# Some thermodynamic properties of black body radiation in an n-dimensional Euclidean space


**Jennifer Pérez Oregon, Julián González Ayala, Fernando Angulo Brown.**

**Escuela Superior de Física y Matemáticas del Instituto Politécnico Nacional.**



## Abstract.

In this work we present the generalization of some thermodynamical properties of the black body radiation (BBR) towards a n-dimensional Euclidean space. The Planck function and the Stefan-Boltzmann law has already been given by Landsberg and De Vos [1] and some adjustments by Menon and Agrawal [2]. However, since then no much more has been done on this subject and we believe there are some relevant aspects yet to explore. In addition to the results previously found we calculate the thermodynamical potentials, the efficiency of the Carnot and Otto engines, the law for adiabatic processes and the calorific capacity at constant volume. Finally we consider a possible application to cosmology by making the assumption that the Universe could have "preferred" to be 3-dimensional in space by minimizing or maximizing a thermodynamical property since the very beginning of its existence.


## Generalization to n dimensions.

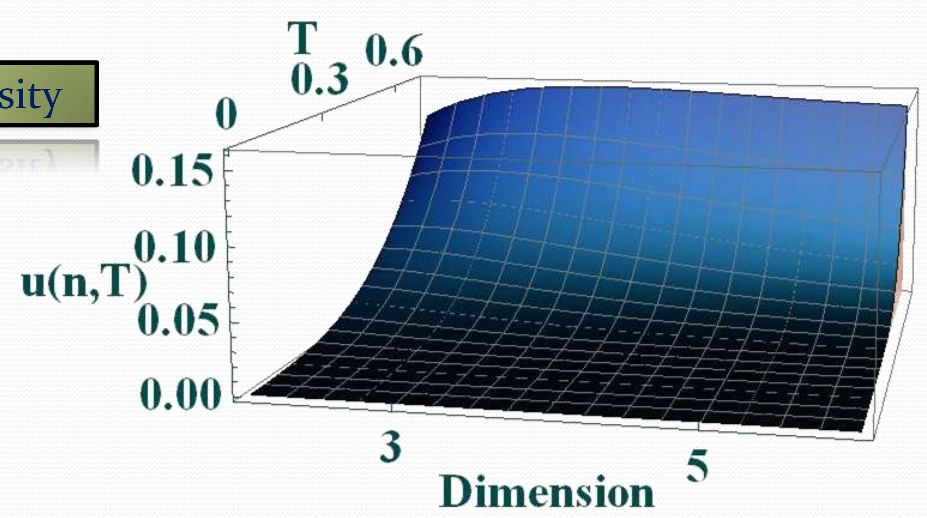
For a black body in an n-dimensional space it is known [1, 2] that the Planck's function for BBR is

$$f(\nu,n,V)\,d\nu = \frac{2(n-1)\pi^{\frac{n}{2}}V\nu^{n-1}}{\Gamma\left(\frac{n}{2}\right)c^n}d\nu, \qquad \overline{n_\nu} = \frac{1}{\exp(\beta h\nu)-1}$$

$$dU_\nu(\nu,T,V,n) = \frac{2(n-1)\pi^{\frac{n}{2}}Vh\nu^n}{\Gamma\left(\frac{n}{2}\right)c^n\left(e^{\frac{h\nu}{kT}}-1\right)}d\nu$$

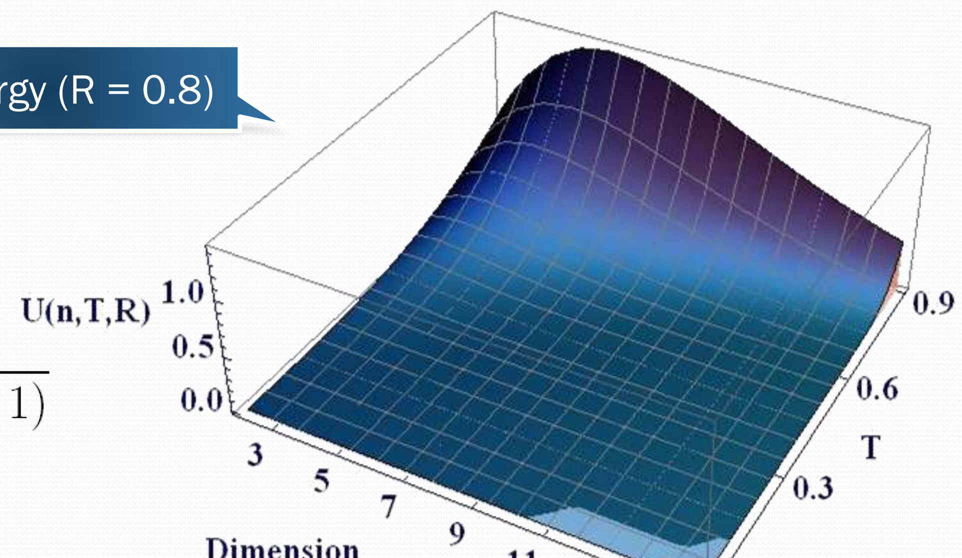
$$u(T,n) = \frac{2(n-1)\pi^{\frac{n}{2}}(kT)^{n+1}\zeta(n+1)\Gamma(n+1)}{c^n h^n\Gamma\left(\frac{n}{2}\right)}$$

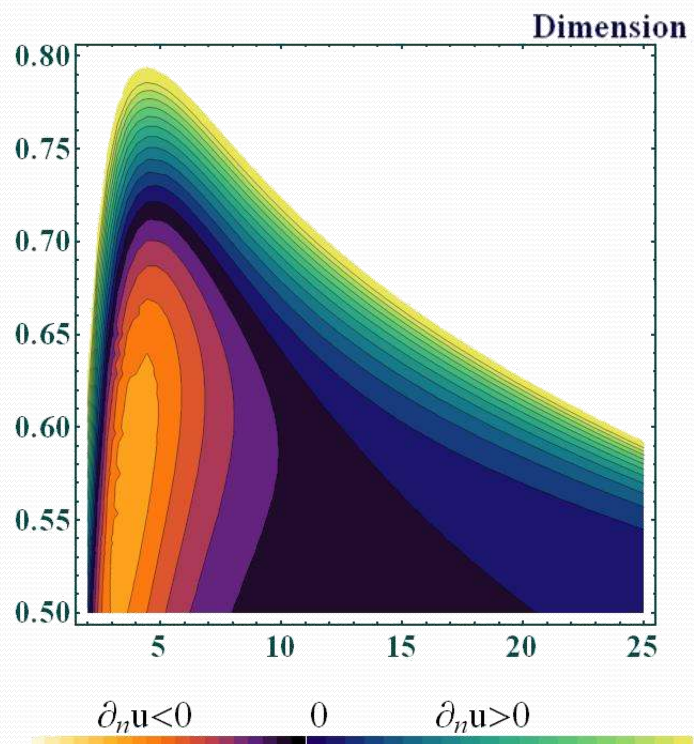
Energy density

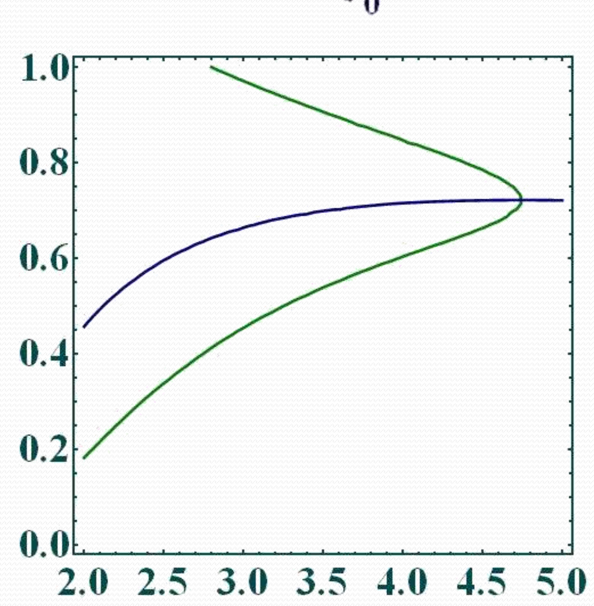



Total Energy (R = 0.8)

$$V_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{1}{2}n+1\right)}$$



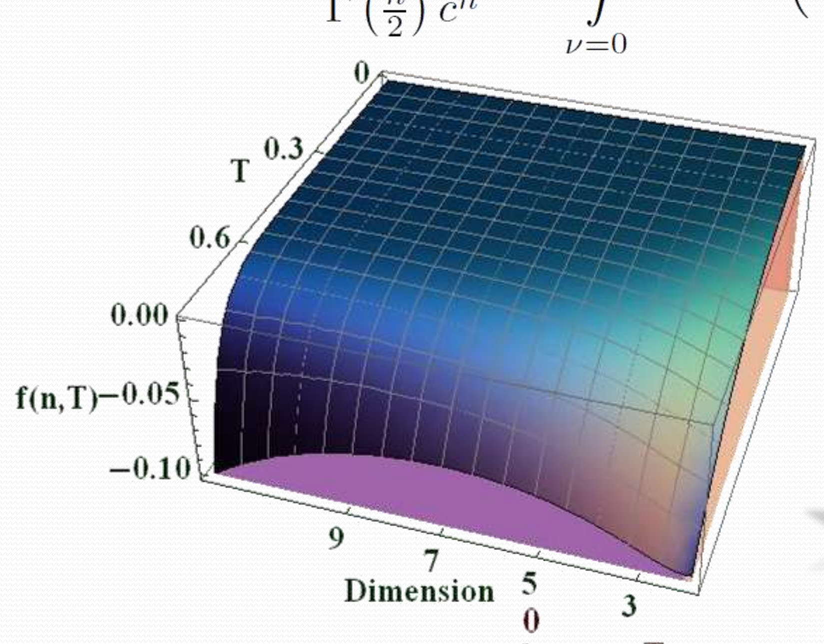



$$F(n,V,T) = -kT\int_{\nu=0}^{\infty} ln\left(Z_{ph}(\nu,T)\right)f(\nu,n,V)\,d\nu$$

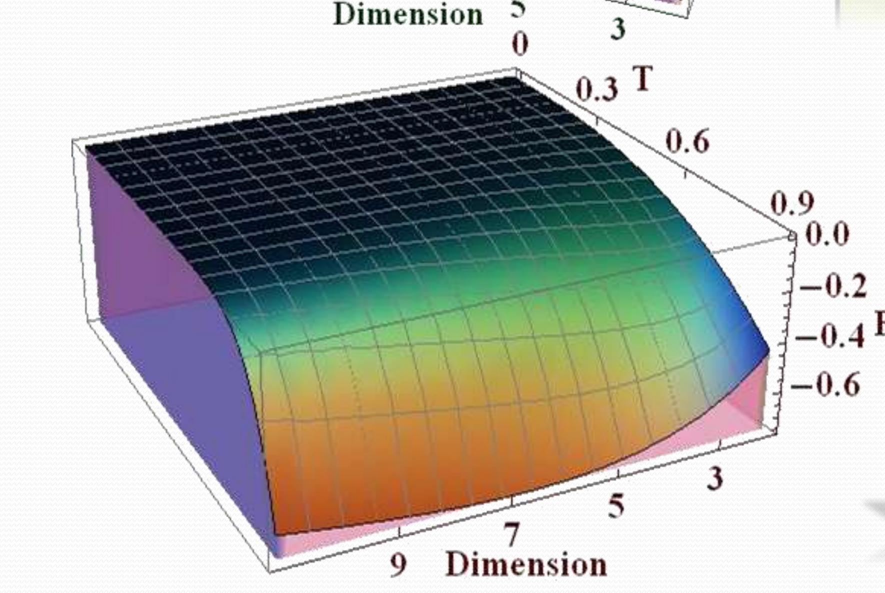
$$Z_{ph}(\nu,T) = \frac{1}{1-e^{-\frac{h\nu}{kT}}}$$

$$F = \frac{kT2(n-1)\pi^{\frac{n}{2}}V}{\Gamma\left(\frac{n}{2}\right)c^n}\int_{\nu=0}^{\infty}\nu^{n-1}ln\left(1-e^{-\frac{h\nu}{kT}}\right)d\nu$$

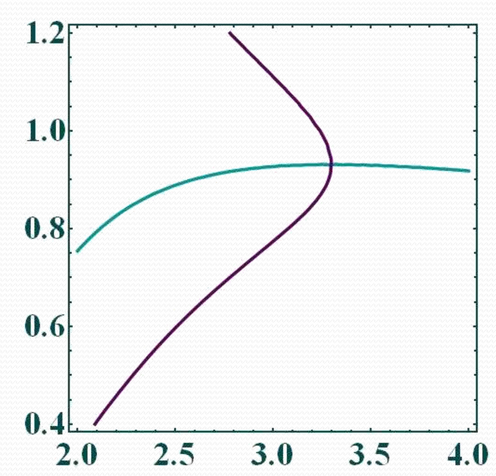
$$F = -\frac{V}{n}u$$




Helmholtz free energy density

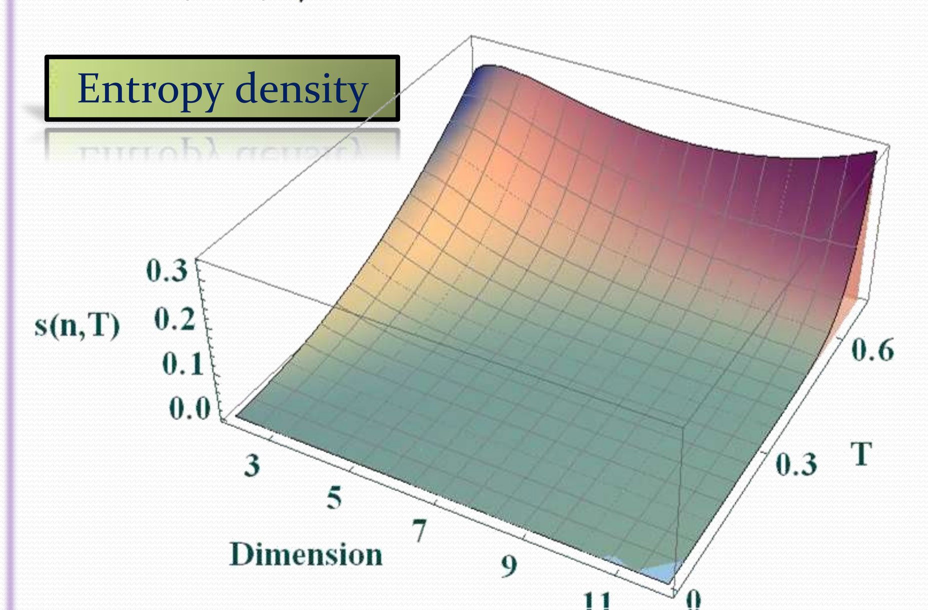



Total Helmholtz free energy

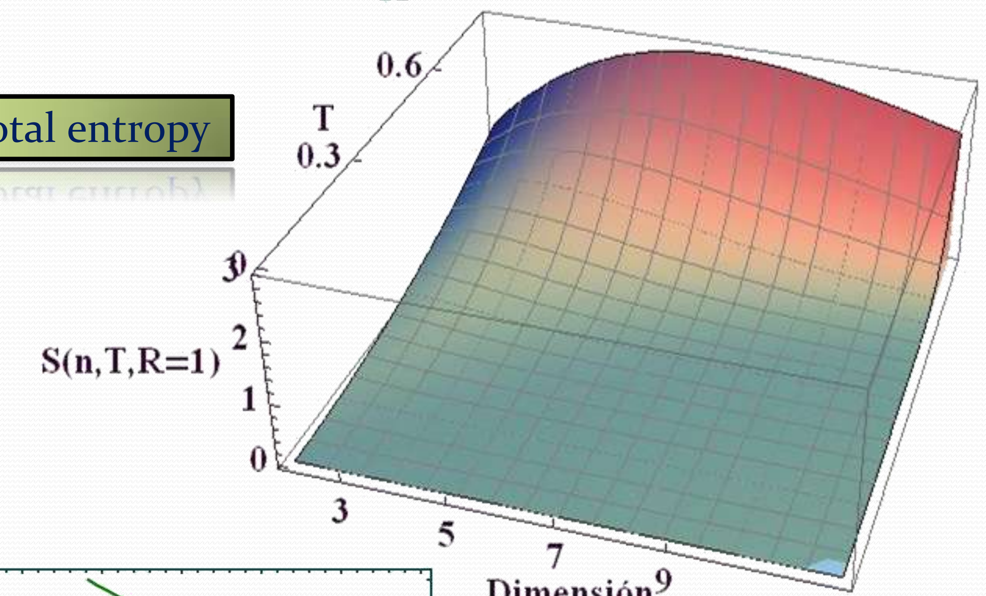


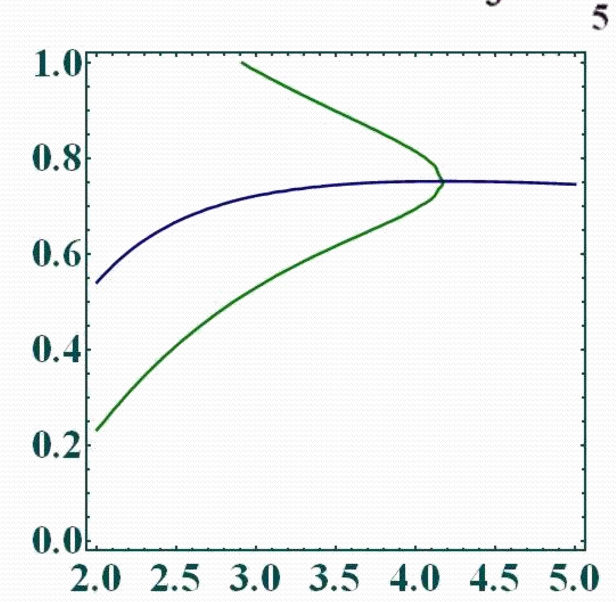
$$S = -\left(\frac{\partial F}{\partial T}\right)_V \longrightarrow S = \frac{V}{n}\left(\frac{\partial u}{\partial T}\right)_V \longrightarrow S = \left(\frac{n+1}{n}\right)\frac{U}{T}$$

Entropy density




Total entropy



$$p = -\left(\frac{\partial F}{\partial V}\right)_T \longrightarrow p = \left(\frac{1}{n}\right)u$$

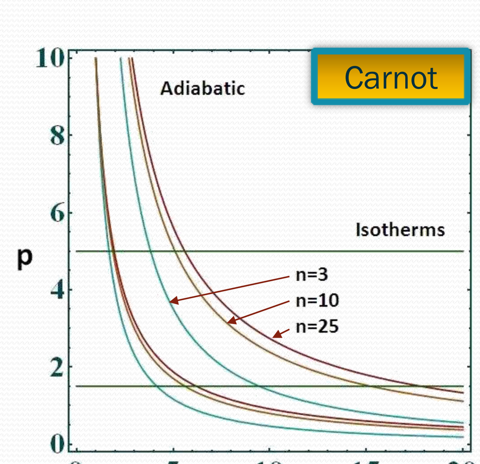
$$H \equiv U + pV \longrightarrow H = \left(\frac{n+1}{n}\right)U$$

$$G \equiv F + pV \longrightarrow G = -\frac{1}{n}U + \frac{1}{n}U = 0$$

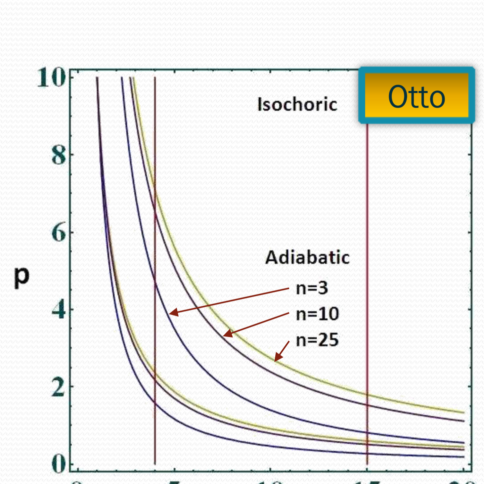
$$\mu_i = \left(\frac{\partial G}{\partial N_i}\right)_{T,p,n_{j\neq i}} = 0$$

Adiabatic processes

$$pV^{\frac{n+1}{n}} = k = cte$$




$$\eta_c = 1 - \frac{T_2}{T_1}$$




$$\eta = 1 - r^{\frac{1}{n}} = 1 - r^{\gamma-1}$$

$$r = \frac{V_2}{V_1} \qquad \gamma = \frac{n+1}{n}$$

## A simple application to cosmology

It is known that at that time the Universe was dominated by the radiation energy density from $t = 10^{-44}s$ to $t \approx 10^{10}s$ . So, considering the whole primeval Universe as a black body in an Euclidean space is a good approach.

**Could a 3-dimensional space be thermodynamically more convenient?**

Table 1: Location of maxima and minima of energy, entropy and Helmholtz free energy densities are restricted to certain dimensionalities.

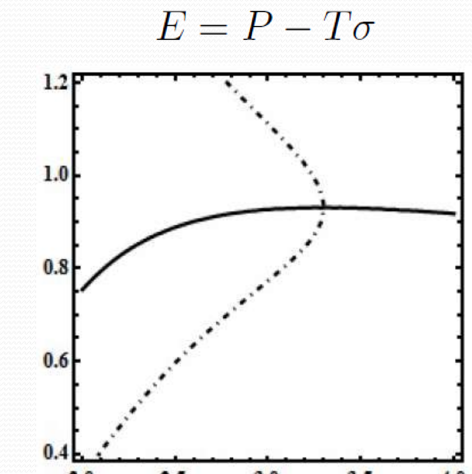
| | $u(n,T)$ | $f(n,T)$ | $s(n,T)$ |
|---|---|---|---|
| Maximum appears at | $n \leq 4$ | $n > 3$ | $n \leq 4$ |
| Mínimum appears at | $n > 4$ | $n \leq 3$ | $n > 4$ |

In some textbooks it is referred that the critical energy density at the time of $t \approx 1t_P$ was about $u \approx 1E_P$ and the temperature aproximatelly $T \approx 1T_P$ . Wich, surprisingly, they are near of the energy density and temperature obtained when the maximum of the energy density is located at $n = 3$.

According to string theories at some point (at the end of the Planck epoch) the rest of the dimensions collapsed and only the 3-dimensional space grew bigger[falta]. The remaining question is why do we live in a 3-dim Universe? We can be reckless at this point and say that perhaps the conditions of energy and temperature were the ideal to maximize the energy density. Thus maybe the Universe preferred to maximize this quantity, or some other, like the Helmholtz Potential.

However a principle of maximization of energy density or entropy is not already established. We suspect that a good candidate as a property to optimize is the so called Ecological function [6]. This function represents a good compromise between high power and low entropy production. It works well to describe a variety of systems (thermal engines, atmospheric convective cells, glucose synthesis, among other processes). This function is defined as

$$E = P - T\sigma$$



The temperature at which there are no maxima or minima is approximately $T \approx .93T_P$ and $n = 3$

Let's suppose that by some unknown mechanism the dimension of space was reduced from a given value to 3. In this scenario a change of energy can be obtained as it is shown below.

| $n_{initial}$ | $E_p$ | $E\ [GeV]$ |
|---|---|---|
| 3 | 0.4922 | 6.02E18 |
| 4 | 0.6578 | 8.04E18 |
| 9 | 2.7356 | 3.34E19 |
| 25 | 15000 | 1.83E23 |

Table 2: Energy density at $T = 0.93$.

| $n_{initial}$ | $n_{final}$ | $\Delta E_p$ | $\%E_{lost}$ |
|---|---|---|---|
| 4 | 3 | 0.16 | 25.177 |
| 9 | 3 | 2.24 | 82 |
| 10 | 3 | 3.42 | 87.42 |
| 25 | 3 | 14999.7 | 99.9967 |

Change of energy at the temperature of $T = 0.93$


## References

[1] **P.T. Landsberg and De Vos** (1989). The Stefan Boltzman constant in an n-dimensional space. Journal of Physics A: Mathematics and General, volume 22, pp 1073-1084.

[2] **V J Menon and D C Agrawal** (1998). Comment on "The Stefan-Boltzmann constant in n-dimensional space". J. Phys. A: Math. Gen. 31, pp 1109-1110.

[3] **F Angulo Brown** (1991). An Ecological Optimization Criterion for Finite-time Heat Engines. Journal of Applied Physics, vol. 69.

[4] **Greiner, Walter; Neise, Ludwig; Stöcker, Horst.** Thermodynamics and Statistical Mechanics. Springer, E.U.A, 1995, pp 130-131.

[5] **Jordi Cepa.** Cosmología Física. Ed. Akal, Madrid España (2007).