Wackelsatz and Stechkin's inequality for discrete Muckenhoupt weights

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The purpose of this paper is to present full proofs for two important results on discrete Muckenhoupt weights. The first states that if w is a weight in the Muckenhoupt class A_p for ℓ^p, then w^r belongs to A_p for all r sufficiently close to 1 ("Wackelsatz"). The second result is Stechkin's inequality, which gives an upper estimate for the multiplier norm on $\ell^p(w)$ ($w \in A_p$) through the L^∞ norm and the total variation of the multiplier. Although both results are certainly well-known to specialists, we have not found self-contained proofs in the literature.

1 Introduction

Throughout this paper we assume that $1 < p < \infty$. A weight on $\mathbf{Z}_+ = \{0, 1, 2, \ldots\}$ is a sequence $w = \{w_n\}_{n=0}^\infty$ of positive real numbers. Given a weight w, we denote by $\ell^p(w) := \ell^p(\mathbf{Z}_+, w)$ the Banach space of all complex-valued sequences $x = \{x_n\}_{n=0}^\infty$ such that

$$\|x\|_{\ell^p(w)} := \left(\sum_{n=0}^\infty |x_n|^p w_n^p \right)^{1/p} < \infty.$$

We write $w \in A_p$ and say that w is a Muckenhoupt weight for ℓ^p if there is a constant $C < \infty$ such that

$$\frac{1}{n-m+1} \left(\sum_{k=m}^n w_k^p \right)^{1/p} \left(\sum_{k=m}^n w_k^{-q} \right)^{1/q} < C \tag{1}$$

for all m, n with $0 \le m \le n$; here and in what follows, q is given by $1/p + 1/q = 1$. Hunt, Muckenhoupt, and Wheeden [5] showed that the matrix

$$S_+ := \begin{pmatrix} 0 & -1 & -1/2 & -1/3 & \cdots \\ 1 & 0 & -1 & -1/2 & \cdots \\ 1/2 & 1 & 0 & -1 & \cdots \\ 1/3 & 1/2 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{2}$$

generates a bounded operator on $\ell^p(w)$ if and only if $w \in A_p$.

Weights of the form $\{(n+1)^\lambda\}_{n=0}^\infty$ are referred to as power weights. It is readily seen that a power weight belongs to A_p if and only if $-1/p < \lambda < 1/q$. Clearly, if $-1/p < \lambda < 1/q$, then $-1/p < \lambda r < 1/q$ for all r close enough to 1. The following theorem generalizes this trivial observation to general Muckenhoupt weights.

Theorem 1.1 *If $w \in A_p$, then there is an $\varepsilon = \varepsilon_{p,w} > 0$ such that $w^r \in A_p$ for all $r \in (1-\varepsilon, 1+\varepsilon)$.*

We like the German name "Wackelsatz" for this theorem. The literal but less charming English translation is probably "shake theorem".

Theorem 1.1 is well known, although we are not able to give a convenient explicit reference. Strömberg and Torchinsky [7, p. 11] state such a result in an abstract context. We give a full and self-contained proof in Section 2 of this paper.

Given a weight $w = \{w_n\}_{n=0}^{\infty}$ on $\mathbf{Z}_+$, we denote by $\ell^p(\mathbf{Z}, w)$ the Banach space of all complex-valued sequences $x = \{x_n\}_{n=-\infty}^{\infty}$ for which

$$\|x\|_{\ell^p(\mathbf{Z},w)} := \left(\sum_{n=-\infty}^{\infty} |x_n|^p w_{|n|}^p \right)^{1/p} < \infty;$$

thus, we identify w with its continuation to an even weight onto all of $\mathbf{Z}$.

If X and Y are Banach spaces, we let $\mathcal{L}(X,Y)$ and $\mathcal{K}(X,Y)$ denote the bounded and compact linear operators of X to Y, respectively. As usual, we put $\mathcal{L}(X,X) =: \mathcal{L}(X)$, $\mathcal{K}(X,X) =: \mathcal{K}(X)$.

Let $\mathbf{T}$ be the complex unit circle. For $a \in L^\infty := L^\infty(\mathbf{T})$, we denote by $\{a_n\}_{n=-\infty}^{\infty}$ the sequence of the Fourier coefficients of a,

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\pi}) e^{-in\theta} \mathrm{d}\theta,$$

and we let $L(a)$ stand for the Laurent matrix generated by a:

$$L(a) = (a_{j-k})_{j,k=-\infty}^{\infty}.$$

It is well known that $L(a)$ generates a bounded operator on $\ell^2(\mathbf{Z})$. We let $M_{p,w}$ denote the set of all $a \in L^\infty$ for which there is a constant $C_{p,w,a}$ such that

$$\|L(a)x\|_{\ell^p(\mathbf{Z},w)} \le C_{p,w,a} \|x\|_{\ell^p(\mathbf{Z},w)} \text{ for all } x \in \ell^2(\mathbf{Z}) \cap \ell^p(\mathbf{Z}, w).$$

Clearly, if $a \in M_{p,w}$, then $L(a)$ induces a bounded operator on $\ell^p(\mathbf{Z}, w)$. The set $M_{p,w}$ is a Banach algebra with pointwise algebraic operations and the norm

$$\|a\|_{p,w} := \|L(a)\|_{\mathcal{L}(\ell^p(\mathbf{Z},w))}. \tag{3}$$

A function $a : \mathbf{T} \to \mathbf{C}$ is said to be of bounded variation, $a \in BV$, if the function

$$\tilde{a} : [0, 2\pi] \to \mathbf{C}, \quad \theta \mapsto a(e^{i\theta})$$

is of bounded variation, that is, if

$$\mathrm{V}(a) := \sup \sum_{j=1}^{n} \left| a(e^{i\theta_{j+1}}) - a(e^{i\theta_j}) \right| < \infty,$$

the supremum over all partitions $0 \le \theta_1 < \theta_2 < \dots \theta_{n+1} \le 2\pi$.

For spaces without weight, the following result goes back to Stechkin[8] It is also well known for power weights [1], [2], [4], [6]. Specialists certainly know the result for Muckenhoupt weights, but we have not found it in the literature. The proofs of [1], [2], [4], [6] do not immediately extend to the case of general Muckenhoupt weights.

Theorem 1.2 (Stechkin's inequality) *If $w \in A_p$, then there is a constant $C_{p,w} < \infty$ depending only on p and w such that*

$$\|a\|_{p,w} \le C_{p,w}\Big(\|a\|_\infty + \mathrm{V}(a)\Big)$$

for all $a \in BV$. In particular, $BV \subset M_{p,w}$.

A full proof of Theorem 1.2 will be given in Section 3.

2 Proof of the Wackelsatz

This section is devoted to the proof of Theorem 1.1. Our proof is an appropriate modification of the very clear proof given by García-Curva and Rubio de Francia [3] for Muckenhoupt weights on $\mathbf{R}^n$.

Throughout this section, let R be a set of the form $R = \{m, m+1, \dots, n\}$ with $m, n \in \mathbf{Z}_+$ and $m \le n$. The cardinality of a set E will be denoted by $|E|$.

Lemma 2.1 *If $w \in A_p$, then there exists a constant $C < \infty$ depending only on p and w such that*

$$\left(\frac{|E|}{|R|}\right)^p \sum_{k\in R} w_k^p \le C \sum_{k\in E} w_k^p$$

for all R and all $E \subset R$.

Proof. Let $\{x_k\}$ be an arbitrary sequence of complex numbers. Hölder's inequality gives

$$\sum_{k\in R} |x_k| \le \left(\sum_{k\in R} |x_k|^p w_k^p\right)^{1/p} \left(\sum_{k\in R} w_k^{-q}\right)^{1/q},$$

whence

$$\begin{aligned} &\frac{1}{|R|}\left(\sum_{k\in R} w_k^p\right)^{1/p}\left(\sum_{k\in R}|x_k|\right) \\ &\le \frac{1}{|R|}\left(\sum_{k\in R} w_k^p\right)^{1/p}\left(\sum_{k\in R} w_k^{-q}\right)^{1/q}\left(\sum_{k\in R}|x_k|^p w_k^p\right)^{1/p} \\ &\le \tilde{C}\left(\sum_{k\in R}|x_k|^p w_k^p\right)^{1/p}, \end{aligned}$$

where $\tilde{C}$ is the constant from (1). Letting $x_k = 1$ for $k \in E$ and $x_k = 0$ for $k \notin E$, we arrive at the desired estimate. ■

Lemma 2.2 *If $w \in A_p$, then for every $\alpha \in (0,1)$ there exists a $\beta \in (0,1)$ with the following property:*

$$A \subset R, \quad |A| \leq \alpha |R| \quad \Rightarrow \quad \sum_{k \in A} w_k^p \leq \beta \sum_{k \in R} w_k^p$$

Proof. Lemma 2.1 for $E = R \setminus A$ tells us that

$$\left(\frac{|R \setminus A|}{|R|} \right)^p \sum_{k \in R} w_k^p \leq C \left(\sum_{k \in R} w_k^p - \sum_{k \in A} w_k^p \right),$$

and since obviously

$$\frac{|R \setminus A|}{|R|} = \frac{|R| - |A|}{|R|} = 1 - \frac{|A|}{|R|} \geq 1 - \alpha$$

it follows that

$$C \sum_{k \in A} w_k^p \leq (C - (1-\alpha)^p) \sum_{k \in R} w_k^p.$$

This is the assertion with $\beta = 1 - (1-\alpha)^p / C$. ■

A simple application of Hölder's inequality shows that if w is *any* weight, then

$$\left(\frac{1}{|R|} \sum_{k \in R} w_k^p \right)^{1+\varepsilon} \leq \frac{1}{|R|} \sum_{k \in R} w_k^{p(1+\varepsilon)} \tag{4}$$

for all $\varepsilon > 0$. If w is a *Muckenhoupt* weight, then, up to a constant, the "$\leq$" can be reversed provided that ε is sufficiently small. We confine ourselves to proving this in the case where $|R|$ is a power of 2.

Lemma 2.3 (The reverse Hölder inequality) *If $w \in A_p$, then there exist constants $\delta > 0$ and $M < \infty$ depending only on p and w such that*

$$\frac{1}{|R|} \sum_{k \in R} w_k^{p(1+\varepsilon)} \leq M \left(\frac{1}{|R|} \sum_{k \in R} w_k^p \right)^{1+\varepsilon}$$

for all $\varepsilon \in [0, \delta]$ and all R of the form $|R| = 2^r$ with $r \in \mathbf{N}$.

Proof. Fix any R such that $|R| = 2^r$. Pick a number $\alpha \in (0,1)$ and put

$$\lambda_0 = \frac{1}{|R|} \sum_{k \in R} w_k^p, \quad \lambda_k = \left(\frac{2}{\alpha}\right)^k \lambda_0 \quad (k \geq 1).$$

Obviously, $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$

Let $\mathcal{D}$ be the following set of subsets of R:

$$\begin{aligned}\mathcal{D} := \Big\{ &\{m\}, \{m+1\}, \{m+2\}, \{m+3\}, \ldots, && \{n\},\\ &\{m, \quad m+1\}, \{m+2, \quad m+3\}, \ldots, && \{n-1, \quad n\},\\ &\{m, \quad m+1, \quad m+2, \quad m+3\}, \ldots, \{n-3, \quad n-2, && n-1, \quad n\},\\ &\qquad\qquad\qquad \cdots \\ &\{m, \quad m+1, \quad \ldots, && n\} \Big\}.\end{aligned}$$

Define $F : \mathcal{D} \to \mathbf{R}$ by

$$F(A) = \frac{1}{|A|} \sum_{l \in A} w_l^p,$$

and, for $k \in \mathbf{Z}_+$, consider the sets

$$\mathcal{G}_k := \{A \in \mathcal{D} : F(A) > \lambda_k\}.$$

Because of the structure of $\mathcal{D}$, every set $A \in \mathcal{G}_k$ is either properly contained in another set of $\mathcal{G}_k$ or disjoint to all sets of $\mathcal{G}_k$ the cardinality of which is greater than $|A|$. We call the sets of the latter kind maximal sets of $\mathcal{G}_k$ and denote them by $Q_{k,1}, Q_{k,2}, \ldots, Q_{k,n_k}$. Put

$$D_k = \bigcup_j Q_{k,j}. \tag{5}$$

As every set of $\mathcal{G}_k$ is either maximal or contained in some maximal set, we have

$$D_k = \bigcup_{A \in \mathcal{G}_k} A. \tag{6}$$

Our choice of λ_0 implies that $F(R) = \lambda_0$. Hence $R \notin \mathcal{G}_0$. Because $\lambda_k > \lambda_0$ for $k \geq 1$, we have $R \notin \mathcal{G}_k$ for $k \geq 1$ as well. Consequently, every maximal set of $\mathcal{D}$ is *properly* contained in another set of $\mathcal{D}$ (at least in R). Let now $Q_{k,j}$ be a maximal set and $A \in \mathcal{D}$ be the set which is (uniquely) determined by the requirements $Q_{k,j} \subset A$ and $|A| = 2|Q_{k,j}|$. Then

$$F(A) = \frac{1}{|A|} \sum_{l \in A} w_l^p > \frac{1}{|A|} \sum_{l \in Q_{k,j}} w_l^p = \frac{1}{2|Q_{k,j}|} \sum_{l \in Q_{k,j}} w_l^p = \frac{1}{2} F(Q_{k,j}).$$

If A belonged to $\mathcal{G}_k$ then $Q_{k,j}$ would not be maximal. Hence $A \notin \mathcal{G}_k$ and thus $F(A) \leq \lambda_k$. Consequently,

$$\lambda_k < F(Q_{k,j}) < 2\lambda_k.$$

Because $\lambda_{k+1} > \lambda_k$, each $Q_{k+1,m}$ also belongs to $\mathcal{G}_k$ and is therefore contained in some $Q_{k,\varphi(m)}$. This in conjunction with (5) implies that $D_{k+1} \subset D_k$. Furthermore, $D_{k+1} \cap Q_{k,j}$ is the union of those sets $Q_{k+1,m}$ for which $\varphi(m) = j$. As the maximal sets are pairwise disjoint, we get

$$\begin{aligned} 2\lambda_k &> \frac{1}{|Q_{k,j}|} \sum_{l\in Q_{k,j}} w_l^p \geq \frac{1}{|Q_{k,j}|} \sum_{l\in Q_{k,j}\cap D_{k+1}} w_l^p \\ &= \frac{1}{|Q_{k,j}|} \sum_{m:Q_{k+1,m}\subset Q_{k,j}} \sum_{l\in Q_{k+1,m}} w_l^p \\ &> \frac{1}{|Q_{k,j}|} \sum_{m:Q_{k+1,m}\subset Q_{k,j}} |Q_{k+1,m}|\ \lambda_{k+1} \\ &= \frac{1}{|Q_{k,j}|} |Q_{k,j}\cap D_{k+1}|\ \lambda_{k+1}. \end{aligned}$$

Thus,

$$\frac{|Q_{k,j}\cap D_{k+1}|}{|Q_{k,j}|} < 2\frac{\lambda_k}{\lambda_{k+1}} = 2\frac{\alpha}{2} = \alpha.$$

Lemma 2.2 now ensures the existence of a number $\beta \in (0,1)$ such that

$$\sum_{l\in Q_{k,j}\cap D_{k+1}} w_l^p \leq \beta \sum_{l\in Q_{k,j}} w_l^p.$$

This holds for all $Q_{k,j}$. Adding these inequalities for all j we obtain

$$\sum_{l\in D_{k+1}} w_l^p \leq \beta \sum_{l\in D_k} w_l^p,$$

whence

$$\sum_{l\in D_k} w_l^p \leq \beta^k \sum_{l\in D_0} w_l^p. \tag{7}$$

Let N be the greatest $k \in \mathbf{Z}_+$ for which $\mathcal{G}_k \neq \emptyset$; in case $\mathcal{G}_k = \emptyset$ for all $k \in \mathbf{Z}_+$, we put $N = -1$ and $\sum_{k=0}^{-1} \ldots = 0$. If $l \in D_k \setminus D_{k+1}$ then $w_l^p \leq \lambda_{k+1}$, since otherwise the set $\{l\}$ would belong to $\mathcal{G}_{k+1}$ (recall (6)). Thus, for every $\varepsilon > 0$,

$$\begin{aligned} \sum_{l\in R} w_l^{p\,(1+\varepsilon)} &= \sum_{l\in R\setminus D_0} w_l^{p\,(1+\varepsilon)} + \sum_{k=0}^{N} \sum_{l\in D_k\setminus D_{k+1}} w_l^{p\,(1+\varepsilon)} \\ &\leq \lambda_0^\varepsilon \sum_{l\in R\setminus D_0} w_l^p + \sum_{k=0}^{N} \left(\lambda_{k+1}^\varepsilon \sum_{l\in D_k\setminus D_{k+1}} w_l^p \right) \\ &= \lambda_0^\varepsilon \left(\sum_{l\in R\setminus D_0} w_l^p + \sum_{k=0}^{N} \left(\frac{2}{\alpha}\right)^{(k+1)\varepsilon} \sum_{l\in D_k\setminus D_{k+1}} w_l^p \right). \end{aligned}$$

From (7) we see that

$$\sum_{l\in D_k\setminus D_{k+1}} w_l^p \le \sum_{l\in D_k} w_l^p \le \beta^k \sum_{l\in D_0} w_l^p.$$

In summary,

$$\sum_{l\in R} w_l^{p\,(1+\varepsilon)} \le \lambda_0^\varepsilon \left(\sum_{l\in R\setminus D_0} w_l^p + \left(\frac{2}{\alpha}\right)^\varepsilon \sum_{k=0}^{N} \left(\frac{2^\varepsilon \beta}{\alpha^\varepsilon}\right)^k \sum_{l\in D_0} w_l^p \right).$$

Evidently, there is a $\delta > 0$ such that $2^\varepsilon\beta/\alpha^\varepsilon < 1$ for all $\varepsilon \in [0,\delta]$. For these ε we have

$$M := \max\left\{1, \left(\frac{2}{\alpha}\right)^\varepsilon \sum_{k=0}^{\infty} \left(\frac{2^\varepsilon \beta}{\alpha^\varepsilon}\right)^k\right\} < \infty,$$

which gives

$$\sum_{l\in R} w_l^{p\,(1+\varepsilon)} \le \lambda_0^\varepsilon M \sum_{l\in R} w_l^p.$$

Taking into account the definition of λ_0, we finally obtain

$$\frac{1}{|R|}\sum_{l\in R} w_l^{p\,(1+\varepsilon)} \le M \left(\frac{1}{|R|}\sum_{l\in R} w_l^p\right)^\varepsilon \left(\frac{1}{|R|}\sum_{l\in R} w_l^p\right) = M \left(\frac{1}{|R|}\sum_{l\in R} w_l^p\right)^{1+\varepsilon}. \blacksquare$$

Proof of Theorem 1.1. Let w in A_p. Then $w^{-1} \in A_q$. Applying Lemma 2.3 to w and w^{-1} we get the existence of constants $\delta > 0$ and $M < \infty$ such that if $\varepsilon \in [0,\delta]$ and $|R| = 2^r$, then

$$\begin{aligned} \frac{1}{|R|}\sum_{k\in R} w_k^{p\,(1+\varepsilon)} &\le M \left(\frac{1}{|R|}\sum_{k\in R} w_k^p\right)^{1+\varepsilon}, \\ \frac{1}{|R|}\sum_{k\in R} w_l^{-q\,(1+\varepsilon)} &\le M \left(\frac{1}{|R|}\sum_{k\in R} w_k^{-q}\right)^{1+\varepsilon}. \end{aligned}$$

Consequently,

$$\begin{aligned} &\frac{1}{|R|}\left(\sum_{k\in R} w_k^{(1+\varepsilon)p}\right)^{1/p} \left(\sum_{k\in R} w_k^{-(1+\varepsilon)q}\right)^{1/q} \\ &\le M^{\frac{1}{p}+\frac{1}{q}} \left(\frac{1}{|R|}\sum_{k\in R} w_k^p\right)^{\frac{1+\varepsilon}{p}} \left(\frac{1}{|R|}\sum_{k\in R} w_k^{-q}\right)^{\frac{1+\varepsilon}{q}} \\ &\le M\, C^{1+\varepsilon}, \end{aligned}$$

where C is the constant of (1).

Now let A be an arbitrary subset of $\mathbf{Z}_+$ with $|A| \geq 2$. Choose any subset R of $\mathbf{Z}_+$ such that $|R| = 2^r$ for some $r \in \mathbf{N}$, $A \subset R$, and $|R| \leq 2|A|$. With M, C, ε as above, we have

$$\begin{aligned} & \frac{1}{|A|} \left(\sum_{k \in A} w_k^{(1+\varepsilon)p} \right)^{1/p} \left(\sum_{k \in A} w_k^{-(1+\varepsilon)q} \right)^{1/q} \\ & \leq \frac{2}{|R|} \left(\sum_{k \in R} w_k^{(1+\varepsilon)p} \right)^{1/p} \left(\sum_{k \in R} w_k^{-(1+\varepsilon)q} \right)^{1/q} \leq 2M\, C^{1+\varepsilon}. \end{aligned}$$

This proves that $w^{1+\varepsilon} \in A_p$ for all $\varepsilon \in [0, \delta]$.

Since $w^r \in A_p$ if and only if the operator given by the matrix (2) is bounded on $\ell^p(w^r)$, we finally see from the Stein-Weiss interpolation theorem that $w^r \in A_p$ for all $r \in [0, 1 + \delta]$. This completes the proof of Theorem 1.1. ■

3 Proof of the Stechkin inequality

The purpose of this section is to prove Theorem 1.2.

Let $S = L(a)$ where $a(e^{i\theta}) = i(\pi - \theta)$ for $\theta \in [0, 2\pi)$. Thus,

$$S = \left(\frac{1 - \delta_{jk}}{j - k} \right)_{j,k=-\infty}^{\infty}$$

where δ_{jk} is the Kronecker delta and $(1-\delta_{jk})/(j-k)$ is defined to be zero for $j = k$. Hunt, Muckenhoupt, and Wheeden [5] proved that S is bounded on $\ell^p(\mathbf{Z}, w)$ if and only if $w \in A_p(\mathbf{Z})$, which means that (1) is satisfied for all $m, n \in \mathbf{Z}$ such that $m \leq n$.

Given $x \in (0, 2\pi)$, we define $\chi_x : \mathbf{T} \to \{0, 1\}$ by

$$\chi_x(e^{i\theta}) = \begin{cases} 0 & \text{if } \theta \in [0, x], \\ 1 & \text{if } \theta \in (x, 2\pi). \end{cases}$$

The Fourier series of χ_x is

$$\frac{2\pi - x}{2\pi} + \sum_{n \neq 0} \frac{i}{2\pi n} \left(1 - e^{-inx}\right) e^{in\theta}, \tag{8}$$

which gives

$$L(\chi_x) = \frac{2\pi - x}{2\pi} I + \frac{i}{2\pi}(S - R_{-x} S R_x).$$

with the diagonal operator

$$R_x = \operatorname{diag}\left(e^{ikx}\right)_{k=-\infty}^{\infty}.$$

Thus, if $w \in A_p(\mathbf{Z})$ then $L(\chi_x)$ is bounded on $\ell^p(\mathbf{Z}, w)$ and

$$\|L(\chi_x)\| \le 1 + \frac{1}{2\pi} 2\|S\| =: \sigma. \tag{9}$$

Let a be a real-valued and bounded function on $\mathbf{T}$ and suppose the function

$$\tilde{a} : [0, 2\pi] \to \mathbf{R}, \quad \theta \mapsto a(e^{i\theta})$$

is monotonously increasing. We then can approximate a in the L^∞ norm by monotonously increasing piecewise constant functions with only finitely many jumps as closely as desired. In formulas: for each $m \in \mathbf{N}$ there exist points

$$0 < x_1^{(m)} < x_2^{(m)} < \ldots x_m^{(m)} < 2\pi$$

and real numbers

$$\tilde{a}(0+0) \le \alpha_1^{(m)} < \alpha_2^{(m)} < \ldots < \alpha_{m+1}^{(m)} \le \tilde{a}(2\pi - 0)$$

such that $\|a - a_m\|_\infty \to 0$ for

$$a_m = \alpha_1^{(m)} + \sum_{j=1}^{m} \left(\alpha_{j+1}^{(m)} - \alpha_j^{(m)} \right) \chi_{x_j^{(m)}}.$$

From (9) we infer that

$$\begin{aligned} \|L(a_m)\| &\le |\alpha_1^{(m)}| + \sum_{j=1}^{m} \left(\alpha_{j+1}^{(m)} - \alpha_j^{(m)} \right) \sigma \\ &\le \|a\|_\infty + \mathrm{V}(a)\sigma \le \sigma\Big(\|a\|_\infty + \mathrm{V}(a) \Big). \end{aligned} \tag{10}$$

Lemma 3.1 *If $w \in A_p(\mathbf{Z})$ and $x \in \ell^p(\mathbf{Z}, w)$, then $\{L(a_m)x\}_{m=0}^{\infty}$ is a Cauchy sequence in $\ell^p(\mathbf{Z}, w)$.*

Proof. By virtue of (10), it suffices to consider the case where x is the lth element e_l of the standard basis of $\ell^p(\mathbf{Z}, w)$. Since (8) is the Fourier series of χ_x, we see that $|(L(a_m)e_l)_k|$ equals

$$\left| \alpha_1^{(m)} \delta_{kl} + \sum_{j=1}^{m} \left(\alpha_{j+1}^{(m)} - \alpha_j^{(m)} \right) \left(\delta_{kl} \frac{2\pi - x_j^{(m)}}{2\pi} + \frac{(1 - \delta_{kl}) i}{2\pi\,(k-l)} \left(1 - e^{i(l-k)x_j^{(m)}} \right) \right) \right|,$$

and in case $k \ne l$ this does not exceed

$$\begin{aligned} &\sum_{j=1}^{m} \left(\alpha_{j+1}^{(m)} - \alpha_j^{(m)} \right) |(Se_l)_k| \frac{2}{2\pi} \\ &\le \frac{1}{\pi} \Big(\tilde{a}(2\pi - 0) - \tilde{a}(0+0) \Big) |(Se_l)_k| = \frac{\mathrm{V}(a)}{\pi} |(Se_l)_k|. \end{aligned}$$

Let $\varepsilon > 0$. We know that if $k \neq l$, then

$$|(L(a_m)e_l - L(a_n)e_l)_k| \leq \frac{2\mathrm{V}(a)}{\pi} |(Se_l)_k|,$$

and as $\sum_{k=-\infty}^{\infty} |(Se_l)_k|^p w_k^p < \infty$, there is an $N > |l|$ such that

$$\sum_{|k| \geq N+1} |(Se_l)_k|^p w_k^p < \left(\frac{\pi}{2\mathrm{V}(a)}\right)^p \frac{\varepsilon^p}{2}$$

(the requirement $N > |l|$ guarantees that $k \neq l$). It follows that

$$\sum_{|k| \geq N+1} |(L(a_m)e_l - L(a_n)e_l)_k|^p w_k^p < \frac{\varepsilon^p}{2}. \tag{11}$$

Since all norms on $\mathbf{C}^{2N+1}$ are equivalent, we have

$$\left(\sum_{|k| \leq N} |x_k|^p w_k^p\right)^{1/p} \leq C_N \left(\sum_{|k| \leq N} |x_k|^2\right)^{1/2} \quad \text{for all } x \in \mathbf{C}^{2N+1}$$

with some constant C_N independent of x. Because $\{a_m\}_{m=0}^{\infty}$ is a Cauchy sequence in L^∞, we obtain that if $m \geq n$ are sufficiently large, then

$$\begin{aligned} \sum_{|k| \leq N} |(L(a_m)e_l - L(a_n)e_l)_k|^p w_k^p &\leq C_N^p \left(\sum_{|k| \leq N} |(L(a_m)e_l - L(a_n)e_l)_k|^2\right)^{p/2} \\ &\leq C_N^p \|a_m - a_n\|_\infty^p < \varepsilon^p/2. \end{aligned} \tag{12}$$

Adding (11) and (12) we arrive at the conclusion that $\{L(a_m)e_l\}_{m=0}^{\infty}$ is a Cauchy sequence. ■

The lemma in conjunction with the Banach-Steinhaus theorem and (10) implies that the operators $L(a_m)$ converge strongly to an operator $A \in \mathcal{L}(\ell^p(\mathbf{Z}, w))$ for which $\|A\| \leq \sigma(\|a\|_\infty + \mathrm{V}(a))$. As $(L(a_m)e_l)_k \to a_{k-l}$, we see that $A = L(a)$. Hence, $L(a)$ is bounded on $\ell^p(\mathbf{Z}, w)$ and

$$\|L(a)\| \leq \sigma\Big(\|a\|_\infty + \mathrm{V}(a)\Big). \tag{13}$$

Now let $a : \mathbf{T} \to \mathbf{C}$ be an arbitrary function of bounded variation. Put $u = \mathrm{Re}\, a$. Clearly, $\|u\|_\infty \leq \|a\|_\infty$ and $\mathrm{V}(u) \leq \mathrm{V}(a)$. For $\theta \in (0, 2\pi]$, denote by $\mathrm{V}_{[0,\theta]}(u)$ the total variation of $\tilde{u}$ on $[0, \theta]$, and let $\mathrm{V}_{[0,0]}(u) = 0$. Then set

$$u_1(e^{i\theta}) = \mathrm{V}_{[0,\theta]}(u), \quad u_2 = u_1 - u.$$

The function u_1 is monotonously increasing from 0 to $\mathrm{V}(u)$. Hence

$$\|u_1\|_\infty \le \mathrm{V}(u_1) = \mathrm{V}(u) \le \mathrm{V}(a),$$

and (13) gives

$$\|L(u_1)\| \le \sigma\Big(\|u_1\|_\infty + \mathrm{V}(u_1)\Big) \le 2\sigma\mathrm{V}(a).$$

For u_2 we have

$$\begin{aligned}\|u_2\|_\infty &\le \|u_1\|_\infty + \|u\|_\infty \le \mathrm{V}(a) + \|a\|_\infty,\\ \mathrm{V}(u_2) &\le \mathrm{V}(u_1) + \mathrm{V}(u) \le \mathrm{V}(a) + \mathrm{V}(a) = 2\mathrm{V}(a),\end{aligned}$$

and again invoking (13) we get

$$\|L(u_2)\| \le \sigma\left(3\mathrm{V}(a) + \|a\|_\infty\right).$$

In summary,

$$\|L(u)\| \le \|L(u_1)\| + \|L(u_2)\| \le \sigma\left(5\mathrm{V}(a) + \|a\|_\infty\right).$$

Analogously one can show that if $v = \operatorname{Im} a$, then

$$\|L(v)\| \le \sigma\left(5\mathrm{V}(a) + \|a\|_\infty\right).$$

In the end we obtain

$$\|L(a)\| \le \|L(u)\| + \|L(v)\| \le 10\sigma\Big(\|a\|_\infty + \mathrm{V}(a)\Big),$$

which completes the proof of Theorem 1.2. ■

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