MIMO ℓ_1-OPTIMAL CONTROL VIA BLOCK TOEPLITZ OPERATORS

ZDENĚK HURÁK* AND ALBRECHT BÖTTCHER†

Abstract. The paper is devoted to the standard problem of ℓ_1-optimal control for MIMO systems, that is, to the design of a feedback compensator that minimizes peaks in the amplitude of the regulated variables under the assumption that the exogenous variables are bounded in magnitude and persistent. The solution proposed avoids interpolation completely and is therefore distinguished by fine numerical properties. An alternative proof for the finiteness of the impulse response in the square (one-block) case is given, while both lower and upper bounds for the optimal value of the norm are established in the general multiblock case.

Key words. Toeplitz operator, banded, lower-triangular, range, minimum distance problem, ℓ^1-optimal control.

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1. Introduction. The objective of ℓ_1-optimal control is, loosely speaking, to minimize worst-case peaks in the amplitudes of regulated variables that are induced by exogenous variables. The only assumption about the exogenous variables is that they are persistent and bounded in magnitude.

The ℓ_1-optimal control problem was formulated and contrasted with the already established $\mathcal{H}_\infty$-optimal control by Vidyasagar [20] in the late 1980s.

A complete solution to SISO and square MIMO problems (also called one-block problems) was proposed by Dahleh and Pearson in [7]. The primal problem is cast as a linear program with an infinite number of variables subject to a finite number of constraints. Using standard duality results and thinking of ℓ_1 as the dual of c_0, they were able to state the dual problem with a finite number of variables and an infinite number of constraints. By showing that only finitely many constraints are active, they arrived at a finite-dimensional linear program. Moreover, by employing the alignment condition, they showed that the optimal closed-loop impulse response is finite.

This procedure was extended to the general four-block case in [8] and [16]. The primal optimization problem now has infinitely many constraints and therefore it is not possible to compute the solution precisely. It is necessary to approximate an optimal impulse response by a solution of a truncated problem with only finitely many nonzero entries (in the primal domain). The sequence of the optimal solutions of the truncated problems gives a converging upper bound for the optimal cost to the original minimization problem. This approach has later been named the Finitely Many Variables (FMV) method.

A lower bound for the optimal cost was obtained independently by Staffans [19] and Dahleh [5] by solving a truncated dual problem (with a finite number of dual variables and thus a finite number of primal constraints). This was called the Finitely Many Equations (FME) method. The corresponding algorithm is based on an iterative FMV/FME scheme in which two finite linear programs are solved at each truncation.

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A different solution to the general multiblock problem, referred to as the Delay Augmentation (DA) method and providing both lower and upper bounds converging to the optimum, was proposed by Diaz-Bobillo [9]. It converts a multiblock problem into a one-block problem by introducing delays (right-shifts). The idea is to attempt at reducing the order inflation caused by straightforward truncation as in FMV/FME and to reduce some computational burdens by solving only one linear program at each iteration. On the other hand, additional effort is required for reordering the inputs and outputs of the system because this has significant impact on convergence.

The results mentioned above were obtained in the late 1980s and early 1990s and are perfectly documented in the comprehensive book [6] by Dahleh and Diaz-Bobillo. A common feature of these methods is that both the derivation of theoretical properties and the actual numerical computation are based on numerically tricky interpolation in order to come up with the solution of ill-conditioned Vandermonde systems. There are two additional sources of numerical troubles. First, one has to compute the zeros and the so-called zero directions of polynomial matrices and secondly and perhaps even more importantly, one has to extract an optimal controller from the closed-loop impulse response. The latter problem is particularly serious in case one has achieved a suboptimal solution only.

In the late 1990s, the research into ℓ_1-optimal control was strongly stimulated by attempts to reformulate the standard problem of ℓ_1-optimal control and to obviate the interpolation step. This endeavor led to two distinguished new approaches that share the angle of attack with this paper as they consider the Youla-Kučera parameter $Q(\lambda)$ as a component of the optimization variable.

The first of these approaches was proposed by Khammash in [12] and [13] and is called the $\mathcal{Q}$-scaled method. Khammash solves an auxiliary (regularized) problem that includes a scaled norm of $Q(\lambda)$ in the objective function, provides lower and upper bounds, and then relates the result to the solution to the original problem.

The second has the flavor of a polynomial approach to ℓ_1-optimal control and was proposed by Cassavola [4], with preliminary results in [2] and [3]. In this method, both the closed-loop transfer function and Youla-Kučera parameter $Q(\lambda)$ are parameterized by a free term, which results in a sequence of unconstrained and redundancy-free linear programs.

Only very recently, Šebek and the authors [11] presented a third approach to SISO ℓ_1-optimal control that avoided any recourse to interpolation. The original problem was approximated by appropriately constructed finite problems with lower-triangular Toeplitz matrices. These finite problems are overdetermined and the task is to find a solution that minimizes the ℓ_1 norm of the residue. The advantages of the method of [11] include that the algorithm is conceptually extremely simple, that the lower-triangular Toeplitz matrices involved are well-conditioned, and that the Youla-Kučera parameter is a direct output of the algorithm. The purpose of this paper is to extend the approach of [11] to the MIMO case.

2. Block Toeplitz operators. By a proper transfer function we mean a function $g(\lambda) = \sum_{k=0}^{\infty} g_k \lambda^k$ with real coefficients g_k that is analytic in a neighborhood of the origin. The Toeplitz matrix $T(g)$ associated with a proper transfer function g is the infinite lower-triangular matrix

$$T(g) = \begin{pmatrix} g_0 & & & \\ g_1 & g_0 & & \\ g_2 & g_1 & g_0 & \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

For a real sequence $s = \{s_j\}_{j=0}^{\infty}$, we define the sequence $T(g)s$ as the sequence $\sigma = \{\sigma_j\}_{j=0}^{\infty}$ given by

$$\begin{pmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \\ \dots \end{pmatrix} = \begin{pmatrix} g_0 & & & \\ g_1 & g_0 & & \\ g_2 & g_1 & g_0 & \\ \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ \dots \end{pmatrix}.$$

We denote by ℓ_∞ the real Banach space of all real sequences $s = \{s_j\}_{j=0}^{\infty}$ for which

$$\|s\|_\infty := \sup_{j\geq 0} |s_j| < \infty.$$

It is well known that $T(g)$ induces a bounded operator on ℓ_∞ if and only if

$$\|g\|_W := \sum_{k=0}^{\infty} |g_k| < \infty. \tag{2.1}$$

The set of all proper transfer functions g satisfying (2.1) is called the (real and analytic) Wiener algebra and is denoted by W_+. Clearly, functions in W_+ are analytic for $|\lambda| < 1$ and continuous for $|\lambda| \leq 1$. If $g \in W_+$, then the norm of $T(g)$ on ℓ_∞ is known to be just $\|g\|_W$.

Now let $G(\lambda) = \sum_{k=0}^{\infty} G_k \lambda^k$ be a matrix function with coefficients in $\mathbf{R}^{m\times n}$ that is analytic in a neighborhood of the origin. We refer to such matrix functions as proper transfer functions as well. We write $G = (G_{ij})_{i=1,j=1}^{m,n}$ and define the block Toeplitz matrix $T(G)$ by

$$T(G) = \begin{pmatrix} T(G_{11}) & \dots & T(G_{1n}) \\ \vdots & & \vdots \\ T(G_{m1}) & \dots & T(G_{mn}) \end{pmatrix}.$$

This matrix transforms n-tuples $(s^1, \dots, s^n)$ of real sequences into m-tuples $(\sigma^1, \dots, \sigma^m)$ of real sequences in the natural fashion. We denote by ℓ_∞^k the real Banach space of all k-tuples $(s^1, \dots, s^k)$ of sequences $s^1, \dots, s^k \in \ell_\infty$ with the norm

$$\|s\|_\infty := \max(\|s^1\|_\infty, \dots, \|s^k\|_\infty).$$

From the preceding paragraph we know that $T(G)$ generates a bounded operator of ℓ_∞^n to ℓ_∞^m if and only if $G \in W_+^{m\times n}$. In that case the norm of $T(G)$ as an operator of ℓ_∞^n to ℓ_∞^m,

$$\|T(G)\|_{\mathcal{B}(\ell_\infty^n, \ell_\infty^m)} := \sup_{\|s\|_\infty \leq 1} \|T(G)s\|_\infty,$$

equals the row-sum norm

$$\|G\|_W := \max_{1\leq i\leq m} \sum_{j=1}^{n} \|G_{ij}\|_W. \tag{2.2}$$

3. The ℓ_1-optimal control problem. We consider the configuration of Figure 3.1 with discrete-time linear time-invariant systems. The inputs and outputs are real sequences $\{s_j\}_{j=0}^{\infty}$. We suppose that we have n_w exogenous inputs, n_u control inputs, n_z regulated outputs, and n_y measured variables. The entire feedback system can be written in the form

$$\begin{aligned} z &= T(P_{wz})w + T(P_{uz})u, \\ y &= T(P_{wy})w + T(P_{uy})u, \\ u &= T(C)y. \end{aligned}$$

Here $P_{wz}, P_{uz}, P_{wy}, P_{uy}$ are given proper transfer functions of appropriate sizes and C is a proper transfer function of appropriate size that has to be designed.

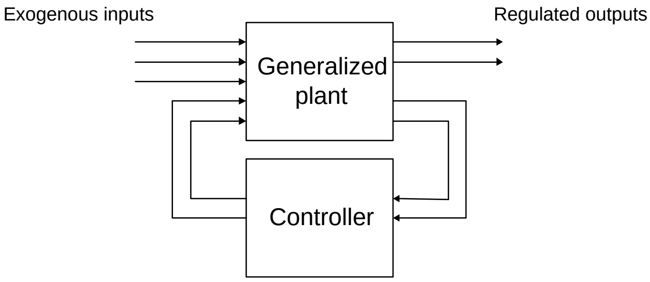

FIG. 3.1. *Standard feedback control configuration*

We have $z = T(G)w$ with the proper transfer function

$$G = P_{wz} + P_{uz}C(I - P_{uy}C)^{-1}P_{wy}. \tag{3.1}$$

The objective of ℓ_1-control is to find the C's such that $T(G)$ is a bounded linear operator of $\ell_\infty^{n_w}$ to $\ell_\infty^{n_z}$, and in ℓ_1-optimal control we look for the C's for which

$$\|T(G)\|_{\mathcal{B}(\ell_\infty^{n_z}, \ell_\infty^{n_w})} = \|G\|_W$$

is minimal or close to the infimum.

We make the usual assumptions. Thus, we assume that P_{wz}, P_{uz}, P_{wy} are rational matrix functions with entries in W_+ and that P_{uy} is a rational matrix function that may have poles in the closed unit disk. Then there exist (real) matrix polynomials B_R, A_R, B_L, A_L, Y_R^0, X_R^0, Y_L^0, X_L^0 such that

$$P_{uy} = B_R A_R^{-1} = A_L^{-1} B_L \tag{3.2}$$

and

$$\begin{pmatrix} X_L^0 & -Y_L^0 \\ -B_L & A_L \end{pmatrix} \begin{pmatrix} A_R & Y_R^0 \\ B_R & X_R^0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{3.3}$$

Under these assumptions, the set of all controllers C for which $\|G\|_W$ is finite is given by the Youla-Kučera parametrization:

$$C = Y_R X_R^{-1} = X_L^{-1} Y_L \tag{3.4}$$

where

$$X_R = X_R^0 + B_R\widetilde{Q}, \quad Y_R = Y_R^0 + A_R\widetilde{Q}, \tag{3.5}$$
$$X_L = X_L^0 + \widetilde{Q}B_L, \quad Y_L = Y_L^0 + \widetilde{Q}A_L, \tag{3.6}$$

and $\widetilde{Q}$ is an arbitrary matrix function in $W_+^{n_u \times n_y}$. Inserting (3.5) in (3.4) and taking into account (3.2) and (3.3) we obtain

$$C(I - P_{uy}C)^{-1} = (Y_R^0 + A_R\widetilde{Q})A_L,$$

and inserting this in (3.1) we arrive at the representation[1]

$$G = H + \widetilde{U}\widetilde{Q}\widetilde{V}$$

with

$$H = P_{wz} + P_{uz}Y_R^0 A_L P_{wy},$$
$$\widetilde{U} = P_{uz}A_R, \quad \widetilde{V} = A_L P_{wy}.$$

The rational matrix functions $\widetilde{U}$ and $\widetilde{V}$ can be written as

$$\widetilde{U} = -UU_R^{-1}, \quad \widetilde{V} = V_L^{-1}V$$

with (real) matrix polynomials U, U_R, V_L, V such that $\det U_R(\lambda) \neq 0$ and $\det V_L(\lambda) \neq 0$ for $|\lambda| \leq 1$. We put $Q = U_R^{-1}\widetilde{Q}V_L^{-1}$ and have

$$G = H - UQV.$$

Thus, our problem is as follows: we are given a rational matrix function $H \in W_+^{n_z \times n_w}$ and two matrix polynomials $U \in W_+^{n_z \times n_u}$ and $V \in W_+^{n_y \times n_w}$, and we look for a matrix function $Q \in W_+^{n_u \times n_y}$ such that $\|H - UQV\|_W$ is minimal or close to the infimum. Furthermore, it is desirable to find a rational matrix function Q with this property.

4. Existence of the solution. In practically relevant control problems we always have $n_z \geq n_u$ and $n_w \geq n_y$. Throughout what follows we assume that these two inequalities are satisfied. Our main assumption is that the two matrix polynomials U and V have full rank on the unit circle $\mathbf{T}$, that is,

$$\operatorname{rank} U(\lambda) = n_u \quad \text{and} \quad \operatorname{rank} V(\lambda) = n_y \quad \text{for all } \lambda \in \mathbf{T}. \tag{4.1}$$

The following lemma is well known and can be easily proved using the Smith normal form. For the reader's convenience, we cite it with an absolutely elementary proof.

LEMMA 4.1. *Under assumption (4.1), there exist rational (real) matrix functions L and K without poles on $\mathbf{T}$ such that $LU = I$ and $VK = I$.*

Proof. To avoid heavy notation, let us consider the case where

$$V = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ v_5 & v_6 & v_7 & v_8 \end{pmatrix}.$$

[1] We really have the plus sign in the formula for G. The minus sign in (6.116) of [15] is actually false, because already (6.60) has the wrong sign.

We denote by $V_1, \ldots, V_6$ the 2×2 submatrices of V. From (4.1) we infer that

$$\sum_{k=1}^{6} |\det V_k(\lambda)|^2 > 0 \quad \text{for all} \;\; \lambda \in \mathbf{T}.$$

Put

$$h_j(\lambda) = \frac{\overline{\det V_j(\lambda)}}{\sum_{k=1}^{6} |\det V_k(\lambda)|^2},$$

the bar denoting complex conjugation. Then h_j is a rational function without poles on $\mathbf{T}$ and

$$\sum_{j=1}^{6} (\det V_j) h_j = 1.$$

The following trick is from [1, Proposition 13.9] and [17, Lemma 3.1]. Let $\operatorname{adj} V_j$ denote the adjugate matrix of V_j. Then

$$(\det V_j) I = V_j (\operatorname{adj} V_j)$$

and letting $K_j = (\operatorname{adj} V_j) h_j$ we get

$$\sum_{j=1}^{6} V_j K_j = \sum_{j=1}^{6} V_j (\operatorname{adj} V_j) h_j = \sum_{j=1}^{6} (\det V_j) h_j I = I.$$

Write

$$K_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}.$$

We have

$$\begin{aligned} &\begin{pmatrix} v_1 & v_2 \\ v_5 & v_6 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} v_1 & v_3 \\ v_5 & v_7 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \\ &+ \begin{pmatrix} v_1 & v_4 \\ v_5 & v_8 \end{pmatrix} \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} + \begin{pmatrix} v_2 & v_3 \\ v_6 & v_7 \end{pmatrix} \begin{pmatrix} a_4 & b_4 \\ c_4 & d_4 \end{pmatrix} \\ &+ \begin{pmatrix} v_2 & v_4 \\ v_6 & v_8 \end{pmatrix} \begin{pmatrix} a_5 & b_5 \\ c_5 & d_5 \end{pmatrix} + \begin{pmatrix} v_3 & v_4 \\ v_7 & v_8 \end{pmatrix} \begin{pmatrix} a_6 & b_6 \\ c_6 & d_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

and hence

$$\begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ v_5 & v_6 & v_7 & v_8 \end{pmatrix} \begin{pmatrix} a_1 + a_2 + a_3 & b_1 + b_2 + b_3 \\ c_1 + a_4 + a_5 & d_1 + b_4 + b_5 \\ c_2 + c_4 + a_6 & d_2 + d_4 + b_6 \\ c_3 + c_5 + c_6 & d_3 + d_5 + d_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which completes the proof. □

It will be convenient to write the matrix $H - UQV$ as a column. This can be done in a standard fashion, or better in two standards manners, namely by column stacking on the one hand and by row stacking on the other. As we are concerned with the row-sum norm (2.2), we stack matrices by rows. For example, the equality

$$\begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix} - \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} = \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}$$

is equivalent to the equality

$$\begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \end{pmatrix} - \begin{pmatrix} u_1v_1 & u_1v_3 \\ u_1v_2 & u_1v_4 \\ u_2v_1 & u_2v_3 \\ u_2v_2 & u_2v_4 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{pmatrix}. \tag{4.2}$$

Let ℓ_1 be the usual real Banach space of real sequences $s = \{s_j\}_{j=0}^{\infty}$ with

$$\|s\|_1 := \sum_{j=0}^{\infty} |s_j| < \infty.$$

We denote by $h_j \in \ell_1$ the sequence of the Taylor coefficients of H_j and define q_j and e_j analogously. (By Taylor coefficients we always mean the Taylor coefficients at the origin.) Then (4.2) can also be written in the form

$$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} - T \begin{pmatrix} u_1v_1 & u_1v_3 \\ u_1v_2 & u_1v_4 \\ u_2v_1 & u_2v_3 \\ u_2v_2 & u_2v_4 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}. \tag{4.3}$$

We have

$$\begin{aligned} \left\| \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix} \right\|_W &= \max(\|E_1\|_W + \|E_2\|_W, \|E_3\|_W + \|E_4\|_W) \\ &= \max(\|e_1\|_1 + \|e_2\|_1, \|e_3\|_1 + \|e_4\|_1) \end{aligned} \tag{4.4}$$

Thus, when viewing (4.3) as an equality in ℓ_1^4, we must define the norm in ℓ_1^4 by (4.4).

In the general case we write $H - UQV = E$ as

$$\operatorname{vec} H - (U \otimes V^\top) \operatorname{vec} Q = \operatorname{vec} E,$$

where $\otimes$ is the Kronecker product of matrices and vec is defined in the obvious way, and then we pass to $\ell_1^{n_z n_w}$ by writing

$$h - T(F)q = e$$

with $F = U \otimes V^\top$. The norm in $\ell_1^{n_z n_w}$ is defined in analogy to (4.4), that is, if

$$e = \begin{pmatrix} e_{11} & \dots & e_{1n_z} & \dots & e_{n_w 1} & \dots & e_{n_w n_z} \end{pmatrix}^\top \in \ell_1^{n_z n_w}$$

then

$$\|e\|_1 := \max_{1 \le i \le n_w} (\|e_{i1}\|_1 + \ldots + \|e_{in_z}\|_1). \tag{4.5}$$

Clearly,

$$\|H - UQV\|_W = \|h - T(F)q\|_1. \tag{4.6}$$

The block Toeplitz operator $T(F)$ acts from $\ell_1^{n_u n_y}$ to $\ell_1^{n_z n_w}$. While the norm in $\ell_1^{n_z n_w}$ is given by (4.5) we need not and do not specify a concrete norm in $\ell_1^{n_u n_y}$ - we equip this space with any vector norm of the ℓ_1 norms of the components.

THEOREM 4.2. *Under assumption (4.1), the operator $T(F) : \ell_1^{n_u n_y} \to \ell_1^{n_z n_w}$ has closed range.*

Proof. Suppose $\|z - T(F)x_n\|_1 \to 0$. There are $Z \in W_+^{n_z \times n_w}$ and $X_n \in W_+^{n_u \times n_y}$ such that z and x_n are the Taylor coefficients of $\operatorname{vec} Z$ and $\operatorname{vec} X_n$, respectively. By (4.6), $\|Z - UX_nV\|_W \to 0$. We denote by W the (real and full) Wiener algebra of all function g on $\mathbf{T}$ with real Fourier coefficients and absolutely convergent Fourier series, Thus, $g \in W$ if and only if

$$g(\lambda) = \sum_{k=-\infty}^{\infty} g_k \lambda^k \ (|\lambda| = 1), \quad g_k \in \mathbf{R}, \quad \|g\|_W := \sum_{k=-\infty}^{\infty} |g_k| < \infty.$$

Let L and K be the rational matrix functions of Lemma 4.1. Obviously, $L \in W^{n_u \times n_z}$ and $K \in W^{n_w \times n_y}$. Therefore,

$$\|LZK - X_n\|_W = \|L(Z - UX_nV)K\|_W \le \|L\|_W \|Z - UX_nV\|_W \|K\|_W \to 0,$$

which implies that $X_n \to LZK =: X$ in $W^{n_u \times n_y}$. As $X_n \in W_+^{n_u \times n_y}$, it follows that $X \in W_+^{n_u \times n_y}$. Let $x \in \ell_1^{n_u n_y}$ be the sequence of the Taylor coefficients of $\operatorname{vec} X$. Since $x_n \to x$ in $\ell_1^{n_u n_y}$, we obtain that $z = T(F)x$ is in the range of $T(F)$. ◻

COROLLARY 4.3. *Under assumption (4.1), there exists a $Q \in W_+^{n_u \times n_y}$ at which $\|H - UQV\|_W$ attains its minimum.*

Proof. Let $d = \inf\{\|H - UQV\|_W : Q \in W_+^{n_u \times n_y}\}$. From (4.6) we infer that

$$d = \inf\{\|h - m^*\|_1 : m^* \in \mathcal{R}(T(F))\}, \tag{4.7}$$

where $\mathcal{R}(T(F))$ is the range of $T(F)$. Let c_0 be the real Banach space of all real sequences that converge to zero. The norm in c_0 is the ℓ_∞ norm and c_0 is known to be a closed subspace of ℓ_∞. Since $\ell_1^{n_z n_w}$ is the dual space of $c_0^{n_z n_w}$, the norm in $c_0^{n_z n_w}$ being,

$$\left\| \begin{pmatrix} e_{11} & \dots & e_{1n_z} & \dots & e_{n_w 1} & \dots & e_{n_w n_z} \end{pmatrix}^\top \right\|_{c_0} := \sum_{i=1}^{n_w} \max(\|e_{i1}\|_\infty, \dots, \|e_{in_z}\|_\infty), \tag{4.8}$$

and, by Theorem 4.2, the range of $T(F)$ is closed, the infimum in (4.7) is attained by virtue of a well known duality result (see, e.g., Theorem 2 on page 121 of [14]). ◻

5. The square case. We now consider the case where U and V are square. Thus, suppose $n_z = n_u$ and $n_y = n_w$. Condition (4.1) is then equivalent to the requirement

$$\det U(\lambda) \neq 0 \quad \text{and} \quad \det V(\lambda) \neq 0 \quad \text{for all } \lambda \in \mathbf{T}. \tag{5.1}$$

THEOREM 5.1. *Let (5.1) be satisfied. If $Q \in W_+^{n_u \times n_y}$ is any matrix function at which $\|H - UQV\|$ attains its minimum, then Q is a rational matrix function and the residue $H - UQV$ is a matrix polynomial.*

Proof. Given a scalar transfer function $g(\lambda) = \sum_{k=0}^{\infty} g_k \lambda^k$, we define the Toeplitz matrix $T(g^*)$ as the upper-triangular matrix

$$T(g^*) = \begin{pmatrix} g_0 & g_1 & g_2 & \cdots \\ & g_0 & g_1 & \cdots \\ & & g_0 & \cdots \\ & & & \cdots \end{pmatrix}.$$

Let $F = U \otimes V^\top$ and write $F = (F_{ij})_{i=1,j=1}^{M,N}$ with scalar functions F_{ij}. Notice that $M = n_z n_w$ and $N = n_u n_y$. We define the block Toeplitz matrix $T(F^*)$ by

$$T(F^*) = \begin{pmatrix} T(F_{11}^*) & \ldots & T(F_{M1}^*) \\ \vdots & & \vdots \\ T(F_{1N}^*) & \ldots & T(F_{MN}^*) \end{pmatrix}.$$

Obviously, $T(F) : \ell_1^N \to \ell_1^M$ is the adjoint of $T(F^*) : c_0^M \to c_0^N$. Let $\mathcal{N}(T(F^*))$ be the null space of $T(F^*)$ on c_0^M. Due to Theorem 4.2, $\mathcal{R}(T(F)) = \mathcal{N}(T(F^*))^\perp$. Consequently, by Theorem 2 on page 121 of [14], the number (4.7) equals

$$d = \sup\{\langle z, h\rangle : z \in \mathcal{N}(T(F^*)), \|z\|_{c_0} \le 1\}. \tag{5.2}$$

We have

$$\det F(\lambda) = \det (U(\lambda) \otimes V^\top(\lambda)) = (\det U(\lambda))^{n_y} (\det V(\lambda))^{n_u}$$

and hence $\det F(\lambda) \ne 0$ for $\lambda \in \mathbf{T}$. This implies that $\mathcal{N}(T(F^*))$ is finite-dimensional (see, e.g., [11, Proposition 13.3] or [10, Section VIII.4]). It follows that the supremum in (5.2) is a maximum. Assume this maximum is attained at z_0. Again by Theorem 2 on page 121 of [14], z_0 and $e := h - T(F)q$ are aligned, that is, $\langle z_0, e\rangle = \|z_0\|_{c_0} \|e\|_1$. Taking into account definitions (4.5) and (4.8), this easily gives that e is finitely supported. Hence $E = H - UQV$ is a matrix polynomial, which shows that Q is rational. □

6. A numerical algorithm. By (4.6), the problem $\|H - UQV\|_W \to \min$ is equivalent to the problem

$$\|h - T(F)q\|_1 \to \min \tag{6.1}$$

with a matrix polynomial

$$F(\lambda) = F_0 + F_1\lambda + \ldots + F_r\lambda^r, \quad F_j \in \mathbf{R}^{n_z n_w \times n_u n_y} =: \mathbf{R}^{M \times N}.$$

We replace (6.1) by the finite problem

$$\|P_n h - P_n T(F) P_{n-r} q\|_1 \to \min. \tag{6.2}$$

Here $P_n : \ell_1 \to \ell_1$ is projection onto the first coordinates, that is,

$$P_n : \{s_0, s_1, s_2, \ldots\} \mapsto \{s_0, s_1, \ldots, s_{n-1}, 0, \ldots\}.$$

For a k-tuple $s = (s^1, \ldots, s^k) \in \ell_1^k$, we define $P_n s = (P_n s^1, \ldots, P_n s^k)$. Let $Q_n = I - P_n$. In the special case (4.3), for example, problem (6.2) amounts to minimizing

$$\left\| \begin{pmatrix} P_n h_1 \\ P_n h_2 \\ P_n h_3 \\ P_n h_4 \end{pmatrix} - \begin{pmatrix} P_n T(u_1 v_1) P_{n-r} & P_n T(u_1 v_3) P_{n-r} \\ P_n T(u_1 v_2) P_{n-r} & P_n T(u_1 v_4) P_{n-r} \\ P_n T(u_2 v_1) P_{n-r} & P_n T(u_2 v_3) P_{n-r} \\ P_n T(u_2 v_2) P_{n-r} & P_n T(u_2 v_4) P_{n-r} \end{pmatrix} \begin{pmatrix} P_{n-r} q_1 \\ P_{n-r} q_2 \end{pmatrix} \right\|_1.$$

Notice that $P_n T(u_i v_j) P_{n-r}$ may be identified with an $n \times (n-r)$ matrix, and hence we may regard the problem as the problem of finding an ℓ_1^4 optimal solution of an overdetermind system with $4n$ equations and $2(n-r)$ variables. In the general case, (6.2) involves Mn linear expressions (= "equations") and $N(n-r)$ variables.

We denote the minima in (6.1) and (6.2) by d and d_n, respectively.

THEOREM 6.1. *We have*

$$d \le d_n + \|Q_n h\|_1$$

for all $n \ge 1$. *If (4.1) is satisfied, then* $d_n \to d$ *as* $n \to \infty$.

Proof. Let

$$d_n = \|P_n h - P_n T(F) P_{n-r} q_n^*\|_1.$$

We may think of $P_{n-r} q_n^*$ as the Taylor coefficients of a matrix polynomial D of degree at most $n-r-1$ and we know that F is a matrix polynomial of degree r. This implies that DF has at most the degree $n-1$ and hence $Q_n T(F) P_{n-r} q_n^* = 0$. It follows that

$$\begin{aligned} d_n &= \|Ph - P_n T(F) P_{n-r} q_n^*\|_1 \\ &= \|P_n h - T(F) P_{n-r} q_n^*\|_1 \\ &\ge \|h - T(F) P_{n-r} q_n^*\|_1 - \|Q_n h\|_1 \\ &\ge d - \|Q_n h\|_1, \end{aligned} \tag{6.3}$$

as claimed.

If (4.1) holds, we deduce from Corollary 4.3 that there is a q_0 such that

$$d = \|h - T(F) q_0\|_1. \tag{6.4}$$

Since $\|P_n y\|_1 \le \|y\|_1$ for every y, we get

$$\begin{aligned} d_n &= \|P_n(h - T(F) P_{n-r} q_0)\|_1 \\ &\le \|h - T(F) P_{n-r} q_0\|_1 \\ &\le \|h - T(F) q_0\|_1 + \|T(F) Q_{n-r} q_0\|_1 \\ &\le d + \|T(F)\| \, \|Q_{n-r} q_0\|_1. \end{aligned} \tag{6.5}$$

Combining (6.3) and (6.5) we arrive at the conclusion that $d_n \to d$. □

We remark that since H is a rational matrix function with entries in W_+, the term $\|Q_n h\|_1$ goes to zero exponentially fast. In the square case, we can say even more.

COROLLARY 6.2. *If U and V are square matrix polynomials satisfying (5.1), then there are constants* $C < \infty$ *and* $\delta > 0$ *such that*

$$d_n - Ce^{-\delta n} \le d \le d_n + Ce^{-\delta n}$$

for all $n \ge 1$.

Proof. Let $\|H - UQV\|_W = d$ and, accordingly, $\|h - T(F) q_0\|_1 = d$. We already noted that $h \in \ell_1^M$ is exponentially decaying, and hence the estimate $d \le d_n + Ce^{-\delta n}$ follows from Theorem 6.1. By Theorem 5.1, the Taylor coefficients of Q are exponentially decaying, which implies that $q_0 \in \ell_1^M$ is also exponentially decaying. This in conjunction with (6.5) gives the estimate $d_n - Ce^{-\delta n} \le d$. □

It is clear that (6.2) is the simpler to handle the smaller the defect r between n and $n-r$ is. In [11], we considered the SISO case and showed that then Corollary 6.2 is true with (6.2) replaced by

$$\|P_n h - P_n T(F) P_{n-\kappa} q\|_1 \to \min,$$

where κ is the number of zeros (counted with multiplicities) of the scalar polynomial F in the open unit disk. Clearly, $\kappa \le r$. However, the proof of this result is much more involved than the proofs of Theorems 6.1 and Corollary 6.2.

Theorem 6.1 provides us with pretty good upper bounds for d. The search for tight lower bounds motivates the following modification of problem (6.2). Note that (6.2) is equivalent to finding

$$d_n := \min_{Q_{n-r} P_n q = 0} \|P_n h - P_n T(F) P_n q\|_1,$$

because the constraint $Q_{n-r} P_n q = 0$ forces the last r components of P_n to be zero. We now fix a number $\varepsilon > 0$ and consider the problem of determining

$$\tilde{d}_n := \min_{\|Q_{n-r} P_n q\| \le \varepsilon} \|P_n h - P_n T(F) P_n q\|_1. \tag{6.6}$$

As, obviously, $\tilde{d}_n \le d_n$ and $d_n + \|Q_n h\|_1$ is only slightly larger than d, there is some hope that $\tilde{d}_n$ is a close lower bound for d. The vector-valued sequence q is living in $\ell_1^{n_u n_y}$ and as said in Section 4, there is no need for specifying a concrete norm in $\ell_1^{n_u n_y}$. We now may take advantage of this freedom. In fact, the requirement $\|Q_{n-r} P_n q\| \le \varepsilon$ is a constraint for Nr variables and we may choose $\|\cdot\|$ to be any vector norm on $\mathbf{R}^{Nr}$. For example, in the context of (4.3), we can take

$$\|Q_{n-r} P_n q\| = \max(\|Q_{n-r} P_n q_1\|_\infty, \|Q_{n-r} P_n q_2\|_\infty),$$

and if we write

$$P_n q_i = (q_0^{(i)}, q_1^{(i)}, \dots, q_{n-1}^{(i)}),$$

then $\|Q_{n-r} P_n q\| \le \varepsilon$ is the constraint

$$|q_{n-r}^{(1)}| \le \varepsilon, \quad \dots, \quad |q_{n-1}^{(1)}| \le \varepsilon, \quad |q_{n-r}^{(2)}| \le \varepsilon, \quad \dots, \quad |q_{n-1}^{(2)}| \le \varepsilon.$$

Since $Q_n T(F) P_{n-r} q = 0$, we have the estimate

$$\|Q_n T(F) P_n q\|_1 = \|Q_n T(F) Q_{n-r} P_n q\|_1 \le \|T(F)\| \, \|Q_{n-r} P_n q\|, \tag{6.7}$$

and $\|T(F)\|$ depends on the norm on $\mathbf{R}^{Nr}$ we have chosen but not on n.

Let q_0 satisfy $\|h - T(F) q_0\|_1 = d$ and put $e = h - T(F) q_0$.

THEOREM 6.3. *Suppose (4.1) holds. If* $\|Q_{n-r} P_n q_0\| \le \varepsilon$ *then*

$$\tilde{d}_n - \|Q_n e\|_1 \le d,$$

and for arbitrary $n \ge 1$ *we have*

$$d \le \tilde{d}_n + \|Q_n h\|_1 + \|T(F)\| \, \varepsilon.$$

Proof. Our assumptions guarantee that

$$\begin{aligned}\tilde{d}_n &\le \|P_n h - P_n T(F) P_n q_0\|_1\\ &= \|P_n h - P_n T(F) q_0\|_1\\ &= \|P_n(h - T(F)q_0)\|_1\\ &= \|P_n e\|_1 = \|e - Q_n e\|_1\\ &\le \|e\|_1 + \|Q_n e\|_1 = d + \|Q_n e\|_1.\end{aligned}$$

On the other hand, let q_n^* be a solution of the minimum problem (6.6):

$$\|Q_{n-r} P_n q_n^*\| \le \varepsilon, \quad \|P_n h - P_n T(F) P_n q_n^*\|_1 = \tilde{d}_n.$$

Then

$$\begin{aligned}d &\le \|h - T(F) P_n q_n^*\|_1\\ &= \|P_n h + Q_n h - P_n T(F) P_n q_n^* - Q_n T(F) P_n q_n^*\|_1\\ &\le \|P_n h - P_n T(F) P_n q_n^*\|_1 + \|Q_n h\|_1 + \|Q_n T(F) P_n q_n^*\|_1\\ &\le \tilde{d}_n + \|Q_n h\|_1 + \|T(F)\|\varepsilon,\end{aligned}$$

the last estimate resulting from (6.7). □

We know that $\|Q_n h\|_1 \to 0$ (exponentially fast) and $\|Q_n e\|_1 \to 0$ (in the square case even $\|Q_n e\|_1 = 0$ for all sufficiently large n). Consequently, we certainly have $\|Q_n h\|_1 \le \varepsilon$ and $\|Q_n e\|_1 \le \varepsilon$ if only n is large enough. Thus, Theorem 6.3 shows that $\tilde{d}_n - \varepsilon$ is a lower bound for d whenever n is large enough and, moreover, that this bound is at a distance of at most $2\varepsilon + \|T(F)\|\varepsilon$ to d. If e (or equivalently, q_0) decays exponentially, then the n's for which the lower bound $\tilde{d}_n - \varepsilon$ is applicable are certainly not of astronomic dimensions.

7. Linear equation with polynomials instead of truncated rational functions. Very often in applications, H is a rational function and the above outlined truncation scheme becomes overly inefficient. It is however, always possible to convert the equation $E = H - UQV$ with rational matrix functions H, U and V into an equivalent equation with polynomial matrix functions. The next paragraph gives conditions under which one such a simple transformation is possible and the section devoted to an example of mixed sensitivity design elaborates on a practical instance of this.

Recall from the section 3 that the closed-loop (matrix) transfer function is

$$G = P_{wz} + P_{uz} Y_R^0 A_L P_{wy} + P_{uz} A_R Q A_L P_{wy}$$

Let the rational functions P_{uz} and P_{wy} be expressed as ratios of polynomial matrices $P_{uz} = D_{uz}^{-1} N_{uz}$ and $P_{wy} = N_{wy} D_{wy}^{-1}$, respectively. A simple way to transform this equation into an equation where all the known terms are polynomial matrices is to multiply both sides of the equation by the polynomial matrix D_{uz} from the left and D_{wy} from the right. The condition under which this procedure will successfully yield a *polynomial* equation is then obvious: the product $D_{uz} P_{zw} D_{wy}$ must be a polynomial matrix. This is satisfied in most control problems like mixed sensitivity minimization (see the section with control design example).

8. A numerical example for square case. We consider the triple of polynomial matrices U, V, and H given by

$$U(\lambda) = \begin{pmatrix} 4.2 + 6.5\lambda + 0.0025\lambda^2 + 0.39\lambda^3 & -1.4 - 2.1\lambda + 0.11\lambda^2 - 0.17\lambda^3 \\ 2.1 - 1.3\lambda + 0.32\lambda^2 & -0.79 + 0.66\lambda - 0.13\lambda^2 \end{pmatrix}$$
$$V(\lambda) = \begin{pmatrix} 0.2 + 4.5\lambda + 4.1\lambda^2 + 0.92\lambda^3 & 0.51 + 5.8\lambda + 7.7\lambda^2 + 2.4\lambda^3 \\ 0.27 + 3.5\lambda + 1.4\lambda^2 & 0.7 + 4.5\lambda + 3.6\lambda^2 \end{pmatrix}$$
$$H(\lambda) = \begin{pmatrix} -3 + \lambda + 4\lambda^2 & -5 + 3\lambda + 9\lambda^2 \\ 13\lambda - 3\lambda^2 & 14 + \lambda^2 \end{pmatrix}.$$

We look for a pair of matrix functions Q and E in $W_+^{2\times 2}$ that solve the equation $UQV + E = H$ and minimize the row-sum norm of the matrix function E.

For a matrix function $S(\lambda) = \sum_{j\geq 0} S_j\lambda^j$, we define the matrix polynomial $\pi_n S$ by

$$(\pi_n S)(\lambda) = \sum_{j=0}^{n-1} S_j\lambda^j.$$

With this notation, the equation

$$P_n T(F) P_{n-r} q + P_n e = P_n h$$

is equivalent to the equation

$$(\pi_n U)(\pi_{n-r} Q)(\pi_n V) + \pi_n E = \pi_n H. \tag{8.1}$$

We solve (8.1) successively for $n = r, r+1, r+2, \ldots$. In the case at hand, the matrix polynomial $F = U \otimes V^\top$ has the degree $r = 6$ and since U, V, H are of degree 3, they are not affected by π_n for $n \geq r = 6$. Thus,we have to find matrix polynomials $\pi_{n-6}Q$ and $\pi_n E$ such that

$$U(\pi_{n-6}Q)V + \pi_n E = H, \quad \|\pi_n E\|_W \to \min. \tag{8.2}$$

We call n the expected degree of E and $n-6$ the expected degree of Q. If (8.2) is uniquely solvable, we refer to the degrees of the solutions $\pi_n E$ and $\pi_{n-6}Q$ as the true degrees of E and Q, respectively. Table 8.1 reports the results. The relative error is defined as

$$\|U(\pi_{n-6}Q)V + \pi_n E - H\|_W / \|(U, V, H)\|_W.$$

Thus, as seen from Table 8.1 and the Figure 8, the optimal solution pair can be found within 6 steps. This pair is

$$Q(\lambda) = \begin{pmatrix} -6.4 - 15\lambda - 0.51\lambda^2 + 0.085\lambda^3 - 0.63\lambda^4 & 7.6 + 23\lambda + 11\lambda^2 + 0.11\lambda^3 + 0.77\lambda^4 + 0.41\lambda^5 \\ -26 - 59\lambda - 6.9\lambda^2 - 0.82\lambda^3 - 1.5\lambda^4 & 29 + 92\lambda + 48\lambda^2 + 5.3\lambda^3 + 2.5\lambda^4 + 0.97\lambda^5 \end{pmatrix}$$
$$E(\lambda) = \begin{pmatrix} -2.6 - 6.6\lambda - 9.8\lambda^2 & -4 + 3.3\lambda^4 \\ 0.51 + 7.8\lambda & 15 - 2.6\lambda \end{pmatrix}.$$

9. Control design example - mixed sensitivity minimization. Design a digital feedback controller for an unstable continuous-time plant described by the transfer function

$$P(s) = \frac{\omega_n^2}{s^2 - 2\zeta\omega_n s + \omega_n^2}$$

for $\zeta = 0.05$ and $\omega_n = 20 \times 2\pi$. A practically reasonable formulation of a control

TABLE 8.1
Sequence of the optimal solutions to (8.2) for increasing $n \geq 6$.

Expected deg E	True deg E	Expected deg Q	True deg Q	$\Vert\pi_n E\Vert_W$	Relative error
6	3	0	0	30.5730	4.45e-15
7	4	1	1	30.0714	7.33e-13
8	4	2	2	27.1000	8.76e-15
9	5	3	3	26.3444	8.42e-14
10	7	4	4	26.2335	3.52e-12
11	4	5	5	26.2141	1.52e-12
12	4	6	5	26.2141	4.43e-12
13	4	7	5	26.2141	1.36e-13
14	4	8	5	26.2141	1.84e-12
15	4	9	5	26.2141	2.21e-11
16	4	10	5	26.2141	1.71e-12
17	4	11	5	26.2141	1.98e-12
18	4	12	5	26.2141	9.25e-12
19	4	13	5	26.2141	1.62e-12
20	4	14	5	26.2141	1.06e-11

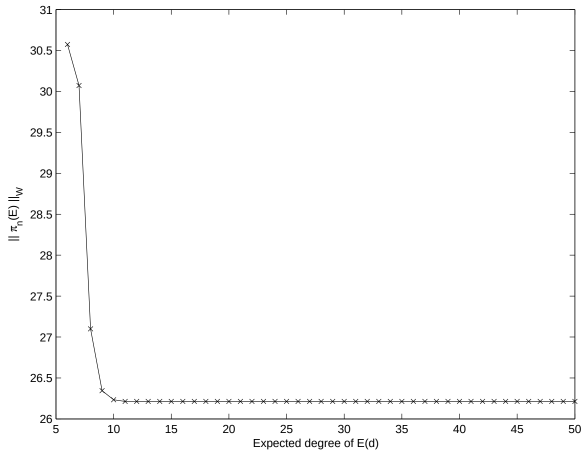

FIG. 8.1. *Norm of the residue E dependent on the degree of the polynomial E.*

design objective is to designing discrete-time controller for a discretized model $P(\lambda) = \frac{b(\lambda)}{a(\lambda)}$ such that the weighted mixed sensitivity function is minimized

$$\min_{C \text{ stabilizing}} \left\Vert \begin{bmatrix} W_1 S \\ W_2 T \end{bmatrix} \right\Vert_W \tag{9.1}$$

where $S = 1/(1 + PC)$ and $T = PC/(1 + PC)$ and the two weighting filters $W_1(\lambda) = \frac{n_{W_1}(\lambda)}{d_{W_1}(\lambda)}$ and $W_2(\lambda) = \frac{n_{W_2}(\lambda)}{d_{W_2}(\lambda)}$ characterize the knowledge of the disturbance and the uncertainty in the model, respectively. Generalized plant with 2 inputs (disturbance

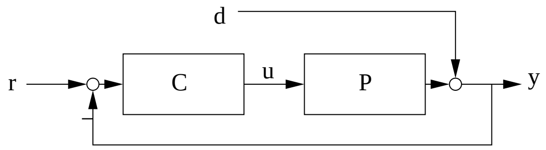

FIG. 9.1. *Standard feedback configuration*

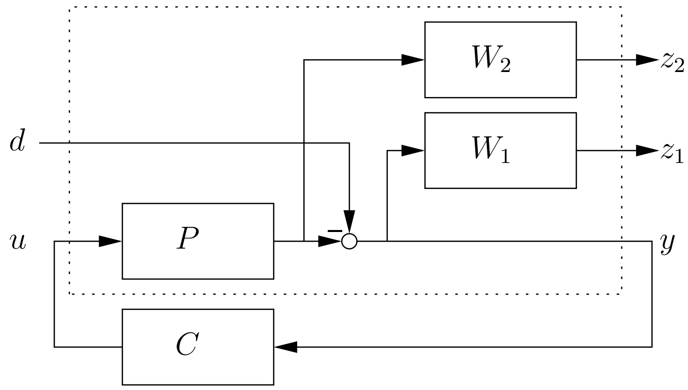

FIG. 9.2. *Generalized plant (in the dashed box) for mixed sensitivity minimization problem.*

and control) and three outputs (weighted error, weighted output, plant output) is

$$P = \begin{bmatrix} W_1 & -W_1 P \\ 0 & W_3 P \\ 1 & -P \end{bmatrix} \tag{9.2}$$

The affine parameterization of all achievable closed-loop transfer functions is

$$G = \begin{bmatrix} W_1 a x_0 \\ W_3 b y_0 \end{bmatrix} + \begin{bmatrix} -W_1 b \\ W_3 b \end{bmatrix} Qa \tag{9.3}$$

where x_0 and y_0 satisfy $ax_0 + by_0 = 1$. Multiplying the equation (9.3) on both sides by diag$\{d_{W_1}, d_{W_2}\}$ yields a linear equations where all the given terms are polynomial matrices

$$\begin{bmatrix} d_{W_1} & 0 \\ 0 & d_{W_2} \end{bmatrix} G = \begin{bmatrix} n_{W_1} a x_0 \\ n_{W_3} b y_0 \end{bmatrix} + \begin{bmatrix} -n_{W_1} b \\ n_{W_3} b \end{bmatrix} Qa \tag{9.4}$$

A sequence of linear equations with polynomial matrices minimizing the row sum norm of $G(\lambda)$ is solved for increasing degree of $G(\lambda)$. For a concrete couple of weighting filters: $W_1 = (2 - 0.089\lambda + 0.35\lambda^2)/(1 - 1.8\lambda + 0.8\lambda^2)$ and $W_3 = (0.094 - 0.19\lambda + 0.094\lambda^2)/(1 + 1.8\lambda + 0.85\lambda^2)$, the Figure 9.3 shows how the upper and lower bounds on the best achievable ℓ_1 norm of the closed-loop impulse response evolve with the increasing degree of the closed-loop FIR transfer function. The Figure 9.4 then shows how the relative error between the two bounds evolves. From this we can conclude that to guarantee 5% accuracy, we can stop the computation at the degree of $G(\lambda)$ equal to 100. The (sub)optimal controller is then of correspondingly high order. This high order is a well-known curse of all existing methods for ℓ_1-optimal control that attack the ℓ_1 norm directly from definition as sum of absolute values. Fortunately, in our

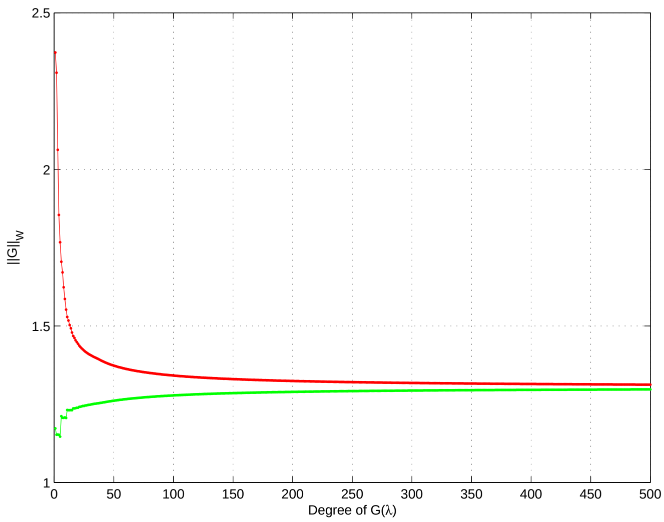

FIG. 9.3. *Converging upper and lower bounds on the optimal ℓ_1 norm of the closed-loop impulse response.*

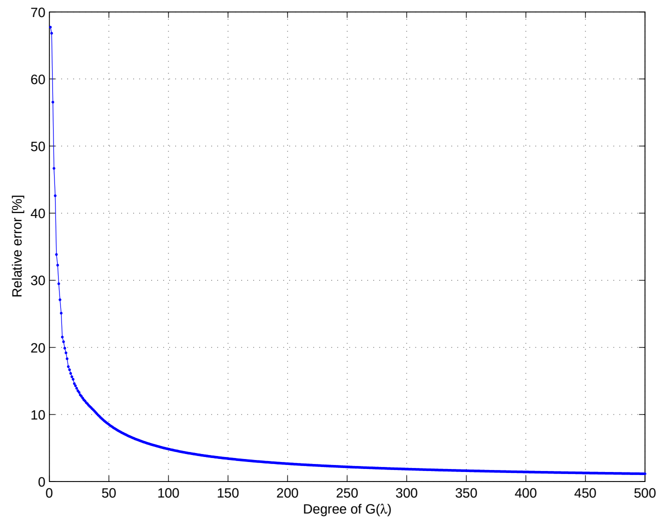

FIG. 9.4. *Difference between the upper and lower bound on the achievable ℓ_1 norm of the closed loop impulse response related to the mean of the two.*

method (as well as in other non-interpolation methods), when it comes to a controller implementation, common parametric identification methods can be used to get a rational approximation of the Youla-Kučera parameter and therefore an acceptably low order controler (with quantified error introduced by this approximation). The Figures 9.5 and 9.6 show some comparison between the responses of ℓ_1-(sub)optimal and $\mathcal{H}_\infty$-optimal controllers.

10. Conclusions. A new theoretical framework is built for formulating and solving the standard problem of designing an ℓ_1-optimal controller for a MIMO systems.

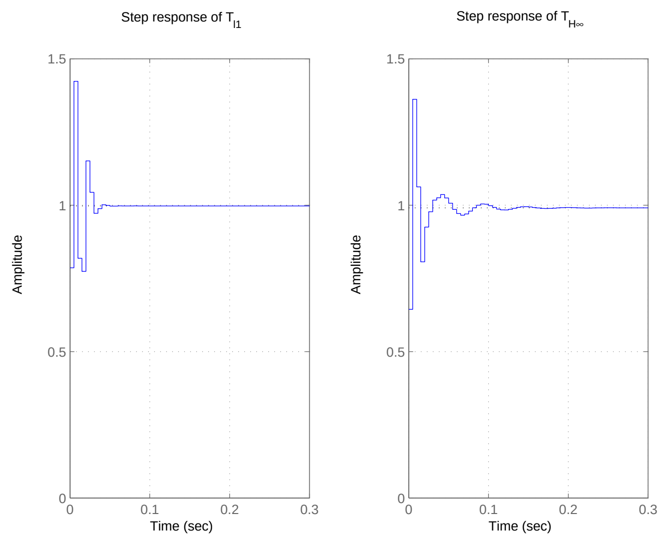

FIG. 9.5. *Unit step responses of the closed loops with ℓ_1-optimal and $\mathcal{H}_\infty$-optimal controllers.*

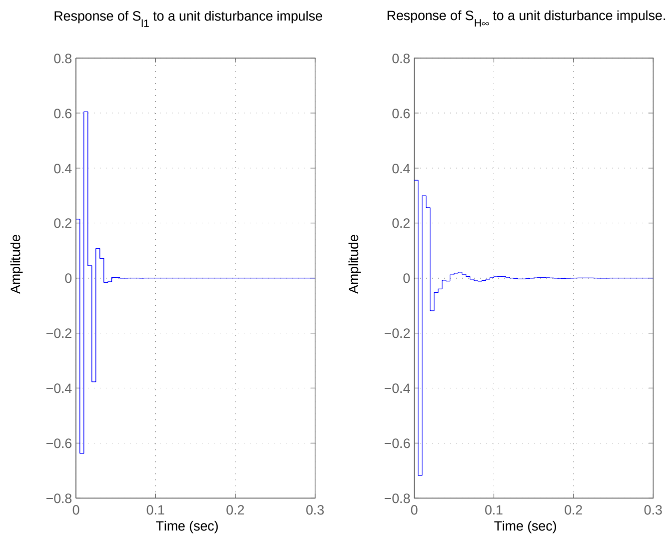

FIG. 9.6. *Responses to unit impulse disturbances with ℓ_1-optimal and $\mathcal{H}_\infty$-optimal controllers.*

It is based on the well-established theory of block Toeplitz operators and eludes the use of interpolation that is present in most currently available solutions to the problem. A numerical algorithm is proposed for solving the one-block case exactly in a finite number of truncation steps. Converging lower and upper bounds are supplied for the general multiblock case. A key advantage of this new method is that there is no need for computing zeros and zero directions of polynomial matrices and that the optimal Youla-Kučera parameter is returned immediately as an outcome from the optimization procedure. The numerical troubles with the extraction of a controller from the optimal closed-loop transfer function are thus avoided. Further research will include tighter lower bounds with faster convergence and possibly mixed optimization criteria ($\ell_1/\mathcal{H}_2$, $\ell_1/\mathcal{H}_\infty$), which has already been done for interpolation-based methods [18].

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