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Numerical Computation of Deflating Subspaces of Embedded Hamiltonian Pencils*

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Abstract

We discuss the numerical solution of structured generalized eigenvalue problems that arise from linear-quadratic optimal control problems, H_∞ optimization, multibody systems and many other areas of applied mathematics, physics, and chemistry. The classical approach for these problems requires computing invariant and deflating subspaces of matrices and matrix pencils with Hamiltonian and/or skew-Hamiltonian structure. We extend the recently developed methods for Hamiltonian matrices and matrix pencils to the general case of embedded matrix pencils. The rounding error and perturbation analysis of the resulting algorithms is favorable.

Keywords. eigenvalue problem, deflating subspace, algebraic Riccati equation, Hamiltonian matrix, skew-Hamiltonian matrix, skew-Hamiltonian/Hamiltonian matrix pencil, embedded matrix pencils.
AMS subject classification. 49N10, 65F15, 93B40, 93B36, 93C60.

1 Introduction and Preliminaries

In this paper we study eigenvalue and invariant subspace computations involving matrices and matrix pencils with the following algebraic structures.

Definition 1 *Let* $\mathcal{J} := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, *where* I_n *is the* $n \times n$ *identity matrix.*

a) A matrix $\mathcal{H} \in \mathbb{C}^{2n,2n}$ *is* Hamiltonian *if* $(\mathcal{H}\mathcal{J})^H = \mathcal{H}\mathcal{J}$. *The Lie Algebra of Hamiltonian matrices in* $\mathbb{C}^{2n,2n}$ *is denoted by* $\mathbb{H}_{2n}$.

b) A matrix $\mathcal{H} \in \mathbb{C}^{2n,2n}$ *is* skew-Hamiltonian *if* $(\mathcal{H}\mathcal{J})^H = -\mathcal{H}\mathcal{J}$. *The Jordan algebra of skew-Hamiltonian matrices in* $\mathbb{C}^{2n,2n}$ *is denoted by* $\mathbb{SH}_{2n}$.

c) A matrix pencil $\alpha\mathcal{S} - \beta\mathcal{H} \in \mathbb{C}^{2n,2n}$ *is* skew-Hamiltonian/Hamiltonian *if* $\mathcal{S} \in \mathbb{SH}_{2n}$ *and* $\mathcal{H} \in \mathbb{H}_{2n}$.

d) A matrix $\mathcal{S} \in \mathbb{C}^{2n,2n}$ *is* symplectic *if* $\mathcal{S}\mathcal{J}\mathcal{S}^H = \mathcal{J}$. *The Lie group of symplectic matrices in* $\mathbb{C}^{2n,2n}$ *is denoted by* $\mathbb{S}_{2n}$.

e) A matrix $\mathcal{U}_d \in \mathbb{C}^{2n,2n}$ *is* unitary symplectic *if* $\mathcal{U}_d\mathcal{J}\mathcal{U}_d^H = \mathcal{J}$ *and* $\mathcal{U}_d\mathcal{U}_d^H = I_{2n}$. *The compact Lie group of unitary symplectic matrices in* $\mathbb{C}^{2n,2n}$ *is denoted by* $\mathbb{US}_{2n}$.

f) A subspace $\mathcal{L}$ *of* $\mathbb{C}^{2n}$ *is called* Lagrangian *if it has dimension* n *and* $x^H\mathcal{J}y = 0$ *for all* $x, y \in \mathcal{L}$.

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There is little difference between the structure of complex skew-Hamiltonian matrices and complex Hamiltonian matrices. If $\mathcal{S} \in \mathbb{SH}_{2n}$ and $\mathcal{H} \in \mathbb{H}_{2n}$, then $i\mathcal{S} \in \mathbb{H}_{2n}$ and $i\mathcal{H} \in \mathbb{SH}_{2n}$. Similarly, there is little difference between the structure of complex skew-Hamiltonian/Hamiltonian matrix pencils, complex skew-Hamiltonian/skew-Hamiltonian and complex Hamiltonian/Hamiltonian matrix pencils. For $\mathcal{S} \in \mathbb{SH}_{2n}$ and $\mathcal{H} \in \mathbb{H}_{2n}$ the matrix pencil $\alpha\mathcal{S} - \beta\mathcal{H}$ is skew-Hamiltonian/Hamiltonian, the matrix pencil $\alpha\mathcal{S} - \tilde{\beta}(i\mathcal{H})$ is skew-Hamiltonian/skew-Hamiltonian, and the matrix pencil $\tilde{\alpha}(i\mathcal{S}) - \beta\mathcal{H}$ is Hamiltonian/Hamiltonian. (Here, $\tilde{\alpha} = -i\alpha$ and $\tilde{\beta} = -i\beta$.) However, *real* skew-Hamiltonian matrices are not real scalar multiples of Hamiltonian matrices. There is a greater difference in the structure of real skew-Hamiltonian and real Hamiltonian matrices than in the complex case.

The structures in Definition 1 arise in linear-quadratic optimal control [35, 40, 43], H_∞ optimization [20, 48] and several other areas of applied mathematics, computational physics and chemistry, e.g., gyroscopic systems [28], numerical simulation of elastic deformation [42], and linear response theory [37]. Here, we focus on applications in linear-quadratic optimal control and H_∞ optimization.

First, we consider the continuous time, infinite horizon, linear-quadratic optimal control problem:

Choose a control function $u(t)$ to minimize the cost functional

$$\mathcal{S}_c = \int_{t_0}^{\infty} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^H \begin{bmatrix} Q & S \\ S^H & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \tag{1}$$

subject to the linear differential-algebraic system (descriptor control system)

$$E\dot{x} = Ax + Bu, \qquad x(t_0) = x^0. \tag{2}$$

Here, $u(t) \in \mathbb{C}^m$, $x(t) \in \mathbb{C}^n$, A, $E \in \mathbb{C}^{n,n}$, $B \in \mathbb{C}^{n,m}$, $Q = Q^H \in \mathbb{C}^{n,n}$, $R = R^H \in \mathbb{C}^{m,m}$ and $S \in \mathbb{C}^{n,m}$. If the $(m+n) \times (m+n)$ weighting matrix

$$\mathcal{R} = \begin{bmatrix} Q & S \\ S^H & R \end{bmatrix}$$

is Hermitian and positive semidefinite, then the problem is well-posed. Note that $\mathcal{R}$, Q and/or R may be singular. Typically, in addition to minimizing (1), the control $u(t)$ must make $x(t)$ asymptotically stable. (Of course, if $\mathcal{R}$ is positive definite, then asymptotic stability of $x(t)$ is automatically achieved by any control for which the cost functional (1) is finite.)

Application of the maximum principle [35, 41] yields as a necessary optimality condition that the control u satisfies the two-point boundary value problem of Euler-Lagrange equations

$$\mathcal{E}_c \begin{bmatrix} \dot{x} \\ \dot{\mu} \\ \dot{u} \end{bmatrix} = \mathcal{A}_c \begin{bmatrix} x \\ \mu \\ u \end{bmatrix}, \qquad x(t_0) = x^0, \qquad \lim_{t\to\infty} E^H \mu(t) = 0, \tag{3}$$

with the matrix pencil

$$\alpha\mathcal{E}_c - \beta\mathcal{A}_c := \alpha \begin{bmatrix} E & 0 & 0 \\ 0 & -E^H & 0 \\ 0 & 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} A & 0 & B \\ Q & A^H & S \\ S^H & B^H & R \end{bmatrix}. \tag{4}$$

The matrix pencil (4) does not have matrices with one of the structures of Definition 1. Nevertheless, many of the properties of Hamiltonian matrices carry over [35]. As reviewed below, if E and/or R are nonsingular, then (3) and (4) reduce to an ordinary differential equation with Hamiltonian matrix coefficients or a differential-algebraic equation with a skew-Hamiltonian/Hamiltonian matrix pencil coefficients. In Section 2, we circumvent the nonsingularity assumptions by embedding the matrix pencil (4) into a skew-Hamiltonian/Hamiltonian matrix pencil of larger dimension.

If both E and R are nonsingular, then with $\eta := -E^H\mu$, (3) reduces to the two-point boundary value problem

$$\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = \mathcal{H}\begin{bmatrix} x \\ \eta \end{bmatrix}, \qquad x(t_0) = x^0, \qquad \lim_{t\to\infty}\eta(t) = 0 \tag{5}$$

with the Hamiltonian matrix

$$\mathcal{H} = \begin{bmatrix} F & G \\ H & -F^H \end{bmatrix} := \begin{bmatrix} E^{-1}(A - BR^{-1}S^H) & E^{-1}BR^{-1}B^HE^{-H} \\ Q - SR^{-1}S^H & -(E^{-1}(A - BR^{-1}S^H))^H \end{bmatrix}. \tag{6}$$

This is the classical formulation found in many textbooks on linear-quadratic optimal control like [25, 40, 43]. The solution of this boundary value problem can be obtained in many different ways [35]. For example, let Y be a symmetric solution (if it exists) of the associated (continuous-time) *algebraic Riccati equation*

$$0 = H + YF + F^HY - YGY.$$

Multiplying (6) from the left by the matrix $\left[\begin{smallmatrix} I & 0 \\ Y & I \end{smallmatrix}\right]$ and changing variables to $\left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ Y & I \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ \eta \end{smallmatrix}\right]$ one obtains the decoupled system

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} F - GY & G \\ 0 & -F^H + YG \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix}, \qquad x(t_0) = x_0, \qquad \lim_{t\to\infty}\xi(t) = 0.$$

In the desired solution, ξ is identically zero. In that case, x is the solution to $\dot{x} = (F - GY)x$ (with initial condition $x(0) = x^0$), $\eta = -Yx$, $\mu = -E^{-H}\eta$, and the control that minimizes (1) is $u = -R^{-1}(S^H + B^HE^{-H}Y)x$.

If E is singular and R is nonsingular, then (3) does not simplify quite so much, because it is a differential-algebraic equation with nontrivial linear constraints. Substituting $u(t) = -R^{-1}(S^Hx(t) + B^H\mu(t))$, system (3) does simplify to

$$\mathcal{S}\begin{bmatrix} \dot{x} \\ \dot{\mu} \end{bmatrix} = \mathcal{H}\begin{bmatrix} x \\ \mu \end{bmatrix}, \qquad x(t_0) = x^0, \qquad \lim_{t\to\infty}E^H\mu(t) = 0, \tag{7}$$

with the reduced skew-Hamiltonian/Hamiltonian matrix pencil

$$\alpha\mathcal{S} - \beta\mathcal{H} := \alpha\begin{bmatrix} E & 0 \\ 0 & E^H \end{bmatrix} - \beta\begin{bmatrix} A - BR^{-1}S^H & -BR^{-1}B^H \\ SR^{-1}S^H - Q & -(A - BR^{-1}S)^H \end{bmatrix}. \tag{8}$$

In this case, the corresponding generalized algebraic Riccati equation is

$$\begin{aligned} 0 \;=\; & Q - SR^{-1}S^H + E^HY(A - BR^{-1}S) \\ & + (A - BR^{-1}S)^HYE - E^HY(BR^{-1}B^H)YE, \end{aligned} \tag{9}$$

but, in general, the relationship with solutions of the optimal control problem is lost or hidden. See [3, 26, 27, 35] for details. But even for E nonsingular, the formulation in (7)–(9) is preferable from a numerical point of view if E is ill-conditioned with respect to inversion.

If R is singular and E is nonsingular then the situation becomes still more complicated. Although the boundary value problem remains well defined, the Riccati equation does not. This case has been recently studied [22, 23, 24].

At this writing, the case in which both E and R are singular has not been analyzed in full generality.

The Euler-Lagrange and Riccati equations, their solvability and their numerical solution have been the subject of numerous publications in recent years, see, e.g., [9, 29, 35, 40, 43] and the references therein. In most numerical methods, the Riccati solutions are obtained through the computation of deflating or invariant subspaces of associated matrix pencils, see [35]. A key observation in this context, is that the Riccati solutions need not be formed explicitly [45]. It suffices to work with bases of the deflating subspaces of the matrix pencil.

For example, suppose $\alpha\mathcal{E}_c - \beta\mathcal{A}_c$ in (4) has an n-dimensional deflating subspace associated with eigenvalues in the left half plane. (This can only be the case if E is invertible.) Let this subspace be spanned by the columns of a matrix $\mathcal{U}$, partitioned conformally with (4) as

$$\mathcal{U} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}.$$

If U_1 is invertible, then the optimal control is a linear feedback of the form $u(t) = U_3U_1^{-1}x(t)$ and the solution of the associated Riccati equation is $Y = U_2U_1^{-1}E^{-1}$, see [35]. Observe that the optimal control may be obtained without explicitly inverting E (or solving equations with coefficient E). If E is "nearly singular", i.e., ill-conditioned, then explicitly forming the Riccati solution may introduce unnecessarily large rounding errors. If E is singular, then an n-dimensional deflating subspace associated with left half plane eigenvalues may not exist. In some circumstances, the deflating subspace can be augmented in dimension by enlarging a basis with appropriately chosen eigenvectors and principal vectors associated with the infinite eigenvalue, see [35]. In this case, the optimal control law may still be obtained as $u(t) = U_3U_1^{-1}x(t)$, but the Riccati solution need not exist. Examples appear in [26, 36].

Matrices and Riccati equations of a similar structure occur in H_∞ optimization, see [20, 48]. Consider the linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + B_1u + B_2w, \\ z &= C_1x + D_{11}u + D_{12}w, \\ y &= C_2x + D_{21}u + D_{22}w, \end{aligned} \tag{10}$$

where $A \in \mathbb{C}^{n,n}$, $B_k \in \mathbb{C}^{n,m_k}$, $C_k \in \mathbb{C}^{p_k,n}$ for $k = 1, 2$, and $D_{ij} \in \mathbb{C}^{p_i,m_j}$ for $i, j = 1, 2$. Here $u \in \mathbb{C}^{m_1}$ denotes the control inputs, $w \in \mathbb{C}^{m_2}$ the exogenous inputs, $y \in \mathbb{C}^{p_2}$ the measured outputs, and $z \in \mathbb{C}^{p_1}$ the output error signals to be minimized. The H_∞ optimization problem is to determine a stabilizing controller that minimizes the closed-loop transfer function T_{zw} from w to z in the H_∞ norm. Under mild hypotheses [48, p. 419], for any $\gamma > 0$, there exists an admissible controller such that $\|T_{zw}\|_\infty < \gamma$ if and only if the following three conditions hold.

(i) There is a Hermitian nonnegative semidefinite solution $X_\infty \in \mathbb{C}^{n,n}$ to the algebraic Riccati equation

$$C_1C_1^H + A^H X_\infty + X_\infty A + X_\infty(\gamma^{-2}B_1B_1^H - B_2B_2^H)X_\infty = 0$$

and $A + (\gamma^{-2}B_1B_1^H - B_2B_2^H)X_\infty$ has no eigenvalue with nonnegative real part.

(ii) There is a Hermitian nonnegative semidefinite solution $Y_\infty \in \mathbb{C}^{n,n}$ to the algebraic Riccati equation

$$B_1B_1^H + Y_\infty A^H + AY_\infty + Y_\infty(\gamma^{-2}C_1C_1^H - C_2C_2^H)Y_\infty = 0$$

and $A + Y_\infty(\gamma^{-2}C_1C_1^H - C_2C_2^H)$ has no eigenvalue with nonnegative real part.

(iii) $\gamma^2 > \rho(X_\infty Y_\infty)$, where $\rho(\cdot)$ represents the spectral radius.

Moreover, a controller that satisfies (i)–(iii) is determined by the linear time invariant system having the state-space representation

$$\begin{array}{rcl} \dot{q} &=& \hat{A}q + \hat{B}y, \\ u &=& \hat{C}q + \hat{D}y, \end{array}$$

where

$$\begin{array}{rcl} \hat{A} &:=& A + \gamma^{-2}B_1B_1^H X_\infty + B_2\hat{C} - \hat{B}C_2, \\ \hat{B} &:=& (I - \gamma^{-2}Y_\infty X_\infty)^{-1}Y_\infty C_2^H, \\ \hat{C} &:=& -B_2^H X_\infty, \\ \hat{D} &:=& 0. \end{array}$$

The (suboptimal) H_∞ controller determined by the above formulae is called the *central controller*.

The H_∞ optimization problem is equivalent to determining the supremum of $\gamma > 0$ for which at least one of the three conditions (i)–(iii) fails.

A difficulty in this optimization is that in the limit as γ approaches the minimal H_∞ norm the Riccati solutions X_∞ and Y_∞ may have infinite norm. Hence, typically with this approach only suboptimal controllers can be computed numerically. The essential role played by X_∞ and Y_∞ is to represent particular right and left invariant subspaces of the Hamiltonian matrices

$$\mathcal{H}_\infty := \begin{bmatrix} A & \gamma^{-2}B_1B_1^H - B_2B_2^H \\ -C_1C_1^H & -A^H \end{bmatrix}$$

and

$$\mathcal{K}_\infty := \begin{bmatrix} A & -B_1B_1^H \\ \gamma^{-2}C_1C_1^H - C_2^H C_2 & -A^H \end{bmatrix}, \tag{11}$$

respectively. The columns of $[I,\, X_\infty]^H$ span a right invariant subspace of $\mathcal{H}_\infty$ and the rows of $[I,\, Y_\infty]$ span a left invariant subspace of $\mathcal{K}_\infty$. In [48, p. 445], conditions (i)–(iii) are generalized in terms of an arbitrary basis of the same invariant subspaces as follows.

(i') There are matrices $Q_1,\, Q_2,\, T_x \in \mathbb{C}^{n,n}$ such that $Q_1^H Q_2 = Q_2^H Q_1$,

$$\mathcal{H}_\infty \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} T_x, \tag{12}$$

and T_x has no eigenvalue with positive real part.

(ii') There are matrices $U_1, U_2, T_y \in \mathbb{C}^{n,n}$ such that $U_1^H U_2 = U_2^H U_1$,

$$\begin{bmatrix} U_1^H & U_2^H \end{bmatrix} \mathcal{K}_\infty = T_y \begin{bmatrix} U_1^H & U_2^H \end{bmatrix},$$

and T_y has no eigenvalue with positive real part.

(iii') The $2n \times 2n$ matrix $\begin{bmatrix} Q_2^H Q_1 & \gamma^{-1} Q_2^H U_2 \\ \gamma^{-1} U_2^H Q_2 & U_2^H U_1 \end{bmatrix}$ is positive semidefinite.

In this formulation the computation of the optimal γ can be obtained numerically by computing the largest γ at which one of the conditions (i'), (ii') or (iii') fails. This approach is more appropriate in finite precision arithmetic. See also [48, Remark 6.14]. For the optimal γ, an admissible controller is determined by the descriptor system having generalized state-space realization

$$\begin{aligned} \hat{E}\dot{q} &= \hat{A}q + \hat{B}y, \\ u &= \hat{C}q + \hat{D}y, \end{aligned}$$

where

$$\begin{aligned} \hat{E} &= U_1^H Q_1 - \gamma^{-1} U_2^H Q_2, \\ \hat{B} &= U_2^H C_2^H, \\ \hat{C} &= -B_2^H Q_2, \\ \hat{D} &= 0, \\ \hat{A} &= \hat{E} T_x - \hat{B} C_2 Q_1 = T_y \hat{E} + U_1^H B_2 \hat{C}. \end{aligned}$$

This form avoids explicit solutions of algebraic Riccati equations and inversion of the potentially ill-conditioned matrix $I - \gamma^{-2} Y_\infty X_\infty$. We close the introduction with some remarks on the numerical solution of the eigenvalue problems for matrices and matrix pencils involving the structures in Definition 1. Although the numerical computation of n-dimensional Lagrangian invariant subspaces of Hamiltonian matrices and the solution of algebraic Riccati equations have been extensively studied (see [12, 30, 35, 43] and the references therein), completely satisfactory methods for general Hamiltonian matrices and extended matrix pencils are still an open problem. Such methods would be numerically backward stable, have complexity $\mathcal{O}(n^3)$ and preserve the given structure. There are several reasons for this difficulty all of which are well demonstrated in the context of algorithms for Hamiltonian matrices. (Similar difficulties arise in the extended matrix pencil case.) First of all, an algorithm based upon structure preserving similarity transformations (including QR-like algorithms) would require a triangular-like Hamiltonian Schur form that displays the desired deflating subspaces. (We summarize the definitions and basic results on Schur-like forms in Section 3.) A Hamiltonian Schur form under unitary symplectic similarity transformations is presented in [38]. Unfortunately, not every Hamiltonian matrix has this kind of Hamiltonian Schur form. For example, the Hamiltonian matrix $\mathcal{J}$ in Definition 1 is invariant under arbitrary symplectic similarity transformations but is not in the Hamiltonian Schur form described in [38]. A characterization of Hamiltonian matrices that do admit a Hamiltonian Schur form under unitary symplectic similarity transformations was conjectured in [31] and proved in [32]. (We summarize that result in Section 3.) Schur-like forms are characterized for skew-Hamiltonian/Hamiltonian matrix pencils in [33, 34] and for the other structures in [32]. A second difficulty comes from the fact that even when a Hamiltonian Schur form exists, there is no known structure preserving, numerical method to compute it. It has been argued in [2] that, except in special cases [13, 14], QR-like algorithms are impractically expensive because of the lack of a Hamiltonian Hessenberg-like form. For this reason other methods like the multishift-method of [1], the structured implicit product methods of [4, 5, 6, 46] do not follow the QR-algorithm paradigm. (The implicit product methods [5, 6] do come quite close to optimality. We extend the method of [5] to skew-Hamiltonian/Hamiltonian matrix pencils in Section 5.) A third difficulty arises when the Hamiltonian matrix or the skew-Hamiltonian/Hamiltonian matrix pencil has eigenvalues on the imaginary axis. In that case, the desired Lagrangian subspace is, in general, not unique [36]. Furthermore, if finite precision arithmetic or other errors perturb the matrix off the Lie algebra of Hamiltonian matrices, then it is typically the case that the perturbed matrix has no Lagrangian subspace or does not have the expected eigenvalue pairings, see, e.g., [6, 46].

To simplify notation, the term *eigenvalue* is used both for eigenvalues of matrices and for pairs $(\alpha, \beta) \neq (0,0)$ for which $\det(\alpha E - \beta A) = 0$. These pairs are not unique. If $\beta \neq 0$ then we identify (α, β) with $(\alpha/\beta, 1)$ and $\lambda = \alpha/\beta$. Pairs $(\alpha, 0)$ with $\alpha \neq 0$ are called *infinite eigenvalues*.

By $\Lambda(E, A)$, we denote the set of eigenvalues of $\alpha E - \beta A$ including finite and infinite eigenvalues both counted according to multiplicity.

We will denote by $\Lambda_-(E, A)$, $\Lambda_0(E, A)$ and $\Lambda_+(E, A)$ the set of finite eigenvalues of $\alpha A - \beta E$ with negative, zero and positive real parts, respectively. The set of infinite eigenvalues is denoted by $\Lambda_\infty(E, A)$. Multiple eigenvalues are repeated in $\Lambda_-(E, A)$, $\Lambda_0(E, A)$, $\Lambda_+(E, A)$ and $\Lambda_\infty(E, A)$ according to algebraic multiplicity. The set of all eigenvalues counted according to multiplicity is $\Lambda(E, A) := \Lambda_-(E, A) \cup \Lambda_0(E, A) \cup \Lambda_+(E, A) \cup \Lambda_\infty(E, A)$. Similarly, we denote by $\mathrm{Def}_-(E, A)$, $\mathrm{Def}_0(E, A)$, $\mathrm{Def}_+(E, A)$ and $\mathrm{Def}_\infty(E, A)$ the right deflating subspaces corresponding to $\Lambda_-(E, A)$, $\Lambda_0(E, A)$, $\Lambda_+(E, A)$ and $\Lambda_\infty(E, A)$, respectively.

Throughout this paper, the imaginary number $\sqrt{-1}$ is denoted by i. The inertia of a Hermitian matrix A consists of the triple $\mathrm{In}(A) = (\pi, \omega, \nu)$, where $\pi = \pi(A)$, $\omega = \omega(A)$ and $\nu = \nu(A)$ represent the number of positive, zero and negative eigenvalues, respectively.

2 Embedding of Extended Matrix Pencils

It is important to exploit and preserve algebraic structures (like symmetries in the matrix blocks or symmetries in the spectrum) as much as possible. Such algebraic structures typically arise from physical properties of the problem. If rounding errors or other perturbations destroy the algebraic structures, then the results may be physically meaningless. Not coincidentally, numerical methods that preserve algebraic structures are typically more efficient as well as more accurate. Preserving and exploiting structure recommends using the Euler-Lagrange equations in the form of (5) and (6) or (7) and (8).

On the other hand, reducing to the form of (5) or (7) using finite precision arithmetic may be ill-advised. For the purpose of finite precision computation, it is desirable to obtain the solution directly from the original data without explicitly forming matrix products and inverses. Otherwise, there is a real danger of numerical instability by transforming the problem to an equivalent but more ill-conditioned one, i.e., transforming the problem to one that is more sensitive to perturbations. The matrices E and/or R may be singular. Even if they are nominally nonsingular, they may be "nearly singular", i.e., ill-conditioned. Even if E and R are not ill-conditioned, forming "matrix-times-its-transpose" products like $BR^{-1}B^H$ suffers from the same kind of well-known numerical instability as forming the normal equations to solve least squares problems. (See, for example, [19, Example 5.3.2].) Hence, it may happen that the transformed coefficient matrices in (6) or (8) are so corrupted by rounding errors that the control $u(t)$ computed from them is of limited value. This recommends using the Euler-Lagrange equations in the form of (3) and (4) in case small rounding errors can not be guaranteed *a priori* when forming $BR^{-1}B^T$.

In this section, we show how to reconcile the superficially contradictory requirement to avoid explicit products and inverses (by working directly with (3) and (4)) with the requirement to preserve and exploit special structure (by working directly with (5) and (6) or (7) and (8)). This is accomplished by embedding (3) and (4) into an extended differential-algebraic boundary value problem involving a skew-Hamiltonian/Hamiltonian matrix pencil of larger dimension. Solutions of the extended system display solutions of (3) and (4). For this approach, no nonsingularity assumption is required and only unitary matrix products are formed explicitly.

The extension requires an even number of controls. If the number of controls m is odd, then the control system (2) must be extended with one or more artificial control variables. Let $v \in \mathbb{C}^k$ be a vector of artificial controls with k chosen so that $m+k$ is even. If m is even, then k may be set to zero and v becomes void. We will see that even if $k > 0$, changing the linear-quadratic optimal control problem appropriately, v will have no influence on the optimal solution u. Introduce the control matrix $\tilde{B} \in \mathbb{C}^{n,k}$ corresponding to v. (If $k > 0$, then we may take $\tilde{B} = 0$.) With the artificial control vector, the descriptor system (2) becomes the extended system

$$E\dot{x} = Ax + \left[\begin{array}{cc} B & \tilde{B} \end{array}\right] \left[\begin{array}{c} u \\ v \end{array}\right], \qquad x(t_0) = x^0. \tag{13}$$

Introduce an artificial Hermitian positive definite weighting matrix $\tilde{R} \in \mathbb{C}^{k,k}$ in (1) to obtain the extended cost functional

$$\mathcal{S}_e = \int_{t_0}^{\infty} \left[\begin{array}{c} x(t) \\ u(t) \\ v(t) \end{array}\right]^H \left[\begin{array}{ccc} Q & S & 0 \\ S^H & R & 0 \\ 0 & 0 & \tilde{R} \end{array}\right] \left[\begin{array}{c} x(t) \\ u(t) \\ v(t) \end{array}\right] dt, \tag{14}$$

where minimization is now performed with respect to $(u, v) \in \mathbb{R}^m \times \mathbb{R}^k$. (If $k > 0$, then $\tilde{R}$ may be taken to be the identity matrix.)

Now we repartition $[u^H,\, v^H]^H$ into two parts u_1 and u_1 of equal dimension $\ell := \frac{m+k}{2}$. Then we may repartition $[B,\, \tilde{B}]$ and the cost functional weighting matrix conformally as

$$\begin{array}{cc} & 1 \\ \begin{array}{c} m \\ k \end{array} & \left[\begin{array}{c} u \\ v \end{array}\right] \end{array} := \begin{array}{cc} & 1 \\ \begin{array}{c} \ell \\ \ell \end{array} & \left[\begin{array}{c} u_1 \\ u_2 \end{array}\right] \end{array}, \tag{15}$$

$$\begin{array}{cc} & \begin{array}{cc} m & k \end{array} \\ n & \left[\begin{array}{cc} B & \tilde{B} \end{array}\right] \end{array} := \begin{array}{cc} & \begin{array}{cc} \ell & \ell \end{array} \\ n & \left[\begin{array}{cc} B_1 & B_2 \end{array}\right] \end{array} \tag{16}$$

and

$$\begin{array}{c} \\ n \\ m \\ k \end{array}\begin{array}{c} \begin{array}{ccc} n & m & k \end{array} \\ \left[\begin{array}{ccc} Q & S & 0 \\ S^H & R & 0 \\ 0 & 0 & \tilde{R} \end{array}\right] \end{array} := \begin{array}{c} \\ n \\ \ell \\ \ell \end{array}\begin{array}{c} \begin{array}{ccc} n & \ell & \ell \end{array} \\ \left[\begin{array}{ccc} Q & S_1 & S_2 \\ S_1^H & R_{11} & R_{12} \\ S_2^H & R_{21} & R_{22} \end{array}\right] \end{array}. \tag{17}$$

In this notation, the extended descriptor system (13) becomes

$$E\dot{x} = Ax + \left[\begin{array}{cc} B_1 & B_2 \end{array}\right] \left[\begin{array}{c} u_1(t) \\ u_2(t) \end{array}\right], \qquad x(t_0) = x^0,$$

and the cost functional becomes

$$\mathcal{S}_e = \int_{t_0}^{\infty} \left[\begin{array}{c} x(t) \\ u_1(t) \\ u_2(t) \end{array}\right]^H \left[\begin{array}{ccc} Q & S_1 & S_2 \\ S_1^H & R_{11} & R_{12} \\ S_2^H & R_{12}^H & R_{22} \end{array}\right] \left[\begin{array}{c} x(t) \\ u_1(t) \\ u_2(t) \end{array}\right] dt.$$

After a reordering of variables and equations, the Euler-Lagrange equations (3) for this extended linear-quadratic optimal control problem become

$$\mathcal{E}_c^e \left[\begin{array}{c} \dot{x} \\ \dot{u}_1 \\ \dot{\mu} \\ \dot{u}_2 \end{array}\right] = \mathcal{A}_c^e \left[\begin{array}{c} x \\ u_1 \\ \mu \\ u_2 \end{array}\right], \qquad x(t_0) = x^0, \qquad \lim_{t\to\infty} E^H \mu(t) = 0, \tag{18}$$

with the skew-Hamiltonian/Hamiltonian matrix pencil

$$\alpha\mathcal{E}_c^e - \beta\mathcal{A}_c^e := \alpha \left[\begin{array}{cc|cc} E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & E^H & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] - \beta \left[\begin{array}{cc|cc} A & B_1 & 0 & B_2 \\ S_2^H & R_{12}^H & B_2^H & R_{22} \\ \hline -Q & -S_1 & -A^H & -S_2 \\ -S_1^H & -R_{11} & -B_1{}^H & -R_{12} \end{array}\right]. \tag{19}$$

Numerical methods working directly with (18) and (19) can preserve and exploit the skew-Hamiltonian/Hamiltonian structure while avoiding the explicit matrix inverses and products required to form (5) and (6) or (7) and (8).

There is some freedom in the choice of k, $\tilde{B}$ and $\tilde{R}$. How best to use the freedom is an open question, but some guidelines are appropriate.

- To avoid increasing the complexity of the problem much, it may be best to choose k as small as possible, i.e., $k = 0$ if m is even and $k = 1$ if m is odd. However, if the dimension of the problem is small, then $k = m$ may also be a suitable choice.

- If $\tilde{B} = 0$ and $\tilde{R}$ is positive definite, then for $(u_*, v_*) \in \mathbb{R}^m \times \mathbb{R}^k$ minimizing $\mathcal{S}_e$ in (14), it is clear that $v_*(t) \equiv 0$ and u_* is a solution to the original problem.

- It is important that $\tilde{R}$ and $\tilde{B}$ be chosen so that the matrix pencil (19) is regular, i.e., for some $(\alpha, \beta) \in \mathbb{C} \times \mathbb{C}$, $\det(\alpha\mathcal{E}_c^e - \beta\mathcal{A}_c^e) \neq 0$. Otherwise, the two-point boundary value problem of differential-algebraic equations (18) may not have a unique solution for all consistently chosen initial values x^0, see [27]. If the original extended Hamiltonian matrix pencil (4) is regular and $\tilde{B} = 0$, then (18) is also regular. Note that (4) can be made to be regular by appropriate preprocessing of the system [15, 27, 35].

- The matrices $\tilde{R}$ and $\tilde{B}$ should be chosen so that the matrix pencil (19) has a structured Schur-like form. Such forms are characterized for skew-Hamiltonian/Hamiltonian matrix pencils in [33, 34] and for the other structures in [32].
- If possible, $\tilde{R}$ and $\tilde{B}$ should be chosen so that the problem of computing the desired invariant subspace should not be more ill-conditioned than that for the original matrix pencil (4). If $\tilde{B} = 0$ and (4) has well-conditioned deflating subspaces, (e.g., if E and R are nonsingular and well-conditioned, and the reduced Hamiltonian matrix in (6) has no purely imaginary eigenvalues), then the deflating subspaces of (19) are likely to have well-conditioned deflating subspaces also.

There is a certain philosophy behind this embedding. First of all, the extension is an approximate "dual" operation to the reduction of the extended problem (3) to the problem (7). Furthermore, from a behavioral point of view [47, 27], i.e., if control variables and state variables are interchangeable, then the partitioning (15)–(17) is not unnatural [15, 16, 27].

A similar embedding may also be constructed for the Hamiltonian matrices in H_∞ optimization.

3 Hamiltonian Triangular Forms

In this section we briefly review the results on the existence of structured Schur forms for Hamiltonian matrices and skew-Hamiltonian/Hamiltonian matrix pencils.

We call a matrix *Hamiltonian block triangular* if it is Hamiltonian and has the form

$$\begin{bmatrix} F & G \\ 0 & -F^H \end{bmatrix}.$$

If, furthermore, F is triangular then we call the matrix *Hamiltonian triangular*. The terms *skew-Hamiltonian block triangular* and *skew-Hamiltonian triangular* are defined analogously. If a Hamiltonian (skew-Hamiltonian) matrix $\mathcal{H}$ can be transformed into Hamiltonian (skew-Hamiltonian) triangular form by a similarity transformation with a unitary symplectic matrix $\mathcal{U} \in \mathcal{US}_{2n}$, then we say that $\mathcal{U}^H \mathcal{H} \mathcal{U}$ has *Hamiltonian Schur form* (*skew-Hamiltonian Schur form*).

Not all Hamiltonian matrices have a Hamiltonian Schur form. All real skew-Hamiltonian matrices (but not all complex skew-Hamiltonian matrices) have a skew-Hamiltonian Schur form [46]. For Hamiltonian matrices that have no purely imaginary eigenvalues the existence of a Hamiltonian Schur form was proved in [38]. The general result was suggested in [31] and a proof based on a structured Hamiltonian Jordan form was recently given in [32]. The results were extended in [33, 34] to skew-Hamiltonian/Hamiltonian matrix pencils. In this section we summarize the results from [32, 33, 34] needed for the analysis and development of the numerical methods below.

The following theorem gives necessary and sufficient conditions for the existence of a Hamiltonian triangular form under unitary symplectic similarity transformations. If this form exists, then we have also a Lagrangian invariant subspace that may be used to decouple the boundary value problem (5) [36]. Note that here and in the following, by abuse of notation, we identify a subspace and a matrix whose columns span this subspace by the same symbol.

Theorem 2 *[32] Let $\mathcal{H}$ be a Hamiltonian matrix with ν pairwise distinct, nonzero, purely imaginary eigenvalues $i\alpha_1$, $i\alpha_2$, ..., $i\alpha_\nu$ and associated invariant subspaces $\mathcal{U}_1$, $\mathcal{U}_2$, ..., $\mathcal{U}_\nu$, respectively. The following are equivalent.*

(i) There exists a symplectic matrix $\mathcal{S}$ such that $\mathcal{S}^{-1}\mathcal{H}\mathcal{S}$ is Hamiltonian triangular.

(ii) There exists a unitary symplectic matrix $\mathcal{U}$ such that $\mathcal{U}^H\mathcal{H}\mathcal{U}$ is Hamiltonian triangular.

(iii) For $k = 1, 2, \ldots, \nu$, $\mathcal{U}_k^H\mathcal{J}\mathcal{U}_k$ is congruent to a copy of $\mathcal{J}$ of appropriate dimension. (If $\nu = 0$, i.e., if $\mathcal{H}$ has no nonzero, purely imaginary eigenvalue, then this statement holds vacuously.)

For regular skew-Hamiltonian/Hamiltonian matrix pencils the situation is similar, see [33, 34]. The structure of skew-Hamiltonian/Hamiltonian matrix pencils is preserved by *J-congruence transformations* [33, 34], i.e., if $\alpha\mathcal{S} - \beta\mathcal{H}$ is skew-Hamiltonian/Hamiltonian then for any nonsingular matrix $\mathcal{Y}$, $\mathcal{J}\mathcal{Y}^H\mathcal{J}^T(\alpha\mathcal{S} - \beta\mathcal{H})\mathcal{Y}$ is also skew-Hamiltonian/Hamiltonian.

Theorem 3 *[33, 34] Let $\alpha\mathcal{S} - \beta\mathcal{H}$ be a regular skew-Hamiltonian/Hamiltonian matrix pencil, with ν pairwise distinct, finite, nonzero, purely imaginary eigenvalues $i\alpha_1$, $i\alpha_2$, ..., $i\alpha_\nu$ of algebraic multiplicity p_1, p_2, ..., p_ν, and associated right deflating subspaces $\mathcal{Q}_1$, $\mathcal{Q}_2$, ..., $\mathcal{Q}_\nu$. Let p_∞ be the algebraic multiplicity of the eigenvalue infinity and let $\mathcal{Q}_\infty$ be its associated deflating subspace. The following are equivalent.*

(i) There exists a nonsingular matrix $\mathcal{Y}$, such that

$$\mathcal{J}\mathcal{Y}^H\mathcal{J}^T(\alpha\mathcal{S} - \beta\mathcal{H})\mathcal{Y} = \alpha\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^H \end{bmatrix} - \beta\begin{bmatrix} H_{11} & H_{12} \\ 0 & -H_{11}^H \end{bmatrix}, \tag{20}$$

where S_{11} and H_{11} are upper triangular while S_{12} is skew-Hermitian and H_{12} is Hermitian.

(ii) There exists a unitary matrix $\mathcal{Q}$ such that $\mathcal{J}\mathcal{Q}^H\mathcal{J}^T(\alpha\mathcal{S} - \beta\mathcal{H})\mathcal{Q}$ is of the form on the right-hand-side of (20).

(iii) For $k = 1, 2, \ldots, \nu$, $\mathcal{Q}_k^H\mathcal{J}\mathcal{S}\mathcal{Q}_k$ is congruent to a $p_k \times p_k$ copy of $\mathcal{J}$. (If $\nu = 0$, i.e., if $\alpha\mathcal{S} - \beta\mathcal{H}$ has no finite, nonzero, purely imaginary eigenvalue, then this statement holds vacuously.)

Furthermore if $p_\infty \neq 0$ then $\mathcal{Q}_\infty^H\mathcal{J}H\mathcal{Q}_\infty$ is congruent to a $p_\infty \times p_\infty$ copy of $i\mathcal{J}$.

The results covering real Schur-like forms of real Hamiltonian matrices and skew-Hamiltonian/Hamiltonian matrix pencils are similar [32, 33, 34].

Theorems 2 and 3 give necessary and sufficient conditions for the existence of structured triangular-like forms. They also demonstrate that whenever a structured triangular-like form exists, then it also exists under unitary transformations. This fact gives hope that these forms and the eigenvalues and deflating subspaces that they display can be computed with structure preserving, numerically stable, unitary transformations. The following sections propose such numerical methods for the computation of eigenvalues of Hamiltonian matrices and skew-Hamiltonian/Hamiltonian matrix pencils.

4 Skew-Hamiltonian/Hamiltonian Matrix Pencils

The skew-Hamiltonian/Hamiltonian matrix pencils (8) and (19) have the common characteristic that the skew-Hamiltonian matrix $\mathcal{S}$ is block diagonal. In this case (and also other cases), the matrix $\mathcal{S}$ factors in the form

$$\mathcal{S} = \mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z} \tag{21}$$

where $\mathcal{J}$ is as in Definition 1. For example, if $\mathcal{S} = \begin{bmatrix} E & 0 \\ 0 & E^H \end{bmatrix}$ where $E \in \mathbb{C}^{n,n}$, then $\mathcal{Z} = \mathrm{diag}(I, E^H)$ in (21).

Consider the indefinite inner product on $\mathbb{C}^{2n} \times \mathbb{C}^{2n}$ defined as $\langle x, y\rangle = x^H\mathcal{J}y$. If $\mathcal{Z} \in \mathbb{C}^{2n,2n}$, then for all x, $y \in \mathbb{C}^{2n}$, $\langle(\mathcal{Z}x), y\rangle = \langle x, (\mathcal{J}\mathcal{Z}^H\mathcal{J}^T)y\rangle$, i.e., the adjoint of $\mathcal{Z}$ with respect to $\langle\,.\,,\,.\,\rangle$ is $\mathcal{J}\mathcal{Z}^H\mathcal{J}^T$. Because $\mathcal{J}^T = -\mathcal{J}$, the adjoint may also be expressed as $\mathcal{J}\mathcal{Z}^H\mathcal{J}^T$. From this point of view, (21) is a symmetric-like factorization of $\mathcal{S}$ into the product of $\mathcal{Z}$ and its adjoint $\mathcal{J}\mathcal{Z}\mathcal{J}^T$. By analogy with the factorization of symmetric matrices, we will call (21) a *skew-Hamiltonian $\mathcal{J}$-Hermitian factorization* and we will use the term *$\mathcal{J}$-semidefinite* to refer to skew-Hamiltonians matrices which have a skew-Hamiltonian $\mathcal{J}$-Hermitian factorization (21). A *$\mathcal{J}$-definite* skew-Hamiltonian matrix is a skew Hamiltonian matrix that is both $\mathcal{J}$-semidefinite and non-singular.

As seen in the skew-Hamiltonian/Hamiltonian matrix pencils (8) and (19) $\mathcal{J}$-semidefiniteness arises frequently in applications. We show below that all real skew-Hamiltonian matrices are $\mathcal{J}$-semidefinite. We also show that if a skew-Hamiltonian/Hamiltonian matrix pencil has a skew-Hamiltonian/Hamiltonian form as in Theorem 3, then the skew-Hamiltonian part is $\mathcal{J}$-semidefinite.

Although $\mathcal{J}$-semidefiniteness is a common property of skew-Hamiltonian matrices it is not universal. The following lemma shows that neither $i\mathcal{J}$ nor any nonsingular, skew-Hamiltonian matrix of the form $i\mathcal{J}LL^T$ is $\mathcal{J}$-semidefinite. (Later, Lemma 5 will show that $i\mathcal{J}LL^T$ fails to be $\mathcal{J}$-semidefinite for any $L \neq 0$.)

Lemma 4 *A nonsingular skew-Hamiltonian matrix $\mathcal{S}$ is $\mathcal{J}$-definite if and only if $i\mathcal{J}\mathcal{S}$ is Hermitian with n positive and n negative eigenvalues.*

Proof. If $\mathcal{S}$ is $\mathcal{J}$-definite, then $\mathcal{Z}$ in (21) is nonsingular and the Hermitian matrix $i\mathcal{J}\mathcal{S}$ is congruent to $-i\mathcal{J}^T = i\mathcal{J}$. It follows from Sylvester's law of inertia [18, p. 296] that $i\mathcal{J}\mathcal{S}$ is a Hermitian matrix with n positive eigenvalues and n negative eigenvalues.

Conversely, suppose that $i\mathcal{J}\mathcal{S}$ is Hermitian with n positive and n negative eigenvalues. The matrix $i\mathcal{J}^T$ also has n positive and n negative eigenvalues, so, by an immediate consequence of Sylvester's law of inertia, there is a nonsingular matrix $\mathcal{Z} \in \mathbb{C}^{2n,2n}$ for which $i\mathcal{J}\mathcal{S} = \mathcal{Z}^H(i\mathcal{J}^T)\mathcal{Z}$. It follows that (21) holds with this matrix $\mathcal{Z}$. $\square$

Lemma 4 suggests that $\mathcal{J}$-semidefiniteness might be a characteristic of the inertia of $i\mathcal{J}\mathcal{S}$. The next lemma shows that this is indeed the case.

Lemma 5 *A matrix $\mathcal{S} \in \mathbb{SH}_{2n}$ is $\mathcal{J}$-semidefinite if and only if $i\mathcal{J}\mathcal{S}$ satisfies both $\pi(i\mathcal{J}\mathcal{S}) \leq n$ and $\nu(i\mathcal{J}\mathcal{S}) \leq n$.*

Proof. Suppose that $\mathcal{S} \in \mathbb{SH}_{2n}$ is $\mathcal{J}$-semidefinite. For some $\mathcal{Z}$ satisfying (21), define $\mathcal{S}(\epsilon)$ by $\mathcal{S}(\epsilon) = \mathcal{J}(Z + \epsilon I)^H\mathcal{J}^T(Z + \epsilon I)$. For ϵ small enough, $\mathcal{Z} + \epsilon I$ is nonsingular, and, by Lemma 4, $\pi(i\mathcal{J}\mathcal{S}(\epsilon)) = n$ and $\nu(i\mathcal{J}\mathcal{S}(\epsilon)) = n$. Because eigenvalues are continuous functions of matrix elements and $\mathcal{S} = \lim_{\epsilon\to 0}\mathcal{S}(\epsilon)$, it follows that $\pi(i\mathcal{J}\mathcal{S}) \leq n$ and $\nu(i\mathcal{J}\mathcal{S}) \leq n$.

For the converse, if $\pi(i\mathcal{J}\mathcal{S}) = p \leq n$ and $\nu(i\mathcal{J}\mathcal{S}) = q \leq n$, then, there exists a nonsingular matrix $\mathcal{W}$ for which $i\mathcal{J}\mathcal{S} = \mathcal{W}^H \mathcal{L} \mathcal{W}$ with signature matrix

$$\mathcal{L} = \begin{array}{c} \\ p \\ n-p \\ q \\ n-q \end{array} \!\!\begin{array}{c} \begin{array}{cccc} p & n-p & q & n-q \end{array} \\ \left[\begin{array}{cccc} I_p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -I_q & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}.$$

Because $p \leq n$ and $q \leq n$, $\mathcal{L}$ factors as $\mathcal{L} = \mathcal{L}\,\mathrm{diag}(I_n, -I_n)\mathcal{L}$ (where I_n is the $n \times n$ identity matrix). The matrix $\mathrm{diag}(I_n, -I_n)$ is the diagonal matrix of eigenvalues of $i\mathcal{J}^T$, so $\mathcal{L} = \mathcal{L}(\mathcal{U}^H(i\mathcal{J}^T)\mathcal{U})\mathcal{L}$, where $\mathcal{U} = (1/\sqrt{2})\left[\begin{smallmatrix} I_n & I_n \\ iI_n & -iI_n \end{smallmatrix}\right]$ is the unitary matrix of eigenvectors of $i\mathcal{J}^T$. Hence, (21) holds with $\mathcal{Z} = \mathcal{U}\mathcal{L}\mathcal{W}$. $\square$

The following immediate corollary also follows from [17].

Corollary 6 *Every real skew-Hamiltonian matrix $\mathcal{S}$ is $\mathcal{J}$-semidefinite.*

Proof. If $\mathcal{S}$ is real, then $\mathcal{J}\mathcal{S}$ is real and skew-symmetric. The eigenvalues of $\mathcal{J}\mathcal{S}$ appear in complex conjugate pairs with zero real part. Hence, the eigenvalues of $i\mathcal{J}\mathcal{S}$ lie on the real axis in $\pm$ pairs. In particular, $\pi(i\mathcal{J}\mathcal{S}) = \nu(i\mathcal{J}\mathcal{S})$. It follows from the trivial identity $\pi(i\mathcal{J}\mathcal{S}) + \omega(i\mathcal{J}\mathcal{S}) + \nu(i\mathcal{J}\mathcal{S}) = 2n$ that $\pi(i\mathcal{J}\mathcal{S}) \leq n$ and $\nu(i\mathcal{J}\mathcal{S}) \leq n$. $\square$

The next lemma and its corollary show that $\mathcal{J}$-semidefiniteness of both $\mathcal{S}$ and $i\mathcal{H}$ are necessary conditions for a skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{S} - \beta\mathcal{H}$ to have the skew-Hamiltonian/Hamiltonian Schur form of Theorem 3.

Lemma 7 *If $\mathcal{S} \in \mathbb{SH}_{2n}$ and there exists a nonsingular matrix $\mathcal{Y}$ such that*

$$\mathcal{J}\mathcal{Y}^H\mathcal{J}^T\mathcal{S}\mathcal{Y} = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^H \end{bmatrix}$$

with S_{11}, $S_{12} \in \mathbb{C}^{n,n}$, then $\mathcal{S}$ is $\mathcal{J}$-semidefinite.

Proof. Let $\mathcal{T}$ be the Hermitian matrix

$$\mathcal{T} = \mathcal{Y}^H(i\mathcal{J}\mathcal{S})\mathcal{Y} = \begin{bmatrix} 0 & iS_{11}^H \\ -iS_{11} & -iS_{12} \end{bmatrix},$$

and set $\mathcal{T}(\epsilon) = \mathcal{T} + \epsilon \begin{bmatrix} 0 & I_n \\ I_n & I_n \end{bmatrix}$. For ϵ sufficiently small, both $\epsilon I_n - iS_{12}$ and $\epsilon I_n - iS_{11}$ are nonsingular and $\mathcal{T}(\epsilon)$ is congruent to

$$\begin{bmatrix} -(\epsilon I_n - iS_{11})(\epsilon I_n - iS_{12})^{-1}(\epsilon I_n - iS_{11})^H & 0 \\ 0 & (\epsilon I_n - iS_{12}) \end{bmatrix}.$$

By Sylvester's law, the inertia of the negative of the $(1,1)$ block is equal to the inertia of the $(2,2)$ block. This implies $\pi(\mathcal{T}(\epsilon)) = \nu(\mathcal{T}(\epsilon)) = n$. Continuity of eigenvalues as $\epsilon \to 0$ implies $\pi(\mathcal{T}) \leq n$ and $\nu(\mathcal{T}) \leq n$. The lemma now follows from Lemma 5. $\square$

Corollary 8 *If $\mathcal{H} \in \mathbb{H}_{2n}$ and there exists a nonsingular matrix $\mathcal{Y}$ such that*

$$\mathcal{J}\mathcal{Y}^H\mathcal{J}^T\mathcal{H}\mathcal{Y} = \begin{bmatrix} H_{11} & H_{12} \\ 0 & -H_{11}^H \end{bmatrix}$$

with H_{11}, $H_{12} \in \mathbb{C}^{n,n}$, then $i\mathcal{H}$ is $\mathcal{J}$-semidefinite.

Proof. Apply Lemma 7 to the skew-Hamiltonian matrix $i\mathcal{H}$. $\square$

It follows from Lemma 7, Corollary 8, and Theorem 3 part (ii) that if $\alpha\mathcal{S}-\beta\mathcal{H}$ is a skew-Hamiltonian/Hamiltonian matrix pencil that has a skew-Hamiltonian/Hamiltonian Schur form, then $\mathcal{S}$ and $i\mathcal{H}$ are $\mathcal{J}$-semidefinite. As noted above, the factor $\mathcal{Z}$ in (21) is often either given explicitly as part of the problem statement or can be obtained as in the proof of Lemma 5. The next theorem shows that if $\mathcal{S}$ is nonsingular, then the skew-Hamiltonian/Hamiltonian Schur form (if it exists) can be expressed in terms of block triangular factorizations of $\mathcal{Z}$ and $\mathcal{H}$ without explicitly using $\mathcal{S}$. This opens the possibility of designing numerical methods that work directly on $\mathcal{Z}$ and $\mathcal{H}$ and avoid forming $\mathcal{S}$ explicitly.

Theorem 9 *Let $\alpha\mathcal{S}-\beta\mathcal{H}$ be a skew-Hamiltonian/Hamiltonian matrix pencil with nonsingular, $\mathcal{J}$-semidefinite skew-Hamiltonian part $\mathcal{S}=\mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z}$. If any of the equivalent conditions of Theorem 3 holds, then there exists a unitary matrix $\mathcal{Q}$ and a unitary symplectic matrix $\mathcal{U}$ such that*

$$\mathcal{U}^H\mathcal{Z}\mathcal{Q} = \begin{bmatrix} Z_{11} & Z_{12} \\ 0 & Z_{22} \end{bmatrix}, \tag{22}$$

$$\mathcal{J}\mathcal{Q}^H\mathcal{J}^T\mathcal{H}\mathcal{Q} = \begin{bmatrix} H_{11} & H_{12} \\ 0 & -H_{11}^H \end{bmatrix}, \tag{23}$$

where Z_{11}, Z_{22}^H and H_{11} are $n\times n$ and upper triangular.

Proof. With $\mathcal{Q}$ as in Theorem 3 part (ii) we obtain (23) and $\mathcal{J}\mathcal{Q}^H\mathcal{J}^T\mathcal{S}\mathcal{Q}=\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}\end{bmatrix}^H$. Partition $\tilde{\mathcal{Z}}=\mathcal{Z}\mathcal{Q}$ as $\tilde{\mathcal{Z}}=[Z_1,\, Z_2]$, where $Z_1,\, Z_2\in\mathbb{C}^{2n,n}$. Using $\mathcal{S}=\mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z}$, we obtain

$$\tilde{\mathcal{Z}}^H\mathcal{J}\tilde{\mathcal{Z}} = \begin{bmatrix} 0 & S_{11}^H \\ -S_{11} & -S_{12} \end{bmatrix}. \tag{24}$$

In particular, $Z_1^H\mathcal{J}Z_1=0$, i.e., the columns of Z_1 form a basis of a Lagrangian subspace and therefore the columns of Z_1 form the first n columns of a symplectic matrix. (It is easy to verify from Definition 1 that, using the non-negative definite square root, $[Z_1,\, -\mathcal{J}Z_1(Z_1^HZ_1)^{-1/2}]$ is symplectic.) It is shown in [11] that Z_1 has a unitary symplectic QR factorization

$$\mathcal{U}^HZ_1 = \begin{bmatrix} Z_{11} \\ 0 \end{bmatrix},$$

where $\mathcal{U}\in\mathbb{US}_{2n}$ is unitary symplectic and $Z_{11}\in\mathbb{C}^{n,n}$ is upper triangular. Setting

$$\mathcal{U}^H\mathcal{Z}\mathcal{Q}=\mathcal{U}^H\tilde{\mathcal{Z}} = \begin{bmatrix} Z_{11} & Z_{12} \\ 0 & Z_{22} \end{bmatrix}$$

we obtain from (24) that $Z_{22}^HZ_{11}=S_{11}$. Since S_{11} and Z_{11} are both upper triangular and Z_{11} is nonsingular, we conclude that Z_{22}^H is also upper triangular. $\square$

Note that the invertibility of $\mathcal{Z}$ is only a sufficient condition for the existence of $\mathcal{U}$ as in (22) and (23). However, there is no particular pathology associated with $\mathcal{Z}$ being singular. The algorithms described below do not require $\mathcal{Z}$ to be nonsingular. (The unitary symplectic matrix $\mathcal{U}$ is closely related to the Hermitian solution of the generalized algebraic Riccati equation (9), see [7]. However, when $\mathcal{Z}$ is singular, the relationship to Riccati equations is complicated [27].)

If both $\mathcal{S}$ and $\mathcal{H}$ are nonsingular, then the following stronger form of Theorem 9 holds.

Corollary 10 *Let $\alpha\mathcal{S} - \beta\mathcal{H}$ be a skew-Hamiltonian/Hamiltonian matrix pencil with nonsingular $\mathcal{J}$-semidefinite skew-Hamiltonian part $\mathcal{S} = \mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z}$ and nonsingular $\mathcal{J}$-semidefinite Hamiltonian part $i\mathcal{H} = \mathcal{J}\mathcal{W}^H\mathcal{J}^T\mathcal{W}$. If any of the equivalent conditions of Theorem 3 holds, then there exists a unitary matrix $\mathcal{Q}$ and a unitary symplectic matrix $\mathcal{U}$ such that*

$$\mathcal{U}^H\mathcal{Z}\mathcal{Q} = \begin{bmatrix} Z_{11} & Z_{12} \\ 0 & Z_{22} \end{bmatrix}, \qquad \mathcal{U}^H\mathcal{W}\mathcal{Q} = \begin{bmatrix} W_{11} & W_{12} \\ 0 & W_{22} \end{bmatrix},$$

where Z_{11}, Z_{22}^H and W_{11}, W_{22}^H are $n \times n$ and upper triangular.

Proof. Similar to the proof of Theorem 9. □

We will obtain the structured Schur form of a complex skew-Hamiltonian/Hamiltonian matrix pencil from the structured Schur form of a real skew-Hamiltonian/skew-Hamiltonian matrix pencil of double dimension. Consequently, we need the following theorem that establishes the existence of a structured real Schur form for these matrix pencils.

Theorem 11 *If $\alpha\mathcal{S} - \beta\mathcal{N}$ is a real, regular skew-Hamiltonian/skew-Hamiltonian matrix pencil with $\mathcal{S} = \mathcal{J}\mathcal{Z}^T\mathcal{J}^T\mathcal{Z}$, then there exists a real orthogonal matrix $\mathcal{Q} \in \mathbb{R}^{2n,2n}$ and a real orthogonal symplectic matrix $\mathcal{U} \in \mathbb{R}^{2n,2n}$ such that*

$$\mathcal{U}^T\mathcal{Z}\mathcal{Q} = \begin{bmatrix} Z_{11} & Z_{12} \\ 0 & Z_{22} \end{bmatrix}, \tag{25}$$

$$\mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{N}\mathcal{Q} = \begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{11}^T \end{bmatrix}, \tag{26}$$

where Z_{11} and Z_{22}^T are upper triangular, N_{11} is quasi upper triangular and N_{12} is skew-symmetric.

Moreover,

$$\mathcal{J}\mathcal{Q}^T\mathcal{J}^T(\alpha\mathcal{S} - \beta\mathcal{N})\mathcal{Q} = \alpha\begin{bmatrix} Z_{22}^T Z_{11} & Z_{22}^T Z_{12} - Z_{12}^T Z_{22} \\ 0 & Z_{11}^T Z_{22} \end{bmatrix} - \beta\begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{11}^T \end{bmatrix} \tag{27}$$

is a $\mathcal{J}$-congruent skew-Hamiltonian/skew-Hamiltonian matrix pencil.

Proof. A constructive proof for the existence of $\mathcal{Q}$ and $\mathcal{U}$ satisfying (25) and (26) is Algorithm 3 in Appendix A. To show (27), recall that $\mathcal{U}$ is orthogonal symplectic and therefore commutes with $\mathcal{J}$. Hence,

$$\begin{aligned} \mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{S}\mathcal{Q} &= \mathcal{J}\mathcal{Q}^T\mathcal{J}^T(\mathcal{J}\mathcal{Z}^T\mathcal{J}^T\mathcal{Z})\mathcal{Q} \\ &= \mathcal{J}\mathcal{Q}^T\mathcal{J}^T(\mathcal{J}\mathcal{Z}^T\mathcal{J}^T\mathcal{U})(\mathcal{U}^T\mathcal{Z}\mathcal{Q}) \\ &= \mathcal{J}(\mathcal{U}^T\mathcal{Z}\mathcal{Q})^T\mathcal{J}^T(\mathcal{U}^T\mathcal{Z}\mathcal{Q}). \end{aligned}$$

Equation (27) now follows from the block triangular form of (25). □

Note that this theorem does not easily extend to complex skew-Hamiltonian/skew-Hamiltonian matrix pencils. In the complex case, there is little difference in the structure of Hamiltonian and skew-Hamiltonian matrices, because a skew-Hamiltonian matrix is just a scalar multiple (by i) of a Hamiltonian matrix. Real skew-Hamiltonian matrices have a fundamentally different structure than real Hamiltonian matrices.

A method for computing the structured Schur form (27) for real matrices was proposed in [39]. If $\mathcal{S}$ is given in factored form, then Algorithm 3 in Appendix A is more robust in finite precision arithmetic, because it avoids forming $\mathcal{S}$ explicitly.

Neither the method in [39] nor Algorithm 3 in Appendix A applies to complex skew-Hamiltonian/Hamiltonian matrix pencils because those algorithms depend on the fact that real diagonal skew-symmetric matrices are identically zero. This property is also crucial for the structured Schur form algorithms in [5, 46].

Algorithm 1 given below, computes the eigenvalues of a complex skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{S} - \beta\mathcal{H}$ using an unusual embedding of $\mathbb{C}$ into $\mathbb{R}^2$ that was recently proposed in [7]. Let $\alpha\mathcal{S} - \beta\mathcal{H}$ be a complex skew-Hamiltonian/Hamiltonian matrix pencil with $\mathcal{J}$-semidefinite skew-Hamiltonian part $\mathcal{S} = \mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z}$. Split the skew-Hamiltonian matrix $\mathcal{N} = i\mathcal{H} \in \mathbb{SH}_{2n}$ as $i\mathcal{H} = \mathcal{N} = \mathcal{N}_1 + i\mathcal{N}_2$, where $\mathcal{N}_1$ is real skew-Hamiltonian and $\mathcal{N}_2$ is real Hamiltonian, i.e.,

$$\begin{aligned} \mathcal{N}_1 &= \begin{bmatrix} F_1 & G_1 \\ H_1 & F_1^T \end{bmatrix}, \quad G_1 = -G_1^T, \quad H_1 = -H_1^T, \\ \mathcal{N}_2 &= \begin{bmatrix} F_2 & G_2 \\ H_2 & -F_2^T \end{bmatrix}, \quad G_2 = G_2^T, \quad H_2 = H_2^T, \end{aligned}$$

and $F_j, G_j, H_j \in \mathbb{R}^{n\times n}$ for $j = 1, 2$. Setting

$$\begin{aligned} \mathcal{Y}_c &= \frac{\sqrt{2}}{2}\begin{bmatrix} I_{2n} & iI_{2n} \\ I_{2n} & -iI_{2n} \end{bmatrix}, \\ \mathcal{P} &= \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix}, \qquad (28) \\ \mathcal{X}_c &= \mathcal{Y}_c\mathcal{P}, \qquad (29) \end{aligned}$$

and using the embedding $\mathcal{B}_\mathcal{N} = \operatorname{diag}(\mathcal{N}, \bar{\mathcal{N}})$ we obtain that

$$\mathcal{B}_\mathcal{N}^c := \mathcal{X}_c^H\mathcal{B}_\mathcal{N}\mathcal{X}_c = \left[\begin{array}{cc|cc} F_1 & -F_2 & G_1 & -G_2 \\ F_2 & F_1 & G_2 & G_1 \\ \hline H_1 & -H_2 & F_1^T & F_2^T \\ H_2 & H_1 & -F_2^T & F_1^T \end{array}\right] \qquad (30)$$

is a real skew-Hamiltonian matrix in $\mathbb{SH}_{4n}$. Similarly, set

$$\begin{aligned} \mathcal{B}_\mathcal{Z} &:= \begin{bmatrix} \mathcal{Z} & 0 \\ 0 & \bar{\mathcal{Z}} \end{bmatrix}, \qquad (31) \\ \mathcal{B}_\mathcal{T} &:= \begin{bmatrix} \mathcal{J}\mathcal{Z}^H\mathcal{J}^T & 0 \\ 0 & \overline{\mathcal{J}\mathcal{Z}^H\mathcal{J}^T} \end{bmatrix}, \qquad (32) \\ \mathcal{B}_\mathcal{S} &:= \begin{bmatrix} \mathcal{S} & 0 \\ 0 & \bar{\mathcal{S}} \end{bmatrix} = \mathcal{B}_\mathcal{T}\mathcal{B}_\mathcal{Z}. \qquad (33) \end{aligned}$$

Hence,

$$\alpha\mathcal{B}_\mathcal{S} - \beta\mathcal{B}_\mathcal{N} = \begin{bmatrix} \alpha\mathcal{S} - \beta\mathcal{N} & 0 \\ 0 & \alpha\bar{\mathcal{S}} - \beta\bar{\mathcal{N}} \end{bmatrix}.$$

We can easily verify that

$$\begin{aligned}
\mathcal{B}^c_{\mathcal{Z}} &:= \mathcal{X}_c^H \mathcal{B}_{\mathcal{Z}} \mathcal{X}_c, && (34)\\
\mathcal{B}^c_{\mathcal{T}} &:= \mathcal{X}_c^H \mathcal{B}_{\mathcal{T}} \mathcal{X}_c = \mathcal{J}(\mathcal{B}^c_{\mathcal{Z}})^T \mathcal{J}^T, \\
\mathcal{B}^c_{\mathcal{S}} &:= \mathcal{X}_c^H \mathcal{B}_{\mathcal{S}} \mathcal{X}_c = \mathcal{J}(\mathcal{B}^c_{\mathcal{Z}})^T \mathcal{J}^T \mathcal{B}^c_{\mathcal{Z}} && (35)
\end{aligned}$$

are all real. Therefore,

$$\begin{aligned}
\alpha\mathcal{B}^c_{\mathcal{S}} - \beta\mathcal{B}^c_{\mathcal{N}} &= \mathcal{X}_c^H(\alpha\mathcal{B}_{\mathcal{S}} - \beta\mathcal{B}_{\mathcal{N}})\mathcal{X}_c \\
&= \mathcal{X}_c^H \begin{bmatrix} \alpha\mathcal{S} - \beta\mathcal{N} & 0 \\ 0 & \alpha\bar{\mathcal{S}} - \beta\bar{\mathcal{N}} \end{bmatrix} \mathcal{X}_c && (36)
\end{aligned}$$

is a real $4n \times 4n$ skew-Hamiltonian/skew-Hamiltonian matrix pencil. For this matrix pencil we can employ Algorithm 3 in Appendix A to compute the structured factorization (26), i.e., we can determine an orthogonal symplectic matrix $\mathcal{U}$ and an orthogonal matrix $\mathcal{Q}$ such that

$$\begin{aligned}
\hat{\mathcal{B}}^c_{\mathcal{Z}} &:= \mathcal{U}^T \mathcal{B}^c_{\mathcal{Z}} \mathcal{Q} = \begin{bmatrix} \mathcal{Z}_{11} & \mathcal{Z}_{12} \\ 0 & \mathcal{Z}_{22} \end{bmatrix}, && (37)\\
\hat{\mathcal{B}}^c_{\mathcal{N}} &:= \mathcal{J}\mathcal{Q}^T\mathcal{J}^T \mathcal{B}^c_{\mathcal{N}} \mathcal{Q} = \begin{bmatrix} \mathcal{N}_{11} & \mathcal{N}_{12} \\ 0 & \mathcal{N}_{11}^T \end{bmatrix}. && (38)
\end{aligned}$$

Thus, if $\hat{\mathcal{B}}^c_{\mathcal{S}} := \mathcal{J}(\hat{\mathcal{B}}^c_{\mathcal{Z}})^T \mathcal{J}^T \hat{\mathcal{B}}^c_{\mathcal{Z}}$, then

$$\alpha\hat{\mathcal{B}}^c_{\mathcal{S}} - \beta\hat{\mathcal{B}}^c_{\mathcal{N}} = \alpha(\mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{B}^c_{\mathcal{S}}\mathcal{Q}) - \beta(\mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{B}^c_{\mathcal{N}}\mathcal{Q})$$

is a $\mathcal{J}$-congruent skew-Hamiltonian/skew-Hamiltonian matrix pencil in Schur form. By (36) and the fact that the finite eigenvalues of $\alpha\mathcal{S} - \beta\mathcal{N}$ are symmetric with respect to the real axis, we get

$$\Lambda(\mathcal{S}, \mathcal{H}) = \Lambda(\mathcal{S}, -i\mathcal{N}) = \Lambda(\mathcal{Z}_{22}^T\mathcal{Z}_{11}, -i\mathcal{N}_{11}).$$

In this way, Algorithm 1 below computes the eigenvalues of the complex skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{S} - \beta\mathcal{H} = \alpha\mathcal{S} + i\beta\mathcal{N}$.

We can also extend the complex skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{S} - \beta\mathcal{H}$ to a double size complex skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{B}_{\mathcal{S}} - \beta\mathcal{B}_{\mathcal{H}}$, where

$$\mathcal{B}_{\mathcal{H}} = \begin{bmatrix} \mathcal{H} & 0 \\ 0 & -\bar{\mathcal{H}} \end{bmatrix} \qquad (39)$$

and $\mathcal{B}_{\mathcal{S}}$ is as in (35). The spectrum of the extended matrix pencil $\alpha\mathcal{B}_{\mathcal{S}} - \beta\mathcal{B}_{\mathcal{H}}$ consists of two copies of the spectrum of $\alpha\mathcal{S} - \beta\mathcal{H}$ [5]. If

$$\mathcal{B}^c_{\mathcal{H}} = \mathcal{X}_c^H \mathcal{B}_{\mathcal{H}} \mathcal{X}_c, \qquad (40)$$

then it follows from (37) and (38) that

$$\begin{aligned}
\tilde{\mathcal{B}}^c_{\mathcal{Z}} &:= \mathcal{U}^T \mathcal{B}^c_{\mathcal{Z}} \mathcal{Q} = \begin{bmatrix} \mathcal{Z}_{11} & \mathcal{Z}_{12} \\ 0 & \mathcal{Z}_{22} \end{bmatrix}, \\
\tilde{\mathcal{B}}^c_{\mathcal{H}} &:= \mathcal{J}\mathcal{Q}^T\mathcal{J}^T \mathcal{B}^c_{\mathcal{H}} \mathcal{Q} = \begin{bmatrix} -i\mathcal{N}_{11} & -i\mathcal{N}_{12} \\ 0 & -(-i\mathcal{N}_{11})^H \end{bmatrix},
\end{aligned}$$

and the matrix pencil $\alpha\tilde{\mathcal{B}}^c_{\mathcal{S}} - \beta\tilde{\mathcal{B}}^c_{\mathcal{H}} := \alpha\mathcal{J}(\tilde{\mathcal{B}}^c_{\mathcal{Z}})^H\mathcal{J}^T\tilde{\mathcal{B}}^c_{\mathcal{Z}} - \beta\tilde{\mathcal{B}}^c_{\mathcal{H}}$ is in skew-Hamiltonian/Hamiltonian Schur form. We have thus obtained the structured Schur form of the extended complex skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{B}^c_{\mathcal{S}} - \beta\mathcal{B}^c_{\mathcal{H}}$. Moreover,

$$\begin{aligned}
\alpha\tilde{\mathcal{B}}^c_{\mathcal{S}} - \beta\tilde{\mathcal{B}}^c_{\mathcal{H}} &= \mathcal{J}\mathcal{Q}^H\mathcal{J}^T(\alpha\mathcal{B}^c_{\mathcal{S}} - \beta\mathcal{B}^c_{\mathcal{H}})\mathcal{Q} \\
&= (\mathcal{X}_c\mathcal{J}\mathcal{Q}\mathcal{J}^T)^H \begin{bmatrix} \alpha\mathcal{S} - \beta\mathcal{H} & 0 \\ 0 & \alpha\bar{\mathcal{S}} + \beta\bar{\mathcal{H}} \end{bmatrix} (\mathcal{X}_c\mathcal{Q}) \qquad (41)
\end{aligned}$$

is in skew-Hamiltonian/Hamiltonian Schur form.

We have seen so far that we can compute structured Schur forms and thus are able to compute the eigenvalues of the structured matrix pencils under consideration using the embedding technique into a structured matrix pencil of double size.

To get the desired subspaces we generalize the techniques developed in [5]. For this we need a structure preserving method to reorder the eigenvalues in the structured Schur form. This reordering method is described in detail in Appendix A. The method shows that the eigenvalues can be ordered along the diagonal of structured Schur form so that all eigenvalues with negative real part appear in the $(1,1)$ block and eigenvalues with positive real part appear in the $(2,2)$ block.

The following theorem uses this eigenvalue ordering to determine the desired deflating subspaces of the matrix pencil $\alpha\mathcal{S} - \beta\mathcal{H}$ from the real structured Schur form (41).

Theorem 12 *Let $\alpha\mathcal{S} - \beta\mathcal{H} \in \mathbb{C}^{2n,2n}$ be a skew-Hamiltonian/Hamiltonian matrix pencil with $\mathcal{J}$-semidefinite skew-Hamiltonian matrix $\mathcal{S} = \mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z}$. Consider the extended matrices*

$$\begin{aligned}
\mathcal{B}_{\mathcal{Z}} &= \operatorname{diag}(\mathcal{Z}, \bar{\mathcal{Z}}), \\
\mathcal{B}_{\mathcal{T}} &= \operatorname{diag}(\mathcal{J}\mathcal{Z}^H\mathcal{J}^T, \overline{\mathcal{J}\mathcal{Z}^H\mathcal{J}^T}), \\
\mathcal{B}_{\mathcal{S}} &= \mathcal{B}_{\mathcal{T}}\mathcal{B}_{\mathcal{Z}} = \operatorname{diag}(\mathcal{S}, \bar{\mathcal{S}}), \\
\mathcal{B}_{\mathcal{H}} &= \operatorname{diag}(\mathcal{H}, -\bar{\mathcal{H}}).
\end{aligned}$$

Let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be unitary matrices such that

$$\begin{aligned}
\mathcal{U}^H\mathcal{B}_{\mathcal{Z}}\mathcal{V} &= \begin{bmatrix} \mathcal{Z}_{11} & \mathcal{Z}_{12} \\ 0 & \mathcal{Z}_{22} \end{bmatrix} =: \mathcal{R}_{\mathcal{Z}}, \\
\mathcal{W}^H\mathcal{B}_{\mathcal{T}}\mathcal{U} &= \begin{bmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ 0 & \mathcal{T}_{22} \end{bmatrix} =: \mathcal{R}_{\mathcal{T}}, \qquad (42) \\
\mathcal{W}^H\mathcal{B}_{\mathcal{H}}\mathcal{V} &= \begin{bmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ 0 & \mathcal{H}_{22} \end{bmatrix} =: \mathcal{R}_{\mathcal{H}},
\end{aligned}$$

where $\Lambda_-(\mathcal{B}_{\mathcal{S}}, \mathcal{B}_{\mathcal{H}}) \subset \Lambda(\mathcal{T}_{11}\mathcal{Z}_{11}, \mathcal{H}_{11})$ and $\Lambda(\mathcal{T}_{11}\mathcal{Z}_{11}, \mathcal{H}_{11}) \cap \Lambda_+(\mathcal{B}_{\mathcal{S}}, \mathcal{B}_{\mathcal{H}}) = \emptyset$. Suppose $\Lambda_-(\mathcal{S}, \mathcal{H})$ contains p eigenvalues. If $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \in \mathbb{C}^{4n,m}$ are the first m columns of $\mathcal{V}$, $2p \le m \le 2n - 2p$, then there are subspaces $\mathbb{L}_1$ and $\mathbb{L}_2$ such that

$$\begin{aligned}
\operatorname{range} V_1 &= \operatorname{Def}_-(\mathcal{S}, \mathcal{H}) + \mathbb{L}_1, \quad \mathbb{L}_1 \subseteq \operatorname{Def}_0(\mathcal{S}, \mathcal{H}) + \operatorname{Def}_\infty(\mathcal{S}, \mathcal{H}), \\
\operatorname{range} \overline{V_2} &= \operatorname{Def}_+(\mathcal{S}, \mathcal{H}) + \mathbb{L}_2, \quad \mathbb{L}_2 \subseteq \operatorname{Def}_0(\mathcal{S}, \mathcal{H}) + \operatorname{Def}_\infty(\mathcal{S}, \mathcal{H}). \qquad (43)
\end{aligned}$$

If $\Lambda(\mathcal{T}_{11}\mathcal{Z}_{11}, \mathcal{H}_{11}) = \Lambda_-(\mathcal{B}_{\mathcal{S}}, \mathcal{B}_{\mathcal{H}})$, and $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$, $\begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$ are the first m columns of $\mathcal{U}$, $\mathcal{W}$, respectively, then there exist unitary matrices Q_U, Q_V, Q_W such that

$$\begin{aligned}
U_1 &= [P_U^-, 0]Q_U, & U_2 &= [0, P_U^+]Q_U, \\
V_1 &= [P_V^-, 0]Q_V, & V_2 &= [0, P_V^+]Q_V, \\
W_1 &= [P_W^-, 0]Q_W, & W_2 &= [0, P_W^+]Q_W
\end{aligned}$$

and the columns of P_V^- *and* $\overline{P_V^+}$ *form orthogonal bases of* $\mathrm{Def}_-(\mathcal{S},\mathcal{H})$ *and* $\mathrm{Def}_+(\mathcal{S},\mathcal{H})$*, respectively. Moreover, the matrices* P_U^-, P_U^+, P_W^- *and* P_W^+ *have orthonormal columns and the following relations are satisfied*

$$\begin{array}{rclrclrcl} \mathcal{Z}\overline{P_V^-} & = & \overline{P_U^-}\,\overline{\tilde{Z}_{11}}, & \mathcal{J}\mathcal{Z}^H\mathcal{J}^T\overline{P_U^-} & = & \overline{P_W^-}\,\overline{\tilde{T}_{11}}, & \mathcal{H}\overline{P_V^-} & = & \overline{P_W^-}\,\overline{\tilde{H}_{11}}, \\ \mathcal{Z}\overline{P_V^+} & = & \overline{P_U^+}\,\tilde{Z}_{22}, & \mathcal{J}\mathcal{Z}^H\mathcal{J}^T\overline{P_U^+} & = & \overline{P_W^+}\,\tilde{T}_{22}, & \mathcal{H}\overline{P_V^+} & = & -\overline{P_W^+}\,\tilde{H}_{22}. \end{array} \tag{44}$$

Here, $\tilde{Z}_{kk}$, $\tilde{T}_{kk}$ *and* $\tilde{H}_{kk}$, $k=1,2$, *satisfy* $\Lambda(\tilde{T}_{11}\tilde{Z}_{11},\tilde{H}_{11}) = \Lambda(\tilde{T}_{22}\tilde{Z}_{22},\tilde{H}_{22}) = \Lambda_-(\mathcal{S},\mathcal{H})$.

Proof.
The factorizations in (42) imply that $\mathcal{B}_\mathcal{S}\mathcal{V} = \mathcal{W}\mathcal{R}_\mathcal{T}\mathcal{R}_\mathcal{Z}$ and $\mathcal{B}_\mathcal{H}\mathcal{V} = \mathcal{W}\mathcal{R}_\mathcal{H}$. Comparing the first m columns and making use of the block forms we have

$$\begin{array}{rclrcl} \mathcal{S}V_1 & = & W_1(\mathcal{T}_{11}\mathcal{Z}_{11}), & \mathcal{H}V_1 & = & W_1\mathcal{H}_{11}, \\ \mathcal{S}\overline{V_2} & = & \overline{W_2}\,(\overline{\mathcal{T}_{11}\mathcal{Z}_{11}}), & \mathcal{H}\overline{V_2} & = & -\overline{W_2}\,\overline{\mathcal{H}_{11}}. \end{array} \tag{45}$$

Clearly $\mathrm{range}\, V_1$ and $\mathrm{range}\,\overline{V_2}$ are both deflating subspaces of $\alpha\mathcal{S} - \beta\mathcal{H}$. Since

$$\Lambda_-(\mathcal{S},\mathcal{H}) \subseteq \Lambda_-(\mathcal{B}_\mathcal{S},\mathcal{B}_\mathcal{H}) \subseteq \Lambda(\mathcal{T}_{11}\mathcal{Z}_{11},\mathcal{H}_{11})$$

and $\Lambda(\mathcal{T}_{11}\mathcal{Z}_{11},\mathcal{H}_{11})$ contains no eigenvalue with positive real part, we get

$$\begin{array}{rcll} \mathrm{range}\, V_1 & \subseteq & \mathrm{Def}_-(\mathcal{S},\mathcal{H}) + \mathbb{L}_1, & \mathbb{L}_1 \subseteq \mathrm{Def}_0(\mathcal{S},\mathcal{H}) + \mathrm{Def}_\infty(\mathcal{S},\mathcal{H}), \\ \mathrm{range}\,\overline{V_2} & \subseteq & \mathrm{Def}_+(\mathcal{S},\mathcal{H}) + \mathbb{L}_2, & \mathbb{L}_2 \subseteq \mathrm{Def}_0(\mathcal{S},\mathcal{H}) + \mathrm{Def}_\infty(\mathcal{S},\mathcal{H}). \end{array}$$

We still need to show that

$$\mathrm{Def}_-(\mathcal{S},\mathcal{H}) \subseteq \mathrm{range}\, V_1, \quad \mathrm{Def}_+(\mathcal{S},\mathcal{H}) \subseteq \mathrm{range}\,\overline{V_2}. \tag{46}$$

Let $\tilde{V}_1$ and $\tilde{V}_2$ be full rank matrices whose columns form bases of $\mathrm{Def}_-(\mathcal{S},\mathcal{H})$ and $\mathrm{Def}_+(\mathcal{S},\mathcal{H})$, respectively. It is easy to show that the columns of $\left[\begin{smallmatrix} \tilde{V}_1 & 0 \\ 0 & \overline{\tilde{V}_2} \end{smallmatrix}\right]$ span $\mathrm{Def}_-(\mathcal{B}_\mathcal{S},\mathcal{B}_\mathcal{H})$. This implies that

$$\mathrm{range}\begin{bmatrix} \tilde{V}_1 & 0 \\ 0 & \overline{\tilde{V}_2} \end{bmatrix} \subseteq \mathrm{range}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}.$$

Therefore,

$$\mathrm{range}\begin{bmatrix} \tilde{V}_1 \\ 0 \end{bmatrix}, \ \mathrm{range}\begin{bmatrix} 0 \\ \overline{\tilde{V}_2} \end{bmatrix} \subseteq \mathrm{range}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix},$$

and from this we obtain (46) and hence (43).

If $\Lambda(\mathcal{T}_{11}\mathcal{Z}_{11},\mathcal{H}_{11}) = \Lambda_-(\mathcal{B}_\mathcal{S},\mathcal{B}_\mathcal{H})$, where p is the number of eigenvalues in $\Lambda_-(\mathcal{S},\mathcal{H})$, then from (43) we have $m = 2p$ and

$$\mathrm{range}\, V_1 = \mathrm{Def}_-(\mathcal{S},\mathcal{H}), \quad \mathrm{range}\,\overline{V_2} = \mathrm{Def}_+(\mathcal{S},\mathcal{H}).$$

Hence, $\mathrm{rank}\, V_1 = \mathrm{rank}\, V_2 = p$ and furthermore $\mathcal{T}_{11}$, $\mathcal{Z}_{11}$ and $\mathcal{H}_{11}$ must be nonsingular. Using (45) we get

$$\begin{array}{rcl} \mathcal{H}V_1 & = & \mathcal{S}V_1((\mathcal{T}_{11}\mathcal{Z}_{11})^{-1}\mathcal{H}_{11}), \\ \mathcal{H}\overline{V_2} & = & -\mathcal{S}\overline{V_2}\,(\overline{(\mathcal{T}_{11}\mathcal{Z}_{11})^{-1}\mathcal{H}_{11}}). \end{array}$$

Let $V_1 = [P_V^-, 0]Q_V$ be an RQ decomposition [19] with P_V^- of full column rank. Since $\operatorname{rank} V_1 = p$ we have $\operatorname{rank} P_V^- = p$. Partition $V_2 Q_V^H = [P_V, P_V^+]$ conformally with $V_1 Q_V^H$. Since the columns of $\left[\begin{smallmatrix} V_1 \\ V_2 \end{smallmatrix}\right]$ are orthonormal, we obtain $(P_V^+)^H P_V^+ = I_p$ and hence $\operatorname{rank} P_V^+ = p$. Furthermore, since $\operatorname{rank} V_2 = p$ we have

$$\operatorname{range} P_V \subseteq \operatorname{range} P_V^+ = \operatorname{range} V_2,$$

and using orthonormality, we obtain $P_V = 0$. Therefore, the columns of P_V^- and $\overline{P_V^+}$ form orthogonal bases of $\operatorname{Def}_-(\mathcal{S}, \mathcal{H})$ and $\operatorname{Def}_+(\mathcal{S}, \mathcal{H})$, respectively.

From (42) we have

$$\mathcal{Z} V_1 = U_1 \mathcal{Z}_{11}, \quad \mathcal{J} \mathcal{Z}^H \mathcal{J}^T U_1 = W_1 \mathcal{T}_{11}, \quad \mathcal{H} V_1 = W_1 \mathcal{H}_{11}, \tag{47}$$

and

$$\mathcal{Z} \overline{V_2} = \overline{U_2}\, \overline{\mathcal{Z}_{11}}, \quad \mathcal{J} \mathcal{Z}^H \mathcal{J}^T \overline{U_2} = \overline{W_2}\, \overline{\mathcal{T}_{11}}, \quad \mathcal{H} \overline{V_2} = -\overline{W_2}\, \overline{\mathcal{H}_{11}}. \tag{48}$$

Let $U_1 = [P_U^-, 0]Q_U$ and $W_1 = [P_W^-, 0]Q_W$ be RQ decompositions, with P_U^-, P_W^- of full column rank. Using $V_1 = [P_V^-, 0]Q_V$ and the fact that $\mathcal{Z} P_V^-$, $\mathcal{S} P_V^-$ and $\mathcal{H} P_V^-$ are of full rank (otherwise there would be a zero or infinite eigenvalue associated with the deflating subspace $\operatorname{range} P_V^-$), from the first and third identity in (47) we obtain

$$\operatorname{rank} P_U^- = \operatorname{rank} P_W^- = \operatorname{rank} P_V^- = p.$$

Moreover, setting

$$\tilde{Z} = Q_U \mathcal{Z}_{11} Q_V^H, \qquad \tilde{T} = Q_W \mathcal{T}_{11} Q_U^H, \qquad \tilde{H} = Q_W \mathcal{H}_{11} Q_V^H,$$

we obtain

$$\tilde{Z} = \begin{bmatrix} \tilde{Z}_{11} & 0 \\ \tilde{Z}_{21} & \tilde{Z}_{22} \end{bmatrix}, \quad \tilde{T} = \begin{bmatrix} \tilde{T}_{11} & 0 \\ \tilde{T}_{21} & \tilde{T}_{22} \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} \tilde{H}_{11} & 0 \\ \tilde{H}_{21} & \tilde{H}_{22} \end{bmatrix},$$

where all diagonal blocks are $p \times p$.

Set $U_2 Q_U^H =: [P_U, P_U^+]$, $W_2 Q_W^H =: [P_W, P_W^+]$ and take $V_2 Q_V^H =: [0, P_V^+]$. The block forms of $\tilde{Z}$, $\tilde{T}$ and $\tilde{H}$ together with the first identity of (48) imply that $\overline{P_U}\, \tilde{Z}_{11} = \overline{P_U^+}\, \tilde{Z}_{21}$. Since the columns of $\left[\begin{smallmatrix} U_1 \\ U_2 \end{smallmatrix}\right]$ are orthonormal, we have $(P_U^+)^H P_U^+ = I_p$ and $(P_U^+)^H P_U = 0$. Hence, $\tilde{Z}_{21} = 0$, and consequently $P_U = 0$. Similarly, from the third identity of (48) we get $P_W = 0$, $\tilde{H}_{21} = 0$ and from the second identity we obtain $\tilde{T}_{21} = 0$. Combining all these observations, we obtain

$$\begin{aligned}
\begin{bmatrix} \mathcal{Z} & 0 \\ 0 & \bar{\mathcal{Z}} \end{bmatrix} \begin{bmatrix} P_V^- & 0 \\ 0 & P_V^+ \end{bmatrix} &= \begin{bmatrix} P_U^- & 0 \\ 0 & P_U^+ \end{bmatrix} \begin{bmatrix} \tilde{Z}_{11} & 0 \\ 0 & \tilde{Z}_{22} \end{bmatrix}, \\
\begin{bmatrix} \mathcal{J} \mathcal{Z}^H \mathcal{J}^T & 0 \\ 0 & \overline{\mathcal{J} \mathcal{Z}^H \mathcal{J}^T} \end{bmatrix} \begin{bmatrix} P_U^- & 0 \\ 0 & P_U^+ \end{bmatrix} &= \begin{bmatrix} P_W^- & 0 \\ 0 & P_W^+ \end{bmatrix} \begin{bmatrix} \tilde{T}_{11} & 0 \\ 0 & \tilde{T}_{22} \end{bmatrix}, \\
\begin{bmatrix} \mathcal{H} & 0 \\ 0 & -\bar{\mathcal{H}} \end{bmatrix} \begin{bmatrix} P_V^- & 0 \\ 0 & P_V^+ \end{bmatrix} &= \begin{bmatrix} P_W^- & 0 \\ 0 & P_W^+ \end{bmatrix} \begin{bmatrix} \tilde{H}_{11} & 0 \\ 0 & \tilde{H}_{22} \end{bmatrix},
\end{aligned}$$

which gives (44). $\square$

(We remark that (42) can be constructed from (41) by reordering the eigenvalues properly.)

Theorem 12 gives a way to obtain the stable deflating subspace of a skew-Hamiltonian/Hamiltonian matrix pencil form the deflating subspaces of an embedded skew-Hamiltonian/Hamiltonian matrix pencil of double size. This will be used by the algorithms formulated in the next section.

5 Algorithms

The results of Theorem 12 together with the embedding technique lead to the following algorithm to compute the eigenvalues and the deflating subspaces $\mathrm{Def}_-(\mathcal{S},\mathcal{H})$ and $\mathrm{Def}_+(\mathcal{S},\mathcal{H})$ of a complex skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{S} - \beta\mathcal{H}$.

In summary, Algorithm 1 proposed below transforms a $2n \times 2n$ complex skew-Hamiltonian/Hamiltonian matrix pencil with $\mathcal{J}$-semidefinite skew-Hamiltonian part into a $4n \times 4n$ complex skew-Hamiltonian/Hamiltonian matrix pencil in Schur form. The process passes through intermediate matrix pencils of the following types.

$$
\begin{gathered}
2n \times 2n \text{ complex skew-Hamiltonian/Hamiltonian matrix pencil} \\
\alpha\mathcal{S} - \beta\mathcal{H} \text{ with } \mathcal{S} = \mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z}. \\
\Downarrow \\
\text{Equation (36)} \\
\Downarrow \\
4n \times 4n \text{ real skew-Hamiltonian/skew-Hamiltonian matrix pencil} \\
\alpha\mathcal{B}_{\mathcal{S}}^c - \beta\mathcal{B}_{\mathcal{N}}^c \text{ with } \mathcal{B}_{\mathcal{S}}^c = \mathcal{J}(\mathcal{B}_{\mathcal{Z}}^c)^T\mathcal{J}^T\mathcal{B}_{\mathcal{Z}} \\
\Downarrow \\
\text{Algorithm 3 in Appendix A} \\
\Downarrow \\
4n \times 4n \text{ real skew-Hamiltonian/skew-Hamiltonian matrix pencil in Schur form} \\
\alpha\hat{\mathcal{B}}_{\mathcal{S}}^c - \beta\hat{\mathcal{B}}_{\mathcal{N}}^c \text{ with } \hat{\mathcal{B}}_{\mathcal{S}}^c = \mathcal{J}(\hat{\mathcal{B}}_{\mathcal{Z}}^c)^T\mathcal{J}^T\hat{\mathcal{B}}_{\mathcal{Z}}^c \\
\text{and } \hat{\mathcal{B}}_{\mathcal{Z}}^c = \mathcal{U}^T\mathcal{B}_{\mathcal{Z}}^c\mathcal{Q} = \begin{bmatrix} \mathcal{Z}_{11} & \mathcal{Z}_{12} \\ 0 & \mathcal{Z}_{22} \end{bmatrix} \text{ as in (37) and (38)} \\
\Downarrow \\
\text{Algorithm 4 in Appendix A} \\
\Downarrow \\
4n \times 4n \text{ complex skew-Hamiltonian/skew-Hamiltonian matrix pencil in Schur form} \\
\text{with ordered eigenvalues.}
\end{gathered}
$$

The required deflating subspaces of the original skew-Hamiltonian/Hamiltonian matrix pencil are then obtained from the deflating subspaces of the final $4n \times 4n$ complex skew-Hamiltonian/skew-Hamiltonian matrix pencil. (Unfortunately, if there are non-real eigenvalues, then Algorithm 4 in Appendix A (the eigenvalue sorting algorithm) reintroduces complex entries into the $4n \times 4n$ extended real matrix pencil.)

Algorithm 1 *Given a complex skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{S} - \beta\mathcal{H}$ with $\mathcal{J}$-semidefinite skew-Hamiltonian part $\mathcal{S} = \mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z}$, this algorithm computes the structured Schur form of the extended skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{B}_{\mathcal{S}}^c - \beta\mathcal{B}_{\mathcal{H}}^c$, the eigenvalues of $\alpha\mathcal{S} - \beta\mathcal{H}$, and orthonormal bases of the deflating subspace* $\mathrm{Def}_-(\mathcal{S},\mathcal{H})$ *and the companion subspace* range P_U^-.

Input: Hamiltonian matrix $\mathcal{H}$ and the factor $\mathcal{Z}$ of $\mathcal{S}$.

Output: P_V^-, P_U^- as defined in Theorem 12.

Step 1:

Set $\mathcal{N} = i\mathcal{H}$ and form matrices $\mathcal{B}_{\mathcal{Z}}^c$, $\mathcal{B}_{\mathcal{N}}^c$ as in (34) and (30), respectively. Find the structured Schur form of the skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{B}_{\mathcal{S}}^{c} - \beta\mathcal{B}_{\mathcal{N}}^{c}$ using Algorithm 3 in Appendix A to compute the factorization

$$\begin{aligned}\tilde{\mathcal{B}}_{\mathcal{Z}}^{c} &= \mathcal{U}^{T}\mathcal{B}_{\mathcal{Z}}^{c}\mathcal{Q} = \begin{bmatrix} \mathcal{Z}_{11} & \mathcal{Z}_{12} \\ 0 & \mathcal{Z}_{22} \end{bmatrix}, \\ \tilde{\mathcal{B}}_{\mathcal{N}}^{c} &= \mathcal{J}\mathcal{Q}^{T}\mathcal{J}^{T}\mathcal{B}_{\mathcal{N}}^{c}\mathcal{Q} = \begin{bmatrix} \mathcal{N}_{11} & \mathcal{N}_{12} \\ 0 & \mathcal{N}_{11}^{T} \end{bmatrix},\end{aligned}$$

where $\mathcal{Q}$ is real orthogonal, $\mathcal{U}$ is real orthogonal symplectic, $\mathcal{Z}_{11}$, $\mathcal{Z}_{22}^{T}$ are upper triangular and $\mathcal{N}_{11}$ is quasi upper triangular.

Step 2:

Reorder the eigenvalues using Algorithm 4 in Appendix A to determine a unitary matrix $\tilde{\mathcal{Q}}$ and a unitary symplectic matrix $\tilde{\mathcal{U}}$ such that

$$\begin{aligned}\tilde{\mathcal{U}}^{H}\tilde{\mathcal{B}}_{\mathcal{Z}}^{c}\tilde{\mathcal{Q}} &= \begin{bmatrix} \tilde{\mathcal{Z}}_{11} & \tilde{\mathcal{Z}}_{12} \\ 0 & \tilde{\mathcal{Z}}_{22} \end{bmatrix}, \\ \mathcal{J}\tilde{\mathcal{Q}}^{H}\mathcal{J}^{T}(-i\tilde{\mathcal{B}}_{\mathcal{N}}^{c})\tilde{\mathcal{Q}} &= \begin{bmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ 0 & -\mathcal{H}_{11}^{H} \end{bmatrix},\end{aligned}$$

with $\tilde{\mathcal{Z}}_{11}$, $\tilde{\mathcal{Z}}_{22}^{H}$, $\mathcal{H}_{11}$ upper triangular such that $\Lambda_{-}(\mathcal{J}(\tilde{\mathcal{B}}_{\mathcal{Z}}^{c})^{H}\mathcal{J}^{T}\tilde{\mathcal{B}}_{\mathcal{Z}}^{c}, -i\tilde{\mathcal{B}}_{\mathcal{N}}^{c})$ is contained in the spectrum of the $2p \times 2p$ leading principal subpencil of $\alpha\tilde{\mathcal{Z}}_{22}^{H}\tilde{\mathcal{Z}}_{11} - \beta\mathcal{H}_{11}$.

Step 3:

Set $V = [I_{2n}, 0]\mathcal{X}_{c}\mathcal{Q}\tilde{\mathcal{Q}}\left[\begin{smallmatrix} I_{2p} \\ 0 \end{smallmatrix}\right]$, $U = [I_{2n}, 0]\mathcal{X}_{c}\mathcal{U}\tilde{\mathcal{U}}\left[\begin{smallmatrix} I_{2p} \\ 0 \end{smallmatrix}\right]$ (where $\mathcal{X}_{c}$ is as in (29)) and compute P_{V}^{-}, P_{U}^{-}, orthogonal bases of range V and range U, respectively, using any numerically stable orthogonalization scheme.

End

A detailed flop count shows that this algorithm needs approximately the same number of flops as the periodic QZ algorithm [10, 21] applied to $\alpha\mathcal{J}\mathcal{Z}^{H}\mathcal{J}^{T}\mathcal{Z} - \beta\mathcal{H}$.

If $\mathcal{S}$ is not factored, then the algorithm can be simplified by using the method of [39] to compute the real skew-Hamiltonian/skew-Hamiltonian Schur form of $\alpha\mathcal{B}_{\mathcal{S}}^{c} - \beta\mathcal{B}_{\mathcal{H}}^{c}$ directly.

Algorithm 2 *Given a complex skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{S} - \beta\mathcal{H}$. This algorithm computes the structured Schur form of the extended skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{B}_{\mathcal{S}}^{c} - \beta\mathcal{B}_{\mathcal{H}}^{c}$, the eigenvalues of $\alpha\mathcal{S} - \beta\mathcal{H}$, and an orthogonal basis of the deflating subspace* $\mathrm{Def}_{-}(\mathcal{S}, \mathcal{H})$.

Input: A complex skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{S} - \beta\mathcal{H}$, with $\mathcal{S} \in \mathbb{SH}_{2n}$, $\mathcal{H} \in \mathbb{H}_{2n}$.

Output: P_{V}^{-} as defined in Theorem 12.

Step 1:

Set $\mathcal{N} = i\mathcal{H}$ and form the matrices $\mathcal{B}_{\mathcal{S}}^{c}$, $\mathcal{B}_{\mathcal{N}}^{c}$ as in (35) and (30), respectively.

Find the structured Schur form of the skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{B}_{\mathcal{S}}^c - \beta\mathcal{B}_{\mathcal{N}}^c$ using Algorithm 5 in Appendix A to compute the factorization

$$\begin{aligned}
\tilde{\mathcal{B}}_{\mathcal{S}}^c &= \mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{B}_{\mathcal{S}}^c\mathcal{Q} = \begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} \\ 0 & \mathcal{S}_{11}^T \end{bmatrix}, \\
\tilde{\mathcal{B}}_{\mathcal{N}}^c &= \mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{B}_{\mathcal{N}}^c\mathcal{Q} = \begin{bmatrix} \mathcal{N}_{11} & \mathcal{N}_{12} \\ 0 & \mathcal{N}_{11}^T \end{bmatrix},
\end{aligned}$$

where $\mathcal{Q}$ is real orthogonal, $\mathcal{S}_{11}$ is upper triangular and $\mathcal{N}_{11}$ is quasi upper triangular.

Step 2:

Reorder the eigenvalues using Algorithm 6 in Appendix A to determine a unitary matrix $\tilde{\mathcal{Q}}$ such that

$$\begin{aligned}
\mathcal{J}\tilde{\mathcal{Q}}^H\mathcal{J}^T\tilde{\mathcal{B}}_{\mathcal{S}}^c\tilde{\mathcal{Q}} &= \begin{bmatrix} \tilde{\mathcal{S}}_{11} & \tilde{\mathcal{S}}_{12} \\ 0 & \tilde{\mathcal{S}}_{11}^H \end{bmatrix}, \\
\mathcal{J}\tilde{\mathcal{Q}}^H\mathcal{J}^T(-i\tilde{\mathcal{B}}_{\mathcal{N}}^c)\tilde{\mathcal{Q}} &= \begin{bmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ 0 & -\mathcal{H}_{11}^H \end{bmatrix},
\end{aligned}$$

with $\tilde{\mathcal{S}}_{11}$, $\mathcal{H}_{11}$ upper triangular and such that $\Lambda_-(\tilde{\mathcal{B}}_{\mathcal{S}}^c, -i\tilde{\mathcal{B}}_{\mathcal{N}}^c)$ is contained in the spectrum of the $2p \times 2p$ leading principal subpencil of $\alpha\tilde{\mathcal{S}}_{11} - \beta\mathcal{H}_{11}$.

Step 3:

Set $V = [I_{2n}, 0]\mathcal{X}_c\mathcal{Q}\tilde{\mathcal{Q}}\begin{bmatrix} I_{2p} \\ 0 \end{bmatrix}$ (where $\mathcal{X}_c$ is as in (29)) and compute P_V^-, the orthogonal basis of range V, using any numerically stable orthogonalization scheme.

End

The algorithm can also compute range P_U^- if $\mathcal{S} = \mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z}$, by computing the orthogonal basis of $\mathcal{Z}P_V^-$. However, if $\mathcal{Z}$ is near singular, then in finite arithmetic, the isotropy of the subspace, i.e., that $(P_U^-)^H\mathcal{J}P_U^- = 0$, may be lost or poor.

Algorithm 2 needs roughly 60% of the flops required by the QZ algorithm applied to $\alpha\mathcal{S} - \beta\mathcal{H}$ as suggested in [45].

In this section we have presented (complex) structured triangular forms and numerical algorithms for the computation of these forms. In the next section we give an error analysis for these methods. The analysis is a generalization of the analysis for Hamiltonian matrices in [5, 6, 7].

6 Error and Perturbation Analysis

In this section we will give the perturbation analysis for eigenvalues and deflating subspaces of skew-Hamiltonian/Hamiltonian matrix pencils. Variables marked with a circumflex denote perturbed quantities.

We begin with the perturbation analysis for the eigenvalues of $\alpha\mathcal{S} - \beta\mathcal{H}$ and $\alpha\mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z} - \beta\mathcal{H}$. In principle, we could multiply out $\mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z}$ and apply the classical perturbation analysis of matrix pencils using the chordal metric [44], but this may give pessimistic bounds and would display neither the effects of perturbing each factor separately nor the effects of structured perturbations. Therefore, we make use of the perturbation analysis for formal products of matrices developed in [8].

If Algorithm 2 is applied to the skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{S} - \beta\mathcal{H}$, then we compute the structured Schur form of the extended skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{B}^c_{\mathcal{S}} - \beta\mathcal{B}^c_{\mathcal{H}}$. The well-known backward error analysis of orthogonal matrix computations implies that rounding errors in Algorithm 2 are equivalent to perturbing $\alpha\mathcal{B}^c_{\mathcal{S}} - \beta\mathcal{B}^c_{\mathcal{H}}$ to a nearby matrix pencil $\alpha\hat{\mathcal{B}}^c_{\mathcal{S}} - \beta\hat{\mathcal{B}}^c_{\mathcal{H}}$, where

$$\begin{aligned} \hat{\mathcal{B}}^c_{\mathcal{S}} &= \mathcal{B}^c_{\mathcal{S}} + \mathcal{E}_{\mathcal{S}}, && (49)\\ \hat{\mathcal{B}}^c_{\mathcal{H}} &= \mathcal{B}^c_{\mathcal{H}} + \mathcal{E}_{\mathcal{H}}, && (50) \end{aligned}$$

with $\mathcal{E}_{\mathcal{S}} \in \mathbb{SH}_{4n}$, $\mathcal{E}_{\mathcal{H}} \in \mathbb{H}_{4n}$ and

$$\begin{aligned} \|\mathcal{E}_{\mathcal{S}}\|_2 &< c_{\mathcal{S}}\varepsilon \|\mathcal{B}^c_{\mathcal{S}}\|_2, && (51)\\ \|\mathcal{E}_{\mathcal{H}}\|_2 &< c_{\mathcal{H}}\varepsilon \|\mathcal{B}^c_{\mathcal{H}}\|_2. && (52) \end{aligned}$$

Here ε is the unit round of the floating point arithmetic and $c_{\mathcal{S}}$ and $c_{\mathcal{H}}$ are modest constants depending on the details of the implementation and arithmetic. Let x and y be unit norm vectors such that

$$\mathcal{H}x = \alpha_1 y, \quad \mathcal{S}x = \beta_1 y, \tag{53}$$

and let $\lambda = \alpha_1/\beta_1$ be a simple eigenvalue of $\alpha\mathcal{S} - \beta\mathcal{H}$. If λ is finite and $\operatorname{Re}\lambda \neq 0$, then $-\bar{\lambda}$ is also a simple eigenvalue of $\alpha\mathcal{S} - \beta\mathcal{H}$. Let u, v be unit norm vectors such that

$$\mathcal{H}u = \alpha_2 v, \quad \mathcal{S}u = \beta_2 v, \tag{54}$$

and $\alpha_2/\beta_2 = -\bar{\lambda}$. Then we have

$$-\bar{\mathcal{H}}\bar{u} = -\bar{\alpha}_2\bar{v}, \quad \bar{\mathcal{S}}\bar{u} = \bar{\beta}_2\bar{v}. \tag{55}$$

Using the equivalence of the matrix pencils $\alpha\mathcal{B}^c_{\mathcal{S}} - \beta\mathcal{B}^c_{\mathcal{H}}$ and $\alpha\mathcal{B}_{\mathcal{S}} - \beta\mathcal{B}_{\mathcal{H}}$, and setting

$$\mathcal{U}_1 = \mathcal{X}_c^H \begin{bmatrix} y & 0 \\ 0 & \bar{v} \end{bmatrix}, \qquad \mathcal{U}_2 = \mathcal{X}_c^H \begin{bmatrix} x & 0 \\ 0 & \bar{u} \end{bmatrix}, \tag{56}$$

we obtain from (53) and (55) that

$$\mathcal{B}^c_{\mathcal{H}}\mathcal{U}_2 = \mathcal{U}_1 \begin{bmatrix} \alpha_1 & 0 \\ 0 & -\bar{\alpha}_2 \end{bmatrix}, \quad \mathcal{B}^c_{\mathcal{S}}\mathcal{U}_2 = \mathcal{U}_1 \begin{bmatrix} \beta_1 & 0 \\ 0 & \bar{\beta}_2 \end{bmatrix},$$

which implies that λ is a double eigenvalue of $\alpha\mathcal{B}^c_{\mathcal{S}} - \beta\mathcal{B}^c_{\mathcal{H}}$ with a complete set of linearly independent eigenvectors. Similarly, $-\bar{\lambda}$ is a double eigenvalue of $\alpha\mathcal{B}^c_{\mathcal{S}} - \beta\mathcal{B}^c_{\mathcal{H}}$ with a complete set of linearly independent eigenvectors and

$$\mathcal{B}^c_{\mathcal{H}}\mathcal{V}_2 = \mathcal{V}_1 \begin{bmatrix} \alpha_2 & 0 \\ 0 & -\bar{\alpha}_1 \end{bmatrix}, \quad \mathcal{B}^c_{\mathcal{S}}\mathcal{V}_2 = \mathcal{V}_1 \begin{bmatrix} \beta_2 & 0 \\ 0 & \bar{\beta}_1 \end{bmatrix},$$

where

$$\mathcal{V}_1 = \mathcal{X}_c^H \begin{bmatrix} v & 0 \\ 0 & \bar{y} \end{bmatrix}, \qquad \mathcal{V}_2 = \mathcal{X}_c^H \begin{bmatrix} u & 0 \\ 0 & \bar{x} \end{bmatrix}. \tag{57}$$

Note that the finite eigenvalues with non-zero real part appear in pairs as in (53) and (54), but infinite and purely imaginary eigenvalues may not appear in pairs. Consequently, in the following perturbation theorem, the bounds for purely imaginary and infinite eigenvalues are different from the bounds for finite eigenvalues with non-zero real part.

Theorem 13 *Consider the skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{S}-\beta\mathcal{H}$ along with the corresponding extended matrix pencils $\alpha\mathcal{B}_{\mathcal{S}}^c-\beta\mathcal{B}_{\mathcal{H}}^c=\mathcal{X}_c^H(\alpha\mathcal{B}_{\mathcal{S}}-\beta\mathcal{B}_{\mathcal{H}})\mathcal{X}_c$, where $\mathcal{B}_{\mathcal{S}}$ is given by (33), $\mathcal{B}_{\mathcal{H}}$ by (40), $\mathcal{X}_c$ by (29) and $\mathcal{B}_{\mathcal{S}}^c$ by (35). Let $\alpha\hat{\mathcal{B}}_{\mathcal{S}}^c-\beta\hat{\mathcal{B}}_{\mathcal{H}}^c$ be a perturbed extended matrix pencil satisfying (49)–(52) with constants $c_{\mathcal{H}}$, $c_{\mathcal{S}}$ and let ε be equal to the unit round of the floating point arithmetic.*

If λ is a simple eigenvalue of $\alpha\mathcal{S}-\beta\mathcal{H}$ with vectors x and y as in (53) and vectors u and v as in (54), then the corresponding double eigenvalue of $\alpha\mathcal{B}_{\mathcal{S}}^c-\beta\mathcal{B}_{\mathcal{H}}^c$ may split into two eigenvalues $\hat\lambda_1$ and $\hat\lambda_2$ of the perturbed matrix pencil $\alpha\hat{\mathcal{B}}_{\mathcal{S}}^c-\beta\hat{\mathcal{B}}_{\mathcal{H}}^c$, each of which satisfies the following bounds.

1. *If λ is finite and $\operatorname{Re}\lambda\neq 0$, then*
$$\left|\frac{\hat\lambda_k-\lambda}{\lambda}\right|\leq\frac{\varepsilon}{|u^H\mathcal{J}y|}\left(\frac{c_{\mathcal{H}}}{|\alpha_1|}\|\mathcal{H}\|_2+\frac{c_{\mathcal{S}}}{|\beta_1|}\|\mathcal{S}\|_2\right)+O(\varepsilon^2),\qquad k=1,\,2.$$

2. *If λ is finite and $\operatorname{Re}\lambda=0$, then*
$$|\hat\lambda_k-\lambda|\leq\frac{\varepsilon}{|\beta_1||x^H\mathcal{J}y|}(c_{\mathcal{H}}\|\mathcal{H}\|_2+c_{\mathcal{S}}|\lambda|\,\|\mathcal{S}\|_2)+O(\varepsilon^2),\qquad k=1,\,2.$$

3. *If $\lambda=\infty$, then*
$$\frac{1}{|\hat\lambda_k|}\leq\varepsilon\frac{c_{\mathcal{S}}\|\mathcal{S}\|_2}{|\alpha_1||x^H\mathcal{J}y|}+O(\varepsilon^2),\qquad k=1,\,2.$$

Proof. We first consider the case that λ is finite and $\operatorname{Re}\lambda\neq 0$. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be defined by (56) and $\mathcal{V}_1$ and $\mathcal{V}_2$ be defined by (57). Using the perturbation theory for formal products of matrices (see [8]), we obtain

$$\begin{aligned}\left|\frac{\hat\lambda-\lambda}{\lambda}\right| &\leq \min\Bigg(\left\|(\mathcal{V}_2^H\mathcal{J}\mathcal{U}_1C_{\mathcal{S}})^{-1}\mathcal{V}_2^H\mathcal{J}(\frac{1}{\lambda}\mathcal{E}_{\mathcal{H}}-\mathcal{E}_{\mathcal{S}})\mathcal{U}_2\right\|_2,\\ &\qquad\left\|(\mathcal{V}_2^H\mathcal{J}\mathcal{U}_1)^{-1}\mathcal{V}_2^H\mathcal{J}(\frac{1}{\lambda}\mathcal{E}_{\mathcal{H}}-\mathcal{E}_{\mathcal{S}})\mathcal{U}_2C_{\mathcal{S}}^{-1}\right\|_2\Bigg)+O(\varepsilon^2).\end{aligned}$$

Here, $C_{\mathcal{S}}=\begin{bmatrix}\beta_1&0\\0&\bar\beta_2\end{bmatrix}$ and $\mathcal{V}_2^H\mathcal{J}\mathcal{U}_1=\begin{bmatrix}u&0\\0&\bar x\end{bmatrix}^H\mathcal{X}_c\mathcal{J}\mathcal{X}_c^H\begin{bmatrix}y&0\\0&\bar v\end{bmatrix}=\begin{bmatrix}u^H\mathcal{J}y&0\\0&x^T\mathcal{J}\bar v\end{bmatrix}$. The second equation in (54) implies $u^H\mathcal{J}\mathcal{S}=\bar\beta_2v^H\mathcal{J}$. Combining this with the second equation of (53) we get $\bar\beta_2v^H\mathcal{J}x=\beta_1u^H\mathcal{J}y$. Hence,

$$\begin{aligned}\left|\frac{\hat\lambda-\lambda}{\lambda}\right| &\leq \left\|(\mathcal{V}_2^H\mathcal{J}\mathcal{U}_1C_{\mathcal{S}})^{-1}\mathcal{V}_2^H\mathcal{J}(\frac{1}{\lambda}\mathcal{E}_{\mathcal{H}}-\mathcal{E}_{\mathcal{S}})\mathcal{U}_2\right\|_2+O(\varepsilon^2)\\ &\leq \|(\mathcal{V}_2^H\mathcal{J}\mathcal{U}_1C_{\mathcal{S}})^{-1}\|_2\left\|\frac{1}{\lambda}\mathcal{E}_{\mathcal{H}}-\mathcal{E}_{\mathcal{S}}\right\|_2+O(\varepsilon^2)\\ &\leq \frac{1}{|u^H\mathcal{J}y|}\left(\frac{\|\mathcal{E}_{\mathcal{H}}\|_2}{|\beta_1\lambda|}+\frac{\|\mathcal{E}_{\mathcal{S}}\|_2}{|\beta_1|}\right)+O(\varepsilon^2)\\ &\leq \frac{\varepsilon}{|u^H\mathcal{J}y|}\left(\frac{c_{\mathcal{H}}}{|\alpha_1|}\|\mathcal{H}\|_2+\frac{c_{\mathcal{S}}}{|\beta_1|}\|\mathcal{S}\|_2\right)+O(\varepsilon^2).\end{aligned}$$

If λ is purely imaginary or infinite, then the bounds are obtained by adapting the classical perturbation theory in [44] to a formal product of matrices (for details see [8]) and by replacing

(55) with $-\bar{\mathcal{H}}\bar{x} = -\bar{\alpha}_1\bar{y}$ and $\bar{\mathcal{S}}\bar{x} = \bar{\beta}_1\bar{y}$ as well as replacing u, v, α_2 and β_2 by x, y, α_1 and β_1, respectively. □

The bound in part 1 appears to involve only u, y, α_1 and β_1 but not v, x, α_2 and β_2. However, note in the proof that $\bar{\beta}_2 v^H \mathcal{J} x = \beta_1 u^H \mathcal{J} y$, so the bound implicitly involves all the parameters.

If $\mathcal{S}$ is given in factored form, Algorithm 1 computes the triangular form of the perturbed matrix pencil

$$\alpha\hat{\mathcal{B}}^c_{\mathcal{Z}} - \beta\hat{\mathcal{B}}^c_{\mathcal{H}} = \alpha(\mathcal{B}^c_{\mathcal{Z}} + \mathcal{E}_{\mathcal{Z}}) - \beta(\mathcal{B}^c_{\mathcal{H}} + \mathcal{E}_{\mathcal{H}}) \tag{58}$$

where

$$\|\mathcal{E}_{\mathcal{Z}}\|_2 \le c_{\mathcal{Z}}\varepsilon \|\mathcal{B}^c_{\mathcal{Z}}\|_2, \quad \|\mathcal{E}_{\mathcal{H}}\|_2 \le c_{\mathcal{H}}\varepsilon \|\mathcal{B}^c_{\mathcal{H}}\|_2 \tag{59}$$

and ε is the machine precision and $c_{\mathcal{Z}}$ and $c_{\mathcal{H}}$ are constants. The eigenvalue perturbation bounds then are essentially the same as in Theorem 13.

Theorem 14 *Consider the skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{S} - \beta\mathcal{H}$ with $\mathcal{J}$-semidefinite skew-Hamiltonian part $\mathcal{S} = \mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z}$. Let $\alpha\mathcal{B}^c_{\mathcal{S}} - \beta\mathcal{B}^c_{\mathcal{H}} = \mathcal{X}_c^H(\alpha\mathcal{B}_{\mathcal{S}} - \beta\mathcal{B}_{\mathcal{H}})\mathcal{X}_c$ be the corresponding extended matrix pencils, where $\mathcal{B}^c_{\mathcal{S}} = \mathcal{J}(\mathcal{B}^c_{\mathcal{Z}})^H\mathcal{J}^T\mathcal{B}^c_{\mathcal{Z}}$, $\mathcal{B}_{\mathcal{Z}}$ is given by (31), $\mathcal{B}_{\mathcal{H}}$ by (39), and $\mathcal{X}_c$ by (29). Let $\alpha\hat{\mathcal{B}}^c_{\mathcal{S}} - \beta\hat{\mathcal{B}}^c_{\mathcal{H}}$ be the perturbed extended matrix pencil in (49)–(52) with constants $c_{\mathcal{H}}$, $c_{\mathcal{S}}$ and let ε be equal to the unit round of the floating point arithmetic.*

Let λ be a simple eigenvalue of $\alpha\mathcal{S} - \beta\mathcal{H} = \alpha\mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z} - \beta\mathcal{H}$ with $\operatorname{Re}\lambda \ne 0$, and let x, y, z, u, v, w be unit norm vectors such that

$$\mathcal{J}\mathcal{Z}^H\mathcal{J}^T x = \alpha_1 y, \quad \mathcal{H}z = \beta_1 y, \quad \mathcal{Z}z = \gamma_1 x, \tag{60}$$

with $\lambda = \frac{\beta_1}{\alpha_1\gamma_1}$, and

$$\mathcal{J}\mathcal{Z}^H\mathcal{J}^T u = \alpha_2 v, \quad \mathcal{H}w = \beta_2 v, \quad \mathcal{Z}w = \gamma_2 u, \tag{61}$$

with $-\bar{\lambda} = \frac{\beta_2}{\alpha_2\gamma_2}$.

The corresponding double eigenvalue of $\alpha\mathcal{B}^c_{\mathcal{S}} - \beta\mathcal{B}^c_{\mathcal{H}}$ may split into two eigenvalues $\hat{\lambda}_1$ and $\hat{\lambda}_2$ of the perturbed matrix pencil $\alpha\hat{\mathcal{B}}^c_{\mathcal{S}} - \beta\hat{\mathcal{B}}^c_{\mathcal{H}}$, each of which satisfies the following bounds.

1. *If λ is finite and $\operatorname{Re}\lambda \ne 0$, then*

$$\left|\frac{\hat{\lambda} - \lambda}{\lambda}\right| \le \varepsilon\left(\frac{c_{\mathcal{H}}}{|\beta_1 w^H\mathcal{J}y|}\|\mathcal{H}\|_2 + 2\frac{c_{\mathcal{Z}}}{\min\{|\gamma_1 u^H\mathcal{J}x|, |\alpha_1 w^H\mathcal{J}y|\}}\|\mathcal{Z}\|_2\right) + O(\varepsilon^2).$$

2. *If λ is purely imaginary, then*

$$|\hat{\lambda} - \lambda| \le \varepsilon\left(\frac{c_{\mathcal{H}}}{|\alpha_1\gamma_1 y^H\mathcal{J}z|}\|\mathcal{H}\|_2 + \frac{2|\lambda|c_{\mathcal{Z}}}{|\gamma_1 u^H\mathcal{J}x|}\|\mathcal{Z}\|_2\right) + O(\varepsilon^2).$$

3. *If $\lambda = \infty$, then $|\hat{\lambda}|^{-1} = O(\varepsilon^2)$.*

Proof. The perturbation analysis follows [8]. If λ is finite and $\operatorname{Re}\lambda \ne 0$, then

$$\left|\frac{\hat{\lambda} - \lambda}{\lambda}\right| \le \left\|(\mathcal{V}_2^H\mathcal{J}\mathcal{U}_3)^{-1}(\tilde{C}_1\tilde{C}_3)^{-1}(\mathcal{V}_3^H\mathcal{E}_{\mathcal{Z}}^H\mathcal{J}\mathcal{U}_1 C_3 + \tilde{C}_3^H\mathcal{U}_1^H\mathcal{J}\mathcal{E}_{\mathcal{Z}}\mathcal{U}_3 - \frac{1}{\lambda}\mathcal{V}_3^H\mathcal{J}\mathcal{E}_{\mathcal{H}}\mathcal{U}_3)\right\|_2 + O(\varepsilon^2).$$

From $\mathcal{V}_2^H \mathcal{J} \mathcal{U}_3 = \begin{bmatrix} v^H \mathcal{J} z & 0 \\ 0 & y^T \mathcal{J} \bar{w} \end{bmatrix}$, it follows that

$$\left| \frac{\hat{\lambda} - \lambda}{\lambda} \right| \leq \frac{\max\{|\gamma_1|, |\gamma_2|\} \, \|\mathcal{E}_{\mathcal{Z}}\|_2 + \frac{1}{|\lambda|} \|\mathcal{E}_{\mathcal{H}}\|_2}{\min\{|\bar{\alpha}_2 \bar{\gamma}_2 v^H \mathcal{J} z|, |\alpha_1 \gamma_1 w^H \mathcal{J} y|\}} + \frac{\|\mathcal{E}_{\mathcal{Z}}\|_2}{\min\{|\bar{\alpha}_2 v^H \mathcal{J} z|, |\alpha_1 w^H \mathcal{J} y|\}} + O(\varepsilon^2).$$

From (60) and (61), we also have

$$\bar{\alpha}_2 v^H \mathcal{J} z = \gamma_1 u^H \mathcal{J} x, \quad \bar{\gamma}_2 u^H \mathcal{J} x = \alpha_1 w^H \mathcal{J} y, \quad \bar{\beta}_2 v^H \mathcal{J} z = -\beta_1 w^H \mathcal{J} y. \tag{62}$$

It follows that

$$|\bar{\alpha}_2 \bar{\gamma}_2 v^H \mathcal{J} z| = |\bar{\gamma}_2 \gamma_1 u^H \mathcal{J} x| = |\gamma_1 \alpha_1 w^H \mathcal{J} y|.$$

Hence,

$$\begin{aligned} \frac{\max\{|\gamma_1|, |\gamma_2|\}}{\min\{|\bar{\alpha}_2 \bar{\gamma}_2 v^H \mathcal{J} z|, |\alpha_1 \gamma_1 w^H \mathcal{J} y|\}} &= \frac{1}{\min\{|\bar{\alpha}_2 v^H \mathcal{J} z|, |\alpha_1 w^H \mathcal{J} y|\}}, \\ |\lambda| \min\{|\bar{\alpha}_2 \bar{\gamma}_2 v^H \mathcal{J} z|, |\alpha_1 \gamma_1 w^H \mathcal{J} y|\} &= |\beta_1 w^H \mathcal{J} y|, \end{aligned}$$

and

$$\left| \frac{\hat{\lambda} - \lambda}{\lambda} \right| \leq \varepsilon \left(\frac{c_{\mathcal{H}}}{|\beta_1 w^H \mathcal{J} y|} \|\mathcal{H}\|_2 + \frac{2 c_{\mathcal{Z}}}{\min\{|\bar{\alpha}_2 v^H \mathcal{J} z|, |\alpha_1 w^H \mathcal{J} y|\}} \|\mathcal{Z}\|_2 \right) + O(\varepsilon^2).$$

Equation (62) implies that $\bar{\alpha}_2 v^H \mathcal{J} z = \gamma_1 u^H \mathcal{J} x$. The first part of the theorem follows.

If λ is purely imaginary, the proof is analogous.

If $\lambda = \infty$, then $\alpha_1 = 0$ or $\gamma_1 = 0$. Using the first equation of (62), we have $\bar{\alpha}_1 y^H \mathcal{J} z = \gamma_1 x^H \mathcal{J} x$, where we have replaced u, v and α_2 by x, y and α_1, respectively. Since λ is simple, i.e., $y^H \mathcal{J} z \neq 0$ and $x^H \mathcal{J} x \neq 0$, we have $\alpha_1 = \gamma_1 = 0$ and hence, $C_1 = C_3 = 0$. Using the same argument as in Theorem 13 gives $\frac{1}{\hat{\lambda}} = O(\varepsilon^2)$. $\square$

To study the perturbations in the computed deflating subspaces we need to study the perturbations for the extended matrix pencil in more detail. As mentioned before, by applying Algorithm 2 to $\alpha \mathcal{B}_{\mathcal{S}}^c - \beta \mathcal{B}_{\mathcal{H}}^c$ we actually compute a unitary matrix $\hat{\mathcal{Q}}$ such that

$$\begin{aligned} \mathcal{J} \hat{\mathcal{Q}}^H \mathcal{J}^T (\alpha \hat{\mathcal{B}}_{\mathcal{S}}^c - \beta \hat{\mathcal{B}}_{\mathcal{H}}^c) \hat{\mathcal{Q}} &= \alpha \hat{\mathcal{R}}_{\mathcal{S}} - \beta \hat{\mathcal{R}}_{\mathcal{H}} \\ &=: \alpha \begin{bmatrix} \hat{\mathcal{S}}_{11} & \hat{\mathcal{S}}_{12} \\ 0 & \hat{\mathcal{S}}_{11}^H \end{bmatrix} - \beta \begin{bmatrix} \hat{\mathcal{H}}_{11} & \hat{\mathcal{H}}_{12} \\ 0 & -\hat{\mathcal{H}}_{11}^H \end{bmatrix}, \end{aligned} \tag{63}$$

where $\hat{\mathcal{B}}_{\mathcal{S}}^c$ and $\hat{\mathcal{B}}_{\mathcal{H}}^c$ are defined in (49) and (50), and $\Lambda(\hat{\mathcal{S}}_{11}, \hat{\mathcal{H}}_{11}) = \Lambda_-(\hat{\mathcal{B}}_{\mathcal{S}}^c, \hat{\mathcal{B}}_{\mathcal{H}}^c)$. If we assume that the matrix pencil $\alpha \mathcal{S} - \beta \mathcal{H}$ has no purely imaginary eigenvalues, then by Theorem 3 there exist unitary matrices $\mathcal{Q}_1$, $\mathcal{Q}_2$ such that

$$\mathcal{J} \mathcal{Q}_1^H \mathcal{J}^T (\alpha \mathcal{S} - \beta \mathcal{H}) \mathcal{Q}_1 = \alpha \begin{bmatrix} S_{11}^- & S_{12}^- \\ 0 & (S_{11}^-)^H \end{bmatrix} - \beta \begin{bmatrix} H_{11}^- & H_{12}^- \\ 0 & -(H_{11}^-)^H \end{bmatrix}$$

with $\Lambda(S_{11}^-, H_{11}^-) = \Lambda_-(\mathcal{S}, \mathcal{H})$, and

$$\mathcal{J} \mathcal{Q}_2^H \mathcal{J}^T (\alpha \mathcal{S} - \beta \mathcal{H}) \mathcal{Q}_2 = \alpha \begin{bmatrix} S_{11}^+ & S_{12}^+ \\ 0 & (S_{11}^+)^H \end{bmatrix} - \beta \begin{bmatrix} H_{11}^+ & H_{12}^+ \\ 0 & -(H_{11}^+)^H \end{bmatrix}$$

with $\Lambda(S_{11}^{+}, H_{11}^{+}) = \Lambda_{+}(\mathcal{S}, \mathcal{H})$, respectively. Set $\mathcal{Q} = \mathcal{X}_c^H \operatorname{diag}(\mathcal{Q}_1, \bar{\mathcal{Q}}_2)\mathcal{P}$ with $\mathcal{P}$ and $\mathcal{X}_c$ as in (28) and (29). Then $\mathcal{Q}$ is unitary and

$$
\begin{aligned}
&\mathcal{J}\mathcal{Q}^H\mathcal{J}^T(\alpha\mathcal{B}_{\mathcal{S}}^c - \beta\mathcal{B}_{\mathcal{H}}^c)\mathcal{Q} \\
&= \alpha\left[\begin{array}{cc|cc} S_{11}^{-} & 0 & S_{12}^{-} & 0 \\ 0 & \overline{S_{11}^{+}} & 0 & \overline{S_{12}^{+}} \\ \hline 0 & 0 & (S_{11}^{-})^H & 0 \\ 0 & 0 & 0 & (\overline{S_{11}^{+}})^H \end{array}\right] - \beta\left[\begin{array}{cc|cc} H_{11}^{-} & 0 & H_{12}^{-} & 0 \\ 0 & -\overline{H_{11}^{+}} & 0 & -\overline{H_{12}^{+}} \\ \hline 0 & 0 & -(H_{11}^{-})^H & 0 \\ 0 & 0 & 0 & (\overline{H_{11}^{+}})^H \end{array}\right] \\
&=: \alpha\begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} \\ 0 & \mathcal{S}_{11}^H \end{bmatrix} - \beta\begin{bmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ 0 & -\mathcal{H}_{11}^H \end{bmatrix} \qquad (64) \\
&=: \alpha\mathcal{R}_{\mathcal{S}} - \beta\mathcal{R}_{\mathcal{H}}.
\end{aligned}
$$

This is the structured Schur form of the extended skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{B}_{\mathcal{S}}^c - \beta\mathcal{B}_{\mathcal{H}}^c$. Moreover, $\Lambda(\mathcal{S}_{11}, \mathcal{H}_{11}) = \Lambda_{-}(\mathcal{B}_{\mathcal{S}}^c, \mathcal{B}_{\mathcal{H}}^c)$.

In the following, we will use the linear space $\mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ endowed with the norm

$$\|(X, Y)\| = \max\{\|X\|_2, \|Y\|_2\}.$$

Theorem 15 *Let $\alpha\mathcal{S} - \beta\mathcal{H}$ be a regular skew-Hamiltonian/Hamiltonian matrix pencil with neither infinite nor purely imaginary eigenvalues. Let $\mathcal{P}_V^{-}$ be the orthogonal basis of the deflating subspace of $\alpha\mathcal{S} - \beta\mathcal{H}$ corresponding to $\Lambda_{-}(\mathcal{S}, \mathcal{H})$, and let $\hat{\mathcal{P}}_V^{-}$ be the perturbation of $\mathcal{P}_V^{-}$ obtained by Algorithm 2 in finite precision arithmetic. Denote by $\Theta \in \mathbb{C}^{n,n}$ the diagonal matrix of canonical angles between $\mathcal{P}_V^{-}$ and $\hat{\mathcal{P}}_V^{-}$.*

Using the structured Schur form of the extended skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{B}_{\mathcal{S}}^c - \beta\mathcal{B}_{\mathcal{H}}^c$ (as in (35) and (40)) given by (64), define δ by

$$\delta = \min_{Y \in \mathbb{C}^{2n,2n}\setminus\{0\}} \frac{\left\|(\mathcal{H}_{11}^H Y + Y^H\mathcal{H}_{11}, \mathcal{S}_{11}^H Y - Y^H\mathcal{S}_{11})\right\|}{\|Y\|_2}. \qquad (65)$$

If

$$8\,\|(\mathcal{E}_{\mathcal{S}}, \mathcal{E}_{\mathcal{H}})\|\,(\delta + \|(\mathcal{S}_{12}, \mathcal{H}_{12})\|) < \delta^2, \qquad (66)$$

then

$$\|\Theta\|_2 < c_b\frac{\|(\mathcal{E}_{\mathcal{S}}, \mathcal{E}_{\mathcal{H}})\|}{\delta} < c_b\varepsilon\frac{\|(c_{\mathcal{S}}\mathcal{S}, c_{\mathcal{H}}\mathcal{H})\|}{\delta}, \qquad (67)$$

where $c_{\mathcal{S}}$ and $c_{\mathcal{H}}$ are the modest constants in (51)–(52) and $c_b = 8(\sqrt{10}+4)/(\sqrt{10}+2) \approx 11.1$.

Proof. Let $\alpha\hat{\mathcal{R}}_{\mathcal{S}} - \beta\hat{\mathcal{R}}_{\mathcal{H}}$, $\hat{\mathcal{Q}}$ be the output of Step 2 in Algorithm 2 in finite precision arithmetic, where $\hat{\mathcal{B}}_{\mathcal{S}}^c$, $\hat{\mathcal{B}}_{\mathcal{H}}^c$ satisfy (49) and (50). Let $\tilde{\mathcal{Q}}$ be the unitary matrix computed by Algorithm 2 in exact arithmetic such that

$$
\begin{aligned}
\mathcal{J}\tilde{\mathcal{Q}}^H\mathcal{J}^T(\alpha\mathcal{B}_{\mathcal{S}}^c - \beta\mathcal{B}_{\mathcal{H}}^c)\tilde{\mathcal{Q}} &= \alpha\tilde{\mathcal{R}}_{\mathcal{S}} - \beta\tilde{\mathcal{R}}_{\mathcal{H}} \\
&= \alpha\begin{bmatrix} \tilde{\mathcal{S}}_{11} & \tilde{\mathcal{S}}_{12} \\ 0 & \tilde{\mathcal{S}}_{11}^H \end{bmatrix} - \beta\begin{bmatrix} \tilde{\mathcal{H}}_{11} & \tilde{\mathcal{H}}_{12} \\ 0 & -\tilde{\mathcal{H}}_{11}^H \end{bmatrix},
\end{aligned}
$$

with $\Lambda(\tilde{\mathcal{S}}_{11}, \tilde{\mathcal{H}}_{11}) = \Lambda_{-}(\mathcal{B}_{\mathcal{S}}^c, \mathcal{B}_{\mathcal{H}}^c)$. Since (64) is another structured Schur form with the same eigenvalue ordering, there exits a unitary diagonal matrix $\mathcal{G} = \operatorname{diag}(G_1, G_2)$ such that $\mathcal{Q} = \tilde{\mathcal{Q}}\mathcal{G}$. Therefore, we have

$$\left\|(\tilde{\mathcal{S}}_{12}, \tilde{\mathcal{H}}_{12})\right\| = \|(\mathcal{S}_{12}, \mathcal{H}_{12})\|,$$

and for δ given in (65) we also have

$$\delta = \min_{Y \in \mathbb{C}^{2n,2n} \setminus \{0\}} \frac{\left\| (\tilde{\mathcal{H}}_{11}^H Y + Y^H \tilde{\mathcal{H}}_{11}, \tilde{\mathcal{S}}_{11}^H Y - Y^H \tilde{\mathcal{S}}_{11}) \right\|}{\|Y\|_2}.$$

Let

$$\tilde{\mathcal{E}}_{\mathcal{S}} := \mathcal{J} \tilde{\mathcal{Q}}^H \mathcal{J}^T \mathcal{E}_{\mathcal{S}} \tilde{\mathcal{Q}} =: \left[\begin{array}{cc} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_{11}^H \end{array} \right], \quad \tilde{\mathcal{E}}_{\mathcal{H}} := \mathcal{J} \tilde{\mathcal{Q}}^H \mathcal{J}^T \mathcal{E}_{\mathcal{H}} \tilde{\mathcal{Q}} =: \left[\begin{array}{cc} \mathcal{F}_{11} & \mathcal{F}_{12} \\ \mathcal{F}_{21} & -\mathcal{F}_{11}^H \end{array} \right]$$

and set $\gamma = \|(\mathcal{E}_{21}, \mathcal{F}_{21})\|$, $\eta = \left\|(\tilde{\mathcal{S}}_{12} + \mathcal{E}_{12}, \tilde{\mathcal{H}}_{12} + \mathcal{F}_{12})\right\|$ and $\tilde{\delta} = \delta - 2\|(\mathcal{E}_{11}, \mathcal{F}_{11})\|$. Since we have $\left\|(\tilde{\mathcal{E}}_{\mathcal{S}}, \tilde{\mathcal{E}}_{\mathcal{H}})\right\| = \|(\mathcal{E}_{\mathcal{S}}, \mathcal{E}_{\mathcal{H}})\|$, condition (66) implies that

$$\tilde{\delta} \geq \delta - 2\|(\mathcal{E}_{\mathcal{S}}, \mathcal{E}_{\mathcal{H}})\| > \frac{3}{4}\delta,$$

and clearly,

$$4\|(\mathcal{E}_{\mathcal{S}}, \mathcal{E}_{\mathcal{H}})\| \|(\mathcal{S}_{12}, \mathcal{H}_{12})\| < \delta^2 - 4\delta \|(\mathcal{E}_{\mathcal{S}}, \mathcal{E}_{\mathcal{H}})\|.$$

Hence

$$\begin{aligned} \frac{\gamma\eta}{\tilde{\delta}^2} &\leq \frac{\|(\mathcal{E}_{\mathcal{S}}, \mathcal{E}_{\mathcal{H}})\| \left\{ \left\|(\tilde{\mathcal{S}}_{12}, \tilde{\mathcal{H}}_{12})\right\| + \|(\mathcal{E}_{\mathcal{S}}, \mathcal{E}_{\mathcal{H}})\| \right\}}{(\delta - 2\|(\mathcal{E}_{\mathcal{S}}, \mathcal{E}_{\mathcal{H}})\|)^2} \\ &< \frac{\|(\mathcal{E}_{\mathcal{S}}, \mathcal{E}_{\mathcal{H}})\|^2 + (\delta^2 - 4\delta\|(\mathcal{E}_{\mathcal{S}}, \mathcal{E}_{\mathcal{H}})\|)/4}{(\delta - 2\|(\mathcal{E}_{\mathcal{S}}, \mathcal{E}_{\mathcal{H}})\|)^2} = \frac{1}{4}. \end{aligned}$$

Following the perturbation analysis for a formal product of matrices in [8], it can be shown that there exists a unitary matrix

$$\mathcal{W} = \left[\begin{array}{cc} (I + W^H W)^{-\frac{1}{2}} & -W^H (I + W W^H)^{-\frac{1}{2}} \\ W(I + W^H W)^{-\frac{1}{2}} & (I + W W^H)^{-\frac{1}{2}} \end{array} \right]$$

with

$$\|W\|_2 < 2\frac{\gamma}{\tilde{\delta}} < \frac{8}{3}\frac{\gamma}{\delta} < \frac{1}{3} \tag{68}$$

such that

$$\mathcal{J}(\tilde{\mathcal{Q}}\mathcal{W})^H \mathcal{J}^T (\alpha \hat{\mathcal{B}}_{\mathcal{S}}^c - \beta \hat{\mathcal{B}}_{\mathcal{H}}^c)(\tilde{\mathcal{Q}}\mathcal{W})$$

is another structured Schur form of the perturbed matrix pencil. Since there are neither infinite nor purely imaginary eigenvalues, (63) implies that $\hat{\mathcal{Q}}^H \tilde{\mathcal{Q}}\mathcal{W}$ is unitary block diagonal.

Without loss of generality we may take $\hat{\mathcal{Q}} = \tilde{\mathcal{Q}}\mathcal{W}$. If $\mathcal{X}_c$ as in (29) and $\mathcal{X}_c \tilde{\mathcal{Q}} = \left[\begin{smallmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{smallmatrix} \right]$, then it follows from Theorem 12 that $\mathcal{P}_V^- = \text{range}\, \mathcal{Q}_{11}$. Clearly $\hat{\mathcal{P}}_V^- = \text{range}\{(\mathcal{Q}_{11} + \mathcal{Q}_{12}W)(I + W^H W)^{-\frac{1}{2}}\}$. The upper bound (67) then can be derived from (68), by using the same argument as in the proof of Theorem 4.4 in [5]. $\square$

If $\mathcal{S}$ is given in factored form, then we obtain a similar result. In this case by using Algorithm 1 we compute a unitary matrix $\hat{\mathcal{Q}}$ and a unitary symplectic matrix $\hat{\mathcal{U}}$ such that

$$\begin{aligned} \hat{\mathcal{U}}^H \hat{\mathcal{B}}_{\mathcal{Z}}^c \hat{\mathcal{Q}} = \hat{\mathcal{R}}_{\mathcal{Z}} &=: \left[\begin{array}{cc} \hat{\mathcal{Z}}_{11} & \hat{\mathcal{Z}}_{12} \\ 0 & \hat{\mathcal{Z}}_{22} \end{array} \right], \\ \mathcal{J}\hat{\mathcal{Q}}^H \mathcal{J}^T \hat{\mathcal{B}}_{\mathcal{H}}^c \hat{\mathcal{Q}} = \hat{\mathcal{R}}_{\mathcal{H}} &=: \left[\begin{array}{cc} \hat{\mathcal{H}}_{11} & \hat{\mathcal{H}}_{12} \\ 0 & -\hat{\mathcal{H}}_{11}^H \end{array} \right], \end{aligned} \tag{69}$$

where $\hat{\mathcal{B}}^c_{\mathcal{Z}}$ and $\hat{\mathcal{B}}^c_{\mathcal{H}}$ are defined in (58) and (59), and $\Lambda(\hat{\mathcal{Z}}_{22}^H\hat{\mathcal{Z}}_{11}, \hat{\mathcal{H}}_{11}) = \Lambda_-(\hat{\mathcal{B}}^c_{\mathcal{S}}, \hat{\mathcal{B}}^c_{\mathcal{H}})$, where $\hat{\mathcal{B}}^c_{\mathcal{S}} = \mathcal{J}(\hat{\mathcal{B}}^c_{\mathcal{Z}})^H\mathcal{J}^T\hat{\mathcal{B}}^c_{\mathcal{Z}}$.

Analogous to Theorem 9, if $\alpha\mathcal{S} - \beta\mathcal{H}$ has no purely imaginary eigenvalues, then there exist unitary matrices Q_1, Q_2 and unitary symplectic matrices U_1, U_2 such that

$$U_1^H\mathcal{Z}Q_1 = \begin{bmatrix} Z_{11}^- & Z_{12}^- \\ 0 & Z_{22}^- \end{bmatrix}, \quad \mathcal{J}Q_1^H\mathcal{J}^T\mathcal{H}Q_1 = \begin{bmatrix} H_{11}^- & H_{12}^- \\ 0 & -(H_{11}^-)^H \end{bmatrix},$$

with $\Lambda((Z_{22}^-)^H Z_{11}^-, H_{11}^-) = \Lambda_-(\mathcal{S},\mathcal{H})$, and

$$U_2^H\mathcal{Z}Q_2 = \begin{bmatrix} Z_{11}^+ & Z_{12}^+ \\ 0 & Z_{22}^+ \end{bmatrix}, \quad \mathcal{J}Q_2^H\mathcal{J}^T\mathcal{H}Q_2 = \begin{bmatrix} H_{11}^+ & H_{12}^+ \\ 0 & -(H_{11}^+)^H \end{bmatrix},$$

with $\Lambda((Z_{22}^+)^H Z_{11}^+, H_{11}^+) = \Lambda_+(\mathcal{S},\mathcal{H})$, respectively. Set

$$\mathcal{Q} = \mathcal{X}_c^H \operatorname{diag}(Q_1, \bar{Q}_2)\mathcal{P}, \qquad \mathcal{U} = \mathcal{X}_c^H \operatorname{diag}(U_1, \bar{U}_2)\mathcal{P},$$

where $\mathcal{P}$ and $\mathcal{X}_c$ are as in (28) and (29). Then $\mathcal{Q}$ is unitary and $\mathcal{U} \in \mathbb{US}_{4n}$, and a simple calculation yields

$$\mathcal{U}^H\mathcal{B}^c_{\mathcal{Z}}\mathcal{Q} = \left[\begin{array}{cc|cc} Z_{11}^- & 0 & Z_{12}^- & 0 \\ 0 & \overline{Z_{11}^+} & 0 & \overline{Z_{12}^+} \\ \hline 0 & 0 & Z_{22}^- & 0 \\ 0 & 0 & 0 & \overline{Z_{22}^+} \end{array}\right] =: \begin{bmatrix} \mathcal{Z}_{11} & \mathcal{Z}_{12} \\ 0 & \mathcal{Z}_{22} \end{bmatrix} =: \mathcal{R}_{\mathcal{Z}}, \tag{70}$$

$$\mathcal{J}\mathcal{Q}^H\mathcal{J}^T\mathcal{B}^c_{\mathcal{H}}\mathcal{Q} = \left[\begin{array}{cc|cc} H_{11}^- & 0 & H_{12}^- & 0 \\ 0 & -\overline{H_{11}^+} & 0 & -\overline{H_{12}^+} \\ \hline 0 & 0 & -(H_{11}^-)^H & 0 \\ 0 & 0 & 0 & (\overline{H_{11}^+})^H \end{array}\right] =: \begin{bmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ 0 & -\mathcal{H}_{11}^H \end{bmatrix} =: \mathcal{R}_{\mathcal{H}}. \tag{71}$$

This leads to the structured Schur form of the extended skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{J}(\mathcal{B}^c_{\mathcal{Z}})^H\mathcal{J}^T\mathcal{B}^c_{\mathcal{Z}} - \beta\mathcal{B}^c_{\mathcal{H}}$ with $\Lambda(\mathcal{Z}_{22}^H\mathcal{Z}_{11}, \mathcal{H}_{11}) = \Lambda_-(\mathcal{B}^c_{\mathcal{S}}, \mathcal{B}^c_{\mathcal{H}})$.

Theorem 16 *Consider the regular skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{S} - \beta\mathcal{H}$ with nonsingular, $\mathcal{J}$-definite skew-Hamiltonian part $\mathcal{S} = \mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z}$. Suppose that $\alpha\mathcal{S} - \beta\mathcal{H}$ has no eigenvalue with zero real part. Let the extended skew-Hamiltonian and Hamiltonian matrix $\mathcal{B}^c_{\mathcal{Z}}$ and $\mathcal{B}^c_{\mathcal{H}}$ be as in (34) and (40), respectively with structured triangular form given by (70) and (71). Define δ_p as*

$$\delta_p = \min_{(X,Y)\in\mathbb{C}^{2n,2n}\times\mathbb{C}^{2n,2n}\setminus\{(0,0)\}} \frac{\left\|(\mathcal{H}_{11}^H Y + Y^H\mathcal{H}_{11}, X\mathcal{Z}_{11} - \mathcal{Z}_{22}Y)\right\|}{\|(X,Y)\|_2}.$$

Define errors $\mathcal{E}_{\mathcal{Z}}$ and $\mathcal{E}_{\mathcal{H}}$ by (58) and (59). Let $\mathcal{P}_V^-$, $\mathcal{P}_U^-$, $\hat{\mathcal{P}}_V^-$ and $\hat{\mathcal{P}}_U^-$ be the deflating subspaces computed by Algorithm 1 in exact and finite precision arithmetic, respectively. Denote by $\Theta_V, \Theta_U \in \mathbb{C}^{n,n}$ the diagonal matrices of canonical angles between $\mathcal{P}_V^-$ and $\hat{\mathcal{P}}_V^-$, $\mathcal{P}_U^-$ and $\hat{\mathcal{P}}_U^-$, respectively.

If

$$8\,\|(\mathcal{E}_{\mathcal{Z}}, \mathcal{E}_{\mathcal{H}})\|\,(\delta_p + \|(\mathcal{Z}_{12}, \mathcal{H}_{12})\|) < \delta_p^2,$$

then

$$\|\Theta_V\|_2, \|\Theta_U\|_2 < c_b\frac{\|(\mathcal{E}_{\mathcal{Z}}, \mathcal{E}_{\mathcal{H}})\|}{\delta_p} < c_b\varepsilon\frac{\|(c_{\mathcal{Z}}\mathcal{Z}, c_{\mathcal{H}}\mathcal{H})\|}{\delta_p},$$

with c_b as in Theorem 15.

Proof. The proof is analogous to the proof of Theorem 15. □

It follows that all the described numerical algorithms are numerically backwards stable. The different methods compute the eigenvalues and the deflating subspaces $\mathrm{Def}_{-}(\mathcal{S},\mathcal{H})$, $\mathrm{Def}_{+}(\mathcal{S},\mathcal{H})$ of a skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{S} - \beta\mathcal{H}$. These algorithms can also be used to compute deflating subspaces which contain eigenvectors associated with infinite or purely imaginary eigenvalues. By Theorem 12 we get partial information also in these cases, but we face the difficulty that the desired deflating subspace may not be unique or may not exist. (See the recent analysis for Hamiltonian matrices [36]).

7 Conclusion

We have presented numerical procedures for the computation of structured Schur forms, eigenvalues, and deflating subspaces of matrix pencils with matrices having a Hamiltonian and/or skew-Hamiltonian structure. These methods generalize the recently developed methods for Hamiltonian matrices which use an extended, double dimension Hamiltonian matrix that always has a Hamiltonian Schur form.

The algorithms circumvent problems with skew-Hamiltonian/Hamiltonian matrix pencils that lack structured Schur form by embedding them in extended matrix pencils that always admit a structured Schur form. For the extended matrix pencils, the algorithms use structure preserving unitary matrix computations and are strongly backwards stable, i.e., they compute the exact structured Schur form of a nearby matrix pencil with the same structure. Such structured Schur forms can always be computed regardless of the regularity of the original matrix pencil.

It is still somewhat unsatisfactory that the algorithms do not efficiently exploit the micro structures of the extended matrix pencils, as for example in the matrix $\mathcal{B}^c_{\mathcal{N}}$ in (30). How best to use these micro structures is still an open question.

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A Algorithmic Details

In the following we describe the details of the factorization algorithms. We employ elementary orthogonal and orthogonal symplectic transformation matrices.

We denote an $m \times m$ Givens rotation matrix by

$$G(i,j,\theta,\phi) = \begin{bmatrix} I_{i-1} & & & & \\ & \cos(\theta) & & \sin(\theta)e^{-i\phi} & \\ & & I_{j-i-1} & & \\ & -\sin(\theta)e^{i\phi} & & \cos(\theta) & \\ & & & & I_{m-j} \end{bmatrix},$$

where $1 \leq i < j \leq m$, $\theta \in [0, 2\pi)$ and $\phi \in [-\pi/2, \pi/2)$. If $\phi = 0$, then the rotation matrix is a real orthogonal matrix. In this case we drop the fourth argument ϕ, and use the three argument notation $G(i,j,\theta) := G(i,j,\theta,0)$. If $m = 2n$, $j = n+i$ and $\phi = 0$, then the rotation matrix is a real orthogonal *and symplectic*. In this case we also drop the second argument j, and use the two argument notation $G_s(i,\theta) := G(i,n+i,\theta) = G(i,n+i,\theta,0)$. If $z \in \mathbb{C}^n$, then the j-th component of the product $G(i,j,\arctan(|x_j/x_i|),(x_j|x_i|)/(|x_j|x_i))x$ is zero. (Here we use the convention that $0/0 = 0$.) If $x \in \mathbb{R}^n$, then the j-th component of the product $G(i,j,\arctan(x_j/x_i))x$ is zero. For brevity, we will use the arctangent formula when describing algorithms. In practice, no transcendental functions are needed. For efficiency and numerical stability, rotations should be constructed as described, for example, in [19].

Another elementary orthogonal symplectic matrix that we will use is

$$G_d(i,j,\theta,\phi) = \begin{bmatrix} G(i,j,\theta,\phi) & 0 \\ 0 & G(i,j,\theta,\phi) \end{bmatrix},$$

where $G(i,j,\theta,\phi)$ is $n \times n$.

For $0 \neq w \in \mathbb{C}^k$, we denote an $n \times n$ Householder matrix ($n \geq k$) by

$$H(k,w) = I_n - 2\frac{\tilde{w}\tilde{w}^H}{\tilde{w}^H\tilde{w}} \quad \tilde{w} = [0, w^T]^T,$$

where $\tilde{w}$ is obtained from w by prepending $n-k$ zeros. If $w=0$, then we take $H(k,0) = I_n$.

For ease of explication, the algorithms below explicitly assemble and multiply rotations and reflections as full size matrices. In actual implementation, one would store and use these elementary unitary matrices in the efficient way described, for example, in [19].

A.1 Schur Forms for Skew-Hamiltonian/Hamiltonian Matrix Pencils and Skew-Hamiltonian/skew-Hamiltonian Matrix Pencils

In this subsection we describe the computation of the structured Schur forms for complex skew-Hamiltonian/Hamiltonian and real skew-Hamiltonian/skew-Hamiltonian matrix pencils. We first give the method for computing the structured factorization of a real skew-Hamiltonian/skew-Hamiltonian matrix pencil as in (25)–(27).

Algorithm 3 *Given a real skew-Hamiltonian/skew-Hamiltonian matrix pencil $\alpha\mathcal{S} - \beta\mathcal{N}$ with $\mathcal{S} = \mathcal{J}\mathcal{Z}^T\mathcal{J}^T\mathcal{Z}$. The algorithm computes an orthogonal matrix $\mathcal{Q}$ and an orthogonal symplectic matrix $\mathcal{U}$ such that $\mathcal{U}^T\mathcal{Z}\mathcal{Q}$ and $\mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{N}\mathcal{Q}$ are in the block triangular forms as in (26).*

Input: A real matrix $\mathcal{N} \in \mathbb{SH}_{2n}$ and a real matrix $\mathcal{Z} \in \mathbb{R}^{2n,2n}$.

Output: A real orthogonal matrix $\mathcal{Q}$, a real orthogonal symplectic matrix $\mathcal{U}$ and the structured factorization (26).

Step 0 Set $\mathcal{U} = \mathcal{Q} = I_{2n}$.

Step 1 By changing the elimination order in the classical RQ decomposition, determine an orthogonal matrix $\mathcal{Q}_1$ such that

$$\mathcal{Z} \ := \ \mathcal{Z}\mathcal{Q}_1 =: \begin{bmatrix} Z_{11} & Z_{12} \\ 0 & Z_{22} \end{bmatrix},$$

where Z_{11}, Z_{22}^T are upper triangular. Update $\mathcal{Q} \ := \ \mathcal{Q}\mathcal{Q}_1$ and $\mathcal{N} \ := \ \mathcal{J}\mathcal{Q}_1^T\mathcal{J}^T\mathcal{N}\mathcal{Q}_1$.

Step 2 FOR $k = 1, \ldots, n-1$

% *I. Annihilate $\mathcal{N}(n+k, k+1:n-1)$ as well as $\mathcal{N}(n+k+1:2n-1, k)$.*

FOR $j = k+1, \ldots, n-1$

a) Use $G(j, j+1, \theta_1)$ to eliminate $\mathcal{N}_{n+k,j}$ from the right. Set

$\mathcal{N} \ := \ \mathcal{J}G(j,j+1,\theta_1)^T\mathcal{J}^T\mathcal{N}G(j,j+1,\theta_1)$,

$$\begin{aligned} \mathcal{Z} &:= \mathcal{Z}G(j, j+1, \theta_1), \\ \mathcal{Q} &:= \mathcal{Q}G(j, j+1, \theta_1). \end{aligned}$$

b) Use $G_d(j, j+1, \theta_2)$ to eliminate $\mathcal{Z}_{j+1,j}$ from the left. Set

$$\begin{aligned} \mathcal{Z} &:= G_d(j, j+1, \theta_2)^T \mathcal{Z}, \\ \mathcal{U} &:= \mathcal{U}G_d(j, j+1, \theta_2). \end{aligned}$$

c) Use $G(n+j, n+j+1, \theta_3)$ to eliminate $\mathcal{Z}_{n+j,n+j+1}$ from the right. Set

$$\begin{aligned} \mathcal{Z} &:= \mathcal{Z}G(n+j, n+j+1, \theta_3), \\ \mathcal{N} &:= \mathcal{J}G(n+j, n+j+1, \theta_3)^T \mathcal{J}^T \mathcal{N} G(n+j, n+j+1, \theta_3), \\ \mathcal{Q} &:= \mathcal{Q}G(n+j, n+j+1, \theta_3). \end{aligned}$$

END FOR j

% *II. Annihilate $\mathcal{N}_{n+k,n}$ (and, due to the skew-Hamiltonian*
% *structure, simultaneously annihilate $\mathcal{N}_{2n,k}$).*

a) Use $G_s(n, \phi_1)$ to eliminate $\mathcal{N}_{n+k,n}$ from the right. Set

$$\begin{aligned} \mathcal{N} &:= \mathcal{J}G_s(n, \phi_1)^T \mathcal{J}^T \mathcal{N} G_s(n, \phi_1), \\ \mathcal{Z} &:= \mathcal{Z}G_s(n, \phi_1), \\ \mathcal{Q} &:= \mathcal{Q}G_s(n, \phi_1). \end{aligned}$$

b) Use $G_s(n, \phi_2)$ to eliminate $\mathcal{Z}_{2n,n}$ from the left. Set

$$\begin{aligned} \mathcal{Z} &:= G_s(n, \phi_2)^T \mathcal{Z}, \\ \mathcal{U} &:= \mathcal{U}G_s(n, \phi_2). \end{aligned}$$

% *III. Annihilate $\mathcal{N}(n+k, n+k+2 : 2n)$ (and, due to the skew-*
% *Hamiltonian structure, simultaneously annihilate $\mathcal{N}(k+2 : n, k)$).*

FOR $j = n, n-1, \ldots, k+2$

a) Use $G(n+j-1, n+j, \psi_1)$ to eliminate $\mathcal{N}_{n+k,n+j}$ from the right. Set

$$\begin{aligned} \mathcal{N} &:= \mathcal{J}G(n+j-1, n+j, \psi_1)^T \mathcal{J}^T \mathcal{N} G(n+j-1, n+j, \psi_1), \\ \mathcal{Z} &:= \mathcal{Z}G(n+j-1, n+j, \psi_1), \\ \mathcal{Q} &:= \mathcal{Q}G(n+j-1, n+j, \psi_1). \end{aligned}$$

b) Use $G_d(j-1, j, \psi_2)$, to eliminate $\mathcal{Z}_{n+j-1,n+j}$ from the left. Set

$$\begin{aligned} \mathcal{Z} &:= G_d(j-1, j, \psi_2)^T \mathcal{Z}, \\ \mathcal{U} &:= \mathcal{U}G_d(j-1, j, \psi_2). \end{aligned}$$

c) Use $G(j-1, j, \psi_3)$ to eliminate $\mathcal{Z}_{j,j-1}$ from the right. Set

$$\begin{aligned} \mathcal{Z} &:= \mathcal{Z}G(j-1, j, \psi_3), \\ \mathcal{N} &:= \mathcal{J}G(j-1, j, \psi_3)^T \mathcal{J}^T \mathcal{N} G(j-1, j, \psi_3), \\ \mathcal{Q} &:= \mathcal{Q}G(j-1, j, \psi_3). \end{aligned}$$

END FOR j

END FOR k

% *Now* $\mathcal{Z} = \begin{bmatrix} Z_{11} & Z_{12} \\ 0 & Z_{22} \end{bmatrix}$, $\mathcal{N} = \begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{11}^T \end{bmatrix}$.

Step 3 Apply the periodic QZ algorithm [10, 21] to the matrix pencil $\alpha Z_{22}^T Z_{11} - \beta N_{11}$ to determine orthogonal matrices Q_1, Q_2, U such that $U^T Z_{11} Q_1$, $Q_2^T Z_{22}^T U$ are both upper triangular and $Q_2^T N_{11} Q_1$ is quasi upper triangular.

Set $\mathcal{U}_1 = \mathrm{diag}(U, U)$, $\mathcal{Q}_2 = \mathrm{diag}(Q_1, Q_2)$.

Update $\mathcal{Z} := \mathcal{U}_1^T \mathcal{Z} \mathcal{Q}_2$, $\mathcal{N} := \mathcal{J} \mathcal{Q}_2^T \mathcal{J}^T \mathcal{N} \mathcal{Q}_2$, $\mathcal{Q} := \mathcal{Q}\mathcal{Q}_2$, $\mathcal{U} := \mathcal{U}\mathcal{U}_1$.

END

The next subroutine is the eigenvalue reordering method for a complex skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{S} - \beta\mathcal{H}$, with $\mathcal{S} = \mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z}$, which is in structured triangular form. We first introduce some subroutines to deal with 2×2 problems.

Subroutine 1 *Given a regular* 2×2 *matrix pencil* $\alpha TZ - \beta H$ *with* T, Z, H *upper triangular. This subroutine determines unitary matrices* Q_1, Q_2, Q_3 *such that* $Q_3^H T Q_2$, $Q_2^H Z Q_1$, $Q_3^H H Q_1$ *are still upper triangular, but the eigenvalues are in the reversed order.*

Input: 2×2 upper triangular matrices T, Z, H.

Output: The 2×2 unitary matrices Q_1, Q_2, Q_3 described above.

Set $\gamma = t_{11} z_{11} h_{22} - t_{22} z_{22} h_{11}$

IF $\gamma = 0$

% $\alpha TZ - \beta H$ *has a double eigenvalue*

Set $Q_1 = Q_2 = Q_3 = I_2$.

ELSE

Compute $Q_1 = G(1, 2, \arctan(\delta/\gamma), \arg(\gamma) - \arg(\delta))$
with $\delta = t_{11} z_{12} h_{22} + t_{12} z_{22} h_{22} - t_{22} z_{22} h_{12}$.

Compute $Q_2 = G(1, 2, \arctan(\delta/\gamma), \arg(\gamma) - \arg(\delta))$
with $\delta = t_{12} z_{11} h_{22} - t_{22} z_{11} h_{12} + t_{22} z_{12} h_{11}$.

Compute $Q_3 = G(1, 2, \arctan(\delta/\gamma), \arg(\gamma) - \arg(\delta))$
with $\delta = -t_{11} z_{11} h_{12} + t_{11} z_{12} h_{11} + t_{12} z_{22} h_{11}$.

END IF

END

The second subroutine deals with 2×2 triangular skew-Hamiltonian/Hamiltonian matrix pencils.

Subroutine 2 *Given a regular* 2×2 *complex skew-Hamiltonian/Hamiltonian matrix pencil* $\alpha\mathcal{S} - \beta\mathcal{H}$ *with* $\mathcal{S} = \mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z}$, *where* $\mathcal{Z} = \begin{bmatrix} z_{11} & z_{12} \\ 0 & z_{22} \end{bmatrix}$, $\mathcal{H} = \begin{bmatrix} h_{11} & h_{12} \\ 0 & -\bar{h}_{11} \end{bmatrix}$. *The subroutine determines a unitary matrix* $\mathcal{Q}$ *and a unitary symplectic matrix* $\mathcal{U}$ *such that* $\mathcal{U}^H\mathcal{Z}\mathcal{Q}$, $(\mathcal{J}\mathcal{Q}\mathcal{J}^T)^H\mathcal{H}\mathcal{Q}$ *are both upper triangular but the eigenvalues of* $(\mathcal{J}\mathcal{Q}\mathcal{J}^T)^H(\alpha\mathcal{S} - \beta\mathcal{H})\mathcal{Q}$ *are in reversed order.*

Input: 2×2 upper triangular matrices $\mathcal{Z}$, $\mathcal{H}$ with $\mathcal{H} \in \mathbb{H}_2$.

Output: The 2×2 unitary matrix $\mathcal{Q}$ and a 2×2 real orthogonal symplectic matrix $\mathcal{U}$ described above.

Set $\gamma = 2\,\mathrm{Re}\,(h_{11}\bar{z}_{11}z_{22})$.

IF $\gamma = 0$

% *$\alpha\mathcal{S} - \beta\mathcal{H}$ has a double purely imaginary eigenvalue*

Set $\mathcal{Q} = \mathcal{U} = I_2$.

ELSE

Compute $\mathcal{U} = G(1, 2, \arctan(\delta/\gamma))$ with $\delta = |z_{11}|^2 h_{12} - 2\,\mathrm{Re}\,(h_{12}\bar{z}_{11}z_{22})$.

Set $\begin{bmatrix} \tilde{z}_{11} & \tilde{z}_{12} \\ \tilde{z}_{21} & \tilde{z}_{22} \end{bmatrix} = \mathcal{U}^H \mathcal{Z}$.

Compute $\mathcal{Q} = G(1, 2, \arctan(\tilde{z}_{22}/\tilde{z}_{11}), \arg(\tilde{z}_{11}) - \arg(\tilde{z}_{22}))$.

END IF

END

Now we are ready to formulate the algorithm.

Algorithm 4 *Given a regular $2n \times 2n$ complex skew-Hamiltonian/Hamiltonian matrix pencil $\alpha\mathcal{S} - \beta\mathcal{H}$ with $\mathcal{J}$-semidefinite skew Hamiltonian part $\mathcal{S} = \mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z}$,*

$$\mathcal{Z} = \begin{bmatrix} Z & W \\ 0 & T \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} H & D \\ 0 & -H^H \end{bmatrix}, \tag{72}$$

and Z, T^H and H upper triangular. This algorithm determines a unitary matrix $\mathcal{Q}$ and a unitary symplectic matrix $\mathcal{U}$ such that $\mathcal{U}^H\mathcal{Z}\mathcal{Q}$ and $\mathcal{J}\mathcal{Q}^H\mathcal{J}^T\mathcal{H}\mathcal{Q}$ still have the same triangular form as $\mathcal{Z}$ and $\mathcal{H}$, respectively, but the eigenvalues in $\Lambda_-(\mathcal{S}, \mathcal{H})$ are reordered so that they occur in the leading principal subpencil of $\mathcal{J}\mathcal{Q}^H\mathcal{J}^T(\alpha\mathcal{S} - \beta\mathcal{H})\mathcal{Q}$.

Input: A Matrix $\mathcal{Z}$ and a Hamiltonian matrix $\mathcal{H}$ in triangular form (72).

Output: A unitary matrix $\mathcal{Q}$ and a unitary symplectic matrix $\mathcal{U}$. The matrices $\mathcal{S}$ and $\mathcal{H}$ are overwritten by $\mathcal{U}^H\mathcal{Z}\mathcal{Q}$ and $\mathcal{J}\mathcal{Q}^H\mathcal{J}^T\mathcal{H}\mathcal{Q}$, respectively. that the eigenvalues of $\mathcal{J}\mathcal{Q}^H\mathcal{J}^T(\alpha\mathcal{S} - \beta\mathcal{H})\mathcal{Q}$ are in the desired order.

Step 0 Set $\mathcal{Q} = \mathcal{U} = I_{2n}$.

Step 1 % *Reorder the eigenvalues in the subpencil $\alpha T^H Z - \beta H$.*

Set $m_- = 0$, $m_+ = n + 1$.

% *I. Reorder the eigenvalues with negative real parts to the top.*

Set $k = 1$.

WHILE $k \leq n$ DO

IF $\bar{t}_{kk}z_{kk}) \neq 0$ and $\mathrm{Re}\,(h_{kk}/(\bar{t}_{kk}z_{kk})) < 0$

FOR $j = k - 1, \ldots, m_- + 1$

a) Apply Subroutine 1 to the matrix pencil

$$\alpha \begin{bmatrix} \bar{t}_{jj} & \bar{t}_{j+1,j} \\ 0 & \bar{t}_{j+1,j+1} \end{bmatrix} \begin{bmatrix} z_{jj} & z_{j,j+1} \\ 0 & z_{j+1,j+1} \end{bmatrix} - \beta \begin{bmatrix} h_{jj} & h_{j,j+1} \\ 0 & h_{j+1,j+1} \end{bmatrix}$$

to determine unitary matrices Q_1, Q_2, Q_3.

b) Set $\tilde{\mathcal{Q}} = \mathrm{diag}(I_{j-1}, Q_1, I_{n-j-1}, I_{j-1}, Q_3, I_{n-j-1})$,
$\tilde{\mathcal{U}} = \mathrm{diag}(I_{j-1}, Q_2, I_{n-j-1}, I_{j-1}, Q_2, I_{n-j-1})$.

c) Update $\mathcal{Z} := \tilde{\mathcal{U}}^H \mathcal{Z} \tilde{\mathcal{Q}}$, $\mathcal{H} := \mathcal{J} \tilde{\mathcal{Q}}^H \mathcal{J}^T \mathcal{H} \tilde{\mathcal{Q}}$, $\mathcal{U} := \mathcal{U}\tilde{\mathcal{U}}$, $\mathcal{Q} := \mathcal{Q}\tilde{\mathcal{Q}}$.

END FOR j

$m_- := m_- + 1$

END IF

$k := k + 1$

END WHILE

% *II. Reorder the eigenvalues with positive real parts to the bottom.*

Set $k = n$.

WHILE $k \geq m_- + 1$ DO

IF $\bar{t}_{kk} z_{kk}) \neq 0$ and $\mathrm{Re}\,(h_{kk}/(\bar{t}_{kk} z_{kk})) > 0$

FOR $j = k, \ldots, m_+ - 1$

a) Apply Subroutine 1 to the matrix pencil

$$\alpha \begin{bmatrix} \bar{t}_{jj} & \bar{t}_{j+1,j} \\ 0 & \bar{t}_{j+1,j+1} \end{bmatrix} \begin{bmatrix} z_{jj} & z_{j,j+1} \\ 0 & z_{j+1,j+1} \end{bmatrix} - \beta \begin{bmatrix} h_{jj} & h_{j,j+1} \\ 0 & h_{j+1,j+1} \end{bmatrix}$$

to determine unitary matrices Q_1, Q_2, Q_3.

b) Set $\tilde{\mathcal{Q}} = \mathrm{diag}(I_{j-1}, Q_1, I_{n-j-1}, I_{j-1}, Q_3, I_{n-j-1})$,
$\tilde{\mathcal{U}} = \mathrm{diag}(I_{j-1}, Q_2, I_{n-j-1}, I_{j-1}, Q_2, I_{n-j-1})$.

c) Update $\mathcal{Z} := \tilde{\mathcal{U}}^H \mathcal{Z} \tilde{\mathcal{Q}}$, $\mathcal{H} := \mathcal{J} \tilde{\mathcal{Q}}^H \mathcal{J}^T \mathcal{H} \tilde{\mathcal{Q}}$, $\mathcal{U} := \mathcal{U}\tilde{\mathcal{U}}$, $\mathcal{Q} := \mathcal{Q}\tilde{\mathcal{Q}}$.

END FOR j

$m_+ := m_+ - 1$

END IF

$k := k - 1$

END WHILE

% *The remaining $n - m_+ + 1$ eigenvalues with negative real part are now in the bottom right subpencil of $\alpha\mathcal{S} - \beta\mathcal{H}$.*

Step 2 % *Reorder the remaining $n - m_+ + 1$ eigenvalues*

FOR $k = n, \ldots, m_+$

% *I. Exchange the eigenvalues between two diagonal blocks*

a) Apply Subroutine 2 to the 2×2 matrix pencil

$$\alpha \begin{bmatrix} \bar{t}_{nn} & -\bar{w}_{nn} \\ 0 & \bar{z}_{nn} \end{bmatrix} \begin{bmatrix} z_{nn} & w_{nn} \\ 0 & t_{nn} \end{bmatrix} - \beta \begin{bmatrix} h_{nn} & d_{nn} \\ 0 & -\bar{h}_{nn} \end{bmatrix}$$

to determine a unitary matrix $Q = \begin{bmatrix} c & \bar{s} \\ -s & \bar{c} \end{bmatrix}$ and a unitary symplectic matrix $U = \begin{bmatrix} u_1 & u_2 \\ -u_2 & u_1 \end{bmatrix}$.

b) Let $Q_1 = \mathrm{diag}(1, \ldots, 1, c), \quad Q_2 = \mathrm{diag}(0, \ldots, 0, s),$
$U_1 = \mathrm{diag}(1, \ldots, 1, u_1), \quad U_2 = \mathrm{diag}(0, \ldots, 0, u_2).$

Set $\tilde{\mathcal{Q}} = \begin{bmatrix} Q_1 & \bar{Q}_2 \\ -Q_2 & \bar{Q}_1 \end{bmatrix}, \tilde{\mathcal{U}} = \begin{bmatrix} U_1 & U_2 \\ -U_2 & U_1 \end{bmatrix}$.

c) Update $\mathcal{Z} := \tilde{\mathcal{U}}^H \mathcal{Z} \tilde{\mathcal{Q}}, \mathcal{H} := \mathcal{J} \tilde{\mathcal{Q}}^H \mathcal{J}^T \mathcal{H} \tilde{\mathcal{Q}}, \mathcal{U} := \mathcal{U} \tilde{\mathcal{Q}}, \mathcal{Q} := \mathcal{Q} \tilde{\mathcal{Q}}$.

% *II. Move the eigenvalue in the n-th diagonal position to the* $(m_- + 1)$ *position*

$m_- := m_- + 1$

FOR $j = n-1, \ldots, m_-$

a) Apply Subroutine 1 to the matrix pencil

$$\alpha \begin{bmatrix} \bar{t}_{jj} & \bar{t}_{j+1,j} \\ 0 & \bar{t}_{j+1,j+1} \end{bmatrix} \begin{bmatrix} z_{jj} & z_{j,j+1} \\ 0 & z_{j+1,j+1} \end{bmatrix} - \beta \begin{bmatrix} h_{jj} & h_{j,j+1} \\ 0 & h_{j+1,j+1} \end{bmatrix}$$

to determine unitary matrices Q_1, Q_2, Q_3.

b) Set $\tilde{\mathcal{Q}} = \mathrm{diag}(I_{j-1}, Q_1, I_{n-j-1}, I_{j-1}, Q_3, I_{n-j-1}),$
$\tilde{\mathcal{U}} = \mathrm{diag}(I_{j-1}, Q_2, I_{n-j-1}, I_{j-1}, Q_2, I_{n-j-1}).$

c) Update $\mathcal{Z} := \tilde{\mathcal{U}}^H \mathcal{Z} \tilde{\mathcal{Q}}, \mathcal{H} := \mathcal{J} \tilde{\mathcal{Q}}^H \mathcal{J}^T \mathcal{H} \tilde{\mathcal{Q}}, \mathcal{U} := \mathcal{U} \tilde{\mathcal{U}}, \mathcal{Q} := \mathcal{Q} \tilde{\mathcal{Q}}$.

END FOR j

END FOR k

END

A.2 Subroutines Required by Algorithm 2

In this appendix we present the subroutines used in Algorithm 2. First we recall the following algorithm from [39].

Algorithm 5 *Given a real skew-Hamiltonian/skew-Hamiltonian matrix pencil* $\alpha\mathcal{S} - \beta\mathcal{N}$*. This algorithm computes an orthogonal matrix* $\mathcal{Q}$ *such that* $\mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{S}\mathcal{Q}$ *and* $\mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{N}\mathcal{Q}$ *are in skew-Hamiltonian triangular form.*

Input: Real matrices $\mathcal{S}, \mathcal{N} \in \mathbb{SH}_{2n}$.

Output: A real orthogonal matrix $\mathcal{Q}$ which reduces $\mathcal{S}$ and $\mathcal{N}$ to skew-Hamiltonian triangular form. $\mathcal{S}$ and $\mathcal{N}$ are overwritten by $\mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{S}\mathcal{Q}$ and $\mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{N}\mathcal{Q}$, respectively.

Step 0 Set $\mathcal{Q} = I_{2n}$.

Step 1 % *Reduce $\mathcal{S}$ to skew-Hamiltonian triangular form*

FOR $k = 1, \ldots, n-1$

a) Determine a Householder matrix $H(n-k, x)$ to eliminate $\mathcal{S}(n+k, k+2 : n)$ (as well as $\mathcal{S}(n+k+2 : 2n, k)$) from the right. Set

$\tilde{\mathcal{Q}} = \mathrm{diag}(H(n-k, x), I_n)$, $\mathcal{S} := \mathcal{J}\tilde{\mathcal{Q}}^T\mathcal{J}^T\mathcal{S}\tilde{\mathcal{Q}}$, $\mathcal{N} := \mathcal{J}\tilde{\mathcal{Q}}^T\mathcal{J}^T\mathcal{N}\tilde{\mathcal{Q}}$, $\mathcal{Q} := \mathcal{Q}\tilde{\mathcal{Q}}$.

b) Determine $G_s(k+1, \zeta)$ to eliminate $\mathcal{S}_{n+k,k+1}$ (as well as $\mathcal{S}_{n+k+1,k}$) from the right. Set

$$\begin{aligned}\mathcal{S} &:= G_s(k+1,\zeta)^T\mathcal{S}G_s(k+1,\zeta),\\ \mathcal{N} &:= G_s(k+1,\zeta)^T\mathcal{N}G_s(k+1,\zeta),\\ \mathcal{Q} &:= \mathcal{Q}G_s(k+1,\zeta).\end{aligned}$$

c) Determine $H(n-k+1, y)$ to eliminate $\mathcal{S}(n+k, n+k+1 : 2n)$ (as well as $\mathcal{S}(k+1 : n, k)$) from the right.

Set $\tilde{\mathcal{Q}} = \mathrm{diag}(I_n, H(n-k+1, y))$.

Update $\mathcal{S} := \mathcal{J}\tilde{\mathcal{Q}}^T\mathcal{J}^T\mathcal{S}\tilde{\mathcal{Q}}$, $\mathcal{N} := \mathcal{J}\tilde{\mathcal{Q}}^T\mathcal{J}^T\mathcal{N}\tilde{\mathcal{Q}}$, $\mathcal{Q} := \mathcal{Q}\tilde{\mathcal{Q}}$.

END FOR k

Step 2 % *Reduce $\mathcal{N}$ to skew-Hamiltonian triangular form*

FOR $k = 1, \ldots, n-1$

% *I. Annihilate $\mathcal{N}(n+k, k+1 : n-1)$ as well as $\mathcal{N}(n+k+1 : 2n-1, k)$.*

FOR $j = k+1, \ldots, n-1$

a) Use $G(j, j+1, \theta_1)$ to eliminate $\mathcal{N}_{n+k,j}$ from the right. Set

$$\begin{aligned}\mathcal{N} &:= \mathcal{J}G(j,j+1,\theta_1)^T\mathcal{J}^T\mathcal{N}G(j,j+1,\theta_1),\\ \mathcal{S} &:= \mathcal{J}G(j,j+1,\theta_1)^T\mathcal{J}^T\mathcal{S}G(j,j+1,\theta_1),\\ \mathcal{Q} &:= \mathcal{Q}G(j,j+1,\theta_1).\end{aligned}$$

b) Use $G(n+j, n+j+1, \theta_2)$ to eliminate $\mathcal{S}_{n+j,n+j+1}$ from the right. Set

$$\begin{aligned}\mathcal{N} &:= \mathcal{J}G(n+j,n+j+1,\theta_2)^T\mathcal{J}^T\mathcal{N}G(n+j,n+j+1,\theta_2),\\ \mathcal{S} &:= \mathcal{J}G(n+j,n+j+1,\theta_2)^T\mathcal{J}^T\mathcal{S}G(n+j,n+j+1,\theta_2),\\ \mathcal{Q} &:= \mathcal{Q}G(n+j,n+j+1,\theta_2).\end{aligned}$$

END FOR j

% *II. Annihilate $\mathcal{N}_{n+k,n}$ (and simultaneously $\mathcal{N}_{2n,k}$).*

Use $G_s(n, \phi_1)$ to eliminate $\mathcal{N}_{n+k,n}$ from the right. Set

$$\begin{aligned}\mathcal{N} &:= G_s(n,\phi_1)^T\mathcal{N}G_s(n,\phi_1),\\ \mathcal{S} &:= G_s(n,\phi_1)^T\mathcal{S}G_s(n,\phi_1),\\ \mathcal{Q} &:= \mathcal{Q}G_s(n,\phi_1).\end{aligned}$$

% *III. Annihilate $\mathcal{N}(n+k, n+k+2 : 2n)$ as well as $\mathcal{N}(k+2 : n, k)$.*

FOR $j = n, n-1, \ldots, k+2$

a) Use $G(n+j-1, n+j, \psi_1)$ to eliminate $\mathcal{N}_{n+k,n+j}$ from the right.
Set

$$\begin{aligned}
\mathcal{N} &:= \mathcal{J}G(n+j-1, n+j, \psi_1)^T \mathcal{J}^T \mathcal{N} G(n+j-1, n+j, \psi_1), \\
\mathcal{S} &:= \mathcal{J}G(n+j-1, n+j, \psi_1)^T \mathcal{J}^T \mathcal{S} G(n+j-1, n+j, \psi_1), \\
\mathcal{Q} &:= \mathcal{Q}G(n+j-1, n+j, \psi_1).
\end{aligned}$$

b) Use $G_d(j-1, j, \psi_2)$ to eliminate $\mathcal{S}_{j,j-1}$ from the right. Set

$$\begin{aligned}
\mathcal{N} &:= \mathcal{J}G(j-1, j, \psi_2)^T \mathcal{J}^T \mathcal{N} G(j-1, j, \psi_2), \\
\mathcal{S} &:= \mathcal{J}G(j-1, j, \psi_2)^T \mathcal{J}^T \mathcal{S} G(j-1, j, \psi_2), \\
\mathcal{Q} &:= \mathcal{Q}G(j-1, j, \psi_2).
\end{aligned}$$

END FOR j

END FOR k

% *Now we have* $\mathcal{S} = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^T \end{bmatrix}$, $\mathcal{N} = \begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{11}^T \end{bmatrix}$.

Step 3 Apply the QZ algorithm [19] to the matrix pencil $\alpha S_{11} - \beta N_{11}$ to determine orthogonal matrices Q_1, Q_2 such that $Q_2^T S_{11} Q_1$ is upper triangular and $Q_2^T N_{11} Q_1$ is quasi upper triangular.

Set $\tilde{\mathcal{Q}} = \mathrm{diag}(Q_1, Q_2)$.

Update $\mathcal{S} := \mathcal{J}\tilde{\mathcal{Q}}^T \mathcal{J}^T \mathcal{S} \tilde{\mathcal{Q}}$, $\mathcal{N} := \mathcal{J}\tilde{\mathcal{Q}}^T \mathcal{J}^T \mathcal{N} \tilde{\mathcal{Q}}$, $\mathcal{Q} := \mathcal{Q}\tilde{\mathcal{Q}}$.

END

We also need an eigenvalue reordering method for Algorithm 2. We first introduce some subroutines.

Subroutine 3 *Given a regular* 2×2 *matrix pencil* $\alpha S - \beta H$ *with* S, H *upper triangular. This subroutine determines unitary matrices* Q_1, Q_2, *such that* $Q_2^H(\alpha S - \beta H)Q_1$ *is still upper triangular, but the eigenvalues are in reversed order.*

Input: 2×2 upper triangular matrices S, H.

Output: The 2×2 unitary matrices Q_1 and Q_2 described above.

Set $\gamma = s_{11}h_{22} - s_{22}h_{11}$.

IF $\gamma = 0$

% $\alpha S - \beta H$ *has a double eigenvalue*

Set $Q_1 = Q_2 = I_2$.

ELSE

Compute $Q_1 = G(1, 2, \arctan(\delta/\gamma), \arg(\gamma) - \arg(\delta))$ where $\delta = s_{12}h_{22} - s_{22}h_{12}$.

Compute $Q_2 = G(1, 2, \arctan(\delta/\gamma), \arg(\gamma) - \arg(\delta))$ where $\delta = -s_{11}h_{12} + s_{12}h_{11}$.

END IF

END

Subroutine 4 *Given a regular* 2×2 *complex skew-Hamiltonian/Hamiltonian matrix pencil* $\alpha\mathcal{S} - \beta\mathcal{H}$ *with* $\mathcal{S} = \begin{bmatrix} s_{11} & s_{12} \\ 0 & \bar{s}_{11} \end{bmatrix}$, $\mathcal{H} = \begin{bmatrix} h_{11} & h_{12} \\ 0 & -\bar{h}_{11} \end{bmatrix}$. *This subroutine determines a unitary matrix* $\mathcal{Q}$ *such that* $\mathcal{J}\mathcal{Q}^H\mathcal{J}^T(\alpha\mathcal{S}-\beta\mathcal{H})\mathcal{Q}$ *is upper triangular but the eigenvalues are in reversed order.*

Input: 2×2 upper triangular matrices $\mathcal{S} \in \mathbb{SH}_2$, $\mathcal{H} \in \mathbb{H}_2$.

Output: The 2×2 unitary matrix $\mathcal{Q}$ described above.

> Set $\gamma = 2\,\mathrm{Re}\,(h_{11}\bar{s}_{11})$.
>
> IF $\gamma = 0$
>
> % *$\alpha S - \beta H$ has a double purely imaginary eigenvalue*
>
> set $Q = I_2$.
>
> ELSE
>
> Compute $Q = G(1, 2, \arctan(\arctan(\delta/\gamma), \arg(\gamma) - \arg(\delta))$ with $\gamma = \tilde{z}_{11}$ and $\delta = \bar{s}_{11}h_{12} + s_{12}\bar{h}_{11}$.
>
> END IF

END

Using these subroutines, we now present the eigenvalue reordering algorithm for a complex skew-Hamiltonian/Hamiltonian matrix pencil.

Algorithm 6 *Given a regular* $2n \times 2n$ *complex skew-Hamiltonian/Hamiltonian matrix pencil* $\alpha\mathcal{S} - \beta\mathcal{H}$ *of the form*

$$\mathcal{S} = \begin{bmatrix} S & W \\ 0 & S^H \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} H & D \\ 0 & -H^H \end{bmatrix}, \tag{73}$$

with S *and* H *upper triangular. This algorithm determines a unitary matrix* $\mathcal{Q}$ *such that the matrix pencil* $\mathcal{J}\mathcal{Q}^H\mathcal{J}^T(\alpha\mathcal{S} - \beta\mathcal{H})\mathcal{Q}$ *remains in triangular form but the eigenvalues in* $\Lambda_-(\mathcal{S}, \mathcal{H})$ *are reordered in the leading principal subpencil.*

Input: Matrices $\mathcal{S} \in \mathbb{SH}_{2n}$ and $\mathcal{H} \in \mathbb{H}_{2n}$ in triangular form (73).

Output: Unitary matrix $\mathcal{Q}$ such that the eigenvalues of $\mathcal{J}\mathcal{Q}^H\mathcal{J}^T(\alpha\mathcal{S} - \beta\mathcal{H})\mathcal{Q}$ are in the desired order. $\mathcal{S}$ and $\mathcal{H}$ are overwritten by $\mathcal{J}\mathcal{Q}^H\mathcal{J}^T\mathcal{S}\mathcal{Q}$ and $\mathcal{J}\mathcal{Q}^H\mathcal{J}^T\mathcal{H}\mathcal{Q}$, respectively.

Step 0 Set $\mathcal{Q} = I_{2n}$.

Step 1 % *Reorder the eigenvalues in the subpencil* $\alpha S - \beta H$

> Set $m_- = 0$, $m_+ = n + 1$.
>
> % *I. Reorder the eigenvalues with negative real parts to the top*
>
> Set $k = 1$.
>
> WHILE $k \leq n$ DO
>
> IF $s_{kk} \neq 0$ and $\mathrm{Re}\,(h_{kk}/s_{kk})) < 0$
>
> FOR $j = k - 1, \ldots, m_- + 1$

a) Apply Subroutine 3 to the matrix pencil

$$\alpha \begin{bmatrix} s_{jj} & s_{j,j+1} \\ 0 & s_{j+1,j+1} \end{bmatrix} - \beta \begin{bmatrix} h_{jj} & h_{j,j+1} \\ 0 & h_{j+1,j+1} \end{bmatrix}$$

to determine unitary matrices Q_1, Q_2.

b) Set $\tilde{\mathcal{Q}} = \mathrm{diag}(I_{j-1}, Q_1, I_{n-j-1}, I_{j-1}, Q_2, I_{n-j-1})$.

c) Update $\mathcal{S} := \mathcal{J}\tilde{\mathcal{Q}}^H\mathcal{J}^T\mathcal{S}\tilde{\mathcal{Q}}$, $\mathcal{H} := \mathcal{J}\tilde{\mathcal{Q}}^H\mathcal{J}^T\mathcal{H}\tilde{\mathcal{Q}}$, $\mathcal{Q} := \mathcal{Q}\tilde{\mathcal{Q}}$.

END FOR j

$m_- := m_- + 1$

END IF

$k := k + 1$

END WHILE

% *II. Reorder the eigenvalues with positive real parts to the bottom*

Set $k = n$.

WHILE $k \geq m_- + 1$ DO

IF $s_{kk} \neq 0$ and $\mathrm{Re}\,(h_{kk}/s_{kk}) > 0$

FOR $j = k, \ldots, m_+ - 1$

a) Apply Subroutine 3 to the matrix pencil

$$\alpha \begin{bmatrix} s_{jj} & s_{j,j+1} \\ 0 & s_{j+1,j+1} \end{bmatrix} - \beta \begin{bmatrix} h_{jj} & h_{j,j+1} \\ 0 & h_{j+1,j+1} \end{bmatrix}$$

to determine unitary matrices Q_1, Q_2.

b) Set $\tilde{\mathcal{Q}} = \mathrm{diag}(I_{j-1}, Q_1, I_{n-j-1}, I_{j-1}, Q_2, I_{n-j-1})$.

c) Update $\mathcal{S} := \mathcal{J}\tilde{\mathcal{Q}}^H\mathcal{J}^T\mathcal{S}\tilde{\mathcal{Q}}$, $\mathcal{H} := \mathcal{J}\tilde{\mathcal{Q}}^H\mathcal{J}^T\mathcal{H}\tilde{\mathcal{Q}}$, $\mathcal{Q} := \mathcal{Q}\tilde{\mathcal{Q}}$.

END FOR j

$m_+ := m_+ - 1$

END IF

$k := k - 1$

END WHILE

% *The remaining $n - m_+ + 1$ eigenvalues with negative real part are now in the bottom right subpencil of $\alpha\mathcal{S} - \beta\mathcal{H}$.*

Step 2 % *Reorder the remaining $n - m_+ + 1$ eigenvalues*

FOR $k = n, \ldots, m_+$

% *I. Exchange the eigenvalues between two diagonal blocks*

a) Apply Subroutine 4 to the 2×2 matrix pencil

$$\alpha \begin{bmatrix} s_{nn} & w_{nn} \\ 0 & \bar{s}_{nn} \end{bmatrix} - \beta \begin{bmatrix} h_{nn} & d_{nn} \\ 0 & -\bar{h}_{nn} \end{bmatrix}$$

to determine a unitary matrix $Q = \begin{bmatrix} c & \bar{s} \\ -s & \bar{c} \end{bmatrix}$.

b) Let $Q_1 = \mathrm{diag}(1, \ldots, 1, c), \quad Q_2 = \mathrm{diag}(0, \ldots, 0, s)$ and set $\tilde{\mathcal{Q}} = \begin{bmatrix} Q_1 & \bar{Q}_2 \\ -Q_2 & \bar{Q}_1 \end{bmatrix}$.

c) Update $\mathcal{S} := \mathcal{J}\tilde{\mathcal{Q}}^H \mathcal{J}^T \mathcal{S}\tilde{\mathcal{Q}}$, $\mathcal{H} := \mathcal{J}\tilde{\mathcal{Q}}^H \mathcal{J}^T \mathcal{H}\tilde{\mathcal{Q}}$, $\mathcal{Q} := \mathcal{Q}\tilde{\mathcal{Q}}$.

% *II. Move the eigenvalue in n-th diagonal position to the $(m_- + 1)$ position*

$m_- := m_- + 1$

FOR $j = n-1, \ldots, m_-$

a) Apply Subroutine 3 to the matrix pencil

$$\alpha \begin{bmatrix} s_{jj} & s_{j,j+1} \\ 0 & s_{j+1,j+1} \end{bmatrix} - \beta \begin{bmatrix} h_{jj} & h_{j,j+1} \\ 0 & h_{j+1,j+1} \end{bmatrix}$$

to determine unitary matrices Q_1, Q_2.

b) Set $\tilde{\mathcal{Q}} = \mathrm{diag}(I_{j-1}, Q_1, I_{n-j-1}, I_{j-1}, Q_2, I_{n-j-1})$.

c) Update $\mathcal{S} := \mathcal{J}\tilde{\mathcal{Q}}^H \mathcal{J}^T \mathcal{S}\tilde{\mathcal{Q}}$, $\mathcal{H} := \mathcal{J}\tilde{\mathcal{Q}}^H \mathcal{J}^T \mathcal{H}\tilde{\mathcal{Q}}$, $\mathcal{Q} := \mathcal{Q}\tilde{\mathcal{Q}}$.

END FOR j

END FOR k

END