MODEL REDUCTION FOR LINEAR DESCRIPTOR SYSTEMS WITH MASSIVE PORTS

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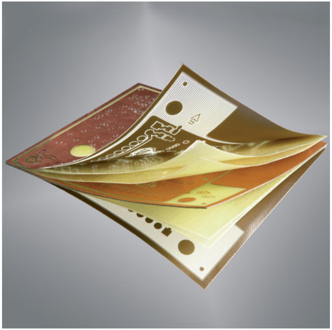
Outline

Wafer Development – Power Grid Model

Typical applications are the simulation of power grids and clock distribution networks.

Recently,

- use of nanometer-scale chip manufacturing process (30nm-level),
- increasing number of (parasitic) elements (*Intel Xeon*, 2–4 kernels, 820 millionen transistors, 45 nm),
- and production of multi-layered ICs, Intel ≥ 12 layers.



Source: http://digi-tarashe.com.

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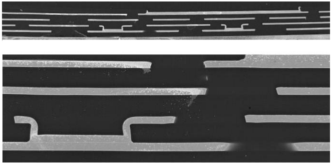

Multilayer WLP (Wafer level package) structure.

Source: http://www.i-micronews.com.

Wafer Development – Power Grid Model

Project setting

Problem:

- Modeling of power grid for voltage supply of the electronic devices in an IC,
- $\rightsquigarrow$ extremly high number of input/output (terminals).

Goals:

- Generation of virtual I/O operators using ESVDMOR approach,
- efficient numerical algorithm for terminal and model order reduction,
- implementation in circuit simulator TITAN.

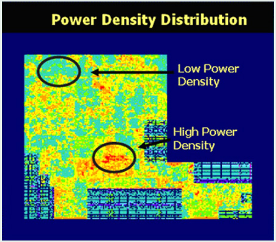

Figure: power density distribution on an IC. Source: `http://www.eetimes.com`

Descriptor system

Using the modified nodal analysis (MNA) in general leads to a differential-algebraic equation (DAE) of the implicit form

$$f(x, \dot{x}, t) = 0,$$

with $\det(\frac{\partial f}{\partial \dot{x}}) \equiv 0$.
Linear (sub-)networks or linearization $\rightsquigarrow$ linear DAE in semi-explicit form $\rightsquigarrow$ linear time-invariant descriptor system,

$$\begin{aligned} C\dot{x}(t) &= -Gx(t) + Bu(t), \quad x(0) = x_0, \\ y(t) &= Lx(t), \end{aligned} \tag{1}$$

with $C, G \in \mathbb{R}^{n \times n}$, $B, L^T \in \mathbb{R}^{n \times m}$, $x \in \mathbb{R}^n$ containing the internal state variables, $u \in \mathbb{R}^m$ the vector of input variables, $y \in \mathbb{R}^m$ the output vector, $x_0 \in \mathbb{R}^n$ the initial value and n the number of state variables, called the order of the system.

Transfer function

Applying the Laplace transform to the descriptor system leads to rational matrix-valued function:

Definition (Transfer function)

The rational matrix-valued function

$$H(s) = L(sC + G)^{-1}B \tag{2}$$

with $s \in \mathbb{C}$ is called the transfer function of the continuous-time descriptor system (1).

If $s = i\omega$, then $\omega \in \mathbb{R}$ is called the frequency.
Matrices L, C, G, B are called a *realization* of $H(s)$.

Keep in mind, that B and L define the input and output terminals.
The goal is to reduce the number of terminals.

Idea [Feldmann/Liu 2004]

We search for an approximation

$$\begin{aligned} H(s) \approx \widehat{H}(s) &= V_L W_L^T L(sC+G)^{-1} B V_B W_B^T \\ &= V_L \tilde{H}(s) W_B^T, \end{aligned}$$

with V_L, W_L, V_B, $W_B \in \mathbb{R}^{m \times p}$ and

$$W_L^T V_L = I_p \qquad \text{and} \qquad W_B^T V_B = I_p.$$

Desired:

$$p \ll m,$$

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$\Rightarrow$ The idea is to use a singular value decomposition (SVD) to reveal the circuit response correlations.

ESVDMOR [Liu/Tan/Yan/McGaughy 2006]

ESVDMOR idea:

- apply SVD low-rank approximation separately to "input and output moment matrices",
- use higher order moment information ⇒ ensure accuracy of reduced model.

The i-th block moment of the system in general is defined as

$$m_i(s_0) = L(-(s_0C + G)^{-1}C)^i(s_0C + G)^{-1}B.$$

Remark: m_i are equal to the coefficients of the Taylor series expansion of the transfer function.

ESVDMOR [Liu/Tan/Yan/McGaughy 2006]

Moment m_i is a $m_{out} \times m_{in}$ matrix

$$m_i = \begin{bmatrix} m^i_{1,1} & m^i_{1,2} & \cdots & m^i_{1,m_{in}} \\ m^i_{2,1} & m^i_{2,2} & \cdots & m^i_{2,m_{in}} \\ \vdots & \vdots & \vdots & \vdots \\ m^i_{m_{out},1} & m^i_{m_{out},2} & \cdots & m^i_{m_{out},m_{in}} \end{bmatrix},$$

with m_{in} number of inputs and m_{out} number of outputs.

We define the output moment response matrix M_O and the input moment response matrix M_I as

$$M_O = \begin{bmatrix} m_0^T \\ m_1^T \\ \vdots \\ m_{r-1}^T \end{bmatrix}, \qquad M_I = \begin{bmatrix} m_0 \\ m_1 \\ \vdots \\ m_{r-1} \end{bmatrix}.$$

ESVDMOR [Liu/Tan/Yan/McGaughy 2006]

We apply SVD to both and just keep the significant singular values, such that

$$M_I = U_I \Sigma_I V_I^T \approx U_{I_{r_i}} \Sigma_{I_{r_i}} V_{I_{r_i}}^T,$$

$$M_O = U_O \Sigma_O V_O^T \approx U_{O_{r_o}} \Sigma_{O_{r_o}} V_{O_{r_o}}^T,$$

where

- r_i and r_o are the numbers of the significant singular values,
- Σ_{r_i} is an $r_i \times r_i$ diagonal matrix,
- Σ_{r_o} is an $r_o \times r_o$ diagonal matrix,
- $V_{I_{r_i}}^T$ is an $r_i \times m_{in}$ isometry,
- $V_{O_{r_o}}^T$ is an $r_o \times m_{out}$ isometry.

ESVDMOR [Liu/Tan/Yan/McGaughy 2006]

Now, the following steps are performed,

$$B \approx B_r V_{I_{r_i}}^T$$

and

$$L \approx V_{O_{r_o}} L_r,$$

where

$$B_r = B V_{I_{r_i}}^{T+} = B V_{I_{r_i}} (V_{I_{r_i}}^T V_{I_{r_i}})^{-1} = B V_{I_{r_i}}$$

and

$$L_r = V_{O_{r_o}}^{+} L = (V_{O_{r_o}}^T V_{O_{r_o}})^{-1} V_{O_{r_o}}^T L = V_{O_{r_o}}^T L,$$

where $B_r \in \mathbb{R}^{n \times r_i}$ and $L_r \in \mathbb{R}^{r_o \times n}$.

ESVDMOR [Liu/Tan/Yan/McGaughy 2006]

The approximation of the transfer function is

$$H(s) \approx \widehat{H}(s) = V_{O_{r_o}} L_r (G + sC)^{-1} B_r V_{I_{r_i}}^T .$$

The new "internal" transfer function is

$$H_r(s) = L_r (G + sC)^{-1} B_r ,$$

which is reduced with a standard MOR algorithm to $\tilde{H}_r(s)$.
The final result is a very compact terminal and order reduced model:

Result

$$H(s) \overset{\text{terminal reduction}}{\longrightarrow} \widehat{H}(s) \overset{\text{model reduction}}{\longrightarrow} \widehat{H}_r(s) = V_{O_{r_o}} \tilde{H}_r(s) V_{I_{r_i}}^T .$$

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- Therefore we use truncated SVD (TSVD): two alternatives
  - TSVD using implicitly restarted Lanczos ($\rightsquigarrow$ `svds` in MATLAB®),
  - Jacobi-Davidson SVD.

  Both methods compute the eigenvalues and eigenvectors of an augmented matrix, e.g.,

$$A = \begin{pmatrix} 0 & M_I \\ M_I^T & 0 \end{pmatrix}.$$

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  Positive eigenvalues of A are the singular values of M_I.
- Requires efficient implementation of matrix-vector products with moment matrices, employing recursive structures!

Moment Computation

We just have to provide a function which calculates the matrix vector product of A with a vector x

$$\mathbf{y} = \begin{pmatrix} 0 & \begin{bmatrix} \mathbf{m_0} \\ \mathbf{m_1} \\ \vdots \\ \mathbf{m_{r-1}} \end{bmatrix} \\ \begin{bmatrix} \mathbf{m_0}^T & \mathbf{m_1}^T & \cdots & \mathbf{m_{r-1}^T} \end{bmatrix} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{r \cdot m_{out} + m_{in}} \end{pmatrix}.$$

The explicit computation of the single moments is not necessary here.

Moment Computation

We compute the $r+1$ parts of $\mathbf{y}$ by repeatedly applying the same factors to parts of $\mathbf{x}$, depending on whether it is one of the first r block rows, or the last one.

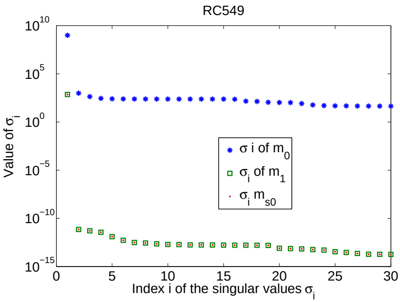
The following algorithm demonstrates this for the first r block rows (recall: $m_i(s_0) = L(-(s_0C+G)^{-1}C)^i(s_0C+G)^{-1}B$):

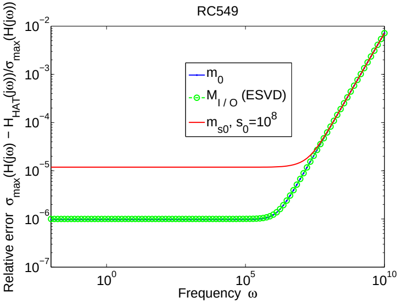
Algorithm 1 Computation of the blocks y^i

$a = Bx^{r+1}$
$a = (s_0C+G)^{-1}a$
for $i = 1$ to r **do**
  $y^i = La$
  $a = Ca$
  $a = -(s_0C+G)^{-1}a$
end for

Numerical Example

- Terminal reduction of Qimonda circuit *rc549*.
- 70 I/O terminals, only 141 nodes.





Stability

Definition

The descriptor system (1) is *asymptotically stable* if $\lim_{t\to\infty} x(t) = 0$ for all solutions x(t) of $C\dot{x(t)} = -Gx(t)$.

Lemma (e.g., Mehrmann/Stykel 2005)

Consider a descriptor system (1) with a regular matrix pencil $\lambda C + G$. The following statements are equivalent:

1. *System (1) is asymptotically stable.*
2. *All finite eigenvalues of the pencil $\lambda C + G$ lie in the open left half-plane.*

Stability

Using these results we are able to formulate the following Theorem:

Theorem

Consider an asymptotically stable descriptor system (1) with transfer function (2). The ESVDMOR reduced system is asymptotically stable iff the inner reduction to $\tilde{H}_r(s)$ is stability preserving.

Proof.

It is obvious that none of the approximations in ESVDMOR changes the eigenvalues of the system. With the assumption that the inner reduction is stability preserving it directly follows that the ESVDMOR approach is stability preserving. □

Passivity

The descriptor system (1) is passive if its transfer function is positive real.

Definition

The transfer function (2) is *positive real* iff the following three assumptions hold:

1. $H(s)$ has no poles in $\mathbb{C}_+ = \{s \in \,|\, \mathrm{Re}\, s > 0\}$,
2. $H(\bar{s}) = \overline{H(s)}$ for all $s \in \mathbb{C}$,
3. $\mathrm{Re}(x^H H(s)x) \geq 0$ for all $s \in \mathbb{C}_+$ and $x \in \mathbb{C}^m$.

We assume

- the number of inputs to be equal to the number of outputs,
- and $L = B^T$, such that

$$H(s) = B^T (sC + G)^{-1} B.$$

Passivity

Further assumptions:

- block structure of the system (as for typical MNA models of RLC circuits)

$$\begin{aligned} \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \dot{x} + \begin{bmatrix} G_1 & G_2 \\ -G_2^T & 0 \end{bmatrix} x &= \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u, \\ y &= \begin{bmatrix} B_1 & 0 \end{bmatrix} x, \end{aligned} \qquad (3)$$

- G_1, C_1, C_2 symmetric,
- $G_1 \geq 0$ and $C_1 \geq 0$, that means both matrices are positive semidefinite,
- $C_2 > 0$, i.e., positive definite,
- matrix pencil $\lambda C + G$ regular.

Passivity

Theorem

Consider a passive system of form (3). The ESVDMOR reduced system is passive iff the inner reduction to $\tilde{H}_r(s)$ is passivity preserving.

Proof.

If we can show that the reduced transfer function is positive real, we have shown that the reduced system is passive, see definition of passivity.

This follows from $\mathbf{m_i}^T(s_0) = \mathbf{m_i}(s_0)$ and thus equality of left and right projectors in terminal reduction plus positive realness of inner reduced transfer function. □

Passivity – Example

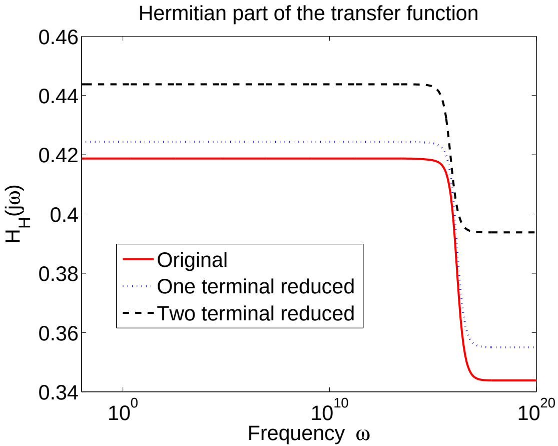


Figure: Smallest eigenvalues of the Hermitian part of the transfer function of *rc549* depending on the frequency ω.

Passivity – Example

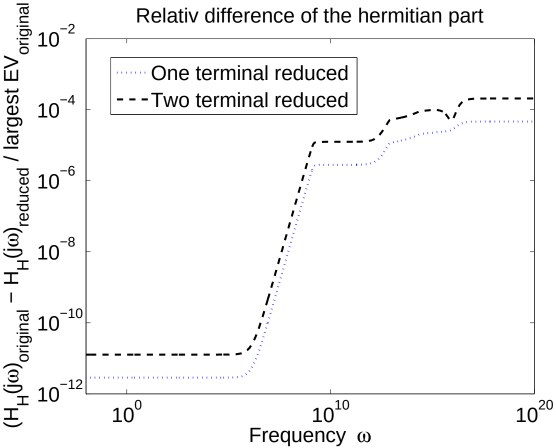


Figure: Relative difference correlated to the largest eigenvalue of the original transfer function.

Reciprocity

Reciprocity is a requirement for synthesis of the reduced order model as circuit. We assume the same setting as in the section about passivity.

Definition

A transfer function (2) is *reciprocal* if there exist $m_1, m_2 \in \mathbb{N}$ with $m_1 + m_2 = m$, such that for $\Sigma_e = \mathrm{diag}(I_{m_1}, -I_{m_2})$ and all $s \in \mathbb{C}$, where $H(s)$ has no pole, it holds

$$H(s)\Sigma_e = \Sigma_e H^T(s).$$

The matrix Σ_e is called *external signature* of the system. A descriptor system is called reciprocal if its transfer function is reciprocal.

As a consequence, a transfer function of a reciprocal system has the form

$$H(s) = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ -H_{12}^T(s) & H_{22}(s) \end{bmatrix}, \tag{4}$$

where $H_{11}(s) = H_{11}^T(s) \in \mathbb{R}^{m_1,m_1}$ and $H_{22}(s) = H_{22}^T(s) \in \mathbb{R}^{m_2,m_2}$.

Reciprocity

Theorem

Consider a reciprocal system of the form (3). *The ESVDMOR reduced system is reciprocal iff the inner reduction to* $\tilde{H}_r(s)$ *is reciprocity preserving.*

Proof.

Due to the reciprocity of the original system, the corresponding transfer function (2) has the structure given in (4). Equation

$$\widehat{H}(s) = V_r B_r^T (G + sC)^{-1} B_r V_r^T,$$

shows that none of the steps in ESVDMOR destroy this symmetric structure if the inner reduction to $\tilde{H}_r(s)$ preserves reciprocity. □

Outlook

We have reviewed the ideas of ESVDMOR and that this approach is stability, passivity, and reciprocity preserving under reasonable assumptions.

Future work:

- error estimation ($\rightsquigarrow$ SCEE 2010),
- implementation of the algorithm in TITAN (in progress),
- alternative applications,
- balanced truncation for massive I/O-ports ($\rightsquigarrow$ MTNS 2010).

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Many thanks for your attention.