Comparison of Truncated SVD and Jacobi-Davidson SVD within ESVDMOR

Peter Benner, Patrick Kürschner and André Schneider

Chemnitz University of Technologies
Department of Mathematics

30th October 2008

Department of Mathematics
CHEMNITZ UNIVERSITY OF TECHNOLOGY

Overview

Überblick

Introductory example

Microelectronics are the core technology for numerous industrial innovations, e. g., 1 GB Micro DIMM (dual in-line memory module)

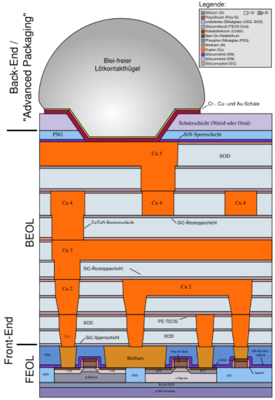
- highly developed
- very compact
- used as main storage in notebooks

source: Qimonda AG

Wafer Development

Recently,

- use of nanometer-scale chip manufacturing process (45nm),
- increasing number of (parasitic) elements (Intel Core 2 Duo 291 million transistors),
- and production of multi-layered ICs (Intel 12 layers).



source: wikipedia.org

Examples are power grids and clock distribution networks.

Motivation

The assumption, that the number of inputs/outputs is much smaller then the number of generalized states, is violated. That leads to methods with the aim:

- reduce the large number of terminals,
- reduce the size of the system

$\Rightarrow$ minimize simulation time and memory requirements,

- no or small errors,
- unchanged input/output variables,
- maintain original system behavior.

Descriptor system

Using the modified nodal analysis (MNA) in general leads to a differential-algebraic equation (DAE) of the implicit form

$$f(x, \dot{x}, t) = 0,$$

with $\det(\frac{\partial f}{\partial \dot{x}}) \equiv 0$.
The DAE in semi-explicit form leads to a linear time-invariant continuous-time system, called descriptor system,

$$\begin{array}{rcl} C\dot{x}(t) & = & -Gx(t) + Bu(t), \quad x(0) = x_0 \\ y(t) & = & Lx(t) \end{array}, \qquad (1)$$

with $C, G \in \mathbb{R}^{n \times n}$, $B, L^T \in \mathbb{R}^{n \times m}$, $x \in \mathbb{R}^n$ containing the internal state variables, $u \in \mathbb{R}^m$ the vector of input variables, $y \in \mathbb{R}^m$ the output vector, $x_0 \in \mathbb{R}^n$ the initial value and n the number of state variables, called the order of the system.

Descriptor system

Using the modified nodal analysis (MNA) in general leads to a differential-algebraic equation (DAE) of the implicit form

$$f(x, \dot{x}, t) = 0,$$

with $\det(\frac{\partial f}{\partial \dot{x}}) \equiv 0$.
The DAE in semi-explicit form leads to a linear time-invariant continuous-time system, called descriptor system,

$$\begin{array}{rcl} C\dot{x}(t) & = & -Gx(t) + Bu(t), \quad x(0) = x_0 \\ y(t) & = & Lx(t) \end{array}, \qquad (1)$$

with $C, G \in \mathbb{R}^{n \times n}$, $B, L^T \in \mathbb{R}^{n \times m}$, $x \in \mathbb{R}^n$ containing the internal state variables, $u \in \mathbb{R}^m$ the vector of input variables, $y \in \mathbb{R}^m$ the output vector, $x_0 \in \mathbb{R}^n$ the initial value and n the number of state variables, called the order of the system.

Descriptor system

Using the modified nodal analysis (MNA) in general leads to a differential-algebraic equation (DAE) of the implicit form

$$f(x, \dot{x}, t) = 0,$$

with $\det(\frac{\partial f}{\partial \dot{x}}) \equiv 0$.
The DAE in semi-explicit form leads to a linear time-invariant continuous-time system, called descriptor system,

$$\begin{array}{rcl} C\dot{x}(t) & = & -Gx(t) + Bu(t), \quad x(0) = x_0 \\ y(t) & = & Lx(t) \end{array}, \tag{1}$$

with $C, G \in \mathbb{R}^{n \times n}$, $B, L^T \in \mathbb{R}^{n \times m}$, $x \in \mathbb{R}^n$ containing the internal state variables, $u \in \mathbb{R}^m$ the vector of input variables, $y \in \mathbb{R}^m$ the output vector, $x_0 \in \mathbb{R}^n$ the initial value and n the number of state variables, called the order of the system.

Descriptor system

Using the modified nodal analysis (MNA) in general leads to a differential-algebraic equation (DAE) of the implicit form

$$f(x, \dot{x}, t) = 0,$$

with $\det(\frac{\partial f}{\partial \dot{x}}) \equiv 0$.
The DAE in semi-explicit form leads to a linear time-invariant continuous-time system, called descriptor system,

$$\begin{array}{rcl} C\dot{x}(t) & = & -Gx(t) + Bu(t), \quad x(0) = x_0 \\ y(t) & = & Lx(t) \end{array}, \tag{1}$$

with $C, G \in \mathbb{R}^{n \times n}$, $B, L^T \in \mathbb{R}^{n \times m}$, $x \in \mathbb{R}^n$ containing the internal state variables, $u \in \mathbb{R}^m$ the vector of input variables, $y \in \mathbb{R}^m$ the output vector, $x_0 \in \mathbb{R}^n$ the initial value and n the number of state variables, called the order of the system.

Descriptor system

Using the modified nodal analysis (MNA) in general leads to a differential-algebraic equation (DAE) of the implicit form

$$f(x, \dot{x}, t) = 0,$$

with $\det(\frac{\partial f}{\partial \dot{x}}) \equiv 0$.
The DAE in semi-explicit form leads to a linear time-invariant continuous-time system, called descriptor system,

$$\begin{array}{rcl} C\dot{x}(t) & = & -Gx(t) + Bu(t), \quad x(0) = x_0 \\ y(t) & = & Lx(t) \end{array}, \qquad (1)$$

with $C, G \in \mathbb{R}^{n \times n}$, $B, L^T \in \mathbb{R}^{n \times m}$, $x \in \mathbb{R}^n$ containing the internal state variables, $u \in \mathbb{R}^m$ the vector of input variables, $y \in \mathbb{R}^m$ the output vector, $x_0 \in \mathbb{R}^n$ the initial value and n the number of state variables, called the order of the system.

Descriptor system

Using the modified nodal analysis (MNA) in general leads to a differential-algebraic equation (DAE) of the implicit form

$$f(x, \dot{x}, t) = 0,$$

with $\det(\frac{\partial f}{\partial \dot{x}}) \equiv 0$.
The DAE in semi-explicit form leads to a linear time-invariant continuous-time system, called descriptor system,

$$\begin{array}{rcl} C\dot{x}(t) & = & -Gx(t) + Bu(t), \quad x(0) = x_0 \\ y(t) & = & Lx(t) \end{array}, \tag{1}$$

with $C, G \in \mathbb{R}^{n\times n}$, $B, L^T \in \mathbb{R}^{n\times m}$, $x \in \mathbb{R}^n$ containing the internal state variables, $u \in \mathbb{R}^m$ the vector of input variables, $y \in \mathbb{R}^m$ the output vector, $x_0 \in \mathbb{R}^n$ the initial value and n the number of state variables, called the order of the system.

Descriptor system

Using the modified nodal analysis (MNA) in general leads to a differential-algebraic equation (DAE) of the implicit form

$$f(x, \dot{x}, t) = 0,$$

with $\det(\frac{\partial f}{\partial \dot{x}}) \equiv 0$.
The DAE in semi-explicit form leads to a linear time-invariant continuous-time system, called descriptor system,

$$\begin{array}{rcl} C\dot{x}(t) & = & -Gx(t) + Bu(t), \quad x(0) = x_0 \\ y(t) & = & Lx(t) \end{array}, \qquad (1)$$

with $C, G \in \mathbb{R}^{n \times n}$, $B, L^T \in \mathbb{R}^{n \times m}$, $x \in \mathbb{R}^n$ containing the internal state variables, $u \in \mathbb{R}^m$ the vector of input variables, $y \in \mathbb{R}^m$ the output vector, $x_0 \in \mathbb{R}^n$ the initial value and n the number of state variables, called the order of the system.

Descriptor system

Using the modified nodal analysis (MNA) in general leads to a differential-algebraic equation (DAE) of the implicit form

$$f(x, \dot{x}, t) = 0,$$

with $\det(\frac{\partial f}{\partial \dot{x}}) \equiv 0$.
The DAE in semi-explicit form leads to a linear time-invariant continuous-time system, called descriptor system,

$$\begin{array}{rcl} C\dot{x}(t) & = & -Gx(t) + Bu(t), \quad x(0) = x_0 \\ y(t) & = & Lx(t) \end{array}, \qquad (1)$$

with $C, G \in \mathbb{R}^{n \times n}$, $B, L^T \in \mathbb{R}^{n \times m}$, $x \in \mathbb{R}^n$ containing the internal state variables, $u \in \mathbb{R}^m$ the vector of input variables, $y \in \mathbb{R}^m$ the output vector, $x_0 \in \mathbb{R}^n$ the initial value and n the number of state variables, called the order of the system.

Transfer function

Applying the Laplace transform to the descriptor system leads to rational matrix-valued function:

Definition (Transfer function)

The rational matrix-valued function

$$H(s) = L(sC + G)^{-1}B \tag{2}$$

with $s \in \mathbb{C}$ is called the transfer function of the continuous-time descriptor system (1).

If $s = i\omega$, then $\omega \in \mathbb{R}$ is called the frequency.
Matrices L, C, G, B are called a *realization* of $H(s)$. Keep in mind, that

B and L define the input and output terminals. We want to reduce their number.

Transfer function

Applying the Laplace transform to the descriptor system leads to rational matrix-valued function:

Definition (Transfer function)

The rational matrix-valued function

$$H(s) = L(sC + G)^{-1}B \qquad (2)$$

with $s \in \mathbb{C}$ *is called the transfer function of the continuous-time descriptor system (1).*

If $s = i\omega$, then $\omega \in \mathbb{R}$ is called the frequency.
Matrices L, C, G, B are called a *realization* of $H(s)$. Keep in mind, that

B and L define the input and output terminals. We want to reduce their number.

Transfer function

Applying the Laplace transform to the descriptor system leads to rational matrix-valued function:

Definition (Transfer function)

The rational matrix-valued function

$$H(s) = L(sC + G)^{-1}B \tag{2}$$

with $s \in \mathbb{C}$ is called the transfer function of the continuous-time descriptor system (1).

If $s = i\omega$, then $\omega \in \mathbb{R}$ is called the frequency.
Matrices L, C, G, B are called a *realization* of $H(s)$. Keep in mind, that

B and L define the input and output terminals. We want to reduce their number.

Transfer function

Applying the Laplace transform to the descriptor system leads to rational matrix-valued function:

Definition (Transfer function)

The rational matrix-valued function

$$H(s) = L(sC + G)^{-1}B \tag{2}$$

with $s \in \mathbb{C}$ is called the transfer function of the continuous-time descriptor system (1).

If $s = i\omega$, then $\omega \in \mathbb{R}$ is called the frequency.
Matrices L, C, G, B are called a *realization* of $H(s)$. Keep in mind, that

B and L define the input and output terminals. We want to reduce their number.

Überblick

Idea [Feldmann/Liu 2004]

SyreNe

We search for a decomposition

$$\widehat{H}(s) = V_L W_L^T L(sC+G)^{-1} B V_B W_B^T$$

with V_L, W_L, V_B, $W_B \in \mathbb{R}^{m \times p}$ and

$$W_L^T V_L = I_p \qquad \text{and} \qquad W_B^T V_B = I_p.$$

Desired:

$$p \ll m$$

$\Rightarrow$ The idea is to use the singular value decomposition (SVD) to reveal the circuit response correlation.

Idea [Feldmann/Liu 2004]

We search for a decomposition

$$\widehat{H}(s) = V_L W_L^T L(sC+G)^{-1} B V_B W_B^T$$

with V_L, W_L, V_B, $W_B \in \mathbb{R}^{m \times p}$ and

$$W_L^T V_L = I_p \qquad \text{and} \qquad W_B^T V_B = I_p.$$

Desired:

$$p \ll m$$

$\Rightarrow$ The idea is to use the singular value decomposition (SVD) to reveal the circuit response correlation.

Idea [Feldmann/Liu 2004]

We search for a decomposition

$$\widehat{H}(s) = V_L W_L^T L(sC+G)^{-1} B V_B W_B^T$$

with V_L, W_L, V_B, $W_B \in \mathbb{R}^{m \times p}$ and

$$W_L^T V_L = I_p \qquad \text{and} \qquad W_B^T V_B = I_p.$$

Desired:

$$p \ll m$$

$\Rightarrow$ The idea is to use the singular value decomposition (SVD) to reveal the circuit response correlation.

ESVDMOR [Liu/Tan/Yan/McGaughy 2006]

- apply SVD low-rank approximation separately to input and output matrices,
- use higher order moment information ⇒ ensure accuracy of reduced model.

The i-th block moment of the system is defined as

$$m_i = L(-G^{-1}C)^i G^{-1}B.$$

Remark: m_i can be directly computed in a recursive way, but that also causes numerical stability problems.

ESVDMOR [Liu/Tan/Yan/McGaughy 2006]

Moment m_i is a $q \times p$ matrix

$$m_i = \begin{bmatrix} m^i_{1,1} & m^i_{1,2} & \cdots & m^i_{1,p} \\ m^i_{2,1} & m^i_{2,2} & \cdots & m^i_{2,p} \\ \vdots & \vdots & \vdots & \vdots \\ m^i_{q,1} & m^i_{q,2} & \cdots & m^i_{q,p} \end{bmatrix},$$

with p number of inputs and q number of outputs.

- Column j represents the moment vector of all output terminals due to input terminal j.
- Row k represents the moment vector of output terminal k due to all inputs.

ESVDMOR [Liu/Tan/Yan/McGaughy 2006]

We define the output moment response matrix M_O as

$$M_O = \begin{bmatrix} m_0^T \\ m_1^T \\ \vdots \\ m_{r-1}^T \end{bmatrix},$$

and the input moment response matrix M_I as

$$M_I = \begin{bmatrix} m_0 \\ m_1 \\ \vdots \\ m_{r-1} \end{bmatrix}.$$

ESVDMOR [Liu/Tan/Yan/McGaughy 2006]

We apply SVD to both and just keep the significant singular values, such that

$$M_I = U_I \Sigma_I V_I^T \approx U_{I_{r_i}} \Sigma_{I_{r_i}} V_{I_{r_i}}^T,$$

$$M_O = U_O \Sigma_O V_O^T \approx U_{O_{r_o}} \Sigma_{O_{r_o}} V_{O_{r_o}}^T,$$

where

- r_i and r_o are the numbers of the significant singular values,
- Σ_{r_i} is a $r_i \times r_i$ diagonal matrix,
- Σ_{r_o} is a $r_o \times r_o$ diagonal matrix,
- $V_{I_{r_i}}^T$ is a $r_i \times p$ matrix,
- $V_{O_{r_o}}^T$ is a $r_o \times q$ matrix.

ESVDMOR [Liu/Tan/Yan/McGaughy 2006]

We apply SVD to both and just keep the significant singular values, such that

$$M_I = U_I \Sigma_I V_I^T \approx U_{I_{r_i}} \Sigma_{I_{r_i}} V_{I_{r_i}}^T,$$

$$M_O = U_O \Sigma_O V_O^T \approx U_{O_{r_o}} \Sigma_{O_{r_o}} V_{O_{r_o}}^T,$$

where

- r_i and r_o are the numbers of the significant singular values,
- Σ_{r_i} is a $r_i \times r_i$ diagonal matrix,
- Σ_{r_o} is a $r_o \times r_o$ diagonal matrix,
- $V_{I_{r_i}}^T$ is a $r_i \times p$ matrix,
- $V_{O_{r_o}}^T$ is a $r_o \times q$ matrix.

ESVDMOR [Liu/Tan/Yan/McGaughy 2006]

Now, the following steps are performed,

$$B \approx B_r V_{I_{r_i}}^T$$

and

$$L \approx V_{O_{r_o}} L_r,$$

where

$$B_r = B V_{I_{r_i}}^{T+} = B V_{I_{r_i}} (V_{I_{r_i}}^T V_{I_{r_i}})^{-1} = B V_{I_{r_i}}$$

and

$$L_r = V_{O_{r_o}}^{+} L = (V_{O_{r_o}}^T V_{O_{r_o}})^{-1} V_{O_{r_o}}^T L = V_{O_{r_o}}^T L,$$

where $B_r \in \mathbb{R}^{n \times r_i}$ and $L_r \in \mathbb{R}^{r_o \times n}$.

ESVDMOR [Liu/Tan/Yan/McGaughy 2006]

Now, the following steps are performed,

$$B \approx B_r V_{I_{r_i}}^T$$

and

$$L \approx V_{O_{r_o}} L_r,$$

where

$$B_r = B V_{I_{r_i}}^{T+} = B V_{I_{r_i}} (V_{I_{r_i}}^T V_{I_{r_i}})^{-1} = B V_{I_{r_i}}$$

and

$$L_r = V_{O_{r_o}}^{+} L = (V_{O_{r_o}}^T V_{O_{r_o}})^{-1} V_{O_{r_o}}^T L = V_{O_{r_o}}^T L,$$

where $B_r \in \mathbb{R}^{n \times r_i}$ and $L_r \in \mathbb{R}^{r_o \times n}$.

ESVDMOR [Liu/Tan/Yan/McGaughy 2006]

Now, the following steps are performed,

$$B \approx B_r V_{I_{r_i}}^T$$

and

$$L \approx V_{O_{r_o}} L_r,$$

where

$$B_r = B V_{I_{r_i}}^{T+} = B V_{I_{r_i}} (V_{I_{r_i}}^T V_{I_{r_i}})^{-1} = B V_{I_{r_i}}$$

and

$$L_r = V_{O_{r_o}}^+ L = (V_{O_{r_o}}^T V_{O_{r_o}})^{-1} V_{O_{r_o}}^T L = V_{O_{r_o}}^T L,$$

where $B_r \in \mathbb{R}^{n \times r_i}$ and $L_r \in \mathbb{R}^{r_o \times n}$.

ESVDMOR [Liu/Tan/Yan/McGaughy 2006]

The approximation of the transfer function is

$$H(s) \approx \widehat{H}(s) = V_{O_{r_o}} L_r (G + sC)^{-1} B_r V_{I_{r_i}}^T .$$

The new "internal" transfer function is

$$H_r(s) = L_r (G + sC)^{-1} B_r ,$$

which is reduced with a standard MOR algorithm to $\tilde{H}_r(s)$.
The final result is a very compact terminal and order reduced model.

Result

$$H(s) \approx V_{O_{r_o}} \tilde{H}_r(s) V_{I_{r_i}}^T$$

Überblick

Basic Idea

We want to find just the important singular values, **not all**!
Therefore we apply the Truncated SVD or the Jacobi-Davidson SVD to the particular ansatz matrix, e. g. M_I.
Both methods compute the eigenvalues and eigenvectors of an augmented matrix

$$A = \begin{pmatrix} 0 & M_I \\ M_I^T & 0 \end{pmatrix}. \tag{3}$$

Positive eigenvalues of A are the singular values of M_I.

Truncated SVD

We just have to provide a function which calculates the matrix vector product of A with a vector x

$$\mathbf{y} = \begin{pmatrix} 0 & \begin{bmatrix} \mathbf{m_0} \\ \mathbf{m_1} \\ \vdots \\ \mathbf{m_{r-1}} \end{bmatrix} \\ \begin{bmatrix} \mathbf{m_0}^T & \mathbf{m_1}^T & \cdots & \mathbf{m_{r-1}^T} \end{bmatrix} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{r \cdot m_{out} + m_{in}} \end{pmatrix}. \quad (4)$$

to build the needed Krylov subspace for the used implicitly restarted Arnoldi method.

The explicit computation of the single moments is not necessary here.

JDSVD [Hochstenbach 2001]

JDSVD uses two search spaces. In fact, it uses the structure of the augmented matrix A. Consider:

- search spaces $\mathcal{U} \subset \mathbb{R}^m, \ \mathcal{V} \subset \mathbb{R}^n$, test spaces $\mathcal{X} \subset \mathbb{R}^m, \ \mathcal{Y} \subset \mathbb{R}^n$
- the double Galerkin condition:

$$r(\theta, \ \eta) := \begin{pmatrix} M_I v - \theta u \\ M_I^T u - \eta v \end{pmatrix} \perp\perp \begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \end{pmatrix}, \ u \in \mathcal{U}, \ v \in \mathcal{V}$$

$$U \in \mathbb{R}^{m \times k}, \ V \in \mathbb{R}^{n \times k} \Rightarrow u = Uc, \ v = Vd, \ c, d \in \mathbb{R}^k$$

$$X \in \mathbb{R}^{m \times k}, \ Y \in \mathbb{R}^{n \times k}$$

$$X^T M_I V d = \theta X^T U c, Y^T M_I^T U c = \eta Y^T V d$$

- test vectors $x \in \mathcal{X} \ , y \in \mathcal{Y}, \ x^T u \neq 0, \ y^T v \neq 0$ lead to approximations $\theta = \frac{x^T M_I v}{x^T u}$ and $\eta = \frac{y^T M_I^T u}{y^T v}$

JDSVD Standard Subspace Extraction

Given: start vectors $(u_1,\ v_1)$ $(=: (s,\ t))$, tolerance ϵ

For $k = 1, \dots$ do:

1. expand search spaces $\mathcal{U}_k,\ \mathcal{V}_k$ with normalized vectors s, t (MGS, RMGS)
2. compute largest singular triple $(\theta,\ c,\ d)$ of $H_k = U_k^T M_I V_k$
   $u = U_k c,\ \ v = V_k d$
3. compute residual $r = \begin{pmatrix} M_I v - \theta u \\ M_I^T u - \theta v \end{pmatrix}$
4. Stop if $\|r\| \leq \epsilon$
5. (approximate) solution of correction equation for $(s, t) \perp\perp (u, v)$

$$\begin{pmatrix} I_m - uu^T & 0 \\ 0 & I_n - vv^T \end{pmatrix} \begin{pmatrix} -\theta I_m & M_I \\ M_I^T & -\theta I_n \end{pmatrix} \begin{pmatrix} I_m - uu^T & 0 \\ 0 & I_n - vv^T \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = -r$$

Überblick

Outlook

The Future requires investigations in the following aspects:

- ensure numerical stability of the computation of the required moments
- analyse alternative methods to (E)SVDMOR (TermMerge, McPack)
- ensure passivity preservation,
- find error bounds/estimation,
- find alternative matrix decomposition approaches.

SyreNe

System Reduction for Nanoscale IC Design

This will be done within the research network *System Reduction for Nanoscale IC Design* (SyreNe) within the program *Mathematics for Innovations in Industry and Services* funded by the German Federal Ministry of Education and Science (BMBF).