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# The robustness of the hierarchical a posteriori error estimator for reaction-diffusion equation on anisotropic meshes

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**Abstract**

Singularly perturbed reaction-diffusion problems exhibit in general solutions with anisotropic features, e.g. strong boundary and/or interior layers. This anisotropy is reflected in the discretization by using meshes with anisotropic elements. The quality of the numerical solution rests on the robustness of the a posteriori error estimator with respect to both the perturbation parameters of the problem and the anisotropy of the mesh.

The simplest local error estimator from the implementation point of view is the so-called hierarchical error estimator. The reliability proof is usually based on two prerequisites: the saturation assumption and the strengthened Cauchy-Schwarz inequality. The proofs of these facts are extended in the present work for the case of the singularly perturbed reaction-diffusion equation and of the meshes with anisotropic elements. It is shown that the constants in the corresponding estimates do neither depend on the aspect ratio of the elements, nor on the perturbation parameters. Utilizing the above arguments the concluding reliability proof is provided as well as the efficiency proof of the estimator, both independent of the aspect ratio and perturbation parameters. A numerical example confirms the theory.

**Key words.** a posteriori error estimation, singular perturbations, robustness, stretched elements, saturation assumption.

**AMS(MOS) subject classification:** 65N15, 65N30, 65N50

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# Contents



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# 1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open domain with polygonal boundary $\partial\Omega$. Consider the reaction-diffusion problem with homogeneous Dirichlet boundary conditions

$$-\varepsilon^2 \Delta u + \kappa^2 u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where $\kappa$ is a nonnegative constant.

If $\varepsilon/\kappa \ll 1$, then we have a singularly perturbed problem. Many physical phenomena lead to singularly perturbed problems, for instance, boundary value problems formulated on thin domains [16], where $\varepsilon/\kappa$ is proportional to the domain thickness. They also arise in mathematical models of physical problems, where diffusion is small compared with reaction and convection.

Such problems yield solutions with local anisotropic behavior, e.g. boundary and/or interior layers. In these cases special mesh adaptivity is desirable. Triangles should not only adapt in size but also in shape, to better fit the function to be approximated. While standard finite element meshes consist of isotropic elements, in the current work so-called anisotropic elements are investigated. They are characterized by a large aspect ratio (the ratio of the diameters of the circumscribed and inscribed spheres). The singularly perturbed reaction diffusion problem typically requires triangles stretched along the boundary or in the direction of the interior layer [3, 4, 5].

Local error estimators have found much use in finite element computations. This paper is concerned with the error in the energy norm, which was shown to be appropriate in adaptive procedures [13]. One of the easiest techniques for a posteriori error estimation is the hierarchical approach [6, 7]. The purpose of the current work is to consider this approach on anisotropic meshes and to construct upper and lower error bounds that are uniform with respect to both the large aspect ratio and the perturbation parameters $\kappa$ and $\varepsilon$.

The paper is organized as follows. After describing the model problem and its discretization in Section 2, and after introducing the special functions for the space enrichment in Section 3, we state in Section 4 an a posteriori residual error estimator that is shown to be robust by Kunert [11].

In Section 5 we give a proof for a saturation assumption. The saturation assumption signifies that using the quadratic finite element basis we achieve strictly higher accuracy than with linear ones. Namely, in some norm $\|\cdot\|$:

$$\|u - u_2\| \leq \alpha \|u - u_1\|, \quad \text{where } \alpha < 1,$$

$u_1$ is the usual linear finite element solution, $u_2$ is the solution using the enriched finite element space. However, as it was shown in the paper by Dörfler and Nochetto [9], there are examples that the saturation assumption fails in this form (the equation $f = -\Delta u$ was set under consideration). The modification done there concerns on additional term – the so-called data oscillation appeares in the right hand side.

For more details on data oscillation see [9]. Their proof of the saturation assumption was based on the residual a posteriori error estimator. More recently Agouzal [1] has given a proof for the saturation assumption for the reaction-diffusion equation (1.1). The proof in this case does not involve any theory of residual a posteriori error estimators. The proof of the current work mainly follows the lines of the work [9], but appears to be much more technical. The estimate obtained (Theorem 5.2) is not only uniform with respect to the mesh size, but also with respect to the aspect ratio and the perturbation parameters $\kappa$ and $\varepsilon$. As in the forementioned works the saturation assumption makes sense if the data oscillation is comparatively small. The main difference with the isotropic case is in the matching function $m_1(\cdot,\cdot)$ which naturally appears in the right hand side of the saturation assumption. The moderate size of the matching function together with the small data oscillation guarantees the saturation assumption.

In Section 6 the validity of the strengthened Cauchy-Schwarz inequality is confirmed. Namely, it is shown that

$$(x, y) \leq \gamma \|x\| \, \|y\|, \quad \forall x \in V_1, \; y \in \widetilde{V}_2,$$

where $V_1$ is the original piecewise linear finite element space, $\widetilde{V}_2$ is the enrichment space, described in section 5. We emphasize that the constant $\gamma$ in the strengthened Cauchy-Schwarz inequality for the chosen pair of spaces is always strictly smaller then 1 independently of aspect ratio.

Furthermore, in Section 7 the saturation assumption and the strengthened Cauchy-Schwarz inequality are utilized in order to show the reliability and the efficiency of the proposed estimator. The final estimates are in accordance with Kunert [12] and Grosman [10]. The numerical experiments presented in Section 8 confirm our formulas for the robustness of the error estimator and show the validity of the saturation assumption.

# 2 The model problem, its discretization and some notation

Assume $f \in L_2(\Omega)$. The Sobolev space of functions from $H^1(\Omega)$ that vanish on $\partial\Omega$ is denoted by $H_0^1(\Omega)$ as usual. The corresponding variational formulation for (1.1) becomes:

$$\text{Find } u \in H_0^1(\Omega): \quad B(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega), \tag{2.1}$$

where

$$\begin{aligned} B(u, v) &:= \int_\Omega \left(\varepsilon^2 \nabla^\top u \nabla v + \kappa^2 uv\right) dx, \\ (f, v) &:= \int_\Omega fv \, dx. \end{aligned}$$

We utilize a family $\mathcal{F} = \{\mathcal{T}\}$ of triangulations $\mathcal{T}$ of $\Omega$. Let $V_1 \subset H^1_0(\Omega)$ be the space of continuous, piecewise linear functions over $\mathcal{T}$ that vanish on $\partial\Omega$. Then the finite element solution $u_1 \in V_1$ is uniquely defined by

$$B(u_1, v) = (f, v) \quad \forall v \in V_1. \tag{2.2}$$

Due to the Lax-Milgram Lemma both problems (2.1) and (2.2) admit unique solutions.

We also will use some notation (for any $\omega \subset \Omega$)

| | | | |
|---|---|---|---|
| $L_2$-norm: | $\|v\|$ | $:=$ | $\left(\int_\Omega v^2\,dx\right)^{1/2}$, |
| Energy norm: | $\|\|\|v\|\|\|$ | $:=$ | $(\varepsilon^2\|\nabla u\|^2 + \kappa^2\|u\|^2)^{1/2}$, |
| Local energy scalar product: | $B_\omega(u,v)$ | $:=$ | $\int_\omega \left(\varepsilon^2 \nabla^\top u \nabla v + \kappa^2 uv\right)\,dx$, |
| Local $L_2$-norm: | $\|v\|_{L_2(\omega)}$ | $:=$ | $\int_\omega v^2\,dx$, |
| Local energy norm: | $\|\|\|v\|\|\|_\omega$ | $:=$ | $\left(\varepsilon^2\|\nabla u\|^2_{L_2(\omega)} + \kappa^2\|u\|^2_{L_2(\omega)}\right)^{1/2}$, |
| Length of an edge $\gamma$ | $\|\gamma\|$ | $:=$ | $\mathrm{meas}_1(\gamma)$, |
| Area of subdomain $\omega$ | $\|\omega\|$ | $:=$ | $\mathrm{meas}_2(\omega)$, |
| Patch of an edge $\gamma$ | $\widetilde{\gamma}$ | $:=$ | $\mathrm{int}\left\{\cup \mathrm{closure}(K),\ K \in \mathcal{T} : \gamma \in \partial K\right\}$. |

We will require the trivial extension operator $F_{ext} : \mathbb{P}^0(\gamma) \mapsto \mathbb{P}^0(K)$ defined by

$$F_{ext}(\varphi)(x) := \varphi|_\gamma \equiv const.$$

Now we introduce so-called bubble functions which are defined as usual, cf. [14]. Denote by $\lambda_{K,1}, \lambda_{K,2}, \lambda_{K,3}$ the barycentric coordinates of an arbitrary triangle $K$. The *element bubble function* $b_K$ is defined by

$$b_K := 27\lambda_{K,1} \cdot \lambda_{K,2} \cdot \lambda_{K,3} \text{ on } K$$

Let $\gamma = \mathrm{int}(\overline{K}_1 \cap \overline{K}_2)$ be an inner face (edge) of $\mathcal{T}_h$. Enumerate the vertices of $K_1$ and $K_2$ such that the vertices of $\gamma$ are numbered first. Define the *face bubble function* $b_\gamma$ by

$$b_\gamma := 4\lambda_{K_i,1} \cdot \lambda_{K_i,2} \text{ on } K_i,\ i = 1, 2,$$

with the obvious modification for a boundary face $\gamma \subset \partial\Omega$. For simplicity assume that $b_K$ and $b_\gamma$ are extended by zero outside their original domain of definition. There holds $0 \le b_K(x), b_\gamma(x) \le 1$ and $\|b_K\|_\infty = \|b_\gamma\|_\infty = 1$.

We will also use the following notation

$$\begin{aligned} a &\succeq b \Leftrightarrow a \ge Cb, \\ a &\preceq b \Leftrightarrow a \le Cb, \\ a &\sim b \Leftrightarrow a \succeq b \ \&\ a \preceq b, \end{aligned}$$

where $C$ does not depend on $\kappa$ and triangulation $\mathcal{T}$.

# 3 Special bubble functions

## 3.1 Special edge bubble functions

Following Kunert [11] we define special edge bubble functions, and state the corresponding inverse inequalities. They play a crucial role in the enrichment of the linear finite element space as well as in the proof of the saturation assumption and a posteriori error estimation before (see [11] and [12]). The definition is given first for the standard triangle $\overline{K}$ and then for the actual triangle $K$.

Consider the standard triangle $\overline{K}$ and the face $\overline{\gamma}$ thereof (by $\gamma$ we denote the corresponding face on the boundary of actual triangle $K$). Without loss of generality we assume that $\overline{\gamma}$ lies on the axis $Oy$. For a real number $\delta \in (0, 1]$ define a linear mapping $F_\delta : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F_\delta(x, y) := (\delta \cdot x, \frac{1-\delta}{2} \cdot y)^\top$$

or,

$$F_\delta(\mathbf{x}) := B_\delta \cdot \mathbf{x} \quad \text{with } B_\delta = \operatorname{diag}\{\delta, \frac{1-\delta}{2}\} \in \mathbb{R}^{2\times 2}.$$

Set $\overline{K}_\delta := F_\delta(\overline{K})$, i.e $\overline{K}_\delta$ is the triangle with the face $\overline{\gamma}$ and a vertex at $\delta \cdot \mathbf{e_1}$.

Let $b_{\overline{\gamma}}$ be the usual face bubble function of $\overline{\gamma}$ on $\overline{K}$. Define the special bubble function $\overline{b}_{\gamma,\delta}$ by

$$\overline{b}_{\gamma,\delta} := b_{\overline{\gamma}} \circ F_\delta^{-1}$$

i.e. $\overline{b}_{\gamma,\delta}$ is the usual face bubble function of $\overline{\gamma}$ on the triangle $\overline{K}_\delta$. For clarity we recall that $\overline{b}_{\gamma,\delta} = 0$ on $\overline{K} \setminus \overline{K}_\delta$.

Consider now an actual triangle $K$. The special edge bubble function $b_{\gamma,\delta} \in H^1(K)$ of a face $\gamma$ of $K$ is defined by

$$b_{\gamma,\delta} := \overline{b}_{\gamma,\delta} \circ F_A^{-1}$$

The actual value of parameter $\delta$ will be specified later.

**Lemma 3.1.** ***(Inverse inequalities for bubble functions and special edge bubble functions).*** *Let $\gamma$ be an arbitrary face of $K$. Assume that $\varphi_K \in \mathbb{P}^0(K)$ and $\varphi_\gamma \in \mathbb{P}^0(\gamma)$. Then the following inverse inequalities hold:*

$$\|\nabla(b_K \cdot \varphi_K)\|_{L_2(K)} \sim h_{min,K}^{-1} \cdot \|\varphi_K\|_{L_2(K)} \tag{3.1}$$

$$\|F_{ext}(\varphi_\gamma) \cdot b_{\gamma,\delta}\|_{L_2(K)} \sim \left(\frac{|K|}{|\gamma|}\right)^{1/2} \cdot \delta^{1/2} \cdot \|\varphi_\gamma\|_{L_2(\gamma)} \tag{3.2}$$

$$\|\nabla(F_{ext}(\varphi_\gamma) \cdot b_{\gamma,\delta})\|_{L_2(K)} \sim \left(\frac{|K|}{|\gamma|}\right)^{1/2} \cdot \delta^{1/2} \cdot \min\left\{\delta\frac{|K|}{|\gamma|}, h_{min,K}\right\}^{-1} \cdot \|\varphi_\gamma\|_{L_2(\gamma)}. \tag{3.3}$$

*Proof.* See [11]. □

We are in a position to specify our parameter $\delta = \delta(\gamma)$. From now on we use

$$\delta_\gamma := \frac{1}{3}\frac{|\gamma|}{|\widetilde{\gamma}|}\varepsilon \min(\varepsilon^{-1}h_{min,\gamma}, \kappa^{-1}). \tag{3.4}$$

Note that if $\gamma = \partial K \bigcap \partial K'$, then

$$\delta_\gamma \sim \frac{|\gamma|}{|K|}\varepsilon \min(\varepsilon^{-1}h_{min,K}, \kappa^{-1}) \sim \frac{|\gamma|}{|K'|}\varepsilon \min(\varepsilon^{-1}h_{min,K'}, \kappa^{-1}).$$

We should mention that the definition (3.4) differs from the original definition in Kunert [11] by a factor of $\frac{1}{3}$, which however does not disturb the estimates. This modification is done in order to avoid overlapping supports of special edge bubble functions.

## 3.2 Notation of the triangle

Let a triangulation $\mathcal{T}$ be given which satisfies the usual conformity condition (see [8], Chapter 2). Following Kunert [11] we introduce the following notation. The three vertices of an arbitrary triangle $K \in \mathcal{T}_h$ are denoted by $P_0, P_1, P_2$ such that $P_0P_1$ is the longest edge of $K$. Additionally define two orthogonal vectors $p_i$ with lengths $h_{i,K} := |p_i|$, see Figure 2. Observe that $h_{1,K} > h_{2,K}$ and set $h_{max,K} := h_{1,K}$, $h_{min,K} := h_{2,K}$.

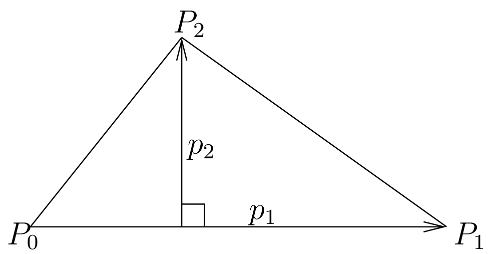


Figure 1: Notation of a triangle $K$.

In addition to the usual conformity conditions of the mesh we demand the following two assumptions.

1. The number of triangles containing a node $x_n$ is bounded uniformly.
2. The dimensions of adjacent triangles must not change rapidly, i.e.

$$h_{i,K'} \sim h_{i,K} \quad \forall K, K' \text{ with } \overline{K} \cap \overline{K'} \neq \emptyset,\ i = 1, 2.$$

Define the matrices $A_K$ and $C_K \in \mathbb{R}^{2\times 2}$ by

$$A_K := (\overrightarrow{P_0P_1}, \overrightarrow{P_0P_2}) \quad \text{and} \quad C_K := (p_1, p_2)$$

and introduce affine linear mappings

$$F_A(\mu) := A_K \cdot \mu + \overrightarrow{P}_0 \quad \text{and} \quad F_C(\mu) := C_K \cdot \mu + \overrightarrow{P}_0,\ \mu \in \mathbb{R}^2.$$

These mappings implicitly define the so-called *standard triangle* $\overline{K} := F_A^{-1}(K)$ and the *reference triangle* $\hat{K} := F_C^{-1}(K)$. Variables that are related to the standard triangle $\overline{K}$ and reference triangle $\hat{K}$ are referred to with a bar and hat, respectively (e.g. $\overline{\nabla}, \hat{v}$). The determinants of both mappings are $|\det(A_K)| = |\det(C_K)| = 6|K|$, and the transformed derivatives satisfy $\overline{\nabla}\overline{v} = A_K^\top \nabla v$ and $\hat{\nabla}\hat{v} = C_K^\top \nabla v$.

Furthermore, for any interior face $\gamma = \overline{K} \bigcap \overline{K'}$ define the quantity $h_{min,\gamma}$ by

$$h_{min,\gamma} := \frac{h_{min,K} + h_{min,K'}}{2}.$$

The advantage of this notation is clear, we need a value that relates to the edge, in contrast with others related to triangles. Note that $h_{min,\gamma} \sim h_{min,K} \sim h_{min,K'}$ due to the mesh assumptions.

# 4 A posteriori residual error estimator

After some additional notation and definition of the matching function we formulate an upper error bound for the error measured in the energy norm. The jump discontinuity in the approximation of the normal flux at an interelement boundary is defined by

$$\left[\frac{\partial u_1}{\partial n}\right] = n_K \cdot (\nabla u_1)_K + n_{K'} \cdot (\nabla u_1)_{K'},$$

and the usual interior and boundary residuals $r$ and $R$ are given by

$$r = f + \varepsilon^2 \Delta u_1 - \kappa^2 u_1$$

and

$$R = \begin{cases} \varepsilon^2 \left[\frac{\partial u_1}{\partial n}\right] & \text{on } \partial K \cap \partial K' \\ 0 & \text{on } \partial K \cap \partial \Omega \end{cases}$$

which are defined as usually (see [2]). Define by $r_K := \frac{1}{|K|}\int_K r\,dx$ the mean value of $r$ over an element $K$.

In addition we introduce the following notation:

$$\begin{aligned} \alpha_K &:= \min(\varepsilon^{-1} h_{min,K}, \kappa^{-1}), \\ \alpha_\gamma &:= \min(\varepsilon^{-1} h_{min,\gamma}, \kappa^{-1}). \end{aligned}$$

**Definition 4.1.** ***(Matching function*** $\mathrm{m}_1$***).*** *Let $v \in H^1(\Omega)$ be any arbitrary non-constant function, and $\mathcal{F}$ be a family of triangulations of $\Omega$. Define the matching function $m_1(\cdot,\cdot) : H^1(\Omega) \times \mathcal{F} \mapsto \mathbb{R}$ by*

$$m_1(v, \mathcal{T}) := \frac{\left(\sum_{K\in\mathcal{T}} h_{min,K}^{-2} \cdot \|C_K^\top \nabla v\|_{L_2(K)}^2\right)^{1/2}}{\|\nabla v\|} \tag{4.1}$$

The matching function satisfies the following property:

$$1 \le m_1(v, \mathcal{T}) \preceq \max_{K \in \mathcal{T}} \frac{h_{max,K}}{h_{min,K}}$$

The definition means that small value of $m_1(v, \mathcal{T})$ is reached for the meshes $\mathcal{T}$ well aligned with an anisotropic function $v$.

**Theorem 4.2.**

$$\begin{aligned} |||u - u_1||| &\preceq m_1(u - u_1, \mathcal{T}) \left\{ \sum_{K \in \mathcal{T}} \alpha_K^2 \|r_K\|^2_{L_2(K)} \right. \\ &+ \left. \sum_{\gamma \in \partial \mathcal{T}} \varepsilon^{-1} \alpha_\gamma \|R\|^2_{L_2(\gamma)} + \sum_{K \in \mathcal{T}} \alpha_K^2 \|r - r_K\|^2_{L_2(K)} \right\}^{1/2}, \end{aligned}$$

*where $\partial \mathcal{T}$ denote the collection of all edges in the triangulation $\mathcal{T}$.*

*Proof.* See the proof for the anisotropic case in Kunert [11]. For the isotropic case it appeared first in Verfürth [15]. □

# 5 Saturation assumption

In the case of a singularly perturbed problem the choice of space enrichment is crucial. First, recall the definition of the space $V_1$:

$$V_1 := \{v_h \in H_0^1(\Omega) : \forall K \in \mathcal{T},\, v_h|_K \in P_1(K)\}.$$

We enrich the space $V_1$ by the squeezed bubble functions for all edges and the interior bubbles. Namely,

$$V_2 := \left\{v_h \in H_0^1(\Omega) : \forall K \in \mathcal{T},\, v_h|_K \in P_1(K) + span\{b_K, b_{\gamma,\delta_\gamma} : \gamma \in \partial K \setminus \partial\Omega\}\right\}.$$

Then the finite element solution $u_2 \in V_2$ is uniquely defined by

$$B(u_2, v) = (f, v) \quad \forall v \in V_2. \tag{5.1}$$

It is not clear at the moment wether we get the estimate similar to the estimate of Theorem 5.2 using the usual bubbles as it was done for example in [9] for the Laplace problem.

The proof of the saturation assumption is based on the following lemma.

**Lemma 5.1.**

$$|||u - u_1|||^2 \le C m_1(u - u_1, \mathcal{T})^2 \left( |||u_1 - u_2|||^2 + \sum_{K \in \mathcal{T}} \alpha_K^2 \|r - r_K\|^2_{L_2(K)} \right).$$

*Proof.* Using the Theorem 4.2 we estimate the terms involving boundary and interior residual subsequently.
1. *Boundary residual.* Due to the fact that $R$ is constant over each edge $\gamma$ applying partial integration we get:

$$\begin{aligned}
&\frac{2}{3}|\gamma|R = \int_\gamma R b_{\gamma,\delta_\gamma}\,ds = -\int_{\widetilde{\gamma}} \nabla u_1 \cdot \nabla b_{\gamma,\delta_\gamma}\,dx \\
&= \varepsilon^2 \int_{\widetilde{\gamma}} \nabla(u_2-u_1)\cdot\nabla b_{\gamma,\delta_\gamma}\,dx + \kappa^2\int_{\widetilde{\gamma}}(u_2-u_1)\cdot b_{\gamma,\delta_\gamma}\,dx - \int_{\widetilde{\gamma}} f\cdot b_{\gamma,\delta_\gamma}\,dx + \kappa^2\int_{\widetilde{\gamma}} u_1\cdot b_{\gamma,\delta_\gamma}\,dx \\
&= B_{\widetilde{\gamma}}(u_2-u_1, b_{\gamma,\delta_\gamma}) - \int_{\widetilde{\gamma}} r b_{\gamma,\delta_\gamma}\,dx,
\end{aligned}$$

where $\widetilde{\gamma}$ is the union of two triangles sharing the edge $\gamma$ (see Section 2). Squaring and integrating over $\gamma$ we get

$$\begin{aligned}
|\gamma|\|R\|^2_{L_2(\gamma)} &\preceq B_{\widetilde{\gamma}}(u_2-u_1, b_{\gamma,\delta_\gamma})^2 + \left(\int_{\widetilde{\gamma}} r b_{\gamma,\delta_\gamma}\,dx\right)^2 \\
&\leq |||u_2-u_1|||^2_{\widetilde{\gamma}}|||b_{\gamma,\delta_\gamma}|||^2_{\widetilde{\gamma}} + \|r\|^2_{L_2(\widetilde{\gamma})}\|b_{\gamma,\delta_\gamma}\|^2_{L_2(\widetilde{\gamma})}
\end{aligned}$$

Estimate the first term using the inequalities for the special bubble functions (3.2), (3.3) and the definition of $\delta_\gamma$ (3.4) as follows:

$$\begin{aligned}
|||b_{\gamma,\delta_\gamma}|||^2_{\widetilde{\gamma}} &= \varepsilon^2\|\nabla b_{\gamma,\delta_\gamma}\|^2_{L_2(\widetilde{\gamma})} + \kappa^2\|b_{\gamma,\delta_\gamma}\|^2_{L_2(\widetilde{\gamma})} \\
&\preceq \sum_{K\subset\widetilde{\gamma}}\left(\varepsilon^2\frac{|K|}{|\gamma|}\delta_\gamma\min\left\{\delta_\gamma\cdot\frac{|K|}{|\gamma|}, h_{min,K}\right\}^{-2}|\gamma| + \kappa^2\frac{|K|}{|\gamma|}\delta_\gamma|\gamma|\right) \\
&\sim \sum_{K\subset\widetilde{\gamma}}\left(\varepsilon^3\alpha_K\min\left\{\varepsilon\alpha_K, h_{min,K}\right\}^{-2}|\gamma| + \kappa^2\varepsilon\alpha_K|\gamma|\right) \\
&\preceq \varepsilon|\gamma|\alpha_\gamma^{-1}
\end{aligned}$$

Estimate the second term using (3.2):

$$\|b_{\gamma,\delta_\gamma}\|^2_{L_2(\widetilde{\gamma})} \preceq \frac{|K|}{|\gamma|}\delta_\gamma|\gamma| \sim |\gamma|\varepsilon\alpha_\gamma$$

Combining three previous estimates we come to the following:

$$\begin{aligned}
\varepsilon^{-1}\alpha_\gamma\|R\|^2_{L_2(\gamma)} &\preceq |||u_2-u_1|||^2_{\widetilde{\gamma}} + \alpha_\gamma^2\|r\|^2_{L_2(\widetilde{\gamma})} \\
&\preceq |||u_2-u_1|||^2_{\widetilde{\gamma}} + \sum_{K\subset\widetilde{\gamma}}\alpha_K^2\|r_K\|^2_{L_2(K)} + \sum_{K\subset\widetilde{\gamma}}\alpha_K^2\|r-r_K\|^2_{L_2(K)}.
\end{aligned}$$

2. *Interior residual.* It remains to estimate the term $\alpha_K^2 \|r_K\|^2_{L_2(\widetilde{\gamma})}$. We have

$$\begin{aligned}
r_K \frac{|K|}{5!} &= \int_K r_K b_K \, dx \\
&= \int_K f b_K \, dx - \kappa^2 \int_K u_1 b_K \, dx - \int_K r b_K \, dx + \int_K r_K b_K \, dx \\
&= B_K(u_2, b_K) - B_K(u_1, b_K) - \int_K (r - r_K) b_K \, dx \\
&= B_K(u_2 - u_1, b_K) - \int_K (r - r_K) b_K \, dx,
\end{aligned}$$

because $\int_K \nabla u_1 \cdot \nabla b_K \, dx = -\int_K \Delta u_1 \cdot b_K \, dx = 0$. Squaring and integrating over an element $K$ we get:

$$\begin{aligned}
|K| \|r_K\|^2_{L_2(K)} &\preceq B_K(u_2 - u_1, b_K)^2 + \left( \int_K (r - r_K) b_K \, dx \right)^2 \\
&\leq |||u_2 - u_1|||^2_K |||b_K|||^2_K + \|r - r_K\|^2_{L_2(K)} \|b_K\|^2_{L_2(K)}
\end{aligned}$$

Now we use (3.1) for $|||b_K|||_K$ as follows

$$\begin{aligned}
|||b_K|||^2_K &= \varepsilon^2 \|\nabla(b_K)\|^2_{L_2(K)} + \kappa^2 \|b_K\|^2_{L_2(K)} \\
&\preceq \left( \varepsilon^2 h^{-2}_{min,K} + \kappa^2 \right) \|b_K\|^2_{L_2(K)} \\
&\preceq \left( \varepsilon^2 h^{-2}_{min,K} + \kappa^2 \right) |K|
\end{aligned}$$

or,

$$|||b_K|||^2_K \preceq \alpha_K^{-2} |K|.$$

Thus, it follows that

$$\alpha_K^2 \|r_K\|^2_{L_2(K)} \preceq |||u_2 - u_1|||^2_K + \alpha_K^2 \|r - r_K\|^2_{L_2(K)} \tag{5.2}$$

Now, applying the inequalities (5.2) and (5.2) to the estimate of the Theorem 4.2 we get the result claimed. □

**Theorem 5.2 (Saturation assumption on anisotropic meshes).** *The following inequality takes place:*

$$|||u - u_2||| \leq \sqrt{1 - \frac{1}{C m_1(u - u_1, \mathcal{T})^2}} |||u - u_1||| + \left( \sum_{K \in \mathcal{T}} \alpha_K^2 \|r - r_K\|^2_K \right)^{1/2}. \tag{5.3}$$

*Proof.* Using the identity

$$|||u - u_1|||^2 = |||u - u_2|||^2 + |||u_1 - u_2|||^2$$

we get

$$|||u-u_2|||^2 \le \left(1-\frac{1}{Cm_1(u-u_1,\mathcal{T})^2}\right)|||u-u_1|||^2+\sum_{K\in\mathcal{T}}\alpha_K^2\|r-r_K\|^2_{L_2(K)}.$$

Taking the square root we finish the proof. □

The estimate (5.3) we call the saturation assumption on anisotropic meshes. As it could be mentioned the constant in (5.3) depends strongly on the value of the matching function $m_1$, and only bounding $m_1$ one can claim that the error reduces its value significantly while using refined finite element space.

# 6 The strengthened Cauchy-Schwarz Inequality

## 6.1 Theoretical background

**Definition 6.1.** *Let $X, Y$ be two subspaces of a Hilbert space equipped with a scalar product $(\cdot,\cdot)$ and induced norm $\|\cdot\|$. A saturation assumption is said to hold for this couple of spaces if there exist a non-negative constant $\gamma < 1$ such that:*

$$(x,y) \le \gamma\|x\|\,\|y\|, \quad \forall x\in X,\ y\in Y. \tag{6.1}$$

Let $X, Y$ be finite dimensional spaces. Consider a stiffness matrix B corresponding to the space $X^* = X\oplus Y$,

$$\mathbf{B}=\begin{bmatrix}\mathbf{B}_{XX} & \mathbf{B}_{XY}\\ \mathbf{B}_{YX} & \mathbf{B}_{YY}\end{bmatrix}.$$

We state here without a proof the following theorem from [2].

**Theorem 6.2.** *The constant $\gamma$ in the Cauchy-Schwarz inequality 6.1 may be expressed in the following way:*

$$\gamma^2=\max_{\mathbf{x}\in\mathbb{R}^{\dim(X)}}\frac{\mathbf{x}^T\mathbf{B}_{XY}\mathbf{B}_{YY}^{-1}\mathbf{B}_{YX}\mathbf{x}}{\mathbf{x}^T\mathbf{B}_{XX}\mathbf{x}}$$

Now come back to our problem. Suppose that for each element $K$ of triangulation the constant $\gamma_K$ is known:

$$B_K(u,v)\le\gamma_K|||u|||_K|||v|||_K, \quad \forall u\in X_K,\ v\in Y_K,$$

where $X_K$ and $Y_K$ are restrictions of corresponding spaces to the element $K$. Now, prescribing $\gamma=\max_K\gamma_K$, we obtain the constant $\gamma$ for the whole mesh:

$$B(u,v)=\sum_K B_K(u,v)\le\gamma\sum_K|||u|||_K|||v|||_K\le\gamma|||u|||\cdot|||v|||, \tag{6.2}$$

where we utilized the discrete Cauchy-Schwarz inequality.

## 6.2 Pure Laplace problem $\kappa = 0$

We state this result here because it could be used in other applications. In the case of a pure Laplace problem the Cauchy-Schwarz constant has a nice structure and may be expressed explicitly (see below). We assume here that $\delta_\gamma = 1$, for all edges $\gamma$ in triangulation. In other words $\widetilde{V}_2$ is the space consisting of three usual edge bubble functions together with interior bubble function. The following decomposition holds $V_2 = V_1 \oplus \widetilde{V}_2$.

Let a triangle have angles $\phi_1, \phi_2, \phi_3$. After straight forward maple calculations we get the matrices needed to obtain the strengthened Cauchy-Schwarz constant for $H_1$ semi-norm:

$$B_{V_1V_1} = \frac{1}{2}\begin{pmatrix} \frac{\sin\phi_1}{\sin\phi_2\sin\phi_3} & -\cot\phi_3 & -\cot\phi_2 \\ -\cot\phi_3 & \frac{\sin\phi_2}{\sin\phi_3\sin\phi_1} & -\cot\phi_1 \\ -\cot\phi_2 & -\cot\phi_1 & \frac{\sin\phi_3}{\sin\phi_1\sin\phi_2} \end{pmatrix},$$

$$B_{\widetilde{V}_2\widetilde{V}_2} = \frac{1}{2}\begin{pmatrix} D & -\cot\phi_3 & -\cot\phi_2 & \frac{\cot\phi_1}{5} \\ -\cot\phi_3 & D & -\cot\phi_1 & \frac{\cot\phi_2}{5} \\ -\cot\phi_2 & -\cot\phi_1 & D & \frac{\cot\phi_3}{5} \\ \frac{\cot\phi_1}{5} & \frac{\cot\phi_2}{5} & \frac{\cot\phi_3}{5} & \frac{D}{15} \end{pmatrix},$$

$$\text{where } D = \frac{\cos\phi_1\cos\phi_2\cos\phi_3 + 1}{\sin\phi_1\sin\phi_2\sin\phi_3}.$$

$$B_{\widetilde{V}_2V_1} = -\frac{1}{6}\begin{pmatrix} \frac{\sin\phi_1}{\sin\phi_2\sin\phi_3} & -\cot\phi_3 & -\cot\phi_2 \\ -\cot\phi_3 & \frac{\sin\phi_2}{\sin\phi_3\sin\phi_1} & -\cot\phi_1 \\ -\cot\phi_2 & -\cot\phi_1 & \frac{\sin\phi_3}{\sin\phi_1\sin\phi_2} \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_{V_1\widetilde{V}_2} = B^T_{\widetilde{V}_2V_1}.$$

Compute the matrix $L = B_{V_1\widetilde{V}_2}B^{-1}_{\widetilde{V}_2\widetilde{V}_2}B_{\widetilde{V}_2V_1} \in \mathbb{R}^{3\times3}$:

$$\begin{pmatrix} A_{1,2,3} & B_{3,1,2} & B_{2,3,1} \\ B_{3,1,2} & A_{2,3,1} & B_{1,2,3} \\ B_{2,3,1} & B_{1,2,3} & A_{3,1,2} \end{pmatrix},$$

where we used the notation:

$$\begin{aligned} A_{i,j,k} &:= \frac{(2-\cos^2\phi_i - 2\cos\phi_i\cos\phi_j\cos\phi_k)\sin\phi_i}{\sin\phi_j\sin\phi_k}, \\ B_{i,j,k} &:= \frac{\cos\phi_j\cos\phi_k - 2\cos\phi_i}{\sin\phi_i}. \end{aligned}$$

What remains is to find

$$\gamma^2 = \max_{\mathbf{x}\in\mathbb{R}^3} \frac{\mathbf{x}^T L\mathbf{x}}{\mathbf{x}^T B_{V_1V_1}\mathbf{x}}.$$

If $B_{V_1V_1}$ was a non-singular matrix then we would solve an eigenvalue problem $L\mathbf{x} = \lambda B_{V_1V_1}\mathbf{x}$ and maximal eigenvalue $\lambda_{max}$ would be our wanted constant $\gamma^2$. However, it is not possible in this case since $B_{V_1V_1}$ has a vector $e = [1,1,1]^T$ in its kernel. It turns out that $e$ lies in the kernel of the matrix $L$ as well. There exist two vectors $e_1$ and $e_2$ such that $\{e, e_1, e_2\}$ is the basis in $\mathbb{R}^3$. We have

$$\begin{aligned} \gamma^2 &= \max_{\alpha,\beta_1,\beta_2\in\mathbb{R}} \frac{(\alpha e+\beta_1 e_1+\beta_2 e_2)^T L(\alpha e+\beta_1 e_1+\beta_2 e_2)}{(\alpha e+\beta_1 e_1+\beta_2 e_2)^T B_{V_1V_1}(\alpha e+\beta_1 e_1+\beta_2 e_2)} \\ &= \max_{\alpha,\beta_1,\beta_2\in\mathbb{R}} \frac{(\beta_1 e_1+\beta_2 e_2)^T L(\beta_1 e_1+\beta_2 e_2)}{(\beta_1 e_1+\beta_2 e_2)^T B_{V_1V_1}(\beta_1 e_1+\beta_2 e_2)} \\ &= \max_{\beta\in\mathbb{R}^2} \frac{\beta^T E^T L E\beta}{\beta^T E^T B_{V_1V_1} E\beta}, \end{aligned}$$

where the matrix $E = (e_1, e_2) \in \mathbb{R}^{3\times 2}$ can be chosen in such a way that $\{e, e_1, e_2\}$ is the basis in $\mathbb{R}^3$. Choosing for simplicity

$$E := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and solving eigenvalue problem $E^T LE\beta = \lambda E^T B_{V_1V_1}E\beta$ we get two eigenvalues

$$\beta_{1,2} = \frac{1}{2} \pm \frac{1}{6}\sqrt{1-8\cos\phi_1\cos\phi_2\cos\phi_3}.$$

Choosing the maximal of these two numbers we formulate the following lemma.

**Theorem 6.3.** *The constant $\gamma$ in the strengthened Cauchy-Schwarz inequality for the spaces $V_1$ and $\widetilde{V}_2$ is expressed by*

$$\gamma^2 = \frac{1}{2} + \frac{1}{6}\sqrt{1-8\cos\phi_1\cos\phi_2\cos\phi_3}.$$

*Proof.* For the proof see above arguments. □

In the inequality of Theorem 6.3 maximum angle condition naturally appears in the sense that, if one angle of a triangle goes to $\pi$ then the strengthened Cauchy-Schwarz constant goes to $1$. It leads to the fact that induced hierarchical error estimator will fail in general on meshes where the maximum angle condition is not satisfied.

### 6.3 Squeezed case

Divide a triangle $K$ into three parts $K_{\frac{1}{3}}$, where the central point is the center of mass. Mention that support of each special edge bubble function lies inside exactly one part $K_{\frac{1}{3}}$. Evaluate (estimate from above) the Cauchy-Schwarz constant for each part independently assuming that the squeezing parameter $\delta$ of the special edge bubble function can be any number from $0$ to $1/3$. The constant for the whole triangle and subsequently for the whole mesh may be chosen as the maximum value of three corresponding constants as we did in (6.2).

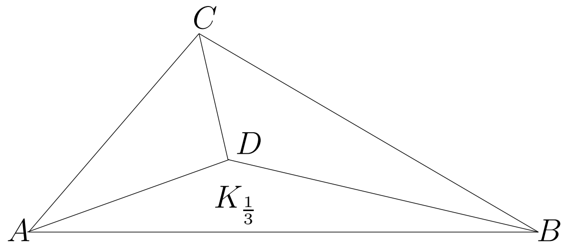


Figure 2: Notation of a trangle, the vertices have the following coordinates: $A = (0,0)$, $B = (a,0)$, $C = ((a+\epsilon)/2, b)$, $D = (A+B+C)/3$.

**Lemma 6.4.**

$$(u,v)_{H^1(K_{\frac{1}{3}})} \leq \frac{2\sqrt{2}}{3}|u|_{H^1(K_{\frac{1}{3}})}|v|_{H^1(K_{\frac{1}{3}})}, \quad \forall u \in V_1, v \in \widetilde{V}_2.$$

*Proof.* Write down matrices in notation from Figure 2.

$$B_{V_1V_1} = \begin{pmatrix} \frac{4b^2+(1-\epsilon)^2}{24b} & -\frac{4b^2-1+\epsilon^2}{24b} & -\frac{1-\epsilon}{12b} \\ -\frac{4b^2-1+\epsilon^2}{24b} & \frac{4b^2+(1-\epsilon)^2}{24b} & -\frac{1+\epsilon}{12b} \\ -\frac{1-\epsilon}{12b} & -\frac{1+\epsilon}{12b} & \frac{1}{6b} \end{pmatrix},$$

$$B_{\widetilde{V}_2\widetilde{V}_2} = \begin{pmatrix} \frac{\delta^2\epsilon^2+4\delta^2b^2+3}{48b\delta} & \frac{\delta^2\epsilon^2-3\delta^2+4\delta^2b^2+10\delta-8}{240b} \\ \frac{\delta^2\epsilon^2-3\delta^2+4\delta^2b^2+10\delta-8}{240b} & \frac{\epsilon^2+4b^2+51}{19440b} \end{pmatrix},$$

$$B_{\widetilde{V}_2V_1} = \begin{pmatrix} \frac{1-\epsilon}{12b} & \frac{1+\epsilon}{12b} & -\frac{1}{6b} \\ -\frac{1-\epsilon}{108b} & -\frac{1+\epsilon}{108b} & \frac{1}{54b} \end{pmatrix}.$$

Compute the matrix $L = B_{V_1\widetilde{V}_2}B^{-1}_{\widetilde{V}_2\widetilde{V}_2}B_{\widetilde{V}_2V_1} \in \mathbb{R}^{3\times 3}$ and using the same trick as in the proof of Theorem 6.3 we solve eigenvalue problem $E^TLE\beta = \lambda E^TB_{V_1V_1}E\beta$. One of two eigenvalues of this problem equals zero, the second one is the constant of interest:

$$\begin{aligned}
\gamma^2 &= \frac{8}{9} - \frac{8}{9}[-90 - 4860\delta^4 + 9558\delta^3 + 729\delta^5 - 4860\delta^2 + 999\delta \\
&+ (-5\delta^2 + 81\delta^5)\epsilon^4 \\
&+ (-1944\delta^3 + 6480\delta^4 + 180\delta - 60 - 1944\delta^5 - 120\delta^2)b^2 \\
&+ (-30\delta^2 + 1620\delta^4 - 15 + 45\delta - 486\delta^5 - 486\delta^3)\epsilon^2 \\
&+ (1296\delta^5 - 80\delta^2)b^4 + (648\delta^5 - 40\delta^2)\epsilon^2 b^2] \\
&/ [5184\delta - 12960\delta^2 - 4860\delta^4 - 765 + 729\delta^5 + 11988\delta^3 \\
&+ (-5\delta^2 + 81\delta^5)\epsilon^4 + (6480\delta^4 - 60 - 1020\delta^2 - 5184\delta^3 - 1944\delta^5)b^2 \\
&+ (-1296\delta^3 - 255\delta^2 + 1620\delta^4 - 486\delta^5 - 15)\epsilon^2 + (1296\delta^5 - 80\delta^2)b^4 + (648\delta^5 - 40\delta^2)\epsilon^2 b^2]
\end{aligned}$$

Observing that for $\delta \in [0, \frac{1}{3}]$

$$\begin{aligned}
&-90 - 4860\delta^4 + 9558\delta^3 + 729\delta^5 - 4860\delta^2 + 999\delta \\
&= 9(3\delta - 1)\left(243\left(\delta - \frac{1}{6}\right)^2 + \frac{13}{4} + 9\delta^2(3\delta - 1)(\delta - 6)\right) \leq 0 \\
&-5\delta^2 + 81\delta^5 = \delta^2(81\delta^3 - 5) \leq -2\delta^2 \leq 0 \\
&-1944\delta^3 + 6480\delta^4 + 180\delta - 60 - 1944\delta^5 - 120\delta^2 \\
&= -\frac{40}{3} - \frac{4}{3}(3\delta - 1)\left(5(15\delta + 7)(3\delta - 1) + 9\delta^2\left(54\left(\frac{3}{2} - \delta\right)^2 - \frac{293}{2}\right)\right) \\
&\leq -\frac{40}{3} - \frac{4}{3}(3\delta - 1)\left(5(15\delta + 7)(3\delta - 1) - 25 \cdot 9\delta^2\right) \leq 0 \\
&-30\delta^2 + 1620\delta^4 - 15 + 45\delta - 486\delta^5 - 486\delta^3 \\
&= -\frac{10}{3} - \frac{1}{3}(3\delta - 1)\left(5(15\delta + 7)(3\delta - 1) + 9\delta^2\left(54\left(\frac{3}{2} - \delta\right)^2 - \frac{293}{2}\right)\right) \leq 0 \\
&1296\delta^5 - 80\delta^2 = 16\delta^2(81\delta^3 - 5) \leq 0 \\
&648\delta^5 - 40\delta^2 = 8\delta^2(81\delta^3 - 5) \leq 0 \\
&5184\delta - 12960\delta^2 - 4860\delta^4 - 765 + 729\delta^5 + 11988\delta^3 \\
&= 9(3\delta - 1)\left(477\left(\delta - \frac{107}{318}\right)^2 + \frac{6571}{212}\right) + 81\delta^3\left(9\left(\frac{10}{3} - \delta\right)^2 - 111\right) \leq 0 \\
&-5\delta^2 + 81\delta^5 \leq 0
\end{aligned}$$

$$
\begin{aligned}
& 6480\delta^4 - 60 - 1020\delta^2 - 5184\delta^3 - 1944\delta^5 \\
&= -60 - 12\delta^2\left(85 + \delta\left(162\left(\frac{5}{3} - \delta\right)^2 - 18\right)\right) \leq 0 \\
& -1296\delta^3 - 255\delta^2 + 1620\delta^4 - 486\delta^5 - 15 \\
&= -15 - 3\delta^2\left(85 + \delta\left(162\left(\frac{5}{3} - \delta\right)^2 - 18\right)\right) \leq 0 \\
& 1296\delta^5 - 80\delta^2 \leq 0 \\
& 648\delta^5 - 40\delta^2 \leq 0
\end{aligned}
$$

we finish the proof. □

**Lemma 6.5.**

$$
(u,v)_{L_2(K_{\frac{1}{3}})} \leq \sqrt{\frac{31927}{35680} + \frac{7\sqrt{193953}}{35680}}\|u\|_{L_2(K_{\frac{1}{3}})}\|v\|_{L_2(K_{\frac{1}{3}})}, \quad \forall u \in V_1, v \in \widetilde{V}_2.
$$

$$
\frac{31927}{35680} + \frac{7\sqrt{193953}}{35680} \approx 0.9812165548 < 1.
$$

*Proof.* The matrices in the $L_2$ case has the following form:

$$
B_{V_1V_1} = a^2b\begin{pmatrix} \frac{13}{324} & \frac{17}{648} & \frac{5}{648} \\ \frac{17}{648} & \frac{13}{324} & \frac{5}{648} \\ \frac{5}{648} & \frac{5}{648} & \frac{1}{324} \end{pmatrix},
$$

$$
B_{\widetilde{V}_2\widetilde{V}_2} = a^2b\begin{pmatrix} \frac{\delta}{180} & \frac{\delta^2(3\delta^2 - 14\delta + 19)}{10080} \\ \frac{\delta^2(3\delta^2 - 14\delta + 19)}{10080} & \frac{1}{15120} \end{pmatrix},
$$

$$
B_{\widetilde{V}_2V_1} = a^2b\begin{pmatrix} \frac{\delta(5-\delta)}{240} & \frac{\delta(5-\delta)}{240} & \frac{\delta^2}{120} \\ \frac{17}{14580} & \frac{17}{14580} & \frac{13}{29160} \end{pmatrix}.
$$

Computing the matrix $L$ and then solving the eigenvalue problem $L\mathbf{x} = \lambda B_{V_1V_1}\mathbf{x}$ we get three eigenvalues, from which one is zero and the maximal is

$$
\begin{aligned}
\gamma^2 &= \frac{1}{2}(-1876770\delta^4 + 408240\delta^5 + 3241350\delta^2 - 302400\delta^3 - 221480 - 1275750\delta \\
&- 70(6123015450\delta^2 - 1146474000\delta + 1201056660\delta^{10} - 12544369560\delta^9 \\
&+ 49789526310\delta^8 - 95941240560\delta^7 + 108536960880\delta^6 - 82792048860\delta^5 \\
&+ 46950550830\delta^4 - 20176086420\delta^3 + 10010896)^{1/2}) \\
&/ \ (54675\delta^7 - 510300\delta^6 + 1883250\delta^5 - 3231900\delta^4 + 2193075\delta^3 - 226800).
\end{aligned}
$$

In Figure 3 we can see dependence of the strengthened Cauchy-Schwarz constant

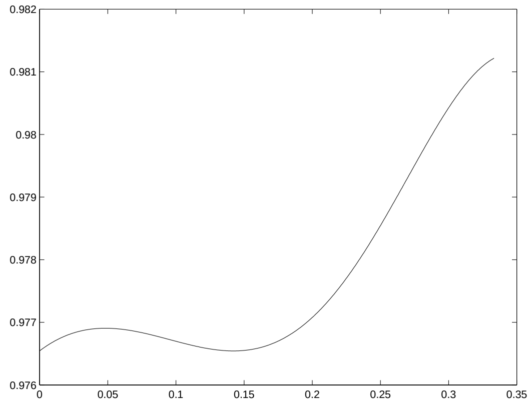


Figure 3: dependence of the strengthened Cauchy-Schwarz constant on the parameter $\delta$.

$\gamma$ on the parameter $\delta$. The maximum is reached for $\delta = 1/3$ and is equal to the forementioned constant. $\square$

**Theorem 6.6.**

$$B_{K_{\frac{1}{3}}}(u,v) \leq \sqrt{\frac{31927}{35680} + \frac{7\sqrt{193953}}{35680}} |||u|||_{K_{\frac{1}{3}}} |||v|||_{K_{\frac{1}{3}}}, \quad \forall u \in V_1, v \in \widetilde{V}_2.$$

*Proof.* This is an immediate consequence of the previous two lemmas. $\square$

**Corollary 6.7.**

$$B(u,v) \leq \sqrt{\frac{31927}{35680} + \frac{7\sqrt{193953}}{35680}} |||u||| \cdot |||v|||, \quad \forall u \in V_1, v \in \widetilde{V}_2.$$

*Proof.* For the proof see (6.2). $\square$

We will also need the Cauchy-Schwarz constant between the space of edge bubble functions and the space of interior bubble functions. Introduce the following notation:

$$\begin{aligned} V_{eb} &= \bigoplus_{\gamma \in \partial \mathcal{T}} span\{b_{\gamma,\delta_\gamma}\}, \\ V_{ib} &= \bigoplus_{K \in \mathcal{T}} span\{b_K\}. \end{aligned}$$

We express the resulting strengthened Cauchy-Schwarz inequality in the following theorem.

**Theorem 6.8.**

$$B_K(u,v) \leq \frac{2\sqrt{2}}{3} |||u|||_K |||v|||_K, \quad \forall u \in V_{eb}, v \in V_{ib}.$$

*Proof.* Prove this inequality for $K_{\frac{1}{3}}$. The statement of the lemma will be a direct consequence.

1. Verify first the inequality for $H_1$ semi-norm and corresponding scalar product. The matrices for computing the strengthened Cauchy-Schwarz constant are now reduced to scalars. Therefore we simply state the constant and estimate it from above.

$$\begin{aligned} \gamma^2 &= \frac{8}{9} - \frac{1}{45}[(40\delta^2 - 729\delta^5)(\epsilon^2 + 4b^2)^2 \\ &+ (4374\delta^5 - 14580\delta^4 + 120 + 11664\delta^3 + 2040\delta^2)(\epsilon^2 + 4b^2) \\ &- 6561\delta^5 + 6120 - 107892\delta^3 + 43740\delta^4 + 116640\delta^2 - 46656\delta] \\ &/ (3 + \delta^2(\epsilon^2 + 4b^2))(\epsilon^2 + 4b^2 + 51) \end{aligned}$$

Observing that for $\delta \in [0, \frac{1}{3}]$

$$\begin{aligned} 40\delta^2 - 729\delta^5 = \delta^2(40 - 729\delta^3) \geq 13\delta^2 &\geq 0 \\ 4374\delta^5 - 14580\delta^4 + 120 + 11664\delta^3 + 2040\delta^2 & \\ = (120 + 7290\delta^3 + 2040\delta^2) + 1458\delta^3(3\delta - 1)(\delta - 3) &\geq 0 \\ -6561\delta^5 + 6120 - 107892\delta^3 + 43740\delta^4 + 116640\delta^2 - 46656\delta & \\ \frac{27}{625}\left\{5\left[213 + 18\sqrt{105} + (1 - 3\delta)(60 + 2\sqrt{105} - 15\delta)\right](1 - 3\delta) + 1755 + 204\sqrt{105}\right\} & \\ \times(15\delta - 15 + \sqrt{105})^2 + \frac{15048}{5} - \frac{36288}{125}\sqrt{105} \geq \frac{15048}{5} - \frac{36288}{125}\sqrt{105} \approx 34.87 &\geq 0 \end{aligned}$$

we conclude

$$(u,v)_{H^1(K_{\frac{1}{3}})} \leq \frac{2\sqrt{2}}{3} |u|_{H^1(K_{\frac{1}{3}})} |v|_{H^1(K_{\frac{1}{3}})}, \quad \forall u \in V_{eb}, v \in V_{ib}. \tag{6.3}$$

2. It remains to verify the inequality for $L_2$ scalar product. In this case the constant of interest has the following structure:

$$\gamma^2 = \frac{3}{112}\delta^3(3\delta^2 - 14\delta + 19)^2.$$

It is easy to verify that the maximum is reached for $\delta = 1/3$ and is equal to $121/567$. Thus,

$$(u,v)_{L_2(K_{\frac{1}{3}})} \leq \sqrt{\frac{121}{567}} \|u\|_{L_2(K_{\frac{1}{3}})} \|v\|_{L_2(K_{\frac{1}{3}})}, \quad \forall u \in V_{eb}, v \in V_{ib}. \tag{6.4}$$

Combining (6.3) and (6.4) and choosing the maximal constant among two we get the result claimed. □

# 7 Hierarchical a posteriori error estimator

Let $V_2 = V_1 \oplus \widetilde{V}_2$. The true error $e$ satisfies the following variational formulation:

$$B(e, v) = (f, v) - B(u_1, v) \quad \forall v \in V.$$

Now let us try to reduce the space $V$ to the space $V_2$, namely consider $e_2$ satisfying

$$B(e_2, v) = (f, v) - B(u_1, v) \quad \forall v \in V_2.$$

It is clear that $e_2 = u_2 - u_1$.

Now by means of the saturation assumption we prove that the approximation of the error $e_2$ is equivalent in the energy norm to the true error $e$.

**Theorem 7.1.**

$$|||e_2||| \le |||e||| \le C m_1(e, \mathcal{T}) \left( |||e_2|||^2 + \sum_{K \in \mathcal{T}} \alpha_K^2 \|r - r_K\|^2_{L_2(K)^2} \right)^{1/2}$$

*Proof.* Let us verify the first inequality:

$$e = u - u_1 = u - u_2 + e_2,$$

and due to orthogonal property we get

$$|||e|||^2 = |||u - u_2|||^2 + |||e_2|||^2,$$

which leads to the first inequality $|||e||| \ge |||e_2|||$.

The second inequality is nothing else, but Lemma 5.1. $\square$

Represent error in the form $e_2 = e_{21} + e_{22}$, where $e_{21} \in V_1$, $e_{22} \in \widetilde{V}_2$, where $e_{21}$, $e_{22}$ satisfy

$$\begin{cases} B(e_{21}, v_1) + B(e_{22}, v_1) = (f, v_1) - B(u_1, v_1) & \forall v_1 \in V_1 \\ B(e_{21}, v_2) + B(e_{22}, v_2) = (f, v_2) - B(u_1, v_2) & \forall v_2 \in \widetilde{V}_2, \end{cases}$$

or,

$$\begin{cases} B(e_{21}, v_1) + B(e_{22}, v_1) = 0 & \forall v_1 \in V_1 \\ B(e_{21}, v_2) + B(e_{22}, v_2) = (f, v_2) - B(u_1, v_2) & \forall v_2 \in \widetilde{V}_2. \end{cases}$$

Ignoring the coupling terms we get

$$\begin{cases} B(\overline{e}_{21}, v_1) = 0 & \forall v_1 \in V_1 \\ B(\overline{e}_{22}, v_2) = (f, v_2) - B(u_1, v_2) & \forall v_2 \in \widetilde{V}_2. \end{cases}$$

From the first equation we immediately get $\overline{e}_{21} = 0$. Denote $\overline{e} := \overline{e}_{22}$. For $\overline{e}$ we have the following equation:

$$B(\overline{e}, v_2) = (f, v_2) - B(u_1, v_2) \quad \forall v_2 \in \widetilde{V}_2.$$

It is useful to know that

$$\begin{cases} B(e_{21}, v_1) + B(e_{22}, v_1) = 0 & \forall v_1 \in V_1 \\ B(e_{21}, v_2) + B(e_{22}, v_2) = B(\overline{e}, v_2) & \forall v_2 \in \widetilde{V}_2, \end{cases}$$

or,

$$\begin{cases} B(e_2, v_1) = 0 & \forall v_1 \in V_1 \\ B(e_2, v_2) = B(\overline{e}, v_2) & \forall v_2 \in \widetilde{V}_2. \end{cases} \tag{7.1}$$

Now by means of strengthened Cauchy-Schwarz inequality we prove that the approximation of the error $\overline{e}$ is equivalent in the energy norm to the approximation $e_2$.

**Theorem 7.2.**

$$|||\overline{e}||| \le |||e_2||| \le \frac{1}{\sqrt{1-\gamma^2}} |||\overline{e}|||,$$

*where* $\gamma = \sqrt{\frac{31927}{35680} + \frac{7\sqrt{193953}}{35680}}$ *is the constant from Corollary 6.7.*

*Proof.* We have

$$\begin{aligned} |||e_2|||^2 &= B(e_{21} + e_{22}, e_{21} + e_{22}) = |||e_{21}|||^2 + 2B(e_{21}, e_{22}) + |||e_{21}|||^2 \\ &\ge |||e_{21}|||^2 - 2\gamma |||e_{21}|||\, |||e_{22}||| + |||e_{22}|||^2. \end{aligned}$$

Utilizing the inequality

$$2\gamma |||e_{21}|||\, |||e_{22}||| \le |||e_{21}|||^2 + \gamma |||e_{22}|||^2,$$

we get

$$|||e_2|||^2 \ge (1-\gamma^2) |||e_{22}|||^2.$$

Applying the first inequality of (7.1) we get

$$|||e_2|||^2 = B(e_2, e_{21}) + B(e_2, e_{22}) = B(e_2, e_{22}).$$

Applying the second inequality of (7.1) we get

$$|||e_2|||^2 = (\overline{e}, e_{22}) \le |||\overline{e}|||\, |||e_{22}||| \le \frac{1}{\sqrt{1-\gamma^2}} |||\overline{e}|||\, |||e_2|||.$$

So we get

$$|||e_2||| \le \frac{1}{\sqrt{1-\gamma^2}} |||\overline{e}|||.$$

Second inequality of the theorem is shown as follows:

$$|||\overline{e}|||^2 = B(\overline{e}, \overline{e}) = B(e_2, \overline{e}) \le |||\overline{e}|||\, |||e_2|||,$$

and thus,

$$|||\overline{e}||| \le |||e_2|||,$$

□

**Definition 7.3 (Error estimator).** *For all triangles $K$ and edges $\gamma$ define the following terms*

$$\begin{aligned} a_\gamma &:= \frac{B(u_1, b_{\gamma,\delta_\gamma}) - \int_\Omega f b_{\gamma,\delta_\gamma}\,dx}{|||b_{\gamma,\delta_\gamma}|||^2} &= -\frac{\int_\gamma R b_{\gamma,\delta_\gamma}\,ds + \int_{\widetilde{\gamma}} r b_{\gamma,\delta_\gamma}\,dx}{|||b_{\gamma,\delta_\gamma}|||^2}, \\ c_K &:= \frac{B(u_1, b_K) - \int_\Omega f b_K\,dx}{|||b_K|||^2} &= -\frac{\int_K r b_K\,dx}{|||b_K|||^2}. \end{aligned}$$

*By means of these terms we define approximation function to the error:*

$$\widetilde{e} := \sum_{\gamma\in\partial\mathcal{T}} a_\gamma b_{\gamma,\delta_\gamma} + \sum_{K\in\mathcal{T}} c_K b_K,$$

$|||\widetilde{e}|||$ *is then the hierarchical a posteriori error estimator.*

Let $v, w \in \widetilde{V}_2$.

$$\begin{aligned} v &= \sum_{\gamma\in\partial\mathcal{T}} v_\gamma b_{\gamma,\delta_\gamma} + \sum_{K\in\mathcal{T}} v_K b_K \\ w &= \sum_{\gamma\in\partial\mathcal{T}} w_\gamma b_{\gamma,\delta_\gamma} + \sum_{K\in\mathcal{T}} w_K b_K. \end{aligned}$$

Define a bilinear form $d(\cdot,\cdot) : \widetilde{V}_2^2 \to \mathbb{R}$ as follows:

$$d(v,w) := \sum_{\gamma\in\partial\mathcal{T}} v_\gamma w_\gamma |||b_{\gamma,\delta_\gamma}|||^2 + \sum_{K\in\mathcal{T}} v_K w_K |||b_K|||^2$$

We need also a local analogue of this bilinear form for any triangle $K$:

$$d_K(v,w) := \sum_{\gamma\subset\partial K} v_\gamma w_\gamma |||b_{\gamma,\delta_\gamma}|||_K^2 + v_K w_K |||b_K|||_K^2$$

This bilinear form has the following properties:

$$d(v,w) = \sum_{K\in\mathcal{T}} d_K(v,w) \tag{7.2}$$

$$d(\widetilde{e}, v) = -B(\overline{e}, v) \quad \forall v \in \widetilde{V}_2. \tag{7.3}$$

The first relation is clear, let us prove the second one. Indeed,

$$\begin{aligned} d(\widetilde{e}, v) &= \sum_{\gamma\in\partial\mathcal{T}} \left[B(u_1, b_{\gamma,\delta_\gamma}) - \int_{\widetilde{\gamma}} f b_{\gamma,\delta_\gamma}\,dx\right] v_\gamma + \sum_{K\in\mathcal{T}} \left[B(u_1, b_K) - \int_K f b_K\,dx\right] v_K \\ &= B(u_1, v) - \int_\Omega f v\,dx = -B(\overline{e}, v). \end{aligned}$$

In subsequent analysis we need a kind of stability property for the bilinear form $d(\cdot,\cdot)$ which we formulate in the following lemma.

**Lemma 7.4.**
$$d(v,v) \le C|||v|||^2 \quad \forall v \in \widetilde{V}_2. \tag{7.4}$$

*Proof.*
$$d(v,v) \le C|||v|||^2 \quad \forall v \in \widetilde{V}_2.$$

First we prove the claimed result locally, namely that for any triangle $K$ the following inequality holds:
$$d_K(v,v) \le C|||v|||_K^2 \quad \forall v \in \widetilde{V}_2,$$
from which the inequality (7.4) evidently follows. Let $v \in \widetilde{V}_2$, then $v|_K$ can be represented in the following way:

$$v|_K = \sum_{\gamma \subset \partial K} v_\gamma b_{\gamma,\delta_\gamma} + v_K b_K.$$

We have

$$\begin{aligned}
|||v|||_K^2 &= B_K\left(\sum_{\gamma\subset\partial K} v_\gamma b_{\gamma,\delta_\gamma} + v_K b_K, \sum_{\gamma\subset\partial K} v_\gamma b_{\gamma,\delta_\gamma} + v_K b_K\right) \\
&= \sum_{\gamma\subset\partial K} v_\gamma^2 |||b_{\gamma,\delta_\gamma}|||_K^2 + v_K^2|||b_K|||_K^2 + 2B_K\left(v_K b_K, \sum_{\gamma\subset\partial K} v_\gamma b_{\gamma,\delta_\gamma}\right) \\
&\ge \sum_{\gamma\subset\partial K} v_\gamma^2 |||b_{\gamma,\delta_\gamma}|||_K^2 + v_K^2|||b_K|||_K^2 - 2\,\gamma|||v_K b_K|||_K \cdot |||\sum_{\gamma\subset\partial K} v_\gamma b_{\gamma,\delta_\gamma}|||_K \\
&\ge (1-\gamma)\left(\sum_{\gamma\subset\partial K} v_\gamma^2 |||b_{\gamma,\delta_\gamma}|||_K^2 + v_K^2|||b_K|||_K^2\right) = (1-\gamma)d_K(v,v),
\end{aligned}$$

where $\gamma$ is the strengthened Cauchy-Schwarz constant from Theorem 6.8. Dividing both sides by $1-\gamma$ we get the result claimed. □

We need also the estimates from above for interior and edge residuals. The following lemma is taken from [10].

**Lemma 7.5 (Interior residual).** *Let $K \in \mathcal{T}$. Then*

$$\|r\|_{L_2(K)} \preceq \varepsilon h_{min,K}^{-1}|||e|||_K + \|r - \overline{r}\|_{L_2(K)}$$

*Proof.* For the proof see Grosman [10]. □

The following lemma is an improved version of the one from [10].

**Lemma 7.6 (Face residual).** *Let $\gamma$ be any interior interface. Then,*

$$\|R\|_{L_2(\gamma)} \;\preceq\; \sum_{K'\in\widetilde{\gamma}} \left\{ \varepsilon^{1/2}\alpha_{K'}^{-1/2}|||e|||_{K'} + \left(\frac{|K'|}{|\gamma|}\right)^{1/2} \delta_\gamma^{1/2}\|r-\overline{r}\|_{L_2(K')} \right\}.$$

*Proof.* Let $v \in H_0^1(\Omega)$. Integrating by parts on each element yields

$$B(e,v) = \sum_{K\in\mathcal{T}} \int_K rv\,dx - \sum_{\gamma\in\partial\mathcal{T}} \int_\gamma Rv\,ds, \tag{7.5}$$

where $\partial\mathcal{T}$ denotes the collection of interelement faces.

Let $\gamma \in \partial\mathcal{T}$. Suppose that $\gamma = \overline{K}_1 \cap \overline{K}_2$. Then $\widetilde{\gamma} = \operatorname{int}(\overline{K}_1 \cup \overline{K}_2)$. Choosing $v := F_{ext}(R)b_{\gamma,\delta_\gamma} \in H_0^1(\Omega)$ in (7.5) implies

$$\int_\gamma b_{\gamma,\delta_\gamma} R^2\,ds = \sum_{K\subset\widetilde{\gamma}} \int_K rF_{ext}(R)b_{\gamma,\delta_\gamma}\,dx - B_{\widetilde{\gamma}}(e, F_{ext}(R)b_{\gamma,\delta_\gamma}).$$

Furthermore, applying the Cauchy-Schwarz inequality, one obtains

$$|B_K(e, F_{ext}(R)b_{\gamma,\delta_\gamma})| \le |||e|||_K |||F_{ext}(R)b_{\gamma,\delta_\gamma}|||_K.$$

Using (3.2) and (3.3) one estimates the second factor as follows:

$$\begin{aligned}|||F_{ext}(R)b_{\gamma,\delta_\gamma}|||_K^2 &= \varepsilon^2\|\nabla(F_{ext}(R)b_{\gamma,\delta_\gamma})\|_{L_2(K)}^2 + \kappa^2\|F_{ext}(R)b_{\gamma,\delta_\gamma}\|_{L_2(K)}^2 \\ &\preceq \left(\varepsilon^2 \min\left\{\delta_\gamma \cdot \frac{|K|}{|\gamma|}, h_{min,K}\right\}^{-2} \delta_\gamma \frac{|K|}{|\gamma|} + \kappa^2\delta_\gamma\frac{|K|}{|\gamma|}\right) \|R\|_{L_2(\gamma)}^2.\end{aligned}$$

Thus, we have

$$\begin{aligned}|B_K(e, F_{ext}(R)b_{\gamma,\delta_\gamma})| \ &\preceq \ \left(\frac{|K|}{|\gamma|}\right)^{1/2} |||e|||_K \|R\|_{L_2(\gamma)} \\ &\quad * \left(\varepsilon \min\left\{\delta_\gamma \cdot \frac{|K|}{|\gamma|}, h_{min,K}\right\}^{-1} \delta_\gamma^{1/2} + \kappa\delta_\gamma^{1/2}\right).\end{aligned}$$

Applying the Cauchy-Schwarz inequality, Lemma 7.5 and (3.2) to the second term we have

$$\begin{aligned}&\left|\int_K rF_{ext}(R)b_{\gamma,\delta_\gamma}\,dx\right| \le \|r\|_{L_2(K)}\|F_{ext}(R)b_{\gamma,\delta_\gamma}\|_{L_2(K)} \\ &\preceq \left[\left(\varepsilon h_{min,K}^{-1} + \kappa\right) |||e|||_K + \|r - \overline{r}\|_{L_2(K)}\right] \delta_\gamma^{1/2} \left(\frac{|K|}{|\gamma|}\right)^{1/2} \|R\|_{L_2(\gamma)}.\end{aligned}$$

Combining two previous estimates we get

$$\begin{aligned}\|R\|_{L_2(\gamma)} \preceq \sum_{K'\in\widetilde{\gamma}} \Bigg\{ &\left(\frac{|K'|}{|\gamma|}\right)^{1/2} \delta_\gamma^{1/2}\|r - \overline{r}\|_{L_2(K')} + \left(\frac{|K'|}{|\gamma|}\right)^{1/2} |||e|||_{K'} \\ &* \left(\varepsilon \min\left\{\delta_\gamma \cdot \frac{|K|}{|\gamma|}, h_{min,K}\right\}^{-1} \delta_\gamma^{1/2} + h_{min,K}^{-1}\delta_\gamma^{1/2} + \kappa\delta_\gamma^{1/2}\right)\Bigg\}.\end{aligned}$$

By simple manipulations we get (with $\delta_\gamma$ from (3.4)),

$$\varepsilon \min\left\{\delta_\gamma \cdot \frac{|K|}{|\gamma|}, h_{min,K}\right\}^{-1} \delta_\gamma^{1/2} + h_{min,K}^{-1}\delta_\gamma^{1/2} + \kappa\delta_\gamma^{1/2} \leq 4\delta_\gamma^{1/2}\alpha_K^{-1},$$

which finishes the proof. □

We are now in a position to formulate a main result of the present paper, namely the robustness of the error estimator.

**Theorem 7.7.** *In foregoing notation and assumptions the following inequalities hold*

$$\begin{aligned} |||e||| &\preceq m_1(e,\mathcal{T})\left(|||\widetilde{e}|||^2 + \sum_{K\in\mathcal{T}} \alpha_K^2 \|r - r_K\|_{L_2(K)}^2\right)^{1/2}, \\ |||\widetilde{e}|||_K &\preceq \left(|||e|||_{\widetilde{K}}^2 + \sum_{K\in\widetilde{K}} \alpha_K^2 \|r - r_K\|_{L_2(K)}^2\right)^{1/2}, \end{aligned}$$

*where $\widetilde{K}$ is a unit of four triangles including $K$ itself and three triangles sharing with $K$ its edges.*

*Proof.* 1. First inequality.

$$\begin{aligned} |||e||| &\preceq m_1(e,\mathcal{T})\left(|||e_2|||^2 + \sum_{K\in\mathcal{T}} \alpha_K^2 \|r - r_K\|_{L_2(K)}^2\right)^{1/2} \\ &\preceq m_1(e,\mathcal{T})\left(\frac{1}{\sqrt{1-\gamma^2}}|||\overline{e}|||^2 + \sum_{K\in\mathcal{T}} \alpha_K^2 \|r - r_K\|_{L_2(K)}^2\right)^{1/2} \end{aligned}$$

It remains to show that $|||\overline{e}||| \leq |||\widetilde{e}|||$. Indeed,

$$|||\overline{e}|||^2 = d(\widetilde{e},\overline{e}) \leq \sqrt{d(\widetilde{e},\widetilde{e})} \cdot \sqrt{d(\overline{e},\overline{e})} \preceq |||\widetilde{e}|||\;|||\overline{e}|||$$

2. Second inequality is obtained more or less straight forward as it is shown below.

$$|||\widetilde{e}|||_K \leq \sum_{\gamma\subset\partial K} |a_\gamma|\;|||b_{\gamma,\delta_\gamma}|||_K + |c_K|\;|||b_K|||_K$$

Estimate first and second term subsequently. Utilize (3.2), (3.3), Lemma 7.5 and Lemma 7.6 subsequently.

$$\begin{aligned} |a_\gamma|\, |||b_{\gamma,\delta_\gamma}|||_K &\leq \frac{\|R\|_{L_2(\gamma)}\|b_{\gamma,\delta_\gamma}\|_{L_2(\gamma)} + \|r\|_{L_2(\widetilde{\gamma})}\|b_{\gamma,\delta_\gamma}\|_{L_2(\widetilde{\gamma})}}{|||b_{\gamma,\delta_\gamma}|||_K} \\ &\preceq \frac{\|R\|_{L_2(\gamma)}}{\kappa\alpha_\gamma^{1/2} + \varepsilon\alpha_\gamma^{-1/2}} + \alpha_\gamma\|r\|_{L_2(\widetilde{\gamma})} \\ &\preceq \varepsilon^{-1/2}\alpha_\gamma^{1/2}\|R\|_{L_2(\gamma)} + \alpha_\gamma\|r\|_{L_2(\widetilde{\gamma})} \\ &\preceq |||e|||_{\widetilde{\gamma}} + \sum_{K\in\widetilde{\gamma}} \alpha_\gamma\|r - r_K\|_{L_2(K)}. \end{aligned}$$

Similarly estimating the second term we get the result claimed.

$$\begin{aligned} |c_K|\, |||b_K|||_K &\leq \frac{\|r\|_{L_2(K)}\|b_K\|_{L_2(K)}}{|||b_K|||_K} \\ &\preceq \alpha_K\|r\|_{L_2(K)} \\ &\preceq |||e|||_K + \alpha_K\|r - r_K\|_{L_2(K)}. \end{aligned}$$

□

Combining two previous estimates we finish the proof.

# 8 Numerical experiments

Let us consider the 2D model problem

$$-\Delta u + \kappa^2 u = 0 \text{ in } \Omega := [0,1]^2, \quad u = u_0 \text{ on } \partial\Omega.$$

Prescribe the exact solution

$$u = e^{-\kappa x} + e^{-\kappa y}$$

which displays typical boundary layers along the sides $x = 0$ and $y = 0$. The Dirichlet boundary data $u_0$ are chosen accordingly.

In the first table we use a sequence of Shishkin meshes with transition parameter $\tau = 2\ln(\kappa)/\kappa$.

The second table has the results for the various transition parameters $\tau = 2\gamma\ln(\kappa)/\kappa$, perturbed from the original value by additional factor $\gamma$. $\gamma \sim 1$ corresponds to the appropriate transition parameter $\tau$. $\gamma \gg 1$ means that the mesh is unnecessarily course, while $\gamma \ll 1$ produces an overrefinement leading to the large values of the matching function $m_1$. We can observe the influence of the matching function to the error estimator and the constant in the saturation assumption. It could be argued that moderate values of the matching function yield the constant in the saturation assumption actually smaller than 1, while the large values destroy the error reduction for enlarged finite element space. Thus it demonstrates that the estimate (5.3) is sharp.

| Mesh | Elements | $\frac{\|\|\|\tilde{e}\|\|\|}{\|\|\|u-u_1\|\|\|}$ | $\frac{\|\|\|u-u_2\|\|\|}{\|\|\|u-u_1\|\|\|}$ | $m_1(u-u_1,\mathcal{T})$ | $\frac{\|\|\|u-u_1\|\|\|}{\|\|\|u\|\|\|}$ |
|---|---|---|---|---|---|
| 1 | 8 | 0.80 | 0.60 | 1.42 | 1.92 |
| 2 | 32 | 0.73 | 0.64 | 1.41 | 1.14 |
| 3 | 128 | 0.64 | 0.69 | 1.42 | 0.65 |
| 4 | 512 | 0.56 | 0.65 | 1.42 | 0.34 |
| 5 | 2048 | 0.50 | 0.55 | 1.42 | 0.17 |
| 6 | 8192 | 0.50 | 0.55 | 1.42 | 0.09 |
| 7 | 32768 | 0.51 | 0.54 | 1.42 | 0.04 |
| 8 | 131072 | 0.51 | 0.54 | 1.43 | 0.02 |

Table 1: Results for $\kappa = 1000$ with transition parameter $\tau = 2\ln(\kappa)/\kappa$.

| $\gamma$ | Mesh | Elements | $\frac{\|\|\|\tilde{e}\|\|\|}{\|\|\|u-u_1\|\|\|}$ | $\frac{\|\|\|u-u_2\|\|\|}{\|\|\|u-u_1\|\|\|}$ | $m_1(u-u_1,\mathcal{T})$ | $\frac{\|\|\|u-u_1\|\|\|}{\|\|\|u\|\|\|}$ |
|---|---|---|---|---|---|---|
| 1 | 4 | 512 | 0.56 | 0.65 | 1.42 | 0.34 |
| 0.1 | 4 | 512 | 0.41 | 0.91 | 16.6 | 0.45 |
| 0.01 | 4 | 512 | 0.40 | 0.97 | 24.8 | 2.56 |
| 10 | 4 | 512 | 0.72 | 0.65 | 1.44 | 2.05 |

Table 2: Results for $\kappa = 1000$ with various transition parameters $\tau = 2\gamma\ln(\kappa)/\kappa$.


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