Recent advances in balancing-related model reduction of circuit equations

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Joint work with Timo Reis

Model Reduction in Circuit Simulation, Hamburg, 30-31.10.2008

SyreNe: System reduction for nanoscale IC design

BMBF Research Network: 1.07.2007 – 30.06.2010 http://www.syrene.org

TP1: M. Hinze, M. Kunkel, M. Vierling (Universität Hamburg)
Model order reduction for coupled systems of ICs

TP2: H. Faßbender, J.P. Amorocho Duran (TU Braunschweig)
Passivity-preserving model reduction for nonlinear DAEs

TP3: T. Stykel, M.-S. Hossain (TU Berlin)
Element-based model reduction in circuit simulation

TP4: P. Benner, A. Schneider (TU Chemnitz)
Reduced representation of power grid models

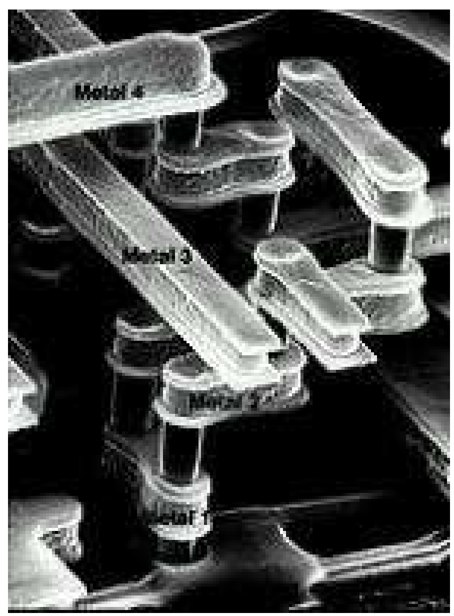
TP5: P. Lang, O. Schmidt (ITWM Kaiserslautern)
Coupling of numeric/symbolic reduction techniques for the generation of parameterized models of nanoelectronic systems

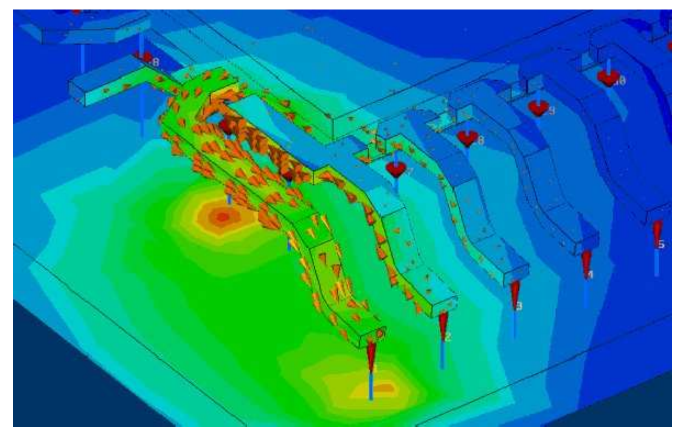
TP6: M. Bollhöfer, A. Eppler (TU Braunschweig)
Numerical solution of systems of equations and coupling of components in model order reduction

IP: Infineon Technologies AG, Qimonda AG, NEC Laboratories Europe

Integrated circuits

	Intel® Core^{TM}2 Processors
increasing chip complexity	$> 4 \cdot 10^8$ transistors
decreasing feature size	65 nm – 45 nm
increasing operating frequency	1.06 GHz – 3.33 GHz
increasing interconnect length	10 km
multilayer structure	9 layers
modelling thermic and electromagnetic effects	





Outline

- Differential-algebraic equations in circuit simulation
- Model order reduction problem
- Balanced truncation for DAEs
- Passivity-preserving balanced truncation method
- Application to circuit equations
- Numerical solution of Riccati and Lyapunov equations
- Numerical example
- Conclusion

Linear circuit equations

Consider a linear DAE system

$$\begin{aligned} E\,\dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + Du(t) \end{aligned}$$

with

$$E = \begin{bmatrix} A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T & 0 & 0 \\ 0 & \mathcal{L} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T & -A_{\mathcal{L}} & -A_{\mathcal{V}} \\ A_{\mathcal{L}}^T & 0 & 0 \\ A_{\mathcal{V}}^T & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -A_{\mathcal{I}} & 0 \\ 0 & 0 \\ 0 & -I \end{bmatrix} = C^T,$$

$$u(t) = \begin{bmatrix} j_{\mathcal{I}}(t) \\ v_{\mathcal{V}}(t) \end{bmatrix} \in \mathbb{R}^m, \qquad x(t) = \begin{bmatrix} \eta(t) \\ j_{\mathcal{L}}(t) \\ j_{\mathcal{V}}(t) \end{bmatrix} \in \mathbb{R}^n, \qquad y(t) = \begin{bmatrix} v_{\mathcal{I}}(t) \\ j_{\mathcal{V}}(t) \end{bmatrix} \in \mathbb{R}^m,$$

$\eta(t)$ – node potentials,

$j_{\mathcal{L}}(t),\ j_{\mathcal{V}}(t),\ j_{\mathcal{I}}(t)$ – currents through inductors, voltage and current sources,

$v_{\mathcal{V}}(t),\ v_{\mathcal{I}}(t)$ – voltages at voltage and current sources,

$A_{\mathcal{R}}, A_{\mathcal{C}}, A_{\mathcal{L}}, A_{\mathcal{V}}, A_{\mathcal{I}}$ – incidence matrices of resistors, capacitors, inductors, voltage and current sources,

$\mathcal{R},\ \mathcal{C},\ \mathcal{L}$ – resistance, capacitance and inductance matrices.

Assumptions

$$\begin{bmatrix} A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T & 0 & 0 \\ 0 & \mathcal{L} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\eta} \\ \dot{j}_{\mathcal{L}} \\ \dot{j}_{\mathcal{V}} \end{bmatrix} = \begin{bmatrix} -A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T & -A_{\mathcal{L}} & -A_{\mathcal{V}} \\ A_{\mathcal{L}}^T & 0 & 0 \\ A_{\mathcal{V}}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \eta \\ j_{\mathcal{L}} \\ j_{\mathcal{V}} \end{bmatrix} - \begin{bmatrix} A_{\mathcal{I}} & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} i_{\mathcal{I}} \\ v_{\mathcal{V}} \end{bmatrix}$$

- $A_{\mathcal{V}}$ has full column rank
- $[\, A_{\mathcal{C}},\, A_{\mathcal{L}},\, A_{\mathcal{R}},\, A_{\mathcal{V}} \,]$ has full row rank
- $\mathcal{R}$, $\mathcal{L}$, $\mathcal{C}$ are symmetric, positive definite

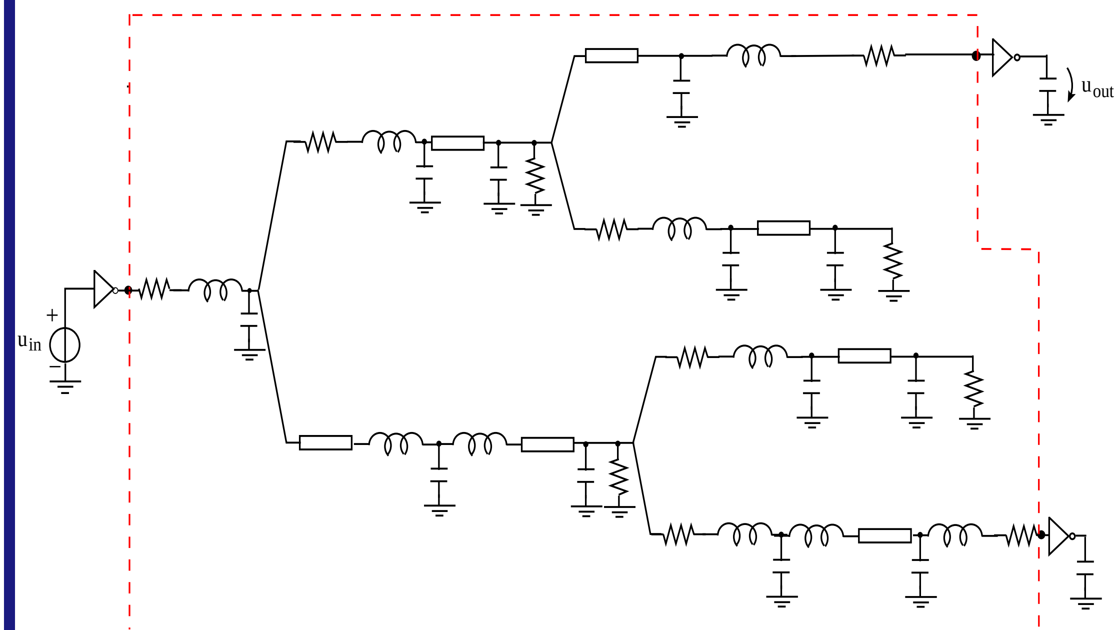
$\Rightarrow$ $\lambda E - A$ is regular ($\det(\lambda E - A) \neq 0$)

$\Rightarrow$ system is passive (= does not generate energy)

$\Rightarrow$ system is reciprocal ($\boldsymbol{G}(s) = C(sE - A)^{-1}B = \Sigma\, \boldsymbol{G}^T(s)\, \Sigma$
with $\Sigma = \mathrm{diag}(I_{n_{\mathcal{I}}},\, -I_{n_{\mathcal{V}}})$)

Linear RLC circuits

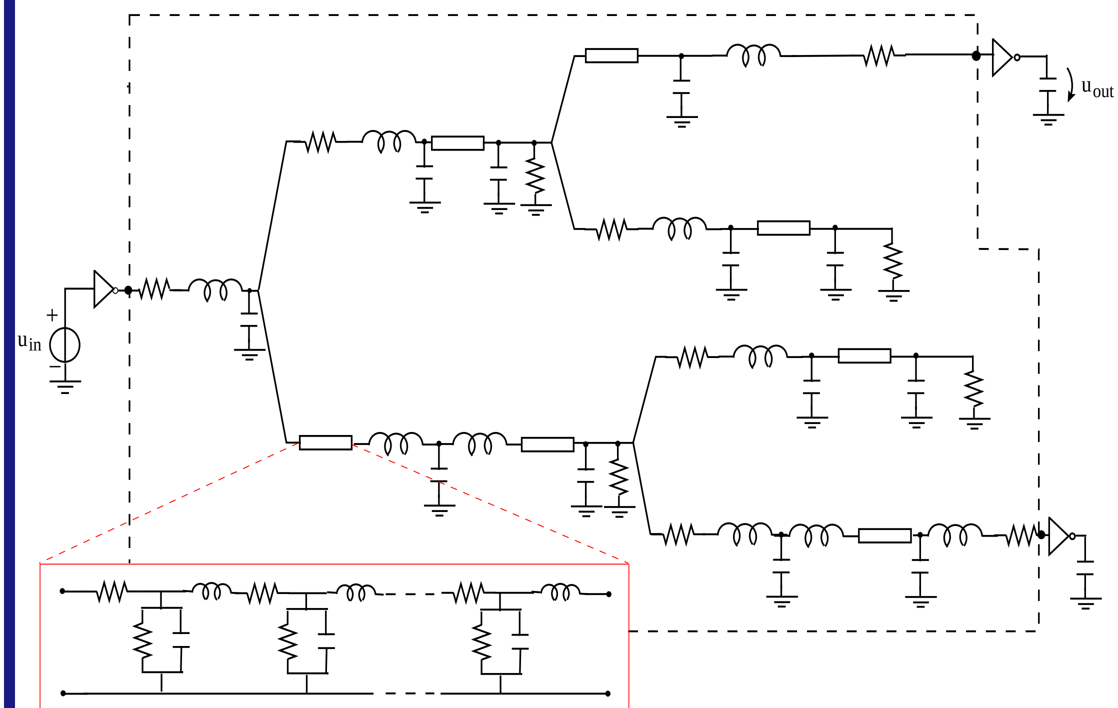
$$\begin{bmatrix} A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T & 0 & 0 \\ 0 & \mathcal{L} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\eta} \\ \dot{j}_{\mathcal{L}} \\ \dot{j}_{\mathcal{V}} \end{bmatrix} = \begin{bmatrix} -A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T & -A_{\mathcal{L}} & -A_{\mathcal{V}} \\ A_{\mathcal{L}}^T & 0 & 0 \\ A_{\mathcal{V}}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \eta \\ j_{\mathcal{L}} \\ j_{\mathcal{V}} \end{bmatrix} - \begin{bmatrix} A_{\mathcal{I}} & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} i_{\mathcal{I}} \\ v_{\mathcal{V}} \end{bmatrix}$$



- decoupling large linear subcircuits

Linear RLC circuits

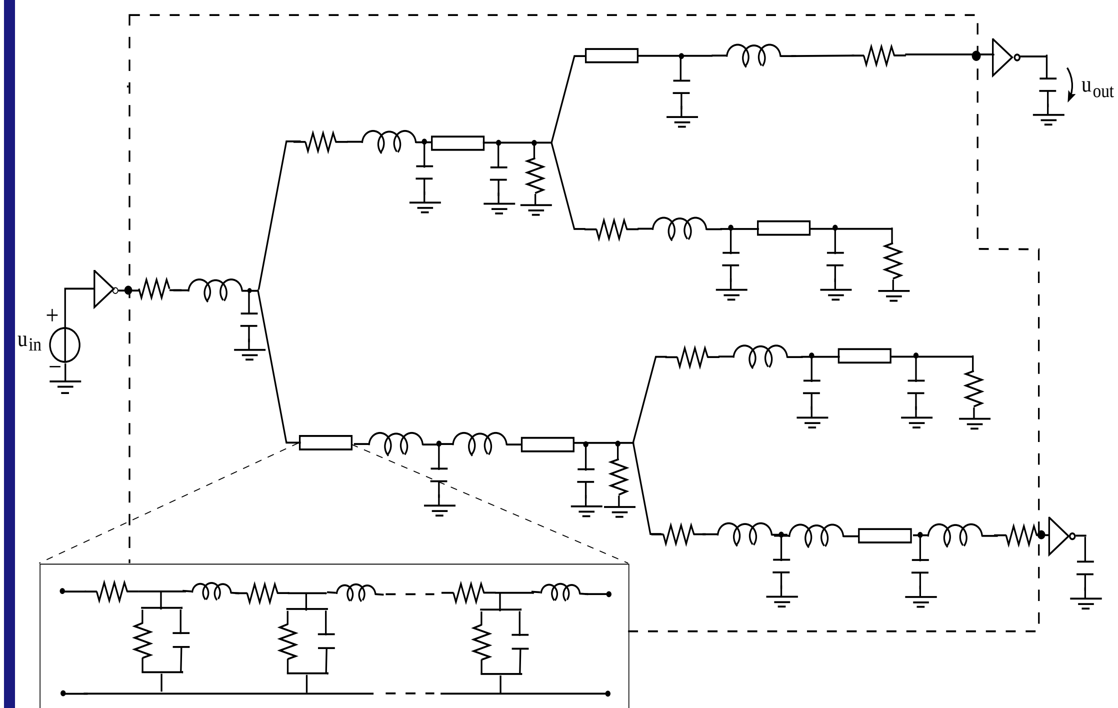
$$\begin{bmatrix} A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T & 0 & 0 \\ 0 & \mathcal{L} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\eta} \\ \dot{j}_{\mathcal{L}} \\ \dot{j}_{\mathcal{V}} \end{bmatrix} = \begin{bmatrix} -A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T & -A_{\mathcal{L}} & -A_{\mathcal{V}} \\ A_{\mathcal{L}}^T & 0 & 0 \\ A_{\mathcal{V}}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \eta \\ j_{\mathcal{L}} \\ j_{\mathcal{V}} \end{bmatrix} - \begin{bmatrix} A_{\mathcal{I}} & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} i_{\mathcal{I}} \\ v_{\mathcal{V}} \end{bmatrix}$$



- decoupling large linear subcircuits
- modelling transmission lines and pin packages

Linear RLC circuits

$$\begin{bmatrix} A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T & 0 & 0 \\ 0 & \mathcal{L} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\eta} \\ \dot{j}_{\mathcal{L}} \\ \dot{j}_{\mathcal{V}} \end{bmatrix} = \begin{bmatrix} -A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T & -A_{\mathcal{L}} & -A_{\mathcal{V}} \\ A_{\mathcal{L}}^T & 0 & 0 \\ A_{\mathcal{V}}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \eta \\ j_{\mathcal{L}} \\ j_{\mathcal{V}} \end{bmatrix} - \begin{bmatrix} A_{\mathcal{I}} & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} i_{\mathcal{I}} \\ v_{\mathcal{V}} \end{bmatrix}$$



- decoupling large linear subcircuits
- modelling transmission lines and pin packages
- modelling circuits elements by Maxwell's equations via partial element equivalent circuits (PEEC)
- small-signal analysis

Model reduction problem

Given a large-scale system

$$E\,\dot{x}(t) = A\,x(t) + B\,u(t),$$
$$y(t) = C\,x(t) + D\,u(t)$$

with $E, A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{p,n}$, $D \in \mathbb{R}^{p,m}$, $n \gg m, p$,

find a reduced-order system

$$\tilde{E}\,\dot{\tilde{x}}(t) = \tilde{A}\,\tilde{x}(t) + \tilde{B}\,u(t),$$
$$\tilde{y}(t) = \tilde{C}\,\tilde{x}(t) + \tilde{D}\,u(t)$$

with $\tilde{E}, \tilde{A} \in \mathbb{R}^{\ell,\ell}$, $\tilde{B} \in \mathbb{R}^{\ell,m}$, $\tilde{C} \in \mathbb{R}^{p,\ell}$, $\tilde{D} \in \mathbb{R}^{p,m}$, $\ell \ll n$.

- preservation of passivity and reciprocity
  - $\hookrightarrow$ synthesis is possible [Reis'08] $\rightsquigarrow$ also Ionutiu's talk
- small approximation error

  $\|\tilde{\boldsymbol{G}} - \boldsymbol{G}\| \le tol$ or $\|\tilde{y} - y\| \le tol \cdot \|u\|$ for all $u \in \mathcal{U}$
  - $\hookrightarrow$ need for computable error bounds
- numerically stable and efficient methods

Model reduction techniques

- Krylov subspace methods (moment matching)
  - passivity-preserving methods

    SyPVL for RC, RL, LC circuits [Freund et al.'96,'97]

    PRIMA, SPRIM, Laguerre-SVD for RLC circuits

    [Odabasioglu et al.'96,'97; Freund'04,'05; Knockaert et al.'00]

    Positive real interpolation [Antoulas'05, Sorensen'05, Ionutiu et al.'08]
  - no global computable error bounds – only local error bounds and error expressions [Bai et al.'99, Gugercin'03]

- Balancing-related model reduction
  - physical properties (stability, passivity, ...) are preserved
  - there exist global computable error bounds
  - numerical solution of Lyapunov / Riccati / Lur'e matrix equations

Balanced truncation: idea

System $\boldsymbol{G} = (\,E,\ A,\ B,\ C,\ D\,)$ is balanced if the controllability and observability Gramians $\mathcal{G}_c$ and $\mathcal{G}_o$ satisfy

$$\mathcal{G}_c = \mathcal{G}_o = \mathrm{diag}(\sigma_1, \ldots, \sigma_n).$$

Idea: balance the system, i.e., find an equivalence transformation

$$\begin{aligned}(\,\hat{E},\ \hat{A},\ \hat{B},\ \hat{C},\ \hat{D}\,) &= (\,W_b E T_b,\ \ W_b A T_b,\ \ W_b B,\ \ C T_b,\ \ D\,)\\ &= \left(\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix},\ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\ \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ [C_1,\ C_2],\ D \right)\end{aligned}$$

such that $\hat{\mathcal{G}}_c = \hat{\mathcal{G}}_o = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ and truncate the states corresponding to small σ_j $\quad\hookrightarrow\quad$ $\widetilde{\boldsymbol{G}} = (\,E_{11},\ A_{11},\ B_1,\ C_1,\ D\,)$.

Generalized Lyapunov equations

- E – nonsingular

  The controllability and observability Gramians $\mathcal{G}_c$ and $\mathcal{G}_o$ solve the generalized Lyapunov equations (GLEs)

  $$E\,\mathcal{G}_c A^T + A\,\mathcal{G}_c E^T = -BB^T, \quad E^T\mathcal{G}_o\,A + A^T\mathcal{G}_o E = -C^TC.$$

  $\hookrightarrow$ σ_j – Hankel singular values

- E – singular
  - The GLEs may have no solutions even if $\lambda_j(E,A) \in \mathbb{C}^-$

    $$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = C^T.$$

  - If a solution exists, it is always nonunique ($X + vv^T$ with $v \in \ker E$ and $Y + ww^T$ with $w \in \ker E^T$).

Projectors

Weierstraß canonical form:

$$E = T_l \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} T_r, \qquad A = T_l \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} T_r,$$

where J – Jordan block ($\lambda_j(J)$ are finite eigenvalues of $\lambda E - A$),
N – nilpotent ($N^{\nu-1} \neq 0$, $N^\nu = 0$ $\rightsquigarrow$ ν is index of $\lambda E - A$).

$$\Longrightarrow \quad P_r = T_r^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_r, \qquad P_l = T_l \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_l^{-1}, \qquad \begin{matrix} Q_r = I - P_r, \\ Q_l = I - P_l \end{matrix}$$

$\Longrightarrow$ $\boldsymbol{G}(s) = C(sE - A)^{-1}B + D = \boldsymbol{G}_{sp}(s) + \boldsymbol{P}(s)$, where

$\boldsymbol{G}_{sp}(s) = C_1(sI - J)^{-1}B_1$ – strictly proper ($\lim\limits_{s\to\infty} \boldsymbol{G}_{sp}(s) = 0$),

$$\boldsymbol{P}(s) = C_2(sN - I)^{-1}B_2 + D = -\sum_{k=0}^{\nu-1} C_2 N^k B_2 s^k + D$$

Proper and improper Gramians

- The proper controllability and observability Gramians $\mathcal{G}_{pc}$ and $\mathcal{G}_{po}$ solve the projected continuous-time Lyapunov equations

$$E\,\mathcal{G}_{pc}A^T + A\,\mathcal{G}_{pc}E^T = -P_l BB^T P_l^T, \qquad \mathcal{G}_{pc} = P_r\mathcal{G}_{pc}P_r^T,$$
$$E^T\mathcal{G}_{po}\,A + A^T\mathcal{G}_{po}E = -P_r^T C^T C P_r, \qquad \mathcal{G}_{po} = P_l^T\mathcal{G}_{po}P_l.$$

- The improper controllability and observability Gramians $\mathcal{G}_{ic}$ and $\mathcal{G}_{io}$ solve the projected discrete-time Lyapunov equations

$$A\,\mathcal{G}_{ic}\,A^T - E\,\mathcal{G}_{ic}E^T = \;Q_l BB^T Q_l^T, \qquad \mathcal{G}_{ic} = Q_r\mathcal{G}_{ic}Q_r^T,$$
$$A^T\mathcal{G}_{io}\,A - E^T\mathcal{G}_{io}\,E = \;Q_r^T C^T C Q_r, \qquad \mathcal{G}_{io} = Q_l^T\mathcal{G}_{io}Q_l.$$

- System $\boldsymbol{G} = (\,E,\,A,\,B,\,C,\,D\,)$ is balanced if

$$\mathcal{G}_{pc} = \mathcal{G}_{po} = \mathrm{diag}(\sigma_1,\ldots,\sigma_{n_f},0,\;\ldots,\;0),$$
$$\mathcal{G}_{ic} = \mathcal{G}_{io} \;=\; \mathrm{diag}(0,\;\ldots,\;0,\;\theta_1,\ldots,\theta_{n_\infty}).$$

- σ_j and θ_j are the proper and improper Hankel singular values

Balanced truncation method

1. Compute

$$\mathcal{G}_{pc} = R_p R_p^T, \quad \mathcal{G}_{po} = L_p L_p^T, \quad \mathcal{G}_{ic} = R_i R_i^T, \quad \mathcal{G}_{io} = L_i L_i^T;$$

2. Compute the SVD $L_p^T E R_p = [U_1, U_2] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} [V_1, V_2]^T$;

3. Compute the SVD $L_i^T A R_i = [U_3, U_4] \begin{bmatrix} \Theta & \\ & 0 \end{bmatrix} [V_3, V_4]^T$;

4. $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) = (W^T E T, W^T A T, W^T B, C T, D)$ with

$$W = [L_p U_1 \Sigma_1^{-1/2}, L_i U_3 \Theta^{-1/2}], \quad T = [R_p V_1 \Sigma_1^{-1/2}, R_i V_3 \Theta^{-1/2}].$$

Error bound:

$$\|\widetilde{y} - y\|_{\mathbb{L}_2} \leq \|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|_{\mathbb{H}_\infty} \|u\|_{\mathbb{L}_2} \leq 2(\sigma_{\ell_f+1} + \ldots + \sigma_{n_f}) \|u\|_{\mathbb{L}_2}$$

Example

$$N\dot{x}(t) = x(t) + Bu(t)$$
$$y(t) = Cx(t)$$

with $N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 10 \\ 0.1 \\ 0 \end{bmatrix}, \quad C^T = \begin{bmatrix} 0.04 \\ 30 \\ 1 \end{bmatrix}$

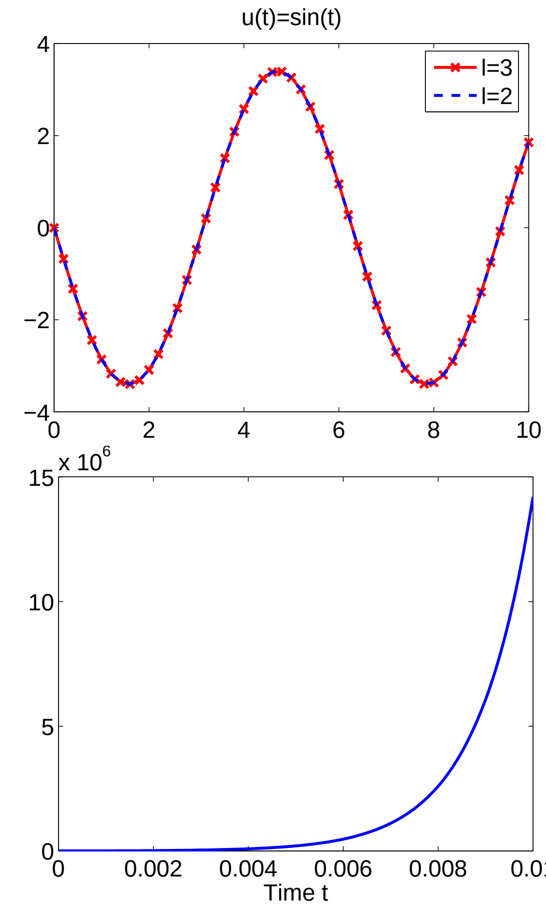
Improper Hankel singular values $\theta_1 = 3.4,\ \theta_2 = 4.7 \cdot 10^{-6},\ \theta_3 = 0$

- Reduced-order system: $\ell = 2$

$$\begin{bmatrix} 1.2 & 1.2 \\ -1.2 & -1.2 \end{bmatrix} \dot{\tilde{x}}(t) = \begin{bmatrix} 10^3 & 0 \\ 0 & 10^3 \end{bmatrix} \tilde{x}(t) + \tilde{B}u(t)$$
$$\tilde{y}(t) = \tilde{C}\,\tilde{x}(t)$$

- Reduced-order system: $\ell = 1$

$$\dot{\tilde{x}}(t) = 850\,\tilde{x}(t) + 1567u(t)$$
$$\tilde{y}(t) = 1.9\,\tilde{x}(t)$$



Passivity and contractivity

- $\boldsymbol{G} = (E, A, B, C, D)$ is passive (= does not generate energy)

  $\Longleftrightarrow \quad \boldsymbol{G}(s) = C(sE - A)^{-1}B + D$ is positive real
  - $\boldsymbol{G}(s)$ is analytic in $\mathbb{C}^+$
  - $\boldsymbol{G}(s) + \boldsymbol{G}^T(\overline{s}) \geq 0$ for all $s \in \mathbb{C}^+$

- $\boldsymbol{G} = (E, A, B, C, D)$ is contractive ($\|y\|_{\mathbb{L}_2} \leq \|u\|_{\mathbb{L}_2}$)

  $\Longleftrightarrow \quad \boldsymbol{G}(s) = C(sE - A)^{-1}B + D$ is bounded real
  - $\boldsymbol{G}(s)$ is analytic in $\mathbb{C}^+$
  - $I - \boldsymbol{G}(s)\,\boldsymbol{G}^T(\overline{s}) \geq 0$ for all $s \in \mathbb{C}^+$

- $\boldsymbol{G}(s)$ is positive real (bounded real) if and only if

  $\boldsymbol{\mathcal{G}}(s) = \big(I - \boldsymbol{G}(s)\big)\big(I + \boldsymbol{G}(s)\big)^{-1}$ is bounded real (positive real).

Passivity-preserving BT

passive
$\boldsymbol{G} = (\,E,\ A,\ B,\ C,\ 0\,)$

positive real

→

balanced truncation

passive
$\tilde{\boldsymbol{G}} = (\,\tilde{E},\ \tilde{A},\ \tilde{B},\ \tilde{C},\ 0\,)$

Passivity-preserving BT

passive

$$\boldsymbol{G} = (\,E,\ A,\ B,\ C,\ 0\,)$$

$$\boldsymbol{\mathcal{G}}(s) = \big(I - \boldsymbol{G}(s)\big)\big(I + \boldsymbol{G}(s)\big)^{-1}$$

contractive

$$\boldsymbol{\mathcal{G}} = (\,\mathcal{E},\ \mathcal{A},\ \mathcal{B},\ \mathcal{C},\ I\,)$$

$$\begin{aligned} \mathcal{E} &= E \\ \mathcal{A} &= A - B\,C \\ \mathcal{B} &= -\sqrt{2}\,B \\ \mathcal{C} &= \sqrt{2}\,C \end{aligned}$$

bounded real

balanced truncation

contractive

$$\tilde{\boldsymbol{\mathcal{G}}} = (\,\tilde{\mathcal{E}},\ \tilde{\mathcal{A}},\ \tilde{\mathcal{B}},\ \tilde{\mathcal{C}},\ I\,)$$

$$\begin{aligned} \tilde{\mathcal{E}} &= W^T \mathcal{E}\, T \\ \tilde{\mathcal{A}} &= W^T \mathcal{A}\, T \\ \tilde{\mathcal{B}} &= W^T \mathcal{B} \\ \tilde{\mathcal{C}} &= \mathcal{C}\, T \end{aligned}$$

$$\tilde{\boldsymbol{G}}(s) = \big(I - \tilde{\boldsymbol{\mathcal{G}}}(s)\big)\big(I + \tilde{\boldsymbol{\mathcal{G}}}(s)\big)^{-1}$$

passive

$$\tilde{\boldsymbol{G}} = (\,\tilde{E},\ \tilde{A},\ \tilde{B},\ \tilde{C},\ 0\,)$$

$$\begin{aligned} \tilde{E} &= \tilde{\mathcal{E}} \\ \tilde{A} &= \tilde{\mathcal{A}} - \tfrac{1}{2}\tilde{\mathcal{B}}\tilde{\mathcal{C}} \\ \tilde{B} &= -\tfrac{\sqrt{2}}{2}\,\tilde{\mathcal{B}} \\ \tilde{C} &= \tfrac{\sqrt{2}}{2}\,\tilde{\mathcal{C}} \end{aligned}$$

Projected Lur'e equations

- If $\boldsymbol{G} = (E, A, B, C, 0)$ is passive, then $\boldsymbol{\mathcal{G}}(s) = \big(I - \boldsymbol{G}(s)\big)\big(I + \boldsymbol{G}(s)\big)^{-1}$ is proper and there exist matrices $\mathcal{X} = \mathcal{X}^T \geq 0$, $\mathcal{J}_c$, $\mathcal{K}_c$ and $\mathcal{Y} = \mathcal{Y}^T \geq 0$, $\mathcal{J}_o$, $\mathcal{K}_o$ that satisfy the projected Lur'e equations

$$\begin{aligned} &(A - BC)\,\mathcal{X}E^T + E\mathcal{X}(A - BC)^T + 2\mathcal{P}_l BB^T\mathcal{P}_l^T = -2\mathcal{K}_c\mathcal{K}_c^T, \quad \mathcal{X} = \mathcal{P}_r\mathcal{X}\mathcal{P}_r^T, \\ &E\,\mathcal{X}C^T - \mathcal{P}_l B\mathcal{M}_0^T = -\mathcal{K}_c\mathcal{J}_c^T, \qquad I - \mathcal{M}_0\mathcal{M}_0^T = \mathcal{J}_c\mathcal{J}_c^T, \end{aligned}$$

$$\begin{aligned} &(A - BC)^T\mathcal{Y}\,E + E^T\mathcal{Y}(A - BC) + 2\mathcal{P}_r^T C^T C\mathcal{P}_r = -2\mathcal{K}_o^T\mathcal{K}_o, \quad \mathcal{Y} = \mathcal{P}_l^T\mathcal{Y}\mathcal{P}_l, \\ &-B^T\mathcal{Y}\,E + \mathcal{M}_0^T C\mathcal{P}_r = -\mathcal{J}_o^T\mathcal{K}_o, \qquad I - \mathcal{M}_0^T\mathcal{M}_0 = \mathcal{J}_o^T\mathcal{J}_o, \end{aligned}$$

  where $\mathcal{M}_0 = \lim\limits_{s\to\infty}\boldsymbol{\mathcal{G}}(s) = I - 2\lim\limits_{s\to\infty} C(sE - A + BC)^{-1}B$.

- $0 \leq \mathcal{X}_{\min} \leq \mathcal{X} \leq \mathcal{X}_{\max}, \qquad 0 \leq \mathcal{Y}_{\min} \leq \mathcal{Y} \leq \mathcal{Y}_{\max}$

  $\mathcal{X}_{\min}$ is bounded real controllability Gramian of $\boldsymbol{\mathcal{G}}$

  $\mathcal{Y}_{\min}$ is bounded real observability Gramian of $\boldsymbol{\mathcal{G}}$

Passivity-preserving BT method

Given a passive system $\boldsymbol{G} = (\,E,\ A,\ B,\ C,\ 0\,)$.

1. Compute $\mathcal{M}_0 = I - 2C(I - \mathcal{P}_r)(s_0 E - A + BC)^{-1}(I - \mathcal{P}_l)B$;

2. Compute $\mathcal{X}_{\min} = RR^T$, $\mathcal{Y}_{\min} = LL^T$ (= solve the Lur'e equations);

3. Compute the SVD $L^T E R = [U_1,\ U_2]\begin{bmatrix} \Pi_1 & \\ & \Pi_2 \end{bmatrix}[V_1,\ V_2]^T$;

4. Compute

$$\tilde{\mathcal{E}} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},\ \tilde{\mathcal{A}} = \begin{bmatrix} W^T(A - BC)T & 0 \\ 0 & I \end{bmatrix},\ \tilde{\mathcal{B}} = \begin{bmatrix} -\sqrt{2}W^T B \\ B_\infty \end{bmatrix},\ \tilde{\mathcal{C}}^T = \begin{bmatrix} \sqrt{2}(CT)^T \\ C_\infty^T \end{bmatrix}$$

with $I - \mathcal{M}_0 = C_\infty B_\infty$, $W = LU_1\Pi_1^{-1/2}$ and $T = RV_1\Pi_1^{-1/2}$;

5. Compute $\tilde{\boldsymbol{G}} = [\,\tilde{E},\ \tilde{A},\ \tilde{B},\ \tilde{C},\ 0\,] = [\,\tilde{\mathcal{E}},\ \tilde{\mathcal{A}} - \frac{1}{2}\tilde{\mathcal{B}}\tilde{\mathcal{C}},\ -\frac{\sqrt{2}}{2}\tilde{\mathcal{B}},\ \frac{\sqrt{2}}{2}\tilde{\mathcal{C}},\ 0\,]$.

Application to circuit equations

$$E=\begin{bmatrix} A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T & 0 & 0 \\ 0 & \mathcal{L} & 0 \\ 0 & 0 & 0 \end{bmatrix},\; A-BC=\begin{bmatrix} -A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T-A_{\mathcal{I}}A_{\mathcal{I}}^T & -A_{\mathcal{L}} & -A_{\mathcal{V}} \\ A_{\mathcal{L}}^T & 0 & 0 \\ A_{\mathcal{V}}^T & 0 & -I \end{bmatrix},\; B=\begin{bmatrix} -A_{\mathcal{I}} & 0 \\ 0 & 0 \\ 0 & -I \end{bmatrix}=C^T$$

- Compute $\mathcal{P}_r$ and $\mathcal{P}_l$ using the canonical projectors technique [März'96]

$$\hookrightarrow \quad \mathcal{P}_r = \begin{bmatrix} H_4(Q_{\mathcal{C}}H_3^{-1}H_2-I) & H_4Q_{\mathcal{C}}H_3^{-1}A_{\mathcal{L}}H_5 & 0 \\ 0 & H_5 & 0 \\ -A_{\mathcal{V}}^T(Q_{\mathcal{C}}H_3^{-1}H_2-I) & -A_{\mathcal{V}}^TQ_{\mathcal{C}}CH_3^{-1}A_{\mathcal{L}}H_5 & 0 \end{bmatrix}$$

with $H_1 = P_{\mathcal{CRIV}}^TP_{\mathcal{CRIV}}+Q_{\mathcal{CRIV}}^TA_{\mathcal{L}}\mathcal{L}^{-1}A_{\mathcal{L}}^TQ_{\mathcal{CRIV}}$, $H_2=\ldots$, $H_3=\ldots$, $H_4=\ldots$,

$H_5 = I-\mathcal{L}^{-1}A_{\mathcal{L}}^TQ_{\mathcal{CRIV}}H_1^{-1}Q_{\mathcal{CRIV}}^TA_{\mathcal{L}}$, $Q_{\mathcal{C}}$ is a projector onto $\ker A_{\mathcal{C}}^T$,

$Q_{\mathcal{CRIV}}$ is a projector onto $\ker[A_{\mathcal{C}},\, A_{\mathcal{R}},\, A_{\mathcal{I}},\, A_{\mathcal{V}}]^T$, $P_{\mathcal{CRIV}} = I - Q_{\mathcal{CRIV}}$.

$$\hookrightarrow \quad \mathcal{P}_l = S\,\mathcal{P}_r^T S^T \quad \text{with} \quad S = \text{diag}(I_{n_\eta}, -I_{n_{\mathcal{L}}}, -I_{n_{\mathcal{V}}})$$

Application to circuit equations

$$E=\begin{bmatrix} A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T & 0 & 0 \\ 0 & \mathcal{L} & 0 \\ 0 & 0 & 0 \end{bmatrix},\; A-BC=\begin{bmatrix} -A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T-A_{\mathcal{I}}A_{\mathcal{I}}^T & -A_{\mathcal{L}} & -A_{\mathcal{V}} \\ A_{\mathcal{L}}^T & 0 & 0 \\ A_{\mathcal{V}}^T & 0 & -I \end{bmatrix},\; B=\begin{bmatrix} -A_{\mathcal{I}} & 0 \\ 0 & 0 \\ 0 & -I \end{bmatrix}=C^T$$

- Compute $\mathcal{P}_r$ and $\mathcal{P}_l = S\,\mathcal{P}_r^T S^T$ with $S = \text{diag}(I_{n_{\mathcal{R}}}, -I_{n_{\mathcal{L}}}, -I_{n_{\mathcal{V}}})$.
- Compute $\mathcal{M}_0 = I - 2C(I-\mathcal{P}_r)(s_0E-A+BC)^{-1}(I-\mathcal{P}_l)B$

$$\hookrightarrow \quad \mathcal{M}_0 = \begin{bmatrix} I-2A_{\mathcal{I}}^TQ_{\mathcal{C}}H_0^{-1}Q_{\mathcal{C}}^TA_{\mathcal{I}} & 2A_{\mathcal{I}}^TQ_{\mathcal{C}}H_0^{-1}Q_{\mathcal{C}}^TA_{\mathcal{V}} \\ -2A_{\mathcal{V}}^TQ_{\mathcal{C}}H_0^{-1}Q_{\mathcal{C}}^TA_{\mathcal{I}} & -I+2A_{\mathcal{V}}^TQ_{\mathcal{C}}H_0^{-1}Q_{\mathcal{C}}^TA_{\mathcal{V}} \end{bmatrix}$$

where $H_0 = Q_{\mathcal{C}}^T(A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T + A_{\mathcal{I}}A_{\mathcal{I}}^T + A_{\mathcal{V}}A_{\mathcal{V}}^T)Q_{\mathcal{C}} + Q_{\mathcal{RIV}-\mathcal{C}}^TQ_{\mathcal{RIV}-\mathcal{C}}$,

$Q_{\mathcal{RIV}-\mathcal{C}}$ is a projector onto $\ker\,[\,A_{\mathcal{R}},\ A_{\mathcal{I}},\ A_{\mathcal{V}}\,]^TQ_{\mathcal{C}}$.

Application to circuit equations

- Compute $\mathcal{M}_0$, $\mathcal{P}_r$, $\mathcal{P}_l = S\,\mathcal{P}_r^T S^T$ with $S = \mathrm{diag}(I_{n_\mathcal{R}}, -I_{n_\mathcal{L}}, -I_{n_\mathcal{V}})$.
- Compute $\mathcal{Y}_{\min} = LL^T$, $\mathcal{X}_{\min} = RR^T$ (solve the projected Lur'e equations)

  If $D_0 = I - \mathcal{M}_0^T \mathcal{M}_0$ is nonsingular, then the projected Lur'e equations are equivalent to the projected Riccati equation

$$(A-BC)^T\mathcal{Y}E + E^T\mathcal{Y}(A-BC) + 2\,\mathcal{P}_r^T C^T C\,\mathcal{P}_r$$
$$+\,2(B^T\mathcal{Y}\,E - \mathcal{M}_0^T C\mathcal{P}_r)^T D_0^{-1}(B^T\mathcal{Y}\,E - \mathcal{M}_0^T C\mathcal{P}_r) = 0, \quad \mathcal{Y} = P_l^T\mathcal{Y}P_l$$

$\hookrightarrow$ compute a low-rank approximation $\mathcal{Y}_{\min} \approx \tilde{L}\tilde{L}^T$, $\tilde{L} \in \mathbb{R}^{n,r}$, $r \ll n$, using the generalized low-rank Newton method [Benner/St.'08]

$\hookrightarrow$ $\mathcal{X}_{\min} = S\,\mathcal{Y}_{\min}\,S^T \approx S\tilde{L}\tilde{L}^T S^T = \tilde{R}\tilde{R}^T$

Application to circuit equations

- Compute $\mathcal{M}_0$, $\mathcal{P}_r$, $\mathcal{P}_l = S\,\mathcal{P}_r^T S^T$ with $S = \text{diag}(I_{n_{\mathcal{R}}}, -I_{n_{\mathcal{L}}}, -I_{n_{\mathcal{V}}})$.
- Compute $\mathcal{Y}_{\min} \approx \tilde{L}\tilde{L}^T$, $\mathcal{X}_{\min} = S\,\mathcal{Y}_{\min}\,S^T \approx S\tilde{L}\tilde{L}^T S^T = \tilde{R}\tilde{R}^T$.
- Compute the SVD of $\tilde{L}^T E \tilde{R}$

$\hookrightarrow$ $\tilde{L}^T E \tilde{R} = \tilde{L}^T E S \tilde{L}$ is symmetric

$\hookrightarrow$ compute the EVD $\tilde{L}^T E S \tilde{L} = [\,U_1, U_2\,]\text{diag}(\Lambda_1, \Lambda_2)[\,U_1, U_2\,]^T$ instead of the SVD

$\hookrightarrow$ $|\lambda_j| = \pi_j$

$\hookrightarrow$ $W = \tilde{L}U_1|\Lambda_1|^{-1/2}$ and $T = S\tilde{L}U_1|\Lambda_1|^{-1/2}\text{sign}(\Lambda_1)$

Application to circuit equations

- Compute $\mathcal{M}_0$, $\mathcal{P}_r$, $\mathcal{P}_l = S\,\mathcal{P}_r^T S^T$ with $S = \text{diag}(I_{n_\mathcal{R}}, -I_{n_\mathcal{L}}, -I_{n_\mathcal{V}})$.
- Compute $\mathcal{Y}_{\min} \approx \tilde{L}\tilde{L}^T$, $\mathcal{X}_{\min} = S\,\mathcal{Y}_{\min}\,S^T \approx S\tilde{L}\tilde{L}^T S^T = \tilde{R}\tilde{R}^T$.
- Compute the EVD $\tilde{L}^T E S \tilde{L} = [\,U_1,\,U_2\,]\text{diag}(\Lambda_1,\Lambda_2)[\,U_1,\,U_2\,]^T$ and $W = \tilde{L}U_1|\Lambda_1|^{-1/2}$, $T = S\tilde{L}U_1|\Lambda_1|^{-1/2}\text{sign}(\Lambda_1)$.
- Compute B_∞ and C_∞ such that $C_\infty B_\infty = I - \mathcal{M}_0$
  - $\hookrightarrow$ $(I - \mathcal{M}_0)\Sigma$ with $\Sigma = \text{diag}(I_{n_\mathcal{I}}, -I_{n_\mathcal{V}})$ is symmetric
  - $\hookrightarrow$ compute the EVD $(I - \mathcal{M}_0)\Sigma = U_0\Lambda_0 U_0^T$
  - $\hookrightarrow$ $B_\infty = \text{sign}(\Lambda_0)|\Lambda_0|^{1/2}U_0^T\Sigma$ and $C_\infty = U_0|\Lambda_0|^{1/2}$

Application to circuit equations

- Compute $\mathcal{M}_0$, $\mathcal{P}_r$, $\mathcal{P}_l = S\,\mathcal{P}_r^T S^T$ with $S = \text{diag}(I_{n_\mathcal{R}}, -I_{n_\mathcal{L}}, -I_{n_\mathcal{V}})$.
- Compute $\mathcal{Y}_{\min} \approx \tilde{L}\tilde{L}^T$, $\mathcal{X}_{\min} = S\,\mathcal{Y}_{\min}\,S^T \approx S\tilde{L}\tilde{L}^T S^T = \tilde{R}\tilde{R}^T$.
- Compute the EVD $\tilde{L}^T E S \tilde{L} = [\,U_1,\,U_2\,]\text{diag}(\Lambda_1, \Lambda_2)[\,U_1,\,U_2\,]^T$ and
  $W = \tilde{L}U_1|\Lambda_1|^{-1/2}$, $T = S\tilde{L}U_1|\Lambda_1|^{-1/2}\text{sign}(\Lambda_1)$.
- Compute the EVD $(I - \mathcal{M}_0)\Sigma = U_0\Lambda_0 U_0^T$ and
  $B_\infty = \text{sign}(\Lambda_0)|\Lambda_0|^{1/2}U_0^T\Sigma$, $C_\infty = U_0|\Lambda_0|^{1/2}$.
- Compute the reduced-order model

$$\tilde{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \qquad \tilde{A} = \frac{1}{2}\begin{bmatrix} 2\,W^T A T & \sqrt{2}\,W^T B C_\infty \\ -\sqrt{2}\,B_\infty C T & 2I - B_\infty C_\infty \end{bmatrix},$$

$$\tilde{B} = \frac{\sqrt{2}}{2}\begin{bmatrix} \sqrt{2}W^T B \\ -B_\infty \end{bmatrix}, \qquad \tilde{C} = \frac{\sqrt{2}}{2}\begin{bmatrix} \sqrt{2}CT, & C_\infty \end{bmatrix}.$$

Properties

- $\tilde{\boldsymbol{G}} = (\,\tilde{E},\ \tilde{A},\ \tilde{B},\ \tilde{C},\ 0\,)$ is passive
- $\tilde{\boldsymbol{G}} = (\,\tilde{E},\ \tilde{A},\ \tilde{B},\ \tilde{C},\ 0\,)$ is reciprocal ($\tilde{\boldsymbol{G}}(s) = \Sigma\,\tilde{\boldsymbol{G}}^T(s)\Sigma$)
- Error bounds: $\|\boldsymbol{G}\|_{\mathbb{H}_\infty} := \sup\limits_{\omega\in\mathbb{R}} \|\boldsymbol{G}(i\omega)\|_2$
  - If $2\,\|I + \boldsymbol{G}\|_{\mathbb{H}_\infty}(\pi_{\ell+1} + \ldots + \pi_{n_f}) < 1$, then

    $$\|\tilde{\boldsymbol{G}} - \boldsymbol{G}\|_{\mathbb{H}_\infty} \le 2\,\|I + \boldsymbol{G}\|^2_{\mathbb{H}_\infty}(\pi_{\ell+1} + \ldots + \pi_{n_f}).$$

  - If $2\,\|I + \tilde{\boldsymbol{G}}\|_{\mathbb{H}_\infty}(\pi_{\ell+1} + \ldots + \pi_{n_f}) < 1$, then

    $$\|\tilde{\boldsymbol{G}} - \boldsymbol{G}\|_{\mathbb{H}_\infty} \le 2\,\|I + \tilde{\boldsymbol{G}}\|^2_{\mathbb{H}_\infty}(\pi_{\ell+1} + \ldots + \pi_{n_f}).$$

Projected Riccati equation

$$\mathcal{R}(Y) \equiv \hat{A}^T Y E + E^T Y \hat{A} + \mathcal{P}_r^T \hat{C}^T \hat{C}\,\mathcal{P}_r + E^T Y \hat{B}\hat{B}^T Y\,E = 0, \quad Y = \mathcal{P}_l^T Y \mathcal{P}_l$$

with $\hat{A} = A - BC - 2\mathcal{P}_l B D_0^{-1} B^T \mathcal{M}_0^T C\,\mathcal{P}_r$, $\hat{B} = \sqrt{2}\,B J_o^{-1}$, $\hat{C} = \sqrt{2}\,J_c^{-1} C$,
$J_o^T J_o = D_0 = I - M_0^T M_0$, $J_c J_c^T = I - M_0 M_0^T$

- Frechét derivative of $\mathcal{R}$ at Y is given by

$$\mathcal{R}'_Y(X) = (A + P_l \hat{B}\hat{B}^T Y E)^T X E + E^T X (A + P_l \hat{B}\hat{B}^T Y E).$$

- Newton's method: $Y_{j+1} = Y_j - \left(\mathcal{R}'_{Y_j}\right)^{-1}(\mathcal{R}(Y_j))$

FOR $j = 0, 1, 2, \ldots$

1. Compute $A_j = \hat{A} + \mathcal{P}_l \hat{B}\hat{B}^T Y_j E$.
2. Solve the projected Lyapunov equation

$$A_j^T X_j E + E^T X_j A_j = -\mathcal{P}_r^T \mathcal{R}(Y_j) \mathcal{P}_r, \quad X_j = \mathcal{P}_l^T X_j \mathcal{P}_l$$

3. Update $Y_{j+1} = Y_j + X_j$.

END FOR

Newton's method: properties

- If $\lambda E - A_0$ is stable, then $\lambda E - A_j$ are stable for $j \geq 1$.
- If $Y_0 = \mathcal{P}_l^T Y_0 \mathcal{P}_l$, then $Y_j = \mathcal{P}_l^T Y_j \mathcal{P}_l$ for $j \geq 1$.
- $Y_1 \leq \ldots \leq Y_j \leq Y_{j+1} \leq \ldots \leq \mathcal{Y}_{\min}$
- $\lim\limits_{k\to\infty} \|\mathcal{R}(Y_j)\|_F = 0$ and $\lim\limits_{j\to\infty} Y_j = \mathcal{Y}_{\min}$
- Since $\mathcal{R}(Y_j) = (E^T X_{j-1} \hat{B})(\hat{B}^T X_{j-1} E) = K_j^T K_j$, we can use the generalized LR-ADI method [St.'08] to solve

$$A_j^T X_j E + E^T X_j A_j = -\mathcal{P}_r^T K_j^T K_j \mathcal{P}_r, \qquad X_j = \mathcal{P}_l^T X_j \mathcal{P}_l$$

for $X_j \approx Z_j Z_j^T$ with $Z_j \in \mathbb{R}^{n,r}$, $r \ll n$.

$\hookrightarrow \quad Y_{j+1} = H_{j+1} H_{j+1}^T$ with $H_{j+1} = [\, H_j,\ Z_j \,]$

[Benner/St.'08]

Projected Lyapunov equations

$$F^T X\, E + E^T X\, F = -\mathcal{P}_r^T K^T K \mathcal{P}_r, \quad X = \mathcal{P}_l^T X\, \mathcal{P}_l$$

- $E = I$ [Wachspress'88, Penzl'00, Li/White'02]
- Generalized ADI method:

$$(E + \tau_k F)^T X_{k-1/2} F = -\mathcal{P}_r^T K^T K \mathcal{P}_r - F^T X_{k-1} (E - \tau_k F)$$
$$(E + \bar{\tau}_k F)^T X_k^T F = -\mathcal{P}_r^T K^T K \mathcal{P}_r - F^T X_{k-1/2}^T (E - \bar{\tau}_k F)$$

  - $X_k = \mathcal{P}_l^T X_k \mathcal{P}_l$
  - If $X_0 = 0$, then $X - X_k = \mathcal{A}_k^* X \mathcal{A}_k$ with
    $$\mathcal{A}_k = \mathcal{P}_l (E - \tau_1 F)(E + \bar{\tau}_1 F)^{-1} \cdot \ldots \cdot (E - \tau_k F)(E + \bar{\tau}_k F)^{-1}.$$
  - If $\tau_k \in \mathbb{C}^-$, then $\lim_{k \to \infty} X_k = X$.
  - X_k – symmetric, positive definite $\rightsquigarrow$ $X_k = Z_k Z_k^T$

Projected Lyapunov equations

SyreNe

$$F^T X E + E^T X F = -\mathcal{P}_r^T K^T K \mathcal{P}_r, \quad X = \mathcal{P}_l^T X \mathcal{P}_l$$

Generalized low-rank ADI method:

$$Z_1 = \sqrt{-2\mathrm{Re}\,(\tau_1)}\,(E + \tau_1 F)^{-T} \mathcal{P}_r^T K^T, \qquad Z_1^{(1)} = Z_1,$$

$$Z^{(k)} = \sqrt{\frac{\mathrm{Re}\,(\tau_k)}{\mathrm{Re}\,(\tau_{k-1})}}\,\big(I - (\overline{\tau}_{k-1} + \tau_k)(E + \tau_k F)^{-T} F^T\big)\, Z^{(k-1)},$$

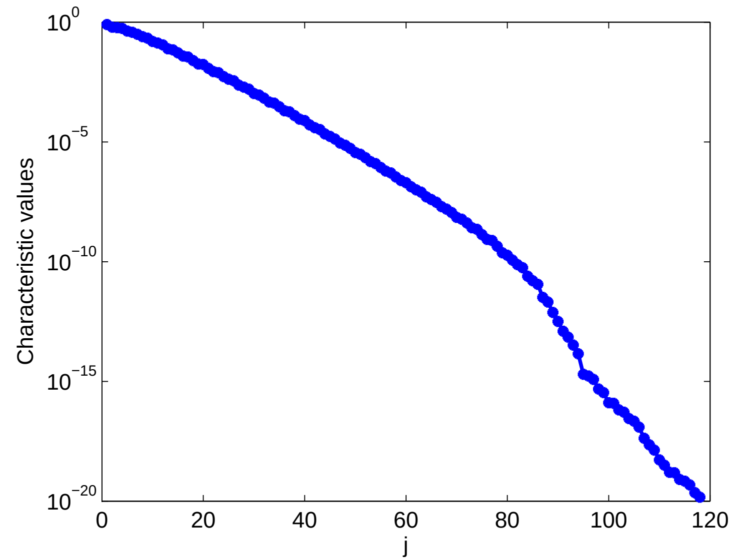
$$Z_k = [\, Z_{k-1}, \quad Z^{(k)} \,]$$

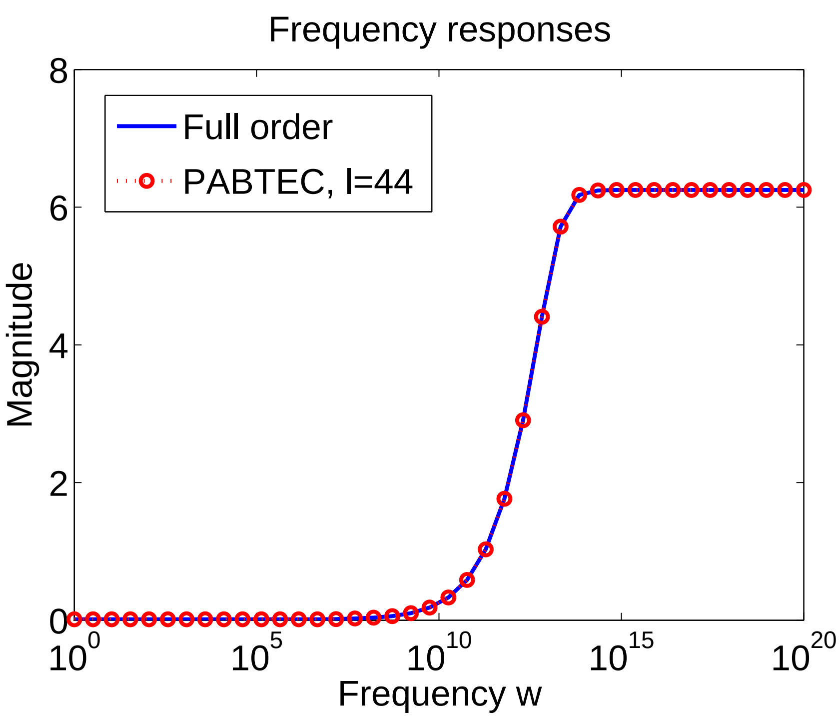
- $X \approx Z_k Z_k^T$ with $Z_k \in \mathbb{R}^{n,km}$
- (sub)optimal ADI parameters [Penzl'00, Sabino'06, Benner et al.'08]
- solve $(E^T + \tau_k F^T)x = b$

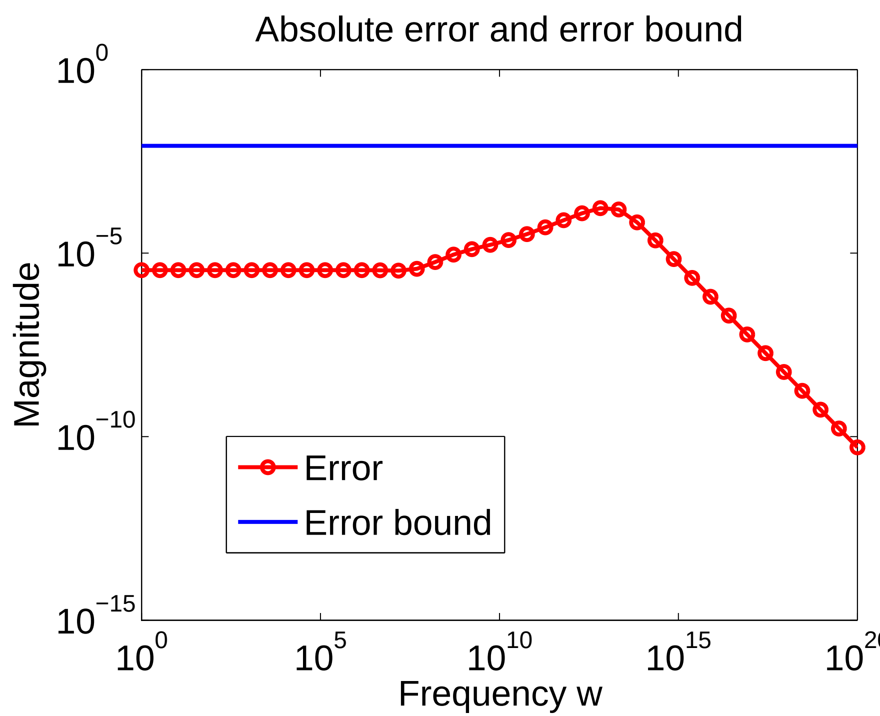
Example: RC circuit

[NEC Laboratories Europe]

- $n = 2007, \ m = 3$
- $\mathcal{Y}_{\min} \approx \tilde{L}\tilde{L}^T, \ \ \tilde{L} \in \mathbb{R}^{n,120}$
- Reduced model: $\ell = 44$







Conclusion

- Balanced truncation model reduction for DAEs
- Model reduction of circuit equations based on bounded real balanced truncation for the Moebius-transformed system
  - passivity and reciprocity are preserved
  - there exist error bounds
- Exploiting the structure of MNA matrices E, A, B and C reduces the computational effort significantly
- Exploiting the circuit topology to compute the projectors (in progress)
- Use modern numerical linear algebra algorithms for solving large-scale projected Riccati and Lyapunov equations
- Numerical solution of projected Lur'e equations (in progress)