Elliptic problems in domains with edges: anisotropic regularity and anisotropic finite element meshes

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Abstract. This paper is concerned with the anisotropic singular behaviour of the solution of elliptic boundary value problems near edges. The paper deals first with the description of the analytic properties of the solution in newly defined, anisotropically weighted Sobolev spaces. The finite element method with anisotropic, graded meshes and piecewise linear shape functions is then investigated for such problems; the schemes exhibit optimal convergence rates with decreasing mesh size. For the proof, new local interpolation error estimates in anisotropically weighted spaces are derived. Moreover, it is shown that the condition number of the stiffness matrix is not affected by the mesh grading. Finally, a numerical experiment is described, that shows a good agreement of the calculated approximation orders with the theoretically predicted ones.

Key Words. Elliptic boundary value problem, singularities, anisotropic regularity, finite element method, anisotropic mesh grading.

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Contents

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1 Motivation and main ideas

1.1 The boundary value problem and analytical results

In this paper we want to study the approximation properties of the finite element method with anisotropic meshes (for an introduction to anisotropic meshes see Subsection 1.2) for certain elliptic boundary value problems over three-dimensional domains.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with non-intersecting edges. Especially we will focus on prismatic domains

$$\Omega = G \times I, \tag{1.1}$$

where $G \subset \mathbb{R}^2$ is a polygonal domain and $I = \,]\,0, z_0\,[\, \subset \mathbb{R}^1$ is an interval. The domain G may have a corner with interior angle $\omega > \pi$ at the origin; thus Ω has an edge which is part of the x_3-axis. The case of more than one edge can be treated similarly because the edge singularities we are interested in, are of local nature only.

Over this domain Ω, we consider the variational form of the boundary value problem which is given by the second order differential equation

$$-\sum_{i,j=1}^{3} a_{ij}\partial_{ij}u = f \quad \text{in } \Omega, \tag{1.2}$$

with either Dirichlet boundary conditions

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

or Newton boundary conditions

$$\sum_{i,j=1}^{3} a_{ij}\partial_i u\, n_j + \sigma u = 0 \quad \text{on } \partial\Omega. \tag{1.4}$$

The constant coefficients $a_{ij} = a_{ji} \in \mathbb{R}$ fulfill

$$\exists \alpha > 0 : \sum_{i,j=1}^{3} a_{ij}\xi_i\xi_j \geq \alpha|\underline{\xi}|^2 \quad \forall \underline{\xi} \in \mathbb{R}^3, \tag{1.5}$$

n_j $(j = 1, 2, 3)$ are the elements of the outward normal vector, and we assume $\sigma \geq 0$ on $\partial\Omega$ and $\sigma = \sigma(\underline{x}) \geq \sigma_0 > 0$ for all $\underline{x}$ in a part $\partial\Omega_T \subset \partial\Omega$ with $\text{meas}_2(\partial\Omega_T) > 0$. Then, the variational form is given for Dirichlet boundary conditions by:

$$\textit{find } u \in V_0 \textit{ such that } a(u, v) = (f, v) \textit{ for all } v \in V_0, \tag{1.6}$$

and in the case of problem (1.2)(1.4) by:

$$\textit{find } u \in V \textit{ such that } a(u, v) = (f, v) \textit{ for all } v \in V. \tag{1.7}$$

The bilinear form $a(.,.)$ and the linear form $(f,.)$ are defined by

$$a(u, v) := \int_\Omega \sum_{i,j=1}^{3} a_{ij}\partial_i u \partial_j v\, d\underline{x} + \int_{\partial\Omega} \sigma u v\, d\Gamma, \tag{1.8}$$

$$(f, v) := \int_\Omega f v\, d\underline{x}, \tag{1.9}$$

where the surface integral in (1.8) disappears in the case of Dirichlet boundary conditions (1.3). We use the abbreviations ∂_i for $\frac{\partial}{\partial x_i}$ and ∂_{ij} for $\partial_i\partial_j$. The spaces are defined by $V := H^1(\Omega)$ and $V_0 := \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$. For the data we consider $f \in L^p(\Omega)$ $(p \geq 2)$ and $\sigma \in W^{1-1/p,p}(\partial\Omega) \cap W^{1-1/s,s}(\partial\Omega)$ for some $s > 3$. $L^p(.)$ $(1 \leq p \leq \infty)$

are the usual Lebesgue spaces, $W^{s,p}(.)$ ($s \geq 0$, $1 \leq p \leq \infty$) the Sobolev(-Slobodetskiĭ) spaces (sometimes we write $W^{0,p}(.)$ for $L^p(.)$), and $H^s(.) := W^{s,2}(.)$. — Note that the conditions of the Lax–Milgram lemma are satisfied; thus the solution $u \in H^1(\Omega)$ of problems (1.6) and (1.7) exists and is unique.

It is well known that for domains with edges with interior angle $\omega > \pi$ the so-called shift theorem ($u \in H^{k+2}(\Omega)$ for $f \in H^k(\Omega)$) does not hold, and there are many papers where the regularity of the solution of these and more general problems is studied. We mention here the papers of Kondrat'ev [15] and Maz'ya/Plamenevskiĭ [18].

In [15], a representation formula for the solution u for $f \in L^2(\Omega)$ is given:

$$u = \xi(r)\gamma(\underline{x})\, r^\lambda \Phi(\varphi) + u_r \quad \text{with} \quad \lambda = \frac{\pi}{\omega_A},\ \gamma \in W^{2,2}_\lambda(\Omega), \tag{1.10}$$

where r, φ are polar coordinates in the plane perpendicular to the edge, $\omega_A \in]\pi, 2\pi[$ is a real number depending on ω and a_{ij} (see Subsection 2.3), $\xi(r)$ is a smooth cut-off function, $\Phi(\varphi) = \sin\lambda\varphi$ for Dirichlet boundary conditions and $\Phi(\varphi) = \cos\lambda\varphi$ for Neumann/Newton boundary conditions, $W^{2,2}_\lambda(\Omega) := \{v \in \mathcal{D}'(\Omega) : r^\lambda D^\alpha v \in L^2(\Omega)\ \forall |\alpha| \leq 2\}$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index and $D^\alpha := \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$.

In [18], the solution is described in the framework of another type of weighted Sobolev spaces: let $f \in L^p(\Omega)$ then

$$u \in V^{2,p}_\beta(\Omega) \quad \text{for } \beta > 2 - \frac{2}{p} - \frac{\pi}{\omega_A}, \tag{1.11}$$

$V^{2,p}_\beta(\Omega) := \{v \in \mathcal{D}'(\Omega) : r^{\beta-2+|\alpha|}D^\alpha v \in L^p(\Omega)\ \forall |\alpha| \leq 2\}$. Note that for problems with more than one edge (with interior angle greater than π) an adequate number of singular terms has to be included in (1.10) and weights corresponding to each edge have to be introduced in the space $V^{2,p}_\beta(\Omega)$.

The anisotropic structure of the edge is reflected by the factor r^λ in (1.10) and the weights in the definition of the spaces $W^{2,2}_\lambda(\Omega)$ and $V^{2,2}_\beta(\Omega)$, because r is the distance to the edge and is independent of the tangential coordinate of the edge. Using these results it has been possible to justify a mesh refinement strategy near edges [2, 6, 17] in order to improve the approximation order (which is in general low because of the low regularity of the solution) of the standard finite element method. In this strategy, isotropic elements (that are elements whose ratio of the diameters of the smallest circumscribed and the largest inscribed balls is bounded independently of the mesh size h) are used, and the size of the elements is determined by their distance to the edge. We remark that isotropic strategies near corners in two dimensions are widely investigated [7, 10, 11, 21, 23, 25].

This result is not really satisfactory because it seems to be natural to treat anisotropic structures like edges with anisotropic finite elements. According to [1] an element is called anisotropic when its diameter in different directions has different asymptotics and, consequently, the ratio of the outer and the inner ball is growing to infinity for $h \to 0$. As shown in that paper for problems with smoother data than we assume here, these elements can be applied successfully in the finite element method with graded meshes near edges.

It was an open problem to justify this anisotropic strategy also for problems with $f \in L^p(\Omega)$ ($p \geq 2$). But the analytic results (1.10) and (1.11) have been insufficient, because the weighted Sobolev spaces used have the disadvantage that all derivatives of the same order have the same weight. This drawback is removed in Section 2 by using more appropriate, anisotropically weighted Sobolev spaces. It is proved that for the solution u from (1.6) or (1.7) the inclusion

$$u \in A^{2,p}_\beta(\Omega) \text{ with } \begin{cases} \beta > 2 - \frac{2}{p} - \frac{\pi}{\omega_A} & \text{for } 2 - \frac{2}{p} \geq \frac{\pi}{\omega_A} > \psi(p) \\ \beta = 0 & \text{for } 2 - \frac{2}{p} < \frac{\pi}{\omega_A} \end{cases} \tag{1.12}$$

holds, where $\psi(p) := 1 - \frac{2}{p}$ in the case if Dirichlet or Neumann boundary conditions, and $\psi(p) := \frac{3}{2} - \frac{3}{p}$ in the case of Newton boundary conditions. The space $A^{2,p}_\beta(\Omega)$ is defined by $A^{2,p}_\beta(\Omega) := \{v \in \mathcal{D}'(\Omega) : \|v; A^{2,p}_\beta(\Omega)\| < \infty\}$,

$$
\begin{aligned}
|v; A^{2,p}_\beta(\Omega)|^p &:= \int_\Omega \left\{ r^{\beta p} \sum_{i,j=1}^{2} |\partial_{ij} u|^p + \sum_{i=1}^{3} |\partial_{3i} u|^p \right\} d\underline{x}, \\
\|v; A^{2,p}_\beta(\Omega)\|^p &:= |v; A^{2,p}_\beta(\Omega)|^p + \int_\Omega \left\{ r^{(\beta-1)p} \sum_{i=1}^{2} |\partial_i u|^p + r^{-p} |\partial_3 u|^p + r^{(\beta-2)p} |u|^p \right\} d\underline{x},
\end{aligned}
$$

and x_3 is the direction of the edge. Particularly, that means $\partial_3 u \in V_0^{1,p}(\Omega) \hookrightarrow W^{1,p}(\Omega)$. For $\frac{\pi}{\omega_A} \le 1 - \frac{2}{p}$ (that means $p \ge 2/(1 - \frac{\pi}{\omega_A})$) we do not have $\partial_3 u \in W^{1,p}(\Omega)$.

Remark 1.1 For general polyhedral domains we have to distinguish between corner and edge singularities. The behaviour of the edge singularities is not different from that described above. The corner singularities are not a problem of anisotropy; they can be treated with isotropic, graded meshes as introduced for example in [6]. The main problem is to construct meshes which are both anisotropic near edges and isotropic near those corners which cause singularities. We will discuss this in a forthcoming paper. — Note that the domain introduced by (1.1) has corners, too. But they do not introduce additional corner singularities [26].

Remark 1.2 It is conjectured that each component of the solution of the Lamé system of linear elasticity has similar properties as described above for the scalar equation, and that our forthcoming approximation results apply also in that case. Indeed, we were able to prove this in the case $p = 2$. The case $p > 2$ has still to be investigated.

1.2 The class of finite element meshes

Assume that we are given a family $\mathcal{T}$ of finite element partitions $\mathcal{T}_h$ with the usual regularity properties:

(a) $\overline{\Omega} = \bigcup_{i=1}^m \overline{\Omega}_i$, where Ω_i are tetrahedra,

(b) $\Omega_i \cap \Omega_j = \emptyset$ for $i \ne j$ $(i, j = 1, \dots, m)$,

(c) any edge or face of Ω_i is either a subset of $\partial\Omega$ or an edge or face of another Ω_j $(i, j = 1, \dots, m)$.

Then we introduce the finite element space V_h of all continuous functions whose restriction to any Ω_i $(i = 1, \dots, m)$ is a polynomial of first degree. Furthermore, we let V_{0h} be defined by $V_{0h} := \{v_h \in V_h : v_h|_{\partial\Omega} = 0\}$. Note that $V_h \subset V$ and $V_{0h} \subset V_0$.

The finite element solutions of problems (1.6) and (1.7) are defined by:

$$\textit{find } u_h \in V_{0h} \textit{ such that } a(u_h, v_h) = (f, v_h) \textit{ for all } v_h \in V_{0h}, \tag{1.13}$$

and

$$\textit{find } u_h \in V_h \textit{ such that } a(u_h, v_h) = (f, v_h) \textit{ for all } v_h \in V_h, \tag{1.14}$$

respectively. The assumptions of the Lax–Milgram lemma are fulfilled; thus these problems have a unique solution.

The investigation of the finite element error $u - u_h$ in the energy norm (here equivalent to the $W^{1,2}(\Omega)$-norm) is usually reduced via Céa's lemma to a general approximation problem. If we want to take advantage of anisotropic finite element meshes (and this kind of mesh seems to be natural near edges, see above), we need an approximation operator for which error estimates are available that take these different asymptotic mesh sizes of the elements into account. As far as we know, such estimates are only available for the interpolation operator [1], see also (3.16). But in order to use these local estimates and to extend them to weighted Sobolev spaces, the mesh must satisfy two more conditions (d) and (e). Moreover, another assumption (f) is necessary for the extension of these estimates to weighted Sobolev spaces. This extension is done in Subsection 3.1 because it is necessary for our global estimate in Subsection 3.2.

For the explanation and for further use we introduce the following notation: Assume we are given a finite element Ω_i. Let e_i be the longest edge of Ω_i and f_i the larger of the two faces of Ω_i with $e_i \subset \overline{f}_i$. Then we denote by $h_{3,i} := \text{meas}_1(e_i)$ the length of e_i, by $h_{2,i} := 2\,\text{meas}_2(f_i)/h_{3,i}$ the diameter of f_i perpendicularly to e_i and by $h_{1,i} := 6\,\text{meas}_3(\Omega_i)/(h_{2,i}h_{3,i})$ the diameter of Ω_i perpendicularly to f_i. Note that $h_{3,i} \geq h_{2,i} \geq h_{1,i}$.

Introduce further local Cartesian coordinate systems $(x_{1,i}, x_{2,i}, x_{3,i})$ such that $(0,0,0)$ is a vertex of Ω_i, e_i is part of the $x_{3,i}$-axis, and f_i is part of the $x_{2,i}, x_{3,i}$-plane. Note that each coordinate system can be transformed via a translation and three rotations around the $x_{j,i}$-axes by an angle $\psi_{j,i}$ $(j = 1,2,3)$ into the original coordinate system (x_1, x_2, x_3). (The angles $\psi_{j,i}$ depend on the order of the three rotations but this influence is of lower order.)

Let the following assumptions be also fulfilled:

(d) all elements Ω_i have to fulfill the maximal angle condition: let $\gamma_{e,i}$ be the maximal angle between faces of Ω_i and $\gamma_{f,i}$ be the maximal interior angle of the four triangular faces of Ω_i $(i = 1, \dots, m)$, then the relations $\gamma_{e,i} \leq \gamma_0 < \pi$ and $\gamma_{f,i} \leq \gamma_0 < \pi$ have to be fulfilled with γ_0 independent of the element counter i and the mesh size parameter h,

(e) the elements are located such that the angles $\psi_{j,i}$ fulfill the following relations:

$$|\tan\psi_{1,i}| \leq C\frac{h_{2,i}}{h_{3,i}}, \ |\tan\psi_{2,i}| \leq C\frac{h_{1,i}}{h_{3,i}}, \ |\tan\psi_{3,i}| \leq C\frac{h_{1,i}}{h_{2,i}}, \ (i = 1, \dots, m),$$

with the exception that the first (respectively the third) inequality is not necessary if $h_{2,i}$ is of order $h_{3,i}$ (respectively $h_{1,i}$ is of order $h_{2,i}$),

(f) all elements Ω_i with distance $r_i = 0$ to the edge (x_3-axis) have two vertices such that the straight line through them is parallel to the x_3-axis.

So we introduce a graded mesh by conditions (a)–(f) and the following choice of the element sizes:

(g) With h being the mesh size parameter, $\mu \in]0,1]$ being the grading parameter, r_i being the distance of Ω_i to the edge ($r_i := \min_{(x_1,x_2,x_3)\in\overline{\Omega}_i}(x_1^2 + x_2^2)^{1/2}$) and some constant $R > 0$ we define real numbers h_i $(i = 1, \dots, m)$

$$h_i := \begin{cases} h^{1/\mu} & \text{for } r_i = 0, \\ h r_i^{1-\mu} & \text{for } 0 < r_i \leq R, \\ h & \text{for } r_i > R, \end{cases} \tag{1.15}$$

and assume that there are positive constants C_1 and C_2 such that for the element sizes $h_{1,i}$, $h_{2,i}$, $h_{3,i}$ the relations

$$\begin{array}{rcccl} C_1 h_i & \leq & h_{j,i} & \leq & C_2 h_i, \quad j = 1,2, \\ C_1 h & \leq & h_{3,i} & \leq & C_2 h, \end{array} \tag{1.16}$$

are fulfilled for $i = 1, \dots, m$.

Corollary 1.3 *For such meshes the following relation holds:*

$$\|v_h; W^{1,2}(\Omega_i)\| \le C h_i^{-1} \|v_h; L^2(\Omega_i)\|, \quad i = 1, \ldots, m, \tag{1.17}$$

which is a special case of the inverse inequality, see [8, Theorem 3.2.6]*. Note that the diameter of the largest ball inscribed in Ω_i has a diameter of order $h_i \le Ch$.*

Note that we use the symbol C for a generic positive constant, that means, C may be of different value at each occurrence. But C is always independent of the function under consideration and of the finite element mesh.

Corollary 1.4 *The volume of any element Ω_i is of order $h_i^2 h$ $(i = 1, \ldots, m)$.*

Because we consider up to now only a domain Ω with the special structure (1.1) it is easy to construct such a mesh:

1. Find a quasiuniform mesh for G with a conventional mesh generator (inner and outer circle shall be of the same order h for all elements).

2. Reproduce this two-dimensional mesh in each plane $x_3 = j\hbar$ ($j = 0, \ldots, J$; $J := \text{int}(z_0/h)$; $\hbar := z_0/J$; $\text{int}(w)$ is the integer part of the real number w), and form a partition of pentahedra (prismatic elements with a triangular basis) using corresponding triangles of two adjacent planes $x_3 = (j-1)h$ and $x_3 = jh$ ($j = 1, \ldots, J$). Divide each pentahedron into three tetrahedra (observing condition (c)).

3. Apply the coordinate transformation

$$\begin{aligned} r &:= (x_{1,\text{old}}^2 + x_{2,\text{old}}^2)^{1/2} \\ x_{1,\text{new}} &:= r^{-1+1/\mu} x_{1,\text{old}} \\ x_{2,\text{new}} &:= r^{-1+1/\mu} x_{2,\text{old}} \end{aligned} \tag{1.18}$$

   to all nodes of the triangulation (at least in a neighbourhood of the edge).

This approach of constructing a graded partition using a coordinate transformation goes back to [20] where it was used in the two-dimensional case. It is explained in detail in [5].

1.3 Outline of the paper

In Section 2 we prove the anisotropic regularity (1.12) of the solution u from (1.6) or (1.7). This is first done for the Laplace operator with Dirichlet boundary conditions in Subsection 2.1, using a representation formula for u from [14]. Furthermore, we prove that the assumption $p < 2/(1 - \frac{\pi}{\omega_A})$ in (1.12) is necessary. In Subsections 2.2–2.4 we extend this result to more general boundary conditions, to a general elliptic second order operator with constant coefficients, and partially (only for $p = 2$) to the Lamé system. These results are then applied in the investigation of the finite element error for the problems introduced above.

In Section 3 we use the standard way, namely the estimation of the interpolation error. One difficulty is that the anisotropic local interpolation error estimate

$$|v - Iv; W^{1,p}(\Omega_i)| \le C \sum_{j=1}^{3} h_{j,i} |\partial_j u; W^{1,p}(\Omega_i)| \tag{1.19}$$

does not hold for $p = 2$, but only for $p > 2$. That is why we restrict our consideration to problems with a right hand side $f \in L^p(\Omega)$ with $p > 2$. Another task is to prove an approximation result for elements Ω_i touching the edge, because the solution does not belong to $W^{2,p}(\Omega_i)$ $(p > 2)$ there, even if we would assume smooth data. Under the conditions (d)–(f) and certain assumptions on p and β (see Theorem 3.11), we get

$$|v - Iv; W^{1,p}(\Omega_i)| \leq C\,(h_{1,i}^{1-\beta} + h_{3,i})|v; A_\beta^{2,p}(\Omega_i)|, \tag{1.20}$$

$$\|v - Iv; L^p(\Omega_i)\| \leq C\,(h_{1,i}^{2-\beta} + h_{3,i}^2)|v; A_\beta^{2,p}(\Omega_i)|. \tag{1.21}$$

The global result is then formulated in Theorem 3.13 and we get for $p \in]2, p_+[$

$$\|u - u_h; W^{1,2}(\Omega)\| \leq Ch^s \|f; L^p(\Omega)\| \tag{1.22}$$

with

$$s = \begin{cases} 1 & \text{for } \mu < \frac{\pi}{\omega_A} \cdot \frac{p}{2p-2}, \\ \frac{2}{p} - 1 + \frac{1}{\mu}\frac{\pi}{\omega_A} - \varepsilon & \text{for } \mu \geq \frac{\pi}{\omega_A} \cdot \frac{p}{2p-2}, \end{cases} \tag{1.23}$$

$$p_+ := \begin{cases} (1 - \frac{\pi}{\omega_A})^{-1} & \text{for Dirichlet or Neumann b.c.,} \\ \min\left\{(1 - \frac{\pi}{\omega_A})^{-1}, (\frac{1}{2} - \frac{\pi}{3\omega_A})^{-1}\right\} & \text{for Newton boundary cond.} \end{cases} \tag{1.24}$$

A similar estimate holds for mixed boundary conditions with some restrictions to ω_A.

In Section 4 we investigate the condition number of the stiffness matrix. We show that it is of the optimal order h^{-2} for the full range of $\mu \in]0, 1]$. Note that in the isotropic case the condition $\mu > \frac{1}{3}$ is required [2, 6].

For test calculations we can refer to another paper. In [4] we documented a test, where one problem was calculated with isotropic as well as with anisotropic graded meshes. We derived approximation orders from the finite element errors for different mesh size parameters h. We observed a good agreement of the calculated approximation orders with the expected ones (see (1.23)). Moreover, it turned out that the same error level can be achieved with less computational effort (smaller number of elements, of nodes, of degrees of freedom) with anisotropic meshes in comparison with isotropic ones.

Another test is documented in the last section. There, the exact solution has a jump in the second derivative in edge direction.

2 Anisotropic regularity near the edge

2.1 The Laplace operator with Dirichlet boundary conditions

Let $D := C \times \mathbb{R}$ be a dihedral cone of $\mathbb{R}^3$, where C is an infinite cone of $\mathbb{R}^2$ of opening ω. As before, we denote by $\underline{x} = (x_1, x_2, x_3)$ the Cartesian coordinates in D, where $x_3 \in \mathbb{R}$ and $(x_1, x_2) \in C$ and by (r, φ) the polar coordinates in C. We are concerned with the edge regularity of the variational solution $v \in \overset{\circ}{H}{}^1(D)$ of the Dirichlet problem

$$-\Delta v = g \in L^p(D), \tag{2.1}$$

for $p \geq 2$. Since we are only interested in the local behaviour of the solution, we suppose that v exists and has a compact support.

For studying the regularity of v, we shall employ some weighted Sobolev spaces of Kondrat'ev type, introduced for instance in [18]. For $l \in \mathbb{N}, \beta \in \mathbb{R}, p \in]1, +\infty[$, we recall that

$$V_\beta^{l,p}(D) := \{v \in \mathcal{D}'(D) : r^{\beta - l + |\alpha|} D^\alpha v \in L^p(D), \forall |\alpha| \leq l\}$$

is a Banach space for the norm

$$\|v; V_\beta^{l,p}(D)\| := \left(\sum_{|\alpha|\le l} \int_D r^{(\beta-l+|\alpha|)p} |D^\alpha v|^p \, d\underline{x} \right)^{1/p}.$$

We start with a weak isotropic regularity result:

Lemma 2.1 *For any $\varepsilon > 0$, we have*

$$v \in V_{1+\varepsilon}^{2,2}(D). \tag{2.2}$$

Proof Using Hardy's inequalities, we find that $v \in V_{-1+\varepsilon}^{0,2}(D)$. This inclusion and Theorem 4.1 of [18] lead to (2.2). □

This allows to use comparison theorems in weighted Sobolev spaces in order to get anisotropic regularity:

Theorem 2.2 *Let $\alpha \ge 0$ be such that*

$$\begin{cases} 2 - \frac{2}{p} - \alpha & < & \frac{\pi}{\omega} & \text{if } 2 - \frac{2}{p} \ge \frac{\pi}{\omega}, \\ \alpha & = & 0 & \text{if } 2 - \frac{2}{p} < \frac{\pi}{\omega}. \end{cases} \tag{2.3}$$

Then

$$v \in V_\alpha^{2,p}(D). \tag{2.4}$$

If moreover

$$1 - \frac{2}{p} < \frac{\pi}{\omega}, \tag{2.5}$$

then

$$\partial_3 v \in V_0^{1,p}(D). \tag{2.6}$$

Proof The inclusion (2.4) is a direct consequence of Theorem 7.2 of [18], using Lemma 2.1 and since $g \in V_{1+\varepsilon}^{0,2}(D) \cap V_\alpha^{0,p}(D)$.

The inclusion (2.6) follows now from Theorem 3.1 of [19] (see also Theorem 30.1 of [16]), since the assumption of that theorem is equivalent to (2.5). □

We shall now improve this theorem using recent results of Grisvard [14]. The obtained inclusions will sometimes recover the above ones, but due to the convenient form of the Laplace operator, their proofs are simpler. Let us first recall Theorem 6.6 of [14]:

Theorem 2.3 *Suppose that $\frac{j\pi}{\omega} \ne 2 - \frac{2}{p}$ for all $j \in \mathbb{Z}$, then the solution $v \in \overset{o}{H}{}^1(D)$ of problem (2.1) admits the decomposition*

$$v = v_r + \sum_{0<\frac{j\pi}{\omega}<2-\frac{2}{p}} (K_j \overset{x_3}{\star} q_j)\, \psi_j, \tag{2.7}$$

where $v_r \in W^{2,p}(D)$ is the regular part of v, $q_j \in B^{2-\frac{2}{p}-\frac{j\pi}{\omega},p}(\mathbb{R})$ (that means in the classical Sobolev space $W^{2-\frac{2}{p}-\frac{j\pi}{\omega},p}(\mathbb{R})$, if $2 - \frac{2}{p} - \frac{j\pi}{\omega} \notin \mathbb{Z}$, otherwise in the Besov space $B^{2-\frac{2}{p}-\frac{j\pi}{\omega},p}(\mathbb{R})$, see [27]), ψ_j are the 2D-singular functions of the Laplace operator in C:

$$\psi_j(r,\varphi) := \xi(r) r^{j\pi/\omega} \sin\left(\frac{j\pi\varphi}{\omega}\right), \tag{2.8}$$

and finally K_j are kernels defined by

$$\begin{aligned} K_j(r,x_3) &:= \frac{r}{\pi(r^2+x_3^2)} \quad \text{if } \frac{j\pi}{\omega} > 1-\frac{2}{p}, \\ K_j(r,x_3) &:= \frac{2r^3}{\pi(r^2+x_3^2)^2} \quad \text{if } \frac{j\pi}{\omega} \leq 1-\frac{2}{p}. \end{aligned}$$

There exists a positive constant C independent of g, such that

$$\|v_r; W^{2,p}(D)\| + \sum_{0<\frac{j\pi}{\omega}<2-\frac{2}{p}} \|q_j; B^{2-\frac{2}{p}-\frac{j\pi}{\omega},p}(\mathbb{R})\| \leq C\,\|g; L^p(\mathbb{R})\|.$$

Here and in the sequel, $K \overset{x_3}{\star} q$ means the convolution with respect to the edge parameter x_3:

$$(K \overset{x_3}{\star} q)(r,x_3) := \int_{\mathbb{R}} K(r,s)q(x_3-s)\,ds.$$

In view of that Theorem, if we want to prove inclusions of type (2.4) or (2.6), it suffices to show that the 3D-singularity function

$$v_j := (K_j \overset{x_3}{\star} q_j)\psi_j \tag{2.9}$$

satisfies such inclusions. Their proofs are based on the next general result concerning convolution with arbitrary kernels, which is inspired from Theorem 6.5 of [14] (notice that this theorem had a different goal).

Theorem 2.4 *Let $K(r,x_3)$ be a kernel satisfying*

$$|K(r,x_3)| \leq C\frac{r^\beta}{(r^2+x_3^2)^\gamma}, \forall r>0, x_3\in\mathbb{R}, \tag{2.10}$$

with some $C>0$ and $\gamma > \frac{1}{2}$ (in order that K would be integrable with respect to x_3) and

$$\int_{\mathbb{R}} K(r,x_3)\,dx_3 = 0. \tag{2.11}$$

For $q \in B^{\sigma,p}(\mathbb{R})$, with $\sigma \in]0,1]$, we set

$$h(r,x_3) := (K \overset{x_3}{\star} q)(r,x_3). \tag{2.12}$$

If $\sigma < 2\gamma - 1$ and $\beta \geq -1-\frac{2}{p}-\sigma+2\gamma$, then there exists a constant $C_1>0$ (independent of q) such that

$$\left(\int_0^1\int_{\mathbb{R}} |h(r,x_3)|^p\, r dr dx_3\right)^{1/p} \leq C_1\|q; B^{\sigma,p}(\mathbb{R})\|. \tag{2.13}$$

Proof From assumption (2.11), we may write

$$h(r,x_3) = \int_{\mathbb{R}} K(r,s)\{q(x_3-s)-q(x_3)\}\,ds,$$

and taking the L^p-norm with respect to x_3, we obtain

$$\|h(r,x_3); L^p_{x_3}(\mathbb{R})\| \leq \int_{\mathbb{R}} |K(r,s)|\cdot\|q(x_3-s)-q(x_3); L^p_{x_3}(\mathbb{R})\|\,ds. \tag{2.14}$$

Let us introduce the functions

$$\begin{aligned} \kappa(s) &:= |s|^{-\sigma-1/p}\|q(x_3-s)-q(x_3); L^p_{x_3}(\mathbb{R})\|, \\ k(t) &:= \frac{|t|^{2\gamma-\sigma-1}}{(1+t^2)^\gamma}, \end{aligned}$$

and the multiplicative convolution I of k with the function $s^{1/p}\kappa$,

$$I(r) := \int_{\mathbb{R}} k(r/s)s^{1/p}\kappa(s)\,\frac{ds}{|s|}.$$

Inserting (2.10) into (2.14) we obtain

$$\|h(r,x_3); L^p_{x_3}(\mathbb{R})\| \leq C r^{\beta+1+\sigma-2\gamma} I(r). \tag{2.15}$$

The assumption $q \in B^{\sigma,p}(\mathbb{R})$ implies that κ belongs to $L^p(\mathbb{R})$ and

$$\|\kappa; L^p(\mathbb{R})\| \leq C_2 \|q; B^{\sigma,p}(\mathbb{R})\|,$$

for some $C_2 > 0$ independent of q. For $\sigma < 1$ this is a direct implication, otherwise we use Theorem 2.5.1 of [27]. Moreover, we readily check that $k \in L^1(\mathbb{R}^+, \frac{dt}{t})$ (this is the space of integrable functions with respect to the measure $\frac{dt}{t}$) iff $-1 < \sigma < 2\gamma - 1$; therefore Young's theorem leads to

$$\left(\int_0^{+\infty} |I(r)|^p \,\frac{dr}{r}\right)^{1/p} \leq C\|\kappa; L^p(\mathbb{R})\| \leq CC_2\|q; B^{\sigma,p}(\mathbb{R})\|. \tag{2.16}$$

Integrating the p-th power of the estimate (2.15) with respect to r on $]0,1[$ and using (2.16), we arrive at (2.13). □

We are now able to prove some anisotropic regularities:

Theorem 2.5 *If* $0 < \frac{j\pi}{\omega} < 2 - \frac{2}{p}$, *then*

$$\partial_{33} v_j \in L^p(D), \tag{2.17}$$

and there exists a positive constant C such that

$$\|\partial_{33} v_j; L^p(D)\| \leq C\,\|g; L^p(\mathbb{R})\|. \tag{2.18}$$

Proof If

$$1 - \frac{2}{p} < \frac{j\pi}{\omega} < 2 - \frac{2}{p}, \tag{2.19}$$

we use Theorem 2.4, with $K(r,x_3) = r^{j\pi/\omega}\partial_{33}K_j(r,x_3)$, since

$$\partial_{33} v_j = (K \overset{x_3}{\star} q_j)(r,x_3)\,\xi(r)\sin\left(\frac{j\pi\varphi}{\omega}\right).$$

This kernel satisfies $|K(r,x_3)| \leq C r^{1+j\pi/\omega}(r^2+x_3^2)^{-2}$ for all $r > 0, x_3 \in \mathbb{R}$. Therefore, we can apply Theorem 2.4 with $\beta = 1 + \frac{j\pi}{\omega}$, $\gamma = 2$ and $\sigma = 2 - \frac{2}{p} - \frac{j\pi}{\omega}$ ($\sigma < 1$ due to assumption (2.19)). Since the hypotheses of that theorem are satisfied, estimate (2.13) can be rephrased as

$$\int_0^1 \int_{\mathbb{R}} |\partial_{33} v_j|^p \;\, r dr dx_3 \leq C\,\|q_j; B^{\sigma,p}(\mathbb{R})\|^p.$$

An integration with respect to φ leads to (2.17) and (2.18).

Conversely, if

$$0 < \frac{j\pi}{\omega} \leq 1 - \frac{2}{p}, \tag{2.20}$$

then we know that $q_j \in B^{\sigma',p}(\mathbb{R})$, with $\sigma' = 2 - \frac{2}{p} - \frac{j\pi}{\omega} \geq 1$. If $\sigma' = 1$, we use Theorem 2.4 as above with $K(r,x_3) = r^{j\pi/\omega}\partial_{33}K_j(r,x_3)$, $q = q_j \in B^{\sigma,p}(\mathbb{R})$, when $\sigma = \sigma'$. On the contrary, if $\sigma' > 1$, we apply Theorem 2.4 with $K(r,x_3) = r^{j\pi/\omega}\partial_3 K_j(r,x_3)$, $q = \partial_3 q_j \in B^{\sigma,p}(\mathbb{R})$, when $\sigma = \sigma' - 1$. □

Analogously, we can consider other derivatives:

Theorem 2.6 *If* $0 < \frac{j\pi}{\omega} < 2 - \frac{2}{p}$, *then*

$$\partial_3 v_j \in L^p(D), \tag{2.21}$$
$$v_j \in L^p(D), \tag{2.22}$$

with norms depending continuously on the L^p*-norm of* g. *If moreover,* $1 - \frac{2}{p} < \frac{j\pi}{\omega}$, *then*

$$\partial_{13} v_j, \partial_{23} v_j \in L^p(D), \tag{2.23}$$
$$r^{\gamma-1}\partial_1 v_j, r^{\gamma-1}\partial_2 v_j \in L^p(D), \tag{2.24}$$
$$r^{\gamma-2} v_j \in L^p(D), \tag{2.25}$$
$$r^{-1}\partial_3 v_j \in L^p(D), \tag{2.26}$$

with $\gamma > 2 - \frac{2}{p} - \frac{j\pi}{\omega}$, *the norms depending continuously on the* L^p*-norm of* g.

Proof Property (2.21) follows from Theorem 2.4 with $K(r, x_3) = r^{j\pi/\omega}\partial_3 K_j$ and $q = q_j$; in the same way, we get (2.26) by multiplying this kernel by r^{-1}. The proof of (2.23) is identical. Indeed, we have

$$\partial_{r3} v_j = \{K_{1j} \overset{x_3}{\star} q_j\}\psi_j + \{K_{2j} \overset{x_3}{\star} q_j\}\partial_r \psi_j,$$

where $K_{2j} = \partial_3 K_j$, $K_{1j} = \partial_r K_{2j}$. For the first term (respectively the second term), we apply Theorem 2.4 with $K(r, x_3) = r^{j\pi/\omega} K_{1j}$ (respectively $K(r, x_3) = r^{-1+j\pi/\omega} K_{2j}$) and $q = q_j$. The derivative $\frac{1}{r}\partial_{\varphi 3} v_j$ is treated similarly.

To establish the other inclusions, we can no more apply Theorem 2.4 because the corresponding kernels do not satisfy (2.11). Therefore, we proceed as follows: for the derivative $\partial_r v_j$, for instance, we must consider a term of the form

$$h(r, x_3) := (\partial_r K_j) \overset{x_3}{\star} q_j\, \psi_j.$$

As $|\partial_r K_j| = \mathcal{O}(\frac{1}{r^2 + x_3^2})$, we get

$$\begin{aligned} \|h(r, x_3); L^p_{x_3}(\mathbb{R})\| &\leq C r^{j\pi/\omega} \int_{\mathbb{R}} |\partial_r K_j(r, s)|\; \|q_j(x_3 - s); L^p_{x_3}(\mathbb{R})\|\, ds \\ &\leq C r^{-1+j\pi/\omega} \|q_j; L^p(\mathbb{R})\|. \end{aligned}$$

Multiplying this estimate by $r^{\gamma-1}$ and integrating with respect to r on $]0, 1[$ and to φ, we obtain the inclusion $r^{\gamma-1} h \in L^p(D)$. Other terms are treated analogously. □

Corollary 2.7 *Let* $u \in \overset{\circ}{H}{}^1(\Omega)$ *be the solution of* $-\Delta u = f$, *with* $f \in L^p(\Omega)$, *then*

$$u \in A^{2,p}_\beta(\Omega) \text{ with } \begin{cases} \beta > 2 - \frac{2}{p} - \frac{\pi}{\omega} & \text{for } 2 - \frac{2}{p} \geq \frac{\pi}{\omega} > 1 - \frac{2}{p} \\ \beta = 0 & \text{for } 2 - \frac{2}{p} < \frac{\pi}{\omega} \end{cases}$$

and

$$\|u; A^{2,p}_\beta(\Omega)\| \leq C\|f; L^p(\Omega)\|.$$

For the definition of $A^{2,p}_\beta(\Omega)$, *see Subsection 1.1.*

Proof Let

$$v := \begin{cases} u & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^3 \setminus \Omega, \end{cases} \quad \text{and} \quad g := \begin{cases} f & \text{in } \Omega, \\ 0 & \text{in } D \setminus \Omega, \end{cases}$$

then v is the variational solution of $-\Delta v = g \in L^p(D)$. Using Theorem 2.6 we get $v \in A^{2,p}_\beta(D)$. The restriction to Ω yields the assertion. □

Let us remark that we cannot improve the conclusions of Theorems 2.5 and 2.6. Indeed, when we apply Theorem 2.4, we get an equality in the condition $\beta \geq -1 - \frac{2}{p} -$

$\sigma + 2\gamma$, in other words we cannot decrease the value of β. This means that, in general, we cannot decrease the power in r in front of the considered derivatives.

Let us also show that the condition $1 - \frac{2}{p} < \frac{j\pi}{\omega}$ in the second part of Theorem 2.6 is necessary, in the sense that without this condition, the conclusion could fail.

Lemma 2.8 *If $0 < \frac{j\pi}{\omega} \leq 1 - \frac{2}{p}$ and $q_j \in B^{2-\frac{2}{p}-\frac{j\pi}{\omega},p}(\mathbb{R})$ is a continuous function such that $q_j \geq 0$, $q_j \not\equiv 0$. Then v_j given by (2.9) satisfies*

$$\frac{1}{r}\partial_\varphi v_j \notin L^p(D). \tag{2.27}$$

Proof By a direct computation, we show that

$$\frac{1}{r}\partial_\varphi v_j = \frac{2j}{\omega} h(r, x_3)\xi(r) r^{-1+j\pi/\omega} \cos(\frac{j\pi\varphi}{\omega}), \tag{2.28}$$

where we have set

$$h(r, x_3) := \int_{\mathbb{R}} \frac{q_j(x_3 - rt)}{(1+t^2)^2}\, dt.$$

Since $q_j \not\equiv 0$, there exist $z_0 \in \mathbb{R}$, $\varepsilon > 0$, $\delta > 0$ such that $q_j(x_3) > \delta$ for all $x_3 \in]z_0 - \varepsilon, z_0 + \varepsilon[$. This implies that for all $x_3 \in]z_0 - \varepsilon/2, z_0 + \varepsilon/2[$, we have $h(r, x_3) \geq \delta\rho$ for all $r < 1$, with $\rho = \int_{-\varepsilon/2}^{\varepsilon/2} (1+t^2)^{-2}\, dt > 0$. Inserting this estimate into (2.28), we get

$$\left|\frac{1}{r}\partial_\varphi v_j\right| \geq \rho' \xi(r) r^{-1+j\pi/\omega} \left|\cos\left(\frac{j\pi\varphi}{\omega}\right)\right|,$$

with some positive constant ρ'. This leads to the conclusion (2.27) because $r^{-1+j\pi/\omega}$ does not belong to L^p with respect to the measure rdr near 0. □

2.2 Extension to general boundary conditions

All the results of this section can be extended in a straightforward manner to Neumann boundary conditions and mixed boundary conditions. This can be seen by replacing $\sin(\frac{j\pi\varphi}{\omega})$ by $\cos(\frac{j\pi\varphi}{\omega})$ (or $\sin(\frac{(j-\frac{1}{2})\pi\varphi}{\omega})$, respectively) everywhere.

Newton boundary conditions need more explanation:

Theorem 2.9 *Let $u \in V$ be the solution of (1.7), then*

$$u \in A^{2,p}_\beta(\Omega) \text{ with } \begin{cases} \beta > 2 - \frac{2}{p} - \frac{\pi}{\omega} & \text{for } 2 - \frac{2}{p} \geq \frac{\pi}{\omega} > \frac{3}{2} - \frac{3}{p} \\ \beta = 0 & \text{for } 2 - \frac{2}{p} < \frac{\pi}{\omega} \end{cases}$$

and

$$\|u; A^{2,p}_\beta(\Omega)\| \leq C\|f; L^p(\Omega)\|.$$

Proof First we transform the boundary conditions (1.4) into

$$\frac{\partial u}{\partial \underline{n}} = -\sigma u \text{ on } \partial\Omega,$$

and use a lifting trace theorem in order to come back to homogeneous Neumann boundary conditions. Indeed, since there exists $s > 3$ such that $\sigma \in W^{1-1/s,s}(\partial\Omega)$, there exists $\tilde{\sigma} \in W^{1,s}(\Omega)$ such that $\tilde{\sigma} = \sigma$ on $\partial\Omega$. Using Theorem 1.4.4.2 of [12] we get $\tilde{\sigma}u \in H^1(\Omega)$, because $u \in H^1(\Omega)$. Consequently, $\sigma u \in H^{1/2}(\partial\Omega)$, and because of the classical trace theorem, there exists $w \in H^2(\Omega)$ such that

$$\frac{\partial w}{\partial \underline{n}} = -\sigma u \text{ on } \partial\Omega.$$

This means that $u_1 := u - w \in V$ is the solution of

$$\int_\Omega \nabla u_1 \cdot \nabla v \, d\underline{x} = \int_\Omega f_1 v \, d\underline{x} \text{ for all } v \in V,$$

where $f_1 := f + \Delta w$. Since $f_1 \in L^2(\Omega)$ and owing to Theorem 23.3 of [9], we conclude that

$$u \in H^{1+\pi/\omega-\varepsilon}(\Omega) \text{ for all } \varepsilon > 0.$$

Using again Theorem 1.4.4.2 of [12] to $u \in H^{1+\pi/\omega-\varepsilon}(\Omega)$ and some $\hat{\sigma} \in W^{1,p}(\Omega)$ such that $\hat{\sigma} = \sigma$ on $\partial\Omega$, we obtain that

$$\hat{\sigma} u \in W^{1,p}(\Omega), \text{ if } \frac{\pi}{\omega} > \frac{3}{2} - \frac{3}{p}.$$

Note that the condition $\frac{\pi}{\omega} > \frac{3}{2} - \frac{3}{p}$ is necessary to have the embedding $H^{1+\pi/\omega-\varepsilon}(\Omega) \hookrightarrow W^{1,p}(\Omega)$. With the help of the classical trace theorem, there exists $w_1 \in W^{2,p}(\Omega)$ such that

$$\frac{\partial w_1}{\partial \underline{n}} = -\sigma u \text{ on } \partial\Omega.$$

Finally, setting $u_2 = u - w_1$, we see that $u_2 \in V$ and that it is a solution of

$$\int_\Omega \nabla u_2 \cdot \nabla v \, d\underline{x} = \int_\Omega f_2 v \, d\underline{x} \text{ for all } v \in V,$$

with $f_2 := f + \Delta w_1 \in L^p(\Omega)$. Applying Corollary 2.7 to u_2 (in the case of homogeneous Neumann boundary conditions), we conclude that $u_2 \in A^{2,p}_\beta(\Omega)$ with β satisfying the conditions of that corollary. Since $w_1 \in W^{2,p}(\Omega)$, we get the assertion. □

2.3 General second order operators with constant coefficients

In this subsection, we briefly consider an elliptic operator L of second order with constant coefficients in a dihedral cone D':

$$L := -\sum_{i,j=1}^{3} a_{ij}\partial_{ij},$$

where $a_{ij} = a_{ji}$ is such that (1.5) is fulfilled.

Since this hypothesis implies that the matrix $\mathcal{A} = (a_{ij})_{i,j=1}^3$ is symmetric and positive definite, there exists a matrix B such that

$$B^T \mathcal{A} B = I.$$

By the change of variables $\underline{x} = B\underline{x}'$, the operator $-L$ is transformed into the Laplace operator, while the dihedral cone D' is mapped to another dihedral cone D. We notice that the transformation matrix B is unique only up to an orthogonal matrix, thus we can choose $B = (b_{ij})_{i,j=1}^3$ such that $b_{13} = b_{23} = 0$ and the dihedral cone $D = C \times \mathbb{R}$, where C is a infinite cone of $\mathbb{R}^2$ of opening ω_A. Moreover, we easily check that the distance to the edge in D' is equivalent to the distance to the edge in D, more precisely, with $r'(\underline{x}')$ being the distance from $\underline{x}'$ to the edge of D', we have

$$\begin{aligned} r'(\underline{x}') &\leq \|B\|\, r(\underline{x}), \\ r(\underline{x}) &\leq \|B^{-1}\|\, r'(\underline{x}') \quad \forall \underline{x}' \in D'. \end{aligned}$$

Therefore, the variational solution $v' \in H^1(D')$ of

$$\begin{aligned} Lv' &= g' \quad \text{in } D', \quad g' \in L^p(D'), \\ v' = 0 \quad \text{or} \quad \sum_{i,j=1}^{3} a_{ij}\partial_i u\, n_j + \sigma u &= 0 \quad \text{on } \partial D', \end{aligned}$$

is mapped by the above change of variables to the solution $v \in H^1(D)$ of

$$\begin{aligned} -\Delta v &= g \quad \text{in } D, \quad g \in L^p(D), \\ v = 0 \quad \text{or} \quad \frac{\partial u}{\partial \underline{n}} + \sigma u &= 0 \quad \text{on } \partial D. \end{aligned}$$

Applying the results of the previous subsections to v and mapping back to D', we get analogous results by replacing ω by ω_A.

2.4 Extension to the Lamé system

As shown in Lemma 4.1 of [13], the variational solution $\underline{v} = (v_1, v_2, v_3)^\tau$ (with a compact support) of the Lamé system

$$-\tilde{\lambda}\Delta \underline{v} - (\tilde{\lambda} + \tilde{\mu}) \text{ grad div } \underline{v} = \underline{f} \quad \text{in } D, \tag{2.29}$$

where $f_i \in L^2(D)$, $i = 1, 2, 3$, with Dirichlet boundary conditions has the tangential regularities

$$\partial_{13}\underline{v}, \partial_{23}\underline{v}, \partial_{33}\underline{v} \in L^2(D)^3.$$

As in Section 4 of [13], this allows to split up the Lamé system into a system involving the first two components and an elliptic equation for the third one. Therefore the techniques developed above lead to the inclusion

$$v_i \in A^{2,2}_\beta(D), \quad i = 1, 2, 3, \tag{2.30}$$

with $\beta > \beta_0$, and $\beta_0 > \frac{1}{2}$ depends on the Lamé coefficients $\tilde{\lambda}, \tilde{\mu}$. Other boundary conditions can also be treated based on further results of [13]. Unfortunately, at present, no result in L^p-spaces is available for the Lamé system.

3 Interpolation error estimates

3.1 Local error estimates in weighted Sobolev spaces

As introduced in Subsection 1.2, we are interested in local approximation error estimates for anisotropic elements. In [1], interpolation error estimates in classical Sobolev spaces were derived. These are useful far from the edge, but unfortunately, we can not apply them for tetrahedrons along the edge. In this subsection, we shall extend these results of [1] to weighted Sobolev spaces and consider particularly the three-dimensional case. We remark that interpolation error estimates for functions from weighted Sobolev spaces were already proved in [22] for the two-dimensional isotropic case.

We consider first estimates on a reference element $\Omega_0 \in \mathcal{R}$ where $\mathcal{R}$ is the set of reference elements discussed later, see Figures 3.1 and 3.2. We notice here that the elements of $\mathcal{R}$ have the following essential property (P):

(P) For each axis x_i $(i = 1, \dots, 3)$ of the coordinate system there exists one edge E_i of the reference element, which is parallel to this axis and, for normalization, which has length $\text{meas}_1(E_i) = 1$.

Using a similar notation as in [1, §2] we denote by P a space of polynomials, and since each monomial $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$ can be identified with the multi-index $\alpha \in \mathbb{N}^3$, we also identify P with the corresponding set of multi-indices. The hull $\overline{P}$ of P is the set $\overline{P} := P \cup \{\alpha + e_i : \alpha \in P, i = 1, 2, 3\}$ ($\{e_i\}_{i=1}^3$ denotes the canonical basis of $\mathbb{R}^3$) and the boundary $\partial\overline{P}$ of P is the set $\overline{P} \setminus P$. Note that $\max_{\alpha \in \overline{P}} |\alpha| = 1 + \max_{\alpha \in P} |\alpha|$.

We introduce now anisotropic weighted Sobolev spaces on Ω_0: For a finite set $P \subset \mathbb{N}^3$ with $0 \in P$ and for $\beta \in \mathbb{R}$ we set

$$V_\beta^{P,p}(\Omega_0) := \{v \in \mathcal{D}'(\Omega_0) : \|v; V_\beta^{P,p}(\Omega_0)\| < \infty\},$$

where

$$\|v; V_\beta^{P,p}(\Omega_0)\|^p := \sum_{\alpha \in P} \int_{\Omega_0} |r^{\beta-k+|\alpha|} D^\alpha v|^p \, d\underline{x},$$

$k := \max_{\alpha \in P} |\alpha|$, $D^\alpha := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}$, and $r(\underline{x}) := (x_1^2 + x_2^2)^{1/2}$. For $v \in V_\beta^{\overline{P},p}(\Omega_0)$ we also introduce the seminorm

$$|v; V_\beta^{\overline{P},p}(\Omega_0)|^p := \sum_{\alpha \in \partial \overline{P}} \int_{\Omega_0} |r^{\beta-k+|\alpha|} D^\alpha v|^p \, d\underline{x}.$$

The space $W^{P,p}(\Omega_0)$ is introduced in analogy to $V_\beta^{P,p}(\Omega_0)$ by omitting the weight.

Lemma 3.1 *Let $P \in \mathbb{N}^3$, P finite with $0 \in P$. Then we have the compact embedding*

$$V_\beta^{\overline{P},p}(\Omega_0) \overset{c}{\hookrightarrow} V_\beta^{P,p}(\Omega_0).$$

Proof For any $v \in V_\beta^{\overline{P},p}(\Omega_0)$ and any fixed $\alpha \in P$, we have

$$\begin{aligned} r^{\beta-k-1+|\alpha|} D^\alpha v &\in L^p(\Omega_0), \\ r^{\beta-k+|\alpha|} D^{\alpha+e_i} v &\in L^p(\Omega_0), \; i = 1, 2, 3. \end{aligned}$$

This implies $r^{\beta-k+|\alpha|} D^\alpha v \in W^{1,p}(\Omega_0)$, since $|r^{\beta-k+|\alpha|} D^\alpha v| \le C |r^{\beta-k+|\alpha|-1} D^\alpha v|$ almost everywhere in Ω_0. Thus there is a constant $C > 0$ such that

$$\|r^{\beta-k+|\alpha|} D^\alpha v; W^{1,p}(\Omega_0)\| \le C \|v; V_\beta^{\overline{P},p}(\Omega_0)\|. \tag{3.1}$$

Let $\{v_m\}_{m \in \mathbb{N}}$ be a sequence in $V_\beta^{\overline{P},p}(\Omega_0)$ such that for some $K > 0$ and for all $m \in \mathbb{N}$ the relation $\|v_m; V_\beta^{\overline{P},p}(\Omega_0)\| < K$ holds. From (3.1) we obtain for all $m \in \mathbb{N}$ and $\alpha \in P$ the bound $\|r^{\beta-k+|\alpha|} D^\alpha v_m; W^{1,p}(\Omega_0)\| \le C$. Owing to the compact embedding $W^{1,p}(\Omega_0) \overset{c}{\hookrightarrow} L^p(\Omega_0)$ (Rellich–Kondrašov theorem), there is a subsequence $\{v_{m_k}\}$ such that for all $\alpha \in P$

$$r^{\beta-k+|\alpha|} D^\alpha v_{m_k} \to w_\alpha \text{ in } L^p(\Omega_0). \tag{3.2}$$

(Since P is finite we can use card(P) times this theorem.) Because $0 \in P$ we obtain in particular

$$r^{\beta-k} v_{m_k} \to w_0 := r^{\beta-k} v \in L^p(\Omega_0),$$

which implies $v_{m_k} \to v$ in $\mathcal{D}'(\Omega_0)$, and $D^\alpha v_{m_k} \to D^\alpha v$ in $\mathcal{D}'(\Omega_0)$ for all $\alpha \in P$. With (3.2), we deduce that $w_\alpha = r^{\beta-k+|\alpha|} D^\alpha v \in L^p(\Omega_0)$ and therefore $v_{m_k} \to v$ in $V_\beta^{P,p}(\Omega_0)$. Thus the embedding is proved. □

We show now that, under some condition on β, elements of $V_\beta^{P,p}(\Omega_0)$ are in $L^1(\Omega_0)$, as well as all derivatives with respect to P.

Lemma 3.2 *Let $P \subset \mathbb{N}^3$, P finite, such that $0 \in P$. If $\beta < 2 - \frac{2}{p}$ then for all $v \in V_\beta^{P,p}(\Omega_0)$ the following relation holds:*

$$D^\alpha v \in L^1(\Omega_0) \text{ for all } \alpha \in P. \tag{3.3}$$

Proof If $\beta \le 0$ the assertion is obvious since $V_\beta^{P,p}(\Omega_0) \hookrightarrow W^{P,p}(\Omega_0)$. If $\beta > 0$, then we have $r^{\beta-k+|\alpha|}D^\alpha v \in L^p(\Omega_0)$ for any $\alpha \in P$. Since $|\alpha| \le k$ we deduce that $r^\beta D^\alpha v \in L^p(\Omega_0)$. Using Hölder's inequality, we show that this implies (3.3): Indeed, we have for $\frac{1}{p} + \frac{1}{q} = 1$

$$\int_{\Omega_0} |D^\alpha v|\, d\underline{x} = \int_{\Omega_0} r^{-\beta}|r^\beta D^\alpha v|\, d\underline{x} \le \|r^{-\beta}; L^q(\Omega_0)\|\, \|r^\beta D^\alpha v; L^p(\Omega_0)\|.$$

The $L^q(\Omega_0)$-norm of $r^{-\beta}$ is finite if and only if $\beta q < 2$ (by using cylindrical coordinates (r, φ, z)). But this is equivalent to $\beta < 2 - \frac{2}{p}$. □

From Lemmas 3.1 and 3.2 and using the same arguments as in [1, Lemma 2], we obtain the following lemma.

Lemma 3.3 *Let $P \in \mathbb{N}^3$ be a finite set of multi-indices with $0 \in P$. If $\beta < 2 - \frac{2}{p}$ then there is a constant $C > 0$ such that*

$$\|v; V_\beta^{\overline{P},p}(\Omega_0)\| \le C|v; V_\beta^{\overline{P},p}(\Omega_0)| \tag{3.4}$$

for all $v \in V_\beta^{\overline{P},p}(\Omega_0)$ satisfying $\int_{\Omega_0} D^\alpha v\, d\underline{x} = 0$ for $\alpha \in P$.

We are now ready to give the interpolation estimate, first in a very general form, then especially for our purposes.

Lemma 3.4 *Let $\beta < 2 - \frac{2}{p}$ be a real number, and let $P, Q \subset \mathbb{N}^3$ and $\gamma \in \mathbb{N}^3$ be such that $0 \in Q$ and $Q + \gamma \subset P$. Further introduce a linear operator $I : C^\mu(\Omega_0) \to P$, $\mu \in \mathbb{N}$, and assume that there are linear functionals $F_i \in \left(V_\beta^{\overline{Q},p}(\Omega_0)\right)'$, $i = 1, \ldots, j$, $j = \dim D^\gamma P$, satisfying*

$$\begin{aligned} &F_i(D^\gamma I v) = F_i(D^\gamma v) \quad (i = 1, \ldots, j) \text{ for all } v \in C^\mu(\Omega_0) \cap V_\beta^{\overline{Q}+\gamma,p}(\Omega_0), \\ &F_i(D^\gamma q) = 0 \text{ for all } i = 1, \ldots, j \Longrightarrow D^\gamma q = 0 \text{ for all } q \in P. \end{aligned} \tag{3.5}$$

Then there is a constant $C > 0$ such that

$$\|D^\gamma(v - Iv); V_\beta^{\overline{Q},p}(\Omega_0)\| \le C|D^\gamma v; V_\beta^{\overline{Q},p}(\Omega_0)|$$

for all $v \in C^\mu(\Omega_0) \cap V_\beta^{\overline{Q}+\gamma,p}(\Omega_0)$.

Proof We follow the proof of Lemma 3 of [1], since Lemma 1 of [1] can be extended to the spaces $V_\beta^{P,p}(\Omega_0)$ (owing to Lemma 3.2), while Lemma 2 of [1] is replaced by Lemma 3.3. □

Theorem 3.5 *Suppose that $0 \le \beta < 1 - \frac{1}{p}$, $p > 2$, and let Iv be the linear Lagrange interpolant of v with respect to the vertices. Then for all $v \in A_\beta^{2,p}(\Omega_0) \cap C(\Omega_0)$ we have*

$$\begin{aligned} &\left\| r^{\beta-1}\partial_i(v - Iv); L^p(\Omega_0)\right\| \le \\ &\le C\left\{\int_{\Omega_0} \left[r^{p\beta}\left(|\partial_{1i}v|^p + |\partial_{2i}v|^p\right) + |\partial_{3i}v|^p\right] d\underline{x}\right\}^{1/p}, \ i = 1, 2, \end{aligned} \tag{3.6}$$

and

$$\left\| r^{-1}\partial_3(v - Iv); L^p(\Omega_0)\right\| \le C\left\{\int_{\Omega_0} \left(|\partial_{13}v|^p + |\partial_{23}v|^p + |\partial_{33}v|^p\right) d\underline{x}\right\}^{1/p}. \tag{3.7}$$

Proof We set $Q := \{(0,0,0)\}, \overline{Q} := \{(0,0,0)\} \cup \{e_i\}_{i=1,2,3}$ and remark that $v \in A^{2,p}_\beta(\Omega_0)$ implies $\partial_i v \in V^{1,p}_\beta(\Omega_0) = V^{\overline{Q},p}_\beta(\Omega_0)$ $(i = 1,2)$ and $\partial_3 v \in V^{1,p}_0(\Omega_0) = V^{\overline{Q},p}_0(\Omega_0)$. To prove the assertion we apply Lemma 3.4 with $P = \overline{Q}$, $\gamma := e_i$ and $F_1(v) := \int_{E_i} v\,dx_i$, where E_i is that edge of Ω_0 which is parallel to the x_i-axis, see Property (P) on page 14. It remains to prove the continuity of F_1.

In the simpler case $i = 3$ we can use the embeddings

$$V^{1,p}_0(\Omega_0) \hookrightarrow W^{1,p}(\Omega_0) \hookrightarrow W^{1-2/p,p}(E_3) \hookrightarrow L^1(E_3)$$

which holds for $1 - \frac{2}{p} > 0$, that means $p > 2$.

For $i = 1,2$ we use that $v \in V^{1,p}_\beta(\Omega_0)$ implies

$$r^\beta v \in W^{1,p}(\Omega_0) \hookrightarrow W^{1-2/p,p}(E_i) \hookrightarrow L^p(E_i),\ i = 1,2.$$

Using Hölder's inequality we conclude for $\frac{1}{p} + \frac{1}{q} = 1$ that

$$\begin{aligned} \int_{E_i} |v|\,dx_i &\leq \|r^{-\beta}; L^q(E_i)\|\,\|r^\beta v; L^p(E_i)\| \\ &\leq \|r^{-\beta}; L^q(E_i)\|\,\|v; V^{1,p}_\beta(\Omega_0)\|. \end{aligned}$$

Using that $r^{-\beta} \in L^q(E_i)$ for $\beta < \frac{1}{q} = 1 - \frac{1}{p}$ the proof is complete. □

Remark 3.6 In applications with the same type of boundary conditions on both faces of the edge, we have $\beta = 2 - \frac{2}{p} - \frac{\pi}{\omega_A} + \varepsilon$ with an arbitrarily small positive real ε. That means $\beta < 1 - \frac{1}{p}$ is equivalent to $1 - \frac{1}{p} < \frac{\pi}{\omega_A}$, so that for p close to 2 this condition always holds.

For mixed boundary conditions we have to replace $\frac{\pi}{\omega_A}$ by $\frac{\pi}{2\omega_A}$, that means we are restricted to $\omega_A < \pi$. This restriction is known from the isotropic case (see [6]); it is equivalent to the condition that u must be contained in $W^{3/2+\varepsilon,2} \hookrightarrow C(\overline{\Omega})$ in order to have well-defined pointwise values of u. Only in that case interpolation makes sense.

Lemma 3.4 can also be applied to prove an L^p-estimate:

Theorem 3.7 *Suppose that $0 \leq \beta < 2 - \frac{3}{p}$, $p \geq 1$, and let Iv be the linear Lagrange interpolant of v with respect to the vertices. Then for all $v \in A^{2,p}_\beta(\Omega_0)$ we have*

$$\|r^{\beta-2}(v - Iv); L^p(\Omega_0)\| \leq C\,|v; V^{2,p}_\beta(\Omega_0)|. \tag{3.8}$$

Proof We set $Q := \{\alpha \in \mathbb{N}^3 : |\alpha| \leq 1\}$, $\overline{Q} := \{\alpha \in \mathbb{N}^3 : |\alpha| \leq 2\}$, and remark that $v \in A^{2,p}_\beta(\Omega_0) \hookrightarrow V^{2,p}_\beta(\Omega_0) = V^{\overline{Q},p}_\beta(\Omega_0)$. We apply Lemma 3.4 with $\gamma = (0,0,0)$ and $F_i(v) := v(\underline{x}^{(i)})$, $i = 1,\ldots,4$, where $\underline{x}^{(i)}$ are the vertices of Ω_0. To prove the continuity of F_i we use the embedding [24]

$$V^{2,p}_\beta(\Omega_0) \hookrightarrow V^{2-\beta,p}_0(\Omega_0) \hookrightarrow W^{2-\beta,p}(\Omega_0) \hookrightarrow C(\overline{\Omega_0})$$

which is valid just for $0 \leq \beta < 2 - \frac{3}{p}$. □

Remark 3.8 The restriction $\beta < 2 - \frac{3}{p}$ does not imply difficulties because for $p \geq 2$ this condition is weaker than that of Theorem 3.5.

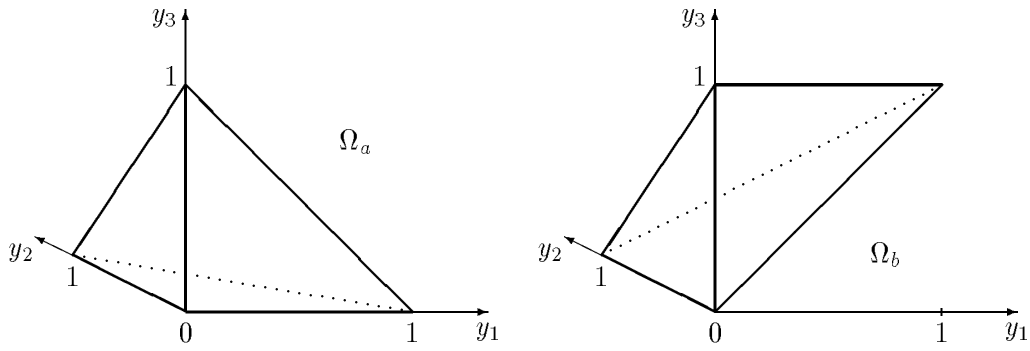

Figure 3.1: Basic reference elements for anisotropic interpolation error estimates in the three-dimensional case.

Corollary 3.9 *For $p \geq 1$, $0 \leq \beta < 2 - \frac{3}{p}$, and $v \in A^{2,p}_\beta(\Omega_0)$ we have*

$$\|v - Iv; L^p(\Omega_0)\| \leq C\,|v; A^{2,p}_\beta(\Omega_0)|, \tag{3.9}$$

and for $p > 2$, $0 \leq \beta < 1 - \frac{1}{p}$, we have for $v \in A^{2,p}_\beta(\Omega_0)$ and $i = 1, 2$ the estimates

$$\|\partial_i(v - Iv); L^p(\Omega_0)\| \leq C\left\{\int_{\Omega_0} \left[r^{p\beta}\left(|\partial_{1i}v|^p + |\partial_{2i}v|^p\right) + |\partial_{3i}v|^p\right] d\underline{x}\right\}^{1/p}, \tag{3.10}$$

$$\|\partial_3(v - Iv); L^p(\Omega_0)\| \leq C\left\{\int_{\Omega_0} \left(|\partial_{13}v|^p + |\partial_{23}v|^p + |\partial_{33}v|^p\right) d\underline{x}\right\}^{1/p}. \tag{3.11}$$

Proof Estimate (3.9) follows from (3.8) since $r^{\beta-2}$ is bounded from below and r^β is bounded from above by some constant $C > 0$. The estimates (3.10) and (3.11) follow with the same arguments from (3.6) and (3.7). □

Now we are going to transform these estimates to the actual finite elements Ω_i. We recall from [8] that for two tetrahedra Ω_i and Ω_0 there is an affine linear transformation

$$\underline{x} = F(\underline{y}) = B\underline{y} + \underline{b} \tag{3.12}$$

with $B = (b_{jk})^3_{j,k=1} \in \mathbb{R}^{3\times3}$, $\underline{b} = (b_j)^3_{j=1} \in \mathbb{R}^3$, such that $\Omega_i = F(\Omega_0)$. We consider two reference elements Ω_a and Ω_b as given in Figure 3.1. Note that anisotropic elements can have three or four edges with length of order h_3, they are mapped to Ω_a and Ω_b, respectively. In Appendix A of [3] it is shown that then conditions (d) and (e) lead to the following relations for the matrix elements b_{jk} and $b^{(-1)}_{jk}$ of B and B^{-1}, respectively:

$$|b_{jk}| \leq C\,\min\{h_{j,i}, h_{k,i}\}, \quad |b^{(-1)}_{jk}| \leq C\,\min\{h^{-1}_{j,i}, h^{-1}_{k,i}\}. \tag{3.13}$$

Using Corollary 3.9 for the special case $\beta = 0$ we get with (3.13) the estimates

$$\|v - Iv; L^p(\Omega_i)\| \leq C\sum_{j,k=1}^{3} h_{j,i}h_{k,i}\,\|\partial_{jk}v; L^p(\Omega_i)\| \quad \text{for } p \geq 1, \tag{3.14}$$

$$\|\partial_j(v - Iv); L^p(\Omega_i)\| \leq C\sum_{k=1}^{3} h_{k,i}\,\|\partial_{jk}v; L^p(\Omega_i)\|,\ j = 1, 2, 3, \quad \text{for } p > 2, \tag{3.15}$$

where (3.15) can be formulated in the following equivalent way:

$$|v - Iv; W^{1,p}(\Omega_i)| \leq C\sum_{k=1}^{3} h_{k,i}|\partial_k v; W^{1,p}(\Omega_i)| \quad \text{for } p > 2. \tag{3.16}$$

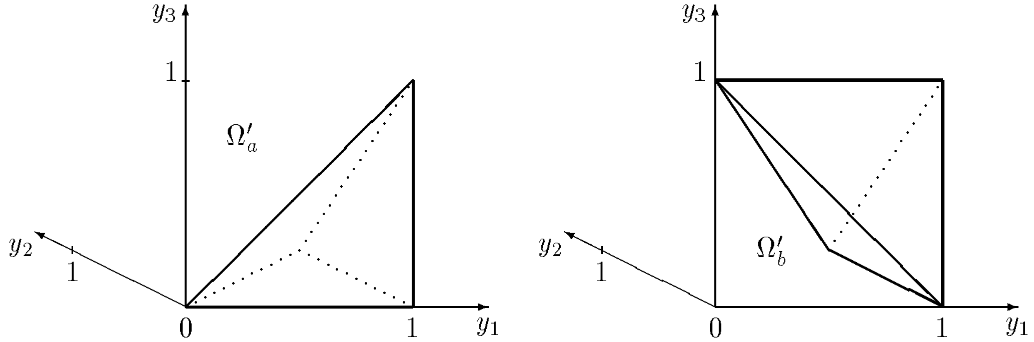

Figure 3.2: Additional reference elements for interpolation error estimates in weighted Sobolev spaces.

This estimate was first proved in [1].

To transform the estimates (3.9)–(3.11) for $\beta > 0$ we can assume that h_1 and h_2 are of the same order, but we need additionally that

$$y_1^2 + y_2^2 \le C h_1^{-2}(x_1^2 + x_2^2) \quad \text{for all } \underline{x} \in \Omega_i, \tag{3.17}$$

which can be concluded from

$$b_{13} = b_{23} = 0 \quad \text{and} \quad b_1 = b_2 = 0. \tag{3.18}$$

The geometrical meaning of this condition is investigated in the following lemma.

Lemma 3.10 *Assume that we are given an element Ω_i satisfying the following condition.*

(i) At least one vertex of Ω_i is contained in the z-axis.

Consider a set $\mathcal{R} = \{\Omega_a, \Omega'_a, \Omega_b, \Omega'_b\}$ of four reference elements, where Ω'_a and Ω'_b are obtained from Ω_a and Ω_b, respectively, by a reflection at the plane $x_1 = \frac{1}{2}$, see Figure 3.2.

Then we can choose one element $\Omega_0 \in \mathcal{R}$ such that the corresponding transformation (3.12) satisfies (3.18) if and only if

(ii) there are two vertices of Ω_i such that the straight line through them is parallel to the z-axis.

Proof Observe that a translation parallel to the x_1, x_2-plane affects property (3.17); therefore we consider four reference elements. One can choose the appropriate reference element by the number of edges with length of order h_3 (three or four) and the number of vertices of Ω_i that are contained in the x_3-axis (one or two).

To realize the necessity of condition (ii) let $\underline{y}^{(i)}$ and $\underline{y}^{(j)}$ $(i, j \in \{0, 1, 2, 3\})$ be those vertices of Ω_0 such that $\underline{y}^{(i)} - \underline{y}^{(j)} = (0, 0, 1)^T$, and let $\underline{x}^{(i)} := B\underline{y}^{(i)} + \underline{b}$ and $\underline{x}^{(j)} := B\underline{y}^{(j)} + \underline{b}$. Then $B^{-1}(\underline{x}^{(i)} - \underline{x}^{(j)}) = (0, 0, 1)^T$, and in particular

$$\begin{pmatrix} b_{11}^{(-1)} & b_{12}^{(-1)} \\ b_{21}^{(-1)} & b_{22}^{(-1)} \end{pmatrix} \begin{pmatrix} x_1^{(i)} - x_1^{(j)} \\ x_2^{(i)} - x_2^{(j)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Observing that $b_{13} = b_{23} = 0$ leads to $b_{13}^{(-1)} = b_{23}^{(-1)} = 0$, we conclude $b_{11}^{(-1)} b_{22}^{(-1)} - b_{12}^{(-1)} b_{21}^{(-1)} = \det B^{(-1)} / b_{33}^{(-1)} \neq 0$ and follow (ii).

With similar arguments we examine that (i) and (ii) are sufficient for (3.18): Assume we are given an element Ω_i satisfying (i) and (ii). The use of four reference elements allows to find one reference element such that (1) each vertex contained in the x_3-axis is mapped to a vertex at the y_3-axis, thus $b_1 = b_2 = 0$, and (2) two vertices $\underline{x}^{(i)}$ and $\underline{x}^{(j)}$ with $x_k^{(i)} = x_k^{(j)}$, $k = 1, 2$, are mapped to $\underline{y}^{(i)}$ and $\underline{y}^{(j)}$ with $y_k^{(i)} = y_k^{(j)}$, $k = 1, 2$, thus from the first two equations of the system $\underline{x}^{(i)} - \underline{x}^{(j)} = B(\underline{y}^{(i)} - \underline{y}^{(j)})$ we get $b_{13} = b_{23} = 0$. □

If we apply the estimates with $\beta > 0$ only for the elements close to the edge, then the conditions (i) and (ii) are satisfied via assumption (f) from page 5.

We remark that it is desirable for treating curved edges that the assumption (ii) is weakened to a condition similar to (e) from page 5. This seems to be possible for special cases but not in general. We will discuss this in a subsequent paper.

Theorem 3.11 *Let $I_h v$ be the linear Lagrange interpolant of $v \in A_\beta^{2,p}(\Omega_i)$ with respect to the vertices. Assume further that for the element Ω_i with $h_{2,i} \le h_{1,i}$ the conditions (d), (e), (i), and (ii) are satisfied. Then for $0 \le \beta < 2 - \frac{3}{p}$, $p \ge 1$, the following local interpolation error estimate holds:*

$$\|v - I_h v; L^p(\Omega_i)\| \le C \left\{ \int_{\Omega_i} \left[h_{1,i}^{2-\beta} \sum_{\substack{|\alpha|=2 \\ \alpha_3=0}} r^{\beta p} |D^\alpha v|^p + C h_{3,i}^2 \sum_{\substack{|\alpha|=2 \\ \alpha_3>0}} |D^\alpha v|^p \right] d\underline{x} \right\}^{1/p}. \tag{3.19}$$

Moreover, if $0 \le \beta < 1 - \frac{1}{p}$, $p > 2$, then for all $v \in A_\beta^{2,p}(\Omega_i)$ the norm of the derivatives of the interpolation error can be estimated by

$$\begin{aligned} &\|\partial_j(v - Iv); L^p(\Omega_i)\| \\ &\quad \le C \left\{ \int_{\Omega_i} \left[h_{1,i}^{p(1-\beta)} r^{p\beta} \left(|\partial_{1j} v|^p + |\partial_{2j} v|^p \right) + h_{3,i}^p |\partial_{3j} v|^p \right] d\underline{x} \right\}^{1/p}, \quad i = 1, 2, \end{aligned} \tag{3.20}$$

$$\|\partial_3(v - Iv); L^p(\Omega_i)\| \le C \left\{ \int_{\Omega_i} \sum_{k=1}^{3} h_{k,i}^p |\partial_{k3} v|^p \, d\underline{x} \right\}^{1/p}. \tag{3.21}$$

Proof The assertion is a direct consequence from Corollary 3.9 using the transformation (3.12) with (3.13) and (3.17). □

Corollary 3.12 *Under the assumptions of Theorem 3.11 the following estimates hold:*

$$\|v - I_h v; L^p(\Omega_i)\| \le C\,(h_{1,i}^{2-\beta} + h_{3,i}^2) |v; A_\beta^{2,p}(\Omega_i)|, \tag{3.22}$$

$$|v - I_h v; W^{1,p}(\Omega_i)| \le C\,(h_{1,i}^{1-\beta} + h_{3,i}) |v; A_\beta^{2,p}(\Omega_i)|. \tag{3.23}$$

3.2 Global error estimates

In this section, we investigate the global interpolation error, that is the difference between the solution u of our boundary value problem (1.6) or (1.7) and its piecewise linear interpolant $I_h u$ on the family of anisotropic graded meshes introduced in Subsection 1.2. The difficulty is that we are interested on one hand in an estimate in the energy norm which is equivalent to $\| \, . \, ; W^{1,2}(\Omega)\|$, in order to apply Céa's lemma for the finite element error. But on the other hand the local interpolation error estimates (3.16) and (3.23) are valid for $\| \, . \, ; W^{1,p}(\Omega_i)\|$ with $p > 2$ only.

Theorem 3.13 *Let u be the solution of the boundary value problem (1.6) or (1.7), and $2 < p < p_+$ (p_+ is defined in (1.24)). Then for the interpolation error $u - I_h u$, I_h defined on the family of meshes in Subsection 1.2, the following estimate holds:*

$$\|u - I_h u; W^{1,2}(\Omega)\| \le C h^s \|f; L^p(\Omega)\|,$$

$$s = \begin{cases} 1 & \text{for } \mu < \frac{\pi}{\omega_A} \cdot \frac{p}{2p-2}, \\ \frac{2}{p} - 1 + \frac{1}{\mu} \cdot \frac{\pi}{\omega_A} - \varepsilon & \text{for } \mu \ge \frac{\pi}{\omega_A} \cdot \frac{p}{2p-2}. \end{cases} \tag{3.24}$$

Proof We reduce the estimation of the global error to the evaluation of the local errors and distinguish between the $m_0 = \mathcal{O}(h^{-1})$ elements whose closure has at least one common point with the edge, and the $m - m_0 = \mathcal{O}(h^{-3})$ elements away from the edge:

$$|u - I_h u; W^{1,2}(\Omega)|^2 = \sum_{i=1}^{m_0} |u - I_h u; W^{1,2}(\Omega_i)|^2 + \sum_{i=m_0+1}^{m} |u - I_h u; W^{1,2}(\Omega_i)|^2. \tag{3.25}$$

For the elements in the first sum we apply the local estimate (3.23). Using Hölder's inequality, we have for $i = 1, \ldots, m_0$

$$\begin{aligned} |u - I_h u; W^{1,2}(\Omega_i)|^p &\leq (\text{meas}\Omega_i)^{-1+p/2} |u - I_h u; W^{1,p}(\Omega_i)|^p \\ &\leq C(hh_i^2)^{-1+p/2}(h_i^{1-\beta} + h)^p |u; A_\beta^{2,p}(\Omega_i)|^p. \end{aligned}$$

Summing up these estimates for all $i = 1, \ldots, m_0$, and using again Hölder's inequality, we can conclude

$$\begin{aligned} \sum_{i=1}^{m_0} |u - I_h u; W^{1,2}(\Omega_i)|^2 &\leq m_0^{1-2/p} \left(\sum_{i=1}^{m_0} |u - I_h u; W^{1,p}(\Omega_i)|^p \right)^{2/p} \\ &\leq C \sum_{i=1}^{m_0} h^{-1+2/p} (hh_i^2)^{1-2/p} (h_i^{1-\beta} + h)^2 |u; A_\beta^{2,p}(\Omega_i)|^2 \\ &\leq C \left(h^{(2-\beta-2/p)/\mu} + h^{1+(1-2/p)/\mu} \right)^2 \|f; L^p(\Omega)\|^2. \end{aligned}$$

Since for $\beta = \max\{0;\ 2 - \frac{2}{p} - \frac{\pi}{\omega_A} + \varepsilon'\}$ there holds $\frac{1}{\mu}(2 - \frac{2}{p} - \beta) > s$, and we have directly $1 + \frac{1}{\mu}(1 - \frac{2}{p}) > 1 \geq s$ (with s from (3.24)), we get

$$\sum_{i=1}^{m_0} |u - I_h u; W^{1,2}(\Omega_i)|^2 \leq C h^{2s} \|f; L^p(\Omega)\|^2. \tag{3.26}$$

For the elements in the second sum of (3.25) we can use that $u \in W^{2,p}(\Omega_i)$, $i = m_0 + 1, \ldots, m$, and thus apply the local estimate (3.16). Again with Hölder's inequality, we have for $i = m_0 + 1, \ldots, m$:

$$\begin{aligned} &|u - I_h u; W^{1,2}(\Omega_i)|^p \leq (\text{meas}\Omega_i)^{-1+p/2} |u - I_h u; W^{1,p}(\Omega_i)|^p \\ &\leq\ C(hh_i^2)^{-1+p/2} \left(h_i^p \sum_{\substack{|\alpha|=2 \\ \alpha_3=0}} \|D^\alpha u; L^p(\Omega_i)\|^p + C h^p \sum_{\substack{|\alpha|=2 \\ \alpha_3>0}} \|D^\alpha u; L^p(\Omega_i)\|^p \right) \end{aligned} \tag{3.27}$$

For $\mu < \frac{\pi}{\omega_A} \cdot \frac{p}{2p-2}$ we can estimate $h_i^{2p-2} \leq C h^{2p-2} r_i^{(2p-2)(1-\mu)} = C h^{2p-2} r_i^{p\beta}$ with $\beta = \frac{1}{p}(2p - 2)(1 - \mu) > 2 - \frac{2}{p} - \frac{\pi}{\omega_A}$.

For $\mu \geq \frac{\pi}{\omega_A} \cdot \frac{p}{2p-2}$ we have to use part of h_i^{2p-2} via $h_i < C r_i$ to get also the power $p\beta$ of r_i on the right hand side:

$$\begin{aligned} h_i^{2p-2} &= h_i^{\frac{p}{\mu} \cdot \frac{\pi}{\omega_A} - p\varepsilon} h_i^{2p-2-\frac{p}{\mu} \cdot \frac{\pi}{\omega_A} + p\varepsilon} \\ &\leq C h^{\frac{p}{\mu} \cdot \frac{\pi}{\omega_A} - p\varepsilon} r_i^{(\frac{p}{\mu} \cdot \frac{\pi}{\omega_A} - p\varepsilon)(1-\mu)} r_i^{2p-2-\frac{p}{\mu} \cdot \frac{\pi}{\omega_A} + p\varepsilon} \\ &= C h^{\frac{p}{\mu} \cdot \frac{\pi}{\omega_A} - p\varepsilon} r_i^{p\beta} \end{aligned}$$

with $\beta = \frac{1}{p} \left[\left(\frac{p}{\mu} \cdot \frac{\pi}{\omega_A} - p\varepsilon \right) (1 - \mu) + 2p - 2 - \frac{p}{\mu} \cdot \frac{\pi}{\omega_A} + p\varepsilon \right] = 2 - \frac{2}{p} - \frac{\pi}{\omega_A} + \frac{\varepsilon}{\mu} > 2 - \frac{2}{p} - \frac{\pi}{\omega_A}$ for $\varepsilon > 0$. Note that $h_i^{2p-2-\frac{p}{\mu} \cdot \frac{\pi}{\omega_A} + p\varepsilon} < C r_i^{2p-2-\frac{p}{\mu} \cdot \frac{\pi}{\omega_A} + p\varepsilon}$ because $2p - 2 - \frac{p}{\mu} \cdot \frac{\pi}{\omega_A} + p\varepsilon > 2p - 2 - \frac{p}{\mu} \cdot \frac{\pi}{\omega_A} \geq 0$ due to the assumption on μ.

Thus we get with (3.27)

$$|u - I_h u; W^{1,2}(\Omega_i)|^p \le C h^{ps+3(p-2)/2} \|u; A^{2,p}_\beta(\Omega_i)\|^p$$

with s from (3.24). Summing up these estimates for all $i = m_0 + 1, \dots, m$, and using again Hölder's inequality, we can conclude with Corollary 2.7

$$\begin{aligned}
\sum_{i=m_0+1}^{m} |u - I_h u; W^{1,2}(\Omega_i)|^2 &\le (m - m_0)^{1-\frac{2}{p}} \left(\sum_{i=m_0+1}^{m} |u - I_h u; W^{1,2}(\Omega_i)|^p \right)^{2/p} \\
&\le C h^{-3(1-\frac{2}{p})} h^{\frac{2}{p}(ps+\frac{3}{2}(p-2))} \|u; A^{2,p}_\beta(\Omega)\|^2 \\
&= C h^{2s} \|f; L^p(\Omega)\|^2. \qquad (3.28)
\end{aligned}$$

From (3.26) and (3.28) we get

$$|u - I_h u; W^{1,2}(\Omega)| \le C h^s \|f; L^p(\Omega)\|. \qquad (3.29)$$

The estimation is much simpler in the case of the L^2-error because the local estimates hold for all $p \ge 1$: For $i = 1, \dots, m_0$ we have by (3.22) with $p = 2$

$$\begin{aligned}
\sum_{i=1}^{m_0} \|u - I_h u; L^2(\Omega_i)\|^2 &\le C \left(h^{(2-\beta)/\mu} + h^2 \right)^2 \|u; A^{2,2}_\beta(\Omega)\|^2 \\
&\le C h^{2s} \|f; L^p(\Omega)\|,
\end{aligned}$$

because $\frac{1}{\mu}(2 - \beta) > \frac{1}{\mu}(1 + \frac{\pi}{\omega_A} - \varepsilon) > 1 + s$. For the second term we have via (3.14)

$$\begin{aligned}
&\sum_{i=m_0+1}^{m} \|u - I_h u; L^2(\Omega_i)|^2 \le \\
&\le C \sum_{i=m_0+1}^{m} \left(h_i^4 \sum_{\substack{|\alpha|=2 \\ \alpha_3=0}} \|D^\alpha u; L^2(\Omega_i)\|^2 + C h^4 \sum_{\substack{|\alpha|=2 \\ \alpha_3>0}} \|D^\alpha u; L^2(\Omega_i)\|^2 \right).
\end{aligned}$$

With the same ideas as above we get

$$h_i^4 \le \begin{cases} h^4 r^{2\beta} & \text{for } \mu < \frac{1}{2}(1 + \frac{\pi}{\omega_A}) \\ h^{2(1+\frac{1}{\mu}\cdot\frac{\pi}{\omega_A}-\varepsilon)} r^{2\beta} & \text{for } \mu \ge \frac{1}{2}(1 + \frac{\pi}{\omega_A}) \end{cases}$$

and thus

$$\sum_{i=m_0+1}^{m} \|u - I_h u; L^2(\Omega_i)\|^2 \le C h^{2s'} \|f; L^2(\Omega)\|^2$$

with $s' = 2$ for $\mu < \frac{1}{2}(1+\frac{\pi}{\omega_A})$, $s' = 1 + \frac{1}{\mu}\cdot\frac{\pi}{\omega_A} - \varepsilon$ for $\mu \ge \frac{1}{2}(1+\frac{\pi}{\omega_A})$, that means $s' > 1 + s$ for all μ, so that (3.29) holds not only for the seminorm $|\,.\,; W^{1,2}(\Omega)|$ but also for the norm $\|\,.\,; W^{1,2}(\Omega)\|$. □

Corollary 3.14 *Let u be the solution of the boundary value problem (1.6) or (1.7), $2 < p < p_+$ (p_+ is defined in (1.24)), and let u_h be the finite element solution of (1.13) or (1.14), respectively, using a family of meshes as defined in Subsection 1.2. Then the error estimate*

$$\|u - u_h; W^{1,2}(\Omega)\| \le C h^s \|f; L^p(\Omega)\|$$

holds, with s from (3.24).

This assertion remains true for mixed boundary conditions and $\omega < \pi$, see Subsection 2.2 and Remark 3.6. However, we have to replace the condition on p by $2 < p < (1 - \frac{\pi}{\omega_A})^{-1}$ and $\frac{\pi}{\omega_A}$ by $\frac{\pi}{2\omega_A}$ in (3.24).

Remark 3.15 Note that the restriction $p < p_+$ is not essential for this estimate, because $f \in L^p(\Omega)$ yields $f \in L^q(\Omega)$ for $q \le p$ and $\|f; L^q(\Omega)\| \le C\|f; L^p(\Omega)\|$. We can always find some $q < p_+$ and apply Theorem 3.13 for q. Nevertheless, we have to replace p in (3.24) by $\min\{p;\ p_+ - \delta\}$, $\delta > 0$ arbitrary.

Remark 3.16 In order to use meshes which are not too much refined, the estimates are most favourable for p close to 2. For $p = 2+\delta$ (δ is an arbitrarily small real number) we have

$$s = \begin{cases} 1 & \text{for } \mu < \frac{\pi}{\omega_A}\left(1 - \frac{\delta}{2+2\delta}\right), \\ \frac{1}{\mu}\cdot\frac{\pi}{\omega_A} - \varepsilon - \frac{\delta}{2+\delta} & \text{for } \mu \ge \frac{\pi}{\omega_A}\left(1 - \frac{\delta}{2+2\delta}\right), \end{cases}$$

so that one can conclude that the approximation order s is

$$s = \begin{cases} 1 & \text{for } \mu < \frac{\pi}{\omega_A}, \\ \frac{1}{\mu}\cdot\frac{\pi}{\omega_A} - \varepsilon & \text{for } \mu \ge \frac{\pi}{\omega_A}, \end{cases} \tag{3.30}$$

(even when f is smoother). On the other hand it is not clear in which way the constant C depends on p; we suspect that $C \to \infty$ for $p \to 2$.

4 Condition number of the stiffness matrix

Consider the basis $\{\phi_i(\underline{x})\}_{i=1}^k$ with $\phi_i(\underline{x}^{(j)}) = \delta_{ij}$ in V_h (or V_{0h}, respectively), with k being the number of degrees of freedom. Thus each function $v_h \in V_h$ (or V_{0h}) can be represented by $v_h(\underline{x}) = \sum_{i=1}^k v_i\phi_i(\underline{x})$, with $v_i = v_h(\underline{x}^{(i)})$.

The stiffness matrix $\mathcal{A} := (a_{ij})_{i,j=1}^k$ has the entries $a_{ij} = a(\phi_i, \phi_j)$. We want to estimate the condition number κ of this matrix:

$$\kappa := \frac{\lambda_{\max}}{\lambda_{\min}} \tag{4.1}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximal and minimal eigenvalues of $\mathcal{A}$.

Using the Rayleigh quotient and the boundedness and coercivity of the bilinear form

$$\underline{C}\|v_h; W^{1,2}(\Omega)\|^2 \le a(v_h, v_h) \le \overline{C}\|v_h; W^{1,2}(\Omega)\|^2 \quad \forall v_h \in V_h\ (V_{0h}),$$

as well as the identity $a(v_h, v_h) = \langle \mathcal{A}\underline{v}, \underline{v}\rangle$ ($\langle\ .\ ,\ .\ \rangle$ is the Euclidean scalar product in $\mathbb{R}^k$, $\underline{v} := (v_i)_{i=1}^k$ is the grid function related to v_h), we get

$$\lambda_{\max} \le \overline{C}\max_{\underline{v}\in\mathbb{R}^k} \frac{\|v_h; W^{1,2}(\Omega)\|^2}{\langle \underline{v}, \underline{v}\rangle}, \tag{4.2}$$

$$\lambda_{\min} \ge \overline{C}\min_{\underline{v}\in\mathbb{R}^k} \frac{\|v_h; W^{1,2}(\Omega)\|^2}{\langle \underline{v}, \underline{v}\rangle}. \tag{4.3}$$

We are now looking for an upper and a lower bound of $\|v_h; W^{1,2}(\Omega)\|^2$ in terms of $\langle \underline{v}, \underline{v}\rangle$.

Using the inverse inequality (Lemma 1.3) we have

$$\|v_h; W^{1,2}(\Omega)\|^2 = \sum_{i=1}^m \|v_h; W^{1,2}(\Omega_i)\|^2 \le C\sum_{i=1}^m h_i^{-2}\|v_h; L^2(\Omega_i)\|^2. \tag{4.4}$$

On the reference element Ω_0 we have

$$\mu_{\min}\sum_{j\in I_i} v_j^2 \le \|v_h; L^2(\Omega_0)\|^2 \le \mu_{\max}\sum_{j\in I_i} v_j^2, \tag{4.5}$$

where $\mu_{\min}$ and $\mu_{\max}$ are the minimal and the maximal eigenvalues of the element mass matrix $\left(\int_{\Omega_0} \phi_i \phi_j \, d\underline{x}\right)_{i,j=1}^4$ and I_i is the set of numbers of the nodes belonging to Ω_i. Transforming (4.5) to Ω_i we get

$$C_1 \mathrm{meas}(\Omega_i) \sum_{j\in I_i} v_j^2 \leq \|v_h; L^2(\Omega_i)\|^2 \leq C_2 \mathrm{meas}(\Omega_i) \sum_{j\in I_i} v_j^2. \tag{4.6}$$

Inserting (4.6) into (4.4) and using $\mathrm{meas}(\Omega_i) \leq C h_i^2 h$ and that each node belongs only to a bounded number of elements we get

$$\|v_h; W^{1,2}(\Omega)\|^2 \leq Ch\langle \underline{v}, \underline{v}\rangle$$

and with (4.2)

$$\lambda_{\max} \leq Ch \tag{4.7}$$

For the lower estimate of $\|v_h; W^{1,2}(\Omega)\|^2$ we use the embedding

$$W^{1,2}(\Omega) \hookrightarrow W^{1,2}_{1-\delta}(\Omega) \hookrightarrow W^{0,2}_{-\delta}(\Omega)$$

which holds for $0 \leq \delta < 1$ [16]. (The weighted W-spaces were introduced in Subsection 1.1.) Consequently, we have

$$\|v_h; W^{1,2}(\Omega)\|^2 \geq C\|r^{-\delta} v_h; L^2(\Omega)\|^2. \tag{4.8}$$

Denoting $R_i := \max_{\underline{x}\in\Omega_i} r(\underline{x})$, and using (4.6) we get from (4.8)

$$\begin{aligned} \|v_h; W^{1,2}(\Omega)\|^2 &\geq C \sum_{i=1}^m R_i^{-2\delta} \|v_h; L^2(\Omega_i)\|^2 && (4.9)\\ &\geq C \sum_{i=1}^m R_i^{-2\delta} h_i^2 h \sum_{j\in I_i} v_j^2 && (4.10) \end{aligned}$$

Using $h_i \geq ChR_i^{1-\mu}$ (note that this estimate holds for all $i = 1, \ldots, m$) and choosing $\delta = 1 - \mu$, we obtain

$$\|v_h; W^{1,2}(\Omega)\|^2 \geq Ch^3\langle \underline{v}, \underline{v}\rangle$$

and with (4.3)

$$\lambda_{\min} \geq Ch^3 \tag{4.11}$$

independent of the choice of μ. In contrast to this we get $\lambda_{\min} \geq Ch^3$ for isotropic elements only in the case $\mu > \frac{1}{3}$, see for example [6].

From (4.7) and (4.11) we get the estimate

$$\kappa \leq Ch^{-2}, \tag{4.12}$$

that means, the order of the condition number has the same order as in the case of smooth solutions and isotropic meshes.

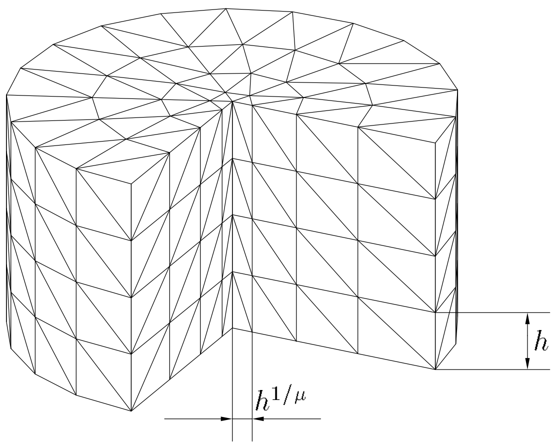

Figure 5.1: Example for an anisotropic mesh.

5 Numerical tests

As an example we consider the Poisson problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega^{(1)} \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega^{(2)} \end{aligned}$$

in the three-dimensional domain $\Omega = \{(x_1, x_2, x_3) = (r\cos\varphi, r\sin\varphi, z) \in \mathbb{R}^3 : \; r < 1, 0 < \varphi < \frac{3}{2}\pi, 0 < z < 1\}$ with the boundary $\partial\Omega = \partial\Omega^{(1)} \cup \partial\Omega^{(2)}$, $\partial\Omega^{(1)} = \{\underline{x} \in \partial\Omega : r = 1\}$. The right hand sides f and g are taken such that

$$\begin{aligned} u &= \gamma r^{2/3} \cos\frac{2}{3}\varphi \\ \gamma &= \begin{cases} z^2 + 1 & \text{for } z \in [0, \frac{1}{2}] \\ -z^2 + 2z + \frac{1}{2} & \text{for } z \in (\frac{1}{2}, 1] \end{cases} \end{aligned}$$

is the exact solution of the problem. It has the typical behaviour in the neighbourhood of the edge for a Neumann problem. We remark that $u \in A^{2,p}_\beta(\Omega)$ if and only if $\beta > 2 - \frac{2}{p} - \frac{2}{3}$, that $f = -\Delta u \in L_p(\Omega)$ for all $p \in [1, \infty]$, and that f has a jump at $z = \frac{1}{2}$.

The meshes used were constructed as described at the end of Subsection 1.2, see also Figure 5.1. In order to investigate the influence of anisotropic mesh grading on the approximation order we varied the mesh size h ($\frac{1}{6}$, $\frac{1}{12}$, $\frac{1}{18}, \ldots, \frac{1}{42}$) and computed numerical solutions for $\mu = 1.0$ and $\mu = 0.5$. From them we calculated the energy norm $\|e\|$ of the finite element error $e = u - u_h$ by numerical integration with an 11-point formula. The relative norms $\|e\|_\% := \|e\|/\|u_h\|$ are plotted against the number N of unknowns using a double logarithmic scale in Figure 5.2.

The computations show that the use of anisotropic meshes leads to the optimal approximation order and diminishes the error even for rather coarse meshes.

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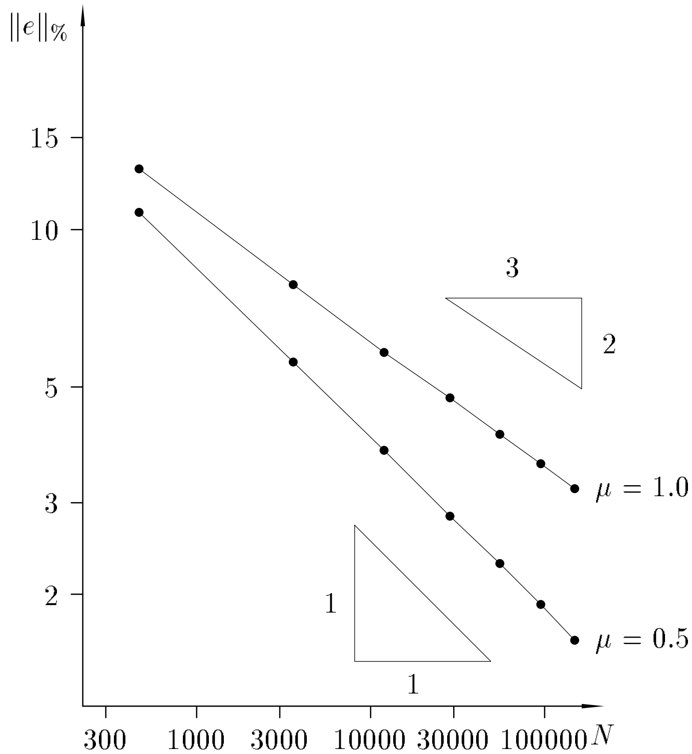

Figure 5.2: Behaviour of the finite element error for ungraded ($\mu = 1.0$) and anisotropically graded meshes ($\mu = 0.5$).

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