Galerkin-based multi-scale time integration for nonlinear structural dynamics

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This paper deals with a GALERKIN-based multi-scale time integration of a viscoelastic rope model. Using HAMILTON's dynamical formulation, NEWTON's equation of motion as a second-order partial differential equation is transformed into two coupled first order partial differential equations in time. The considered finite viscoelastic deformations are described by means of a deformation-like internal variable determined by a first order ordinary differential equation in time. The corresponding multi-scale time-integration is based on a PETROV-GALERKIN approximation of all time evolution equations, leading to a new family of time stepping schemes with different accuracy orders in the state variables. The resulting nonlinear algebraic time evolution equations are solved by a multi-level NEWTON-RAPHSON method. Realizing this transient numerical simulation, we also demonstrates a parallelized solution of the viscous evolution equation in CUDA©.

1 Introduction

We consider finite motions of a viscoelastic rope (cp. KRUPA ET AL. [2] for further details). As shown in Fig. 1, we have to distinguish between the reference configuration $\mathcal{B}_0 \in \mathbb{R}^2$ and the current configuration $\mathcal{B}_t$ at a fixed time $t \geq 0$. Here, $\mathbf{R}$ and $\mathbf{r}$ denote the position vectors in the reference and current configuration, respectively. Considering an open dynamical system with external forces $\overline{\mathbf{q}}$, the local form of NEWTON's equations of motion is given by

$$\varrho(S)A(S)\ddot{\mathbf{r}}(S,t) = \mathbf{N}_{,S}(S,t) + \overline{\mathbf{q}}(S,t)$$
$$\mathbf{N}_{,S}(S,t) = N(S,t)\frac{\mathbf{r}_{,S}(S,t)}{\|\mathbf{r}_{,S}(S,t)\|}$$

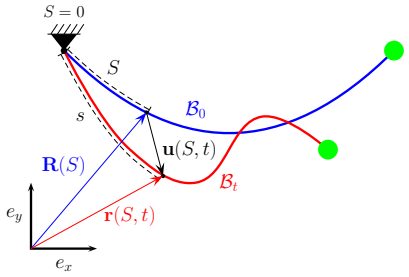

Fig. 1: The viscoelastic rope model

where N describes the viscoelastic constitutive law of the internal forces. According to [4,5], this material model is based on the following viscoelastic coupled free energy function and viscous evolution equation, respectively:

$$\psi(c, c_\mathrm{i}) = \psi^{\mathrm{com}}(c) + \psi^{\mathrm{vis}}(c, c_\mathrm{i}) \qquad \text{and} \qquad \dot{c}_\mathrm{i} = -\frac{4}{V_{\mathrm{vol}}}(c_\mathrm{i})^2 \frac{\partial\, \psi}{\partial\, c_\mathrm{i}}$$

2 Discretization

The equations of motion and the viscous evolution equation are discretized by a space-time FE-approximation (see also [1]). Using the HAMILTONian formulation based on the generalized momentuum $\mathbf{p}$ (see GONZALEZ [3]), the equations of motion arise as a system of first order ordinary differential equations. Denoting by $(\tilde{\bullet})^h$ the approximation of a trial function and by $\delta(\tilde{\bullet})^h$ an approximated test function, we obtain

$$\int_0^T \int_0^L \delta\tilde{\mathbf{p}}^h(S,t) \cdot \left[\dot{\tilde{\mathbf{r}}}^h(S,t) - \frac{1}{\rho(S)A(S)}\tilde{\mathbf{p}}^h(S,t)\right] \mathrm{d}S\,\mathrm{d}t = 0$$
$$\int_0^T \int_0^L \left[\delta\tilde{\mathbf{r}}^h_{,S}(S,t) \cdot \mathbf{N}(S,t) + \delta\tilde{\mathbf{r}}^h(S,t) \cdot \left[\dot{\tilde{\mathbf{p}}}^h(S,t) - \overline{\mathbf{q}}(S,t)\right]\right] \mathrm{d}S\,\mathrm{d}t = 0\,.$$

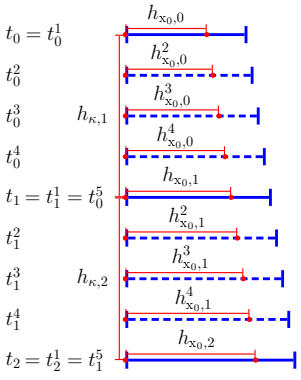

Fig. 2: The space-time FE-mesh

At this point, we apply the BUBNOV-GALERKIN method with LAGRANGE finite elements in space. LAGRANGE's shape functions are also used for the discretization in time, but here we utilize the PETROV-GALERKIN method for an exact fulfillment of the inital conditions. In Fig. 2, an exemplarily obtained space-time FE-mesh is shown. Each integral is transformed to the corresponding reference element, where the interval $[-1,1]$ denotes the reference element in space and $[0,1]$ the reference element in time. The time step size in the physical mesh is designated by $h_{\kappa,i}$.

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The differences $\Delta t_1 := t_1 - t_0 = h_{\kappa,1}$ and $\Delta t_2 := t_2 - t_1 = h_{\kappa,2}$ are so-called macro time steps. However, for higher order shape functions in time, we also arrive at micro time steps like t_0^2, t_0^3 or t_0^4. Starting from t_0, we calculate all micro time steps simultaneously in each macro time step. According to this, the start solution for the next macro time step is the last micro time step of this first macro time step. For instance, we show in Fig. 2 an approximation with a polynomial degree of four.

3 Parallelization

After a consistent linearization, the obtained system of nonlinear algebraic equations is solved by a multi-level NEWTON-RAPHSON method. The global part of the NEWTON-RAPHSON method is the calculation of the equations of motion. For every spatial element, the entries of the tangent matrix can be calculated in parallel. The local part of the NEWTON-RAPHSON method consists of the discrete viscous evolution equation. This calculation is also processed in parallel with CUDA©. As expected, the CPU time difference between the sequential C code and the parallelized CUDA© code depends on the number of spatial elements. For a small number of spatial elements, the C code is faster due to the overhead caused by the data transfer between CPU and GPU. For a relatively large number of elements, the CUDA© code reveals an increased computing performance compared to the C code. The speedup for the calculation of the whole system is about 12% for a chosen number of elements from 500 to about 8000 elements. This relatively small speedup results from the fact that the calculation of the global linear equation system is more time consuming than the parallelized parts. In detail, we performed the calculations on an INTEL© I7 3820 CPU and a NVIDIA© Tesla C2075 GPU.

4 Numerical example

We denote by k_r and k_p the polynomial degrees in the discrete equations of motion, and by k_ci the polynomial degree in the discrete viscous evolution equation. Our example demonstrates the convergence order of the position vector at time $T = 20\,\mathrm{s}$. First, we consider equal polynomial degrees. As it is obvious in Fig. 3, the theoretical convergence order is reached by $2 \cdot k_i \ \forall i \in [\mathrm{r}, \mathrm{p}, \mathrm{ci}]$. It is worth pointing out that larger time steps may be used for reaching the same relative error ε_rel. Now, we show the results for different polynomial degrees. With a fixed polynomial degree $k_\mathrm{ci} = 3$ and variating degrees in the equations of motion, the convergence order for the position vector is shown in Fig. 4. The important point to note here is that the convergence order of the position vector is not bounded by a lower approximation of the evolution equation. The convergence reaches a value of about 8 for the equation of motion with $k_\mathrm{r} = k_\mathrm{p} = 4$ and reaches a value of about 10 with $k_\mathrm{r} = k_\mathrm{p} = 5$.

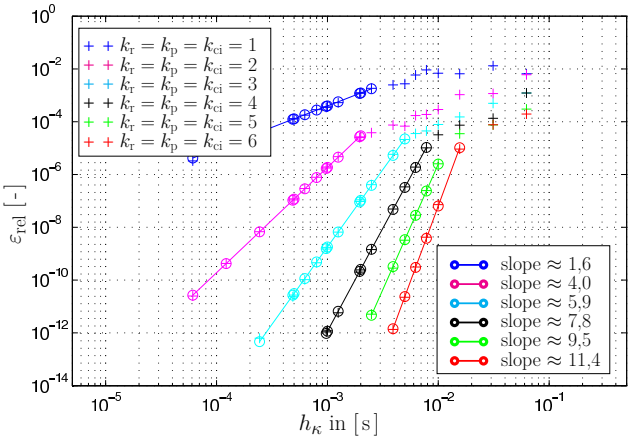

Fig. 3: Convergence plot for equal polynomial degrees

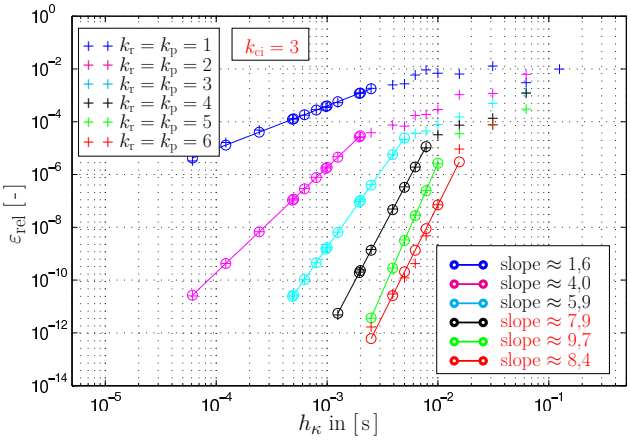

Fig. 4: Convergence plot for unequal polynomial degrees

References

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