On Model Reduction for Periodic Descriptor Systems Using Balanced Truncation

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joint work with
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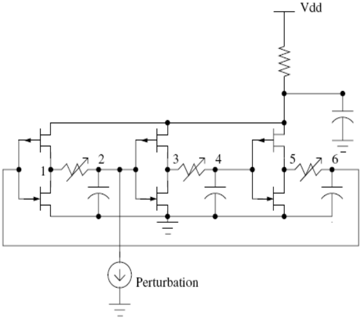
Typical ring oscillator



- Model circuit problem can be described by nonlinear DAEs

$$\begin{aligned}\frac{dq(x(t))}{dt}+f(x(t)) &= u_L(t)+Bu(t),\\ y(t) &= d^T(t)x(t), \quad x(t)\in\mathbb{R}^n.\end{aligned}$$

- Standard MOR techniques for LTI and LTV systems are not applicable.

[image source: Lai and Roychowdhury'06]

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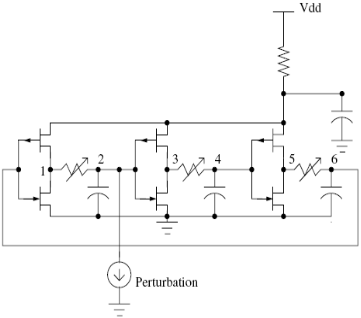

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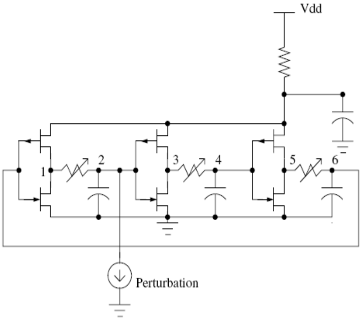

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Model reduction strategy

$$\begin{aligned} \frac{dq(x(t))}{dt} + f(x(t)) = u_L(t) + Bu(t), \\ y(t) = d^T(t)x(t). \end{aligned}$$

Linearization:

- Linearize around the large signal, $u_L(t)$
- Get a linearized periodic time-varying system (LPTV):

$$\overline{G}(t)x + \frac{d}{dt}(\overline{C}(t)x) = Bu(t),$$

where

$$\overline{G}(t) = \frac{\partial f(x)}{\partial x}|_{x_s(t)}, \overline{C}(t) = \frac{\partial q(x)}{\partial x}|_{x_s(t)}.$$

Discretization:

- Discretization of LPTV over time-domain, $[0, K]$
- Get a discrete-time LPTV system :

$$\begin{aligned} \bar{E}(k)\bar{x}(k+1) &= \bar{A}\bar{x}(k) + \bar{B}\bar{u}(k), \\ \bar{y}(k) &= \bar{d}^T\bar{x}(k), \end{aligned}$$

where $\bar{*}$ have specific structures through discretization.

MOR techniques can be applied.

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Overview

Introduction
Periodic Descriptor Systems

The model problem

Consider the periodic LTV descriptor system with time-varying dimensions

$$G : \begin{cases} E_k x_{k+1} &= A_k x_k + B_k u_k, \\ y_k &= C_k x_k, \end{cases} \tag{sys}$$

where $E_k \in \mathbb{R}^{\mu_{k+1} \times n_{k+1}}$, $A_k \in \mathbb{R}^{\mu_{k+1} \times n_k}$, $B_k \in \mathbb{R}^{\mu_{k+1} \times m}$, $C_k \in \mathbb{R}^{p \times n_k}$ are periodic with period $K \geq 1$, and the matrices E_k are allowed to be singular for all k, $\sum_{k=0}^{K-1} \mu_k = \sum_{k=0}^{K-1} n_k = N$, and $n = (n_0, n_1, \ldots, n_{K-1})$.

Model Order Reduction (MOR)

$$\tilde{G} : \begin{cases} \tilde{E}_k \tilde{x}_{k+1} &= \tilde{A}_k \tilde{x}_k + \tilde{B}_k u_k, \\ \tilde{y}_k &= \tilde{C}_k \tilde{x}_k, \end{cases}$$

where $r = (r_0, r_1, \ldots, r_{K-1})$, and $r_k \ll n_k$ for $k = 0, 1, \ldots, K-1$.

Introduction

Periodic Descriptor Systems

Characteristic of reduced model

- preserve regularity, stability, passivity
- small approximation error

Applications

- Multirate signal processing and communication systems.
- Aerospace realm (control of Helicopter rotors, magnetic attitude control for spacecrafts).
- Computer control of industrial processes.

Introduction

Periodic Descriptor Systems

Stability issue

- Assume all E_k are square and invertible. System (sys) is said to be asymptotically stable iff

$$\Lambda(E_{K-1}^{-1}A_{K-1}E_{K-2}^{-1}A_{K-2}\cdots E_0^{-1}A_0) < 1$$

- When E_k are rectangular and singular?
  - We look for a "lifted representation" of (sys)

Literature:

- BT MOR of Periodic systems [A.Varga'99, '00].
- Large periodic Lyapunov equations: Algorithms and Applications [Kressner'03].
- Periodic descriptor systems: Solvability and Conditionability [Sreedhar and Van Dooren'99].

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Introduction

Lifted representation of descriptor systems

- Cyclic lifted representation of (sys):

$$\begin{aligned} \mathcal{E}\mathcal{X}_{k+1} &= \mathcal{A}\mathcal{X}_k + \mathcal{B}\mathcal{U}_k, \\ \mathcal{Y}_k &= \mathcal{C}\mathcal{X}_k, \end{aligned} \qquad \text{(lsys)}$$

where $\mathcal{E} := \mathcal{Z}_n^{-k}\mathrm{E}_k\mathcal{Z}_n^{k} = \mathrm{E}_0$, $\mathcal{A} := \mathcal{Z}_n^{-k}\mathrm{A}_k\mathcal{Z}_n^{k-1} = \mathcal{Z}_n^{-1}\mathrm{A}_1$, $\mathcal{B} := \mathcal{Z}_n^{-k}\mathrm{B}_k\mathcal{Z}_m^{k} = \mathrm{B}_0$, and $\mathcal{C} := \mathcal{Z}_p^{-k}\mathrm{C}_k\mathcal{Z}_n^{k-1} = \mathrm{C}_0\mathcal{Z}_n^{-1}$.

- E_k=diag$(E_k, E_{k+1}, \cdots, E_{k+K-1})$ and same for others.
- Cyclic shift matrix

$$\mathcal{Z}_j = \begin{bmatrix} 0 & I_{(K-1)j} \\ I_j & 0 \end{bmatrix}, \quad j > 0.$$

- $\mathrm{X}_k = \mathcal{Z}_n^{k-1}\mathcal{X}_k, \quad \mathrm{U}_k = \mathcal{Z}_m^{k}\mathcal{U}_k.$

Introduction

Lifted representation of descriptor systems

Transfer function

$$\mathcal{G}(z) = \mathcal{C}(z\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}$$

Stability of lifted system

$$\alpha\mathcal{E} - \beta\mathcal{A} := \begin{bmatrix} \alpha E_0 & 0 & \dots & 0 & -\beta A_0 \\ -\beta A_1 & \alpha E_1 & & & 0 \\ & \ddots & \ddots & & \\ & & & & \\ 0 & & 0 & -\beta A_{K-1} & \alpha E_{K-1} \end{bmatrix}$$

is regular and $\Lambda_f(\mathcal{E}, \mathcal{A}) < 1$, where α, β are complex variables.

Solving projected Lyapunov equations

Kronecker structure of matrix pairs

- For nonsingular X_k, Y_k with $Y_K = Y_0$,

$$X_k E_k Y_{k+1} = \begin{bmatrix} I & 0 \\ 0 & E_k^b \end{bmatrix}, \quad X_k A_k Y_k = \begin{bmatrix} A_k^f & 0 \\ 0 & I \end{bmatrix}. \qquad \text{(kron1)}$$

- $A_{k+K-1}^f A_{k+K-2}^f \cdots A_k^f \equiv J_k$ is $n^f \times n^f$ Jordan matrix corresponding to finite eigenvalues.
- $E_k^b E_{k+1}^b \cdots E_{k+K-1}^b \equiv N_k$ is $n^\infty \times n^\infty$ Nilpotent Jordan matrix corresponding to infinite eigenvalues.
- $n = n^f + n^\infty$.

- ν is the index of the system, where $\nu = \max\{\nu_0, \nu_1, \cdots, \nu_{K-1}\}$, and ν_k is nilpotency of N_k for $k = 0, 1, \ldots, K-1$.

Solving projected Lyapunov equations

Kronecker structure of matrix pairs

Defining Forward-backward subsystems

For $k \in \mathbb{Z}$ and

$$x_k = Y_k \begin{bmatrix} x_k^f \\ x_k^b \end{bmatrix}, \quad X_k B_k = \begin{bmatrix} B_k^f \\ B_k^b \end{bmatrix}, \quad C_k Y_k = \begin{bmatrix} C_k^f & C_k^b \end{bmatrix}, \qquad \text{(kron2)}$$

we decompose the original system (sys) as

- Forward subsystem: $x_{k+1}^f = A_k^f x_k^f + B_k^f u_k$, $y_k^f = C_k^f x_k^f$.
- Backward subsystem: $E_k^b x_{k+1}^b = x_k^b + B_k^b u_k$, $y_k^b = C_k^b x_k^b$.

System output:

$$y_k = y_k^f + y_k^b.$$

Solving projected Lyapunov equations

Periodic Gramians

Define periodic Gramians for forward and backward subsystems

- Causal (Forward) reachability Gramian: $\{G_k^{cr}\}_{k=0}^{K-1}$
- Noncausal (Backward) reachability Gramian: $\{G_k^{nr}\}_{k=0}^{K-1}$
- Causal (Forward) observability Gramian: $\{G_k^{co}\}_{k=0}^{K-1}$
- Noncausal (Backward) observability Gramian: $\{G_k^{no}\}_{k=0}^{K-1}$

Complete Gramians

$$\begin{aligned} G_k^r &\equiv G_k^{cr} + G_k^{nr}, \\ G_k^o &\equiv G_k^{co} + G_k^{no}, \end{aligned}$$

for $k = 0, 1, \ldots, K-1$.

[for details, see Chu, et. al.'07, Van Dooren, et. al.'99]

Solving projected Lyapunov equations

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Complete Gramians

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Solving projected Lyapunov equations

Solutions of generalized discrete-time periodic Lyapunov equations (GDPLEs)

- $\{G_k^{cr}\}_{k=0}^{K-1}$ satisfies GDPLEs

$$\begin{aligned} E_k G_{k+1}^{cr} E_k^T - A_k G_k^{cr} A_k^T &= P_l(k) B_k B_k^T P_l(k)^T \\ G_k^{cr} &= P_r(k) G_k^{cr} P_r(k)^T, \end{aligned} \qquad \text{(re1)}$$

- $\{G_k^{nr}\}_{k=0}^{K-1}$ satisfies GDPLEs

$$\begin{aligned} E_k G_{k+1}^{nr} E_k^T - A_k G_k^{nr} A_k^T &= -(I - P_l(k)) B_k B_k^T (I - P_l(k))^T \\ G_k^{nr} &= (I - P_r(k)) G_k^{nr} (I - P_r(k))^T, \end{aligned} \qquad \text{(re2)}$$

where $k = 0, 1, \dots, K-1$, $G_K^{cr} = G_0^{cr}$, $G_K^{nr} = G_0^{nr}$.

Spectral projections:

$$P_l(k) = X_k^{-1} \begin{bmatrix} I_{n^f} & 0 \\ 0 & 0 \end{bmatrix} X_k, \quad P_r(k) = Y_k \begin{bmatrix} I_{n^f} & 0 \\ 0 & 0 \end{bmatrix} Y_k^{-1}$$

Solving projected Lyapunov equations

Solutions of generalized discrete-time periodic Lyapunov equations (GDPLEs)

- $\{G_k^{co}\}_{k=0}^{K-1}$ satisfies GDPLEs

$$\begin{aligned} E_{k-1}^T G_k^{co} E_{k-1} - A_k^T G_{k+1}^{co} A_k &= P_r(k)^T C_k^T C_k P_r(k) \\ G_k^{co} &= P_l(k-1)^T G_k^{co} P_l(k-1), \end{aligned} \tag{ob1}$$

- $\{G_k^{no}\}_{k=0}^{K-1}$ satisfies GDPLEs

$$\begin{aligned} E_{k-1}^T G_k^{no} E_{k-1} - A_k^T G_{k+1}^{no} A_k &= -(I - P_r(k))^T C_k^T C_k (I - P_r(k)) \\ G_k^{no} &= (I - P_l(k-1))^T G_k^{no} (I - P_l(k-1)), \end{aligned} \tag{ob2}$$

where $G_K^{co} = G_0^{co}$, $G_K^{no} = G_0^{no}$, $E_{-1} = E_{K-1}$, $P_l(-1) = P_l(K-1)$.

Spectral projections:

$$P_l(k) = X_k^{-1} \begin{bmatrix} I_{n^f} & 0 \\ 0 & 0 \end{bmatrix} X_k, \quad P_r(k) = Y_k \begin{bmatrix} I_{n^f} & 0 \\ 0 & 0 \end{bmatrix} Y_k^{-1}$$

Solving projected Lyapunov equations

Solutions of Lyapunov equations

- Lifted structure for (re1):

$$\mathcal{E}\mathcal{G}^{cr}\mathcal{E}^T - \mathcal{A}\mathcal{G}^{cr}\mathcal{A}^T = \mathcal{B}\mathcal{B}^T, \qquad \text{(lre1)}$$

where

- $\mathcal{E} = \text{diag}(E_0, E_1, \cdots, E_{K-1})$,
- $\mathcal{A} = \mathcal{Z}_n^{-1} \,.\, \text{diag}\,(A_1, \cdots, A_{K-1}, A_0)$,
- $\mathcal{B} = \text{diag}(P_l(0)B_0, P_l(1)B_1, \cdots, P_l(K-1)B_{K-1})$,
- $\mathcal{G}^{cr} = \text{diag}\,(G_1^{cr}, \cdots, G_{K-1}^{cr}, G_0^{cr})$.

Remarks

- Notice the structure of the solution
- Lifted structure of (re2) is analogous to (lre1)

Solving projected Lyapunov equations

Solutions of Lyapunov equations

- Lifted structure for (ob1):

$$\sigma\mathcal{E}\mathcal{G}^{co}\sigma\mathcal{E}^T - \sigma\mathcal{A}\mathcal{G}^{co}\sigma\mathcal{A}^T = \sigma\mathcal{C}^T\sigma\mathcal{C}, \qquad \text{(lob1)}$$

where

- $\sigma\mathcal{E} = \text{diag}(E_{K-1}, E_0, \cdots, E_{K-2})$,
- $\sigma\mathcal{A} = \mathcal{Z}_n^{-1}.\text{diag}\,(A_0, A_1, \cdots, A_{K-1})$,
- $\mathcal{C} = \text{diag}\,(C_0, C_1, \cdots, C_{K-1}).\mathcal{Z}_n^{-1}$,
- $\mathcal{G}^{co} = \text{diag}(G_0^{co}, G_1^{co}, \cdots, G_{K-1}^{co})$.

Remarks

- Notice the structure of the solution
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Solving projected Lyapunov equations

Solutions of Lyapunov equations

- Use the canonical structure of matrix pairs.
- Reformulate the Lyapunov equations for forward and backward subsystems.

Reformulated lifted structure of GDPLEs

$$\begin{aligned} \mathcal{G}_f^{cr} - \mathcal{A}_f \mathcal{G}_f^{cr} \mathcal{A}_f^T &= \mathcal{B}_f \mathcal{B}_f^T \\ \mathcal{E}_b \mathcal{G}_\infty^{nr} \mathcal{E}_b^T - \mathcal{G}_\infty^{nr} &= -\mathcal{B}_b \mathcal{B}_b^T \end{aligned}$$

- $\mathcal{A}_f$, $\mathcal{E}_b$, $\mathcal{B}_f$, $\mathcal{B}_b$, $\mathcal{G}_f^{cr}$ and $\mathcal{G}_\infty^{nr}$ preserve the structures of (lre1) along with their forms in (kron1) and (kron2).
- $$Y_k^{-1} G_k^r Y_k^{-T} = Y_k^{-1} \begin{bmatrix} G_k^{cr} & 0 \\ 0 & G_k^{nr} \end{bmatrix} Y_k^{-T} = \begin{bmatrix} G_{k,f}^{cr} & 0 \\ 0 & G_{k,\infty}^{nr} \end{bmatrix}$$

Solving projected Lyapunov equations

Solutions of Lyapunov equations

Reformulated lifted structure of GDPLEs

$$\begin{aligned} \mathcal{G}_f^{co} - \sigma\mathcal{A}_f^T \mathcal{G}_f^{co} \sigma\mathcal{A}_f &= \quad \sigma\mathcal{C}_f^T \sigma\mathcal{C}_f \\ \sigma\mathcal{E}_b^T \mathcal{G}_\infty^{no} \sigma\mathcal{E}_b - \mathcal{G}_\infty^{no} &= \ -\sigma\mathcal{C}_b^T \sigma\mathcal{C}_b \end{aligned}$$

- $\sigma\mathcal{A}_f$, $\sigma\mathcal{E}_b$, $\sigma\mathcal{C}_f$, and $\sigma\mathcal{C}_b$, $\mathcal{G}_f^{co}$, and $\mathcal{G}_\infty^{no}$ preserve the structures of (lob1) along with their forms in (kron1) and (kron2).
- $$X_k^{-T} G_k^o X_k^{-1} = X_k^{-T} \begin{bmatrix} G_k^{co} & 0 \\ 0 & G_k^{no} \end{bmatrix} X_k^{-1} = \begin{bmatrix} G_{k,f}^{co} & 0 \\ 0 & G_{k,\infty}^{no} \end{bmatrix}$$

Solving projected Lyapunov equations

Solutions of matrix equations

- rank (G_k^{cr})=rank $(G_k^{co})=n_k^f$
- rank (G_k^{nr})=rank $(G_k^{no})=n_k^\infty$

Full rank Cholosky factors

$$\begin{aligned} G_k^{cr} &= R_k R_k^T, & G_k^{nr} &= \hat{R}_k \hat{R}_k^T \\ G_k^{co} &= L_k^T L_k, & G_k^{no} &= \hat{L}_k^T \hat{L}_k \end{aligned}$$

Hankel singular values

$$\begin{aligned} \xi_{k,j} &= \delta_j(L_k E_{k-1} R_k) \\ \theta_{k,j} &= \delta_j(\hat{L}_{k+1} A_k \hat{R}_k) \end{aligned} ,$$

where $\delta_j(.)$ denotes the singular values of the corresponding matrices

Balanced Truncation MOR

Balanced realization and truncation

- (E_k, A_k, B_k, C_k) balanced $\rightsquigarrow$

$$G_k^{cr} = G_k^{co} = \begin{bmatrix} D_{k,f} & 0 \\ 0 & 0 \end{bmatrix}, \quad G_k^{nr} = G_{k+1}^{no} = \begin{bmatrix} 0 & 0 \\ 0 & D_{k,\infty} \end{bmatrix}$$

  where $D_{k,f}$, $D_{k,\infty}$ are diagonal and $(k = 0, 1, \ldots, K-1)$.
- Find S_k and T_k $(k = 0, 1, \ldots, K-1)$ with $T_K = T_0$, such that

$$(\hat{E}_k, \hat{A}_k, \hat{B}_k, \hat{C}_k) \equiv (S_k^T E_k T_{k+1}, S_k^T A_k T_k, S_k^T B_k, C_k T_k)$$

  is balanced.

Balanced Truncation MOR

Balanced realization and truncation

- Compute the skinny SVDs

$$\begin{aligned} L_k E_{k-1} R_k &= [U_{k,1}, U_{k,2}] \begin{bmatrix} \Sigma_{k,1} & \\ & \Sigma_{k,2} \end{bmatrix} [V_{k,1}, V_{k,2}]^T, \\ \hat{L}_{k+1} A_k \hat{R}_k &= \hat{U}_k \Theta_k \hat{V}_k^T, \end{aligned}$$

where $\Sigma_{k,1} = \text{diag}\,(\xi_{k,1}, \cdots, \xi_{k,\ell_f})$, $\Sigma_{k,2} = \text{diag}\,(\xi_{k,\ell_f+1}, \cdots, \xi_{k,n^f})$, $\Theta_k = \text{diag}\,(\theta_{k,1}, \cdots, \theta_{k,n^\infty})$.

- Compute $\tilde{S}_k$ and $\tilde{T}_k$ for the truncated model

$$\begin{aligned} \tilde{S}_k &= [L_{k+1}^T U_{k+1,1} \Sigma_{k+1,1}^{-1/2}, \hat{L}_{k+1}^T \hat{U}_k \Theta_k^{-1/2}] \\ \tilde{T}_k &= [R_k V_{k,1} \Sigma_{k,1}^{-1/2}, \hat{R}_k \hat{V}_k \Theta_k^{-1/2}] \end{aligned}$$

Balanced Truncation MOR

Reduced model

- Compute the reduced order model

$$(\tilde{E}_k, \tilde{A}_k, \tilde{B}_k, \tilde{C}_k) \equiv (\tilde{S}_k^T E_k \tilde{T}_{k+1}, \tilde{S}_k^T A_k \tilde{T}_k, \tilde{S}_k^T B_k, C_k \tilde{T}_k)$$

for $(k = 0, 1, \cdots, K-1)$.

Error bound:

$$\| \mathcal{G} - \tilde{\mathcal{G}} \|_\infty \leq 2 \sum_{k=0}^{K-1} \sum_{i=\ell_f+1}^{n^f} \xi_{k,i}$$

Balanced Truncation MOR

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Numerical results and discussion

Numerical results

Numerical example (Chu, et. al.'07)

Model discrete-time descriptor system with $n = 10$, $m = 2$, $p = 3$ and period $K = 3$ $(k = 0, 1, 2)$.

- $\{(E_k, A_k)\}_{k=0}^{K-1}$ are pd-stable.
- $n_k^f = 8$ and $n_k^\infty = 2$ for all k.
- Reduced order model, $r = (7, 8, 8)$.

Numerical results and discussion

Numerical results

Norms and Relative residuals (γ) of Grammians

Norms and relative residuals of reachability Gramians

k	$\| G_k^{cr} \|_2$	γ_k^{cr}	$\| G_k^{nr} \|_2$	γ_k^{nr}
0	5.8182×10^2	6.1727×10^{-12}	1.3946×10^1	1.5444×10^{-14}
1	8.2981×10^4	8.2172×10^{-12}	1.3660×10^1	1.7508×10^{-14}
2	7.1107×10^3	3.0961×10^{-12}	1.4308×10^1	3.3847×10^{-14}

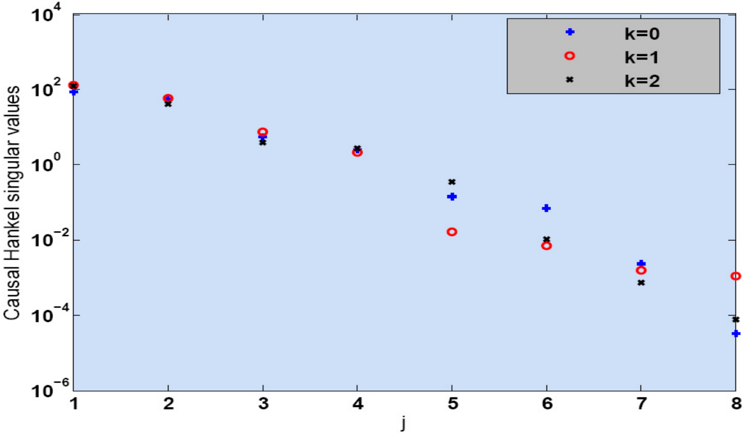
Norms and relative residuals of observability Gramians

k	$\| G_k^{co} \|_2$	γ_k^{co}	$\| G_k^{no} \|_2$	γ_k^{no}
0	9.7353×10^1	2.7678×10^{-13}	1.6866×10^0	1.3372×10^{-15}
1	1.1373×10^3	7.7003×10^{-14}	1.7406×10^0	2.1113×10^{-15}
2	9.6984×10^0	1.7859×10^{-14}	1.6866×10^0	1.1626×10^{-15}

Numerical results and discussion

Numerical results

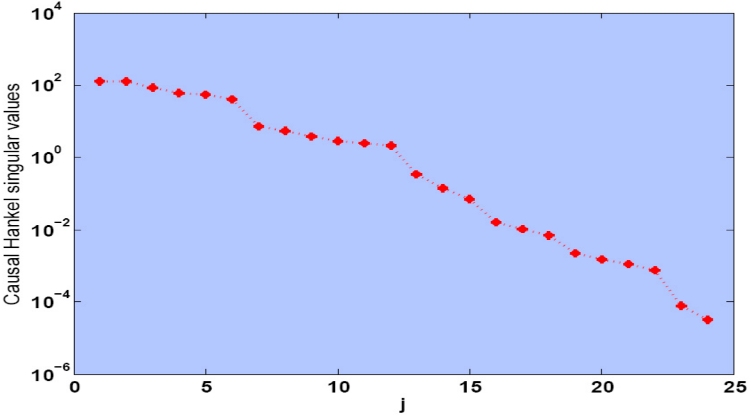
Causal Hankel singular values for the descriptor system



Numerical results and discussion

Numerical results

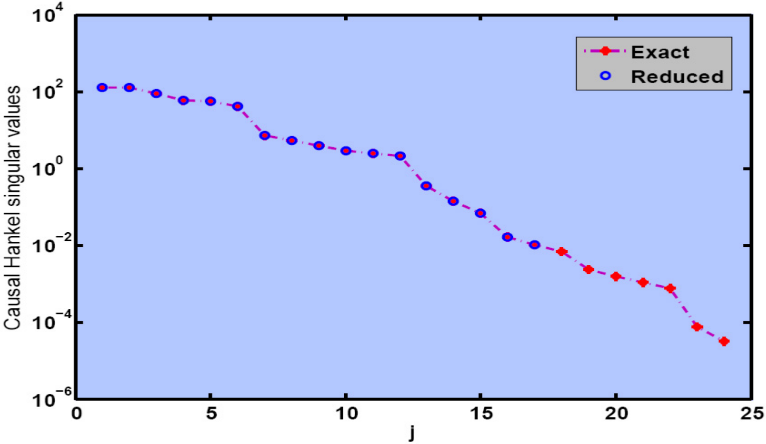
Hankel singular values for the Exact lifted system



Numerical results and discussion

Numerical results

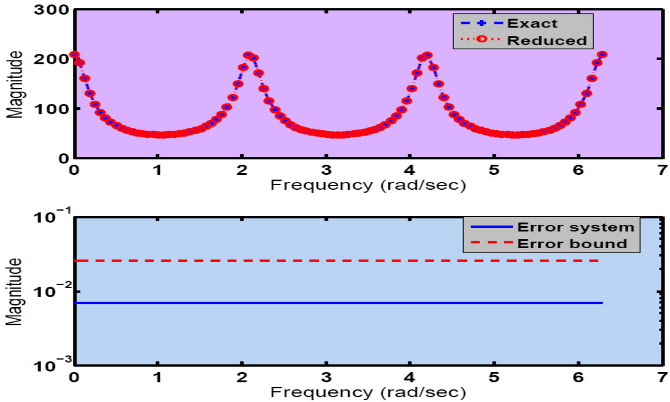
Hankel singular values for the Reduced lifted system



Numerical results and discussion

Numerical results

Norms of Exact and Reduced transfer functions



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- Not suitable for very large dimensional problems.
  - Iterative techniques find low-rank solutions of large Lyapunov equations.(....next goal)

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