TECHNISCHE UNIVERSITÄT CHEMNITZ

Sonderforschungsbereich 393

Parallele Numerische Simulation für Physik und Kontinuumsmechanik

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A local error analysis of the boundary concentrated FEM

Preprint SFB393/04-05

Abstract

The boundary concentrated finite element method is a variant of the hp-version of the FEM that is particularly suited for the numerical treatment of elliptic boundary value problems with smooth coefficients and boundary conditions with low regularity or non-smooth geometries. In this paper we consider the case of the discretization of a Dirichlet problem with exact solution $u \in H^{1+\delta}(\Omega)$ and investigate the local error in various norms. We show that for a $\beta > 0$ these norms behave as $O(N^{-\delta-\beta})$, where N denotes the dimension of the underlying finite element space. Furthermore, we present a new Gauss-Lobatto based interpolation operator that is adapted to the case non-uniform polynomial degree distributions.

Preprintreihe des Chemnitzer SFB 393

ISSN 1619-7178 (Print) ISSN 1619-7186 (Internet)

SFB393/04-05 Mai 2004

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1 Introduction

The boundary concentrated finite element method (bcFEM), introduced in [9], is a numerical method particularly suited for solving elliptic boundary value problems where the differential equation has analytic coefficients but the boundary conditions have low regularity or the geometry is non-smooth. Under these assumptions on the data the solution is analytic on the domain Ω but of low Sobolev regularity globally. The boundary concentrated finite element method exploits the interior regularity of the solution in the framework of the *hp*-version of the FEM by using special types of meshes and polynomial distributions, namely, small elements with low order polynomials near the boundary and large elements with high order polynomials in the interior. In this way it achieves (for two-dimensional problems) a global rate of convergence

$$\|u - u_N\|_{H^1(\Omega)} \leq CN^{-\delta} \quad \text{for } \ u \in H^{1+\delta}(\Omega),$$

where N is the problem size. We refer to [9] for a detailed description. In the present paper, we focus on the local error and we will investigate the behavior of the local error on compact subsets of the domain Ω. We prove the existence of a $\beta > 0$ such that these errors behave as $O(N^{-\delta-\beta})$, up to logarithmic terms. For simplicity of exposition we analyze here as a model problem a Poisson problem in two dimensions. We expect that the local error analysis of Theorem 2.1 can be adapted to a more general class of strongly elliptic operators with analytic coefficients. The restriction to two dimensions is likewise done for simplicity of exposition—the techniques used in this paper are likely to have extentions to higher dimensions. The paper is organized as follows: We start with a brief repetition of the foundations of boundary concentrated FEM. Therafter, in Section 2, we formulate the main theorem concerning the local error behavior of boundary concentrated FEM and in Section 3 we present some numerical examples. In Section 4, we introduce a new *hp*-interpolation operator, which is an essential tool for our local analysis. The remainder of the paper is devoted to the proof of auxiliary results that were used in the proof of our main theorem, and we conclude the paper with an outlook on future work.

1.1 Notation

For ease of notation we introduce the following abbreviations:

- For a Lipschitz domain $\Omega \subset \mathbb{R}^2$ and $\mathbf{x} \in \Omega$ we denote by $r(\mathbf{x}) := \text{dist}(x, \partial\Omega)$ the distance of $\mathbf{x}$ to the boundary of Ω.

- The trace operator $\gamma_0 : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ and the normal derivative $\gamma_1 : H^1(\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ are defined by

  $$\gamma_0 u = u|_{\partial\Omega} \qquad \gamma_1 u = \partial_n u|_{\partial\Omega},$$

  where ∂_n denotes the normal derivative.

- The characteristic function χ is defined by:

$$\chi_A(\mathbf{x}) := \begin{cases} 1 & : \ \mathbf{x} \in A \\ 0 & : \ \text{otherwise} \end{cases} .$$

As is standard, C stands for a generic constant that possibly different in each instance.

1.2 Model problem and regularity of the solution

For a polygonal Lipschitz domain $\Omega \subset \mathbb{R}^2$, we consider the following Dirichlet problem, given in weak formulation:

Problem 1.1. (model problem) *Find $u \in V := \{u \in H^1(\Omega) \mid \gamma_0 u = g\}$ such that*

$$B(u,v) := \int\limits_\Omega \nabla u \cdot \nabla v d\Omega = \int\limits_\Omega f v d\Omega =: F(v) \qquad \forall\, v \in H_0^1(\Omega). \tag{1}$$

Furthermore, we make the following assumptions: Throughout the paper, we will make the following assumption on Problem 1.1:

Assumption 1.2. *The solution u of Problem 1.1 satisfies $u \in H^{1+\delta}(\Omega)$ for a $\delta \in (0,1]$ and the right-hand side $f \in L^2(\Omega)$ is analytic on $\overline{\Omega}$. The boundary conditions must satisfy $g \in H^{\frac{1}{2}}(\partial\Omega)$.*

Our local error analysis will depend on the solution of a dual problem. Concerning solvability and regularity of these dual problems we will assume the following:

Assumption 1.3. *There exists a $\delta_0 \in (0,1]$ such that for fixed compact subset $\Omega' \subset\subset \Omega$ there exists $C > 0$ with the following property: For arbitary $e \in L^2(K)$ the problem: Find $z \in H_0^1(\Omega)$ such that*

$$\int\limits_\Omega \nabla z \nabla v d\Omega = \int\limits_K e v d\Omega \qquad \forall\, v \in H_0^1(\Omega)$$

admits a unique solution $z \in H^{1+\delta_0}(\Omega)$, which satisfies

$$\|z\|_{H^{1+\delta_0}(\Omega)} + \|\gamma_1 z\|_{H^{\delta_0 - \frac{1}{2}}(\partial\Omega)} \leq C\|e\|_{L^2(K)}.$$

Remark 1.4. *For the case of a polygonal domain we have $z \in H^{1+s}(\Omega)$ and $\gamma_1 z \in H^{-1/2+s}(\partial\Omega)$ together with the a priori bound*

$$\|z\|_{H^{1+s}(\Omega)} + \|\gamma_1 z\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \leq C_s\|e\|_{L^2(K)}$$

for any $s \in [0,1] \cap [0, \pi/\alpha_{max})$, where $\alpha_{max} \in (0, 2\pi)$ denotes the largest interior angles of Ω; see [5]. The case of general Lipschitz domains is covered in work by Nečas, where it is shown that Assumption 1.3 is true for any $\delta_0 < \frac{1}{2}$. In fact, [10] shows the bound on $\|z\|_{H^{1+\delta_0}(\Omega)}$ for any $\delta_0 < 1/2$ and [11, Thm. 3.1, Chap. 5] shows the bound on $\|\gamma_1 u\|_{H^{\delta_0 - 1/2}(\partial\Omega)}$ for the limiting case $\delta_0 = 1/2$.

Even though the analyticity of the right-hand side f implies that any solution u of Problem 1.1 is analytic on Ω, due to boundary effects the higher order derivatives are not necessarily bounded as one approaches the boundary. In order to get control of the blow-up of the higher order derivatives, we introduce the countably normed spaces $\tilde{\mathcal{B}}^2_\nu$ defined as follows:

Definition 1.5. (countably normed spaces) *For $\nu \in [0,1)$ and $C, \gamma > 0$ we define*

$$H^2_\nu(\Omega) := \overline{\mathcal{C}^\infty(\overline{\Omega})}^{\|\cdot\|_{H^2_\nu(\Omega)}} \quad \textit{with} \quad \|u\|^2_{H^2_\nu(\Omega)} := \|u\|^2_{H^1(\Omega)} + \left\|r^\nu \nabla^2 u\right\|^2_{L^2(\Omega)}$$

and

$$\tilde{\mathcal{B}}^2_\nu(C,\gamma) = \{u \in H^2_\nu(\Omega) \mid \|u\|_{H^2_\nu(\Omega)} \leq C,\ \left\|r^{\nu+n}\nabla^{n+2}u\right\|_{L^2(\Omega)} \leq C\gamma^n n!\ \forall n \in \mathbb{N}\}.$$

Now we are in position to make precise statements concerning the regularity of the solution u corresponding to Problem 1.1 and to measure the blow-up of the higher order derivatives:

Lemma 1.6. *Let Ω be a Lipschitz domain. Let f be analytic on $\overline{\Omega}$ and assume $u \in H^{1+\delta}(\Omega)$ solves (1). Then u is analytic on Ω, and there exist $C, \gamma > 0$ such that*

$$u \in \tilde{\mathcal{B}}^2_{1-\delta}(C_u, \gamma_u).$$

Proof. See [9, Thm. 1.4]. □

1.3 The geometric mesh, the linear degree vector and the FE-space

We will restrict our considerations to γ-shape-regular triangulations $\mathcal{T}$ of Ω consisting of affine triangles. That is, each element $K \in \mathcal{T}$ is the image $F_K(\hat{K})$ of the reference triangle $\hat{K}$, and we have

$$h_K^{-1}\left\|F'_K\right\|_{L^\infty(K)} + h_K \left\|(F'_K)^{-1}\right\|_{L^\infty(K)} \leq \gamma \qquad \forall\, K \in \mathcal{T},$$

where h_K denotes the diameter of the element K. We furthermore assume that the mesh is a geometric mesh defined as follows:

Definition 1.7. (geometric mesh) *A γ-shape-regular mesh $\mathcal{T}$ is called a geometric mesh with boundary mesh size h if there exist $c_1, c_2 > 0$ such that for all $K \in \mathcal{T}$:*

1. *if $\overline{K} \cap \partial\Omega \neq \emptyset$, then $h \leq h_K \leq c_2 h$,*
2. *if $\overline{K} \cap \partial\Omega = \emptyset$, then $c_1 \inf_{\mathbf{x}\in K} r(\mathbf{x}) \leq h_K \leq c_2 \sup_{\mathbf{x}\in K} r(\mathbf{x})$.*

As a direct conclusion we obtain the following:

Lemma 1.8. *Let $\mathcal{T}$ be a geometric mesh in the sense of Definition 1.7. Then there exist $c_1, c_2 > 0$ depending only on the shape-regularity constant γ and the constants c_1, c_2 of Definition 1.7 such that*

1. $\inf_{\mathbf{x}\in K} r(\mathbf{x}) \geq c_1 h_K \qquad \forall\, K \in \mathcal{T} \;$ *with* $\; \overline{K} \cap \partial\Omega = \emptyset$,

2. $\sup_{\mathbf{x}\in K} r(\mathbf{x}) \leq c_2 h_K \qquad \forall\, K \in \mathcal{T}$.

In order to define *hp*-FEM spaces on a mesh $\mathcal{T}$, we associate a polynomial degree $p_K \in \mathbb{N}$ with each element $K \in \mathcal{T}$ and collect these p_K in the polynomial degree vector $\mathbf{p} := (p_K)_{K\in\mathcal{T}}$. In conjunction with geometric meshes a particularly useful polynomial degree distribution is the linear degree vector:

Definition 1.9. (linear degree vector) *Let $\mathcal{T}$ be a geometric mesh with boundary mesh size h in the sense of Definition 1.7. A polynomial degree vector $\mathbf{p} = (p_K)_{K\in\mathcal{T}}$ is said to be a linear degree vector with slope $\alpha > 0$ if*

$$1 + \alpha c_1 \log \frac{h_K}{h} \leq p_K \leq 1 + \alpha c_2 \log \frac{h_K}{h} \tag{2}$$

for some $c_1, c_2 > 0$.

We furthermore associate with each edge e of the triangulation a polynomial degree

$$p_e := \min \{p_K \,|\, \text{e is an edge of element } K\} \tag{3}$$

and denote by

$$\mathbf{p}(K) := (p_{e1}, p_{e2}, p_{e3}, p_K) \tag{4}$$

the vector containing the polynomial distribution of the triangle $K \in \mathcal{T}$ with edges $\{e_i \mid i = 1, 2, 3\}$. An important property of a linear degree vector $\mathbf{p}$ is:

Lemma 1.10. *Let $\mathcal{T}$ be a geometric mesh and $\mathbf{p}$ a linear degree vector. Then there exists a constant $C > 0$ such that*

$$C^{-1} p_{K'} \leq p_K \leq C p_{K'} \qquad \forall K, K' \quad \text{with } \overline{K} \cap \overline{K'} \neq \emptyset$$

Proof. See [9]. ☐

Now we are in the position to define our *hp*-FEM spaces:

Definition 1.11. (FEM spaces) *Let $\mathcal{T}$ be a geometric mesh and $\mathbf{p}$ be a linear degree vector. Furthermore, for all edges e let p_e be given by (3) and for all $K \in \mathcal{T}$ let $\mathbf{p}(K)$ be given by (4). Then we set*

$$\begin{aligned} S^{\mathbf{p}}(\Omega, \mathcal{T}) &:= \{u \in H^1(\Omega) \mid u \circ F_K \in \mathcal{P}_{\mathbf{p}(K)}(\hat{K}) \quad \forall K \in \mathcal{T}\}, \\ S_0^{\mathbf{p}}(\Omega, \mathcal{T}) &:= S^{\mathbf{p}}(\Omega, \mathcal{T}) \cap H_0^1(\Omega), \\ Y^{\mathbf{p}}(\Omega, \mathcal{T}) &:= \{\gamma_0 u \mid u \in S^{\mathbf{p}}(\Omega, \mathcal{T})\}, \end{aligned}$$

where

$$\mathcal{P}_{\mathbf{p}(K)}(\hat{K}) := \{u \in \mathcal{P}_{p_K}(\hat{K}) \mid u|_{e_i} \in \mathcal{P}_{p_{e_i}}, i = 1, \ldots, 3\}$$

The FE-discretization of Problem 1.1 then reads:

Problem 1.12. (bcFEM approximation) *Find $u_h \in S^{\mathbf{p}}(\Omega,\mathcal{T})$ such that*

$$\gamma_0 u_h = g_h \quad \textit{and} \quad B(u,v) = F(v) \qquad \forall\, v \in S^{\mathbf{p}}_0(\Omega,\mathcal{T}),$$

where $g_h \in Y^{\mathbf{p}}(\Omega,\mathcal{T})$ is the $L^2(\partial\Omega)$-projection of g onto $Y^{\mathbf{p}}(\Omega,\mathcal{T})$ and is given by

$$\int_{\partial\Omega} g_h v\, d\Gamma = \int_{\partial\Omega} g v\, d\Gamma \qquad \forall\, v \in Y^{\mathbf{p}}(\Omega,\mathcal{T}).$$

2 Local error analysis

This section is devoted to the main result of the paper, the analysis of the local error of Problem 1.12 in the framework of the boundary concentrated finite element method. The main theorem is:

Theorem 2.1. (local error bound) *Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and $\Omega' \subset\subset \Omega$ be a compact subset. Let Assumptions 1.2, 1.3 be valid. Let u_h be the solution of Problem 1.12 for a geometric mesh $\mathcal{T}$ with boundary mesh size h and linear degree vector $\mathbf{p}$ with slope α. Then there exists a $\beta \in (0,\delta_0]$ such that for sufficiently large slope α and all elements $\dot{K} \in \mathcal{T}$ with $\dot{K} \subset \Omega'$ we have*

$$\begin{aligned} \|u-u_h\|_{L^2(\dot{K})} &\leq Ch^{\delta+\beta} \leq CN^{-\delta-\beta}, &(5)\\ |u-u_h|_{W^{k,2}(\dot{K})} &\leq Cp_{\dot{K}}^{2k} h^{\delta+\beta} \leq C(\log N)^{2k} N^{-\delta-\beta}, &(6)\\ |u-u_h|_{W^{k,\infty}(\dot{K})} &\leq Cp_{\dot{K}}^{2k+2} h^{\delta+\beta} \leq C(\log N)^{2k+2} N^{-\delta-\beta}. &(7)\end{aligned}$$

Here, $N = O(h^{-1})$ denotes the dimension of the space $S^{\mathbf{p}}(\Omega,\mathcal{T})$. The constants β, α, C are independent of h (and therefore independent of N) but depend on the mesh parameters appearing in Definitions 1.7, 1.9. In addition, β depends on the subdomain Ω'; the slope α depends on the solution u; the constants C depends on Ω' and k.

The proof of Theorem 2.1 relies on several technical results, whose proof is relegated to Sections 4, 5. A major technical tool is a suitable weight function $\omega_{\beta,\mathcal{T}}$, defined below in Definition 2.2. We will make use of several polynomial interpolation operators and approximation results. An auxiliary result of great importance to our analysis is a Hardy inequality, Lemma 5.2, whose use restricts us to considerating Dirichlet boundary conditions and limits the size of our parameter β.
In order to prove Theorem 2.1, we introduce the following weight function $\omega_{\beta,\mathcal{T}}$:

Definition 2.2. (weight function) *Let $\mathcal{T}$ be a geometric mesh in the sense of Definition 1.7 with boundary mesh size h. For a parameter $\beta \in (0,1]$, we define the weight function $\omega_{\beta,\mathcal{T}}$ by*

$$\omega_{\beta,\mathcal{T}}(\mathbf{x}) := I\left[\left(\frac{h}{h+r(\mathbf{x})}\right)^{\beta}\right],$$

where I denotes the standard piecewise linear interpolation operator.

Moreover, we have to make use of the following auxiliary functions:

Definition 2.3. (auxiliary functions) *Under the assumptions of Theorem 2.1 we define* $z \in H_0^1(\Omega)$ *and* $z_h \in S_0^{\mathbf{p}}(\Omega, \mathcal{T})$ *as follows:*

- *Find* $z \in H_0^1(\Omega)$ *such that*

$$-\Delta z = \chi_{\dot{K}}(u - u_h) \text{ on } \Omega \quad \text{and} \quad \gamma_0 z = 0 \text{ on } \partial\Omega,$$

  or in weak formulation

$$\int_{\Omega} \nabla z \cdot \nabla v \, d\Omega = \int_{\dot{K}} (u - u_h)\, v \, d\Omega \quad \forall\, v \in H_0^1(\Omega). \tag{8}$$

- *Find* $z_h \in S_0^{\mathbf{p}}(\Omega, \mathcal{T})$ *such that*

$$\int_{\Omega} \nabla z_h \cdot \nabla v \, d\Omega = \int_{\dot{K}} (u - u_h)\, v \, d\Omega \quad \forall\, v \in S_0^{\mathbf{p}}(\Omega, \mathcal{T}). \tag{9}$$

For a study of the properties of the weight function $\omega_{\beta,\mathcal{T}}$ and the auxiliary functions z and z_h we refer to Section 5. Now we turn to the proof of Theorem 2.1.

Proof of Theorem 2.1. **Inequality (5):** By means of Green's formula (see [5, Lemma 1.5.3.7, Lemma 1.5.3.9]) we obtain

$$\begin{aligned}
\int_{\dot{K}} (u - u_h)^2 \mathrm{d}\Omega &= -\int_{\Omega} \Delta z (u - u_h) \mathrm{d}\Omega \\
&= \int_{\Omega} \nabla z \cdot \nabla (u - u_h) \mathrm{d}\Omega - \langle \gamma_1 z, u - u_h \rangle_{H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)}.
\end{aligned}$$

Inserting the boundary conditions and exploiting orthogonalities gives

$$\begin{aligned}
\int_{\dot{K}} (u - u_h)^2 \mathrm{d}\Omega &= \int_{\Omega} \nabla (z - z_h) \cdot \nabla (u - u_h) \mathrm{d}\Omega - \langle \gamma_1 z, g - g_h \rangle_{H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)} \\
&= \int_{\Omega} \nabla (z - z_h) \cdot \nabla (u - Iu) \mathrm{d}\Omega - \langle \gamma_1 z - q^*, g - g_h \rangle_{H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)}
\end{aligned}$$

for arbitrary $q \in Y^{\mathbf{p}}(\Omega, \mathcal{T}) \subset H^{\frac{1}{2}}(\partial\Omega)$ and a corresponding $q^* \in H^{-\frac{1}{2}}(\partial\Omega)$ uniquely determined by Riesz representation theorem. Thus, inserting the weight function $\omega_{\beta,\mathcal{T}}$ and by making use of the Cauchy-Schwarz-inequality, we get

$$\begin{aligned}
\|u - u_h\|_{L^2(\dot{K})}^2 \;\leq\; & \left\| \sqrt{\omega_{\beta,\mathcal{T}}} \nabla (z - z_h) \right\|_{L^2(\Omega)} \left\| \frac{1}{\sqrt{\omega_{\beta,\mathcal{T}}}} \nabla (u - Iu) \right\|_{L^2(\Omega)} \\
& + \left\| \gamma_1 z - q^* \right\|_{H^{-\frac{1}{2}}(\partial\Omega)} \left\| g - g_h \right\|_{H^{\frac{1}{2}}(\partial\Omega)}.
\end{aligned}$$

For α sufficiently large, the desired bound follows from the observation $g - g_h = \gamma_0(u - u_h)$ and the trace theorem:

$$\|g - g_h\|_{H^{\frac{1}{2}}(\partial\Omega)} = \|\gamma_0(u - u_h)\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C\|u - u_h\|_{H^1(\Omega)} \leq Ch^{\delta}$$

together with Lemma 5.5, Lemma 5.8 and Lemma 5.11 below.

Inequality (6): Let $\hat{K}$ be the reference triangle and let the pull back of a function to the reference element be marked by a hat. Then, since we assume shape regularity and since $\dot{K} \subset \Omega' \subset\subset \Omega$ implies $h_{\dot{K}} \geq C$, we have

$$\begin{aligned} |u - u_h|_{W^{k,2}(\dot{K})} &\leq Ch_{\dot{K}}^{1-k}|\hat{u} - \hat{u}_h|_{W^{k,2}(\hat{K})} \\ &\leq C_k|\hat{u} - \hat{q}|_{W^{k,2}(\hat{K})} + C_k|\hat{q} - \hat{u}_h|_{W^{k,2}(\hat{K})} \end{aligned}$$

for arbitrary $\hat{q} \in \mathcal{P}_{\mathbf{p}(\dot{K})}(\hat{K})$. An inverse inequality (see, e.g., [14, (4.6.5)]) now yields

$$\begin{aligned} |u - u_h|_{W^{k,2}(\dot{K})} &\leq C_k|\hat{u} - \hat{q}|_{W^{k,2}(\hat{K})} + C_k\overline{p}^{2k}\|\hat{q} - \hat{u}_h\|_{L^2(\hat{K})} \\ &\leq C_k|\hat{u} - \hat{q}|_{W^{k,2}(\hat{K})} + C_k\overline{p}^{2k}\|\hat{u} - \hat{u}_h\|_{L^2(\hat{K})} + C_k\overline{p}^{2k}\|\hat{u} - \hat{q}\|_{L^2(\hat{K})} \\ &\leq C_k\overline{p}^{2k}\|\hat{u} - \hat{q}\|_{W^{k,\infty}(\hat{K})} + C_k\overline{p}^{2k}\|\hat{u} - \hat{u}_h\|_{L^2(\hat{K})} \\ &\leq C_k\overline{p}^{2k}\|\hat{u} - \hat{q}\|_{W^{k,\infty}(\hat{K})} + C_kh_{\dot{K}}^{-1}\overline{p}^{2k}\|u - u_h\|_{L^2(\dot{K})}, \end{aligned}$$

where $\overline{p}$ denotes the maximum entry of $\mathbf{p}(\dot{K})$ and $\underline{p}$ the minimal entry of $\mathbf{p}(\dot{K})$. Finally, exploiting $\overline{p} \leq C\underline{p}$ and $h_{\dot{K}} > C$, [12, Corollary 3.2.17] in combination with (5) and α sufficiently large implies

$$|u - u_h|_{W^{k,2}(\dot{K})} \leq C_k\overline{p}^{2k}e^{-b\underline{p}} + C_k\overline{p}^{2k}h^{\delta+\beta} \leq C_kp_{\dot{K}}^{2k}h^{\delta+\beta}$$

for some $\beta \in (0, \delta_0]$.

Inequality (7): We proceed in the same way as in the proof of (6). At first, we pull back to the referenz triangle $\hat{K}$ and insert an arbitrary element $\hat{q} \in \mathcal{P}_{\mathbf{p}(\dot{K})}(\hat{K})$:

$$|u - u_h|_{W^{k,\infty}(\dot{K})} \leq C_k|\hat{u} - \hat{q}|_{W^{k,\infty}(\hat{K})} + C_k|\hat{u}_h - \hat{q}|_{W^{k,\infty}(\hat{K})}.$$

Exploiting inverse inequalities (see, e.g., [14, (4.6.1), (4.6.5)]) gives

$$\begin{aligned} |u - u_h|_{W^{k,\infty}(\dot{K})} &\leq C_k|\hat{u} - \hat{q}|_{W^{k,\infty}(\hat{K})} + C_k\overline{p}^{2k}\|\hat{u}_h - \hat{q}\|_{L^\infty(\hat{K})} \\ &\leq C_k|\hat{u} - \hat{q}|_{W^{k,\infty}(\hat{K})} + C_k\overline{p}^{2k+2}\|\hat{u}_h - \hat{q}\|_{L^2(\hat{K})}. \end{aligned}$$

Since

$$\begin{aligned} \|\hat{u}_h - \hat{q}\|_{L^2(\hat{K})} &\leq \|\hat{u} - \hat{u}_h\|_{L^2(\hat{K})} + \|\hat{u} - \hat{q}\|_{L^2(\hat{K})} \\ &\leq \|\hat{u} - \hat{u}_h\|_{L^2(\hat{K})} + |\hat{K}|\|\hat{u} - \hat{q}\|_{L^\infty(\hat{K})}, \end{aligned}$$

we arrive at

$$|u - u_h|_{W^{k,\infty}(\dot{K})} \leq C_k\overline{p}^{2k+2}\|\hat{u} - \hat{q}\|_{W^{k,\infty}(\hat{K})} + C_k\overline{p}^{2k+2}\|\hat{u} - \hat{u}_h\|_{L^2(\hat{K})}.$$

Now the desired bound follows analogously to the proof of (6). ∎

3 Numerical examples

In this section, we present some numerical examples to confirm the theoretical results of Theorem 2.1. In all examples we start with a coarse grid $\mathcal{T}_0$ of the given domain Ω, and we create a sequence of hierarchically nested geometric meshes $\{\mathcal{T}_l\}_{l=0,1,\ldots}$ with boundary mesh sizes $h_l \sim 2^{-l}h_0$ by applying a suitable mesh refinement strategy (see Figure 1 for an example). Furthermore, we define for each mesh $\mathcal{T}_l$ and a common slope parameter $\alpha > 0$ the polynomial degree distribution via

$$p_{K,l} := \left\lfloor \frac{3}{2} + \alpha \ln\left(\frac{h_K}{\underline{h_l}}\right) \right\rfloor \quad \forall\, K \in \mathcal{T}_l, \quad \underline{h_l} := \min\{\text{length}(e) \mid e \text{ is an edge in} \mathcal{T}_l\}$$

and compute the finite element solution $u_l \in S_0^{\mathbf{p}}(\Omega, \mathcal{T}_l)$.
In order to check the statements of Theorem 2.1, we choose an arbitrary point $\dot{\mathbf{x}} \in \Omega\backslash\{e \mid e \text{ is an edge of } \mathcal{T}_l \text{ for some } l \leq 0\}$ and consider the sequence $\{\dot{K}_l\}$, where $\dot{K}_l$ denotes the triangle uniquely determined by the conditions $\dot{K}_l \in \mathcal{T}_l$ and $\dot{\mathbf{x}} \in \dot{K}_l$. Since it is not difficult to show that there always exists an integer L such that $\dot{K}_m = \dot{K}_n$ for all $n, m \geq L$, we can use $\{\dot{K}_l\}_{l=0,1,\ldots}$ to compute a sequence of local errors $\{\|u - u_l\|_{H^1(\dot{K}_l)}\}_{l=0,1,\ldots}$ which is well suited for pointing out the dependence of the local error on the boundary mesh size h.

Example 3.1. *We consider the L-shaped domain* $\Omega = (-1,1)^2\backslash([0,1] \times [-1,0])$ *as shown in Figure 1 together with the model problem*

$$-\Delta u = f \quad on\ \Omega \qquad u = 0 \quad on\ \partial\Omega,$$

where the right-hand side f *is chosen in such a way that the exact solution* u *given by*

$$u = r^{\frac{2}{3}} \sin\left(\frac{2}{3}\varphi\right)\left(1 - r^2\cos^2\varphi\right)\left(1 - r^2\sin^2\varphi\right).$$

According to [5, Thm. 1.4.5.3]), we have $u \in H^{\frac{5}{3}-\varepsilon}(\Omega)\ \forall \varepsilon > 0$. *Furthermore, we choose* $\dot{\mathbf{x}}_1 = (0.4, 0.3)$ *and* $\dot{\mathbf{x}}_2 = (0.1, 0.2)$.

Our computations are performed with $\alpha = 1$ and the results are collected in Table 1 and plotted in Figure 2. Since we have $u \in H^{\frac{5}{3}-\varepsilon}(\Omega)$, we achieve a global convergence rate of $O(N^{-\frac{2}{3}})$ measured in the energy norm. As Figure 2 shows, the local convergence rates are about twice the rates of the global error, which confirms our theoretical result of an increased local convergence rate. In the second example we want to verify our theoretical results for a domain with a more complicated boundary. To that end, we consider a domain looking like a snow flake (see Figure 3) together with the following Dirichlet problem:

Example 3.2.

$$-\Delta u = 1 \quad on\ \Omega, \qquad u = 0 \quad on\ \partial\Omega.$$

Figure 1: Coarse grid and refinement level 7

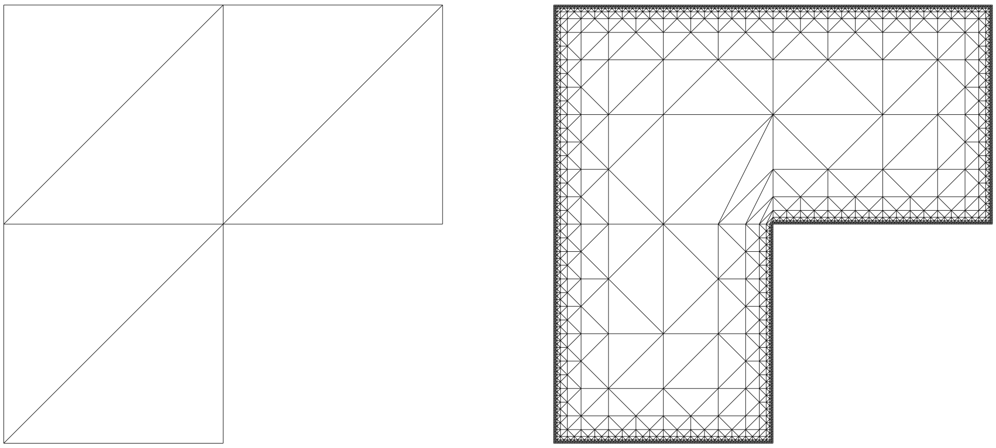


Table 1: Example 3.1 with $e = u - u_h$

			$\mathbf{x}_1 = (0.4, 0.3)$		$\mathbf{x}_2 = (0.1, 0.2)$	
Level	h	p_{max}	$\|e\|_{L^2(\dot{K})}$	$\|e\|_{H^1(\dot{K})}$	$\|e\|_{L^2(\dot{K})}$	$\|e\|_{H^1(\dot{K})}$
1	5.000e-01	1	2.2235e-02	1.8745e-01	3.6191e-02	1.5049e-01
2	2.500e-01	2	2.7989e-03	3.2605e-02	7.5637e-03	6.8149e-02
3	1.250e-01	2	5.5324e-04	3.4199e-03	1.6298e-03	1.2146e-02
4	6.250e-02	3	2.1672e-04	2.0377e-03	5.3150e-04	4.2661e-03
5	3.125e-02	4	7.0529e-05	3.2238e-04	1.8040e-04	1.7937e-03
6	1.562e-02	4	2.2953e-05	9.5572e-05	5.3930e-05	3.8128e-04
7	7.812e-03	5	9.2033e-06	3.8645e-05	2.1980e-05	1.7076e-04
8	3.906e-03	6	3.6312e-06	1.4808e-05	8.6461e-06	4.9317e-05
9	1.953e-03	7	1.4531e-06	5.8801e-06	3.4978e-06	2.0284e-05
10	9.766e-04	7	5.7437e-07	2.3362e-06	1.3837e-06	8.9835e-06
11	4.883e-04	8	2.2889e-07	9.2812e-07	5.5404e-07	6.4693e-06
12	2.441e-04	9	9.0761e-08	3.6813e-07	2.1832e-07	1.7664e-06
13	1.221e-04	9	3.6062e-08	1.4604e-07	8.6847e-08	9.3302e-07
14	6.104e-05	10	1.4305e-08	5.8347e-08	3.4376e-08	2.9232e-07
15	3.052e-05	11	5.6805e-09	2.2980e-08	1.3628e-08	1.6625e-07

Figure 2: Results corresponding to Example 3.1

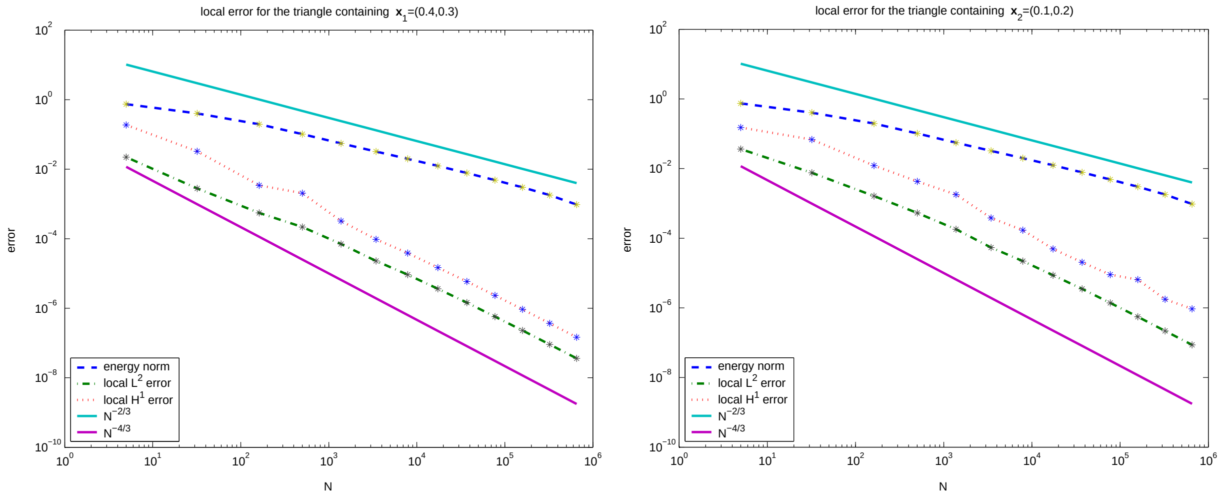
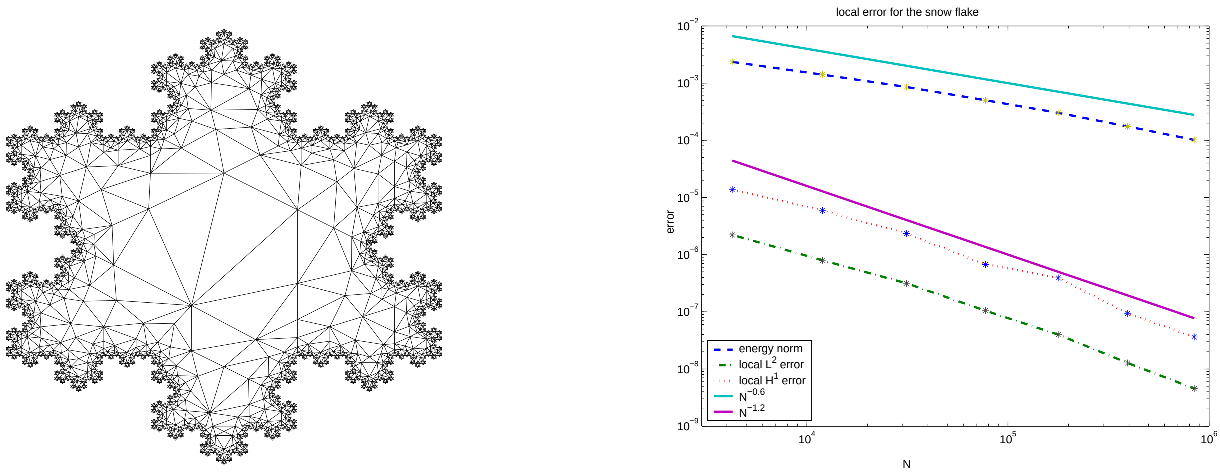


Figure 3: Domain and results corresponding to Example 3.2



We do not know the exact solution of Example 3.2, but extrapolation leads to the results shown in Figure 3. As in the previous example and according to Theorem 2.1, we obtain local convergence rates that are significantly better than the rate of $O(N^{-0.6})$ observed for the global energy norm.

4 An *hp*-interpolation operator

In this section, we present a new variable order *hp*-interpolation operator. The operator is based on Gauss-Lobatto interpolation and is a very useful tool for our local analysis.

4.1 Properties of the Gauss-Lobatto interpolation operator

In order to define our *hp*-interpolation operator, we start with recalling some facts about the one-dimensional Gauss-Lobatto interpolation operator i_p:

Lemma 4.1. *On the interval $I = (-1, 1)$ let i_p be the Gauss-Lobatto interpolation operator. Then for every $k \geq 1$ and $r \in [0, 1]$ there exists $C > 0$ depending solely on k and r such that for every $u \in H^k(I)$*

$$\begin{aligned} \|u - i_p u\|_{H^r(I)} &\leq Cp^{-(k-r)}\|u\|_{H^k(I)}, && (10) \\ \|\frac{1}{\sqrt{1-x^2}}(u - i_p u)\|_{L^2(I)} &\leq Cp^{-k}\|u\|_{H^k(I)}. && (11) \end{aligned}$$

Additionally, for a generic constant $C > 0$, we have the following stability bounds:

$$\begin{aligned} \|i_p u\|_{H^1(I)} &\leq C\|u\|_{H^1(I)} \qquad \forall u \in H^1(I), && (12) \\ \|i_p u\|_{L^2(I)} &\leq C\left[\|u\|_{L^2(I)} + \frac{1}{p}\|u'\|_{L^2(I)}\right] \qquad \forall u \in H^1(I). && (13) \end{aligned}$$

Finally, for p, $p' \in \mathbb{N}$ and $u \in \mathcal{P}_{p'}$, we have the following stability estimates:

$$\begin{aligned} \|i_p u\|_{L^2(I)} &\leq C(1 + p'/p)\|u\|_{L^2(I)}, && (14) \\ \|i_p u\|_{H^{1/2}(I)} &\leq C(1 + p'/p)\|u\|_{H^{1/2}(I)}, && (15) \\ \|u - i_p u\|_{H^{1/2}_{00}(I)} &\leq C(1 + p'/p)\|u\|_{H^{1/2}(I)}. && (16) \end{aligned}$$

Proof. The bounds (10), (11), (13) are proved in [3, Thm. 13.4]. The bound (12) is proved as [3, (13.27)]; (14) is discussed in [3, Remark 13.5]. In order to prove (15), we identify I with the edge $I \times \{-1\}$ of the square $S = (-1, 1)^2$ and let $U \in \mathcal{Q}_{p'}$ be an extension of u, i.e., $U|_\Gamma = u$ and $\|U\|_{H^1(S)} \leq C\|u\|_{H^{1/2}(I)}$; here, the constant $C > 0$ is independent of p' and u, [1, 13]. We denote by i_p^x the Gauss-Lobatto interpolation operator acting solely on the x-variable. Since $i_p u = (i_p^x U)|_\Gamma$, we get with the trace theorem and the one-dimensional stability results (14), (12)

$$\begin{aligned} \|i_p u\|_{H^{1/2}(I)} &\leq C\|i_p^x U\|_{H^1(S)} \leq C\left[(1+p'/p)\|U\|_{L^2(S)} + \|\nabla U\|_{L^2(S)}\right] \\ &\leq C(1+p'/p)\|U\|_{H^1(S)} \leq C(1+p'/p)\|u\|_{H^{1/2}(I)}. \end{aligned}$$

For the last bound, estimate (16), we employ (11) with $k = 1$ and the inverse estimate $\|u\|_{H^1(I)} \leq p'\|u\|_{H^{1/2}(I)}$, which is valid for all polynomials $u \in \mathcal{P}_{p'}$:

$$\|\frac{1}{\sqrt{1-x^2}}(u - i_p u)\|_{L^2(I)} \leq Cp^{-1}\|u\|_{H^1(I)} \leq C\frac{p'}{p}\|u\|_{H^{1/2}(I)}.$$

This estimate together with (15) implies (16). □

By tensorization, the one-dimensional results can be generalized to results on the square:

Lemma 4.2. *Let $S = (-1,1)^2$. For $p \in \mathbb{N}$ denote by $i_p^x \circ i_p^y : C(\overline{S}) \to \mathcal{Q}_p$ the tensor product Gauss-Lobatto interpolation operator of degree p. Then for $k > 3/2$ there exists a constant $C > 0$ depending solely on k such that for any $u \in H^k(S)$ the following holds: Let Γ be an edge of S or one of the diagonals of S. Then:*

$$\|u - i_p^x \circ i_p^y u\|_{H^1(S)} \leq Cp^{-(k-1)}\|u\|_{H^k(S)}, \tag{17}$$

$$\|u - i_p^x \circ i_p^y u\|_{H_{00}^{1/2}(\Gamma)} \leq Cp^{-(k-1)}\|u\|_{H^k(S)}. \tag{18}$$

Proof. Estimate (17) follows from tensor product arguments, [3, Thm. 14.2]. For the estimate (18), we note that $(i_p^x \circ i_p^y u)|_\Gamma$ coincides with the one-dimensional Gauss-Lobatto interpolation $i_{p,\Gamma} u$ of $u|_\Gamma$ if Γ is an edge of S or one of the diagonals (in the case of a diagonal, this follows from the fact that the Gauss-Lobatto points are distributed symmetrically about the midpoint of the interval). By the trace theorem we have $u|_\Gamma \in H^{k-1/2}(\Gamma)$. From (10) with $r = 1/2$ and (11) we then get the desired bound (18). □

4.2 *hp*-interpolation

Let $\widehat{K}$ be the reference square S or the reference triangle T. Denote by Γ_i, $i = 1,\ldots,n$, the edges of $\widehat{K}$; here $n = 3$ if $\widehat{K} = T$ and $n = 4$ if $\widehat{K} = S$. With each edge Γ_i, $i = 1,\ldots,n$, we associate a polynomial degree p_i and with the interior the polynomial degree p_{int}; these are collected in the degree vector $\mathbf{p} = \{p_1,\ldots,p_n,p_{int}\} \in \mathbb{N}^{n+1}$. We will assume that

$$p_i \leq p_{int} \quad \forall i \in \{1,\ldots,n\}. \tag{19}$$

For $p \in \mathbb{N}$ we define

$$\Pi_p(\widehat{K}) = \begin{cases} \mathcal{P}_p & \text{if } \widehat{K} = T, \\ \mathcal{Q}_p & \text{if } \widehat{K} = S. \end{cases}$$

Next, we define $\Pi_{\mathbf{p}}(\widehat{K})$ as

$$\Pi_{\mathbf{p}}(\widehat{K}) := \{u \in \Pi_{p_{int}}(\widehat{K}) \,|\, u|_{\Gamma_i} \in \mathcal{P}_{p_i}, i = 1, \ldots, n\}. \tag{20}$$

For the edges Γ_i of $\widehat{K}$, we denote by i_{p,Γ_i} the Gauss-Lobatto interpolation operator of degree p on that edge.
Before coming to the construction of the interpolation operator, we recall the following polynomial lifting result:

Lemma 4.3. *Let $\widehat{K}$ be the reference square or the reference triangle. Then there exists a bounded linear operator $E : H^{1/2}(\partial\widehat{K}) \to H^1(\widehat{K})$ such that $(Eu)|_{\partial\widehat{K}} = u$ with the following property: if $u \in H^{1/2}(\partial\widehat{K})$ is a polynomial of degree p on each edge, then $Eu \in \Pi_p(\widehat{K})$.*

Proof. See, e.g., [1, 13]. □

Theorem 4.4. *Let $\hat{K}$ be the reference square or the reference triangle. Let $k > 3/2$. Let p_i, $i = 1, \ldots, n$, $p_{int} \in \mathbb{N}$ satisfy (19) and set*

$$\underline{p} := \min_{i=1,\ldots,n} p_i, \qquad \overline{p} := \max_{i=1,\ldots,n} p_i \le p_{int}.$$

Then there exists a generic constant $C > 0$ and a linear operator $I : H^k(\widehat{K}) \to \mathcal{P}_{\mathbf{p}}$ such that

(i) $(Iu)|_{\Gamma_i} = i_{p_i,\Gamma_i} u$ *for* $i \in \{1, \ldots, n\}$;

(ii) $Iu = u$ *for all* $u \in \Pi_{\mathbf{p}}(\widehat{K})$;

(iii) $\|Iu\|_{H^1(T)} \le C(1 + p'/\underline{p})\|u\|_{H^1(T)}$ *for all* $u \in \mathcal{P}_{p'}$;

(iv) $|Iu|_{H^1(T)} \le C(1 + p'/\underline{p})|u|_{H^1(T)}$ *for all* $u \in \mathcal{P}_{p'}$.

Furthermore, the operator I has the following approximation properties for a constant $C_k > 0$ depending solely on k and $\hat{K}$:

$$\|u - Iu\|_{H^1(\widehat{K})} \le C_k \underline{p}^{-(k-1)} \|u\|_{H^k(\widehat{K})}, \tag{21}$$

$$\|u - Iu\|_{H^{1/2}_{00}(\Gamma_i)} \le C_k \underline{p}^{-(k-1)} \|u\|_{H^k(\widehat{K})}, \qquad i = 1, \ldots, n. \tag{22}$$

If $\mathcal{P}_{k-1} \subset \Pi_{\mathbf{p}}(\widehat{K})$, then the full norms on the right-hand sides of (21), (22) may be replaced with $|u|_{H^k(\widehat{K})}$.

Proof. We start with the construction of I and then show that it has the various properties ascertained in the theorem.
Construction of I:
1. step: We assume that the reference triangle T has the form $T = \{(x, y) \,|\, -1 < x < 1,\ -1 < y < -x\}$. Also without loss of generality, we may assume in the case $\widehat{K} = T$ that

the function u is extended to S via the universal extension operator of [15, Chap. VI.3]; hence, we may assume that u is given on S. Next, we may assume $\underline{p} \geq 2$. We then set

$$p := \begin{cases} \underline{p} & \text{if } \widehat{K} = S, \\ \lfloor \frac{\underline{p}}{2} \rfloor & \text{if } \widehat{K} = T. \end{cases}$$

We define $u_1 := i_p^x \circ i_p^y u \in \mathcal{Q}_p \subset \Pi_{\mathbf{p}}(\widehat{K})$ by the choice of p and, in case $\widehat{K} = T$, the additional observation that $\mathcal{Q}_p \subset \mathcal{P}_{2p} \subset \mathcal{P}_{\underline{p}} \subset \Pi_{\mathbf{p}}(\widehat{K})$.
We note that the map $u \mapsto u_1$ is linear and that $u(V) = u_1(V)$ for all vertices V of $\widehat{K}$. In fact, for all edges Γ_i of $\widehat{K}$ we have

$$u_1|_{\Gamma_i} = i_{p,\Gamma_i} u. \tag{23}$$

The assertion (23) is trivial for all edges parallel to the coordinate axes and only non-trival for the diagonal $y = -x$ of the reference triangle. There, it follows from from symmetry properties of the Gauss-Lobatto points.
2. step: We now correct the behavior of u_1 on the edges of $\widehat{K}$. Since $u(V) = u_1(V)$ for all vertices of $\widehat{K}$, we have that $i_{p_i,\Gamma_i} u - u_1 \in H_{00}^{1/2}(\Gamma_i)$ for $i = 1, \ldots, n$. We define $e \in H^{1/2}(\partial\widehat{K})$ edgewise by $e|_{\Gamma_i} = i_{p_i,\Gamma_i} u - u_1 \in H_{00}^{1/2}(\Gamma_i)$ for $i = 1, \ldots, n$. Hence, we may define

$$u_2 := u_1 + Ee \in \Pi_{p_{int}}(\widehat{K}) \tag{24}$$

where the polynomial lifting operator E is given by Lemma 4.3. We note that, so far, the mapping $u \mapsto u_2$ is a linear map and

$$\|Ee\|_{H^1(\widehat{K})} \leq C\|e\|_{H^{1/2}(\partial\widehat{K})} \leq C \sum_{i=1}^{n} \|i_{p_i,\Gamma_i} u - u_1\|_{H_{00}^{1/2}(\Gamma_i)}. \tag{25}$$

3. step: We finally adjust u_2 by a suitable bubble function: we define $u_{min} \in H_0^1(\widehat{K}) \cap \Pi_{p_{int}}(\widehat{K})$ as the $H^1(\widehat{K})$-projection of $u - u_2$ onto $H_0^1(\widehat{K}) \cap \Pi_{p_{int}}(\widehat{K})$, i.e., $u_{min} \in H_0^1(\widehat{K}) \cap \Pi_{p_{int}}(\widehat{K})$ is the unique solution to

$$\langle (u - u_2) - u_{min}, v\rangle_{H^1} = 0 \qquad \forall v \in \Pi_{p_{int}}(\widehat{K}) \cap H_0^1(\widehat{K});$$

here, $\langle \cdot, \cdot, \rangle_{H^1}$ denotes the standard inner product on the Hilbert space $H^1(\widehat{K})$. Setting

$$Iu := u_2 + u_{min},$$

we therefore obtain

$$\langle u - Iu, v\rangle_{H^1} = 0 \qquad v \in H_0^1(\widehat{K}) \cap \Pi_{p_{int}}(\widehat{K}). \tag{26}$$

It is easy to see that $I : H^k(\widehat{K}) \to \Pi_{\mathbf{p}}(\widehat{K})$ is a linear operator.

Analysis of the properties of I:
By construction, $i_{p_i,\Gamma_i} u = u_2|_{\Gamma_i} = (Iu)|_{\Gamma_i}$ for all i, if $u \in \Pi_{\mathbf{p}}(\widehat{K})$. Hence, property (i) is satisfied. From this and (26) is easily ascertained that (ii) holds as well.
We next turn to the approximation properties of I. Let $u \in H^k(\widehat{K})$, $k > 3/2$. Then, from Lemma 4.2

$$\begin{aligned} \|u - u_1\|_{H^1(\widehat{K})} &\leq Cp^{-(k-1)} \|u\|_{H^k(\widehat{K})}, \\ \|u - u_1\|_{H^{1/2}_{00}(\Gamma_i)} &\leq Cp^{-(k-1)} \|u\|_{H^k(\widehat{K})}, \qquad i = 1, \dots, n. \end{aligned}$$

Hence, by (25) we get

$$\|u - u_2\|_{H^1(\widehat{K})} \leq \|u - u_1\|_{H^1(\widehat{K})} + \|Ee\|_{H^1(\widehat{K})} \leq Cp^{-(k-1)} \|u\|_{H^k(\widehat{K})}.$$

Finally, by the orthogonality property (26) satisfied by u_{min}, we get

$$\|u - Iu\|^2_{H^1(\widehat{K})} = \|u - u_2\|^2_{H^1(\widehat{K})} - \|u_{min}\|^2_{H^1(\widehat{K})} \leq \|u - u_2\|^2_{H^1(\widehat{K})} \leq Cp^{-2(k-1)} \|u\|^2_{H^k(\widehat{K})}.$$

This proves the approximation property (21). Estimate (22) follows from combining Property (i), the approximation result Lemma 4.1, and the trace theorem. The claim that the expressions $\|u\|_{H^k(\widehat{K})}$ may be replaced with $|u|_{H^k(\widehat{K})}$ follows in the standard way from Property (i) and a Bramble-Hilbert type argument (see, e.g., [4, Thm. 3.1.1] for the details).
We now turn to the proving the stability results on spaces of polynomials, Properties (iii), (iv). Set $\widetilde{e} := (Iu)|_{\partial\widehat{K}}$. Then by Property (i), we have $\widetilde{e}|_{\Gamma_i} = i_{p_i,\Gamma_i} u$. From the stability result (16) and the assumption $u|_{\Gamma_i} \in \mathcal{P}_{p'}$ we have

$$\|u - \widetilde{e}\|_{H^{1/2}_{00}(\Gamma_i)} \leq C(1 + p'/p_i) \|u\|_{H^{1/2}(\Gamma_i)} \leq C \left(1 + \frac{p'}{\underline{p}}\right) \|u\|_{H^1(\widehat{K})}. \tag{27}$$

Let E be the lifting operator of Lemma 4.3. Then $E\widetilde{e} \in \Pi_{p_{int}(\widehat{K})}$ and $(E\widetilde{e})|_{\partial\widehat{K}} = (Iu)_{\partial\widehat{K}}$. Hence, from the orthogonality property (26)

$$\|u - Iu\|^2_{H^1(\widehat{K})} = \langle u - Iu, u - Iu\rangle_{H^1} = \langle u - Iu, u - E\widetilde{e}\rangle_{H^1} \leq \|u - Iu\|_{H^1(\widehat{K})} \|u - E\widetilde{e}\|_{H^1(\widehat{K})}$$

we conclude together with (27) and the trace theorem

$$\begin{aligned} \|u - Iu\|_{H^1(\widehat{K})} &\leq \|u - E\widetilde{e}\|_{H^1(\widehat{K})} \leq \|u\|_{H^1(\widehat{K})} + C\|\widetilde{e}\|_{H^{1/2}(\partial\widehat{K})} \\ &\leq \|u\|_{H^1(\widehat{K})} + C\|u\|_{H^{1/2}(\partial\widehat{K})} + C\|u - \widetilde{e}\|_{H^{1/2}(\partial\widehat{K})} \\ &\leq C\|u\|_{H^1(\widehat{K})} + C\sum_{i=1}^{n} \|u - \widetilde{e}\|_{H^{1/2}_{00}(\Gamma_i)} \leq C\left(1 + \frac{p'}{\underline{p}}\right) \|u\|_{H^1(\widehat{K})}. \end{aligned}$$

From this, we easily infer Property (iii). Property (iv) follows from the Poincaré inequality as follows: denoting by $\overline{u} \in \mathbb{R}$ the average of u over $\hat{K}$, we get

$$\begin{aligned} |Iu|_{H^1(\hat{K})} &\leq |u|_{H^1(\hat{K})} + \|u - Iu\|_{H^1(\hat{K})} \leq |u|_{H^1(\hat{K})} + \|u - \overline{u} - I(u - \overline{u})\|_{H^1(\hat{K})} \\ &\leq |u|_{H^1(\hat{K})} + C\|u - \overline{u}\|_{H^1(\hat{K})} \leq |u|_{H^1(\hat{K})}. \end{aligned}$$

□

5 Auxiliary Results

This sections is devoted to the proof of all the auxiliary results that were used in the proof of Theorem 2.1.

5.1 The weight function $\omega_{\beta,\mathcal{T}}$

We start with studying the most important properties of the weight function $\omega_{\beta,\mathcal{T}}$ introduced in Definition 2.2.

Lemma 5.1. (properties of $\omega_{\beta,\mathcal{T}}$) *Let $\mathcal{T}$ be a geometric mesh and let $\omega_{\beta,\mathcal{T}}$ be given by Definition 2.2. Then there exist constants $C_1, \cdots, C_4 > 0$ depending only on the shape-regularity constant γ and the constants of Definition 1.7 such that for all $K \in \mathcal{T}$ and arbitrary $\beta \in (0,1]$*

1. $\inf_{\mathbf{x}\in K} |\omega_{\beta,\mathcal{T}}(\mathbf{x})| \geq C_2 \left(\frac{h}{h_K}\right)^{\beta}$,
2. $\sup_{\mathbf{x}\in K} |\omega_{\beta,\mathcal{T}}(\mathbf{x})| \leq C_1 \left(\frac{h}{h_K}\right)^{\beta}$,
3. $|\nabla \omega_{\beta,\mathcal{T}}(\mathbf{x})| = C_{\beta,\mathcal{T},K} \leq C_3\beta \frac{\omega_{\beta,\mathcal{T}}(\mathbf{x})}{h_K} \leq C_4\beta\frac{\omega_{\beta,\mathcal{T}}(\mathbf{x})}{r(\mathbf{x})} \qquad \forall\, \mathbf{x} \in K.$

Proof.

1. Since the restriction of $w_{\beta,\mathcal{T}}(\mathbf{x})$ to the triangle K is the linear function that coincides with the non interpolated weight function $\tilde{\omega}_{\beta,\mathcal{T}}(\mathbf{x})$ in all vertices v_i of K, we have

$$\inf_{\mathbf{x}\in K} |\omega_{\beta,\mathcal{T}}(\mathbf{x})| \geq \inf_{\mathbf{x}\in K} |\tilde{\omega}_{\beta,\mathcal{T}}(\mathbf{x})| = \inf_{\mathbf{x}\in K} \left(\frac{h}{h + r(\mathbf{x})}\right)^{\beta}.$$

   Now, the assertion follows directly from the estimates $r \leq ch_K$ and $h \leq ch_K$.

2. Follows analogous to (1) from the definition of $\omega_{\beta,\mathcal{T}}(\mathbf{x})$ and from the estimates in Definition 1.7 or Lemma 1.8 respectively.

3. First, we consider the function

$$x \mapsto \left(\frac{h}{h + xh_K}\right)^{\beta}.$$

   For $x_2 > x_1 \geq 0$ the mean value theorem guarantees the existence of a $\xi \in (x_1, x_2)$ such that

$$\left(\frac{h}{h + x_1 h_K}\right)^{\beta} - \left(\frac{h}{h + x_2 h_K}\right)^{\beta} = \frac{\beta(x_2 - x_1)h_k}{h + \xi h_k}\left(\frac{h}{h + \xi h_K}\right)^{\beta}. \tag{28}$$

Next, since $\omega_{\beta,\mathcal{T}}(\mathbf{x})$ is linear, we have $|\nabla\omega_{\beta,\mathcal{T}}(\mathbf{x})| = C_{\beta,\mathcal{T},K}$ for all $\mathbf{x} \in K$ and due to the γ-shape-regularity of the mesh we get

$$C_{\beta,\mathcal{T},K} \leq ch_K^{-1} \left| \sup_{\mathbf{x}\in K} \omega_{\beta,\mathcal{T}}(\mathbf{x}) - \inf_{\mathbf{x}\in K} \omega_{\beta,\mathcal{T}}(\mathbf{x}) \right|,$$

where the constant c depends only on γ. Hence, the definition of $\omega_{\beta,\mathcal{T}}(\mathbf{x})$ together with Lemma 1.8 imply:

$$C_{\beta,\mathcal{T}\omega,K} \leq ch_K^{-1} \begin{cases} \left|1 - \left(\frac{h}{h+c_2h_K}\right)^\beta\right| & : \overline{K} \cap \partial\Omega \neq \emptyset \\ \left|\left(\frac{h}{h+c_1h_K}\right)^\beta - \left(\frac{h}{h+c_2h_K}\right)^\beta\right| & : \overline{K} \cap \partial\Omega = \emptyset \end{cases},$$

where $c_1, c_2 > 0$ depend only on the shape-regularity constant γ. Thus, applying inequality (28) leads to

$$C_{\beta,\mathcal{T},K} \leq c\beta h_K^{-1} \begin{cases} c_2 \frac{h_K}{h+\xi_1 h_K} \left(\frac{h}{h+\xi_1 h_K}\right)^\beta & : \overline{K} \cap \partial\Omega \neq \emptyset \\ |c_1 - c_2| \frac{h_K}{h+\xi_2 h_K} \left(\frac{h}{h+\xi_2 h_K}\right)^\beta & : \overline{K} \cap \partial\Omega = \emptyset \end{cases}$$

for a $\xi_1 \in (0, c_2)$ or $\xi_2 \in (c_1, c_2)$ respectively. Since $h_K \leq ch$ and $\xi_2 \geq c_1$ we arrive at

$$C_{\beta,\mathcal{T},K} \leq c\beta h_K^{-1} \begin{cases} 1 & \text{for all } K \in \mathcal{T} \mid \overline{K} \cap \partial\Omega \neq \emptyset \\ \left(\frac{h}{h+ch_K}\right)^\beta & \text{for all } K \in \mathcal{T} \mid \overline{K} \cap \partial\Omega = \emptyset \end{cases}$$

and the assertion follows from $h_K \geq cr(\mathbf{x})$ for all $K \in \mathcal{T}$ and $h \geq ch_K$ in the event of $\overline{K} \cap \partial\Omega \neq \emptyset$.

$\square$

Lemma 5.2. *Let $\Omega \in \mathbb{R}^2$ be a polygonal domain. Let $\mathcal{T}$ be a geometric mesh and let $\omega_{\beta,\mathcal{T}}$ be given by Definition 2.2. Then there exist $\beta' \in (0, 1]$, $C_{\beta'} > 0$ depending only on Ω, the shape regularity constant γ and the constants of Definition 1.7, such that for all $\beta \in (0, \beta']$ and all $f \in H_0^1(\Omega)$*

$$\left\|r^{-1}\sqrt{\omega_{\beta,\mathcal{T}}}f\right\|_{L^2(\Omega)} \leq C_{\beta'} \left\|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla f\right\|_{L^2(\Omega)}, \tag{29}$$

$$\left\|r^{-1}\omega_{\beta,\mathcal{T}}f\right\|_{L^2(\Omega)} \leq C_{\beta'} \left\|\omega_{\beta,\mathcal{T}}\nabla f\right\|_{L^2(\Omega)}. \tag{30}$$

Proof. We will only prove the first inequality as the second one is proved similarly. Since $f \in H_0^1(\Omega)$, we can apply Hardy's inequality (see [5, Thm. 1.4.4.3]):

$$\left\|\sqrt{\omega_{\beta,\mathcal{T}}}\frac{f}{r}\right\|_{L^2(\Omega)} \leq C_1 \left\|\nabla(\sqrt{\omega_{\beta,\mathcal{T}}}f)\right\|_{L^2(\Omega)} \tag{31}$$

$$\leq C_1 \left\|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla f\right\|_{L^2(\Omega)} + \frac{C_1}{2} \left\|\frac{f}{\sqrt{\omega_{\beta,\mathcal{T}}}}\nabla\omega_{\beta,\mathcal{T}}\right\|_{L^2(\Omega)}. \tag{32}$$

Next, using property 3 of Lemma 5.1, we obtain:

$$\left\| \frac{f}{\sqrt{\omega_{\beta,\mathcal{T}}}} \nabla \omega_{\beta,\mathcal{T}} \right\|^2_{L^2(\Omega)} \leq (C_2\beta)^2 \int\limits_\Omega \left(\frac{f}{r} \sqrt{\omega_{\beta,\mathcal{T}}} \right)^2 \mathrm{d}\Omega \tag{33}$$

$$= (C_2\beta)^2 \left\| \sqrt{\omega_{\beta,\mathcal{T}}} \frac{f}{r} \right\|^2_{L^2(\Omega)} . \tag{34}$$

Combining (32) and (34) yields

$$\left\| \sqrt{\omega_{\beta,\mathcal{T}}} \frac{f}{d} \right\|_{L^2(\Omega)} \leq \frac{2C_1}{2 - C_1 C_2 \beta} \left\| \sqrt{\omega_{\beta,\mathcal{T}}} \nabla f \right\|_{L^2(\Omega)} ;$$

hence for arbitrary $0 < \beta' < 2(C_1C_2)^{-1}$ together with $C_{\beta'} := \frac{2C_1}{2 - C_1 C_2 \beta'}$ the desired inequality (29) follows. ☐

Lemma 5.3. *Let $\Omega \in \mathbb{R}^2$ be a polygonal domain. Let $\mathcal{T}$ be a geometric mesh and $\mathbf{p}$ a linear degree vector with slope $\alpha > 0$. Furthermore, let the linear operator $I : S_0^{2\mathbf{p}}(\Omega, \mathcal{T}) \to S_0^{\mathbf{p}}$ be defined by:*

$$\left[I u \right] (\mathbf{x}) := \left[\hat{I} \left(u \circ F_K \right) \right] \left(F_K^{-1} \mathbf{x} \right) \quad \forall \mathbf{x} \in \overline{K} \ \ \forall K \in \mathcal{T},$$

where $\hat{I}$ denotes the operator of Theorem 4.4. Then there exist $\beta' \in (0, 1]$ and $C_{\beta'} > 0$ depending only on Ω, the shape regularity constant γ and the constants of Definitions 1.7, 1.9 such that for all $\beta \in (0, \beta']$ and all $g \in S_0^{\mathbf{p}}(\Omega, \mathcal{T})$

$$\left\| \frac{1}{\sqrt{\omega_{\beta,\mathcal{T}}}} \nabla \left(\omega_{\beta,\mathcal{T}} g - I(\omega_{\beta,\mathcal{T}} g) \right) \right\|_{L^2(\Omega)} \leq C_{\beta'} \beta \left\| \sqrt{\omega_{\beta,\mathcal{T}}} \nabla g \right\|_{L^2(\Omega)} .$$

Proof. We set

$$\overline{\omega}_{\beta,\mathcal{T},K} := \sup_{\mathbf{x} \in K} \omega_{\beta,\mathcal{T}}(\mathbf{x}), \qquad \underline{\omega}_{\beta,\mathcal{T},K} := \inf_{\mathbf{x} \in K} \omega_{\beta,\mathcal{T}}(\mathbf{x})$$

and obtain

$$\left\| \frac{1}{\sqrt{\omega_{\beta,\mathcal{T}}}} \nabla \left(\omega_{\beta,\mathcal{T}} g - I(\omega_{\beta,\mathcal{T}} g) \right) \right\|^2_{L^2(\Omega)} \leq \sum_{K \in \mathcal{T}} \frac{1}{\underline{\omega}_{\beta,\mathcal{T},K}} \left\| \nabla \left(\omega_{\beta,\mathcal{T}} g - q_K - I(\omega_{\beta,\mathcal{T}} g - q_K) \right) \right\|^2_{L^2(K)} ,$$

where for each $K \in \mathcal{T}$ an arbitrary q_K with $q_K \circ F_K \in S^{\mathbf{p}(K)}(\hat{K})$ may be chosen. Now, the stability property (iv) of Theorem 4.4 implies

$$\begin{aligned} &\left\| \frac{1}{\sqrt{\omega_{\beta,\mathcal{T}}}} \nabla \left(\omega_{\beta,\mathcal{T}} g - I(\omega_{\beta,\mathcal{T}} g) \right) \right\|^2_{L^2(\Omega)} \\ &\leq 2 \sum_{K \in \mathcal{T}} \frac{1}{\underline{\omega}_{\beta,\mathcal{T},K}} \left(|\omega_{\beta,\mathcal{T}} g - q_K|^2_{H^1(K)} + |I(\omega_{\beta,\mathcal{T}} g - q_K)|^2_{H^1(K)} \right) \\ &\leq C \sum_{K \in \mathcal{T}} \frac{1}{\underline{\omega}_{\beta,\mathcal{T},K}} |\omega_{\beta,\mathcal{T}} g - q_K|^2_{H^1(K)}; \end{aligned}$$

and by choosing $q_K := \omega_{\beta,\mathcal{T},K} g := w_{\beta,\mathcal{T}}(\mathbf{x}_K) g(\mathbf{x})$ for $\mathbf{x}_K \in K$ arbitrary, we arrive at

$$\begin{aligned}
&\left\| \frac{1}{\sqrt{\omega_{\beta,\mathcal{T}}}} \nabla \left(\omega_{\beta,\mathcal{T}} g - I(\omega_{\beta,\mathcal{T}} g) \right) \right\|^2_{L^2(\Omega)} \\
&\quad \leq \; C \sum_{K\in\mathcal{T}} \frac{1}{\underline{\omega}_{\beta,\mathcal{T},K}} |(\omega_{\beta,\mathcal{T}} - \omega_{\beta,\mathcal{T},K}) g|^2_{H^1(K)} \\
&\quad \leq \; C \sum_{K\in\mathcal{T}} \frac{1}{\underline{\omega}_{\beta,\mathcal{T},K}} \left(\|g \nabla \omega_{\beta,\mathcal{T}}\|^2_{L^2(K)} + \|(\omega_{\beta,\mathcal{T}} - \omega_{\beta,\mathcal{T},K}) \nabla g\|^2_{L^2(K)} \right).
\end{aligned}$$

From Lemma 5.1 we deduce $\overline{\omega}_{\beta,\mathcal{T},K} \leq C \underline{\omega}_{\beta,\mathcal{T},K}$ for all $K \in \mathcal{T}$. Thus, repeated use of Lemma 5.1 leads to

$$\begin{aligned}
&\left\| \frac{1}{\sqrt{\omega_{\beta,\mathcal{T}}}} \nabla \left(\omega_{\beta,\mathcal{T}} g - I(\omega_{\beta,\mathcal{T}} g) \right) \right\|^2_{L^2(\Omega)} \\
&\quad \leq \; C \sum_{K\in\mathcal{T}} \frac{1}{\underline{\omega}_{\beta,\mathcal{T},K}} \left(\beta^2 \left\| g \frac{\omega_{\beta,\mathcal{T}}}{r} \right\|^2_{L^2(K)} + h_K^2 |\nabla \omega_{\beta,\mathcal{T}}|^2_K \|\nabla g\|^2_{L^2(K)} \right) \\
&\quad \leq \; C\beta^2 \sum_{K\in\mathcal{T}} \frac{1}{\underline{\omega}_{\beta,\mathcal{T},K}} \left(\left\| g \frac{\omega_{\beta,\mathcal{T}}}{r} \right\|^2_{L^2(K)} + \overline{\omega}^2_{\beta,\mathcal{T},K} \|\nabla g\|^2_{L^2(K)} \right) \\
&\quad \leq \; C\beta^2 \sum_{K\in\mathcal{T}} \left(\left\| g \frac{\sqrt{\omega_{\beta,\mathcal{T}}}}{r} \right\|^2_{L^2(K)} + \|\sqrt{\omega_{\beta,\mathcal{T}}} \nabla g\|^2_{L^2(K)} \right).
\end{aligned}$$

Finally exploiting Lemma 5.2 gives the desired result. □

5.2 Approximation of $\tilde{\mathcal{B}}^2_{1-\delta}$ functions from $S^{\mathbf{p}}$ in a ω-weighted norm

In this subsection we will use the results of [9, Section 2.3.2] to deduce an approximation result for the ω-weighted H^1-seminorm. We start with recalling from [9] the following approximation result:

Lemma 5.4. *Let $\mathcal{T}$ be a geometric mesh with boundary mesh size h as defined in Definition 1.7. Let $\mathbf{p}$ be a linear degree vector with slope $\alpha > 0$. Let $u \in \tilde{\mathcal{B}}^2_{1-\delta}(C_u, \gamma_u)$, $C_u, \gamma_u > 0$. Then there exists an element $Iu \in S^{\mathbf{p}}(\Omega, \mathcal{T})$ such that*

$$\|u - Iu\|_{H^1(K)} \leq \begin{cases} C C_K h_K^{\delta} & \text{for all } K \in \mathcal{T} \text{ abutting on } \partial\Omega \\ C C_K h_K^{\delta - \alpha b'} h^{\alpha b'} & \text{otherwise} \end{cases},$$

where C, $b' > 0$ depend only on the shape-regularity constant γ, the constants of Definitions 1.7, 1.9, and γ_u; C depends additionally on δ. The constants C_K are given by

$$C_K^2 := \sum_{n=0}^{\infty} \frac{1}{(2\gamma_u)^{2n} (n!)^2} \left\| r^{n+1-\delta} \nabla^{n+2} u \right\|^2_{L^2(K)} \text{ and we have } \sum_{K\in\mathcal{T}} C_K^2 \leq \frac{4}{3} C_u^2.$$

Proof. See [9, Proposition 2.10] for the construction of such an element. □

Now, by means of Lemma 5.4, we are able to prove the following lemma, concerning the approximation of $\tilde{\mathcal{B}}^2_{1-\delta}(C_u, \gamma_u)$ functions from $S^{\mathbf{p}}(\Omega, \mathcal{T})$ in the $\omega_{\beta,\mathcal{T}}$-weighted H^1-seminorm.

Lemma 5.5. *Let $\beta \in (0, 1]$. Let $\mathcal{T}$ be a geometric mesh with boundary mesh size h as defined in Definition 1.7. Let $\mathbf{p}$ be a linear degree vector with slope α. Furthermore, let $u \in \tilde{\mathcal{B}}^2_{1-\delta}(C_u, \gamma_u)$ and $\omega_{\beta,\mathcal{T}}$ be given by Definition 2.2. Then there exists an element $Iu \in S^{\mathbf{p}}(\Omega, \mathcal{T})$ such that for α, C sufficiently large (depending only on the shape-regularity constant γ, the constants of Definitions 1.7, 1.9, and γ_u, δ) we have*

$$\left[\int_\Omega \left(\frac{1}{\sqrt{\omega_{\beta,\mathcal{T}}}} \nabla (u - Iu) \right)^2 d\Omega \right]^{\frac{1}{2}} \leq C C_u h^\delta.$$

Proof. We will show that the element Iu of Lemma 5.4 has the desired property. To that end, we distinguish between two cases:

1. For $K \in \mathcal{T}$ with $\overline{K} \cap \partial\Omega \neq \emptyset$ we have

$$\left\| \frac{1}{\sqrt{\omega_{\beta,\mathcal{T}}}} \nabla (u - Iu) \right\|_{L^2(K)} \leq C C_K h^\delta \left\| \frac{1}{\sqrt{\omega_{\beta,\mathcal{T}}}} \right\|_{L^\infty(K)} \leq C C_K h^\delta.$$

2. For $K \in \mathcal{T}$ with $\overline{K} \cap \partial\Omega = \emptyset$ we have

$$\left\| \frac{1}{\sqrt{\omega_{\beta,\mathcal{T}}}} \nabla (u - Iu) \right\|_{L^2(K)} \leq C C_K \left\| \frac{1}{\sqrt{\omega_{\beta,\mathcal{T}}}} \right\|_{L^\infty(K)} h_K^{\delta - \alpha b'} h^{\alpha b'},$$

   and, since $\omega_{\beta,\mathcal{T}}(\mathbf{x}) \geq c \left(\frac{h}{h_K} \right)^\beta$ for all $\mathbf{x} \in K$, we obtain

$$\left\| \frac{1}{\sqrt{\omega_{\beta,\mathcal{T}}}} \nabla (u - Iu) \right\|_{L^2(K)} \leq C C_K h_K^{\delta - \alpha b' + \frac{\beta}{2}} h^{\alpha b' - \frac{\beta}{2}}.$$

   For α sufficiently large, namely, $\alpha > b'^{-1}(\delta + \beta/2)$, we exploit $h_K \geq ch$ to arrive at:

$$\left\| \frac{1}{\sqrt{\omega_{\beta,\mathcal{T}}}} \nabla (u - Iu) \right\|_{L^2(K)} \leq C C_K h^\delta.$$

Finally, we add up all element contributions and the assertion follows:

$$\begin{aligned} \left[\int_\Omega \left(\frac{1}{\sqrt{\omega_{\beta,\mathcal{T}}}} \nabla (u - Iu) \right)^2 \mathrm{d}\Omega \right]^{\frac{1}{2}} &= \left(\sum_{K \in \mathcal{T}} \left\| \frac{1}{\sqrt{\omega_{\beta,\mathcal{T}}}} \nabla (u - Iu) \right\|^2_{L^2(K)} \right)^{\frac{1}{2}} \\ &\leq C \left(h^{2\delta} \sum_{K \in \mathcal{T}} C_K^2 \right)^{\frac{1}{2}} \leq C C_u h^\delta. \end{aligned}$$

Remark 5.6. *In the proof of Lemma 5.5 we demand $\alpha > b'^{-1}(\delta + \beta/2)$. Since $\delta \in (0, 1]$ and $\beta \in (0, 1]$ this claim will be fulfilled if $\alpha > 3/(2b')$, independent of β and δ.*

□

5.3 Properties of z and z_h

In this section we want to point out the most important properties of z and z_h defined in Definition 2.3.

Lemma 5.7. (basic properties of z and z_h) *Let the assumptions of Theorem 2.1 be valid and let $\dot{K} \subset \Omega'' \subset\subset \Omega' \subset\subset \Omega$. Furthermore, let z and z_h be given by Definition 2.3 Then for constants C_Ω, $C_{\Omega'}$, γ_z depending on Ω, Ω', δ_0 we have:*

1. $\|z\|_{H^1(\Omega)} \leq \|z\|_{H^{1+\delta_0}(\Omega)} \leq C_\Omega \|u - u_h\|_{L^2(\dot{K})}$
2. $z \in H^2(\Omega')$ *and* $\|z\|_{H^2(\Omega')} \leq C_{\Omega'} \|u - u_h\|_{L^2(\dot{K})}$
3. $z|_{\Omega\backslash\Omega'} \in \tilde{\mathcal{B}}^2_{1-\delta_0}(C_{\Omega'}\|u - u_h\|_{L^2(\dot{K})}, \gamma_z)$
4. $\|z - z_h\|_{H^1(\Omega)} \leq c\|u - u_h\|_{L^2(\dot{K})}$.

Proof.

1. This is just a rephrasing of Assumption 1.3.
2. This expresses interior regularity for elliptic problems: From [6, Thm. 9.1.26] we obtain $z \in H^2(\Omega')$ together with

$$\|z\|_{H^2(\Omega')} \leq C_{\Omega'} \left(\|u - u_h\|_{L^2(\dot{K})} + \|z\|_{H^1(\Omega)} \right).$$

   The desired bound now follows from this estimate and the preceding one.

3. This follows from [9, Thm. A.1]: Without loose of generality, we may assume Ω'' to be a smooth domain. Since $z \in H^{1+\delta_0}(\Omega\backslash\Omega'')$ satisfies $-\Delta z = 0$ on $\Omega\backslash\Omega''$ and since $\|z\|_{H^{1+\delta_0}(\Omega\backslash\Omega'')} \leq \|z\|_{H^{1+\delta_0}(\Omega)} \leq C\|u - u_h\|_{L^2(\dot{K})}$, the result now follows from [9, Thm. A.1].
4. Follows from the Lax-Milgram lemma together with the first estimate:

$$\|z - z_h\|_{H^1(\Omega)} \leq C \inf_{v \in S_0^{\mathbf{p}}(\Omega,\mathcal{T})} \|z - v\|_{H^1(\Omega)} \leq C\|z\|_{H^1(\Omega)} \leq C\|u - u_h\|_{L^2(\dot{K})}.$$

□

Lemma 5.8. *Let the assumptions of Theorem 2.1 be valid. Furthermore, let z be given by Definition 2.3. Then there exists an element $q \in Y^{\mathbf{p}}(\Omega, \mathcal{T})$ such that*

$$\|\gamma_1 z - q^*\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C_s h^s \|u - u_h\|_{L^2(\dot{K})} \quad \forall\, s \in (0, \delta_0], \tag{35}$$

where $q^* \in Y^{\mathbf{p}}(\Omega, \mathcal{T})^*$ denotes the representation of q given by the Riesz representation theorem.

Proof. Assumption 1.3 gives us $\delta_0 > 0$ such that $\gamma_1 z \in H^{-1/2+s}(\partial\Omega)$ with

$$\|\gamma_1 z\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \leq C\|\chi_{\dot{K}}(u - u_h)\|_{L^2(\Omega)} = C\|u - u_h\|_{L^2(\dot{K})}$$

for all $0 \leq s \leq \delta_0$. [9, Lemma 2.8] guarantees the existence of an element $q \in Y^{\mathbf{p}}(\Omega, \mathcal{T})$ such that

$$\|\gamma_1 z - q^*\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C_s h^s \|\gamma_1 z\|_{H^{s-\frac{1}{2}}(\partial\Omega)}.$$

Combining these two inequalities yields the desired bound (35). □

Lemma 5.9. *Let the assumptions of Theorem 2.1 be valid. Furthermore, let z be given by Definition 2.3 and let $\omega_{\beta,\mathcal{T}}$ be given by Definition 2.2. Then, for α sufficiently large depending only on the shape-regularity constant γ, the constants of Definitions 1.7, Definition 1.9, and u there exists an element $Iu \in S_0^{\mathbf{p}}(\Omega, \mathcal{T})$ such that*

$$\left\|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla(z - Iz)\right\|_{L^2(\Omega)} \leq Ch^{\beta} \|u - u_h\|_{L^2(\Omega)}$$

for all $\beta \in (0, \delta_0]$ and C independent of h and β.

Proof. We construct Iu as follows:

$$[Iu](\mathbf{x}) := \left[\hat{I}\,(u \circ F_K)\right]\left(F_K^{-1}\mathbf{x}\right) \quad \forall \mathbf{x} \in \overline{K} \;\; \forall K \in \mathcal{T}$$

and distinguish two cases:

- For $K \in \mathcal{T}_1 := \{K \in \mathcal{T} \mid \overline{K} \cap \overline{\dot{K}} = \emptyset\}$ the operator $\hat{I}$ is taken from [9, Lemma 2.9].
- For $K \in \mathcal{T}_2 := \{K \in \mathcal{T} \mid \overline{K} \cap \overline{\dot{K}} \neq \emptyset\}$ the operator $\hat{I}$ is taken as the one constructed in Theorem 4.4.

Furthermore, we assume that there exists no $K \in \mathcal{T}_2$ such that $\overline{K} \cap \partial\Omega \neq \emptyset$. Thus, we have $\tilde{\Omega} \,:\, \bigcup_{K\in\mathcal{T}_2} \overline{K} \subset \tilde{\Omega} \subset\subset \Omega$ independent of h. Because of $\omega_{\beta,\mathcal{T}}(\mathbf{x}) \in (0, 1]$ and in view of Lemma 5.1 we have

$$\begin{aligned}\left\|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla(z - Iz)\right\|^2_{L^2(\Omega)} &\leq \sum_{K\in\mathcal{T}} \left\|\sqrt{\omega_{\beta,\mathcal{T}}}\right\|^2_{L^\infty(K)} |z - Iz|^2_{H^1(K)} \\ &\leq \sum_{K\in\mathcal{T}_1} |z - Iz|^2_{H^1(K)} + C \sum_{K\in\mathcal{T}_2} \left(\frac{h}{h_K}\right)^{\beta} |z - Iz|^2_{H^1(K)}.\end{aligned}$$

Exploiting $z \in H^2(\tilde{\Omega})$ with $\|z\|_{H^2(\tilde{\Omega})} \le C_{\tilde{\Omega}} \|u - u_h\|_{L^2(\dot{K})}$ (see Lemma 5.7), pulling back to the reference triangle and making use of Theorem 4.4 we can bound the second sum as follows:

$$
\begin{aligned}
\sum_{K \in \mathcal{T}_2} \left(\frac{h}{h_K}\right)^\beta |z - Iz|^2_{H^1(K)} &\le C \sum_{K \in \mathcal{T}_2} \left(\frac{h}{h_K}\right)^\beta |\hat{z}|^2_{H^2(\hat{K})} \\
&\le C \sum_{K \in \mathcal{T}_2} \left(\frac{h}{h_K}\right)^\beta h_K^2 \, |z|^2_{H^2(K)} \\
&\le C h^\beta \sum_{K \in \mathcal{T}_2} |z|^2_{H^2(K)} \le C h^\beta \|u - u_h\|^2_{L^2(\dot{K})}, \qquad (36)
\end{aligned}
$$

with a constant C independent of h and β. In order to bound the first sum, we exploit $z|_{\Omega \setminus \tilde{\Omega}} \in \tilde{\mathcal{B}}^2_{1-\delta_0}(C_{\tilde{\Omega}} \|u - u_h\|_{L^2(\dot{K})}, \gamma_z)$. Because of Lemma 5.4 and since no $K \in \mathcal{T}_1$ has a distance less than $c h_{\dot{K}}$ from $\dot{K}$ we obtain

$$
\sum_{K \in \mathcal{T}_1} |z - Iz|^2_{H^1(K)} \le \sum_{K \in \mathcal{T}_1 | K \cap \partial\Omega \neq \emptyset} C^2_{K,z} h^{2\delta_0} + \sum_{K \in \mathcal{T}_1 | K \cap \partial\Omega \ = \emptyset} C^2_{K,z} h_K^{2(\delta_0 - \alpha b')} h^{2\alpha b'}.
$$

That is, for α sufficiently large, we obtain

$$
\sum_{K \in \mathcal{T}_1} |z - Iz|^2_{H^1(K)} \le h^{2\delta_0} C \sum_{K \in \mathcal{T}_1} C^2_{K,z} \le h^{2\delta_0} C \|u - u_h\|^2_{L^2(\dot{K})}. \qquad (37)
$$

Combining (36) and (37) gives us the desired result. □

Remark 5.10. *In order to prove Lemma 5.9 we demand that there exists no $K \in \mathcal{T}_2$ such that $\overline{K} \cap \partial\Omega \neq \emptyset$. However, this is no restriction. Since we assume shape regularity and $\dot{K} \subset \Omega' \subset\subset \Omega$ the claim will automaticly be fulfilled, if only h becomes small enough.*

Lemma 5.11. *Let the assumptions of Theorem 2.1 be valid. Furthermore, let z and z_h be given by Definition 2.3 and let $\omega_{\beta,\mathcal{T}}$ be given by Definition 2.2. Then there exists a $\beta' \in (0, \delta_0]$ depending only on Ω, Ω', the shape regularity constant γ and the constants of Definitions 1.7, 1.9 such that*

$$
\left\| \sqrt{\omega_{\beta,\mathcal{T}}} \nabla (z - z_h) \right\|_{L^2(\Omega)} \le C_{\beta'} h^\beta \, \|u - u_h\|_{L^2(\dot{K})} \qquad (38)
$$

for all $\beta \in (0, \beta']$; $C_{\beta'}$ independent of h and β.

Proof. For simplicity of notation we set $e = z - z_h$ and we proceed in several steps:

- First, we observe

$$
\left\| \sqrt{\omega_{\beta,\mathcal{T}}} \nabla e \right\|^2_{L^2(\Omega)} = \int_\Omega \nabla e \cdot \nabla (\omega_{\beta,\mathcal{T}} e) \mathrm{d}\Omega - \int_\Omega e \nabla e \cdot \nabla \omega \mathrm{d}\Omega.
$$

Now, Lemma 5.2 garantees the existence of $\beta' > 0$ and $C'_{\beta'} > 0$ such that

$$\begin{aligned}\left|\int_\Omega e\nabla e\cdot\nabla w \mathrm{d}\Omega\right| &\leq C\beta\int_\Omega \left|\sqrt{\omega_{\beta,\mathcal{T}}}\frac{e}{r}\right|\,\left|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla e\right|\mathrm{d}\Omega \\ &\leq C\beta\left\|\sqrt{\omega_{\beta,\mathcal{T}}}\frac{e}{r}\right\|_{L^2(\Omega)}\left\|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla e\right\|_{L^2(\Omega)} \\ &\leq CC'_{\beta'}\beta\left\|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla e\right\|^2_{L^2(\Omega)}\end{aligned}$$

for all $\beta \in (0, \beta']$. Since $C_{\beta'}$ is a monotone increasing function of β', we additionally claim $CC'_{\beta'}\beta' < 1$ for a possibly smaller β', and obtain

$$\|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla e\|^2_{L^2(\Omega)} \leq C_{\beta'}\mid\int_\Omega \nabla e\cdot\nabla(\omega_{\beta,\mathcal{T}}e)\mathrm{d}\Omega\mid$$

for all $\beta \in (0, \beta']$ and $C_{\beta'} := \frac{1}{1-CC'_{\beta'}\beta'}$.

- Next, we apply the triangular inequality to insert the element $Iz \in S_0^{\mathbf{p}}(\Omega,\mathcal{T})$ of Lemma 5.9

$$\left\|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla e\right\|^2_{L^2(\Omega)} \leq C_{\beta'}\left|\int_\Omega \nabla e\cdot\nabla(\omega_{\beta,\mathcal{T}}(z-Iz))\mathrm{d}\Omega\right| + C_{\beta'}\left|\int_\Omega \nabla e\cdot\nabla(\omega_{\beta,\mathcal{T}}(Iz-z_h)\mathrm{d}\Omega\right|.$$

Making use of Cauchy-Schwarz inequality together with Lemma 5.1 and Lemma 5.2, the first term can be bounded by

$$\begin{aligned}\left|\int_\Omega \nabla e\cdot\nabla(\omega_{\beta,\mathcal{T}}(z-Iz))\mathrm{d}\Omega\right| &\leq \left\|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla e\right\|_{L^2(\Omega)}\left\|\frac{1}{\sqrt{\omega_{\beta,\mathcal{T}}}}\nabla\left(\omega_{\beta,\mathcal{T}}(z-Iz)\right)\right\|_{L^2(\Omega)} \\ &\leq (1+CC_{\beta'}\beta)\left\|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla e\right\|_{L^2(\Omega)}\left\|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla(z-Iz)\right\|_{L^2(\Omega)}\end{aligned}$$

for all $\beta \in (0, \beta']$. To bound the second term we make use of Galerkin orthogonality and deduce

$$\begin{aligned}\left|\int_\Omega \nabla e\cdot\nabla(\omega(Iz-z_h))\mathrm{d}\Omega\right| &= \left|\int_\Omega \nabla e\cdot\nabla(\omega_{\beta,\mathcal{T}}(Iz-z_h)-\overline{I}(\omega_{\beta,\mathcal{T}}(Iz-z_h)))\mathrm{d}\Omega\right| \\ &\leq CC_{\beta'}\beta\left\|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla e\right\|_{L^2(\Omega)}\left\|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla(Iz-z_h)\right\|_{L^2(\Omega)},\end{aligned}$$

for all $\beta \in (0, \beta']$, where $\overline{I} : S_0^{2\mathbf{p}}(\Omega,\mathcal{T}) \to S_0^{\mathbf{p}}$ denotes the operator defined in Lemma 5.3. Hence, combining these two bounds we arrive at

$$\begin{aligned}\left\|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla e\right\|^2_{L^2(\Omega)} \leq\ & (1+CC_{\beta'}\beta)\left\|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla e\right\|_{L^2(\Omega)}\left\|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla(z-Iz)\right\|_{L^2(\Omega)} + \\ & CC_{\beta'}\beta\left\|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla e\right\|_{L^2(\Omega)}\left\|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla(z-Iz)\right\|_{L^2(\Omega)} + \\ & CC_{\beta'}\beta\left\|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla e\right\|_{L^2(\Omega)}\left\|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla e\right\|_{L^2(\Omega)},\end{aligned}$$

that is, for β' sufficiently small we have

$$\left\|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla e\right\|^2_{L^2(\Omega)} \leq C_{\beta'} \left\|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla e\right\|_{L^2(\Omega)} \left\|\sqrt{\omega_{\beta,\mathcal{T}}}\nabla(z - Iz)\right\|_{L^2(\Omega)};$$

for all $\beta \in (0, \beta']$ and finally, Lemma 5.9 yields (38).

□

6 Outlook

In Theorem 2.1 we proved the existence of some $\beta > 0$ such that the local error estimates (5), (6), and (7) hold. Since all of our numerical experiments achieve $\beta = \delta$ we assume that it is actually possible to prove an improved version of Theorem 2.1, where $\beta > 0$ is replaced by $\beta = \delta$. Numerical evidence such as Example 6.1 below indicates that Theorem 2.1 is not necessarily restricted to Dirichlet problems but is also true for other types of boundary conditions such as Neumann or mixed boundary conditions.
We want to mention that the doubling of the convergence rate can be obtained using the "standard" duality approach if a slightly different mesh is considered as proposed in [7] (see also [8]). There, the mesh size is defined according to $h_K \sim \min\{\sqrt{h}, h + \text{dist}(K, \partial\Omega)\}$ and the polynomial degree $\mathbf{p}$ is defined as in Definition 1.9. The key thing to note is that in the interior of the computational domain a quasi-uniform mesh with mesh size $O(\sqrt{h})$ and fixed polynomial degree is employed. Thus, the standard duality arguments can be used to recover a local L_2-convergence rate of $O(h^{1/2+\delta})$ for $u \in H^{1+\delta}(\Omega)$. It should be noted that the above choice of meshes and polynomial degree distribution also lead to a problem size $N = O(h^{-1})$.

Example 6.1. (mixed boundary conditions) *We consider the L-shaped domain as shown in Figure 1 together with the model problem*

$$\begin{aligned} -\Delta u = f \quad & on \quad \Omega = (-1,1)^2 \backslash ([0,1] \times [-1,0]) \\ \gamma_1 u = 0 \quad & on \quad \partial\Gamma_N = (\{-1\} \times [-1,1]) \cup ([-1,1] \times \{1\}) \\ u = 0 \quad & on \quad \partial\Gamma_D = \partial\Omega \backslash \Gamma_N, \end{aligned}$$

where the right-hand side f is chosen in such a way that the exact solution u is given by

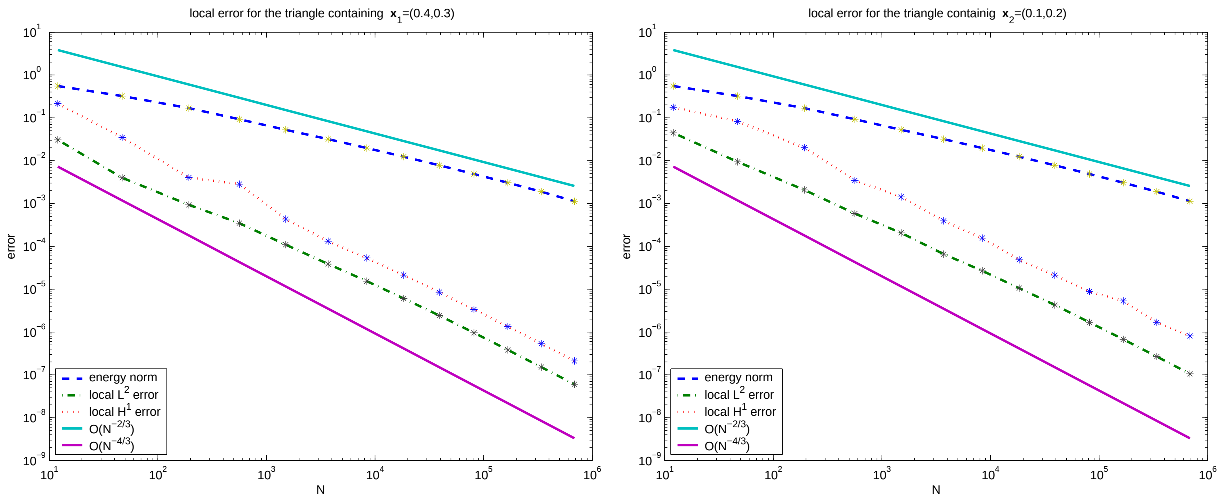
$$u = r^{\frac{2}{3}} \sin\left(\frac{2}{3}\varphi\right) \left(1 - r^2\cos^2\varphi\right)(1 + r\cos\varphi)\left(1 - r^2\sin^2\varphi\right)(1 - r\sin\varphi).$$

As in Example 3.1, we have $u \in H^{\frac{5}{3}-\varepsilon}(\Omega)$ $\forall\varepsilon > 0$, and we choose $\dot{\mathbf{x}}_1 = (0.4, 0.3)$, $\dot{\mathbf{x}}_2 = (0.1, 0.2)$. Our computations are performed with $\alpha = 1$. The results are collected in Table 2 and plotted in Figure 4. We clearly observe that the covergence rate on the fixed element element $\dot{K}$ is better than the convergence rate in the global energy norm.

Table 2: Example 6.1 with $e = u - u_h$

			$\mathbf{x}_1 = (0.4, 0.3)$		$\mathbf{x}_2 = (0.1, 0.2)$	
Level	h	p_{max}	$\|e\|_{L^2(\dot{K})}$	$\|e\|_{H^1(\dot{K})}$	$\|e\|_{L^2(\dot{K})}$	$\|e\|_{H^1(\dot{K})}$
1	5.000e-01	1	3.0616e-02	2.1547e-01	4.4508e-02	1.7582e-01
2	2.500e-01	2	3.9843e-03	3.4705e-02	9.3600e-03	8.2524e-02
3	1.250e-01	2	9.3002e-04	4.0317e-03	2.0784e-03	2.0144e-02
4	6.250e-02	3	3.4768e-04	2.8333e-03	5.8271e-04	3.4600e-03
5	3.125e-02	4	1.0915e-04	4.3509e-04	2.0533e-04	1.4255e-03
6	1.562e-02	4	3.8583e-05	1.3150e-04	6.5801e-05	3.9335e-04
7	7.812e-03	5	1.5366e-05	5.3646e-05	2.6794e-05	1.5621e-04
8	3.906e-03	6	6.0745e-06	2.1389e-05	1.0550e-05	4.8660e-05
9	1.953e-03	7	2.4260e-06	8.5037e-06	4.2667e-06	2.1289e-05
10	9.766e-04	7	9.6054e-07	3.3699e-06	1.6877e-06	8.7675e-06
11	4.883e-04	8	3.8240e-07	1.3399e-06	6.7419e-07	5.3085e-06
12	2.441e-04	9	1.5166e-07	5.3211e-07	2.6653e-07	1.6800e-06
13	1.221e-04	9	6.0242e-08	2.1117e-07	1.0600e-07	8.1111e-07
14	6.104e-05	10	2.3895e-08	8.3872e-08	4.1974e-08	2.1525e-07
15	3.052e-05	11	9.4841e-09	3.3246e-08	1.6644e-08	1.2683e-07

Figure 4: Results corresponding to Example 6.1



References

[1] I. Babuška, A. Craig, J. Mandel, and J. Pitkäranta. Efficient preconditioning for the p version finite element method in two dimensions. SIAM J. Numer. Anal., 28(3):624–661, 1991.

[2] I. Babuška and J.E. Osborn, Eigenvalue Problems, Handbook of Numerical Analysis, Vol. II, 641–789, 1991.

[3] C. Bernardi and Y. Maday, Spectral methods, In P.G. Ciarlet and J.L. Lions, editors, Handbook of Numerical Analysis, Vol. 5. North Holland, 1997.

[4] P. G. Ciarlet The Finite Element Method for Elliptic Problems, North-Holland Publishing Company, 1976.

[5] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Pitman, 1985.

[6] W. Hackbusch, Theorie und Numerik elliptischer Differentialgleichungen, Teubner, 1996

[7] W. Hackbusch, B.N. Khoromskij and S. Sauter, Adaptive Galerkin boundary element methods with panel clustering. Leipzig 2004 (in progress).

[8] B.N. Khoromskij, Hierarchical matrix approximation to Green's function via boundary concentrated FEM, J. Numer. Math. 11, No.3 195-223, 2003

[9] B.N. Khoromskij and J.M. Melenk, Boundary concentrated FEM, SIAM J. Numer. Anal., 41:1-36, 2003.

[10] J. Nečas, Sur la coercivité des formes sesquilinéaires elliptiques, Rev. Roumaine de Math. Pures et App., vol. 9, 1:47-69, 1964

[11] J. Nečas, Les méthodes directes en théorie des équations elliptiques, Masson, 1967.

[12] J.M. Melenk, *hp*-Finite Element Methods for Singular Perturbations, Springer, 2002.

[13] J.M. Melenk, *hp*-interpolation of nonsmooth functions, Technical Report NI03050, Isaac Newton Institute for Mathematics Sciences, 20 Clarkson Road, Cambridge, CB3 0EH, U.K., 2003.

[14] C. Schwab, *p*- and *hp*- Finite Element Methods, Oxford University Press, 1998.

[15] E.M. Stein, Singular integrals and differentiability properties of functions, Princeton University Press, 1970.