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Necessary and sufficient conditions for the regularity of a planar Coons map

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Abstract

We consider the problem of characterizing whether a Coons map is a diffeomorphism from the unit square onto a planar domain delineated by four given curves. We aim primarily at having not only theoretically correct conditions but also practically efficient methods. Throughout the chapter we suppose that the given four boundary curves are presented in Bézier forms. We will prove three sufficient conditions: the first one is based upon the tangents of the boundary curves, the second one exploits the representation of the Jacobian in Bézier surface with a degree elevation when relevant, and the last one invokes the subdivision and polar forms techniques. Further, we will prove that the last condition is also necessary for sufficiently many subdivisions. We present a way of adaptive subdivision so as to make it efficient. Numerical results are reported in order to illustrate the approaches.

1 Introduction

Determining whether a Coons map is a diffeomorphism is not just an interesting problem but it could have interesting applications. We would like to mention the numerical solution of integral equations on CAD objects [1, 2]. If the wavelet Galerkin scheme [13, 9] is used to solve the integral equation then the surface of the CAD object has to be split into patches, each having four sides, and we need a diffeomorphism from the unit square onto each patch. In an earlier work [12], we utilized transfinite interpolation in order to generate a parametric mapping from the unit square to a four-sided domain. It is on that account that we need to have an efficient method to characterize if a Coons map is a diffeomorphism. For given four curves α, β, γ, δ which enclose a planar domain (Fig. 1), the purpose of this paper is to recognize if the corresponding Coons map is a diffeomorphism. We will suppose throughout that the boundary curves are Bézier curves.

In fact, we will prove three sufficient conditions which are mainly expressed with the help of the control points of the boundary curves and the blending functions. This paper is organized as follows. In the next section, we will make some excursus on transfinite interpolation and state our problem more clearly. The first sufficient condition which is based on the tangents of the boundary curves will be described in section 3. We will discuss there also some interesting case in which the blending functions take only positive values. In section 4, we will propose and prove a sufficient condition based on the control points of the Jacobian of the Coons patch. We do not present any necessary condition until section 5 where we use subdivision methods to express our condition and where we have both low computational cost and effectivity as objective . We will propose an adaptive strategy to accomplish adaptive subdivision. Based on that, we will present an alogrithm whose termination is ensured by a theorem which will also investigate. Finally numerical results are presented in the last section to test the performance of the proposed approaches in practice.

2 Transfinite interpolation and problem setting

Let us consider four continuously differentiable parametric curves α, β, γ, δ defined on the interval $[0,1]$ and taking values in $\mathbf{R}^2$. They are supposed to fulfill the compatibility condition (see Fig. 1) at the corners:

$$\alpha(0) = \delta(0)\,, \quad \alpha(1) = \beta(0)\,, \quad \gamma(0) = \delta(1)\,, \quad \gamma(1) = \beta(1). \tag{1}$$

Since our method of generating a mapping from the unit square to the four-sided domain bounded by the four curves is based on transfinite interpolation, we would like now to briefly recall some basic facts about this technique. For a more in-depth understanding regarding transfinite interpolation in general we direct the readers to [5, 6, 7, 8, 14].
We are interested in generating a parametric surface $\mathbf{x}(u,v)$ defined on the unit square $[0,1]^2$ such that the boundary of the image of $\mathbf{x}$ coincides with the given four curves:

$$\begin{array}{lll} \mathbf{x}(u,0) = \alpha(u) & \mathbf{x}(u,1) = \gamma(u) & \forall\, u \in [0,1] \\ \mathbf{x}(0,v) = \delta(v) & \mathbf{x}(1,v) = \beta(v) & \forall\, v \in [0,1]\ . \end{array} \tag{2}$$

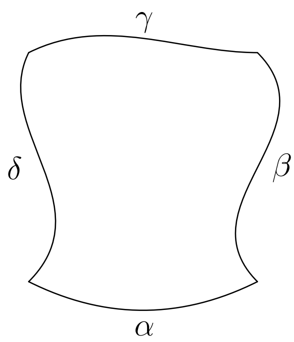


Figure 1: A four sided domain for Coons patch

This transfinite interpolation problem can be solved by a first order Coons patch whose construction involves the operators

$$(\mathcal{P}\mathbf{x})(u,v) \;:=\; F_0(v)\mathbf{x}(u,0) + F_1(v)\mathbf{x}(u,1) \tag{3}$$

$$(\mathcal{Q}\mathbf{x})(u,v) \;:=\; F_0(u)\mathbf{x}(0,v) + F_1(u)\mathbf{x}(1,v) \tag{4}$$

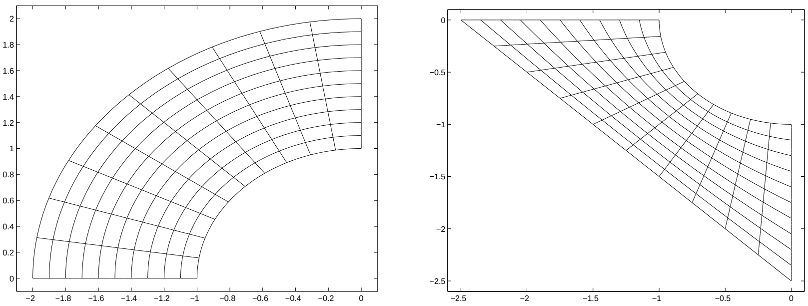
where F_0 and F_1 denote two arbitrary smooth functions satisfying

$$F_i(j) = \delta_{ij} \quad i,j = 0,1 \quad \text{and} \qquad F_0(t) + F_1(t) = 1 \quad \forall\, t \in [0,1]. \tag{5}$$

Now, a Coons patch $\mathbf{x}$ can be defined by the relation

$$\mathcal{P} \oplus \mathcal{Q}(\mathbf{x}) = \mathbf{x}, \qquad \text{where} \tag{6}$$

$$\mathcal{P} \oplus \mathcal{Q} := \mathcal{P} + \mathcal{Q} - \mathcal{P}\mathcal{Q}. \tag{7}$$



(a) (b)

Figure 2: Diffeomorph Coons patches

The functions F_0, F_1 which are better known as blending functions can be chosen in several ways (see [3, 5, 10, 14]). On account of the fact that we need to verify diffeomorphisms, we choose in the sequel blending functions which are sufficiently smooth. The simplest case that one can take as bilinear blending function is

$$F_0(t) = 1 - t, \qquad F_1(t) = t. \tag{8}$$

We will see in our following discussion that the theoretical results that we derive are valid for a large range of blending functions. According to (7) we can express the solution to (2) in matrix form as:

$$\begin{aligned} \mathbf{x}(u,v) &= \begin{bmatrix} F_0(u) & F_1(u) \end{bmatrix} \begin{bmatrix} \delta(v) \\ \beta(v) \end{bmatrix} + \\ & \begin{bmatrix} \alpha(u) & \gamma(u) \end{bmatrix} \begin{bmatrix} F_0(v) \\ F_1(v) \end{bmatrix} - \\ & \begin{bmatrix} F_0(u) & F_1(u) \end{bmatrix} \begin{bmatrix} \alpha(0) & \gamma(0) \\ \alpha(1) & \gamma(1) \end{bmatrix} \begin{bmatrix} F_0(v) \\ F_1(v) \end{bmatrix}. \end{aligned} \tag{9}$$

From (6) it follows that $\mathbf{x}$ is of the form

$$\mathbf{x}(u,v) = -\begin{bmatrix} -1 \\ F_0(u) \\ F_1(u) \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{x}(u,0) & \mathbf{x}(u,1) \\ \mathbf{x}(0,v) & \mathbf{x}(0,0) & \mathbf{x}(0,1) \\ \mathbf{x}(1,v) & \mathbf{x}(1,0) & \mathbf{x}(1,1) \end{bmatrix} \begin{bmatrix} -1 \\ F_0(v) \\ F_1(v) \end{bmatrix}. \tag{10}$$

This construction is due to S. M. Coons and it has been developed for free form surface modeling. The Boolean sum character of a Coons patch has been discovered by W. Gordon. The differentiability of a Coons map is guaranteed if all curves and blending functions involved are themselves differentiable. In Figs. 2(a) and 2(b), we illustrate that for most practical cases a Coons patch is already a diffeomorphism. However, when the boundary curves become too wavy, like in Fig. 3, we observe overlapping isolines indicating that the mapping is not invertible.

The purpose of this paper is to analyse under which conditions the Coons map (10) is a diffeomorphism . For that we need sufficient and necessary conditions which characterize the diffeomorphisms. Our fundamental aim is not only conditions which are theoretically valid. We aim also at having conditions which we can check in a fast and efficient way practically. Throughout the paper we wuppose that the boundary curves α, β, γ, δ for the Coons map are Bézier curves of degree n and that their Bézier points are α_i, β_i, γ_i, δ_i with $i = 0, ..., n$ respectively. That is,

$$\alpha(t) = \sum_{i=0}^{n} \alpha_i B_i^n(t), \qquad \beta(t) = \sum_{i=0}^{n} \beta_i B_i^n(t), \tag{11}$$

$$\gamma(t) = \sum_{i=0}^{n} \gamma_i B_i^n(t), \qquad \delta(t) = \sum_{i=0}^{n} \delta_i B_i^n(t). \tag{12}$$

The blending function is supposed also to be a polynomial which is represented in its Bézier form as

$$F_1(t) = 1 - F_0(t) = \sum_{i=0}^{n} \phi_i B_i^n(t). \tag{13}$$

Further we introduce the following constants.

$$\begin{array}{rcl} q & := & \inf\{F_1(t) : t \in [0,1]\}, \\ Q & := & \sup\{F_1(t) : t \in [0,1]\}, \\ \rho & := & \sup\{|F_1'(t)| : t \in [0,1]\}. \end{array} \tag{14}$$

3 First sufficient condition

Before we introduce our first result, let us adopt some more notations. First, for $u, v \in [0,1]$ and $\zeta, \chi \in [q, Q]$, we will denote the combination of opposite tangents by

$$K_{u,\zeta} := (1-\zeta)\alpha'(u) + \zeta\gamma'(u) \tag{15}$$

$$L_{v,\chi} := (1-\chi)\delta'(v) + \chi\beta'(v). \tag{16}$$

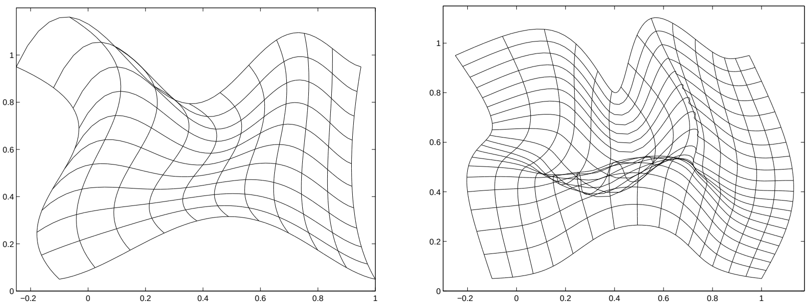
Then we introduce

$$M := \max\left\{ \sup_{(u,\zeta)\in[0,1]\times[q,Q]} \|K_{u,\zeta}\| \ , \sup_{(v,\chi)\in[0,1]\times[q,Q]} \|K_{v,\chi}\| \right\}. \tag{17}$$

Besides, we have the following maxima

$$\begin{array}{rcl} S^1 & := & \max_{i=0,\dots,n} \{\rho \|(\beta_i - \delta_i) + \phi_i(\gamma_0 - \gamma_n + \alpha_n - \alpha_0) + (\alpha_0 - \alpha_n)\|\} \\ S^2 & := & \max_{i=0,\dots,n} \{\rho \|(\gamma_i - \alpha_i) + \phi_i(\alpha_n - \gamma_n + \gamma_0 - \alpha_0) + (\alpha_0 - \gamma_0)\|\}. \end{array}$$

Finally F is defined to be S^1 or S^2, whichever has the largest value.



(a) (b)

Figure 3: Undesired overspill phenomena

Theorem 1 If there exists some $\kappa > 0$ such that

$$\det(K_{u,\zeta}, L_{v,\chi}) \geq \kappa \tag{18}$$

and

$$2MF + F^2 < \kappa, \tag{19}$$

then the Coons patch with respect to α, β, γ, δ is a diffeomorphism.

Proof
Some few computations reveal that the partial derivatives of the Coons patch are

$$\mathbf{x}_u(u,v) = F_1'(u) S_u + C_u\,, \qquad \mathbf{x}_v(u,v) = F_1'(v) S_v + C_v \qquad \text{where} \tag{20}$$

$$\begin{aligned}
S_u &:= \beta(v) - \delta(v) + [F_0(v)\alpha(0) + F_1(v)\gamma(0)] - [F_0(v)\alpha(1) + F_1(v)\gamma(1)] \\
S_v &:= \gamma(u) - \alpha(u) + [F_0(u)\alpha(0) + F_1(u)\alpha(1)] - [F_0(u)\gamma(0) + F_1(u)\gamma(1)] \\
C_u &:= F_0(v)\alpha'(u) + F_1(v)\gamma'(u) \\
C_v &:= F_0(u)\delta'(v) + F_1(u)\beta'(v)\,.
\end{aligned}$$

Therefore we obtain

$$S_u = \sum_{i=0}^{n} (\beta_i - \delta_i) B_i^n(v) + F_1(v)(\gamma_0 - \gamma_n + \alpha_n - \alpha_0) + (\alpha_0 - \alpha_n) \tag{21}$$

$$S_v = \sum_{i=0}^{n} (\gamma_i - \alpha_i) B_i^n(u) + F_1(u)(\alpha_n - \gamma_n + \gamma_0 - \alpha_0) + (\alpha_0 - \gamma_0) \tag{22}$$

After a few rearrangements

$$S_u = \sum_{i=0}^{n} [(\beta_i - \delta_i) + \phi_i(\gamma_0 - \gamma_n + \alpha_n - \alpha_0) + (\alpha_0 - \alpha_n)] B_i^n(v) \tag{23}$$

$$S_v = \sum_{i=0}^{n} [(\gamma_i - \alpha_i) + \phi_i(\alpha_n - \gamma_n + \gamma_0 - \alpha_0) + (\alpha_0 - \gamma_0)] B_i^n(u) \tag{24}$$

By using (C1) one obtains

$$|F_1'(u)|.\|S_u\| \leq F \quad \text{and} \quad |F_1'(v)|.\|S_v\| \leq F. \tag{25}$$

Because of multilinearity of the determinant function we have

$$\begin{aligned}
\det(\mathbf{x}_u, \mathbf{x}_v) &= F_1'(u)F_1'(v)\det(S_u, S_v) + F_1'(u)\det(S_u, C_v) + \\
&\quad +F_1'(v)\det(C_u, S_v) + \det(C_u, C_v). \\
&\geq \det(C_u, C_v) - \{|F_1'(u)F_1'(v)\det(S_u, S_v)| + \\
&\quad +|F_1'(u)\det(S_u, C_v)| + |F_1'(v)\det(C_u, S_v)|\} \\
&\geq \kappa - (F^2 + 2FM) > 0 \quad \text{due to (25) (C2) and (C3).}
\end{aligned}$$

That means the Jacobian is nowhere zero. The inverse function theorem ensures therefore that the Coons patch is a diffeomorphism.

Remark 1 Condition (18) has some geometric interpretation. Suppose the bounds q and Q from relation (14) are 0 and 1 respectively. So, if ones consider any convex combination K of the tangent vectors $\alpha'(u)$ and $\gamma'(u)$ and L of $\delta'(v)$ and $\beta'(v)$, then K and L are bounded away from being collinear and they are never zero in norm. Observe in Figure 4 the angle θ which represents with some respect the scaled determinant. If $(q, Q) \neq (0, 1)$ one can draw a similar figure after some rescalings.

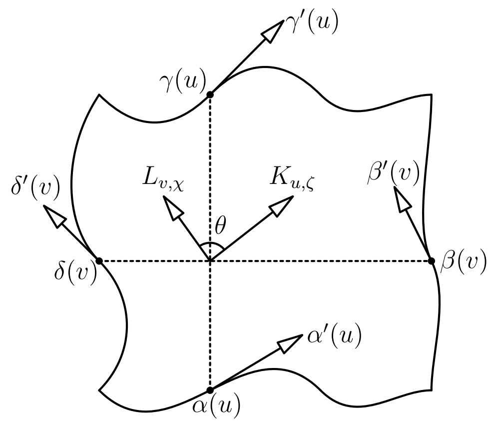


Figure 4: A four sided domain for Coons patch

Remark 2 A direct computation of M from relation (17) could be expensive or inaccurate in practice because it involves non-discrete information. In fact, the constant M could just as well be replaced by another constant that verifies for all $i = 0, ..., n-1$ and $j = 0, ..., n$

$$\begin{aligned}
n\|\phi_j(\gamma_{i+1} - \gamma_i + \alpha_i - \alpha_{i+1}) + (\alpha_{i+1} - \alpha_i)\| &\leq M \\
n\|\phi_j(\beta_{i+1} - \beta_i + \delta_i - \delta_{i+1}) + (\delta_{i+1} - \delta_i)\| &\leq M,
\end{aligned} \tag{26}$$

which is easier to check. The idea of the proof remains fundamentally unchanged. Indeed, relation (26) implies in particular the following bounds

$$\|C_u\| \leq M \qquad \|C_v\| \leq M,$$

where

$$\begin{aligned} C_u &:= F_0(v)\alpha'(u) + F_1(v)\gamma'(u) \\ C_v &:= F_0(u)\delta'(v) + F_1(u)\beta'(v). \end{aligned} \tag{27}$$

Remark 3 In the previous theorem we have treated the very general case in which the blending functions F_0 and F_1 could take any sign. Still, more can be stated if they are to take only positive values as we will see in the next remark. That is for example the case if we choose the following blending functions:

$$\begin{aligned} F_0(t) &:= 2t^3 - 3t^2 + 1 = B_0^3(t) + B_1^3(t) \\ F_1(t) &:= -2t^3 - 3t^2 = B_2^3(t) + B_3^3(t). \end{aligned}$$

Interestingly, we will see that condition (18) could be replaced by another one that takes a discrete form which is of course of practical concern.

Remark 4 Suppose that the blending functions are positive:

$$F_0(t) \geq 0, \qquad F_1(t) \geq 0 \qquad \forall\, t \in [0,1]. \tag{28}$$

If we replace condition (18) of the previous theorem by

$$\begin{aligned} n^2 \det[\alpha_{i+1} - \alpha_i, \delta_{j+1} - \delta_j] &> 0\,, \\ n^2 \det[\alpha_{i+1} - \alpha_i, \beta_{j+1} - \beta_j] &> 0\,, \\ n^2 \det[\gamma_{i+1} - \gamma_i, \delta_{j+1} - \delta_j] &> 0\,, \\ n^2 \det[\gamma_{i+1} - \gamma_i, \beta_{j+1} - \beta_j] &> 0\,, \end{aligned} \tag{29}$$

for all $i, j = 0, ..., n-1$ and define $\kappa > 0$ to be the minimum of them , then we can deduce the same claim.

Proof
According to the multilinearity of the determinant again, we have (see also definitions from relation (27))

$$\begin{aligned} \det[C_u, C_v] = {} & F_0(v)F_0(u)\det[\alpha'(u), \delta'(v)] + F_0(v)F_1(u)\det[\alpha'(u), \beta'(v)] + \\ & F_1(v)F_0(u)\det[\gamma'(u), \delta'(v)] + F_1(v)F_1(u)\det[\gamma'(u), \beta'(v)]. \end{aligned}$$

On account of the fact that

$$\alpha'(u) = \sum_{i=0}^{n-1} n(\alpha_{i+1} - \alpha_i)B_i^{n-1}(u)$$

and similar relations for β, γ, δ, we deduce from (29) that

$$\det[C_u, C_v] \geq \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \kappa B_i^{n-1}(u)B_j^{n-1}(v).$$

Since the Bernstein polynomials form a partition of unity, we deduce the result.

Lemma 1 Suppose the boundary curves α, β, γ, δ and the blending function are given as before. Then the Coons patch is a Bezier surface

$$\mathbf{x}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{n} \mathbf{E}_{ij} B_i^n(u) B_j^n(v) \tag{30}$$

where the control points are

$$\begin{aligned} \mathbf{E}_{ij} \quad := \quad & \delta_j - \alpha_0 + \alpha_i + \phi_j(\gamma_i - \alpha_i + \alpha_0 - \gamma_0) \\ & \phi_i[\alpha_0 - \alpha_n + \beta_j - \delta_j + \phi_j(\alpha_n - \gamma_n + \gamma_0 - \alpha_0)] \end{aligned} \tag{31}$$

Proof
Obvious. See also ([4]) for a similar discussion.

4 Second sufficient condition

Theorem 2 Consider the assumption above and define

$$D(i,j,k,l) := n^2 \det[\mathbf{E}_{i+1,j} - \mathbf{E}_{ij}, \mathbf{E}_{k,l+1} - \mathbf{E}_{kl}] \tag{32}$$

$$\begin{aligned} C(i,j,k,l) \quad := \quad & \tfrac{l}{n}\left[\tfrac{i}{n} D(i-1,j,k,l-1) + (1-\tfrac{i}{n}) D(i,j,k,l-1)\right] + \\ & (1-\tfrac{l}{n})\left[\tfrac{i}{n} D(i-1,j,k,l) + (1-\tfrac{i}{n}) D(i,j,k,l)\right] \end{aligned} \tag{33}$$

If for all $p, q = 0, ..., 2n$

$$J_{pq} := \sum_{\substack{i+k=p \\ j+l=q}} C(i,j,k,l) \frac{\binom{n}{i}\binom{n}{k}}{\binom{2n}{i+k}} \frac{\binom{n}{j}\binom{n}{l}}{\binom{2n}{j+l}} > 0 \tag{34}$$

then the Coons patch is a diffeomorphism.

Proof
The partial derivatives are

$$\mathbf{x}_u(u,v) \quad = \quad \sum_{i=0}^{n-1}\sum_{j=0}^{n} n(\mathbf{E}_{i+1,j} - \mathbf{E}_{ij}) B_i^{n-1}(u) B_j^n(v) \tag{35}$$

$$\mathbf{x}_v(u,v) \quad = \quad \sum_{k=0}^{n}\sum_{l=0}^{n-1} n(\mathbf{E}_{k,l+1} - \mathbf{E}_{kl}) B_k^n(u) B_l^{n-1}(v) \tag{36}$$

Therefore we obtain the determinant

$$\mathbf{det}(\mathbf{x}_u, \mathbf{x}_v) = \sum_{i=0}^{n-1}\sum_{j=0}^{n}\sum_{k=0}^{n}\sum_{l=0}^{n-1} D(i,j,k,l) B_i^{n-1}(u) B_j^n(v) B_k^n(u) B_l^{n-1}(v) \tag{37}$$

After application of degree elevation with respect to the indices i and l we obtain

$$\mathbf{det}(\mathbf{x}_u, \mathbf{x}_v) = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n}\sum_{l=0}^{n} C(i,j,k,l) B_i^n(u) B_j^n(v) B_k^n(u) B_l^n(v) \tag{38}$$

By using the product formula

$$B_i^n(t) B_k^n(t) = \frac{\binom{n}{i}\binom{n}{k}}{\binom{2n}{i+k}} B_{i+k}^{2n}(t) \tag{39}$$

the Jacobian can be written as

$$J(u,v) := \det(\mathbf{x}_u, \mathbf{x}_v) = \sum_{p=0}^{2n}\sum_{q=0}^{2n} J_{pq} B_p^{2n}(u) B_q^{2n}(v) \tag{40}$$

Therefore the Coons patch $\mathbf{x}$ is a diffeomorphism.

5 Sufficient and necessary condition

In the previous theorems we have presented two methods for verifying whether a Coons patch describes a diffeomorphism. In fact, they give only sufficient conditions. Let us describe a short contrast between those two approaches. As far as computational cost is concerned, the second approach is more computationally intensive than the first one as it will be observed in the numerical experiments from section 6. On the other hand, the second approach is also more sensitive. That is, it can provide some response whereas the first one fails. Additionally, as we degree-elevate the boundary Bezier curves, the second approach becomes more and more sensitive. Computational cost is of course a trade-off to consider if one chooses n large during the degree elevation of the second test. In the next discussion, we will propose a method to achieve at the same time low computational cost and effective results. We will demonstrate a condition, based upon subdivision methods, which is both necessary and sufficient.

5.1 Subdivision

Before we see a necessary condition, let us see the following fact. A Bezier surface F defined on $[a,b] \times [c,d]$ can be subdivided into four Bezier surfaces A, B, C, D which are respectively defined on

$$I^A = [a, 0.5(a+b)] \quad \times \quad [c, 0.5(c+d)] \tag{41}$$
$$I^B = [a, 0.5(a+b)] \quad \times \quad [0.5(c+d), d] \tag{42}$$
$$I^C = [0.5(a+b), b] \quad \times \quad [c, 0.5(c+d)] \tag{43}$$
$$I^D = [0.5(a+b), b] \quad \times \quad [0.5(c+d), d] \tag{44}$$

by using the following recursions. Suppose the control points of F are F_{ij} $i,j=0,...,n$. We define

$$\left\{\begin{array}{lll} F_{ij}^{[0]} & := & F_{ij} \quad \text{and} \\ F_{ij}^{[k]} & := & 0.5(F_{i-1,j}^{[k-1]}+F_{ij}^{[k-1]}) \end{array}\right. \tag{45}$$

$$\left\{\begin{array}{lll} P_{ij}^{[0]} & := & F_{ij}^{[i]} \quad \text{and} \\ P_{ij}^{[k]} & := & 0.5(P_{i,j-1}^{[k-1]}+P_{ij}^{[k-1]}) \end{array}\right. \quad \left\{\begin{array}{lll} Q_{ij}^{[0]} & := & F_{nj}^{[n-i]} \quad \text{and} \\ Q_{ij}^{[k]} & := & 0.5(Q_{i,j-1}^{[k-1]}+Q_{ij}^{[k-1]}) \end{array}\right. \tag{46}$$

The control points of A, B, C and D are respectively $A_{ij}:=P_{ij}^{[j]}$, $B_{ij}:=P_{in}^{[n-j]}$, $C_{ij}:=Q_{ij}^{[j]}$, $D_{ij}:=Q_{in}^{[n-j]}$. We have in particular

$$F(u,v)=Q(u,v) \qquad \text{if} \quad (u,v)\in I^Q,$$

where $Q=A,B,C$, or D.

Theorem 3 Let us adopt the same notations as in the previous statement. Suppose that the Coons patch $\mathbf{x}$ defined with α, β, γ, δ is a diffeomorphism. Suppose further that we have subdivided J into σ^2 Bezier surfaces J^{ij} which are defined on

$$I^{ij}:=[(i-1)/\sigma,i/\sigma]\times[(j-1)/\sigma,j/\sigma] \quad i,j=1,...,\sigma. \tag{47}$$

Denote by J_{pq}^{ij}, $p,q=0,...,2n$ the control points of the Bezier surface J^{ij}.
We claim that if σ is sufficiently large then J_{pq}^{ij} is of constant sign uniformly on $i,j=1,...,\sigma$ and on $p,q=0,...,2n$.

Proof
On the one hand, the Jacobian $J(u,v)$ must be of constant sign because it is never zero. Without loss of generality we suppose that it is positive:

$$J(u,v)>0 \qquad \forall (u,v)\in[0,1]\times[0,1]. \tag{48}$$

Since the function J is continuous on the compact $[0,1]\times[0,1]$, there must exist some $\rho>0$ such that

$$J(u,v)\geq\rho \qquad \forall\,(u,v)\in[0,1]\times[0,1]. \tag{49}$$

On the other hand, let us fix i,j and let us denote by $[a,b]\times[c,d]$ the interval I^{ij} in order to simplify the notation. We are going to use the notation $s,..[m]..,s$ in order to stress that s is to be repeated m times. Further let us introduce the blossom [16, 11] function $P^{ij}(u_1,...,u_{2n};v_1,...,v_{2n})$ corresponding to the polynomial J^{ij}:

$$J^{ij}(u,v)=P^{ij}(u,..[2n]..,u;v,..[2n]..,v) \tag{50}$$

Define $h:=1/(2n\sigma)$ and $a_p:=a+ph$, $c_q:=c+qh$ for $p=0,...,2n$ and $q=0,...,2n$. Now we would like to apply multivariate Taylor development of the first order to the blossom P^{ij} at $(a_p,..[2n]..,a_p;c_q,..[2n]..,c_q)$.

$$
\begin{aligned}
&P^{ij}(a,..[2n-p]..,a,b,..[p]..,b;c,..[2n-q]..,c,d,..[q]..,d) = \\
&P^{ij}(a_p,..[2n]..,a_p;c_q,..[2n]..,c_q) + \\
&\sum_{r=1}^{2n-p}(a-a_p)\frac{\partial}{\partial u_r}P^{ij}(a_p,..[2n]..,a_p;c_q,..[2n]..,c_q) + \\
&\sum_{r=2n-p+1}^{2n}(b-a_p)\frac{\partial}{\partial u_r}P^{ij}(a_p,..[2n]..,a_p;c_q,..[2n]..,c_q) + \\
&\sum_{r=1}^{2n-q}(c-c_q)\frac{\partial}{\partial v_r}P^{ij}(a_p,..[2n]..,a_p;c_q,..[2n]..,c_q) + \\
&\sum_{r=2n-q+1}^{2n}(d-c_q)\frac{\partial}{\partial v_r}P^{ij}(a_p,..[2n]..,a_p;c_q,..[2n]..,c_q) + \mathcal{O}(h^2)
\end{aligned}
$$

Due to the symmetry [11, 15] of the blossom function we obtain

$$
\begin{aligned}
&P^{ij}(a,..[2n-p]..,a,b,..[p]..,b;c,..[2n-q]..,c,d,..[q]..d) = \\
&P^{ij}(a_p,..[2n]..,a_p;c_q,..[2n]..,c_q) + \mathcal{O}(h^2)
\end{aligned} \tag{51}
$$

In other words, we have the following relation regarding the control points

$$
J^{ij}_{pq} = J^{ij}(a_p,c_q) + \mathcal{O}(h^2) \tag{52}
$$

Combining (49) and (52), there must exist some constant $C > 0$ such that we have the following relations

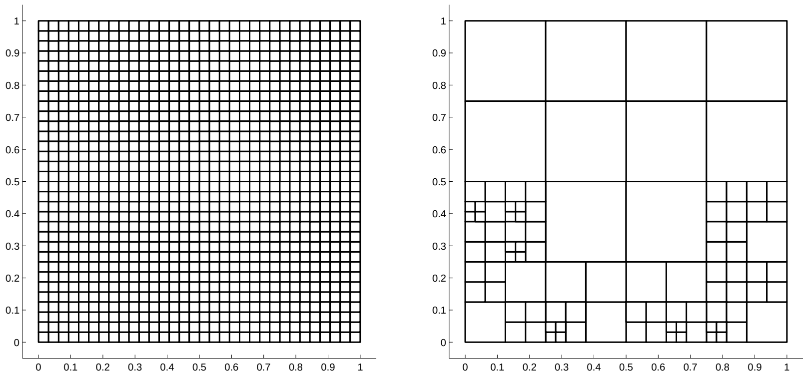
$$
\begin{aligned}
J^{ij}_{pq} &= J^{ij}(a_p,c_q) + J^{ij}_{pq} - J^{ij}(a_p,c_q) && (53)\\
&\geq J(a_p,c_q) - Ch^2 && (54)\\
&\geq \rho - Ch^2 && (55)
\end{aligned}
$$

Since $Ch^2 = \frac{C}{(2n\sigma)^2}$ tends to 0 as σ tends to infinity, we deduce that $J^{ij}_{pq} > 0$ for σ sufficiently large and it concludes the proof.

5.2 Adaptivity

So far, we have always described something which should work if the Coons patch $\mathbf{x}$ is a diffeomorphism. What happens if it is not? On that account, we want to state the following result.

Theorem 4 Adopt the same notations as in the previous theorem but suppose now that the Coons patch is not a diffeomorphism.
There must exist (i_1, j_1) and (i_2, j_2) such that



(a) (b)

Figure 5: Multiple subdivision:(a)uniform (b)adaptive

$$\begin{cases} J_{pq}^{i_1,j_1} > 0 & \forall\, p,q = 0,...,2n \\ J_{pq}^{i_2,j_2} < 0 & \forall\, p,q = 0,...,2n. \end{cases} \tag{56}$$

Proof

This theorem is demonstrated in a very similar way as the preceding one. Therefore we omit the proof.

Remark 5 The condition that is seen in (56) can of course be used in the next algorithm as an abortion criterion. That is, once the condition (56) occurs in the loop (see step 1), we terminate the algorithm and conclude at the same time that the Coons patch is not a diffeomorphism.

In the preceding theorems, we have subdivided the unit square uniformly (see Fig. 5(a)) which is not always essential in practice. It is advisable to apply the former Bézier subdivisions only to those patches which do not give responses (affirmative or negative) as it can be seen in Fig. 5(b). Before we give our algorithm of adaptive subdivision, let us observe the following simple fact. If the Bézier coefficients of the surface F from remark 5.1 are all positive then so are those of the resulting surfaces A, B, C, D. The theoretical results that we derived earlier give rise to the following algorithm.

Algorithm: Adaptive regularity

step 0 : Initialize the grid G to have only one cell $[0,1]\times[0,1]$ and compute the Bezier coefficients J_{pq} according to (34).

step 1 : Traverse the cells $I=[a,b]\times[c,d]$ of the grid G

- Check if all coefficients J^I_{pq} have fixed sign irrespective of the indices $p,q=0,...,2n$
- If not, split I into four cells $I1$, $I2$, $I3$, $I4$ and subdivide the Bezier surface J^I into four Bezier surfaces J^{I1}, J^{I2}, J^{I3}, J^{I4} as in Remark 5.1.
- If there was some cell $\tilde{I}\neq I$ for which $J^{\tilde{I}}_{pq}$ was always positive (resp. negative) for all $p,q=0,...,2n$ and the current J^I_{pq} is always negative (resp. positive), then abort the whole algorithm and conclude that the Coons map is NOT a diffeomorphism as discussed in Theorem 4.

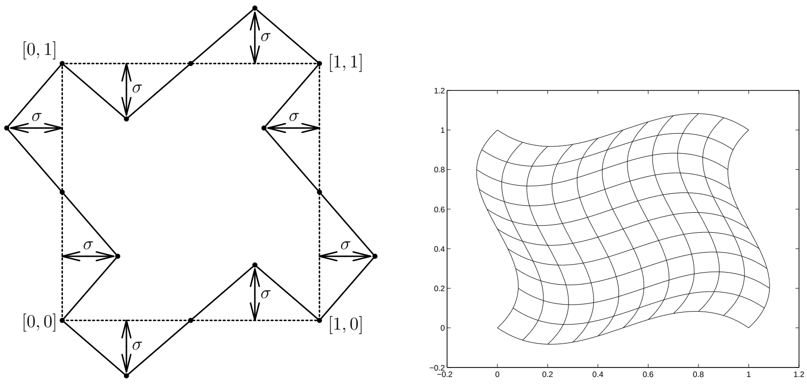
step 2 : If in step 1, all J^I_{pq} have fixed sign irrespective of the cell I and the indices $p,q=0,1,...,2n$ then terminate the algorithm and conclude that the Coons map is a diffeomorphism otherwise go to step 1.

6 Numerical results

This section will be occupied by numerical results which support the formerly described theories as well as algorithm. As a first test, we consider a Coons map whose boundary curves can be controlled by a parameter σ. More precisely, let us consider the control polygons which are seen in Fig. 6(a). A special case of such a map for $\sigma=0.216$ is portrayed in Fig. 6(b). If the parameter σ is zero then we retrieve the unit square. The purpose of this first test is to investigate numerically the theoretical conditions that we discussed earlier. On that account, we want to see the performance of the three sufficient conditions to verify diffeomorphism. For $\sigma>0.36$, the Coons map does not present any diffeomorphism any more. We want therefore to vary the value of σ which will then range from 0 to 0.35.

The numerical data that are in Table 1 have been collected from an Intel Pentium 4 processor 2.66 GHz running Windows XP. For any given σ in the first column, we find the corresponding results of Theorems 1, 2 and 3 in the three last columns respectively.

If the conditions in the theorems are successful then we provide the time needed to run the test. If the map is a diffeomorphism but our condition could not detect that, then we report a failure information in the table ('*fails*'). For the results of the second test, additional information about the required degree n of the boundary curves is provided in



(a) (b)

Figure 6: (a)Control polygons of the boundary curves (b)Coons map for $\sigma = 0.216$.

parentheses. In other words, we degree-elevate the curves until the second test gives some response. The same remark applies to the third test with the number of required cells in parenthesis.

σ	first test	second test	adaptive subdivision
0.000	1.1 E-07 sec	1.2 E-05 sec (n=4)	1.2 E-05 sec (nb cells=1)
0.036	1.1 E-07 sec	1.2 E-05 sec (n=4)	1.2 E-05 sec (nb cells=1)
0.072	1.1 E-07 sec	1.2 E-05 sec (n=4)	1.2 E-05 sec (nb cells=1)
0.216	1.1 E-07 sec	1.2 E-05 sec (n=4)	1.2 E-05 sec (nb cells=1)
0.252	*fails*	1.2 E-05 sec (n=4)	1.2 E-05 sec (nb cells=1)
0.280	*fails*	1.2 E-05 sec (n=4)	1.2 E-05 sec (nb cells=1)
0.324	*fails*	7.8 E-002 sec (n=9)	1.5 E-002 sec(nb cells=4)
0.350	*fails*	5.6 E+010 sec (n=40)	1.5 E-002 sec(nb cells=4)

Table 1: Performance of the three conditions

From the figures in the table, we see that the first test is only successful till the value of σ is 0.216, a case which corresponds to the map in Fig. 6(b). On the other hand, it is also to be noticed that the first test is comparatively less computationally intensive than the other two tests. A closer look at Table 1 reveals that the second test is not any longer efficient when the degree n is too large because one single test lasts approx. one minute.
Our next experiment is to consider some Coons maps and to investigate in which case they present diffeomorphisms. Let us consider the Coons map whose control polygons are seen in Fig. 7(a). Observe that the boundary curve δ is specified by some parameter μ_1 which could be positive or negative. For examples we see in Fig. 7(b) the result if the parameter takes the value $\mu_1 = 0.7$. We have used the former theorems to characterize whether the resulting Coons map is a diffeomorphism: if μ_1 is negative then we have always a diffeomorphism. If $\mu_1 \in [0, 0.88]$ then we still have a diffeomorphism. For $\mu_1 = 0.89$, we do not have any more diffeomorphism.

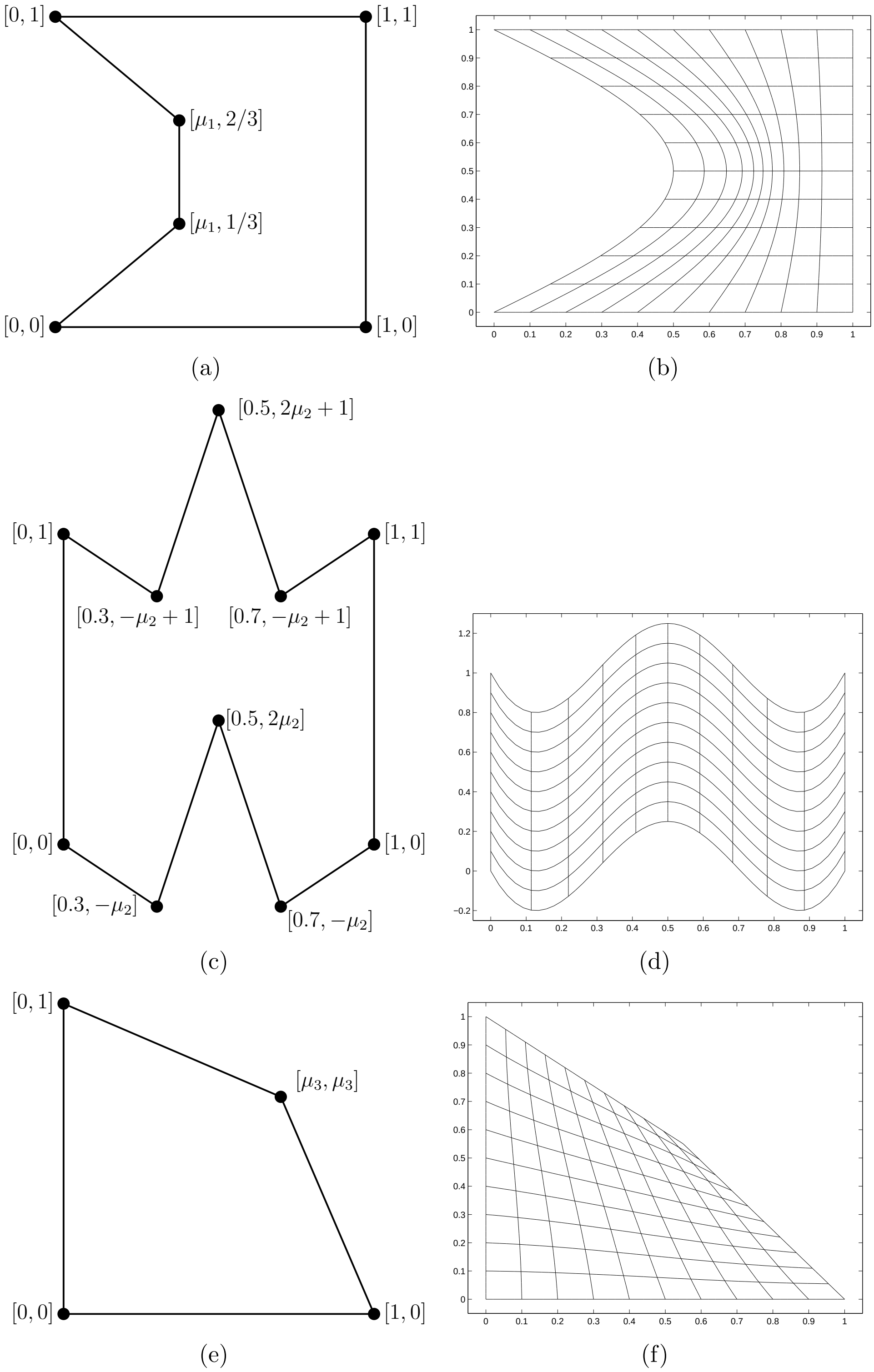


Figure 7: Examples of diffeomorphisms

Now we want to do a similiar test but this time we want to apply it to a control polygon where the boundary curve α is parallel to the boundary curve γ and the other curves are straight lines. As illustrated in Fig. 7(c), the control points of the curved boundaries α and γ are determined by some constant μ_2. In Fig. 7(d), we can see the Coons map in which we chose $\mu_2 = 1$. After applying the former theory in which we let μ_2 vary inside the interval $[-10, 10]$ we have concluded that the resulting Coons map is consistently a diffeomorphism irrespective of the value of μ_2.
Another example is depicted in Fig. 7(e) where the parameter $\mu_3 > 0$ controls the distance of one corner to the origin. Our former theorems allow us to conclude that the correspoding Coons map is a diffeomorphism for $\mu_3 > 0.5$ and it is not a diffeomorphism for $\mu_3 \in [0, 0.5]$.

References

[1] W. Dahmen, Wavelet and multiscale methods for operator equations, Acta Numerica **6** (1997) 55–228.

[2] W. Dahmen, R. Schneider, Wavelets on manifolds. I: Construction and domain decomposition, SIAM J. Math. Anal. **31**, No.1 (1999) 184–230.

[3] G. Farin, Curves and surfaces for computer aided geometric design: a practical guide, Academic Press, Boston, 1997.

[4] G. Farin, Discrete Coons patches, Comput. Aided Geom. Des. **16**, No.7 (1999) 691–700.

[5] A. Forrest, On Coons and other methods for the representation of curved surfaces, Comput. Graph. Img Process. **1** (1972) 341–359.

[6] W. Gordon, C. Hall, Construction of curvilinear co-ordinate systems and applications to mesh generation, Int. J. Numer. Methods Eng. **7** (1973) 461–477.

[7] W. Gordon, C. Hall, Transfinite element methods: Blending-function interpolation over arbitrary curved element domains, Numer. Math. **21** (1973) 109–129.

[8] W. Gordon, Sculptured surface interpolation via blending-function methods, Research Report, Department of Mathematics and Computer Science, Drexel University, Philadelphia, 1982.

[9] H. Harbrecht, R. Schneider, Biorthogonal wavelet bases for the boundary element method, Preprint SFB393/03-10, Technische Universität Chemnitz, Sonderforschungsbereich 393, 2003.

[10] J. Hoschek, D. Lasser, Grundlagen der geometrischen Datenverarbeitung, Teubner, Stuttgart, 1989.

[11] H. Prautzsch, W. Boehm, M. Paluszny, Bézier and B-Spline techniques, Springer, Berlin, 2002.

[12] M. Randrianarivony, G. Brunnett, R. Schneider, Constructing a diffeomorphism between a trimmed domain and the unit square, Sonderforschungsbereich 393, Preprint SFB393/03-20, 2003.

[13] R. Schneider, Multiskalen- und Wavelet-Matrixkompression: Analysisbasierte Methoden zur Lösung grosser vollbesetzter Gleichungssysteme, B.G. Teubner, Stuttgart, 1998.

[14] G. Schulze, Blending-Function-Methoden im CAGD, Diplomarbeit, Universität Dortmund, 1986.

[15] H. Seidel, Computing B-spline control points using polar forms, Comput.-Aided Des. **23**, No.9 (1991) 634–640.

[16] H. Seidel, Polar forms for geometrically continuous spline curves of arbitrary degree, ACM Trans. Graph. **12**, No.1 (1993) 1–34.

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