obtain that

$$\begin{aligned} Q^{-1}AQ &= \begin{bmatrix} A_1 & 0 \\ 0 & -G_2^{-*}H_2A_1H_2^{-1}G_2^* \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & -A_1 \end{bmatrix}, \\ Q^*HQ &= \begin{bmatrix} H_1 & 0 \\ 0 & \frac{1}{|\gamma^2|}G_2H_2^{-1}H_2H_2^{-1}G_2^* \end{bmatrix} = \begin{bmatrix} H_1 & 0 \\ 0 & (\frac{\gamma}{|\gamma|})^2 H_1 \end{bmatrix} \quad \text{and} \\ Q^*GQ &= \begin{bmatrix} 0 & \gamma^{-1}G_2H_2^{-1}G_2^* \\ \overline{\gamma}^{-1}G_2H_2^{-1}G_2^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & \gamma H_1 \\ \overline{\gamma}H_1 & 0 \end{bmatrix}. \end{aligned}$$

Thus, it follows from a proper block permutation that we may assume that

$$A = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 \\ 0 & -\mathcal{J}_p(\lambda) \end{bmatrix}, \quad H = \begin{bmatrix} \varepsilon Z_p & 0 \\ 0 & \varepsilon(\frac{\gamma}{|\gamma|})^2 Z_p \end{bmatrix}, \quad \text{and} \quad G = \begin{bmatrix} 0 & \gamma\varepsilon Z_p \\ \overline{\gamma}\varepsilon Z_p & 0 \end{bmatrix}.$$

Hence, setting

$$\tilde{Q} = \begin{bmatrix} \varepsilon^{-1} I_m & 0 \\ 0 & I_m \end{bmatrix},$$

we find that $\tilde{Q}^{-1} A \tilde{Q}$, $\tilde{Q}^* H \tilde{Q}$, and $\tilde{Q}^* G \tilde{Q}$ have the desired forms.

Subcase (1c): The spectrum of A is $\{\lambda, -\lambda\}$, where $\lambda \in i\mathbb{R}$, $\operatorname{Im}(\lambda) > 0$.

The matrix $-iA$ is G-selfadjoint, H-skew-adjoint and has only a pair of real eigenvalues. Noting that the spectrum of $\varrho G - H$ is contained in $\{\gamma^{-1}, -\gamma^{-1}\}$, we can reduce the problem to Case (1b), i.e., it is sufficient consider the case that $-iA$, G, and H have the form (21).

$$-iA = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 \\ 0 & \mathcal{J}_p(-\lambda) \end{bmatrix}, G = \begin{bmatrix} \tilde{\varepsilon} Z_m & 0 \\ 0 & \tilde{\varepsilon}(\frac{|\gamma|}{\gamma})^2 Z_m \end{bmatrix}, H = \begin{bmatrix} 0 & \gamma^{-1} Z_m \\ \overline{\gamma}^{-1} Z_m & 0 \end{bmatrix},$$

where $\tilde{\varepsilon} \in \{+1, -1\}$. Setting

$$Q = \begin{bmatrix} \overline{\gamma}^{\frac{1}{2}} I_m & 0 \\ 0 & \gamma^{\frac{1}{2}} I_m \end{bmatrix},$$

we obtain that $Q^{-1}(-iA)Q = -iA$,

$$Q^* H Q = \begin{bmatrix} 0 & Z_m \\ Z_m & 0 \end{bmatrix}, \quad \text{and} \quad Q^* G Q = \tilde{\varepsilon}|\gamma| \begin{bmatrix} Z_m & 0 \\ 0 & \tilde{\varepsilon}(\frac{|\gamma|}{\gamma})^2 Z_m \end{bmatrix}.$$

Subcase (1d): The spectrum of A is $\{0\}$.

It follows from Lemma 27 in the appendix that the triple (A, H, G) can be reduced to blocks of the forms (23) or (24).

Case (2): $\operatorname{Re}(\gamma)\operatorname{Im}(\gamma) \neq 0$.

By Corollary 7 we may assume that the pencil $\varrho H - G$ is already in the form

$$\varrho H - G = \varrho \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} - \begin{bmatrix} 0 & \gamma\Sigma \\ \overline{\gamma}\Sigma & 0 \end{bmatrix},$$

where $m = \frac{n}{2} \in \mathbb{N}$ and $\Sigma = \operatorname{diag}(I_p, I_{m-p})$, $1 \le p \le m$ and, furthermore, we have that (16). Then Lemma 19 implies that A has the form

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_1^* \end{bmatrix}.$$

Note, that by Lemma 19 a similarity transformation on A with a corresponding block diagonal matrix and simultaneous congruence transformations on H, G will not change the block structure of A and the identity (16), but it does change the block forms in H and G. Hence we can apply similarity transformations on A_1 and at the same time keep the relation (16). Again, we will consider the following four subcases.

Subcase (2a): The spectrum of A is $\{\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}\}$, where $\mathrm{Re}(\lambda)\mathrm{Im}(\lambda) > 0$.

Again, the eigenvalues $\lambda, -\lambda, \overline{\lambda}$, and $-\overline{\lambda}$ have the same Jordan structures. Moreover, there exists a nonsingular matrix

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_1^{-*} \end{bmatrix} \in \mathbb{C}^{n\times n}$$

such that

$$Q^{-1}AQ = \begin{bmatrix} A_{11} & 0 & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ 0 & 0 & A_{11}^* & 0 \\ 0 & 0 & 0 & A_{22}^* \end{bmatrix} \quad \text{and} \quad Q^*HQ = H,$$

where $A_{11} \in \mathbb{C}^{k\times k}$ has the eigenvalues λ and $-\lambda$ and $A_{22} \in \mathbb{C}^{(\frac{n}{2}-k)\times(\frac{n}{2}-k)}$ has the eigenvalues $\overline{\lambda}$ and $-\overline{\lambda}$. Partitioning Q^*GQ conformably, i.e.,

$$Q^*GQ = \begin{bmatrix} 0 & 0 & G_1 & G_2 \\ 0 & 0 & G_3 & G_4 \\ G_1^* & G_3^* & 0 & 0 \\ G_2^* & G_4^* & 0 & 0 \end{bmatrix},$$

we obtain from the equation $-A^*G = GA$ and the fact that A_{11} and $-A_{22}$ have no common eigenvalues that $G_2 = G_3 = 0$. Thus, after a proper block permutation, we may consider two smaller subproblems. The first one is

$$\tilde{A} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{11}^* \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} 0 & I_k \\ I_k & 0 \end{bmatrix}, \quad \text{and} \quad \tilde{G} = \begin{bmatrix} 0 & G_1 \\ G_1^* & 0 \end{bmatrix};$$

and (16) implies that

$$\tilde{H}^{-1}\tilde{G}\tilde{H}^{-1}\tilde{G} = \begin{bmatrix} \overline{\gamma}^2 I_k & 0 \\ 0 & \gamma^2 I_k \end{bmatrix}.$$

Hence, after applying a similarity transformation on A_{11} , we may assume that $\tilde{A}$, $\tilde{G}$, and $\tilde{H}$ are in the forms (32) and (33), where $(G_1)^2 = \gamma^2 I$. The remainder of the proof then proceeds analogously to Subcase (1a). The second subproblem with respect to A_{22} can be transformed in the same way.

Subcase (2b): The spectrum of A is $\{\lambda, -\lambda\}$, where $\lambda > 0$.

We obtain from Theorem 11 that λ and $-\lambda$ have the same Jordan structures. Hence, both A_1 and A_1^* must have the eigenvalues λ and $-\lambda$ with the same Jordan structures. Thus, there exists a nonsingular matrix

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_1^{-*} \end{bmatrix}$$

such that

$$Q^{-1}AQ = \begin{bmatrix} \mathcal{J}(\lambda) & 0 & 0 & 0 \\ 0 & -\mathcal{J}(\lambda) & 0 & 0 \\ 0 & 0 & \mathcal{J}(\lambda) & 0 \\ 0 & 0 & 0 & -\mathcal{J}(\lambda) \end{bmatrix} \quad \text{and} \quad Q^*HQ = H,$$

where $k = \frac{n}{4}$ and $\mathcal{J}(\lambda)$ is an $k \times k$ matrix in Jordan canonical form associated with only one eigenvalue λ. Partitioning Q^*GQ conformably, i.e.,

$$Q^*GQ = \begin{bmatrix} 0 & 0 & G_1 & G_2 \\ 0 & 0 & G_3 & G_4 \\ G_1^* & G_3^* & 0 & 0 \\ G_2^* & G_4^* & 0 & 0 \end{bmatrix},$$

we obtain from $-A^*G = GA$ and the fact that $\mathcal{J}(\lambda)$ and $-\mathcal{J}(\lambda)$ have no common eigenvalues that $G_1 = G_4 = 0$; and $\mathcal{J}(\lambda)^*G_2 = G_2\mathcal{J}(\lambda)$, $\mathcal{J}(\lambda)^*G_3 = G_3\mathcal{J}(\lambda)$. Moreover, we still have (16), which implies that $G_3G_2 = \gamma^2 I$. Thus we may assume that A, G, and H are in the forms (32) and (33), where $G_3G_2 = \gamma^2 I$. The remainder of the proof then proceeds analogously to Subcase (1a).

Subcase (2c): The spectrum of A is $\{\lambda, -\lambda\}$, where $\lambda \in i\mathbb{R}$.

The proof proceeds analogously to the proof of Subcase (1c).

Subcase (2d): The spectrum of A is $\{0\}$.

This case follows from Lemma 30 in the appendix and by applying the Z-trick.

Uniqueness: Analogous to the proof of Theorem 16, it is sufficient to prove uniqueness for the case that the spectrum of A is $\{\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}\}$ for some $\lambda \in \mathbb{C}$ and that the spectrum of $\varrho H - G$ is contained in $\{\gamma, -\gamma, \overline{\gamma}, -\overline{\gamma}\}$ for some $\gamma \in \mathbb{C}$. Again, the canonical form for the pair (A, H) is unique. In any case except for the case that $\lambda = 0$ and $\gamma^2 \notin \mathbb{R}$, the matrix G is then uniquely determined by the invariants γ with $\mathrm{Re}(\gamma), \mathrm{Im}(\gamma) \geq 0$ (and signs ε or $\tilde{\varepsilon}$ in some cases that are uniquely determined by the canonical form for the pair (A, G)). Only in the case $\lambda = 0$ and $\gamma^2 \notin \mathbb{R}$, we have an additional invariant ε that is not an invariant of the canonical form for the pair (A, G). In this case, the uniqueness follows from Lemma 30 in the appendix.

In all cases (1a)–(2d) it is easy to verify that the H and G-structure indices of each block are as claimed in the theorem. □

5 Singly and doubly structured pencils

In this section, we discuss canonical forms for matrix pencils $\varrho A - B$, where both A and B are matrices that are singly or doubly structured with respect to some indefinite inner product. It turns out that the case of structured pencils can be reduced to the matrix case. This is done in the following theorem.

Theorem 21 *Let the matrices $G, H \in \mathbb{C}^{n\times n}$ be nonsingular and Hermitian or skew-Hermitian, i.e.,*

$$G^* = \eta_G G \quad \text{and} \quad H^* = \eta_H H,$$

where $\eta_G, \eta_H \in \{1, -1\}$. Furthermore, let $\varrho A - B \in \mathbb{C}^{n\times n}$ be a regular pencil such that

$$\begin{array}{ll} A^*H = \varepsilon_A HA, & A^*G = \delta_A GA, \\ B^*H = \varepsilon_B HB, & B^*G = \delta_B GB, \end{array} \tag{37}$$

where $\varepsilon_A, \varepsilon_B, \delta_A, \delta_B \in \{1, -1\}$. Then there exists nonsingular matrices $P, Q \in \mathbb{C}^{n\times n}$ such that

$$\begin{aligned} P^{-1}(\varrho A - B)Q &= \varrho \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} M & 0 \\ 0 & I_{n_2} \end{bmatrix}, \\ Q^*HP &= \begin{bmatrix} H_{11} & 0 \\ 0 & H_{22} \end{bmatrix}, \\ Q^*GP &= \begin{bmatrix} G_{11} & 0 \\ 0 & G_{22} \end{bmatrix}, \end{aligned}$$

where $M, H_{11}, G_{11} \in \mathbb{C}^{n_1\times n_1}$ and $N, H_{22}, G_{22} \in \mathbb{C}^{n_2\times n_2}$. Moreover, M and N are in Jordan canonical form, N is nilpotent and the following conditions are satisfied.

$$\begin{array}{rr} H_{11}^* = \eta_H \varepsilon_A H_{11}, & G_{11}^* = \eta_G \delta_A G_{11}, \\ M^*H_{11} = \varepsilon_A \varepsilon_B H_{11} M, & M^*G_{11} = \delta_A \delta_B G_{11} M, \\ H_{22}^* = \eta_H \varepsilon_B H_{22}, & G_{22}^* = \eta_G \delta_B G_{22}, \\ N^*H_{22} = \varepsilon_A \varepsilon_B H_{22} N, & N^*G_{22} = \delta_A \delta_B G_{22} N. \end{array}$$

Proof. Let $P, Q \in \mathbb{C}^{n\times n}$ be such that the pencil

$$P^{-1}(\varrho A - B)Q = \lambda \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} M & 0 \\ 0 & I_{n_2} \end{bmatrix} \tag{38}$$

is in Kronecker canonical form (see [6]), where M, N are in Jordan canonical form and N is nilpotent. Then (37) and (38) imply in particular that

$$Q^*H(\varrho\varepsilon_A A - \varepsilon_B B) = Q^*(\varrho A^* - B^*)H = \left(\varrho \begin{bmatrix} I_{n_1} & 0 \\ 0 & N^* \end{bmatrix} - \begin{bmatrix} M^* & 0 \\ 0 & I_{n_2} \end{bmatrix} \right) P^*H.$$

From this and (38) we obtain that

$$\begin{aligned} Q^*HP \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} &= Q^*HAQ &= \varepsilon_A \begin{bmatrix} I_{n_1} & 0 \\ 0 & N^* \end{bmatrix} P^*HQ, \\ Q^*HP \begin{bmatrix} M & 0 \\ 0 & I_{n_2} \end{bmatrix} &= Q^*HBQ &= \varepsilon_B \begin{bmatrix} M^* & 0 \\ 0 & I_{n_2} \end{bmatrix} P^*HQ. \end{aligned}$$

Setting $Q^*HP = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$ and noting that $P^*HQ = \eta_H(Q^*HP)^*$, we find that

$$\begin{bmatrix} H_{11} & H_{12}N \\ H_{21} & H_{22}N \end{bmatrix} = \eta_H\varepsilon_A \begin{bmatrix} H_{11}^* & H_{21}^* \\ N^*H_{12}^* & N^*H_{22}^* \end{bmatrix}$$

$$\text{and} \quad \begin{bmatrix} H_{11}M & H_{12} \\ H_{21}M & H_{22} \end{bmatrix} = \eta_H\varepsilon_B \begin{bmatrix} M^*H_{11}^* & M^*H_{21}^* \\ H_{12}^* & H_{22}^* \end{bmatrix}.$$

This implies, in particular, that

$$H_{12} = \eta_H\varepsilon_B M^*H_{21}^* = \varepsilon_A\varepsilon_B M^*H_{12}N = (\varepsilon_A\varepsilon_B)^k(M^*)^kH_{12}N^k \quad \text{for every } k \in \mathbb{N}.$$

Since N is nilpotent, it follows that $H_{12} = 0$ and thus, also $H_{21} = \eta_H\varepsilon_A N^*H_{12}^* = 0$. Moreover, $H_{11} = \eta_H\varepsilon_A H_{11}^*$ and $H_{22} = \eta_H\varepsilon_B H_{22}^*$, and $H_{22}N = \eta_H\varepsilon_A N^*H_{22}^* = \varepsilon_A\varepsilon_B N^*H_{22}$, $H_{11}M = \eta_H\varepsilon_B M^*H_{11}^* = \varepsilon_A\varepsilon_B M^*H_{11}$. Analogously we show that Q^*GP has the structure claimed in the theorem. This concludes the proof. □

We note that M is a doubly structured matrix with structures induced by H_{11} and G_{11} and that N is a nilpotent doubly structured matrix with structured induced by H_{22} and G_{22}, where H_{11}, G_{11}, H_{22}, and G_{22} are all Hermitian or skew-Hermitian. Therefore, Theorem 21 gives a general description about how to obtain the canonical forms for the pencil case from the canonical forms in the matrix case that are given in the previous sections. We only have to further reduce M and N by applying the results from Section 4. Note that Theorem 21 does not require the pencil $\varrho H - G$ to be nondefective. However, canonical forms for the matrix case are known for this case only.

Theorem 21 also describes the case of singly structured pencils. In this case one may choose $H = G$, $\varepsilon_A = \delta_A$, and $\varepsilon_B = \delta_B$. Thus, Theorem 21 gives a general description how to obtain canonical forms for singly and doubly structured pencils from the canonical forms in the matrix case. For obvious reasons, we do not give a list of the canonical forms for all possible cases, but only one example to illustrate the effect of Theorem 21.

Theorem 22 *Let $H \in \mathbb{C}^{n\times n}$ be Hermitian and nonsingular and let $\varrho A - B \in \mathbb{C}^{n\times n}$ be a regular pencil such that A and B are H-selfadjoint. Then there exists nonsingular matrices $P, Q \in \mathbb{C}^{n\times n}$ such that*

$$\begin{aligned} P^{-1}(\varrho A - B)Q &= \varrho \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{bmatrix} - \begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_k \end{bmatrix} \\ Q^*HP &= \begin{bmatrix} H_1 & & 0 \\ & \ddots & \\ 0 & & H_k \end{bmatrix}, \end{aligned}$$

where the blocks A_j, B_j, and H_j have corresponding sizes and are of one and only one of the following forms:

1. *Blocks associated with real eigenvalues:*

$$A_j = I_p, \quad B_j = \mathcal{J}_p(\lambda), \quad \text{and} \quad H_j = \varepsilon Z_p,$$

where $p \in \mathbb{N}$, $\lambda \in \mathbb{R}$, *and* $\varepsilon \in \{1, -1\}$.

2. *Blocks associated with a pair of nonreal eigenvalues:*

$$A_j = I_{2p}, \quad B_j = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 \\ 0 & \mathcal{J}_p(\overline{\lambda}) \end{bmatrix}, \quad \text{and} \quad H_j = Z_{2p},$$

where $p \in \mathbb{N}$ *and* $\lambda \in \mathbb{C}\backslash\mathbb{R}$.

3. *Blocks associated with the eigenvalue* ∞*:*

$$A_j = \mathcal{J}_p(0), \quad B_j = I_p, \quad \text{and} \quad H_j = \varepsilon Z_p,$$

where $p \in \mathbb{N}$ *and* $\varepsilon \in \{1, -1\}$.

Moreover, this form is uniquely determined up to permutation of blocks.

Proof. This follows directly from Theorem 21 and Theorem 9. $\square$

Note that with the assumptions and notation of Theorem 22 the pencil $H(\varrho A - B) = \varrho HA - HB$ is a Hermitian pencil. It turns out that Theorem 22 is a generalization of Theorem 6. Indeed, the pencil $Q^* HPP^{-1}(\varrho A - B)Q$ is a Hermitian pencil in canonical form.

6 Conclusions

We have derived canonical forms for matrices and matrix pencils that are doubly structured in the sense that they are H-selfadjoint (or H-skew-adjoint) and at the same time G-selfadjoint (or G-skew-adjoint), where we have assumed that G, H are nonsingular Hermitian (or skew Hermitian) and $\rho G - H$ is a nondefective pencil. The general case that G or H are singular, or that the pencil $\rho G - H$ is defective is still an open problem. Also the associated real canonical forms, which appear to be much more difficult, are open.

In view of the applications in eigenvalue computations, it is also important to restrict the transformation matrices to be unitary (or orthogonal in the real case). This case will be covered in a forthcoming paper, that will also address numerical methods, in particular for the classes of pencils arising in quantum chemistry that we have discussed in the introduction.

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Appendix

In the appendix we derive some technical Lemmas. Recall the Kronecker product (see, e.g., [9, 12]).

Definition 23 Let $A = [a_{jk}] \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times q}$. Then

$$A \otimes B := \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix} \in \mathbb{C}^{mp\times nq}.$$

This product has the following basic properties (see, e.g., [9, 12]).

Proposition 24 *Let $A, C \in \mathbb{C}^{p_1\times p_2}$, $B, D \in \mathbb{C}^{q_1\times q_2}$, $E \in \mathbb{C}^{p_2\times p_3}$, and $F \in \mathbb{C}^{q_2\times q_3}$. Then the following identities hold.*

1. $A \otimes (B + D) = A \otimes B + A \otimes D, \quad (A + C) \otimes B = A \otimes B + C \otimes B.$
2. $(A \otimes B)(E \otimes F) = (AE) \otimes (BF).$
3. *$A \otimes B$ is invertible if and only if A and B are invertible. In this case we have that $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.*
4. $(A \otimes B)^T = A^T \otimes B^T, \quad (A \otimes B)^* = A^* \otimes B^*.$
5. *$A \otimes B = 0$ if and only if $A = 0$ or $B = 0$.*

We will frequently need the permutation matrix

$$\Omega_{m,n} = [e_1, e_{n+1}, \dots, e_{(m-1)n+1}; e_2, e_{n+2}, \dots, e_{(m-1)n+2}; e_n, e_{2n}, \dots, e_{mn}].$$

If A, B are $m \times n$ and $p \times q$, respectively, then

$$\Omega_{m,p}^*(A \otimes B)\Omega_{n,q} = B \otimes A.$$

In the following we derive the canonical forms for doubly structured matrices that are nilpotent. This case is the most complicated case, since we have least symmetry in the spectrum. Therefore, we have to use a very technical reduction procedure.

For the sake of briefness of notation, let $\mathcal{J}_p$ denote the nilpotent Jordan block $\mathcal{J}_p(0)$ of size p. $\mathcal{O}_{pq}$ is the $p \times q$ zero matrix.

Lemma 25 *Let Z_p, D_p, and F_p be defined as in Section 2 and let $k, l, p, q \in \mathbb{N}$, $(p \geq q)$. Then*

$$Z_p \mathcal{J}_p^l = (\mathcal{J}_p^l)^* Z_p, \quad D_p \mathcal{J}_p^l D_p = (-1)^l \mathcal{J}_p^l, \quad F_p \mathcal{J}_p^l = (-1)^l (\mathcal{J}_p^l)^* F_p. \tag{39}$$

$$Z_p \mathcal{J}_p^k \begin{bmatrix} \mathcal{J}_q^l \\ \mathcal{O}_{p-q,q} \end{bmatrix} = \begin{bmatrix} \mathcal{O}_{p-q,q} \\ Z_q \mathcal{J}_q^{k+l} \end{bmatrix}. \tag{40}$$

$$F_p \mathcal{J}_p^k \begin{bmatrix} \mathcal{J}_q^l \\ \mathcal{O}_{p-q,q} \end{bmatrix} = (-1)^{p-q} \begin{bmatrix} \mathcal{O}_{p-q,q} \\ F_q \mathcal{J}_q^{k+l} \end{bmatrix}. \tag{41}$$

$$D_p \begin{bmatrix} \mathcal{O}_{p-q,q} \\ F_q \end{bmatrix} = (-1)^{p-q} \begin{bmatrix} \mathcal{O}_{p-q,q} \\ D_q F_q \end{bmatrix}. \tag{42}$$

Definition 26 Let $A = (a_{jk})_{nn} \in \mathbb{C}^{n \times n}$. Then the l-th lower anti-diagonal of A or, in short, the l-th anti-diagonal of A is defined by the elements a_{jk}, where $j + k = n + 1 + l$. Here, we allow $l = 0$. The 0-th anti-diagonal is also called the main anti-diagonal. If

$$B = \begin{bmatrix} 0 & \tilde{B} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 \\ \tilde{C} \end{bmatrix},$$

where $\tilde{B}$ and $\tilde{C}$ are square matrices, then the l-th anti-diagonal of $\tilde{B}$ and $\tilde{C}$ is called the l-th anti-diagonal of B and C, respectively. Analogously, we define the l-th block anti-diagonal for square and non-square block matrices.

Lemma 27 *Let $G, H \in \mathbb{C}^{n \times n}$ be Hermitian nonsingular such that the pencil $\varrho H - G$ is nondefective and such that its spectrum is contained in $\{\gamma, -\gamma\}$, where $\gamma^2 \in \mathbb{R}\backslash\{0\}$ and $Re(\gamma), Im(\gamma) \geq 0$. Furthermore, let $A \in \mathbb{C}^{n \times n}$ be nilpotent, H-selfadjoint and G-skew-adjoint. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n \times n}$ such that*

$$\begin{aligned} P^{-1}AP &= A_1 \oplus \ldots \oplus A_k, \\ P^*GP &= G_1 \oplus \ldots \oplus G_k, \\ P^*HP &= H_1 \oplus \ldots \oplus H_k, \end{aligned} \tag{43}$$

where the blocks A_j, G_j, H_j have corresponding sizes and, for each j, are of one and only one of the following forms.

Type (1d1):

$$A_j = \mathcal{J}_p(0), \quad H_j = \varepsilon Z_p, \quad \text{and} \quad G_j = \tilde{\varepsilon}\gamma F_p, \tag{44}$$

where $p \in \mathbb{N}$ is odd if $\gamma \in \mathbb{R}$ and even if $\gamma \in i\mathbb{R}$.

Type (1d2):

$$A_j = \begin{bmatrix} \mathcal{J}_p(0) & 0 \\ 0 & \mathcal{J}_p(0) \end{bmatrix}, \quad H_j = \begin{bmatrix} 0 & Z_p \\ Z_p & 0 \end{bmatrix}, \quad G_j = \begin{bmatrix} 0 & \gamma F_p \\ -\gamma F_p & 0 \end{bmatrix}, \tag{45}$$

where $p \in \mathbb{N}$ is even if $\gamma \in \mathbb{R}$ and odd if $\gamma \in i\mathbb{R}$.

Proof. Applying Theorem 9, we may assume that (A, H) is in canonical form, i.e., collecting blocks of same size and representing them by means of the Kronecker product, we may assume that

$$A = \begin{bmatrix} I_{m_1} \otimes \mathcal{J}_{p_1} & & 0 \\ & \ddots & \\ 0 & & I_{m_k} \otimes \mathcal{J}_{p_k} \end{bmatrix}, \quad H = \begin{bmatrix} \Sigma_{m_1} \otimes Z_{p_1} & & 0 \\ & \ddots & \\ 0 & & \Sigma_{m_k} \otimes Z_{p_k} \end{bmatrix},$$

where $p_1 > \ldots > p_k$ are the sizes of Jordan blocks and Σ_{m_j} are signature matrices for $j = 1, \ldots, k$. Setting

$$F = \begin{bmatrix} I_{m_1} \otimes F_{p_1} & & 0 \\ & \ddots & \\ 0 & & I_{m_k} \otimes F_{p_k} \end{bmatrix},$$

we obtain from $-A^*G = GA$ and (39) that A and FG commute. Thus, the structure of G is implicitly given by the well-known form for matrices that commute with matrices in Jordan canonical form (see [5]). For the sake of better clarity, we will not work directly on A, H, and G, but first apply a permutation. Setting $\Omega = \Omega_{m_1,p_1} \oplus \ldots \oplus \Omega_{m_k,p_k}$ and updating A, H, G by $\Omega^{-1}A\Omega$, $\Omega^*H\Omega$, $\Omega^*G\Omega$, we may consider the following situation.

$$A = \begin{bmatrix} \mathcal{J}_{p_1} \otimes I_{m_1} & & 0 \\ & \ddots & \\ 0 & & \mathcal{J}_{p_k} \otimes I_{m_k} \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} H_{11} & & 0 \\ & \ddots & \\ 0 & & H_{kk} \end{bmatrix}. \tag{46}$$

where $H_{jj} := Z_{p_j} \otimes \Sigma_{m_j}$. Partitioning

$$G = \begin{bmatrix} G_{11} & \ldots & G_{1k} \\ \vdots & \ddots & \vdots \\ G_{1k}^* & \ldots & G_{kk} \end{bmatrix} \tag{47}$$

conformably and using the well-known structures of matrices that commute with matrices in Jordan canonical form [5], we obtain that

$$
\begin{aligned}
G_{qq} &= \sum_{l=0}^{p_q-1} (F_{p_q}\mathcal{J}_{p_q}^l) \otimes G_{q,q}^{(l)} \\
&= \begin{bmatrix} 0 & \dots & \dots & 0 & G_{q,q}^{(0)} \\ \vdots & & \iddots & -G_{q,q}^{(0)} & -G_{q,q}^{(1)} \\ \vdots & \iddots & G_{q,q}^{(0)} & G_{q,q}^{(1)} & \vdots \\ 0 & \iddots & \iddots & & \vdots \\ (-1)^{p_q+1}G_{q,q}^{(0)} & \dots & \dots & \dots & (-1)^{p_q+1}G_{q,q}^{(p_q-1)} \end{bmatrix}
\end{aligned} \tag{48}
$$

for $q = 1, \dots, k$, where $G_{q,q}^{(l)} \in \mathbb{C}^{m_q \times m_q}$ and

$$
G_{qr} = \sum_{l=1}^{p_r-1} \begin{bmatrix} \mathcal{O}_{p_q-p_r,p_r} \\ F_{p_r}\mathcal{J}_{p_r}^l \end{bmatrix} \otimes G_{q,r}^{(l)} \tag{49}
$$

for $1 \le q < r \le k$, with $G_{q,r}^{(l)} \in \mathbb{C}^{m_q \times m_r}$.

We will stepwise reduce the matrix G, while keeping the forms of A and H.

Step (1): We first show that $G_{j,j}^{(0)}$ is nonsingular for $j = 1, \dots, k$.

Since the pencil $\varrho H - G$ is nondefective and has only the eigenvalues $\gamma, -\gamma$, where $\gamma^2 \in \mathbb{R}\backslash\{0\}$, we obtain from Lemma 19 that

$$
GH^{-1}G = \gamma^2 H.
$$

Comparing the j-th diagonal blocks on both sides, this implies in particular that

$$
\gamma^2 H_{jj} = G_{1j}^* H_{11} G_{1j} + \dots + G_{jj} H_{jj} G_{jj} + \dots + G_{jk} H_{kk} G_{jk}^*. \tag{50}
$$

Because of the structure of the blocks G_{qr}, it follows that all the block anti-diagonals of $G_{qr}^* H_{qq} G_{qr}$ and $G_{qr} H_{rr} G_{qr}^*$ are zero for $q < r$, and hence, comparing the main block anti-diagonals on both sides of (50), we obtain that

$$
\begin{aligned}
\gamma^2 Z_{p_j} \otimes \Sigma_{m_j} &= (F_{p_j} \otimes G_{j,j}^{(0)})(Z_{p_j} \otimes \Sigma_{m_j})(F_{p_j} \otimes G_{j,j}^{(0)}) \\
&= (F_{p_j} Z_{p_j} F_{p_j}) \otimes (G_{j,j}^{(0)} \Sigma_{m_j} G_{j,j}^{(0)})
\end{aligned}
$$

Since $F_{p_j} Z_{p_j} F_{p_j} = Z_{p_j}$, this implies that

$$
G_{j,j}^{(0)} \Sigma_{m_j} G_{j,j}^{(0)} = \frac{1}{\gamma^2} \Sigma_{m_j} \tag{51}
$$

and thus, $G_{j,j}^{(0)}$ is nonsingular.

Step (2): Elimination of $G_{12},\ldots,G_{1k}$

Assume that we already have $G_{1,j}^{(l-1)}=0$ for all $j=2,\ldots,k$ and $G_{1,j}^{(l)}=0$ for $j=2,\ldots,r-1$, where $l\geq 0$ and $r\geq 2$. We then show how to eliminate $G_{1,r}^{(l)}$, while keeping the forms of A and H. Let

$$
X=\begin{array}{c c}
 & \begin{array}{ccccc} \phantom{X_{r1}} & & & r & \end{array} \\
\begin{array}{c} \\ \\ r \\ \\ \end{array} &
\left[\begin{array}{ccccc}
I & & & X_{1r} & \\
 & \ddots & & & \\
X_{r1} & & & \ddots & \\
 & & & & I
\end{array}\right]
\end{array}
$$

have a block form analogous to G, where zero blocks of the matrix are indicated by blanks and, moreover,

$$
\begin{aligned}
X_{1r} &= \begin{bmatrix} \mathcal{J}_{p_r}^l \\ \mathcal{O}_{p_1-p_r,p_r}\end{bmatrix}\otimes\left(\frac{1}{2}(-1)^{p_1-p_r+1}(G_{1,1}^{(0)})^{-1}G_{1,r}^{(l)}\right) \quad \left(\hat{=}\left[\begin{array}{ccc} 0 & * & 0\\ & \ddots & \\ 0 & & * \\ \hline 0 & \ldots & 0\end{array}\right]\right)\\
X_{r1} &= \begin{bmatrix} \mathcal{O}_{p_r,p_1-p_r} & \mathcal{J}_{p_r}^l\end{bmatrix}\otimes\left(\frac{1}{2}(-1)^{l+1}(G_{r,r}^{(0)})^{-*}(G_{1,r}^{(l)})^*\right).
\end{aligned}
$$

Substep (2a) X is chosen such that it commutes with A.

Substep (2b) In the updated matrix $\tilde{G}:=X^*GX$ we have $\tilde{G}_{1,r}^{(l)}=0$.

Indeed, it is easy to see that $\tilde{G}$ is again a matrix of the form (47), (48), and (49). The $(1,r)$-block of $\tilde{G}$ satisfies

$$
\tilde{G}_{1r}=G_{11}X_{1r}+X_{r1}^*G_{1r}^*X_{1r}+G_{1r}+X_{r1}^*G_{rr}. \tag{52}
$$

From the structure of G and X, we find immediately that the first $l-1$ block anti-diagonals of all the summands of the right hand side of (52) are zero. Furthermore, the l-th block anti-diagonal of $\tilde{G}_{1r}$ has the form

$$
\begin{aligned}
&(F_{p_1}\otimes G_{1,1}^{(0)})\left(\begin{bmatrix} \mathcal{J}_{p_r}^l \\ \mathcal{O}_{p_1-p_r,p_r}\end{bmatrix}\otimes\left(\frac{1}{2}(-1)^{p_1-p_r+1}(G_{1,1}^{(0)})^{-1}G_{1,r}^{(l)}\right)\right)\\
&+\begin{bmatrix} \mathcal{O}_{p_1-p_r,p_r} \\ F_{p_r}\mathcal{J}_{p_r}^l\end{bmatrix}\otimes G_{1,r}^{(l)}\\
&+\left(\begin{bmatrix} \mathcal{O}_{p_r,p_1-p_r} \\ (\mathcal{J}_{p_r}^l)^*\end{bmatrix}\otimes\left(\frac{1}{2}(-1)^{l+1}G_{1,r}^{(l)}(G_{r,r}^{(0)})^{-1}\right)\right)(F_{p_r}\otimes G_{r,r}^{(0)})\\
=\;&\frac{1}{2}(-1)^{p_1-p_r+1}\left(F_{p_1}\begin{bmatrix} \mathcal{J}_{p_r}^l \\ \mathcal{O}_{p_1-p_r,p_r}\end{bmatrix}\right)\otimes G_{1,r}^{(l)}+\begin{bmatrix} \mathcal{O}_{p_1-p_r,p_r} \\ F_{p_r}\mathcal{J}_{p_r}^l\end{bmatrix}\otimes G_{1,r}^{(l)}\\
&+\frac{1}{2}(-1)^{l+1}\left(\begin{bmatrix} \mathcal{O}_{p_r,p_1-p_r} \\ (\mathcal{J}_{p_r}^l)^*\end{bmatrix}F_{p_r}\right)\otimes G_{1,r}^{(l)}=0,
\end{aligned}
$$

using (39) and (41).

Substep (2c) In the updated matrix $\tilde{G} := X^* G X$ we still have $\tilde{G}_{1,j}^{(l-1)} = 0$ for all $j = 2, \ldots, k$ and $\tilde{G}_{1,j}^{(l)} = 0$ for $j = 2, \ldots, r-1$.

Indeed, the elements of the first block row of $\tilde{G}$ have the form

$$\begin{array}{ll} G_{1q} + X_{r1}^* G_{qr}^* & \text{for } 1 < q < r \quad \text{and} \\ G_{1q} + X_{r1}^* G_{rq} & \text{for } r < q. \end{array}$$

From the block structure of G_{1q}, G_{rq}, G_{qr}, and X_{r1}, we obtain that the first $p_q - p_r + 2l$ block anti-diagonals in $X_{r1}^* G_{qr}^*$ and the first $p_r - p_q + 2l - 1$ block anti-diagonals in $X_{r1}^* G_{rq}$ are zero.

Substep (2d) We show that the matrix $\tilde{H} := X^* H X$ is block diagonal.

The only changes outside the block diagonal can have happened to the $(1,r)$-block $\tilde{H}_{1r}$ and the $(r,1)$-block $\tilde{H}_{r1} = \tilde{H}_{1r}^*$. The $(1,r)$-block has the form

$$\begin{aligned} \tilde{H}_{1r} &= (Z_{p_1} \otimes \Sigma_{m_1}) X_{1r} + X_{1r}^* (Z_{p_r} \otimes \Sigma_{m_r}) && (53) \\ &= \frac{1}{2} \begin{bmatrix} \mathcal{O}_{p_1 - p_r, p_r} \\ Z_{p_r} \mathcal{J}_{p_r}^l \end{bmatrix} && (54) \\ &\otimes \left((-1)^{p_1 - p_r + 1} \Sigma_{m_1} (G_{1,1}^{(0)})^{-1} G_{1,r}^{(l)} + (-1)^{l+1} G_{1,r}^{(l)} (G_{r,r}^{(0)})^{-1} \Sigma_{m_r} \right), && (55) \end{aligned}$$

using (39) and (40). On the other hand, we have $G H^{-1} G = \gamma^2 H$. Noting that $H^{-1} = H$ and comparing the $(1,r)$ blocks of both sides, we obtain that

$$0 = G_{11} H_{11} G_{1r} + \left(\sum_{q=2}^{r-1} G_{1q} H_{qq} G_{qr} \right) + G_{1r} H_{rr} G_{rr} + \left(\sum_{q=r+1}^{k} G_{1q} H_{qq} G_{rq}^* \right). \qquad (56)$$

Clearly, the first $l-1$ anti-diagonals of all the summands in (56) are zero. We now consider the l-th block anti-diagonal. We note that $G_{11} H_{11} G_{1r}$ and $G_{1r} H_{rr} G_{rr}$ are the only summands that have a nonzero l-th block anti-diagonal. For the terms $G_{1q} H_{qq} G_{qr}$, $1 < q < r$ this follows from the fact that the l-th block anti-diagonal of G_{1q} is already zero. For $G_{1q} H_{qq} G_{rq}^*$, $q > r$ this can be seen as follows. If we write the j-th block antidiagonal of $G_{1q} H_{qq} G_{rq}^*$ in the form $S_j \otimes T_j$, then we obtain

$$\begin{aligned} S_j &= \begin{array}{c} \\ p_1 - p_q \\ p_q \end{array} \begin{array}{c} p_q \\ \begin{bmatrix} 0 \\ F_{p_q} \mathcal{J}_{p_q}^j \end{bmatrix} \end{array} Z_{p_q} \left(\begin{array}{c} \\ p_q \end{array} \begin{array}{cc} p_r - p_q & p_q \\ \begin{bmatrix} 0 & F_{p_q}^* \end{bmatrix} \end{array} \right) \\ &= \begin{array}{c} \\ p_1 - p_r \\ p_r - p_q \\ p_q \end{array} \begin{array}{cc} p_r - p_q & p_q \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & F_{p_q} \mathcal{J}_{p_q}^j Z_{p_q} F_{p_q}^* \end{bmatrix} \end{array}. \end{aligned}$$

Having in mind that the first $l-1$ block anti-diagonals of G_{1q} are zero, we find that the first $p_r - p_q + l - 1$ block anti-diagonals of $G_{1q}H_{qq}G_{rq}^*$ are zero.

Finally comparing the l-th block anti-diagonals in (56), we obtain

$$\begin{aligned} 0 &= (F_{p_1} \otimes G_{1,1,0})(Z_{p_1} \otimes \Sigma_{m_1}) \left(\begin{bmatrix} \mathcal{O}_{p_1-p_r,p_r} \\ F_{p_r}\mathcal{J}_{p_r}^l \end{bmatrix} \otimes G_{1,r}^{(l)} \right) \\ &\quad + \left(\begin{bmatrix} \mathcal{O}_{p_1-p_r,p_r} \\ F_{p_r}\mathcal{J}_{p_r}^l \end{bmatrix} \otimes G_{1,r}^{(l)} \right) (Z_{p_r} \otimes \Sigma_{m_r})(F_{p_r} \otimes G_{r,r}^{(0)}) \\ &= \begin{bmatrix} \mathcal{O}_{p_1-p_r,p_r} \\ Z_{p_r}\mathcal{J}_{p_r}^l \end{bmatrix} \otimes \left((-1)^{p_1-p_r} G_{1,1}^{(0)} \Sigma_{m_1} G_{1,r}^{(l)} + (-1)^l G_{1,r}^{(l)} \Sigma_{m_r} G_{r,r}^{(0)} \right), \end{aligned}$$

using (39), (40), and (41). Using then (51) and (54) this implies $\tilde{H}_{1r} = 0$.

Substep (2e): Retrieving H.

Although $\tilde{H}$ is block diagonal, the diagonal blocks may differ from those of H. We now show how to retrieve H from $\tilde{H}$ while keeping the zero block anti-diagonals of $\tilde{G}$. It follows from Theorem 9 that there exists a nonsingular matrix $T \in \mathbb{C}^{p_1 m_1 \times p_1 m_1}$ such that

$$T^{-1}(\mathcal{J}_{p_1} \otimes I_{m_1})T = \mathcal{J}_{p_1} \otimes I_{m_1} \quad \text{and} \quad T^* \tilde{H}_{11} T = Z_{p_1} \otimes \Sigma_{m_1}.$$

Since T commutes with $\mathcal{J}_{p_1} \otimes I_{m_1}$, it has the block structure

$$T = \begin{bmatrix} T_1 & \dots & T_{m_1} \\ & \ddots & \vdots \\ 0 & & T_1 \end{bmatrix}$$

with $T_j \in \mathbb{C}^{p_1 \times p_1}$, $j = 1, \dots, m_1$. Setting $\tilde{T} := T \oplus I_{p_2 m_2} \oplus \dots \oplus I_{p_k m_k}$, we obtain

$$\tilde{T}^{-1} A \tilde{T} = A \quad \text{and} \quad \tilde{T}^* \tilde{H} \tilde{T} = H.$$

Moreover, the $(1,q)$-block of $\tilde{T}^* \tilde{G} \tilde{T}$ has the form $\tilde{T}^* \tilde{G}_{1q}$. Note that the multiplication from the left with $\tilde{T}^*$ does neither change the first $l-1$ block anti-diagonals of $\tilde{G}_{1q}$ for $q = 2, \dots, k$ nor the l-th block anti-diagonals of $\tilde{G}_{1q}$ for $q = 2, \dots, r$.

Substep (2f): By induction, we finally obtain that there exists a nonsingular matrix S, such that

$$S^{-1} A S = \begin{bmatrix} \mathcal{J}_{p_1} \otimes I_{m_1} & 0 \\ 0 & A_2 \end{bmatrix}, \quad S^* H S = \begin{bmatrix} Z_{p_1} \otimes \Sigma_{m_1} & 0 \\ 0 & H_2 \end{bmatrix},$$

$$\text{and} \quad S^* G S = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix},$$

where $G_1 \in \mathbb{C}^{p_1 m_1 \times p_1 m_1}$. Hence, it is sufficient to assume that we are in the following situation.

$$A = \mathcal{J}_p \otimes I_m, \quad H = Z_p \otimes \Sigma \quad \text{and} \tag{57}$$

$$G=\sum_{k=0}^{p-1}(F_p\mathcal{J}_p^k)\otimes G_k=\begin{bmatrix} & 0 & & & G_0 \\ & & & -G_0 & -G_1 \\ & & \iddots & \iddots & \vdots \\ (-1)^{p+1}G_0 & \dots & & \dots & (-1)^{p+1}G_{p-1}\end{bmatrix}, \tag{58}$$

where $k,m,p\in\mathbb{N}$, Σ is a signature matrix and $G_j\in\mathbb{C}^{m\times m}$ for $j=0,\dots,p-1$.

Step (3): Reducing G to block anti-diagonal form.

Assume that we already have $G_1=\ldots=G_{l-1}=0$ for some $l\le p-1$. We then eliminate G_l while keeping the structure of A and H.

Substep (3a): Elimination of G_l.

Since G is Hermitian and F_p is Hermitian for odd p and skew-Hermitian for even p, we obtain that

$$G_k^*=(-1)^{p+k+1}G_k. \tag{59}$$

This implies, in particular, that

$$(G_0^{-1}G_l)^*=(-1)^lG_lG_0^{-1}. \tag{60}$$

Setting

$$X:=I_p\otimes I_m-\frac{1}{2}\mathcal{J}_p^l\otimes(G_0^{-1}G_l),$$

it follows that X commutes with A. Moreover, we obtain that the first $l-1$ block anti-diagonals in $\tilde{G}:=X^*GX$ are still zero. Then using (39), it follows that the l-th block anti-diagonal has the form

$$(I_p\otimes I_m)\Big((F_p\mathcal{J}_p^l)\otimes G_l\Big)(I_p\otimes I_m)-\frac{1}{2}(-1)^l\Big((\mathcal{J}_p^l)^*\otimes(G_lG_0^{-1})\Big)(F_p\otimes G_0)(I_p\otimes I_m)$$
$$-\frac{1}{2}(I_p\otimes I_m)(F_p\otimes G_0)\Big(\mathcal{J}_p^l\otimes(G_0^{-1}G_l)\Big)=0.$$

Substep (3b): Retrieving H.

Comparing the l-th block anti-diagonals on both sides of $GH^{-1}G=\gamma^2H$ and using that $G_1=\ldots=G_{l-1}=0$, by applying (39) and Lemma 4 we obtain that

$$\begin{aligned} 0 &= (F_p\otimes G_0)(Z_p\otimes\Sigma)\Big((F_p\mathcal{J}_p^l)\otimes G_l\Big)+\Big((F_p\mathcal{J}_p^l)\otimes G_l\Big)(Z_p\otimes\Sigma)(F_p\otimes G_0) \\ &= (F_pZ_pF_p\mathcal{J}_p^l)\otimes\Big(G_0\Sigma G_l+(-1)^lG_l\Sigma G_0\Big). \end{aligned}$$

This implies, in particular, that for $l\ge p-1$

$$G_lG_0^{-1}\Sigma+(-1)^l\Sigma G_0^{-1}G_l=0.$$

Here we have used the identity $G_0 \Sigma G_0 = \gamma^2 \Sigma$, which follows from comparing the diagonal blocks in $GH^{-1}G = \gamma^2 H$. Therefore, with this relation and (60) we obtain that

$$
\begin{aligned}
& X^* H X \\
= \quad & Z_p \otimes \Sigma - \frac{1}{2} Z \mathcal{J}_p^l \otimes \left(G_l^* G_0^{-*} \Sigma + \Sigma G_0^{-1} G_l \right) + \frac{1}{4} \left((\mathcal{J}_p^l)^* Z_p \mathcal{J}_p^l \right) \otimes \left(G_l^* G_0^{-*} \Sigma G_0^{-1} G_l \right) \\
= \quad & Z_p \otimes \Sigma - \frac{1}{4} (Z_p \mathcal{J}_p^{2l}) \otimes \left(\Sigma (G_0^{-1} G_l)^2 \right).
\end{aligned}
$$

The $(2l)$-th block anti-diagonal of $X^* H X$ can now be eliminated by a congruence transformation with

$$
Y = I_r \otimes I_m + \frac{1}{8} \mathcal{J}_p^{2l} \otimes (G_0^{-1} G_l)^2.
$$

This transformation does not change the first l block anti-diagonals of $\tilde{G}$ but may change the j-th block anti-diagonal of $X^* H X$ for some $j > 2l$. However, repeating the procedure described above a finite number of times, we can finally retrieve H while keeping the property that the first l block anti-diagonals in $\tilde{G}$ are zero.

Substep (3c): By induction, we finally obtain that there exists a nonsingular matrix S, such that

$$
S^{-1} A S = \mathcal{J}_p \otimes I_m, \quad S^* H S = Z_p \otimes \Sigma = \begin{bmatrix} 0 & & \Sigma \\ & \iddots & \\ \Sigma & & 0 \end{bmatrix},
$$

$$
\text{and} \quad S^* G S = F_p \otimes G_0 = \begin{bmatrix} 0 & & G_0 \\ & \iddots & \\ (-1)^{p+1} G_0 & & 0 \end{bmatrix}.
$$

Step (4): Final reduction of G.

Since the pencil $\varrho H - G$ is nondefective and its spectrum is contained in $\{\gamma, -\gamma\}$, this also holds for each subpencil $\varrho \Sigma - (\pm G_0)$. We will distinguish four cases.

Case (a): $\gamma \in \mathbb{R}$ and p is even.

Identity (59) implies that G_0 is skew-Hermitian. Since the pencil $\varrho \Sigma - (\pm G_0)$ has only real eigenvalues γ and/or $-\gamma$, it follows that $\varrho \Sigma - (\pm G_0)$ has both eigenvalues with equal algebraic multiplicity. This implies, in particular, that m is even and that there exists a nonsingular matrix $R \in \mathbb{C}^{m \times m}$ such that

$$
R^* \Sigma R = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \quad \text{and} \quad R^* G_0 R = \begin{bmatrix} 0 & \gamma I \\ -\gamma I & 0 \end{bmatrix}.
$$

Set $\mathcal{R} = I_p \otimes R$. Then

$$
\mathcal{R}^{-1} A \mathcal{R} = A, \quad \mathcal{R}^* H \mathcal{R} = Z_p \otimes \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \text{and} \quad \mathcal{R}^* G \mathcal{R} = F_p \otimes \begin{bmatrix} 0 & \gamma I \\ -\gamma I & 0 \end{bmatrix}.
$$

Applying a transformation with $\Omega_{p,m}$, the form stated in (45) for the case that p is even, follows from a proper block permutation.

Case (b): $\gamma \in \mathbb{R}$ and p is odd.

In this case, (59) implies that G_0 is Hermitian. Considering the Hermitian pencil $\varrho\Sigma - (\pm G_0)$, there exists a nonsingular matrix $R \in \mathbb{C}^{m\times m}$ such that

$$R^*\Sigma R = \Sigma \quad \text{and} \quad R^* G_0 R = \gamma\tilde{\Sigma},$$

where $\tilde{\Sigma}$ is another signature matrix. Setting $\mathcal{R} := I_p \otimes R$ and applying transformations with $\mathcal{R}$ and $\Omega_{p,m}$, the form stated in (44) for the case that p is odd follows from a proper block permutation.

Case (c): $\gamma \in i\mathbb{R}$ and p is even.

In this case, (59) implies that G_0 is Hermitian. The rest follows as in Case (b).

Case (d): $\gamma \in i\mathbb{R}$ and p is odd.

This case follows analogously to Case (a). This concludes the proof. □

Definition 28 Let $A = (a_{jk})_{nn} \in \mathbb{C}^{n\times n}$. Then the l-th upper diagonal of A or, in short, the l-th diagonal of A is defined by the elements a_{jk}, where $k = j + l$. Here, we allow $l = 0$. If

$$B = \begin{bmatrix} 0 & \tilde{B} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} \tilde{C} \\ 0 \end{bmatrix},$$

where $\tilde{B}$ and $\tilde{C}$ are square matrices, then the l-th diagonal of $\tilde{B}$ and $\tilde{C}$ is called the l-th diagonal of B and C, respectively. Analogously, we define the l-th block diagonal for square and non-square block matrices.

Lemma 29 *Supose that $A_0, G_0 \in \mathbb{C}^{n\times n}$ anti-commute, i.e. $A_0G_0 = -G_0A_0$. Furthermore, let A_0 be nilpotent and G_0 be diagonalizable and nonsingular. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$\begin{aligned} P^{-1}A_0P &= A_1 \oplus \ldots \oplus A_k, \\ P^{-1}G_0P &= G_1 \oplus \ldots \oplus G_k, \end{aligned} \tag{61}$$

where the blocks A_j, G_j have corresponding sizes and, for each j, are of the following form.

$$A_j = \mathcal{J}_p(0) \quad \text{and} \quad G_j = \varepsilon_j \gamma D_p, \tag{62}$$

where $p \in \mathbb{N}$, $\gamma \in \mathbb{C}$ with $Re(\gamma) \geq 0$ and $Im(\gamma) > 0$ if $Re(\gamma) = 0$, and $\varepsilon_j \in \{+1, -1\}$. Moreover, the form (61) is unique up to the permutation of blocks.

Proof. Let $Q \in \mathbb{C}^{n\times n}$ be nonsingular such that

$$Q^{-1}A_0Q = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad Q^{-1}G_0Q = \begin{bmatrix} G_{11} & 0 \\ 0 & G_{22} \end{bmatrix},$$

where the spectrum of G_{11} is contained in $\{\gamma, -\gamma\}$ and the spectrum of G_{22} is disjoint from $\{\gamma, -\gamma\}$. Then $-A_0G_0 = G_0A_0$ implies $A_{12} = A_{21} = 0$. Hence, we may assume w.l.o.g. that G_0 has at most the eigenvalues $\gamma, -\gamma$, where $\gamma \in \mathbb{C}\backslash\mathbb{R}$ with $\mathrm{Re}(\gamma) \geq 0$ and $\mathrm{Im}(\gamma) > 0$ if $\mathrm{Re}(\gamma) = 0$. Since G_0 is diagonalizable, this implies in particular that $G_0^2 = \gamma^2 I_n$. Furthermore, we may assume that A_0 is in Jordan canonical form. Thus, we obtain that

$$A_0 = \begin{bmatrix} \mathcal{J}_{p_1} \otimes I_{m_1} & & 0 \\ & \ddots & \\ 0 & & \mathcal{J}_{p_k} \otimes I_{m_k} \end{bmatrix} \quad \text{and} \quad G_0 = \begin{bmatrix} G_{11} & \dots & G_{1k} \\ \vdots & \ddots & \vdots \\ G_{k1} & \dots & G_{kk} \end{bmatrix}. \tag{63}$$

for integers $p_1 \geq \ldots \geq p_k$, $m_1, \ldots, m_k$ and $G_{qr} \in \mathbb{C}^{m_q\times m_r}$. Setting

$$D := \begin{bmatrix} D_{p_1} \otimes I_{m_1} & & 0 \\ & \ddots & \\ 0 & & D_{p_k} \otimes I_{m_k} \end{bmatrix},$$

and using (39), the fact that A_0 and G_0 anti-commute is equivalent to $A_0(DG_0) = (DG_0)A_0$. Therefore, we obtain the following structures for the blocks of G_0.

$$G_{qq} = \sum_{j=0}^{p_q-1}(D_{p_q}\mathcal{J}_{p_q}^j) \otimes G_{q,q}^{(j)}, \quad G_{qr} = \sum_{j=0}^{p_r-1} \begin{bmatrix} D_{p_r}\mathcal{J}_{p_r}^j \\ \mathcal{O}_{p_q-p_r,p_r} \end{bmatrix} \otimes G_{q,r}^{(j)} \quad \text{for } q < r, \quad \text{and} \tag{64}$$

$$G_{qr} = \sum_{j=0}^{p_q-1} \begin{bmatrix} \mathcal{O}_{p_q,p_r-p_q} & D_{p_q}\mathcal{J}_{p_q}^j \end{bmatrix} \otimes G_{q,r}^{(j)} \quad \text{for } q > r, \tag{65}$$

where $G_{q,q}^{(j)}$ and $G_{q,r}^{(j)}$ are matrices of suitable dimensions. We will now reduce G_0 stepwise to canonical form.

Step (1): Since $G_0^2 = \gamma^2 I$, as in Step (1) in the proof of Lemma 27, it follows that $G_{q,q}^{(l)}$ is nonsingular.

Step (2): Elimination of $G_{12}, \ldots, G_{1k}$ and $G_{21}, \ldots, G_{k1}$.

Assume that we already have $G_{1,j}^{(l-1)} = 0$ for all $j = 2, \ldots, k$ and $G_{1,j}^{(l)} = 0$ for $j = 2, \ldots, r-1$, where $l \geq 0$, $r > 1$. We then eliminate $G_{1,r}^{(l)}$. Note that $G_0^2 = \gamma^2 I$ implies that

$$G_{11}G_{1r} + \ldots + G_{1k}G_{kr} = 0$$

for $r > 1$. From this and using an argument similar to the argument in Step (1) in the proof of Lemma 27, we obtain that only the blocks $G_{11}G_{1r}$ and $G_{1r}G_{rr}$ contribute to the l-th diagonal of the left hand side. Using (39), this implies that

$$
\begin{aligned}
0 &= (D_{p_1} \otimes G^{(0)}_{1,1}) \left(\begin{bmatrix} D_{p_r}\mathcal{J}^l_{p_r} \\ \mathcal{O}_{p_1-p_r,p_r} \end{bmatrix} \otimes G^{(l)}_{1,r} \right) \\
&\quad + \left(\begin{bmatrix} D_{p_r}\mathcal{J}^l_{p_r} \\ \mathcal{O}_{p_1-p_r,p_r} \end{bmatrix} \otimes G^{(l)}_{1,r} \right) (D_{p_r} \otimes G^{(0)}_{r,r}) \\
&= \begin{bmatrix} \mathcal{J}^l_{p_r} \\ \mathcal{O}_{p_1-p_r,p_r} \end{bmatrix} \otimes \left(G^{(0)}_{1,1}G^{(l)}_{1,r} + (-1)^l G^{(l)}_{1,r}G^{(0)}_{r,r} \right). \qquad (66)
\end{aligned}
$$

Setting

$$
X_0 := \begin{array}{c} \begin{array}{cccc} & & r & \end{array} \\ \begin{bmatrix} I & & X_{1r} & \\ & \ddots & & \\ & & \ddots & \\ & & & I \end{bmatrix} \end{array},
$$

where

$$
X_{1r} = -\frac{1}{2} \begin{bmatrix} \mathcal{J}^l_{p_r} \\ \mathcal{O}_{p_1-p_r,p_r} \end{bmatrix} \otimes (G^{(0)}_{1,1})^{-1} G^{(l)}_{1,r},
$$

we obtain that X_0 commutes with A_0. Furthermore, partitioning $\tilde{G}_0 := X_0^{-1} G_0 X_0$ conformably to G_0, we obtain for the $(1, r)$ block $\tilde{G}_{1r}$ that

$$
\tilde{G}_{1r} = G_{1r} - X_{1r}G_{rr} + G_{11}X_{1r} - X_{1r}G_{r1}X_{1r}.
$$

From this and using (39) and (66), we obtain that the l-th diagonal of $\tilde{G}_{1r}$ has the form

$$
\begin{aligned}
&\begin{bmatrix} D_{p_r}\mathcal{J}^l_{p_r} \\ \mathcal{O}_{p_1-p_r,p_r} \end{bmatrix} \otimes G^{(l)}_{1,r} + \frac{1}{2} \left(\begin{bmatrix} \mathcal{J}^l_{p_r} \\ \mathcal{O}_{p_1-p_r,p_r} \end{bmatrix} D_{p_r} \right) \otimes ((G^{(0)}_{1,1})^{-1} G^{(l)}_{1,r} G^{(0)}_{r,r}) \\
&\qquad - \frac{1}{2} \left(D_{p_1} \begin{bmatrix} \mathcal{J}^l_{p_r} \\ \mathcal{O}_{p_1-p_r,p_r} \end{bmatrix} \right) \otimes G^{(l)}_{1,r} = 0.
\end{aligned}
$$

Analogously to the proof of Lemma 27, we can show that we still have $\tilde{G}^{(l-1)}_{1,j} = 0$ for all $j = 2, \ldots, k$ and $\tilde{G}^{(l)}_{1,j} = 0$ for $j = 2, \ldots, r-1$.

By induction, we can analogously eliminate $G_{12}, \ldots, G_{1k}$. Moreover, we can eliminate $G_{21}, \ldots, G_{k1}$ using transformations of the form

$$
X_0 := \begin{array}{c} \\ \\ r \\ \\ \end{array} \begin{bmatrix} I & & & \\ & \ddots & & \\ X_{r1} & & \ddots & \\ & & & I \end{bmatrix},
$$

where

$$X_{r1} = -\frac{1}{2}\left[\begin{array}{cc} \mathcal{O}_{p_r,p_1-p_r} & \mathcal{J}_{p_r}^l \end{array}\right] \otimes (G_{1,r}^{(l)}(G_{1,1}^{(0)})^{-1}).$$

Note that these transformations do not change $G_{12}, \ldots, G_{1k}$.

To complete Step (2), we may finally assume that

$$A = \mathcal{J}_p \otimes I_m, \quad G_0 = \sum_{j=k}^{p-1} (D_p \mathcal{J}_p^k) \otimes G_{0k} = \left[\begin{array}{ccc} G_{00} & \ldots & G_{0,p-1} \\ & \ddots & \vdots \\ 0 & & (-1)^{p+1} G_{00} \end{array}\right]. \tag{67}$$

where $p, m \in \mathbb{N}$.

Step (3): Reducing G_0 to block diagonal form.

Assume that we have $G_{01} = \ldots = G_{0,l-1} = 0$ for some $0 < l \leq p-1$. We then show how to eliminate G_{0l}. The l-th block diagonal of G_0^2 has the form

$$\begin{aligned} 0 &= \Big((D_p \mathcal{J}_p^l) \otimes G_{0l}\Big)(D_p \otimes G_{00}) + (D_p \otimes G_{00})\Big((D_p \mathcal{J}_p^l) \otimes G_{0l}\Big) \\ &= \mathcal{J}_p^l \otimes \Big((-1)^l G_{0l} G_{00} + G_{00} G_{0l}\Big). \end{aligned}$$

Hence

$$G_{0l} G_{00} = (-1)^{l+1} G_{00} G_{01}. \tag{68}$$

The matrix $X_0 := I_p \otimes I_m - \frac{1}{2} \mathcal{J}_p^l \otimes (G_{00}^{-1} G_{0l})$ commutes with A_0. Moreover, X_0^{-1} has the structure

$$X_0^{-1} = I_p \otimes I_m + \frac{1}{2} \mathcal{J}_p^l \otimes (G_{00}^{-1} G_{0l}) + \sum_{k=2}^{\infty} \mathcal{J}_p^{kl} \otimes X_{0k}$$

for some matrices X_{0k}. Hence, setting $\tilde{G}_0 := X_0^{-1} G_0 X_0$, we obtain that the first $l-1$ block diagonals are still zero and that the l-th block diagonal has the form

$$\begin{aligned} & \Big(\frac{1}{2} \mathcal{J}_p^l \otimes (G_{00}^{-1} G_{0l})\Big)(D_p \otimes G_{00})(I_p \otimes I_m) + (D_p \mathcal{J}_p^l) \otimes G_{0l} \\ & +(I_p \otimes I_m)(D_p \otimes G_{00})\Big(-\frac{1}{2} \mathcal{J}_p^l \otimes (G_{00}^{-1} G_{0l})\Big) \\ =\; & 0, \end{aligned}$$

using (39) and (68).

By an induction argument and then applying $\Omega_{m,p}$, we may finally assume that

$$A = I_m \otimes J_p \quad \text{and} \quad G_0 = G_{00} \otimes D_p.$$

Since G_0 is diagonalizable, this also holds for the matrix G_{00}. Moreover, G_{00} has at most the eigenvalues γ and $-\gamma$. Hence, there exists a nonsingular matrix R such that

$$R^{-1} G_{00} R = \left[\begin{array}{cc} \gamma I_q & 0 \\ 0 & -\gamma I_{m-q} \end{array}\right]$$

for some $q \in \mathbb{N}$. Setting $\mathcal{R} := R \otimes I_p$, we obtain that $\mathcal{R}^{-1}A_0\mathcal{R} = A_0$ and

$$\mathcal{R}^{-1}G\mathcal{R} = \begin{bmatrix} \gamma I_q & 0 \\ 0 & -\gamma I_{m-q} \end{bmatrix} \otimes D_p.$$

The assertion then follows by a proper block permutation.

Uniqueness: Analogous to the argument in the proofs of Theorem 16 and 20, it is sufficient to consider uniqueness for the case that G_0 has at most the eigenvalues $\gamma, -\gamma$ with $\mathrm{Re}(\gamma) \geq 0$ and $\mathrm{Im}(\gamma) > 0$ if $\mathrm{Re}(\gamma) = 0$. Assume that

$$A_0 = \begin{bmatrix} I_{m_p} \otimes \mathcal{J}_p & & \\ & \ddots & \\ & & I_{m_1} \otimes \mathcal{J}_1 \end{bmatrix}, \quad G_0 = \gamma \begin{bmatrix} \Sigma_{m_p} \otimes D_p & & \\ & \ddots & \\ & & \Sigma_{m_1} \otimes D_1 \end{bmatrix},$$

$$\text{and} \quad \tilde{G}_0 = \gamma \begin{bmatrix} \tilde{\Sigma}_{m_p} \otimes D_p & & \\ & \ddots & \\ & & \tilde{\Sigma}_{m_1} \otimes D_1 \end{bmatrix},$$

where we allow $m_j = 0$ for some $j = 1, \ldots, p$ and where Σ_{m_j} and $\tilde{\Sigma}_{m_j}$ are signature matrices. To prove the uniqueness of the form (61), we have to show that if $S \in \mathbb{C}^{n \times n}$ is nonsingular such that $S^{-1}A_0S = A_0$ and $S^{-1}G_0S = \tilde{G}_0$, then Σ_{m_j} and $\tilde{\Sigma}_{m_j}$ are similar for $j = 1, \ldots, p$.

Note that for each Jordan block, there exists a Jordan chain $\{x_{\alpha\beta}^{(1)}, \ldots, x_{\alpha\beta}^{(\alpha)}\}$, where $\alpha = p, \ldots, 1$ and $\beta = 1, \ldots, m_p$. Let P be the permutation matrix that reorders these chains in the following way. First, we collect $x_{\alpha\beta}^{(1)}$ for $\alpha = p, \ldots, 1$, $\beta = 1, \ldots, m_p$, then $x_{\alpha\beta}^{(2)}$ for $\alpha = p, \ldots, 2$, $\beta = 1, \ldots, m_p$, and so on. Denote $q_r = \sum_{j=1}^{r} m_j$. Then

$$\hat{A}_0 := P^{-1}A_0P = \begin{array}{c} \\ q_p \\ q_{p_1} \\ q_{p-2} \\ \vdots \\ q_1 \end{array} \begin{array}{c} \begin{array}{ccccc} q_p & q_{p-1} & q_{p-2} & \cdots & q_1 \end{array} \\ \left[\begin{array}{ccccc} 0 & \begin{bmatrix} I_{q_{p-1}} \\ 0 \end{bmatrix} & 0 & & \\ & 0 & \begin{bmatrix} I_{q_{p-2}} \\ 0 \end{bmatrix} & \ddots & \\ & & 0 & \ddots & 0 \\ & & & \ddots & \begin{bmatrix} I_{q_1} \\ 0 \end{bmatrix} \\ & & & & 0 \end{array}\right] \end{array}.$$

Moreover, we have

$$\hat{G}_0 := P^{-1}G_0P = \gamma \begin{bmatrix} G_{11} & & 0 \\ & \ddots & \\ 0 & & G_{pp} \end{bmatrix}$$

$$\text{and}\quad \hat{\tilde{G}}_0 := P^{-1}\tilde{G}_0 P = \gamma \begin{bmatrix} \tilde{G}_{11} & & 0 \\ & \ddots & \\ 0 & & \tilde{G}_{pp} \end{bmatrix},$$

where

$$G_{jj} = (-1)^{j+1} \begin{bmatrix} \Sigma_{m_p} & & 0 \\ & \ddots & \\ 0 & & \Sigma_{m_j} \end{bmatrix} \quad \text{and} \quad \tilde{G}_{jj} = (-1)^{j+1} \begin{bmatrix} \tilde{\Sigma}_{m_p} & & 0 \\ & \ddots & \\ 0 & & \tilde{\Sigma}_{m_j} \end{bmatrix}.$$

Assume that there exists a nonsingular matrix T such that $T^{-1}\hat{A}_0 T = \hat{A}_0$ and $T^{-1}\hat{G}_0 T = \hat{\tilde{G}}_0$. Then the structure of $\hat{A}_0$ implies that T is block upper triangular with a block structure corresponding to $\hat{A}_0$. But then we obtain in particular that G_{jj} and $\tilde{G}_{jj}$ are similar for each j. This implies that Σ_{m_j} and $\tilde{\Sigma}_{m_j}$ are similar for each j. □

Lemma 30 *Let $G, H \in \mathbb{C}^{n\times n}$ be Hermitian nonsingular such that the pencil $\varrho H - G$ is nondefective and such that its spectrum is contained in $\{\gamma, -\gamma, \overline{\gamma}, -\overline{\gamma}\}$, where $\gamma^2 \in \mathbb{C}\backslash\mathbb{R}$ and $Re(\gamma)Im(\gamma) \geq 0$. Furthermore, let $A \in \mathbb{C}^{n\times n}$ be nilpotent, H-selfadjoint and G-selfadjoint. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$\begin{aligned} P^{-1}AP &= A_1 \oplus \ldots \oplus A_k, \\ P^*GP &= G_1 \oplus \ldots \oplus G_k, \\ P^*HP &= H_1 \oplus \ldots \oplus H_k, \end{aligned} \tag{69}$$

where, for each j, the blocks A_j, G_j, H_j have corresponding sizes and are of the following form.

Type (2d)*:*

$$A_j = \begin{bmatrix} \mathcal{J}_p(0) & 0 \\ 0 & \mathcal{J}_p(0)^* \end{bmatrix}, \quad H_j = \begin{bmatrix} 0 & Z_p \\ Z_p & 0 \end{bmatrix},$$
$$\text{and} \quad G_j = \begin{bmatrix} 0 & \varepsilon\gamma F_p \\ \varepsilon(-1)^{p+1}\overline{\gamma}F_p & 0 \end{bmatrix}, \tag{70}$$

where $p \in \mathbb{N}$, and $\varepsilon \in \{+1, -1\}$.

Moreover, the form (69) is unique up to the permutation of blocks.

Proof. Using the same argument as in Case (2) of the proof of Theorem 20, we may assume that A, H, and G have the following forms.

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & A_0^* \end{bmatrix}, \quad H = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \text{and} \quad G = \begin{bmatrix} 0 & G_0^* \\ G_0 & 0 \end{bmatrix}, \tag{71}$$

where

$$H^{-1}GH^{-1}G = \begin{bmatrix} \overline{\gamma}^2 I & 0 \\ 0 & \gamma^2 I \end{bmatrix}. \tag{72}$$

This implies, in particular, that $G_0^2 = \overline{\gamma}^2 I$. From $-A^*G = GA$, we obtain that A_0 and G_0 anti-commute. We will now reduce G by congruence transformations with matrices of the form

$$X = \begin{bmatrix} X_0 & 0 \\ 0 & X_0^{-*} \end{bmatrix}.$$

Then

$$X^{-1}AX = \begin{bmatrix} X_0^{-1}A_0X_0 & 0 \\ 0 & (X_0^{-1}A_0X_0)^* \end{bmatrix}, \quad X^*HX = H, \quad \text{and}$$

$$X^*GX = \begin{bmatrix} 0 & (X_0^{-1}G_0X_0)^* \\ X_0^{-1}G_0X_0 & 0 \end{bmatrix}.$$

Thus, the problem of reducing G, while keeping the forms of A and H, reduces to the problem of finding a canonical form for A_0 and G_0 under simultaneous similarity. This is done in Lemma 29. Hence, the result follows from noting that the spectrum of G_0 is contained in $\{\overline{\gamma}, -\overline{\gamma}\}$, and finally applying the Z-trick.

Uniqueness: Assume that

$$A = \begin{bmatrix} \mathcal{J} & 0 \\ 0 & \mathcal{J} \end{bmatrix}, \; H = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \; G_1 = \begin{bmatrix} 0 & G_{11}^* \\ G_{11} & 0 \end{bmatrix}, \; G_2 = \begin{bmatrix} 0 & G_{22}^* \\ G_{22} & 0 \end{bmatrix},$$

where $\mathcal{J}$ is a nilpotent matrix in Jordan canonical form, G_1, G_2 are Hermitian, and $\sigma(G_{11}) = \sigma(G_{22}) \subseteq \{\overline{\gamma}, -\overline{\gamma}\}$. Furthermore, assume that $T^{-1}AT = A$, $T^*HT = H$, and $T^*G_1T = G_2$ for some nonsingular matrix T. Partitioning

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \quad \text{and} \quad T^{-*} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

conformably with A, H, and G, we obtain that

$$T_{12} = S_{21} \quad \text{and} \quad G_{11}T_{12} = S_{21}G_{22}^* = T_{12}G_{22}^*.$$

This implies $T_{12} = 0$. Analogously, we show that $T_{21} = 0$ and hence, we obtain by symmetry $T_{22} = T_{11}^{-*}$. Hence, the uniqueness of the form (69) follows from the uniqueness property in Lemma 29. $\square$