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Numerische Simulation auf massiv parallelen Rechnern

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A non-conforming finite element method with anisotropic mesh grading for the Stokes problem in domains with edges

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Abstract The solution of the Stokes problem in three-dimensional domains with edges has anisotropic singular behaviour which is treated numerically by using anisotropic finite element meshes. The velocity is approximated by Crouzeix-Raviart (non-conforming $\mathcal{P}_1$) elements and the pressure by piecewise constants. This method is stable for general meshes (without minimal or maximal angle condition). The interpolation and consistency errors are of the optimal order $h \sim N^{-1/3}$ which is proved for tensor product meshes. As a by-product, we analyse also non-conforming prismatic elements with $\mathcal{P}_1 \oplus \operatorname{span}\{x_3^2\}$ as the local space for the velocity where x_3 is the direction of the edge.

Key Words Stokes problem, edge singularity, anisotropic mesh, Crouzeix-Raviart element, non-conforming finite element method, consistency error.

AMS(MOS) subject classification 65N30; 65N15, 65N50

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1 Introduction

The solution of the Stokes system in polygonal or polyhedral domains has in general singular behaviour near corners and edges of the domain. Hence standard numerical methods lose accuracy on quasi-uniform meshes, and locally refined meshes are proposed. Two-dimensional problems with corner singularities have been analyzed by Becker and Rannacher [6], Orlt and Sändig [18], and El Bouzid and Nicaise [7]. In the last reference, [7], the authors extend their results also to polyhedral domains where edge and corner singularities may appear.

In [7, 18] the authors use isotropic (regular in Ciarlet's sense) meshes refined in a neighbourhood of the singular edges and corners in order to compensate the singular behaviour of the solution. In three-dimensional problems this method leads to over-refinement near the edges, as we have seen in the analysis of mesh refinement for the Poisson problem [2]. Therefore we want to use anisotropic meshes with refinement only perpendicularly to the edge. This leads to elements with arbitrarily large aspect ratios, so called anisotropic elements. We remark that in viscous flow problems also laminar boundary layers may appear which can also be resolved favorably by using anisotropic meshes.

For the stability of the method it is required that the discrete spaces satisfy an inf-sup condition with a constant independent of the aspect ratio of the elements. Furthermore, the approximation error and, if the method is non-conforming, the consistency error must be estimated under the assumption of the weak regularity of the singular solution. Let us refer to results from the literature.

Quadrilateral elements have been analyzed by Becker and Rannacher [5, 6] and by Schötzau, Schwab, and Stenberg [21, 22]. In particular, the inf-sup condition with a constant independent of the aspect ratio was proved in [5] for stabilized $\mathcal{Q}_1 - \mathcal{P}_0$ and $\mathcal{Q}_1 - \mathcal{Q}_1$ rectangular elements, and in [6] for the $\tilde{\mathcal{Q}}_1 - \mathcal{P}_0$ rectangular element. By $\mathcal{Q}_1$ we denote, as usual, the space of bilinear functions, and by $\tilde{\mathcal{Q}}_1$ the non-parametric rotated $\mathcal{Q}_1$ element [19]. The consistency error was not analyzed. In [21, 22] quadrilateral and triangular elements have been considered for the *hp*-version of the finite element method, in particular combinations $\mathcal{Q}_n - \mathcal{Q}_{n-2}$ and $\mathcal{P}_n - \mathcal{P}_{n-2}$, $n \geq 2$. The inf-sup constant does not depend on the aspect ratio, but slightly on n^{-1} ($n^{-1/2}$ for the quadrilaterals and n^{-3} for the triangles). This is compensated by the exponentially good approximation. We remark that all these results were proved for the two-dimensional case.

Well-known triangular elements are the mini element ($\mathcal{P}_1 \oplus$ bubble) $- \mathcal{P}_1$, the Taylor-Hood element $\mathcal{P}_2 - \mathcal{P}_1$, and its modified form $\mathcal{P}_{1,h/2} - \mathcal{P}_1$. In standard proofs of the inf-sup condition for the isotropic case, the inverse inequality produces a factor h^{-1} which is compensated by a factor h coming from an approximation property. The same proof leads in the anisotropic case to an inf-sup constant depending on the aspect ratio. It has been reported by Russo that the mini element becomes instable on anisotropic meshes [1].

A non-conforming method on triangular and tetrahedral meshes is obtained by using the Crouzeix-Raviart (nonconforming $\mathcal{P}_1$) element for the velocity in combination with piecewise constant pressure. This element was analyzed by Acosta and Duran [1] for anisotropic meshes. The inf-sup condition is simple to prove, the challenge is the analysis of the consistency error. Acosta and Duran used the connection to Raviart-Thomas interpolation and succeeded in the case of regular solutions $(u,p) \in (H^2(\Omega))^3 \times H^1(\Omega)$. Our analysis of this element was performed independently, by a different approach, and, in particular, for solutions with edge singularities. Basic results, without connection to the Stokes system, have already been published in [4].

Since the analysis of the consistency error is not straightforward we are restricted here to tensor product domains $\Omega = G \times Z$ and tensor product meshes. The inf-sup condition holds for general meshes and the interpolation error can be estimated for non-tensor product meshes under a maximal angle condition and a coordinate system condition. We consider also pentahedral meshes where the elements are triangular prisms, because we needed this as an intermediate step in the basic investigations of the consistency error in [4].

The outline of the paper is as follows. In Section 2 we state the Stokes problem in a domain with an edge, introduce some function spaces, and prove the regularity result in the form appropriate for our further analysis. In Section 3 we describe and ana-lyse the discretization. We obtain the optimal finite element error estimate

$$\|u - u_h\|_{1,h} + \|p - p_h\|_{0,\Omega} \lesssim h\|f\|_{0,\Omega}$$

where $h \sim \max_K \operatorname{diam} K$ and $\|.\|_{m,h}^2 := \sum_K | \, . \, |_{m,K}^2$, $m \geq 0$. The notation $a \lesssim b$ means the existence of a positive constant C (which is independent of $\mathcal{T}_h$ and of the function under consideration) such that $a \leq Cb$. For the assessment of this result it is essential to point out that the number of elements/degrees of freedom is of the order h^{-3}, that means, it is asymptotically not larger than that for uniform meshes where only a reduced convergence order h^λ, $0 < \lambda < 1$, is obtained. A numerical test in Section 4 confirms our theoretical results.

2 Statement of the problem and regularity results

Let $\Omega = G \times Z$ where $G \subset \mathbb{R}^2$ is a polygonal domain and Z is a real interval. By the local nature of corner singularities (and then edge ones for Ω), we may suppose that G has possibly one corner with interior angle $\omega > \pi$ at the origin, the other interior angles being smaller than π. The corresponding edge of Ω is part of the x_3-axis and will be called the singular edge of Ω. Over this domain Ω, we consider the stationary Stokes problem with Dirichlet boundary conditions: Given a vector function $f = (f_1, f_2, f_3)$, find a vector function $u = (u_1, u_2, u_3)$ representing the velocity of the fluid and a scalar function p representing the pressure and satisfying

$$\left\{ \begin{array}{rcll} -\Delta u + \nabla p &=& f & \text{in } \Omega, \\ \nabla \cdot u &=& 0 & \text{in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega. \end{array} \right. \tag{1}$$

Here we use the weak formulation which has a unique solution $(u,p) \in X \times M$,

$$X := \{v \in (H^1(\Omega))^3 : v|_\Gamma = 0\}, \qquad M := \{v \in L^2(\Omega) : \int_\Omega v = 0\},$$

for $f \in L^2(\Omega)^3$ as shown in [11, Theorem I.5.1], namely

$$\left\{ \begin{array}{lcll} a(u,v) + b(v,p) & = & (f,v) & \forall v \in X, \\ b(u,q) & = & 0 & \forall q \in M, \end{array} \right. \tag{2}$$

where

$$a(v,w) = \sum_{i=1}^{3} \int_\Omega \nabla v_i(x) \cdot \nabla w_i(x), \qquad b(v,q) = -\int_\Omega q \, \nabla \cdot v.$$

As usual, we denote by $L^p(.)$ $(1 \le p \le \infty)$ the Lebesgue spaces and by $W^{s,p}(.)$ $(s \ge 0$, $1 \le p \le \infty)$ the Sobolev(-Slobodetskiĭ) spaces. Sometimes we write $W^{0,p}(.)$ for $L^p(.)$ and $H^s(.)$ for $W^{s,2}(.)$. The usual norm and seminorm of $W^{s,p}(\Omega)$ is denoted by $\|\cdot\|_{s,p,\Omega}$ and $|\cdot|_{s,p,\Omega}$. In the case $p = 2$, we will drop the index p. In order to describe the edge regularity of the solution of our problem, we will use weighted Sobolev spaces of Kondrat'ev type:

$$V_\beta^{\ell,p}(\Omega) := \{v \in \mathcal{D}'(\Omega) : \|v\|_{\ell,p;\beta,\Omega} < \infty\}, \quad \ell \in \mathbb{N}, \ p \in (1,\infty), \ \beta \in \mathbb{R},$$

$$\|v\|_{\ell,p;\beta,\Omega}^p := \sum_{i+j+k \le \ell} \|r^{\beta-\ell+i+j+k} \partial_1^i \partial_2^j \partial_3^k v; L^p(\Omega)\|^p,$$

where $r(x) = (x_1^2 + x_2^2)^{1/2}$ is the distance of $x = (x_1, x_2, x_3)$ to the singular edge. Again, we will drop the index p in the case $p = 2$. We use the abbreviations ∂_i for $\frac{\partial}{\partial x_i}$ and ∂_{ij} for $\partial_i \partial_j$.

Theorem 1 *Let $\lambda > 0$ be the smallest positive solution of*

$$\sin(\lambda\omega) = -\lambda \sin\omega \tag{3}$$

and assume that $f \in L^2(\Omega)^3$. Then the solution $(u,p) \in X \times M$ of the Stokes problem (2) *satisfies*

$$u \in V_\beta^{2,2}(\Omega)^3 \quad \textit{and} \quad p \in V_\beta^{1,2}(\Omega) \quad \forall \beta \in (1-\lambda, 1), \tag{4}$$

$$\partial_3 u \in V_0^{1,2}(\Omega)^3 \quad \textit{and} \quad \partial_3 p \in L^2(\Omega). \tag{5}$$

and the a-priori estimate

$$\|u\|_{2;\beta,\Omega} + \|\partial_3 u\|_{1;0,\Omega} + \|p\|_{1;\beta,\Omega} + \|\partial_3 p\|_{0,\Omega} \lesssim \|f\|_{0,\Omega}$$

holds.

Proof Theorem 6.2 of [15] yields the regularity

$$\begin{array}{rcl} r^\beta \partial_{ij} u & \in & L^2(\Omega)^3, \ i,j = 1,2,3, \\ r^\beta \partial_i p & \in & L^2(\Omega), \ i = 1,2,3, \end{array}$$

with β from (4) since there is no vertex singularity in the strip $[-1/2, 1]$, see Section 6.2 of [15]. A localization argument and the application of Hardy's inequalities [12, page 28] yield the regularity $u \in V_\beta^{2,2}(\Omega)^3$ and $p \in V_\beta^{1,2}(\Omega)$. Note that Theorem 6.1 of [13] states the same regularity results because of the absence of vertex singularities in the strip $[-1/2, 1]$.

It remains to prove the extra regularity in the edge direction for u and p. First this extra regularity is satisfied far from the endpoint of the singular edge as a consequence of the general arguments of [14]. Near a fixed vertex S of the singular edge, we use a localization argument as in [3, 15]. Fix a cut-off function χ equal to 1 near S and equal to zero outside a small neighbourhood of S and use spherical coordinates (R, θ, ϕ) centered at S such that θ is the angular distance to the singular edge. Then by Theorems 4.3 and 4.4 of [15] the couple (w, q) defined by

$$\begin{aligned} w(t,\theta,\phi) &= e^{-\frac{t}{2}}(\chi u)(e^t,\theta,\phi), \\ q(t,\theta,\phi) &= e^{\frac{t}{2}}(\chi p)(e^t,\theta,\phi), \end{aligned}$$

is solution of an elliptic problem in $I\!R \times G_S$ (whose principal part frozen at $\theta = 0$ is the Stokes system), where G_S is the intersection of Ω and the unit sphere centered at S. Theorem 3.4 of [15] implies that $w \in V_\beta^{2,2}(I\!R \times G_S)$ with β from (4), its norm depending continuously on the norms of the data; furthermore Theorem 3.1 of [14] guarantees that $\frac{\partial w}{\partial t}$ belongs to $V_0^1(I\!R \times G_S)$ and that $\frac{\partial q}{\partial t}$ belongs to $L^2(I\!R \times G_S)$ with norms depending continuously on the norms of the data, where $V_\beta^{\ell,2}(I\!R \times G_S)$ is the weighted Sobolev space on $I\!R \times G_S$ of Kondrat'ev's type defined as before where r is replaced by θ, the distance to the singular edge of $I\!R \times G_S$. Going back to the spherical coordinates and using the regularity (4), we get the desired regularities (5). □

Remark 1 The leading singularity of u_3 is characterized by $r^{\pi/\omega}$. But the smallest positive solution λ of (3) satisfies

$$1/2 < \lambda < \frac{\pi}{\omega},$$

see for instance [10]. Consequently, the global regularity is dominated by r^λ.

3 Discretization and error estimates

Let us recall the meshes used for the treatment of edge singularities of the Poisson problem [2, 4]. We define families of meshes $\mathcal{T}_h = \{K\}$ by introducing in G the standard mesh grading for two-dimensional corner problems, see for example [16, 20]. Let $\{T\}$ be a regular isotropic triangulation of G; the elements are triangles. With h being the global mesh parameter, $\mu \in (0, 1]$ being the grading parameter, r_T being the distance of T to the corner,

$$r_T := \inf_{(x_1,x_2)\in T} (x_1^2 + x_2^2)^{1/2},$$

and with some constant $R > 0$, we assume that the element size $h_T := \operatorname{diam} T$ satisfies

$$h_T \sim \begin{cases} h^{1/\mu} & \text{for } r_T = 0, \\ h r_T^{1-\mu} & \text{for } 0 < r_T \le R, \\ h & \text{for } r_T > R. \end{cases}$$

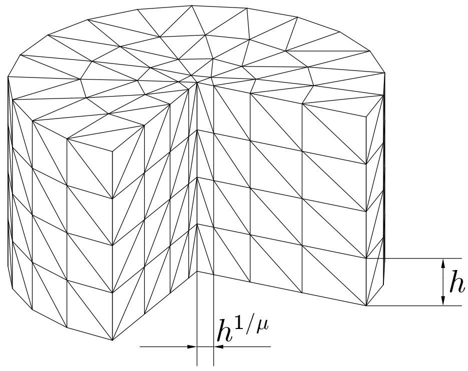

Figure 1: Example for an anisotropic mesh.

This graded two-dimensional mesh is now extended in the third dimension using a uniform mesh size, h. In this way we obtain a pentahedral triangulation or, by dividing each pentahedron, a tetrahedral triangulation of Ω, see Figure 1 for an illustration. Note that the number of elements is of the order h^{-3} for the full range of μ. The notation is extended to the three-dimensional case as follows. Let r_K be the distance of an element K to the edge (x_3-axis) and let $h_{i,K}$ be the length of the projection of K on the x_i-axis. Then these element sizes satisfy

$$h_{3,K} \sim h, \quad h_{1,K} \sim h_{2,K} \sim \begin{cases} h^{1/\mu} & \text{for } r_K = 0, \\ h r_K^{1-\mu} & \text{for } 0 < r_K \le R, \\ h & \text{for } r_K > R. \end{cases} . \tag{6}$$

On tetrahedral meshes $\mathcal{T}_h$ we approximate the velocity in the Crouzeix-Raviart finite element space X_h and the pressure in the space M_h of piecewise constant functions,

$$X_h := \{v_h \in (L^2(\Omega))^3 : v_h|_K \in (\mathcal{P}_1)^3 \ \forall K, \int_F [v_h] = 0 \ \forall F\}, \tag{7}$$

$$M_h := \{q_h \in L^2(\Omega) : q_h|_K \in \mathcal{P}_0 \ \forall K, \int_\Omega q_h = 0\}, \tag{8}$$

where we denote faces of elements by F and by $[v_h]$ the jump of the function v_h on the faces F. For boundary faces we identify $[v_h]$ with v_h. In analogy to [4] we introduce as the corresponding space X_h for pentahedral meshes

$$X_h := \{v_h \in (L^2(\Omega))^3 : v_h|_K \in (\mathcal{P}_1 \oplus \operatorname{span}\{x_3^2\})^3 \ \forall K, \int_F [v_h] = 0 \ \forall F\}. \tag{9}$$

We note that $X_h \not\subset X$. Hence we define the approximate solution by using the weaker bilinear forms $a_h(.,.)$ and $b_h(.,.)$,

$$a_h(u,v) := \sum_K \sum_{i=1}^3 \int_K \nabla u_i \cdot \nabla v_i, \tag{10}$$

$$b_h(u,v) \quad := \quad -\sum_K \int_K q\, \nabla \cdot u. \tag{11}$$

The mixed finite element formulation reads now: Find $u_h \in X_h$, $p_h \in M_h$, such that

$$\begin{cases} a_h(u_h,v_h) + b_h(v_h,p_h) & = & (f,v_h) & \forall v_h \in X_h, \\ b_h(u_h,q_h) & = & 0 & \forall q_h \in M_h. \end{cases} \tag{12}$$

For the analysis of this method it is convenient to introduce the Crouzeix-Raviart interpolant $\mathrm{I}_h : X \to X_h$ which is defined elementwise by

$$\int_F u = \int_F \mathrm{I}_h u \qquad \forall F \subset \partial K, \forall K \in \mathcal{T}_h. \tag{13}$$

In [4] it is analyzed that this interpolant is well defined also for our choice (9) of X_h in the case of pentahedral meshes. In particular, this interpolant is stable in $H^1(\Omega)$,

$$|\mathrm{I}_h u|_{1,K} \lesssim |u|_{1,K}. \tag{14}$$

Hence we can prove the inf-sup condition by the standard proof.

Lemma 1 (inf-sup condition) *There is a constant $\beta > 0$ (independent of h) such that*

$$\inf_{q_h \in M_h} \sup_{v_h \in X_h} \frac{b_h(v_h,q_h)}{\|v_h\|_{1,h}\|q_h\|_{0,\Omega}} \geq \beta. \tag{15}$$

Proof Consider an arbitrary but fixed $q_h \in M_h$. By Corollary I.2.4 of [11] (see also Lemma 6 of [9]), there exists $v \in X$ satisfying

$$\nabla \cdot v = -q_h, \quad |v|_{1,\Omega} \lesssim \|q_h\|_{0,\Omega}. \tag{16}$$

Since by (13) and Green's formula

$$\int_K \nabla \cdot v = \sum_{F \in \partial K} \int_F v = \sum_{F \in \partial K} \int_F \mathrm{I}_h v = \int_K \nabla \cdot \mathrm{I}_h v$$

we get by using $q_h|_K \in \mathcal{P}_0$ and (16)

$$b_h(\mathrm{I}_h v, q_h) = -\sum_K \int_K q_h \nabla \cdot \mathrm{I}_h v = -\sum_K \int_K q_h \nabla \cdot v = \|q_h\|_{0,\Omega}^2. \tag{17}$$

By (14) and (16) we have

$$\|\mathrm{I}_h v\|_{1,h} \lesssim |v|_{1,\Omega} \lesssim \|q_h\|_{0,\Omega}. \tag{18}$$

Combining (17) and (18) we obtain

$$\sup_{v_h \in X_h} \frac{b_h(v_h,q_h)}{\|v_h\|_{1,h}\|q_h\|_{0,\Omega}} \geq \frac{b_h(\mathrm{I}_h v,q_h)}{\|\mathrm{I}_h v\|_{1,h}\|q_h\|_{0,\Omega}} \gtrsim 1.$$

Since q_h was chosen arbitrarily we have proved the assertion. □

Note that the proof works for both tetrahedra and prisms.

Remark 2 In the proof of the inf-sup condition we used only the boundedness (14) of I_h which was proved in [4] for general tetrahedral elements, that means, the inf-sup condition is valid for general tetrahedral meshes.

Lemma 2 (approximation) *Let (u,p) be the solution of the Stokes problem (2). Then the estimates*

$$\inf_{v_h \in X_h} \|u - v_h\|_{1,h} \lesssim h\|f\|_{0,\Omega} \tag{19}$$

$$\inf_{q_h \in M_h} \|u - q_h\|_{1,h} \lesssim h\|f\|_{0,\Omega} \tag{20}$$

hold if the mesh grading parameter μ and the singular exponent λ from (3) satisfy $\mu < \lambda$.

Proof According to Theorem 1 the velocity components u_i satisfy

$$\|\partial_1 u_i\|_{1;\beta,\Omega} + \|\partial_2 u_i\|_{1;\beta,\Omega} + \|\partial_3 u_i\|_{1;0,\Omega} \lesssim \|f\|_{0,\Omega}$$

with $\beta \in (1-\lambda, 1)$. Hence we can apply Theorem 5.1 of [4] and obtain

$$\|u_i - \mathrm{I}_h u_i\|_{1,h} \lesssim h\|f\|_{0,\Omega}.$$

For (20), we estimate $\|p - \mathrm{M}_h p\|_{0,\Omega}$ where $\mathrm{M}_h p|_K := \mathrm{M}_K p := (\mathrm{meas}_3 K)^{-1} \int_K p$. Note that M_K preserves polynomials of degree 0.

For all elements K with $r_K > 0$ we apply the estimate

$$\|p - p\|_{0,K} \lesssim \sum_{i=1}^{3} h_{i,K} \|\partial_i p\|_{0,K}$$

which can be proved by the standard Bramble-Hilbert theory. We can proceed in analogy to the proof for $\|u_i - \mathrm{I}_h u_i\|_{1,h}$ and obtain for $\beta = 1 - \mu$

$$\begin{aligned} \|p - \mathrm{M}_K p\|_{0,K} &\lesssim \sum_{i=1}^{3} h_{i,K} \|\partial_i p\|_{0,K} \\ &\lesssim \sum_{i=1}^{2} h_{i,K} r_K^{-\beta} \|\partial_i p\|_{0;\beta,K} + h_{3,K} \|\partial_3 p\|_{0,K} \\ &\lesssim h \sum_{i=1}^{2} \|\partial_i p\|_{0;\beta,K} + h\|\partial_3 p\|_{0,K}. \end{aligned} \tag{21}$$

Consider now the elements K with $r_K = 0$. We use that $\mathrm{M}_K : L^2(K) \to \mathcal{P}_0$ is bounded and thus for $\beta \le 1$

$$\begin{aligned} \|p - \mathrm{M}_K p\|_{0,K} &\lesssim \|p\|_{0,K} \le \|r^{1-\beta}\|_{0,\infty,K} \|r^{\beta-1} p\|_{0,K} \\ &\lesssim h_{1,K}^{1-\beta} \|r^{\beta-1} p\|_{0,K} \le h\|p\|_{1;\beta,K}. \end{aligned} \tag{22}$$

Summing up the square of the estimates (21) and (22) over all elements we obtain

$$\|p - \mathrm{M}_h p\|_{0,\Omega} \lesssim h\left(\|p\|_{1;\beta,\Omega} + \|\partial_3 p\|_{0,\Omega}\right) \lesssim h\|f\|_{0,\Omega}$$

where we have again used Theorem 1. □

Lemma 3 (consistency) *Let (u,p) be the solution of the Stokes problem* (2), *and let $a_h(.,.)$ and $b_h(.,.)$ be the bilinear forms defined in* (10), (11). *Then the estimate*

$$|a_h(u, v_h) + b_h(v_h, p) - (f, v_h)| \lesssim h\,\|v_h\|_{1,h}\|f\|_{0,\Omega}$$

holds for any $v_h \in X_h$ if $\mu < \lambda$.

Proof Let $(u,v)_h := \sum_K \int_K uv$ be the mesh dependent scalar product and denote by $v_{h,i}$ the components of v_h. We observe that

$$a_h(u, v_h) + b_h(v_h, p) - (f, v_h) = \sum_{i=1}^{3} [(\nabla u_i, \nabla v_{h,i})_h + (p, \partial_i v_{h,i})_h - (f_i, v_{h,i})].$$

For $i = 1$ we set $\eta := \nabla u_1 + (p, 0, 0)^T$. Since $\nabla \cdot \eta = f \in L^2(\Omega)$ and by Theorem 1

$$\eta_1, \eta_2 \in V_\beta^{1,2}(\Omega),\ \beta \in (1-\lambda, 1) \subset [0,1], \qquad \eta_3 \in V_0^{1,2}(\Omega), \tag{23}$$

we can apply [4, Lemma 4.6] and obtain in analogy to [4, Theorem 5.2]

$$\begin{aligned} |(\nabla u_1, \nabla v_{h,1})_h + (p, \partial_1 v_{h,1})_h - (f_1, v_{h,1})| &= |(\eta, \nabla v_{h,1})_h - (f_1, v_{h,1})| \\ &\lesssim h\,\|v_{h,1}\|_{1,h}\|f_1\|_{0,\Omega}. \end{aligned} \tag{24}$$

In the same way we can treat the case $i = 2$.

The case $i = 3$ is different since $\partial_3 u_3 + p \notin V_0^{1,2}(\Omega)$. Here we set $\eta := \nabla u_3$ and get the properties (23) to apply the theory from [4]:

$$|(\eta, \nabla v_{h,3})_h - (\nabla \cdot \eta, v_{h,3})| \lesssim h\,\|v_{h,3}\|_{1,h}\|\nabla \cdot \eta\|_{0,\Omega} = h\,\|v_{h,3}\|_{1,h}\|f_3 + \partial_3 p\|_{0,\Omega}. \tag{25}$$

The desired term is now written as

$$\begin{aligned} &|(\nabla u_3, \nabla v_{h,3})_h + (p, \partial_3 v_{h,3})_h - (f_3, v_{h,3})| \\ &\quad = |(\eta, \nabla v_{h,3})_h + (\nabla \cdot \eta, v_{h,3})_h| + |(p, \partial_3 v_{h,3})_h - (f_3, v_{h,3}) - (\nabla \cdot \eta, v_{h,3})_h| \end{aligned} \tag{26}$$

where the first term is already estimated by (25). The second term is reformulated to

$$\begin{aligned} |(p, \partial_3 v_{h,3})_h - (f_3, v_{h,3}) - (\nabla \cdot \eta, v_{h,3})_h| &= |(p, \partial_3 v_{h,3})_h - (\partial_3 p, v_{h,3})| \\ &= \left| \sum_K \sum_{F \subset \partial K} n_{3,F} \int_F p v_{h,3} \right| \\ &= \left| \sum_K \sum_{F \subset \partial K} n_{3,F} \int_F (p - \mathrm{M}_F p)(v_{h,3} - \mathrm{M}_F v_{h,3}) \right| \end{aligned}$$

where we have used a standard technique. We can apply now [4, Lemma 4.3] and get

$$
\begin{aligned}
&|(p,\partial_3 v_{h,3})_h-(f_3,v_{h,3})-(\nabla\cdot\eta,v_{h,3})_h| \\
&\quad\lesssim \sum_K\sum_{F\subset\partial K} n_{3,F}\frac{\mathrm{meas}_2 F}{\mathrm{meas}_3 K}\left(\sum_{i=1}^3 h_{1,K}^{-2\beta_{i,K}} h_{i,K}^2\|r^{\beta_{i,K}}\partial_i p\|_{0,K}^2\right)^{1/2}\left(\sum_{i=1}^3 h_{i,K}^2\|\partial_i v_{h,3}\|_{0,K}^2\right)^{1/2} \quad (27)
\end{aligned}
$$

with $\beta_{1,K}=\beta_{2,K}=\beta=1-\mu$, $\beta_{3,K}=0$ if $r_K=0$ and $\beta_{1,K}=\beta_{2,K}=\beta_{3,K}=0$ if $r_K>0$. We observe now that $n_{3,F}\cdot\mathrm{meas}_2 F\sim 1\cdot h_{1,K}^2$ for small faces and $n_{3,F}\cdot\mathrm{meas}_2 F\lesssim h_{3,K}^{-1}h_{1,K}\cdot h_{1,K}h_{3,K}\sim h_{1,K}^2$ for large faces, that means

$$
n_{3,F}\frac{\mathrm{meas}_2 F}{\mathrm{meas}_3 K}\lesssim h_{3,K}^{-1}\sim h^{-1}. \quad (28)
$$

Furthermore we get by the known technique

$$
\begin{aligned}
h_{1,K}^{-2\beta_i}h_{i,K}^2\|r^{\beta_{i,K}}\partial_i p\|_{0,K}^2 &\sim h^{2(1-\beta)/\mu}\|r^{\beta_{i,K}}\partial_i p\|_{0,K}^2 \sim h^2\|r^{\beta_{i,K}}\partial_i p\|_{0,K}^2 && \text{for } r_K=0\\
h_{i,K}^2\|\partial_i p\|_{0,K}^2 &\sim h^2 r_K^{2(1-\mu)}\|\partial_i p\|_{0,K}^2 \lesssim h^2\|r^{\beta}\partial_i p\|_{0,K}^2 && \text{for } r_K>0
\end{aligned}
$$

so that

$$
\left(\sum_{i=1}^3 h_{1,K}^{-2\beta_{i,K}}h_{i,K}^2\|r^{\beta_{i,K}}\partial_i p\|_{0,K}^2\right)^{1/2}\lesssim h\left(\sum_{i=1}^2\|r^\beta\partial_i p\|_{0,K}^2+\|\partial_3 p\|_{0,K}^2\right)^{1/2}. \quad (29)
$$

Combining (27), (28) and (29) we derive

$$
\begin{aligned}
|(p,\partial_3 v_{h,3})_h-(f_3,v_{h,3})-(\nabla\cdot\eta,v_{h,3})_h| &\lesssim h\sum_K\left(\sum_{i=1}^2\|r^\beta\partial_i p\|_{0,K}^2+\|\partial_3 p\|_{0,K}^2\right)^{1/2}|v_{h,3}|_{1,K}\\
&\lesssim h\left(\sum_{i=1}^2\|r^\beta\partial_i p\|_{0,\Omega}+\|\partial_3 p\|_{0,\Omega}\right)\|v_{h,3}\|_{1,h}. \quad (30)
\end{aligned}
$$

With (24), (25), (26) and Theorem 1 we obtain the desired estimate. □

Remark 3 We remark that the consistency term can be reformulated by using $\|\sigma-\mathrm{RT}(\sigma)\|_0$, $\sigma:=\nabla u-pI$, RT being the Raviart-Thomas interpolant. This is analyzed for regular solutions $\sigma\in(H^1(\Omega))^{3\times3}$ in [1].

We are now ready to derive our finite element error estimate.

Theorem 2 *Let (u,p) be the solution of the Stokes problem (2), and let (u_h,p_h) be the solution defined by (12). Assume that the mesh is refined according to $\mu<\lambda$, with λ from (3). Then the finite element error can be estimated by*

$$
\|u-u_h\|_{1,h}+\|p-p_h\|_{0,\Omega}\lesssim h\|f\|_{0,\Omega}.
$$

Proof By [8, Proposition 2.16] we get

$$\begin{aligned}\|u-u_h\|_{1,h}+\|p-p_h\|_{0,\Omega} &\lesssim \inf_{v_h\in X_h}\|u-v_h\|_{1,h}+\inf_{q_h\in M_h}\|p-q_h\|_{1,h}+\\ &\quad+\sup_{v_h\in X_h}\frac{|a_h(u,v_h)+b_h(v_h,p)-(f,v_h)|}{\|v_h\|_{1,h}}.\end{aligned}$$

The error estimate follows with Lemmata 1, 2, and 3. □

Remark 4 By analogy one can prove for $\lambda<\mu\le 1$ that

$$\|u-u_h\|_{1,h}+\|p-p_h\|_{0,\Omega}\lesssim h^{\lambda/\mu-\varepsilon}\|f\|_{0,\Omega}$$

for arbitrary small $\varepsilon>0$, compare with [2] where the modifications of the proof are explained for the case of a conforming discretization of the Poisson equation. That means that we get for the unrefined mesh ($\mu=1$) only an approximation order $\lambda-\varepsilon$.

4 Numerical test

Consider the Stokes problem

$$\left\{\begin{array}{rcll}-\Delta u+\nabla p &=& f & \text{in }\Omega,\\ \nabla\cdot u &=& 0 & \text{in }\Omega,\\ u &=& g & \text{on }\partial\Omega\end{array}\right.$$

in the three-dimensional domain

$$\Omega=\{(r\cos\phi, r\sin\phi, x_3)\in I\!R^3 : 0<r<1,\ 0<\phi<\omega,\ 0<x_3<1\}.$$

with $\omega=3\pi/2$. The right hand sides f and g are taken such that the exact solution is

$$u=\begin{pmatrix}x_3 r^\lambda\Phi_1(\phi)\\ x_3 r^\lambda\Phi_2(\phi)\\ r^{2/3}\sin\frac{2}{3}\phi\end{pmatrix},\qquad p=x_3 r^{\lambda-1}\Phi_p(\phi),$$

where $\lambda\approx 0.5445$ is the smallest positive solution of equation (3) and

$$\begin{aligned}\Phi_1(\phi) &= -\sin(\lambda\phi)\cos\omega-\lambda\sin(\phi)\cos(\lambda(\omega-\phi)+\phi)+\lambda\sin(\omega-\phi)\cos(\lambda\phi-\phi)\\ &\quad+\sin(\lambda(\omega-\phi)),\\ \Phi_2(\phi) &= -\sin(\lambda\phi)\sin\omega-\lambda\sin(\phi)\sin(\lambda(\omega-\phi)+\phi)-\lambda\sin(\omega-\phi)\sin(\lambda\phi-\phi),\\ \Phi_p(\phi) &= 2\lambda\,[\sin((\lambda-1)\phi+\omega)+\sin((\lambda-1)\phi-\lambda\omega)].\end{aligned}$$

Since this choice means that $r^\lambda\Phi_1(\phi)$, $r^\lambda\Phi_2(\phi)$, $r^{\lambda-1}\Phi_p(\phi)$ is a solution of the homogeneous Stokes problem over the two-dimensional domain $G=\{(r\cos\phi, r\sin\phi)\in I\!R^2 : 0<r<1,\ 0<\phi<\omega\}$ [17], this solution has the typical singular behaviour near the edge.

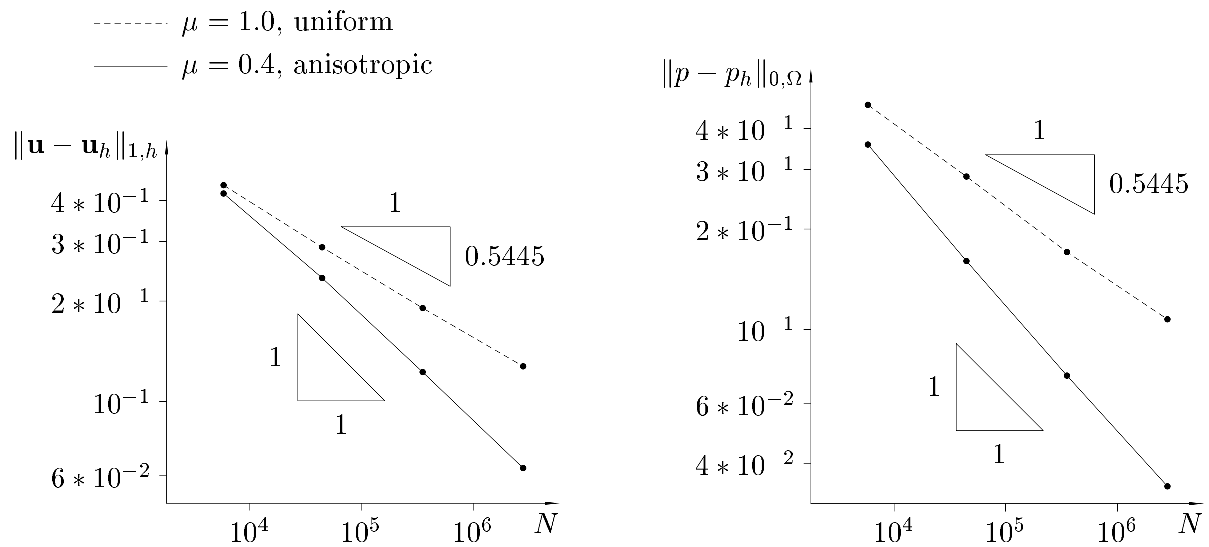

Figure 2: Comparison of uniform vs. graded meshes: error norms for the velocity (left) and the pressure (right).

We constructed tetrahedral meshes as described in Section 3, with $\mu = 1$ (quasi-uniform) and $\mu = 0.4$ (anisotropically refined) and with different numbers of elements. From the numerical solutions $(u_h, p) \in X_h \times M_h$ and the known exact solution, the error norms $\|u - u_h\|_{1,h}$ and $\|p - p_h\|_{0,\Omega}$ were computed. Figure 2 shows the plots of these norms against the number $N = 3N_{\text{face}} + N_{\text{element}}$ of unknowns. A double logarithmic scale was used such that the slope of the curves corresponds to the approximation order. The example verifies the theoretically predicted convergence orders.

Note that the curved boundary at $r = 1$ is approximated by plane triangular faces. As the test has shown, this crime, and also the effect of the non-homogeneous boundary condition on the face $r = 1$, had no influence on our result.

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